

Introduction to Risk Parity and Budgeting

Chapter 2 – Risk Budgeting Approach

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<http://www.crcpress.com/product/isbn/9781482207156>

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May 22, 2013

Figure 2.1, Page 72

Figure: Three budgeting methods with a 30/70 policy rule

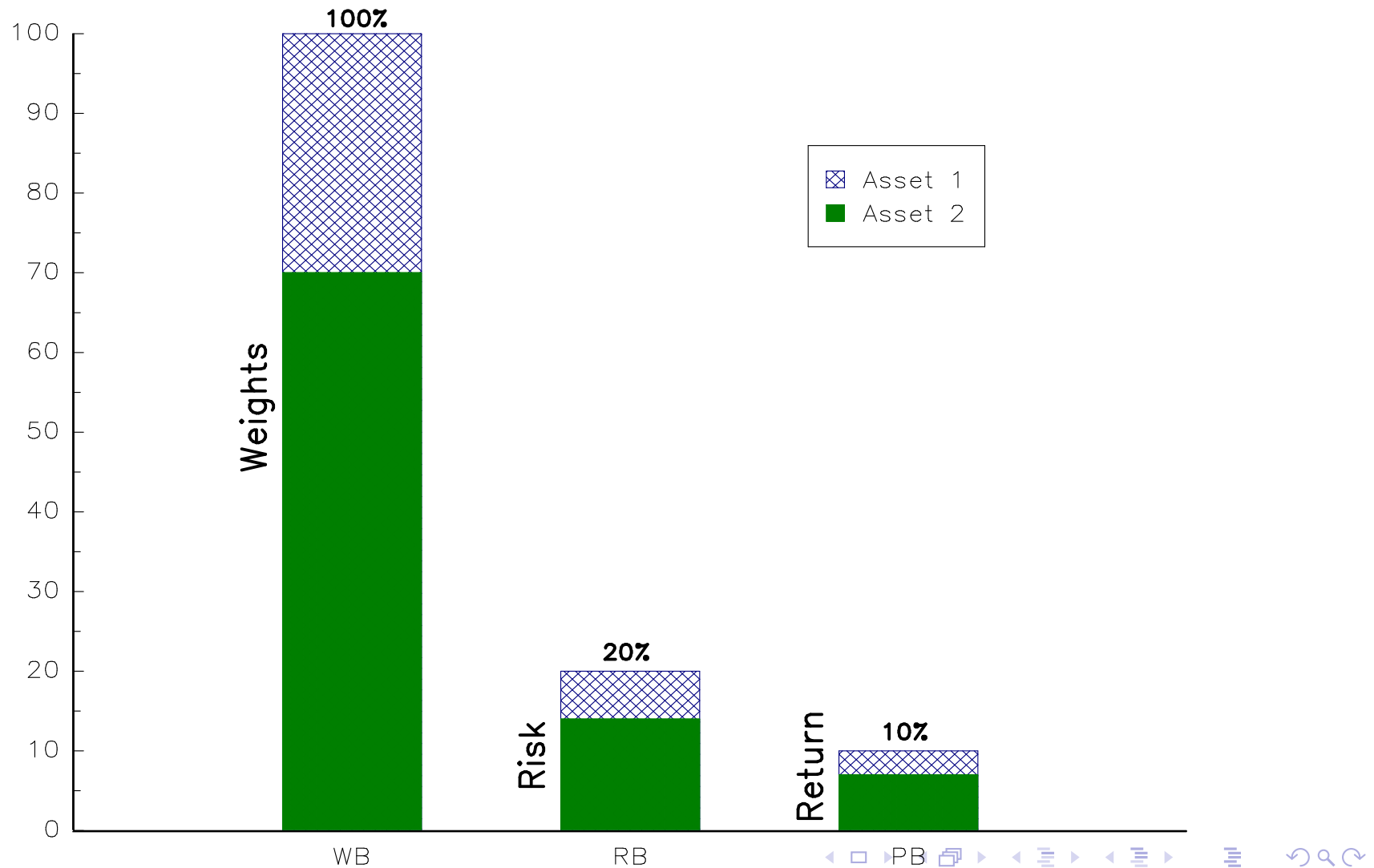


Table 2.1, Page 76

Table: Computation of risk measures $\text{VaR}_\alpha(x)$ and $\text{ES}_\alpha(x)$

Portfolio	$\mathcal{R}(x)$		α			
			90%	95%	99%	99.5%
#1	VaR	(in %)	1.52	2.06	3.06	3.43
		(in \$)	14.27	19.30	28.74	32.20
	ES	(in %)	2.22	2.67	3.56	3.90
		(in \$)	20.83	25.09	33.44	36.58
#2	VaR	(in %)	5.68	7.45	10.76	11.98
		(in \$)	14.94	19.59	28.31	31.50
	ES	(in %)	7.98	9.48	12.41	13.52
		(in \$)	21.00	24.94	32.64	35.54

Table: Risk decomposition of the volatility

Asset	x_i	$\mathcal{M}R_i$	$\mathcal{R}C_i$	$\mathcal{R}C_i^*$
1	50.00	29.40	14.70	70.43
2	20.00	16.63	3.33	15.93
3	30.00	9.49	2.85	13.64
$\mathcal{R}(x)$			20.87	

Tables 2.3 & 2.4, Page 82

Table: Risk decomposition of the value-at-risk

Asset	x_i	MR_i	RC_i	RC_i^*
1	50.00	68.39	34.19	70.43
2	20.00	38.68	7.74	15.93
3	30.00	22.07	6.62	13.64
$\mathcal{R}(x)$			48.55	

Table: Risk decomposition of the expected shortfall

Asset	x_i	MR_i	RC_i	RC_i^*
1	50.00	78.35	39.17	70.43
2	20.00	44.31	8.86	15.93
3	30.00	25.29	7.59	13.64
$\mathcal{R}(x)$			55.62	

Tables 2.5 & 2.6, Page 84

Table: Sensitivity analysis of the volatility with respect to the factor h

Asset	1 bp	10 bp	1%	10%	50%	$-x_i$
1	20.8728	20.8992	21.1639	23.8170	35.6938	6.8593
2	20.8715	20.8865	21.0364	22.5599	29.7077	17.6847
3	20.8708	20.8793	20.9650	21.8495	26.2640	18.3576

Table: Marginal analysis of the volatility with respect to the factor h

Asset	1 bp	10 bp	1%	10%	50%	$-x_i$
1	20.8728	20.8992	21.1638	23.8095	35.5681	6.1716
2	20.8715	20.8865	21.0361	22.5325	29.1833	17.5445
3	20.8708	20.8793	20.9647	21.8186	25.6135	18.0236

Tables, Pages 98 & 99

j	1	2	3	4	5	6	7	8	9	10
$L_1^{(j)}$	14	3	-4	5	6	8	12	25	23	-9
$L_2^{(j)}$	10	-3	8	7	2	17	14	22	-8	-2
j	11	12	13	14	15	16	17	18	19	20
$L_1^{(j)}$	-50	-17	18	-9	-6	-2	0	17	19	1
$L_2^{(j)}$	-10	12	-12	-19	25	-10	4	12	36	-5

j	1	2	3	4	5	6	7	8	9	10
$L_1^{(j:m)}$	-50	-9	-2	-9	-17	1	3	-4	0	18
$L_2^{(j:m)}$	-10	-19	-10	-2	12	-5	-3	8	4	-12
j	11	12	13	14	15	16	17	18	19	20
$L_1^{(j:m)}$	6	5	23	-6	14	8	12	17	25	19
$L_2^{(j:m)}$	2	7	-8	25	10	17	14	12	22	36

