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Introduction to Risk Parity and Budgeting

Introduction

The death of Markowitz optimization?

For a long time, investment theory and practice has been summarized as follows. The capital asset pricing model stated that the market portfolio is optimal. During the 1990s, the development of passive management confirmed the work done by William Sharpe. At that same time, the number of institutional investors grew at an impressive pace. Many of these investors used passive management for their equity and bond exposures. For asset allocation, they used the optimization model developed by Harry Markowitz, even though they knew that such an approach was very sensitive to input parameters, and in particular, to expected returns (Merton, 1980). One reason is that there was no other alternative model. Another reason is that the Markowitz model is easy to use and simple to explain. For expected returns, these investors generally considered long-term historical figures, stating that past history can serve as a reliable guide for the future. Management boards of pension funds were won over by this scientific approach to asset allocation.

The first serious warning shot came with the dot-com crisis. Some institutional investors, in particular defined benefit pension plans, lost substantial amounts of money because of their high exposure to equities (Ryan and Fabozzi, 2002). In November 2001, the pension plan of The Boots Company, a UK pharmacy retailer, decided to invest 100% in bonds (Sutcliffe, 2005). Nevertheless, the performance of the equity market between 2003 and 2007 restored confidence that standard financial models would continue to work and that the dot-com crisis was a non-recurring exception. However, the 2008 financial crisis highlighted the risk inherent in many strategic asset allocations. Moreover, for institutional investors, the crisis was unprecedentedly severe. In 2000, the internet crisis was limited to large capitalization stocks and certain sectors. Small capitalizations and value stocks were not affected, while the performance of hedge funds was flat. In 2008, the subprime crisis led to a violent drop in credit strategies and asset-backed securities. Equities posted negative returns of about −50%. The performance of hedge funds and alternative assets was poor. There was also a paradox. Many institutional investors diversified their portfolios by considering several asset classes and different regions. Unfortunately, this diversification was not enough to protect them. In the end, the 2008 financial crisis was more damaging than the dot-com crisis. This was particularly true for institutional investors in continental Europe, who were relatively well protected against the collapse of the internet bubble because of their low exposure to equities. This is why the 2008 financial crisis was a deep trauma for world-wide institutional investors.

Most institutional portfolios were calibrated through portfolio optimization. In this context, Markowitz's modern portfolio theory was strongly criticized by professionals, and several journal articles announced the death of the Markowitz model[1]. These extreme reactions can be explained by the fact that diversification is traditionally associated with Markowitz optimization, and it failed during the financial crisis. However, the problem was not entirely due to the allocation method. Indeed, much of the failure was caused by the input parameters. With expected returns calibrated to past figures, the model induced an overweight in equities. It also promoted assets that were supposed to have a low correlation to equities. Nonetheless, correlations between asset classes increased significantly during the crisis. In the end, the promised diversification did not occur.

Today, it is hard to find investors who defend Markowitz optimization. However, the criticisms concern not so much the model itself but the way it is used. In the 1990s, researchers began to develop regularization techniques to limit the impact of estimation errors in input parameters and many improvements have been made in recent years. In addition, we now have a better understanding of how this model works. Moreover, we also have a theoretical framework to measure the impact of constraints (Jagannathan and Ma, 2003). More recently, robust optimization based on the lasso approach has improved optimized portfolios (DeMiguel *et al.*, 2009). So the Markowitz model is certainly not dead. Investors must understand that it is a fabulous tool for combining risks and expected returns. The goal of Markowitz optimization is to find arbitrage factors and build a portfolio that will play on them. By construction, this approach is an aggressive model of active management. In this case, it is normal that the model should be sensitive to input parameters (Green and Hollifield, 1992). Changing the parameter values modifies the implied bets. Accordingly, if input parameters are wrong, then arbitrage factors and bets are also wrong, and the resulting portfolio is not satisfied. If investors want a more defensive model, they have to define less aggressive parameter values. This is the main message behind portfolio regularization. In consequence, reports of the death of the Markowitz model have been greatly exaggerated, because it will continue to be used intensively in active management strategies. Moreover, there are no other serious and powerful models to take into account return forecasts.

