Introduction to Risk Parity and Budgeting

Chapter 1 – Modern Portfolio Theory

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http://www.thierry-roncalli.com/RiskParityBook.html

May 22, 2013

Figure 1.1, Page 6

Figure: Optimized Markowitz portfolios

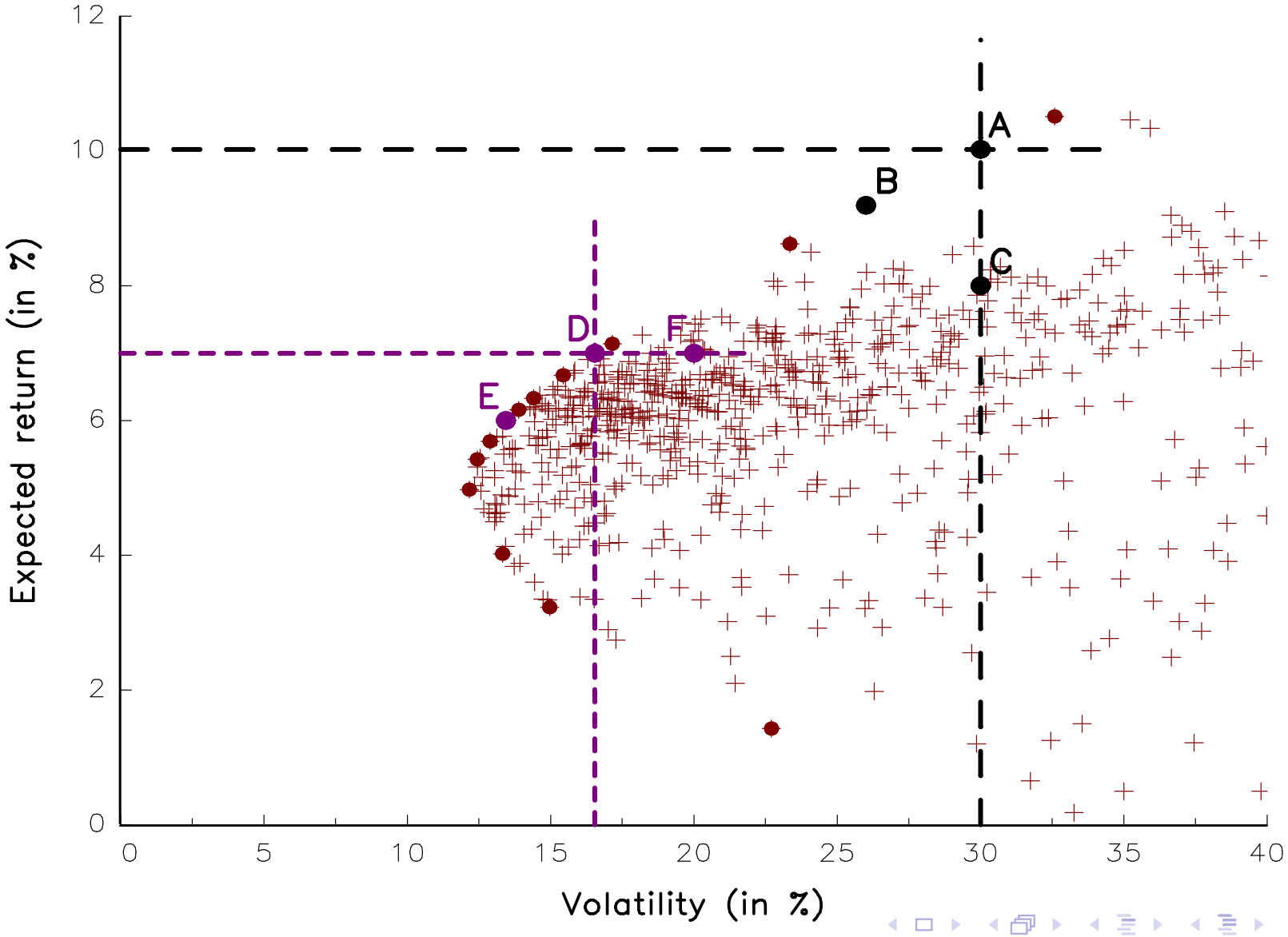


Figure 1.2, Page 8

Figure: The efficient frontier of Markowitz

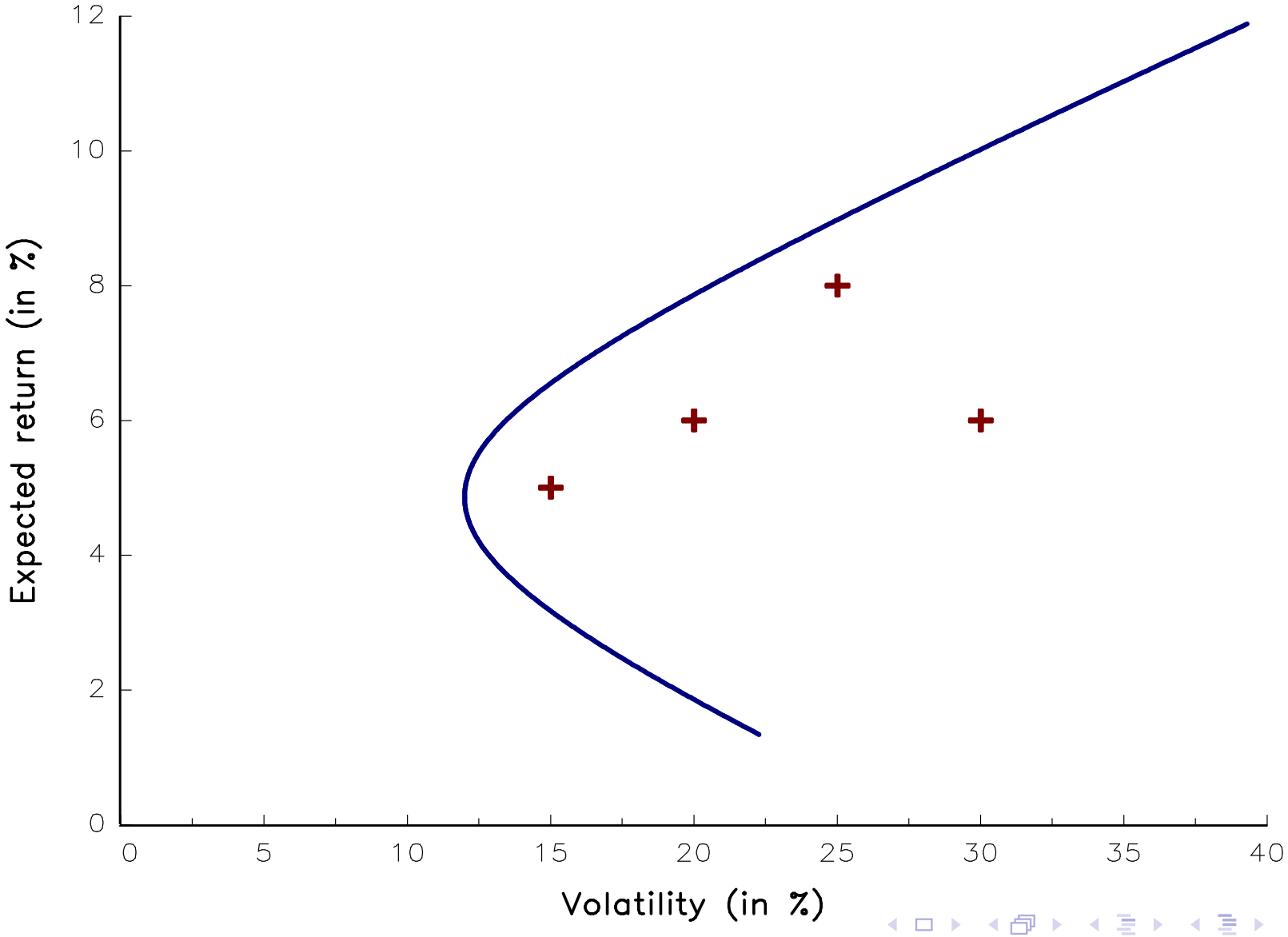


Table 1.1, Page 7

Table: Solving the ϕ-problem

ϕ	$+\infty$	5.00	2.00	1.00	0.50	0.20
$x_1^\star$	72.74	68.48	62.09	51.44	30.15	−33.75
$x_2^\star$	49.46	35.35	14.17	−21.13	−91.72	−303.49
$x_3^\star$	−20.45	12.61	62.21	144.88	310.22	806.22
$x_4^\star$	−1.75	−16.44	−38.48	−75.20	−148.65	−368.99
$\mu(x^\star)$	4.86	5.57	6.62	8.38	11.90	22.46
$\sigma(x^\star)$	12.00	12.57	15.23	22.27	39.39	94.57