Figure 2.2, Page 90

Figure: Density of the risk contribution estimator RC_1

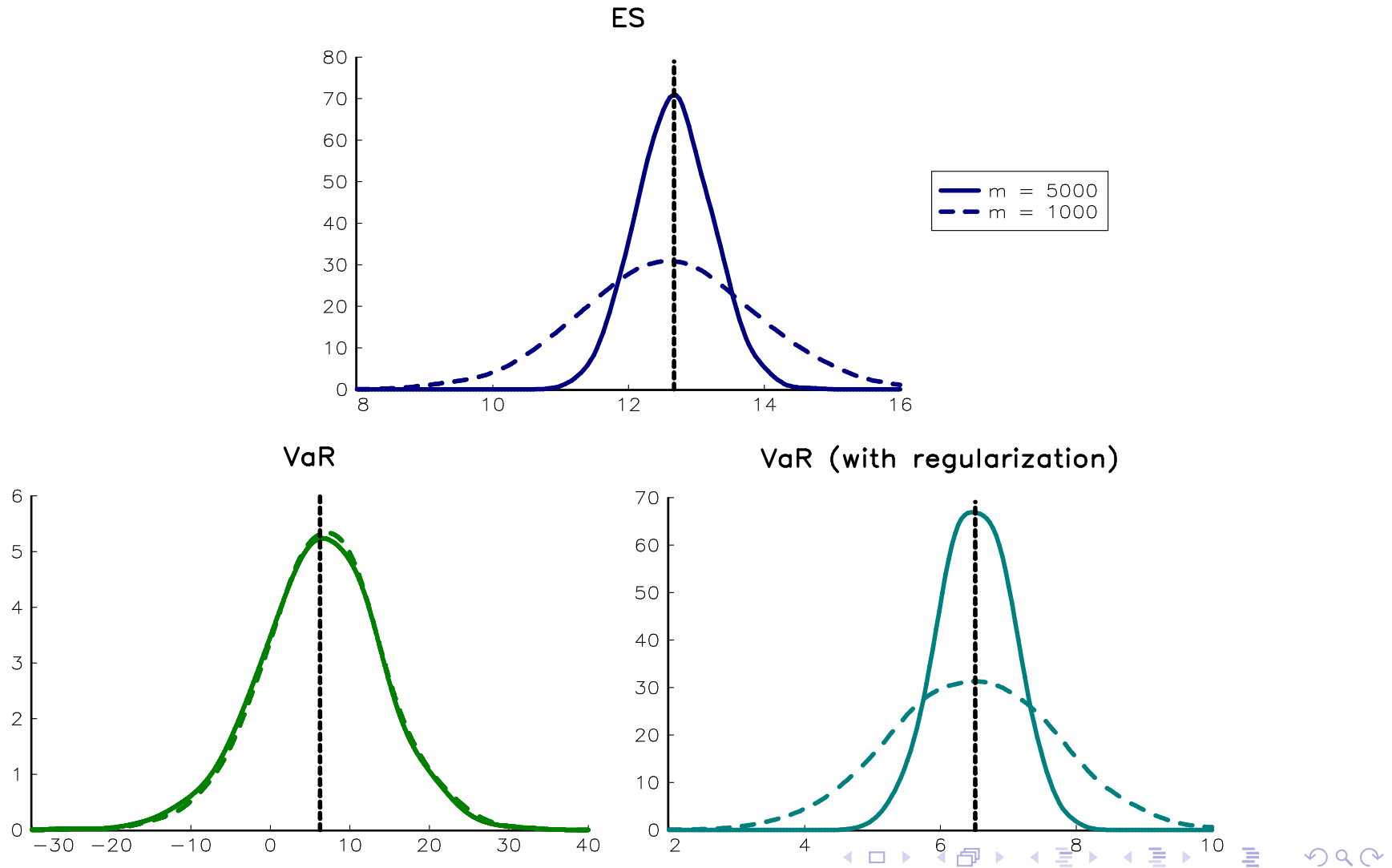


Table 2.7, Page 96

Table: Value-at-risk (in %) when the P&L is skew normal distributed

α		80%	85%	90%	95%	99%
Normal	#1	3.21	4.18	5.41	7.22	10.63
	#2	3.76	5.12	6.84	9.39	14.16
	#3	11.13	13.56	16.61	21.14	29.62
Cornish-Fisher	#1	3.21	4.18	5.41	7.22	10.63
	#2	3.80	4.94	6.34	8.34	11.95
	#3	10.63	13.79	17.90	24.20	36.52
Skew normal	#1	3.21	4.18	5.41	7.22	10.63
	#2	3.86	5.03	6.43	8.41	11.78
	#3	10.67	13.70	17.66	23.80	36.08