[1]See for example the article "*Is Markowitz Dead? Goldman Thinks So*" published in December 2012 by AsianInvestor.

The rise of risk parity portfolios

There are different ways to obtain less aggressive active portfolios. The first one is to use less aggressive parameters. For instance, if we assume that expected returns are the same for all of the assets, we obtain the minimum variance (or MV) portfolio. The second way is to use heuristic methods of asset allocation. The term 'heuristic' refers to experience-based techniques and trial-and-error methods to find an acceptable solution, which does not correspond to the optimal solution of an optimization problem. The equally weighted (or EW) portfolio is an example of such non-optimized 'rule of thumb' portfolio. By allocating the same weight to all the assets of the investment universe, we considerably reduce the sensitivity to input parameters. In fact, there are no active bets any longer. Although these two allocation methods have been known for a long time, they only became popular after the collapse of the internet bubble.

Risk parity is another example of heuristic methods. The underlying idea is to build a balanced portfolio in such a way that the risk contribution is the same for different assets. It is then an equally weighted portfolio in terms of risk, not in terms of weights. Like the minimum variance and equally weighted portfolios, it is impossible to date the risk parity portfolio. The term risk parity was coined by Qian (2005). However, the risk parity approach was certainly used before 2005 by some CTA and equity market neutral funds. For instance, it was the core approach of the All Weather fund managed by Bridgewater for many years (Dalio, 2004). At this point, we note that the risk parity portfolio is used, because it makes sense from a practical point of view. However, it was not until the theoretical work of Maillard *et al.* (2010), first published in 2008, that the analytical properties were explored. In particular, they showed that this portfolio exists, is unique and is located between the minimum variance and equally weighted portfolios.

Since 2008, we have observed an increasing popularity of the risk parity portfolio. For example, Journal of Investing and Investment and Pensions Europe (IPE) ran special issues on risk parity in 2012. In the same year, The Financial Times and Wall Street Journal published several articles on this topic[2]. In fact today, the term risk parity covers different allocation methods. For instance, some professionals use the term risk parity when the asset weight is inversely proportional to the asset return volatility. Others consider that the risk parity portfolio corresponds to the equally weighted risk contribution (or ERC) portfolio. Sometimes, risk parity is equivalent to a risk budgeting (or RB) portfolio. In this case, the risk budgets are not necessarily the same for all of the assets that compose the portfolio. Initially, risk parity

[2] "*New Allocation Funds Redefine Idea of Balance*" (February 2012), "*Same Returns, Less Risk*" (June 2012), "*Risk Parity Strategy Has Its Critics as Well as Fans*" (June 2012), "*Investors Rush for Risk Parity Shield*" (September 2012), etc.

only concerned a portfolio of bonds and equities. Today, risk parity is applied to all investment universes. Nowadays, risk parity is a *marketing term* used by the asset management industry to design a portfolio based on risk budgeting techniques.

More interesting than this marketing operation is the way risk budgeting portfolios are defined. Whereas the objective of Markowitz portfolios is to reach an expected return or to target ex-ante volatility, the goal of risk parity is to assign a risk budget to each asset. Like for the other heuristic approaches, the performance dimension is then absent and the risk management dimension is highlighted. In addition, this last point is certainly truer for the risk parity approach than for the other approaches. We also note that contrary to minimum variance portfolios, which have only seduced equity investors, risk parity portfolios concern not only different traditional asset classes (equities and bonds), but also alternative asset classes (commodities and hedge funds) and multi-asset classes (stock/bond asset mix policy and diversified funds). By placing risk management at the heart of these different management processes, risk parity represents a substantial break with respect to the previous period of Markowitz optimization. Over the last decades, the main objective of institutional investors was to generate performance well beyond the risk-free rate (sometimes approaching double-digit returns). After the 2008 crisis, investors largely revised their expected return targets. Their risk aversion level increased and they do not want to experience another period of such losses. In this context, risk management has become more important than performance management.

