

# Financial Risk Management Tutorial Class — Session 2

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# Single and multi-name credit default swaps

## Question 1

We assume that the default time  $\tau$  follows an exponential distribution with parameter  $\lambda$ . Write the cumulative distribution function  $\mathbf{F}$ , the survival function  $\mathbf{S}$  and the density function  $f$  of the random variable  $\tau$ . How do we simulate this default time?

# Single and multi-name credit default swaps

We have  $\mathbf{F}(t) = 1 - e^{-\lambda t}$ ,  $\mathbf{S}(t) = e^{-\lambda t}$  and  $f(t) = \lambda e^{-\lambda t}$ . We know that  $\mathbf{S}(\tau) \sim \mathcal{U}_{[0,1]}$ . Indeed, we have:

$$\begin{aligned}\Pr\{U \leq u\} &= \Pr\{\mathbf{S}(\tau) \leq u\} \\ &= \Pr\{\tau \geq \mathbf{S}^{-1}(u)\} \\ &= \mathbf{S}(\mathbf{S}^{-1}(u)) \\ &= u\end{aligned}$$

It follows that  $\tau = \mathbf{S}^{-1}(U)$  with  $U \sim \mathcal{U}_{[0,1]}$ . Let  $u$  be a uniform random variate. Simulating  $\tau$  is then equivalent to transform  $u$  into  $t$ :

$$t = -\frac{1}{\lambda} \ln u$$

# Single and multi-name credit default swaps

## Question 2

We consider a CDS 3M with two-year maturity and \$1 mn notional principal. The recovery rate  $\mathcal{R}$  is equal to 40% whereas the spread  $s$  is equal to 150 bps at the inception date. We assume that the protection leg is paid at the default time.

# Single and multi-name credit default swaps

## Question 2.a

Give the cash flow chart. What is the P&L of the protection seller  $A$  if the reference entity does not default? What is the PnL of the protection buyer  $B$  if the reference entity defaults in one year and two months?

# Single and multi-name credit default swaps

The premium leg is paid quarterly. The coupon payment is then equal to:

$$\begin{aligned}\mathcal{PL}(t_m) &= \Delta t_m \times s \times N \\ &= \frac{1}{4} \times 150 \times 10^{-4} \times 10^6 \\ &= \$3\,750\end{aligned}$$

In case of default, the default leg paid by protection seller is equal to:

$$\begin{aligned}\mathcal{DL} &= (1 - \mathcal{R}) \times N \\ &= (1 - 40\%) \times 10^6 \\ &= \$600\,000\end{aligned}$$

# Single and multi-name credit default swaps

The corresponding cash flow chart is given in Figure 1. If the reference entity does not default, the P&L of the protection seller is the sum of premium interests:

$$\Pi^{\text{seller}} = 8 \times 3750 = \$30\,000$$

If the reference entity defaults in one year and two months, the P&L of the protection buyer is<sup>1</sup>:

$$\begin{aligned} \Pi^{\text{buyer}} &= (1 - \mathcal{R}) \times N - \sum_{t_m < \tau} \Delta t_m \times s \times N \\ &= (1 - 40\%) \times 10^6 - \left(4 + \frac{2}{3}\right) \times 3750 \\ &= \$582\,500 \end{aligned}$$

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<sup>1</sup>We include the accrued premium.