Figure 2.3, Page 97

Figure: Density of the P&L with a skew normal distribution

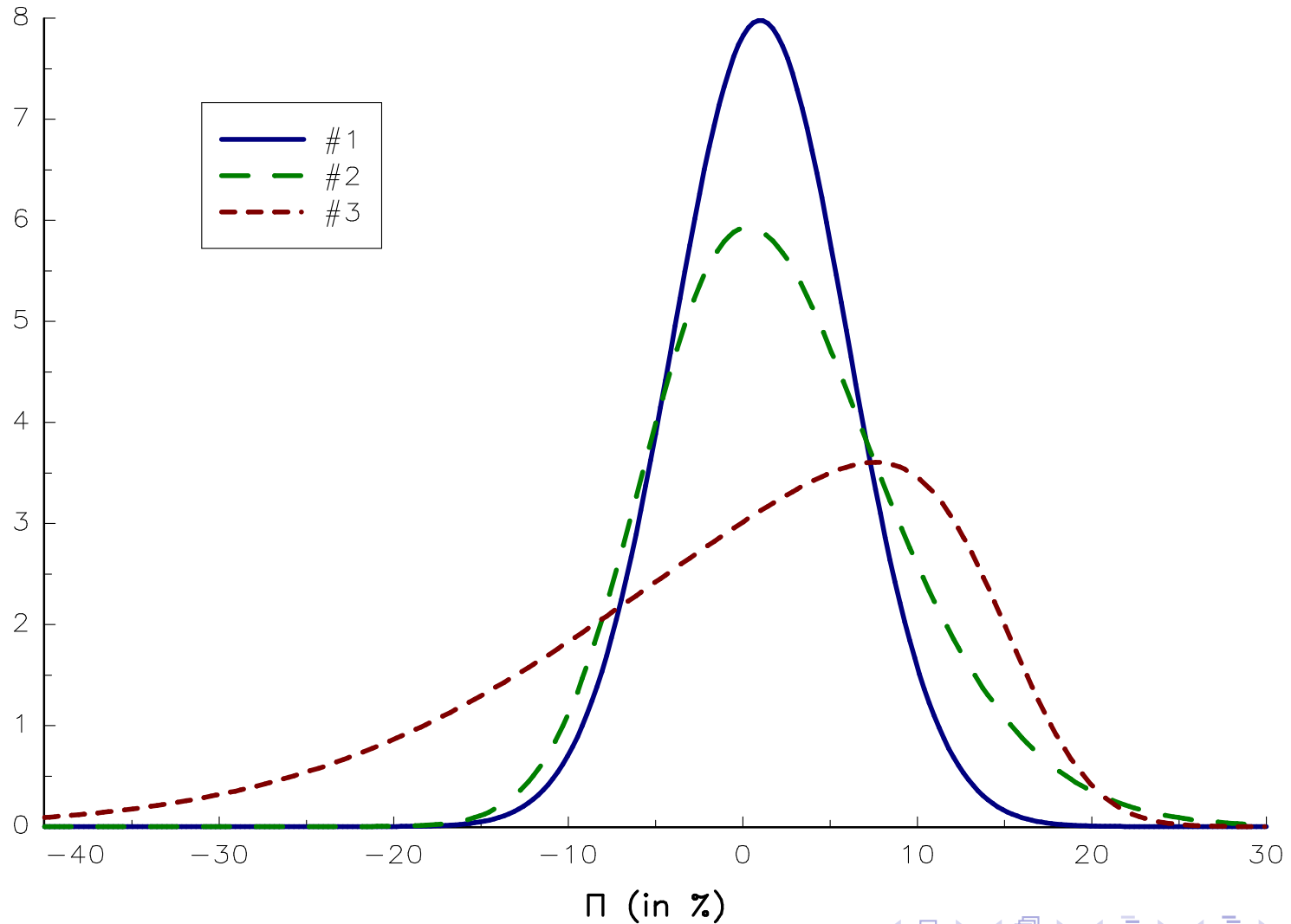


Table 2.8, Page 98

Table: Statistics (in %) to compute the Cornish-Fisher risk contributions

Portfolio	#1	#2	#3	#4	#5	#6
x_1	0.00	10.00	25.00	50.00	75.00	100.00
x_2	100.00	90.00	75.00	50.00	25.00	0.00
$\mathbb{E}[L]$	-0.10	-0.11	-0.13	-0.16	-0.19	-0.23
$\sigma^2(L)$	0.01	0.01	0.01	0.02	0.04	0.07
$\gamma_1(L)$	2.67	7.73	8.42	19.20	29.18	35.13
$\gamma_2(L)$	-13.05	-21.50	-16.08	58.12	103.97	124.21
$\partial_{x_1} \gamma_1$	94.76	11.12	14.46	24.44	7.68	0.00
$\partial_{x_2} \gamma_1$	0.00	-1.24	-4.82	-24.44	-23.05	-18.14
$\partial_{x_1} \gamma_2$	-44.29	-90.59	159.39	131.49	29.58	0.00
$\partial_{x_2} \gamma_2$	0.00	10.07	-53.13	-131.49	-88.74	-53.28
z	231.52	233.07	234.80	258.95	275.19	282.86
$\partial_{x_1} z$	57.41	-13.65	46.98	45.18	10.88	0.00
$\partial_{x_2} z$	0.00	1.52	-15.66	-45.18	-32.63	-21.00
$\text{VaR}_\alpha(L)$	1.92	1.90	2.19	3.59	5.39	7.28
$\mathcal{R}\mathcal{C}_1$	0.00	0.23	1.23	3.49	5.44	7.28
$\mathcal{R}\mathcal{C}_2$	1.92	1.67	0.96	0.10	-0.05	0.00
$\overline{\text{VaR}}_\alpha^*(L)$	1.93	1.89	2.17	3.21	4.53	5.94

Tables 2.9 & 2.10, Pages 99 & 100

Table: Risk budgeting portfolio when the risk measure is the expected shortfall ($\alpha = 95\%$)

Asset	x_i	w_i	MR_i	RC_i	RC_i^*
1	534 430	28.21%	46.78%	250 000	50.00%
2	372 705	19.68%	26.83%	100 000	20.00%
3	987 007	52.11%	15.20%	150 000	30.00%
sum	1 894 142			500 000	

Table: Risk budgeting portfolio when the risk measure is the expected shortfall ($\alpha = 99\%$)

Asset	x_i	w_i	MR_i	RC_i	RC_i^*
1	391 926	29.00%	63.79%	250 000	50.00%
2	273 737	20.26%	36.53%	100 000	20.00%
3	685 779	50.74%	21.87%	150 000	30.00%
sum	1 351 441			500 000	

Table 2.11, Page 103

Table: Weights w^* in the RB portfolio with respect to some values of b and ρ

ρ/b	$\sigma_2 = \sigma_1$				$\sigma_2 = 3 \times \sigma_1$			
	20%	50%	70%	90%	20%	50%	70%	90%
-50%	41.9	50.0	55.2	61.6	68.4	75.0	78.7	82.8
0%	33.3	50.0	60.4	75.0	60.0	75.0	82.1	90.0
25%	29.3	50.0	63.0	80.6	55.5	75.0	83.6	92.6
50%	25.7	50.0	65.5	84.9	51.0	75.0	85.1	94.4
75%	22.6	50.0	67.8	87.9	46.7	75.0	86.3	95.6
90%	21.0	50.0	69.1	89.2	44.4	75.0	87.1	96.1