Nevertheless, like for many other hot topics, there is some exaggeration about risk parity. Although there are people who think that it represents a definitive solution to asset allocation problems, one should remain prudent. Risk parity remains a financial model of investment and its performance also depends on the investor's choice regarding parameters. Choosing the right investment universe or having the right risk budgets is as important as using the right allocation method. As a consequence, risk parity may be useful when defining a reliable allocation, but it cannot free investors of their duty of making their own choices.

About this book

The subject of this book is risk parity approaches. As noted above, risk parity is now a generic term used by the asset management industry to designate risk-based management processes. In this book, the term risk parity is used as a synonym of risk budgeting. When risk budgets are identical, we prefer to use the term ERC portfolio, which is more explicit and less overused by the investment industry. When we speak of a risk parity fund, it corresponds to an equally weighted risk contribution portfolio of equities and bonds.

This book comprises two parts. The first part is more theoretical. Its first chapter is dedicated to modern portfolio theory whereas the second chapter is a comprehensive guide to risk budgeting. The second part contains four chapters, each of which presents an application of risk parity to a specific asset class. The third chapter concerns risk-based equity indexation, also called smart indexing. In the fourth chapter, we show how risk budgeting techniques can be applied to the management of bond portfolios. The fifth chapter deals with alternative investments, such as commodities and hedge funds. Finally, the sixth chapter applies risk parity techniques to multi-asset classes. The book also contains two appendices. The first appendix provides the reader with technical materials on optimization problems, copula functions and dynamic asset allocation. The second appendix contains 30 tutorial exercises. The relevant solutions are not included in this book, but can be accessed at the following web page[3]:

`http://www.thierry-roncalli.com/riskparitybook.html`

This book began with an invitation by Professor Diethelm Würtz to present a tutorial on risk parity at the 6$^{\text{th}}$ R/Rmetrics Meielisalp Workshop & Summer School on Computational Finance and Financial Engineering. This seminar is organized every year at the end of June in Meielisalp, Lake Thune, Switzerland. The idea of tutorial sessions is to offer an overview on a specialized topic in statistics or finance. When preparing this tutorial, I realized that I had sufficient material to write a book on risk parity. First of all, I would like to thank Diethelm Würtz and the participants of the Meielisalp Summer School for their warm welcome and the different discussions we had about risk parity. I would also like to thank all of the people who have invited me to academic and professional conferences in order to speak about risk parity techniques and applications since 2008, in particular Yann Braouezec, Rama Cont, Nathalie Columelli, Felix Goltz, Marie Kratz, Jean-Luc Prigent, Fahd Rachidy and Peter Tankov. I would also like to thank Jérôme Glachant and my other colleagues of the Master of Science in Asset and Risk Management program at the Évry University where I teach the course on Risk Parity. I am also grateful to the CRC editorial staff, in particular Sunil Nair, for their support, encouragement and suggestions.

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[3]This web page also provides readers and instructors other materials related to the book (errata, code, slides, etc.).