# Single and multi-name credit default swaps

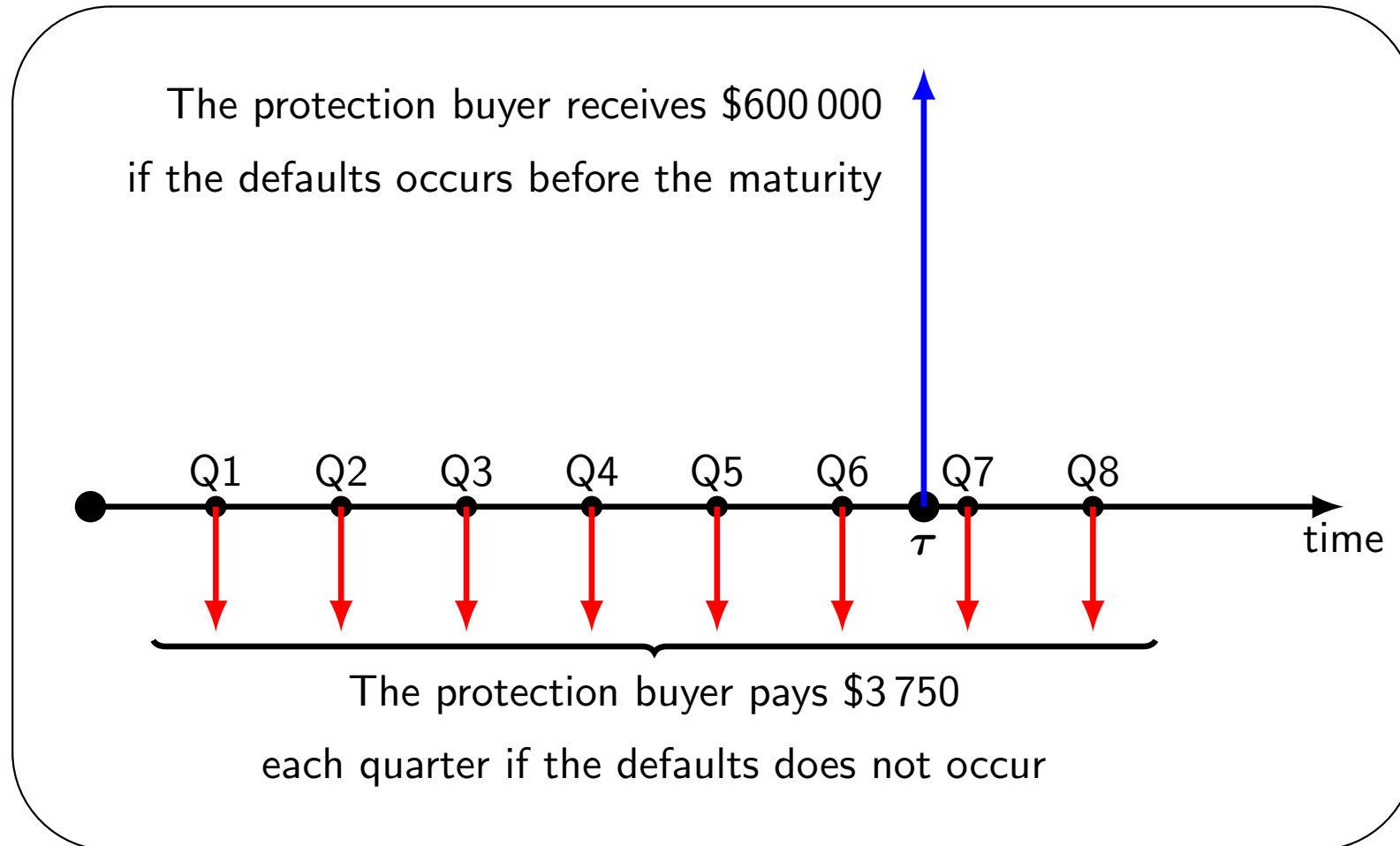


Figure 1: Cash flow chart of the CDS contract



# Single and multi-name credit default swaps

## Question 2.b

What is the relationship between  $s$ ,  $\mathcal{R}$  and  $\lambda$ ? What is the implied one-year default probability at the inception date?

# Single and multi-name credit default swaps

Using the credit triangle relationship, we have:

$$s \simeq (1 - \mathcal{R}) \times \lambda$$

We deduce that<sup>2</sup>:

$$\begin{aligned} \text{PD} &\simeq \lambda \\ &\simeq \frac{s}{1 - \mathcal{R}} \\ &= \frac{150 \times 10^{-4}}{1 - 40\%} \\ &= 2.50\% \end{aligned}$$

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<sup>2</sup>We recall that the one-year default probability is approximately equal to  $\lambda$ :

$$\begin{aligned} \text{PD} &= 1 - \mathbf{S}(1) \\ &= 1 - e^{-\lambda} \\ &\simeq 1 - (1 - \lambda) \\ &\simeq \lambda \end{aligned}$$

# Single and multi-name credit default swaps

## Question 2.c

Seven months later, the CDS spread has increased and is equal to 450 bps. Estimate the new default probability. The protection buyer  $B$  decides to realize his P&L. For that, he reassigns the CDS contract to the counterparty  $C$ . Explain the offsetting mechanism if the risky PV01 is equal to 1.189.

# Single and multi-name credit default swaps

We denote by  $s'$  the new CDS spread. The default probability becomes:

$$\begin{aligned} \text{PD} &= \frac{s'}{1 - \mathcal{R}} \\ &= \frac{450 \times 10^{-4}}{1 - 40\%} \\ &= 7.50\% \end{aligned}$$

The protection buyer is short credit and benefits from the increase of the default probability. His mark-to-market is therefore equal to:

$$\begin{aligned} \Pi^{\text{buyer}} &= N \times (s' - s) \times \text{RPV}_{01} \\ &= 10^6 \times (450 - 150) \times 10^{-4} \times 1.189 \\ &= \$35\,671 \end{aligned}$$

The offsetting mechanism is then the following: the protection buyer  $B$  transfers the agreement to  $C$ , who becomes the new protection buyer;  $C$  continues to pay a premium of 150 bps to the protection seller  $A$ ; in return,  $C$  pays a cash adjustment of \$35 671 to  $B$ .

# Single and multi-name credit default swaps

## Question 3

We consider the following CDS spread curves for three reference entities:

Maturity	#1	#2	#3
6M	130 bps	1 280 bps	30 bps
1Y	135 bps	970 bps	35 bps
3Y	140 bps	750 bps	50 bps
5Y	150 bps	600 bps	80 bps

# Single and multi-name credit default swaps

## Question 3.a

Define the notion of credit curve. Comment the previous spread curves.

# Single and multi-name credit default swaps

For a given date  $t$ , the credit curve is the relationship between the maturity  $T$  and the spread  $s_t(T)$ . The credit curve of the reference entity #1 is almost flat. For the entity #2, the spread is very high in the short-term, meaning that there is a significant probability that the entity defaults. However, if the entity survive, the market anticipates that it will improve its financial position in the long-run. This explains that the credit curve #2 is decreasing. For reference entity #3, we obtain opposite conclusions. The company is actually very strong, but there are some uncertainties in the future<sup>3</sup>. The credit curve is then increasing.

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<sup>3</sup>An example is a company whose has a monopoly because of a strong technology, but faces a hard competition because technology is evolving fast in its domain (e.g. Blackberry at the end of 2000s).

# Single and multi-name credit default swaps

## Question 3.b

Using the Merton Model, we estimate that the one-year default probability is equal to 2.5% for #1, 5% for #2 and 2% for #3 at a five-year horizon time. Which arbitrage position could we consider about the reference entity #2?