Figure 2.4, Page 104

Figure: Evolution of the weight w^* in the RB portfolio with respect to b and ρ

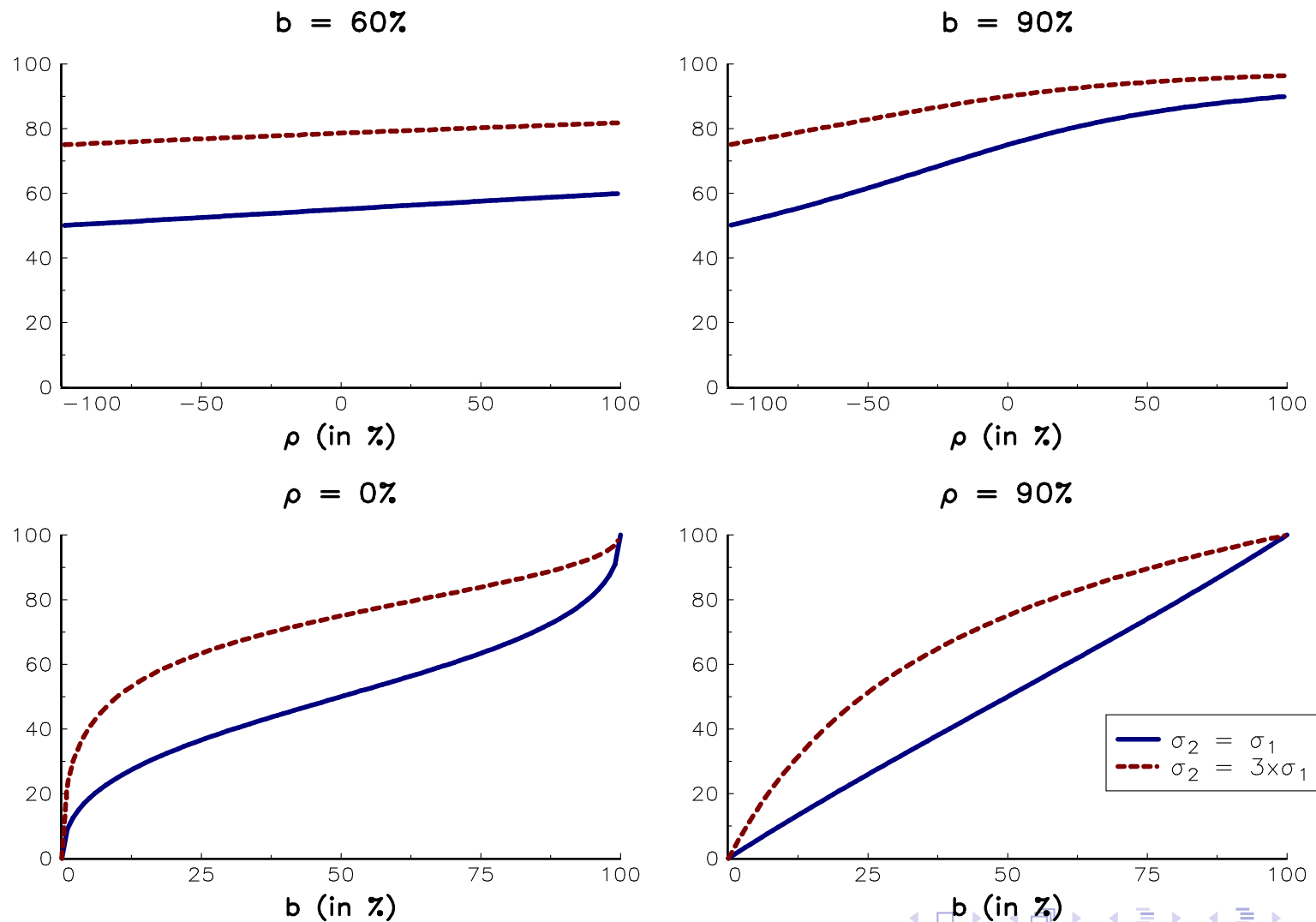


Figure 2.5, Page 107

Figure: Simulation of the weight x_1 when the correlation is constant

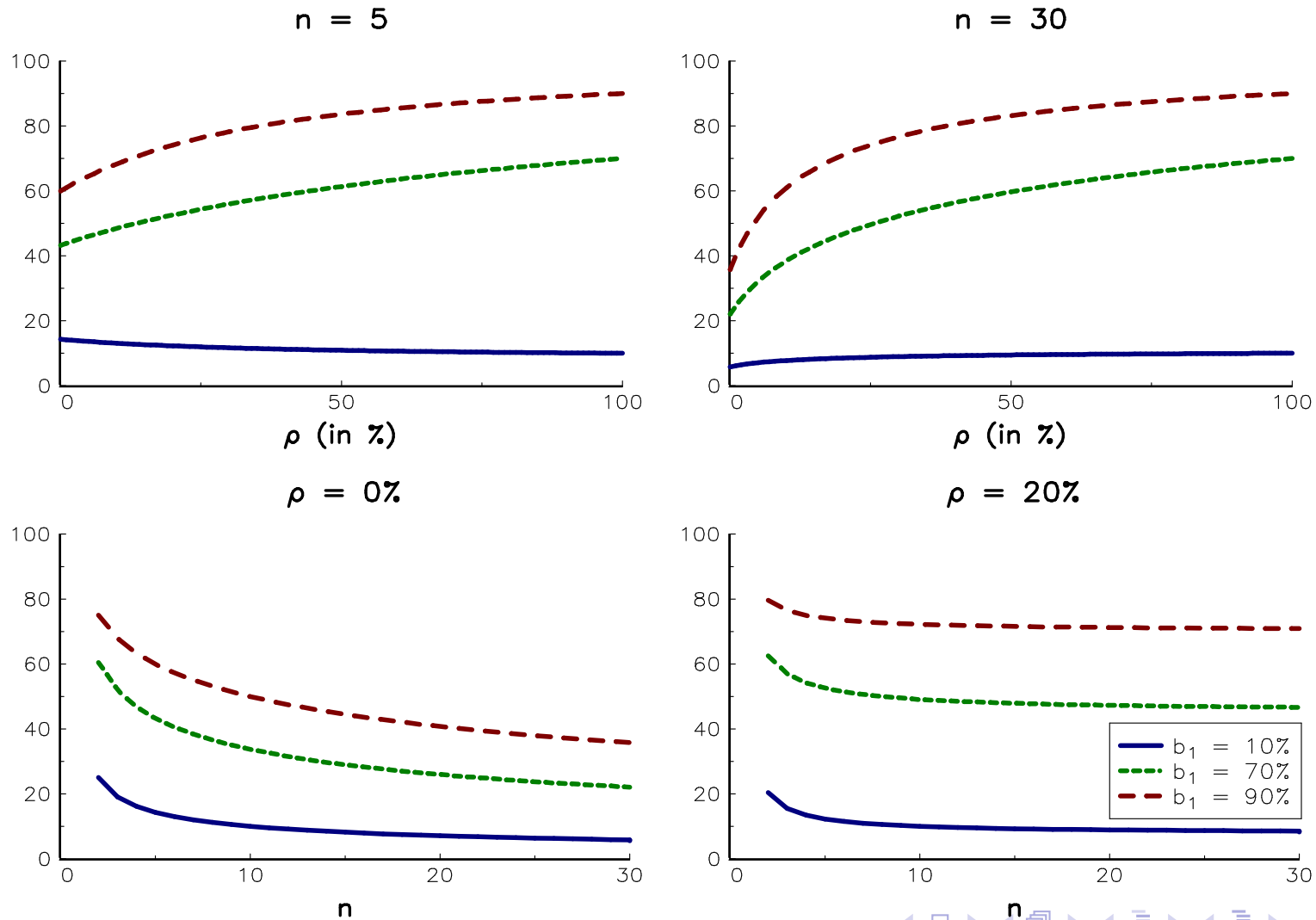


Table 2.12, Page 112

Table: RB solutions when the risk budget b_3 is equal to 0

$\rho_{1,3} = \rho_{2,3}$	Solution	1	2	3	$\sigma(x)$
-25%	x_i	20.00%	40.00%	40.00%	6.63%
	\mathcal{S}_1 MR_i	16.58%	8.29%	0.00%	
	RC_i	50.00%	50.00%	0.00%	
	x_i	33.33%	66.67%	0.00%	11.55%
	\mathcal{S}_2 MR_i	17.32%	8.66%	-1.44%	
	RC_i	50.00%	50.00%	0.00%	
25%	x_i	19.23%	38.46%	42.31%	6.38%
	\mathcal{S}'_1 MR_i	16.42%	8.21%	0.15%	
	RC_i	49.50%	49.50%	1.00%	
25%	x_i	33.33%	66.67%	0.00%	11.55%
	\mathcal{S}_1 MR_i	17.32%	8.66%	1.44%	
	RC_i	50.00%	50.00%	0.00%	

Figure 2.6, Page 112

Figure: Evolution of the portfolio's volatility with respect to x_3

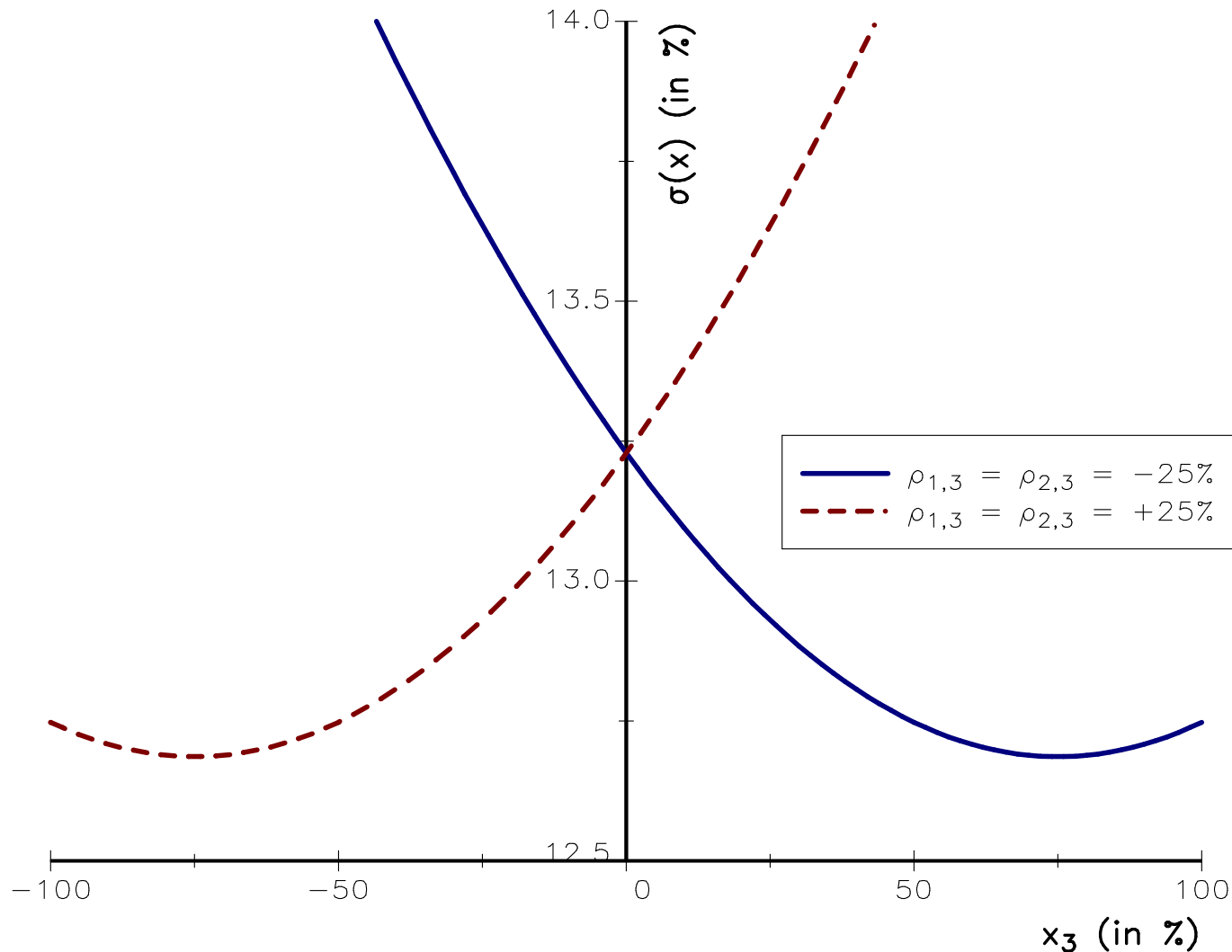


Table 2.13, Page 113

Table: RB solutions when the risk budgets b_3 and b_4 are equal to 0

Solution		1	2	3	4	$\sigma(x)$
\mathcal{S}_1	x_i	20.00%	40.00%	26.67%	13.33%	6.53%
	MR_i	16.33%	8.16%	0.00%	0.00%	
	RC_i	50.00%	50.00%	0.00%	0.00%	
\mathcal{S}_2	x_i	33.33%	66.67%	0.00%	0.00%	11.55%
	MR_i	17.32%	8.66%	-1.44%	-2.89%	
	RC_i	50.00%	50.00%	0.00%	0.00%	
\mathcal{S}_3	x_i	20.00%	40.00%	40.00%	0.00%	6.63%
	MR_i	16.58%	8.29%	0.00%	-1.51%	
	RC_i	50.00%	50.00%	0.00%	0.00%	
\mathcal{S}_4	x_i	25.00%	50.00%	0.00%	25.00%	8.29%
	MR_i	16.58%	8.29%	-0.75%	0.00%	
	RC_i	50.00%	50.00%	0.00%	0.00%	