Tables 1.2 & 1.3, Page 8

Table: Solving the unconstrained μ-problem

$\mu^\star$	5.00	6.00	7.00	8.00	9.00
$x_1^\star$	71.92	65.87	59.81	53.76	47.71
$x_2^\star$	46.73	26.67	6.62	−13.44	−33.50
$x_3^\star$	−14.04	32.93	79.91	126.88	173.86
$x_4^\star$	−4.60	−25.47	−46.34	−67.20	−88.07
$\sigma(x^\star)$	12.02	13.44	16.54	20.58	25.10
ϕ	25.79	3.10	1.65	1.12	0.85

Table: Solving the unconstrained σ-problem

$\sigma^\star$	15.00	20.00	25.00	30.00	35.00
$x_1^\star$	62.52	54.57	47.84	41.53	35.42
$x_2^\star$	15.58	−10.75	−33.07	−54.00	−74.25
$x_3^\star$	58.92	120.58	172.85	221.88	269.31
$x_4^\star$	−37.01	−64.41	−87.62	−109.40	−130.48
$\mu(x^\star)$	6.55	7.87	8.98	10.02	11.03
ϕ	2.08	1.17	0.86	0.68	0.57

Figure 1.3, Page 10

Figure: The efficient frontier with some weight constraints

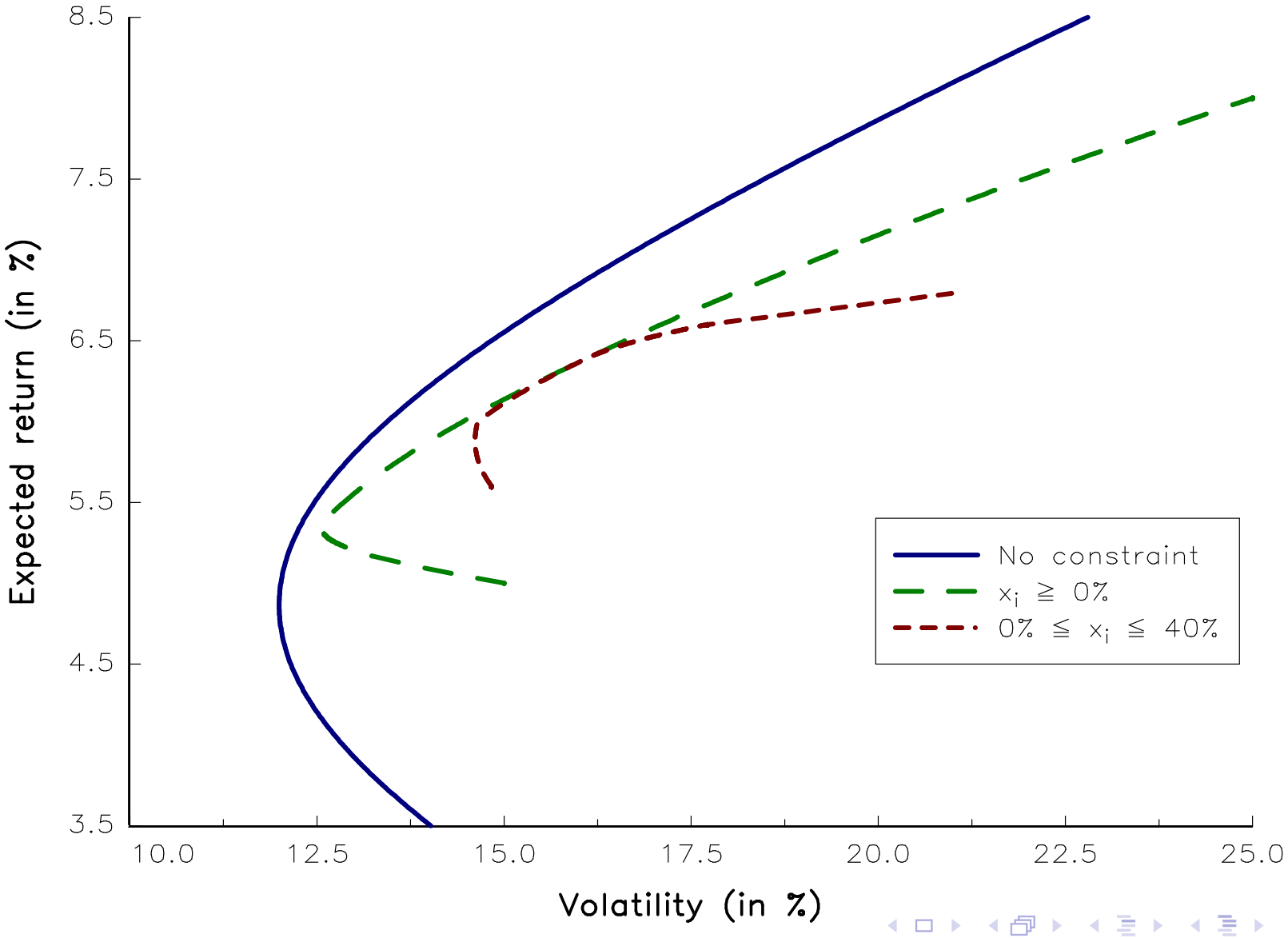


Table 1.4, Page 10

Table: Solving the σ-problem with weight constraints

	$x_i \in \mathbb{R}$		$x_i \geq 0$		$0 \leq x_i \leq 40\%$	
$\sigma^\star$	15.00	20.00	15.00	20.00	15.00	20.00
$x_1^\star$	62.52	54.57	45.59	24.88	40.00	6.13
$x_2^\star$	15.58	−10.75	24.74	4.96	34.36	40.00
$x_3^\star$	58.92	120.58	29.67	70.15	25.64	40.00
$x_4^\star$	−37.01	−64.41	0.00	0.00	0.00	13.87
$\mu(x^\star)$	6.55	7.87	6.14	7.15	6.11	6.74
ϕ	2.08	1.17	1.61	0.91	1.97	0.28