# Single and multi-name credit default swaps

If we consider a standard recovery rate (40%), the implied default probability is 2.50% for #1, 10% for #2 and 1.33% for #3. We can consider a short credit position in #2. In this case, we sell the 5Y protection on #2 because the model tells us that the market default probability is over-estimated. In place of this directional bet, we could consider a relative value strategy: selling the 5Y protection on #2 and buying the 5Y protection on #3.

# Single and multi-name credit default swaps

## Question 4

We consider a basket of  $n$  single-name CDS.

# Single and multi-name credit default swaps

## Question 4.a

What is a first-to-default (FtD), a second-to-default (StD) and a last-to-default (LtD)?

# Single and multi-name credit default swaps

Let  $\tau_{k:n}$  be the  $k^{\text{th}}$  default among the basket. FtD, StD and LtD are three CDS products, whose credit event is related to the default times  $\tau_{1:n}$ ,  $\tau_{2:n}$  and  $\tau_{n:n}$ .

# Single and multi-name credit default swaps

## Question 4.b

Define the notion of default correlation. What is its impact on three previous spreads?

# Single and multi-name credit default swaps

The default correlation  $\rho$  measures the dependence between two default times  $\tau_i$  and  $\tau_j$ . The spread of the FtD (resp. LtD) is a decreasing (resp. increasing) function with respect to  $\rho$ .

# Single and multi-name credit default swaps

## Question 4.c

We assume that  $n = 3$ . Show the following relationship:

$$s_1^{\text{CDS}} + s_2^{\text{CDS}} + s_3^{\text{CDS}} = s^{\text{FtD}} + s^{\text{StD}} + s^{\text{LtD}}$$

where  $s_i^{\text{CDS}}$  is the CDS spread of the  $i^{\text{th}}$  reference entity.

# Single and multi-name credit default swaps

To fully hedge the credit portfolio of the 3 entities, we can buy the 3 CDS. Another solution is to buy the FtD plus the StD and the LtD (or the third-to-default). Because these two hedging strategies are equivalent, we deduce that:

$$s_1^{\text{CDS}} + s_2^{\text{CDS}} + s_3^{\text{CDS}} = s^{\text{FtD}} + s^{\text{StD}} + s^{\text{LtD}}$$



# Single and multi-name credit default swaps

## Question 4.d

Many professionals and academics believe that the subprime crisis is due to the use of the Normal copula. Using the results of the previous question, what could you conclude?

# Single and multi-name credit default swaps

We notice that the default correlation does not affect the value of the CDS basket, but only the price distribution between FtD, StD and LtD. We obtain a similar result for CDO<sup>4</sup>. In the case of the subprime crisis, all the CDO tranches have suffered, meaning that the price of the underlying basket has dropped. The reasons were the underestimation of default probabilities.

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<sup>4</sup>The junior, mezzanine and senior tranches can be viewed as FtD, StD and LtD.

# Risk contribution in the Basel II model

## Question 1

We note  $L$  the portfolio loss of  $n$  credit and  $w_i$  the exposure at default of the  $i^{\text{th}}$  credit. We have:

$$L(w) = w^{\top} \varepsilon = \sum_{i=1}^n w_i \times \varepsilon_i \quad (1)$$

where  $\varepsilon_i$  is the unit loss of the  $i^{\text{th}}$  credit. Let  $\mathbf{F}$  be the cumulative distribution function of  $L(w)$ .

# Risk contribution in the Basel II model

## Question 1.a

We assume that  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \sim \mathcal{N}(\mathbf{0}, \Sigma)$ . Compute the value-at-risk  $\text{VaR}_\alpha(w)$  of the portfolio when the confidence level is equal to  $\alpha$ .

# Risk contribution in the Basel II model

The portfolio loss  $L$  follows a Gaussian probability distribution:

$$L(w) \sim \mathcal{N}\left(0, \sqrt{w^\top \Sigma w}\right)$$

We deduce that:

$$\text{VaR}_\alpha(w) = \Phi^{-1}(\alpha) \sqrt{w^\top \Sigma w}$$

# Risk contribution in the Basel II model

## Question 1.b

Deduce the marginal value-at-risk of the  $i^{\text{th}}$  credit. Define then the risk contribution  $\mathcal{RC}_i$  of the  $i^{\text{th}}$  credit.

# Risk contribution in the Basel II model

We have:

$$\begin{aligned} \frac{\partial \text{VaR}_\alpha(w)}{\partial w} &= \frac{\partial}{\partial w} \left( \Phi^{-1}(\alpha) (w^\top \Sigma w)^{\frac{1}{2}} \right) \\ &= \Phi^{-1}(\alpha) \frac{1}{2} (w^\top \Sigma w)^{-\frac{1}{2}} (2\Sigma w) \\ &= \Phi^{-1}(\alpha) \frac{\Sigma w}{\sqrt{w^\top \Sigma w}} \end{aligned}$$

The marginal value-at-risk of the  $i^{\text{th}}$  credit is then:

$$\mathcal{MR}_i = \frac{\partial \text{VaR}_\alpha(w)}{\partial w_i} = \Phi^{-1}(\alpha) \frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}}$$

The risk contribution of the  $i^{\text{th}}$  credit is the product of the exposure by the marginal risk:

$$\begin{aligned} \mathcal{RC}_i &= w_i \times \mathcal{MR}_i \\ &= \Phi^{-1}(\alpha) \frac{w_i \times (\Sigma w)_i}{\sqrt{w^\top \Sigma w}} \end{aligned}$$

# Risk contribution in the Basel II model

## Question 1.c

Check that the marginal value-at-risk is equal to:

$$\frac{\partial \text{VaR}_\alpha(w)}{\partial w_i} = \mathbb{E}[\varepsilon_i \mid L(w) = \mathbf{F}^{-1}(\alpha)]$$

Comment on this result.