Tables 2.14 & 2.15, Page 116

Table: Implied risk premia when $b = (20\%, 25\%, 40\%, 15\%)$

Asset	x_i	$\mathcal{M}R_i$	$\tilde{\mu}_i$	$\mathcal{P}C_i$	$\mathcal{P}C_i^*$
1	40.91	7.10	3.55	1.45	20.00
2	25.12	14.46	7.23	1.82	25.00
3	25.26	23.01	11.50	2.91	40.00
4	8.71	25.04	12.52	1.09	15.00
Expected return				7.27	

Table: Implied risk premia when $b = (10\%, 10\%, 10\%, 70\%)$

Asset	x_i	$\mathcal{M}R_i$	$\tilde{\mu}_i$	$\mathcal{P}C_i$	$\mathcal{P}C_i^*$
1	35.88	5.27	2.63	0.94	10.00
2	17.94	10.53	5.27	0.94	10.00
3	10.18	18.56	9.28	0.94	10.00
4	35.99	36.75	18.37	6.61	70.00
Expected return				9.45	

Tables 2.16 & 2.17, Page 117

Table: Sensitivity of the MVO portfolio to input parameters

ρ		70%	90%		90%	
σ_2				18%	18%	
μ_1						9%
x_1	38.3	38.3	44.6	13.7	0.0	56.4
x_2	20.2	25.9	8.9	56.1	65.8	0.0
x_3	41.5	35.8	46.5	30.2	34.2	43.6

Table: Sensitivity of the RB portfolio to input parameters

ρ		70%	90%		90%	
σ_2				18%	18%	
μ_1						9%
x_1	38.3	37.7	38.9	37.1	37.7	38.3
x_2	20.2	20.4	20.0	22.8	22.6	20.2
x_3	41.5	41.9	41.1	40.1	39.7	41.5

Tables 2.18 & 2.19, Pages 118 & 119

Table: Shrinkage covariance matrix $\tilde{\Sigma}^{(1)}$ associated to the RB portfolio

Asset	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$		
1	19.13%	100.00%		
2	18.92%	82.54%	100.00%	
3	22.93%	57.69%	68.08%	100.00

Table: Shrinkage covariance matrix $\tilde{\Sigma}^{(3)}$ associated to the RB portfolio

Asset	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$		
1	18.26%	100.00%		
2	17.93%	67.67%	100.00%	
3	24.40%	33.25%	49.39%	100.00

Tables 2.20 & 2.21, Pages 121 & 123

Table: Risk contributions of EW, ERC and MV portfolios

Portfolio	Asset	x_i	MR_i	RC_i	RC_i^*
EW	1	50.00	16.06	8.03	36.84
	2	50.00	27.53	13.76	63.16
ERC	1	60.00	17.32	10.39	50.00
	2	40.00	25.98	10.39	50.00
MV	1	85.71	19.64	16.83	85.71
	2	14.29	19.64	2.81	14.29

Table: Composition of the ERC portfolio

Asset	x_i	MR_i	β_i	RC_i	RC_i^*
1	31.34	8.52	0.80	2.67	25.00
2	17.49	15.27	1.43	2.67	25.00
3	13.05	20.46	1.92	2.67	25.00
4	38.12	7.00	0.66	2.67	25.00
Volatility				10.68	

Figure 2.7, Page 124

Figure: Location of the ERC portfolio in the mean-variance diagram when the Sharpe ratios are the same and the asset correlations are uniform

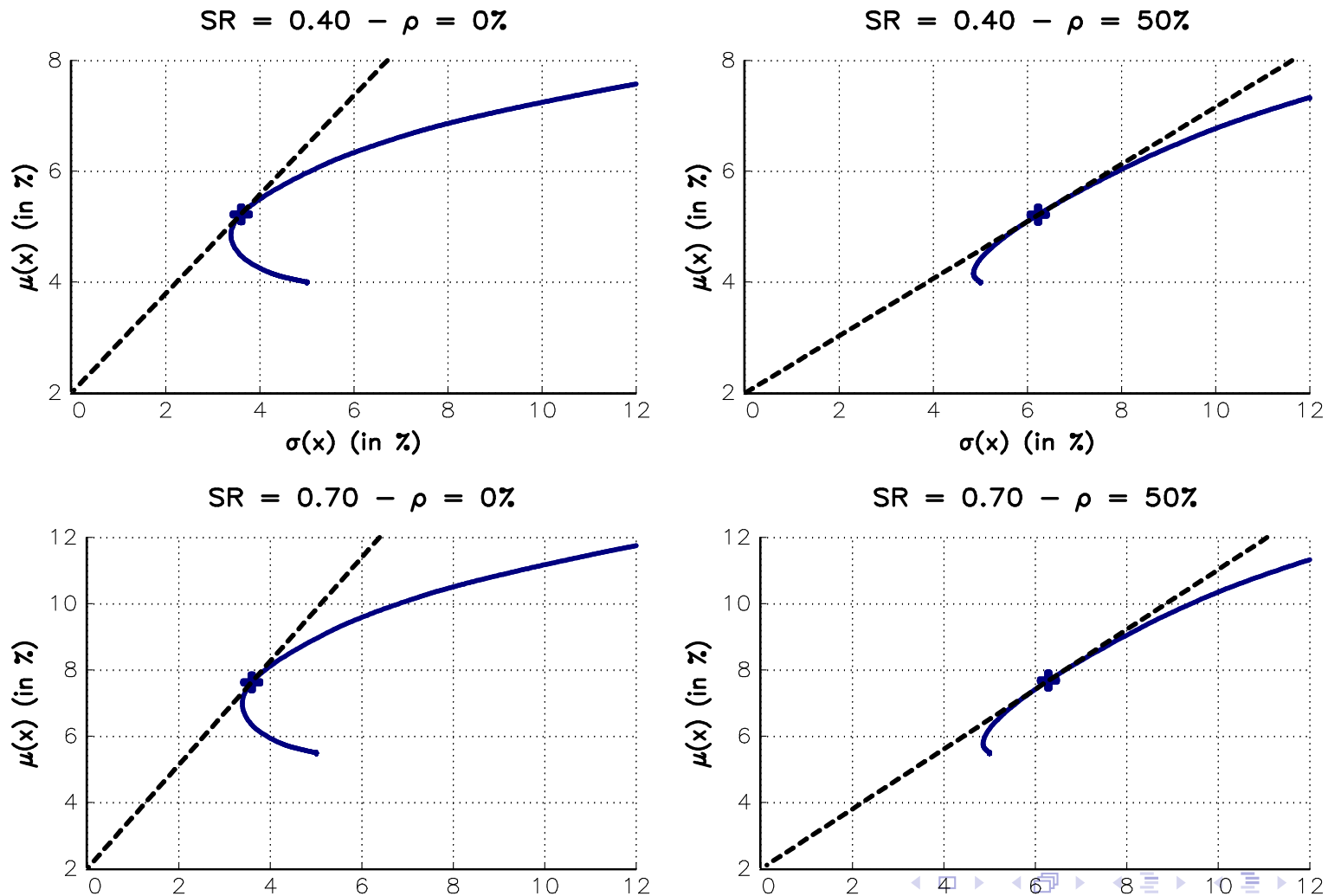


Figure 2.8, Page 125

Figure: Location of the ERC portfolio in the mean-variance diagram when the Sharpe ratios are identical and the asset correlations are not uniform