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Symbol	Description
$\cdot$	Scalar multiplication
$\circ$	Hadamard product: $(x \circ y)_i = x_i y_i$
$\otimes$	Kronecker product $A \otimes B$
$\lvert\mathcal{E}\rvert$	Cardinality of the set $\mathcal{E}$
$\mathbf{1}$	Vector of ones
$\mathbb{1}\{\mathcal{A}\}$	The indicator function is equal to 1 if $\mathcal{A}$ is true, 0 otherwise
$\mathbb{1}_{\mathcal{A}}\{x\}$	The characteristic function is equal to 1 if $x \in \mathcal{A}$, 0 otherwise
$\mathbf{0}$	Vector of zeros
$(A_{i,j})$	Matrix A with entry $A_{i,j}$ in row i and column j
A^{-1}	Inverse of the matrix A
$A^{\top}$	Transpose of the matrix A
A^{+}	Moore-Penrose pseudo-inverse of the matrix A
b	Vector of weights $(b_1, \ldots, b_n)$ for the benchmark b
$B_t(T)$	Price of the zero-coupon bond at time t for the maturity T
β_i	Beta of asset i with respect to portfolio x
$\beta_i(x)$	Another notation for the symbol β_i
$\beta(x \mid b)$	Beta of portfolio x when the benchmark is b
C (or ρ)	Correlation matrix
$\mathbf{C}$	Copula function
$C(t_m)$	Coupon paid at time t_m
$\operatorname{cov}(X)$	Covariance of the random vector X
$C_n(\rho)$	Constant correlation matrix $(n \times n)$ with $\rho_{i,j} = \rho$
D	Covariance matrix of idiosyncratic risks
$\det(A)$	Determinant of the matrix A
$\mathcal{DR}(x)$	Diversification ratio of portfolio x
$\mathbf{e}_i$	The value of the vector is 1 for the row i and 0 elsewhere
$\mathbb{E}[X]$	Mathematical expectation of the random variable X
$\mathcal{E}(\lambda)$	Exponential probability distribution with parameter λ
$\mathrm{ES}_{\alpha}(x)$	Expected shortfall of portfolio x at the confidence level α
$f(x)$	Probability density function (pdf)
$\mathbf{F}(x)$	Cumulative distribution function (cdf)
$\mathcal{F}$	Vector of risk factors $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$
$\mathcal{F}_j$	Risk factor j
$F_t(T)$	Instantaneous forward rate at time t for the maturity T
$F_t(T, m)$	Forward interest rate at

	time t for the period $[T, T+m]$
$\mathcal{G}$	Gini coefficient
γ	Parameter $\gamma = \phi^{-1}$ of the Markowitz γ-problem
γ_1	Skewness
γ_2	Excess kurtosis
$\mathcal{H}$	Herfindahl index
i	Asset i
I_n	Identity matrix of dimension n
$\mathrm{IR}\left(x \mid b\right)$	Information ratio of portfolio x when the benchmark is b
$\ell\left(\theta\right)$	Log-likelihood function with θ the vector of parameters to estimate
ℓ_t	Log-likelihood function for the observation t
$L\left(x\right)$	Loss of portfolio x
$\mathcal{L}\left(x\right)$	Leverage measure of portfolio x
$\mathbb{L}\left(x\right)$	Lorenz function
λ	Parameter of exponential survival times
$\mathcal{MDD}$	Maximum drawdown
$\mathcal{MR}_i$	Marginal risk of asset i
μ	Vector of expected returns $(\mu_1, \ldots, \mu_n)$
μ_i	Expected return of asset i
$\hat{\mu}$	Empirical mean
$\hat{\mu}_{1\mathrm{Y}}$	Annualized return
$\mu\left(x\right)$	Expected return of portfolio x: $\mu\left(x\right) = x^\top \mu$
$\mu\left(x \mid b\right)$	Expected return of the tracking error of portfolio x when the benchmark is b
$\mathcal{N}\left(\mu, \sigma^2\right)$	Probability distribution of a Gaussian random variable with mean μ and standard deviation σ
$\mathcal{N}\left(\mu, \Sigma\right)$	Probability distribution of a Gaussian random vector with mean μ and covariance matrix Σ
Ω	Covariance matrix of risk factors
π	Vector of risk premia $(\pi_1, \ldots, \pi_n)$
$\tilde{\pi}$	Vector of implied risk premia $(\tilde{\pi}_1, \ldots, \tilde{\pi}_n)$
π_i	Risk premium of asset i: $\pi_i = \mu_i - r$
$\tilde{\pi}_i$	Implied risk premium of asset i
$\pi\left(y \mid x\right)$	Risk premium of portfolio y if the tangency portfolio is x: $\pi\left(y \mid x\right) = \beta\left(y \mid x\right)\left(\mu\left(x\right) - r\right)$
Π	P&L of the portfolio
ϕ	Risk aversion parameter of the quadratic utility function
$\phi\left(x\right)$	Probability density function of the standardized normal distribution
$\Phi\left(x\right)$	Cumulative distribution function of the standardized normal distribution
$\Phi^{-1}\left(\alpha\right)$	Inverse of the cdf of the standardized normal distribution
r	Return of the risk-free asset
$r^\star$	Yield to maturity
R	Vector of asset returns $(R_1, \ldots, R_n)$
R_i	Return of asset i
$R_{i,t}$	Return of asset i at time t
$R\left(x\right)$	Return of portfolio x: $R\left(x\right) = x^\top R$
$\mathcal{R}\left(x\right)$	Risk measure of portfolio x
$R_t\left(T\right)$	Zero-coupon rate at time t for the maturity T
$\mathcal{RC}_i$	Risk contribution of asset i
$\mathcal{RC}_i^\star$	Relative risk contribution of asset i
$\Re$	Recovery rate
ρ (or C)	Correlation matrix of asset returns