# Risk contribution in the Basel II model

By construction, the random vector  $(\varepsilon, L(w))$  is Gaussian with:

$$\begin{pmatrix} \varepsilon \\ L(w) \end{pmatrix} \sim \mathcal{N} \left( \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & \Sigma w \\ w^\top \Sigma & w^\top \Sigma w \end{pmatrix} \right)$$

We deduce that the conditional distribution function of  $\varepsilon$  given that  $L(w) = \ell$  is Gaussian and we have:

$$\mathbb{E}[\varepsilon \mid L(w) = \ell] = \mathbf{0} + \Sigma w (w^\top \Sigma w)^{-1} (\ell - 0)$$

We finally obtain:

$$\begin{aligned} \mathbb{E}[\varepsilon \mid L(w) = \mathbf{F}^{-1}(\alpha)] &= \Sigma w (w^\top \Sigma w)^{-1} \Phi^{-1}(\alpha) \sqrt{w^\top \Sigma w} \\ &= \Phi^{-1}(\alpha) \frac{\Sigma w}{\sqrt{w^\top \Sigma w}} \\ &= \frac{\partial \text{VaR}_\alpha(w)}{\partial w} \end{aligned}$$

The marginal VaR of the  $i^{\text{th}}$  credit is then equal to the conditional mean of the individual loss  $\varepsilon_i$  given that the portfolio loss is exactly equal to the value-at-risk.

# Risk contribution in the Basel II model

## Question 2

We consider the Basel II model of credit risk and the value-at-risk risk measure. The expression of the portfolio loss is given by:

$$L = \sum_{i=1}^n \text{EAD}_i \times \text{LGD}_i \times \mathbb{1} \{ \tau_i < M_i \} \quad (2)$$

# Risk contribution in the Basel II model

## Question 2.a

Define the different parameters  $EAD_i$ ,  $LGD_i$ ,  $\tau_i$  and  $M_i$ . Show that Model (2) can be written as Model (1) by identifying  $w_i$  and  $\varepsilon_i$ .

# Risk contribution in the Basel II model

$EAD_i$  is the exposure at default,  $LGD_i$  is the loss given default,  $\tau_i$  is the default time and  $T_i$  is the maturity of the credit  $i$ . We have:

$$\begin{cases} w_i = EAD_i \\ \varepsilon_i = LGD_i \times \mathbb{1} \{ \tau_i < T_i \} \end{cases}$$

The exposure at default is not random, which is not the case of the loss given default.

# Risk contribution in the Basel II model

## Question 2.b

What are the necessary assumptions ( $\mathcal{H}$ ) to obtain this result:

$$\mathbb{E} [\varepsilon_i \mid L = \mathbf{F}^{-1}(\alpha)] = \mathbb{E} [\text{LGD}_i] \times \mathbb{E} [D_i \mid L = \mathbf{F}^{-1}(\alpha)]$$

with  $D_i = \mathbb{1} \{ \tau_i < M_i \}$ .

# Risk contribution in the Basel II model

We have to make the following assumptions:

- (i) the loss given default  $\text{LGD}_i$  is independent from the default time  $\tau_i$ ;
- (ii) the portfolio is infinitely fine-grained meaning that there is no exposure concentration:

$$\frac{\text{EAD}_i}{\sum_{i=1}^n \text{EAD}_i} \simeq 0$$

- (iii) the default times depend on a common risk factor  $X$  and the relationship is monotonic (increasing or decreasing).

In this case, we have:

$$\mathbb{E} [\varepsilon_i \mid L = \mathbf{F}^{-1}(\alpha)] = \mathbb{E} [\text{LGD}_i] \times \mathbb{E} [D_i \mid L = \mathbf{F}^{-1}(\alpha)]$$

with  $D_i = \mathbb{1} \{ \tau_i < T_i \}$ .

# Risk contribution in the Basel II model

## Question 2.c

Deduce the risk contribution  $\mathcal{RC}_i$  of the  $i^{\text{th}}$  credit and the value-at-risk of the credit portfolio.

# Risk contribution in the Basel II model

It follows that:

$$\begin{aligned}\mathcal{RC}_i &= w_i \times \mathcal{MR}_i \\ &= \text{EAD}_i \times \mathbb{E}[\text{LGD}_i] \times \mathbb{E}[D_i \mid L = \mathbf{F}^{-1}(\alpha)]\end{aligned}$$

The expression of the value-at-risk is then:

$$\begin{aligned}\text{VaR}_\alpha(w) &= \sum_{i=1}^n \mathcal{RC}_i \\ &= \sum_{i=1}^n \text{EAD}_i \times \mathbb{E}[\text{LGD}_i] \times \mathbb{E}[D_i \mid L = \mathbf{F}^{-1}(\alpha)]\end{aligned}$$



# Risk contribution in the Basel II model

## Question 2.d

We assume that the credit  $i$  defaults before the maturity  $M_i$  if a latent variable  $Z_i$  goes below a barrier  $B_i$ :

$$\tau_i \leq M_i \Leftrightarrow Z_i \leq B_i$$

We consider that  $Z_i = \sqrt{\rho}X + \sqrt{1 - \rho}\varepsilon_i$  where  $Z_i$ ,  $X$  and  $\varepsilon_i$  are three independent Gaussian variables  $\mathcal{N}(0, 1)$ .  $X$  is the factor (or the systematic risk) and  $\varepsilon_i$  is the idiosyncratic risk.