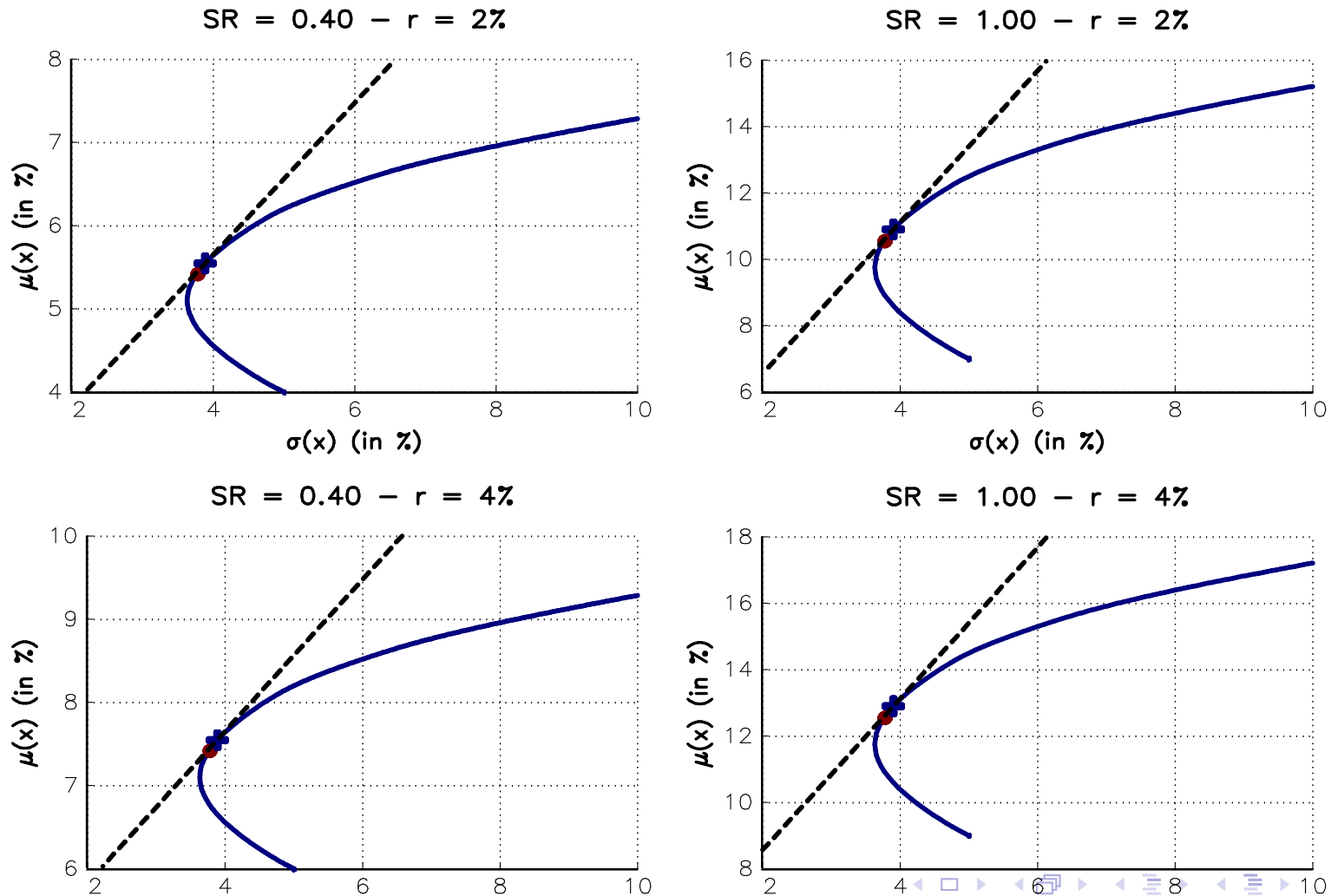


Figure: Geometry of the Lorenz curve

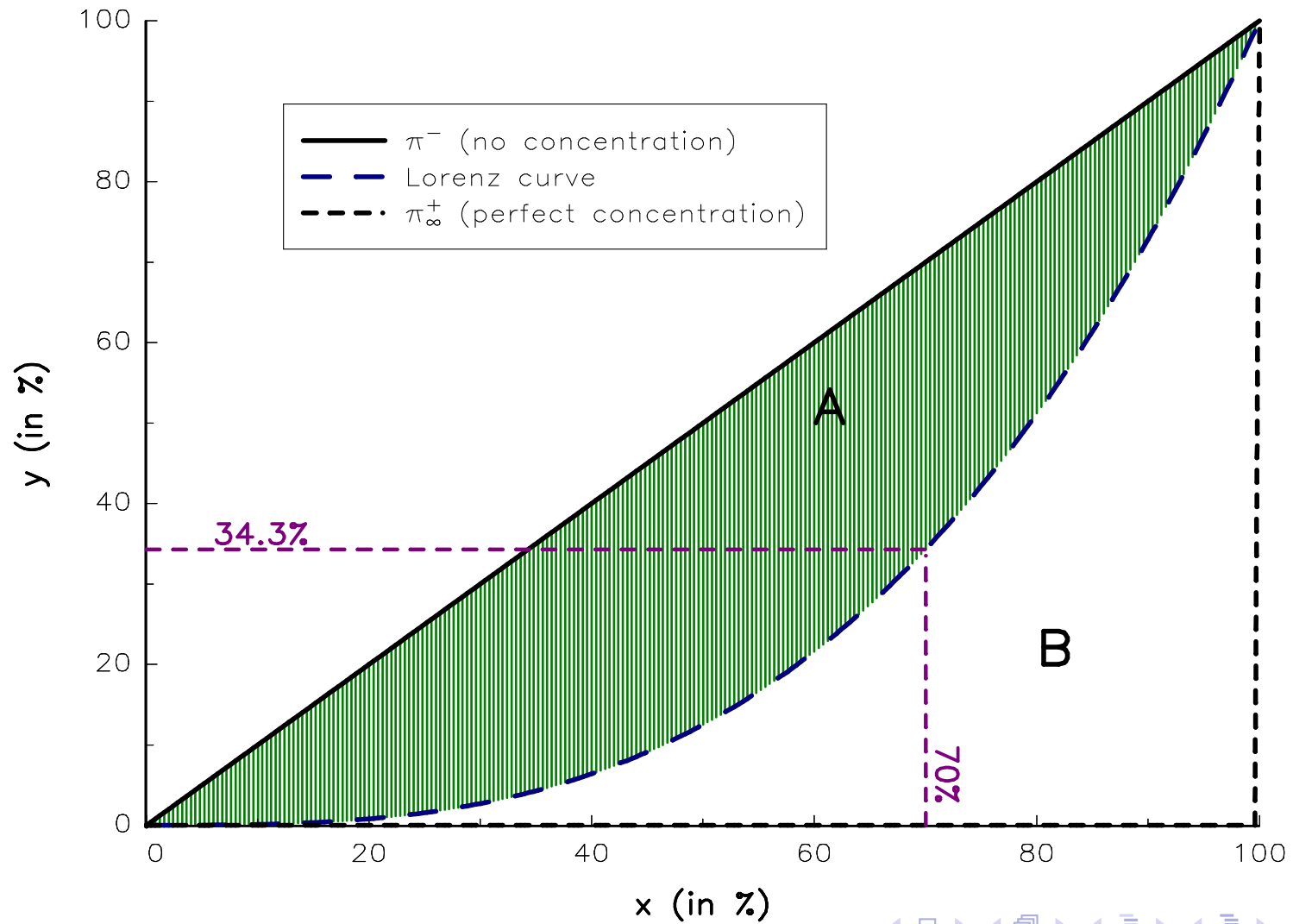


Table 2.22, Page 129

Table: Diversification measures of MV, ERC, MDP and EW portfolios

Asset	MV		ERC		MDP		EW	
	x_i	RC_i^*	x_i	RC_i^*	x_i	RC_i^*	x_i	RC_i^*
1	0.00	0.00	15.70	16.67	0.00	0.00	16.67	16.18
2	3.61	3.61	17.84	16.67	0.00	0.00	16.67	14.08
3	96.39	96.39	28.03	16.67	0.00	0.00	16.67	8.68
4	0.00	0.00	13.08	16.67	0.00	0.00	16.67	19.78
5	0.00	0.00	10.86	16.67	42.86	50.00	16.67	24.43
6	0.00	0.00	14.49	16.67	57.14	50.00	16.67	16.86
$\sigma(x)$	13.99		19.53		26.56		21.39	
$\mathcal{D}(x)$	0.98		0.80		0.77		0.80	
\mathcal{H}^*	0.92	0.92	0.02	0.00	0.41	0.40	0.00	0.02
\mathcal{G}	0.82	0.82	0.17	0.00	0.69	0.67	0.00	0.16
\mathcal{I}^*	1.17	1.17	5.71	6.00	1.98	2.00	6.00	5.74

Table 2.23, Page 132

Table: Risk decomposition of WB, RB and MR portfolios

Portfolio	Asset	x_i	MR_i	RC_i	RC_i^*
WB	1	50.00	41.59	20.80	71.40
	2	20.00	24.18	4.84	16.60
	3	30.00	11.65	3.50	12.00
Expected shortfall				29.13	
RB	1	30.65	39.07	11.97	50.00
	2	21.04	22.76	4.79	20.00
	3	48.32	14.87	7.18	30.00
Expected shortfall				23.94	
MR	1	0.00	29.11	0.00	0.00
	2	30.34	18.81	5.71	30.34
	3	69.66	18.81	13.10	69.66
Expected shortfall				18.81	

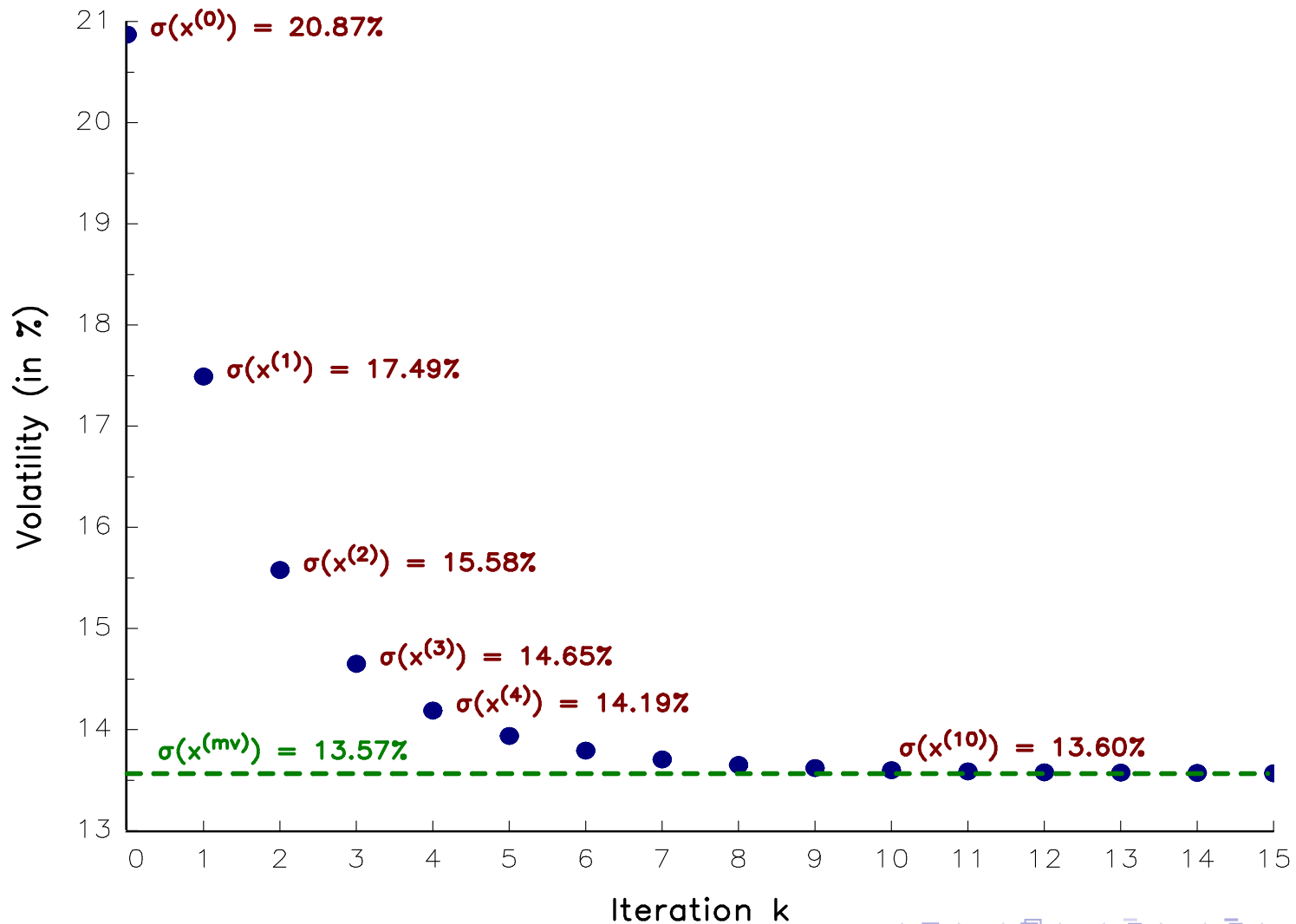
Table 2.24, Page 134

Table: Weights and risk contributions of the iterative RB portfolio $x^{(k)}$