Figure 1.4, Page 13

Figure: The capital market line

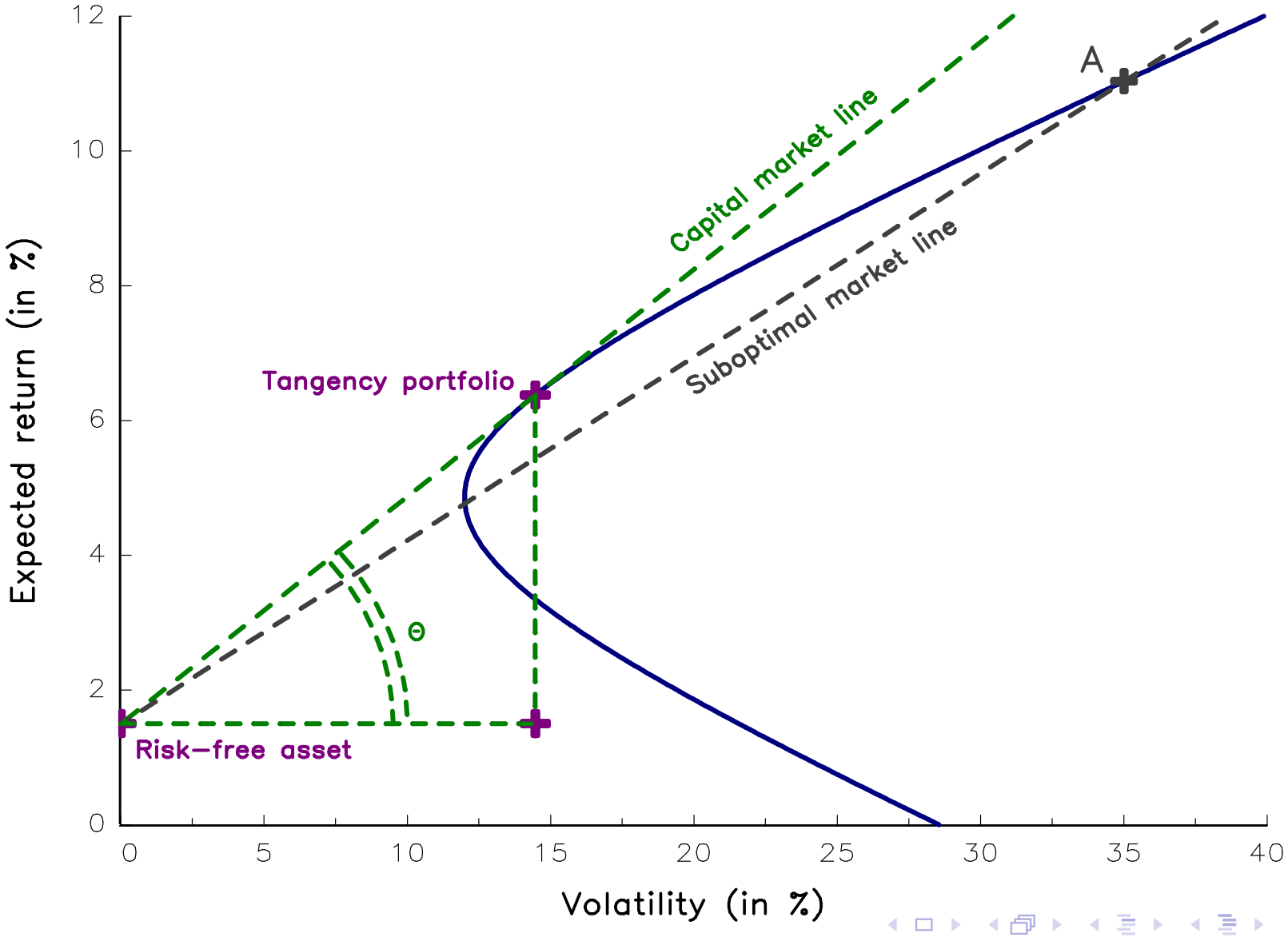
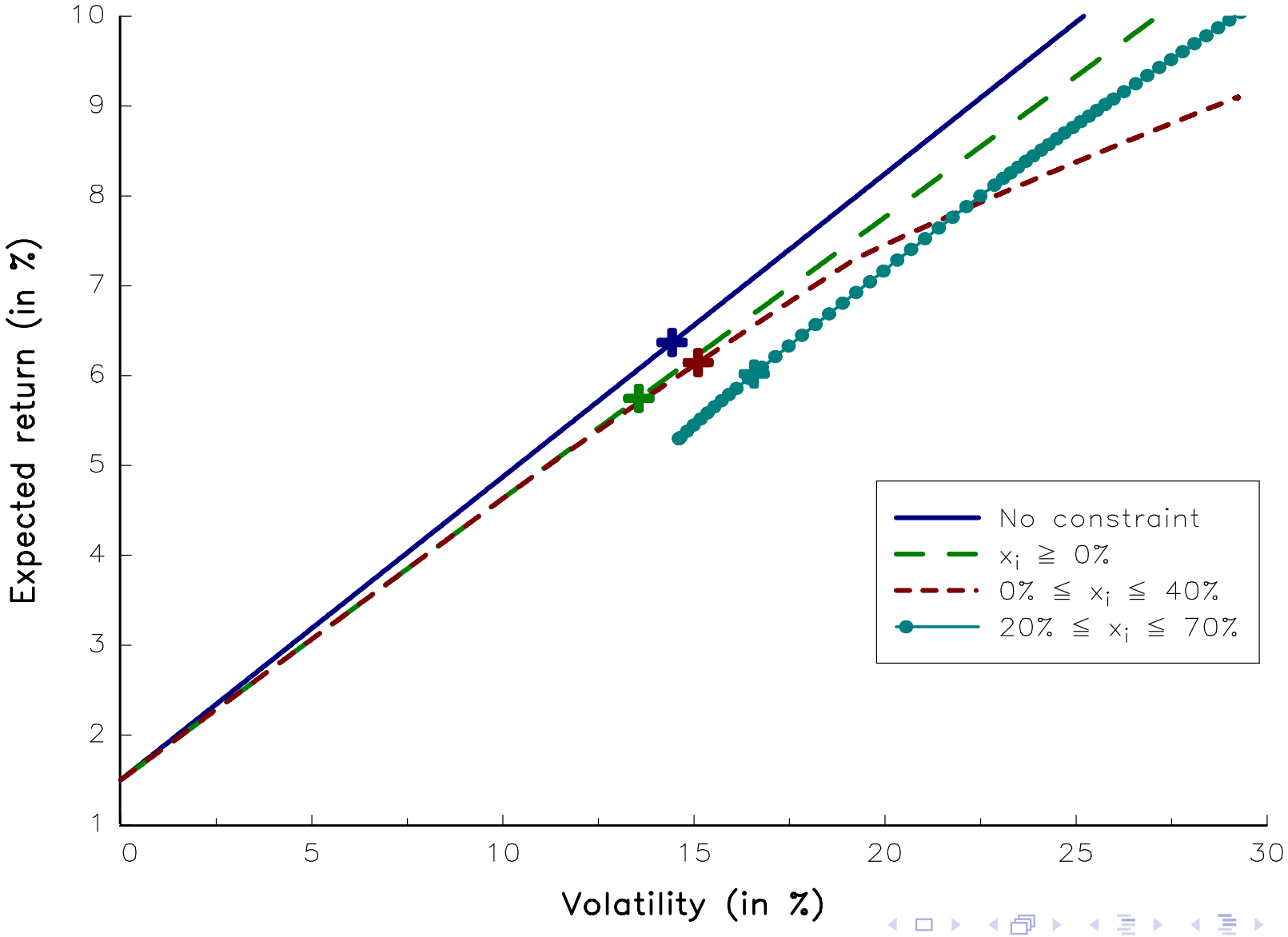