# Risk contribution in the Basel II model

## Question 2.d (i)

Interpret the parameter  $\rho$ .

# Risk contribution in the Basel II model

We have

$$\begin{aligned}\mathbb{E}[Z_i Z_j] &= \mathbb{E}\left[\left(\sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_i\right)\left(\sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_j\right)\right] \\ &= \rho\end{aligned}$$

$\rho$  is the constant correlation between assets  $Z_i$  and  $Z_j$ .

# Risk contribution in the Basel II model

## Question 2.d (ii)

Calculate the unconditional default probability:

$$p_i = \Pr \{ \tau_i \leq M_i \}$$

# Risk contribution in the Basel II model

We have:

$$\begin{aligned} p_i &= \Pr \{ \tau_i \leq T_i \} \\ &= \Pr \{ Z_i \leq B_i \} \\ &= \Phi ( B_i ) \end{aligned}$$

# Risk contribution in the Basel II model

## Question 2.d (iii)

Calculate the conditional default probability:

$$p_i(x) = \Pr\{\tau_i \leq M_i \mid X = x\}$$

# Risk contribution in the Basel II model

It follows that:

$$\begin{aligned} p_i(x) &= \Pr \{ Z_i \leq B_i \mid X = x \} \\ &= \Pr \left\{ \sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_i \leq B_i \mid X = x \right\} \\ &= \Pr \left\{ \varepsilon_i \leq \frac{B_i - \sqrt{\rho}X}{\sqrt{1-\rho}} \mid X = x \right\} \\ &= \Phi \left( \frac{B_i - \sqrt{\rho}x}{\sqrt{1-\rho}} \right) \\ &= \Phi \left( \frac{\Phi^{-1}(p_i) - \sqrt{\rho}x}{\sqrt{1-\rho}} \right) \end{aligned}$$

# Risk contribution in the Basel II model

## Question 2.e

Show that, under the previous assumptions ( $\mathcal{H}$ ), the risk contribution  $\mathcal{RC}_i$  of the  $i^{\text{th}}$  credit is:

$$\mathcal{RC}_i = \text{EAD}_i \times \mathbb{E}[\text{LGD}_i] \times \Phi \left( \frac{\Phi^{-1}(p_i) + \sqrt{\rho} \Phi^{-1}(\alpha)}{\sqrt{1 - \rho}} \right) \quad (3)$$

when the risk measure is the value-at-risk.



# Risk contribution in the Basel II model

Under the assumptions  $(\mathcal{H})$ , we know that:

$$\begin{aligned}
 L &= \sum_{i=1}^n \text{EAD}_i \times \mathbb{E}[\text{LGD}_i] \times p_i(X) \\
 &= \sum_{i=1}^n \text{EAD}_i \times \mathbb{E}[\text{LGD}_i] \times \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho}X}{\sqrt{1-\rho}}\right) \\
 &= g(X)
 \end{aligned}$$

with  $g'(x) < 0$ . We deduce that:

$$\begin{aligned}
 \text{VaR}_\alpha(w) = \mathbf{F}^{-1}(\alpha) &\Leftrightarrow \Pr\{g(X) \leq \text{VaR}_\alpha(w)\} = \alpha \\
 &\Leftrightarrow \Pr\{X \geq g^{-1}(\text{VaR}_\alpha(w))\} = \alpha \\
 &\Leftrightarrow \Pr\{X \leq g^{-1}(\text{VaR}_\alpha(w))\} = 1 - \alpha \\
 &\Leftrightarrow g^{-1}(\text{VaR}_\alpha(w)) = \Phi^{-1}(1 - \alpha) \\
 &\Leftrightarrow \text{VaR}_\alpha(w) = g(\Phi^{-1}(1 - \alpha))
 \end{aligned}$$

# Risk contribution in the Basel II model

It follows that:

$$\begin{aligned} \text{VaR}_\alpha(w) &= g(\Phi^{-1}(1 - \alpha)) \\ &= \sum_{i=1}^n \text{EAD}_i \times \mathbb{E}[\text{LGD}_i] \times p_i(\Phi^{-1}(1 - \alpha)) \end{aligned}$$

The risk contribution  $\mathcal{RC}_i$  of the  $i$ th credit is then:

$$\begin{aligned} \mathcal{RC}_i &= \text{EAD}_i \times \mathbb{E}[\text{LGD}_i] \times p_i(\Phi^{-1}(1 - \alpha)) \\ &= \text{EAD}_i \times \mathbb{E}[\text{LGD}_i] \times \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho}\Phi^{-1}(1 - \alpha)}{\sqrt{1 - \rho}}\right) \\ &= \text{EAD}_i \times \mathbb{E}[\text{LGD}_i] \times \Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1 - \rho}}\right) \end{aligned}$$

# Risk contribution in the Basel II model

## Question 3

We now assume that the risk measure is the expected shortfall:

$$ES_{\alpha}(w) = \mathbb{E}[L \mid L \geq \text{VaR}_{\alpha}(w)]$$

# Risk contribution in the Basel II model

## Question 3.a

In the case of the Basel II framework, show that we have:

$$\text{ES}_\alpha(w) = \sum_{i=1}^n \text{EAD}_i \times \mathbb{E}[\text{LGD}_i] \times \mathbb{E}[p_i(X) \mid X \leq \Phi^{-1}(1 - \alpha)]$$