Portfolio	Asset	x_i	RC_i^*	Portfolio	Asset	x_i	RC_i^*
$x^{(0)}$	1	50.00	70.43	$x^{(1)}$	1	31.15	50.00
	2	20.00	15.93		2	21.90	20.00
	3	30.00	13.64		3	46.96	30.00
	Volatility		20.87		Volatility		17.49
$x^{(2)}$	1	18.52	31.15	$x^{(3)}$	1	11.04	18.52
	2	22.81	21.90		2	23.71	22.81
	3	58.67	46.96		3	65.25	58.67
	Volatility		15.58		Volatility		14.65
$x^{(4)}$	1	6.67	11.04	$x^{(5)}$	1	4.07	6.67
	2	24.76	23.71		2	25.86	24.76
	3	68.57	65.25		3	70.07	68.57
	Volatility		14.19		Volatility		13.94
$x^{(6)}$	1	2.49	4.07	x_{mv}	1	0.00	0.00
	2	26.87	25.86		2	30.34	30.34
	3	70.63	70.07		3	69.66	69.66
	Volatility		13.79		Volatility		13.57

Figure 2.10, Page 134

Figure: Convergence of the iterative RB portfolio $x^{(k)}$ to the MV portfolio



Tables 2.25 & 2.26, Page 139

Table: Risk decomposition of Portfolio #1 with respect to the synthetic assets

Asset i	x_i	$\mathcal{M}R_i$	$\mathcal{R}C_i$	$\mathcal{R}C_i^*$
\mathcal{A}_1	36.00	9.44	3.40	33.33
\mathcal{A}_2	38.00	8.90	3.38	33.17
\mathcal{A}_3	26.00	13.13	3.41	33.50

Table: Risk decomposition of Portfolio #1 with respect to the primary assets

Asset j	y_j	$\mathcal{M}R_j$	$\mathcal{R}C_j$	$\mathcal{R}C_j^*$
\mathcal{A}'_1	9.00	3.53	0.32	3.12
\mathcal{A}'_2	9.00	7.95	0.72	7.02
\mathcal{A}'_3	31.50	19.31	6.08	59.69
\mathcal{A}'_4	31.50	6.95	2.19	21.49
\mathcal{A}'_5	9.50	0.93	0.09	0.87
\mathcal{A}'_6	9.50	8.39	0.80	7.82