$\rho_{i,j}$	Correlation between asset returns i and j
$\rho(x, y)$	Correlation between portfolios x and y
$\mathfrak{s}$	Credit spread
$\mathbf{S}_t(x)$	Survival function at time t
Σ	Covariance matrix
$\hat{\Sigma}$	Empirical covariance matrix
σ_i	Volatility of asset i
σ_m	Volatility of the market portfolio
$\tilde{\sigma}_i$	Idiosyncratic volatility of asset i
$\hat{\sigma}$	Empirical volatility
$\hat{\sigma}_{1\mathrm{Y}}$	Annualized volatility
$\sigma(x)$	Volatility of portfolio x: $\sigma(x) = \sqrt{x^\top \Sigma x}$
$\sigma(x \mid b)$	Standard deviation of the tracking error of portfolio x when the benchmark is b
$\sigma(x, y)$	Covariance between portfolios x and y
$\sigma(X)$	Standard deviation of the random variable X
SR_i	Sharpe ratio of asset i: $\mathrm{SR}_i = \mathrm{SR}(\mathbf{e}_i \mid r)$
$\mathrm{SR}(x \mid r)$	Sharpe ratio of portfolio x when the risk-free asset is r
$\mathbf{t}_v(x)$	Cumulative distribution function of the Student's t distribution with ν the number of degrees of freedom
$\mathbf{t}_v^{-1}(\alpha)$	Inverse of the cdf of the Student's t distribution with ν the number of degrees of freedom
$\mathbf{t}_{\rho,v}(x)$	Cumulative distribution function of the multivariate Student's t distribution with parameters ρ and ν
$\tau(x)$	Turnover of portfolio x
$\mathrm{tr}(A)$	Trace of the matrix A
$\mathrm{TR}(x \mid b)$	Treynor ratio of portfolio x when the benchmark is b
$\mathrm{VaR}_\alpha(x)$	Value-at-risk of portfolio x at the confidence level α
x	Vector of weights $(x_1, \ldots, x_n)$ for portfolio x
x_i	Weight of asset i in portfolio x
$x^\star$	Optimized portfolio

Portfolio Notation

ERC	Equally weighted risk contribution portfolio x_{erc}
EW	Equally weighted portfolio x_{ew}
MDP	Most diversified portfolio x_{mdp}
MSR	Max Sharpe ratio portfolio x_{msr}
MV	Minimum variance portfolio x_{mv}
MVO	Mean-variance optimized (or Markowitz) portfolio x_{mvo}
RB	Risk budgeting portfolio x_{rb}
RFP	Risk factor parity portfolio x_{rfp}
RP	Risk parity portfolio x_{rp}
WB	Weight budgeting portfolio x_{wb}