# Risk contribution in the Basel II model

We note  $\Omega$  the event  $X \leq g^{-1}(\text{VaR}_\alpha(w))$  or equivalently  $X \leq \Phi^{-1}(1 - \alpha)$ . We have:

$$\begin{aligned}
 \text{ES}_\alpha(w) &= \mathbb{E}[L \mid L \geq \text{VaR}_\alpha(w)] \\
 &= \mathbb{E}[L \mid g(X) \geq \text{VaR}_\alpha(w)] \\
 &= \mathbb{E}[L \mid X \leq g^{-1}(\text{VaR}_\alpha(w))] \\
 &= \mathbb{E}\left[\sum_{i=1}^n \text{EAD}_i \times \mathbb{E}[\text{LGD}_i] \times p_i(X) \mid \Omega\right] \\
 &= \sum_{i=1}^n \text{EAD}_i \times \mathbb{E}[\text{LGD}_i] \times \mathbb{E}[p_i(X) \mid \Omega]
 \end{aligned}$$

# Risk contribution in the Basel II model

## Question 3.b

By using the following result:

$$\int_{-\infty}^c \Phi(a + bx)\phi(x) dx = \Phi_2\left(c, \frac{a}{\sqrt{1+b^2}}; \frac{-b}{\sqrt{1+b^2}}\right)$$

where  $\Phi_2(x, y; \rho)$  is the cdf of the bivariate Gaussian distribution with correlation  $\rho$  on the space  $[-\infty, x] \times [-\infty, y]$ , deduce that the risk contribution  $\mathcal{RC}_i$  of the  $i^{\text{th}}$  credit in the Basel II model is:

$$\mathcal{RC}_i = \text{EAD}_i \times \mathbb{E}[\text{LGD}_i] \times \frac{\mathbf{C}(1 - \alpha, p_i; \sqrt{\rho})}{1 - \alpha} \quad (4)$$

when the risk measure is the expected shortfall. Here  $\mathbf{C}(u_1, u_2; \theta)$  is the Normal copula with parameter  $\theta$ .

# Risk contribution in the Basel II model

It follows that:

$$\begin{aligned}
 \mathbb{E}[p_i(X) | \Omega] &= \mathbb{E} \left[ \Phi \left( \frac{\Phi^{-1}(p_i) - \sqrt{\rho}X}{\sqrt{1-\rho}} \right) \middle| \Omega \right] \\
 &= \int_{-\infty}^{\Phi^{-1}(1-\alpha)} \Phi \left( \frac{\Phi^{-1}(p_i)}{\sqrt{1-\rho}} + \frac{-\sqrt{\rho}}{\sqrt{1-\rho}}x \right) \times \\
 &\quad \frac{\phi(x)}{\Phi(\Phi^{-1}(1-\alpha))} dx \\
 &= \frac{\Phi_2(\Phi^{-1}(1-\alpha), \Phi^{-1}(p_i); \sqrt{\rho})}{1-\alpha} \\
 &= \frac{\mathbf{C}(1-\alpha, p_i; \sqrt{\rho})}{1-\alpha}
 \end{aligned}$$

where  $\mathbf{C}$  is the Gaussian copula. We deduce that:

$$\mathcal{RC}_i = \text{EAD}_i \times \mathbb{E}[\text{LGD}_i] \times \frac{\mathbf{C}(1-\alpha, p_i; \sqrt{\rho})}{1-\alpha}$$

# Risk contribution in the Basel II model

## Question 3.c

What become the results (3) and (4) if the correlation  $\rho$  is equal to zero?  
Same question if  $\rho = 1$ .



# Risk contribution in the Basel II model

If  $\rho = 0$ , we have:

$$\begin{aligned} \Phi \left( \frac{\Phi^{-1}(p_i) + \sqrt{\rho} \Phi^{-1}(\alpha)}{\sqrt{1-\rho}} \right) &= \Phi(\Phi^{-1}(p_i)) \\ &= p_i \end{aligned}$$

and:

$$\begin{aligned} \frac{\mathbf{C}(1-\alpha, p_i; \sqrt{\rho})}{1-\alpha} &= \frac{(1-\alpha)p_i}{1-\alpha} \\ &= p_i \end{aligned}$$

The risk contribution is the same for the value-at-risk and the expected shortfall:

$$\begin{aligned} \mathcal{RC}_i &= \text{EAD}_i \times \mathbb{E}[\text{LGD}_i] \times p_i \\ &= \mathbb{E}[L_i] \end{aligned}$$

It corresponds to the expected loss of the credit.

# Risk contribution in the Basel II model

If  $\rho = 1$  and  $\alpha > 50\%$ , we have:

$$\begin{aligned} \Phi \left( \frac{\Phi^{-1}(p_i) + \sqrt{\rho} \Phi^{-1}(\alpha)}{\sqrt{1-\rho}} \right) &= \lim_{\rho \rightarrow 1} \Phi \left( \frac{\Phi^{-1}(p_i) + \Phi^{-1}(\alpha)}{\sqrt{1-\rho}} \right) \\ &= 1 \end{aligned}$$

If  $\rho = 1$  and  $\alpha$  is high ( $\alpha > 1 - \sup_i p_i$ ), we have:

$$\begin{aligned} \frac{\mathbf{C}(1-\alpha, p_i; \sqrt{\rho})}{1-\alpha} &= \frac{\min(1-\alpha; p_i)}{1-\alpha} \\ &= 1 \end{aligned}$$

In this case, the risk contribution is the same for the value-at-risk and the expected shortfall:

$$\mathcal{RC}_i = \text{EAD}_i \times \mathbb{E}[\text{LGD}_i]$$

However, it does not depend on the unconditional probability of default  $p_i$ .