Tables 2.27 & 2.28, Pages 139 & 140

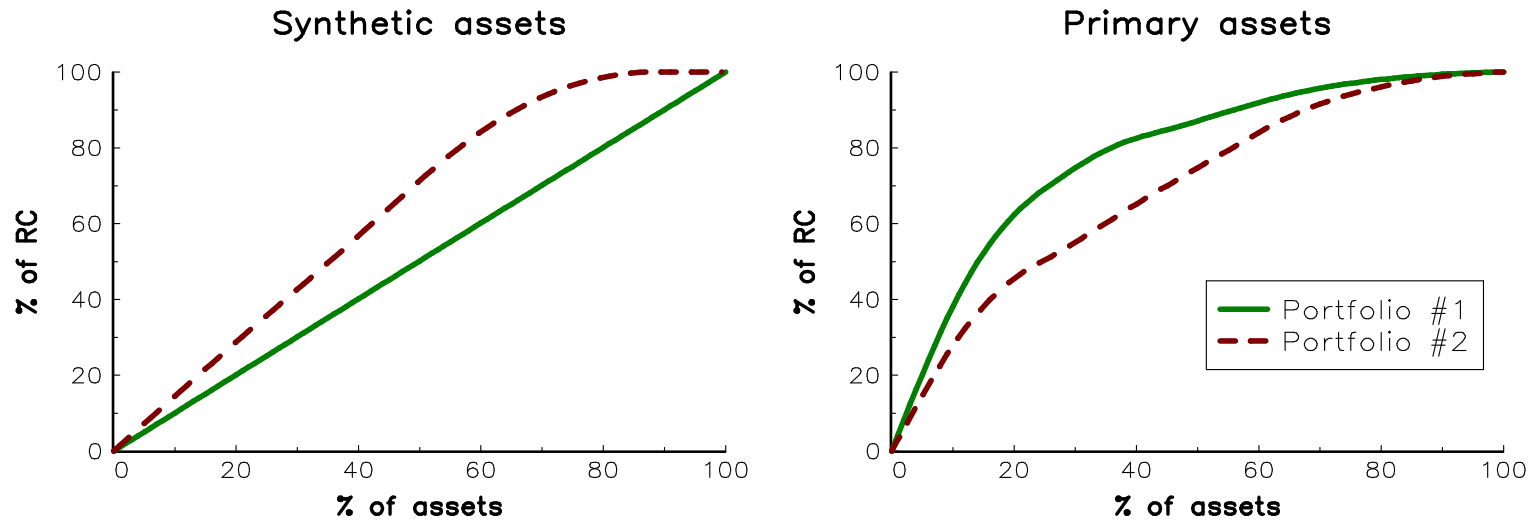
Table: Risk decomposition of Portfolio #2 with respect to the synthetic assets

Asset i	x_i	MR_i	RC_i	RC_i^*
\mathcal{A}_1	48.00	9.84	4.73	49.91
\mathcal{A}_2	50.00	9.03	4.51	47.67
\mathcal{A}_3	2.00	11.45	0.23	2.42

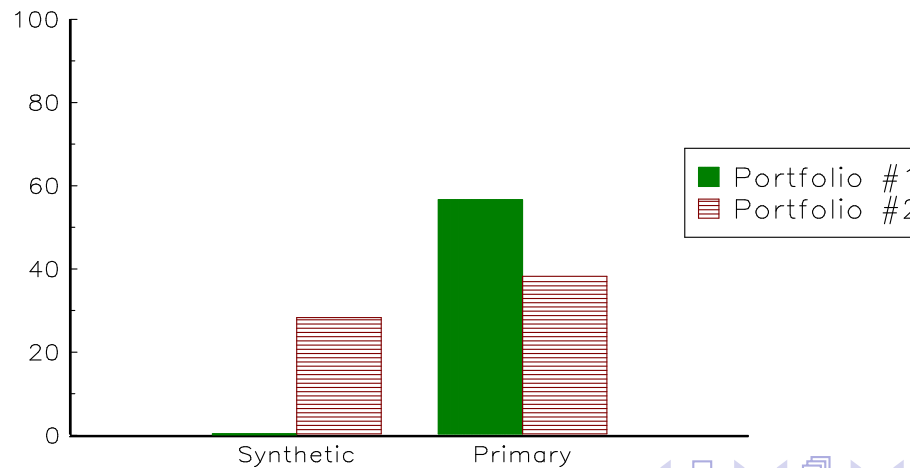
Table: Risk decomposition of Portfolio #2 with respect to the primary assets

Asset j	y_j	MR_j	RC_j	RC_j^*
\mathcal{A}'_1	12.00	5.07	0.61	6.43
\mathcal{A}'_2	12.00	11.41	1.37	14.46
\mathcal{A}'_3	25.50	16.84	4.29	45.35
\mathcal{A}'_4	25.50	6.06	1.55	16.33
\mathcal{A}'_5	12.50	1.32	0.17	1.74
\mathcal{A}'_6	12.50	11.88	1.49	15.69

Figure: Lorenz curve of risk contributions



Gini coefficient (in %)



Tables 2.29 & 2.30, Page 143

Table: Risk decomposition of the EW portfolio with respect to the assets

Asset	x_i	MR_i	RC_i	RC_i^*
1	25.00	18.81	4.70	21.97
2	25.00	23.72	5.93	27.71
3	25.00	24.24	6.06	28.32
4	25.00	18.83	4.71	22.00
Volatility			21.40	

Table: Risk decomposition of the EW portfolio with respect to the risk factors

Factor	y_j	MR_j	RC_j	RC_j^*
\mathcal{F}_1	100.00	17.22	17.22	80.49
\mathcal{F}_2	22.50	9.07	2.04	9.53
\mathcal{F}_3	35.00	6.06	2.12	9.91
$\tilde{\mathcal{F}}_1$	2.75	0.52	0.01	0.07
Volatility			21.40	

Tables 2.31 & 2.32, Pages 144 & 145

Table: Risk decomposition of the RFP portfolio with respect to the risk factors

Factor	y_j	MR_j	RC_j	RC_j^*
\mathcal{F}_1	93.38	11.16	10.42	49.00
\mathcal{F}_2	24.02	22.14	5.32	25.00
\mathcal{F}_3	39.67	13.41	5.32	25.00
$\tilde{\mathcal{F}}_1$	16.39	1.30	0.21	1.00
Volatility				21.27

Table: Risk decomposition of the RFP portfolio with respect to the assets

Asset	x_i	MR_i	RC_i	RC_i^*
1	15.08	17.44	2.63	12.36
2	38.38	23.94	9.19	43.18
3	0.89	21.82	0.20	0.92
4	45.65	20.29	9.26	43.54
Volatility				21.27

Tables 2.33 & 2.34, Pages 145 & 146

Table: Risk decomposition of the balanced RFP portfolio with respect to the risk factors

Factor	y_j	MR_j	RC_j	RC_j^*
F_1	91.97	7.91	7.28	33.26
F_2	25.78	28.23	7.28	33.26
F_3	42.22	17.24	7.28	33.26
\tilde{F}_1	6.74	0.70	0.05	0.21
Volatility				21.88

Table: Risk decomposition of the balanced RFP portfolio with respect to the assets

Asset	x_i	MR_i	RC_i	RC_i^*
1	0.30	16.11	0.05	0.22
2	39.37	23.13	9.11	41.63
3	0.31	20.93	0.07	0.30
4	60.01	21.09	12.66	57.85
Volatility				21.88

Table: Balanced RFP portfolios with $x_i \geq 10\%$

Criterion	$\mathcal{H}(x)$	$\mathcal{G}(x)$	$\mathcal{I}(x)$
x_1	10.00	10.00	10.00
x_2	22.08	18.24	24.91
x_3	10.00	10.00	10.00
x_4	57.92	61.76	55.09
\mathcal{H}^*	0.0436	0.0490	0.0453
\mathcal{G}	0.1570	0.1476	0.1639
\mathcal{I}^*	2.8636	2.8416	2.8643