# Risk contribution in the Basel II model

## Question 4

The risk contributions (3) and (4) were obtained considering the assumptions ( $\mathcal{H}$ ) and the default model defined in Question 2(d). What are the implications in terms of Pillar 2?

# Risk contribution in the Basel II model

Pillar 2 concerns the non-compliance of assumptions ( $\mathcal{H}$ ). In particular, we have to understand the impact on the credit risk measure if the portfolio is not infinitely fine-grained or if asset correlations are not constant.

# Modeling loss given default

## Question 1

What is the difference between the recovery rate and the loss given default?

# Modeling loss given default

The loss given default is equal to:

$$\text{LGD} = 1 - \mathcal{R} + c$$

where  $c$  is the recovery (or litigation) cost. Consider for example a \$200 credit and suppose that the borrower defaults. If we recover \$140 and the litigation cost is \$20, we obtain  $\mathcal{R} = 70\%$  and  $\text{LGD} = 40\%$ , but not  $\text{LGD} = 30\%$ .

# Modeling loss given default

## Question 2

We consider a bank that grants 250 000 credits per year. The average amount of a credit is equal to \$50 000. We estimate that the average default probability is equal to 1% and the average recovery rate is equal to 65%. The total annual cost of the litigation department is equal to \$12.5 mn. Give an estimation of the loss given default?

# Modeling loss given default

The amounts outstanding of credit is:

$$\begin{aligned} \text{EAD} &= 250\,000 \times 50\,000 \\ &= \$12.5 \text{ bn} \end{aligned}$$

The annual loss after recovery is equal to:

$$\begin{aligned} L &= \text{EAD} \times (1 - \mathcal{R}) \times \text{PD} + C \\ &= 43.75 + 12.5 \\ &= \$56.25 \text{ mn} \end{aligned}$$

where  $C$  is the litigation cost.



# Modeling loss given default

We deduce that:

$$\begin{aligned}\text{LGD} &= \frac{L}{\text{EAD} \times \text{PD}} \\ &= \frac{54}{12.5 \times 10^3 \times 1\%} \\ &= 45\%\end{aligned}$$

This figure is larger than 35%, which is the loss given default without taking into account the recovery cost.

# Modeling loss given default

## Question 3

The probability density function of the beta probability distribution  $\mathcal{B}(a, b)$  is:

$$f(x) = \frac{x^{a-1} (1-x)^{b-1}}{\mathbf{B}(a, b)}$$

where  $\mathbf{B}(a, b) = \int_0^1 u^{a-1} (1-u)^{b-1} du$ .

# Modeling loss given default

## Question 3.a

Why is the beta probability distribution a good candidate to model the loss given default? Which parameter pair  $(a, b)$  correspond to the uniform probability distribution?

# Modeling loss given default

The Beta distribution allows to obtain all the forms of LGD (bell curve, inverted-U shaped curve, etc.). The uniform distribution corresponds to the case  $\alpha = 1$  and  $\beta = 1$ . Indeed, we have:

$$\begin{aligned} f(x) &= \frac{x^{\alpha-1} (1-x)^{\beta-1}}{\int_0^1 u^{\alpha-1} (1-u)^{\beta-1} du} \\ &= 1 \end{aligned}$$

# Modeling loss given default

## Question 3.b

Let us consider a sample  $(x_1, \dots, x_n)$  of  $n$  losses in case of default. Write the log-likelihood function. Deduce the first-order conditions of the maximum likelihood estimator.

# Modeling loss given default

We have:

$$\begin{aligned} \ell(\alpha, \beta) &= \sum_{i=1}^n \ln f(x_i) \\ &= -n \ln \mathbf{B}(\alpha, \beta) + (\alpha - 1) \sum_{i=1}^n \ln x_i + (\beta - 1) \sum_{i=1}^n \ln(1 - x_i) \end{aligned}$$

The first-order conditions are:

$$\frac{\partial \ell(\alpha, \beta)}{\partial \alpha} = -n \frac{\partial_{\alpha} \mathbf{B}(\alpha, \beta)}{\mathbf{B}(\alpha, \beta)} + \sum_{i=1}^n \ln x_i = 0$$

and:

$$\frac{\partial \ell(\alpha, \beta)}{\partial \beta} = -n \frac{\partial_{\beta} \mathbf{B}(\alpha, \beta)}{\mathbf{B}(\alpha, \beta)} + \sum_{i=1}^n \ln(1 - x_i) = 0$$

# Modeling loss given default

## Question 3.c

We recall that the first two moments of the beta probability distribution are:

$$\begin{aligned}\mathbb{E}[X] &= \frac{a}{a+b} \\ \sigma^2(X) &= \frac{ab}{(a+b)^2(a+b+1)}\end{aligned}$$

Find the method of moments estimator.

# Modeling loss given default

Let  $\mu_{\text{LGD}}$  and  $\sigma_{\text{LGD}}$  be the mean and standard deviation of the LGD parameter. The method of moments consists in estimating  $\alpha$  and  $\beta$  such that:

$$\frac{\alpha}{\alpha + \beta} = \mu_{\text{LGD}}$$

and:

$$\frac{\alpha\beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)} = \sigma_{\text{LGD}}^2$$

We have:

$$\beta = \alpha \frac{(1 - \mu_{\text{LGD}})}{\mu_{\text{LGD}}}$$

and:

$$(\alpha + \beta)^2 (\alpha + \beta + 1) \sigma_{\text{LGD}}^2 = \alpha\beta$$



# Modeling loss given default

It follows that:

$$\begin{aligned} (\alpha + \beta)^2 &= \left( \alpha + \alpha \frac{(1 - \mu_{\text{LGD}})}{\mu_{\text{LGD}}} \right)^2 \\ &= \frac{\alpha^2}{\mu_{\text{LGD}}^2} \end{aligned}$$

and:

$$\alpha\beta = \frac{\alpha^2}{\mu_{\text{LGD}}^2} \left( \alpha + \alpha \frac{(1 - \mu_{\text{LGD}})}{\mu_{\text{LGD}}} + 1 \right) \sigma_{\text{LGD}}^2 = \alpha^2 \frac{(1 - \mu_{\text{LGD}})}{\mu_{\text{LGD}}}$$

We deduce that:

$$\alpha \left( 1 + \frac{(1 - \mu_{\text{LGD}})}{\mu_{\text{LGD}}} \right) = \frac{(1 - \mu_{\text{LGD}}) \mu_{\text{LGD}}}{\sigma_{\text{LGD}}^2} - 1$$

# Modeling loss given default

We finally obtain:

$$\hat{\alpha}_{\text{MM}} = \frac{\mu_{\text{LGD}}^2 (1 - \mu_{\text{LGD}})}{\sigma_{\text{LGD}}^2} - \mu_{\text{LGD}} \quad (5)$$

$$\hat{\beta}_{\text{MM}} = \frac{\mu_{\text{LGD}} (1 - \mu_{\text{LGD}})^2}{\sigma_{\text{LGD}}^2} - (1 - \mu_{\text{LGD}}) \quad (6)$$

# Modeling loss given default

## Question 4

We consider a risk class  $\mathcal{C}$  corresponding to a customer/product segmentation specific to retail banking. A statistical analysis of 1 000 loss data available for this risk class gives the following results:

LGD <sub>k</sub>	0%	25%	50%	75%	100%
$n_k$	100	100	600	100	100

where  $n_k$  is the number of data corresponding to LGD<sub>k</sub>.

# Modeling loss given default

## Question 4.a

We consider a portfolio of 100 homogeneous credits, which belong to the risk class  $\mathcal{C}$ . The notional is \$10 000 whereas the annual default probability is equal to 1%. Calculate the expected loss of this credit portfolio with a one-year horizon time if we use the previous empirical distribution to model the LGD parameter.

# Modeling loss given default

The mean of the loss given default is equal to:

$$\begin{aligned}\mu_{\text{LGD}} &= \frac{100 \times 0\% + 100 \times 25\% + 600 \times 50\% + \dots}{1000} \\ &= 50\%\end{aligned}$$

The expression of the expected loss is:

$$\text{EL} = \sum_{i=1}^{100} \text{EAD}_i \times \mathbb{E}[\text{LGD}_i] \times \text{PD}_i$$

where  $\text{PD}_i$  is the default probability of credit  $i$ . We finally obtain:

$$\begin{aligned}\text{EL} &= \sum_{i=1}^{100} 10\,000 \times 50\% \times 1\% \\ &= \$5\,000\end{aligned}$$

# Modeling loss given default

## Question 4.b

We assume that the LGD parameter follows a beta distribution  $\mathcal{B}(a, b)$ . Calibrate the parameters  $a$  and  $b$  with the method of moments.

# Modeling loss given default

We have  $\mu_{\text{LGD}} = 50\%$  and:

$$\begin{aligned}
 \sigma_{\text{LGD}} &= \sqrt{\frac{100 \times (0 - 0.5)^2 + 100 \times (0.25 - 0.5)^2 + \dots}{1000}} \\
 &= \sqrt{\frac{2 \times 0.5^2 + 2 \times 0.25^2}{10}} \\
 &= \sqrt{\frac{0.625}{10}} \\
 &= 25\%
 \end{aligned}$$

Using Equations (5) and (6), we deduce that:

$$\begin{aligned}
 \hat{\alpha}_{\text{MM}} &= \frac{0.5^2 \times (1 - 0.5)}{0.25^2} - 0.5 = 1.5 \\
 \hat{\beta}_{\text{MM}} &= \frac{0.5 \times (1 - 0.5)^2}{0.25^2} - (1 - 0.5) = 1.5
 \end{aligned}$$

# Modeling loss given default

## Question 4.c

We assume that the Basel II model is valid. We consider the portfolio described in Question 4(a) and calculate the unexpected loss. What is the impact if we use a uniform probability distribution instead of the calibrated beta probability distribution? Why does this result hold even if we consider different factors to model the default time?



# Modeling loss given default

The previous portfolio is homogeneous and infinitely fine-grained. In this case, we know that the unexpected loss depends on the mean of the loss given default and not on the entire probability distribution. Because the expected value of the calibrated Beta distribution is 50%, there is no difference with the uniform distribution, which has also a mean equal to 50%. This result holds for the Basel model with one factor, and remains true when they are more factors.