Figure 1.5, Page 15

Figure: The efficient frontier with a risk-free asset



Tables 1.5 & 1.6, Pages 17 & 18

Table: Computation of the beta

Portfolio	$\mu(y)$	$\beta(y \mid x^\star)$	$\pi(y \mid x^\star)$
$\mathbf{e}_1$	3.50	0.72	3.50
$\mathbf{e}_2$	4.50	0.92	4.50
$\mathbf{e}_3$	6.50	1.33	6.50
$\mathbf{e}_4$	4.50	0.92	4.50
x_{ew}	4.75	0.98	4.75

Table: Computation of the beta with a constrained tangency portfolio

Portfolio	$\mu(y)$	$\beta(y \mid x^\star)$	$\pi(y \mid x^\star)$
$\mathbf{e}_1$	3.50	0.83	3.50
$\mathbf{e}_2$	4.50	1.06	4.50
$\mathbf{e}_3$	6.50	1.53	6.50
$\mathbf{e}_4$	4.50	1.54	6.53
x_{ew}	4.75	1.24	5.26

Figure 1.6, Page 20

Figure: The efficient frontier with a benchmark

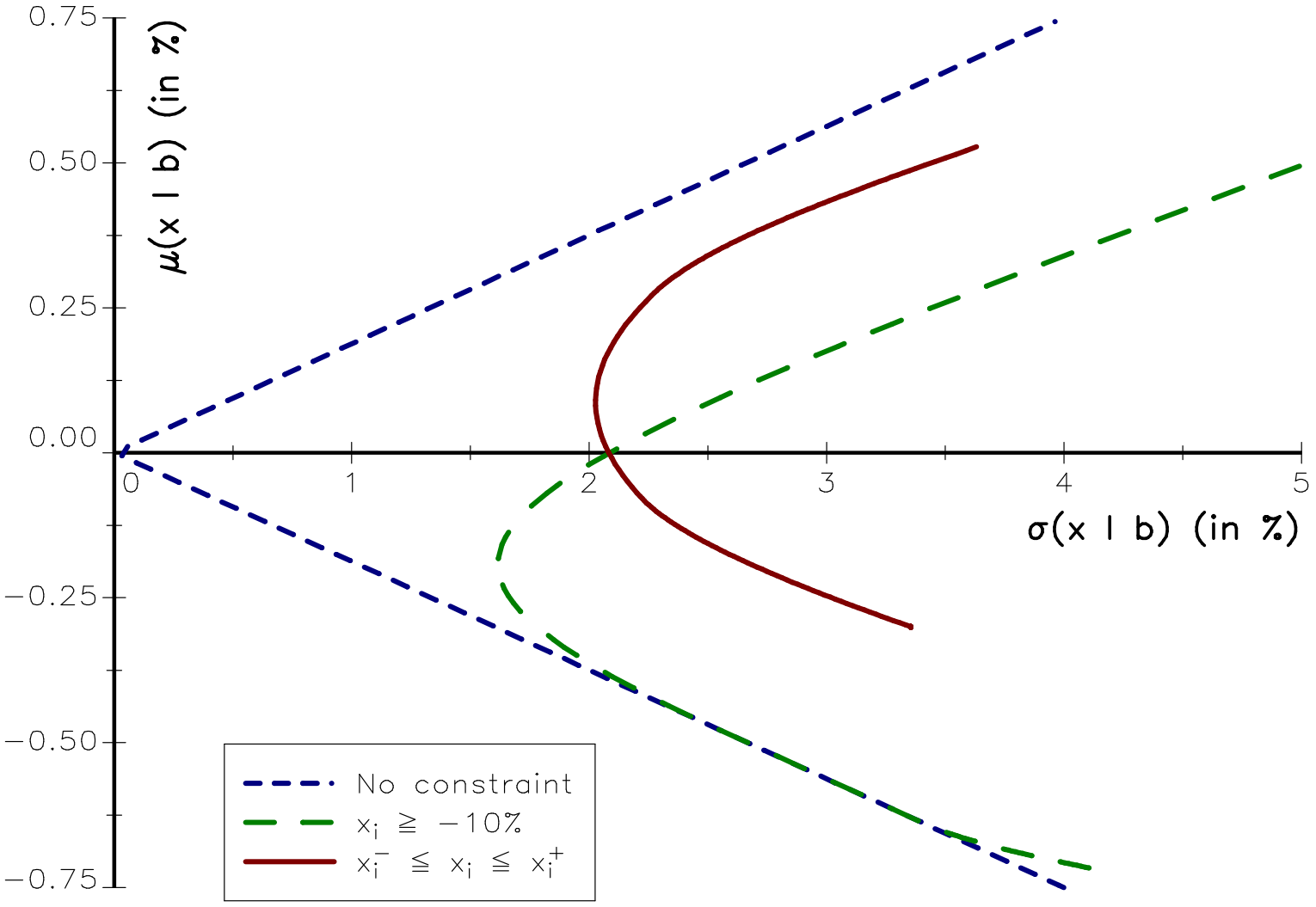


Figure 1.7, Page 22

Figure: The tangency portfolio with respect to a benchmark

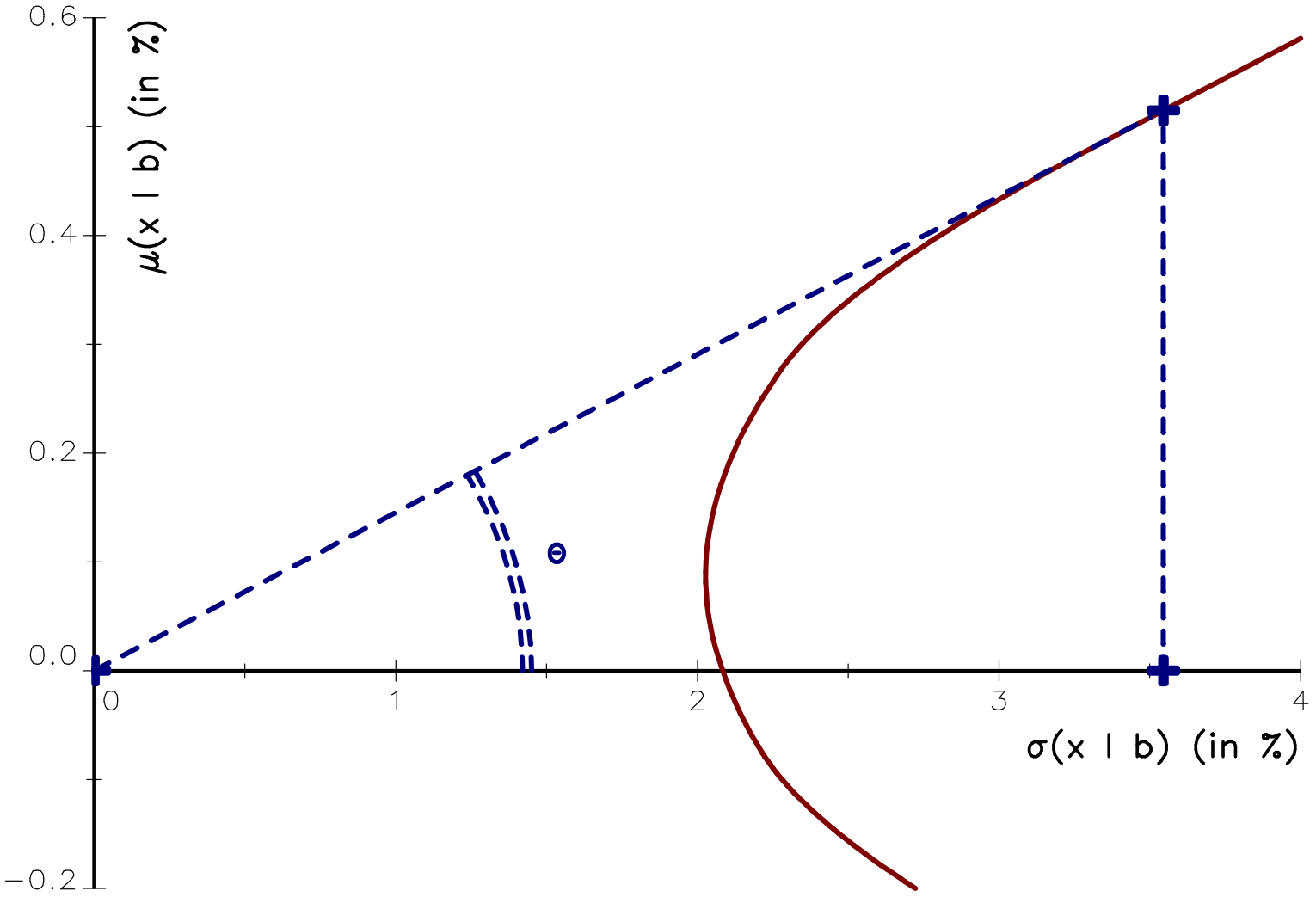


Table 1.7, Page 26

Table: Black-Litterman portfolios

	#0	#1	#2	#3	#4	#5
$x_1^\star$	40.00	33.41	51.16	36.41	38.25	39.77
$x_2^\star$	30.00	51.56	39.91	42.97	42.72	32.60
$x_3^\star$	20.00	5.46	0.00	10.85	9.14	17.65
$x_4^\star$	10.00	9.58	8.93	9.77	9.89	9.98
$\sigma\left(x^\star \mid x_0\right)$	0.00	3.65	3.67	2.19	2.18	0.45

Result (1.16), Page 28

$$\hat{C} = \left(\begin{array}{cccc|ccccc} 1.00 & & & & & & & & \\ 0.88 & 1.00 & & & & & & & \\ 0.88 & 0.94 & 1.00 & & & & & & \\ 0.64 & 0.68 & 0.65 & 1.00 & & & & & \\ \hline 0.77 & 0.76 & 0.78 & 0.61 & 1.00 & & & & \\ 0.56 & 0.61 & 0.61 & 0.50 & 0.64 & 1.00 & & & \\ 0.53 & 0.61 & 0.57 & 0.53 & 0.60 & 0.57 & 1.00 & & \\ 0.64 & 0.68 & 0.67 & 0.68 & 0.68 & 0.60 & 0.66 & 1.00 & \end{array}\right)$$

Figure 1.8, Page 30

Figure: Trading hours of asynchronous markets (UTC time)

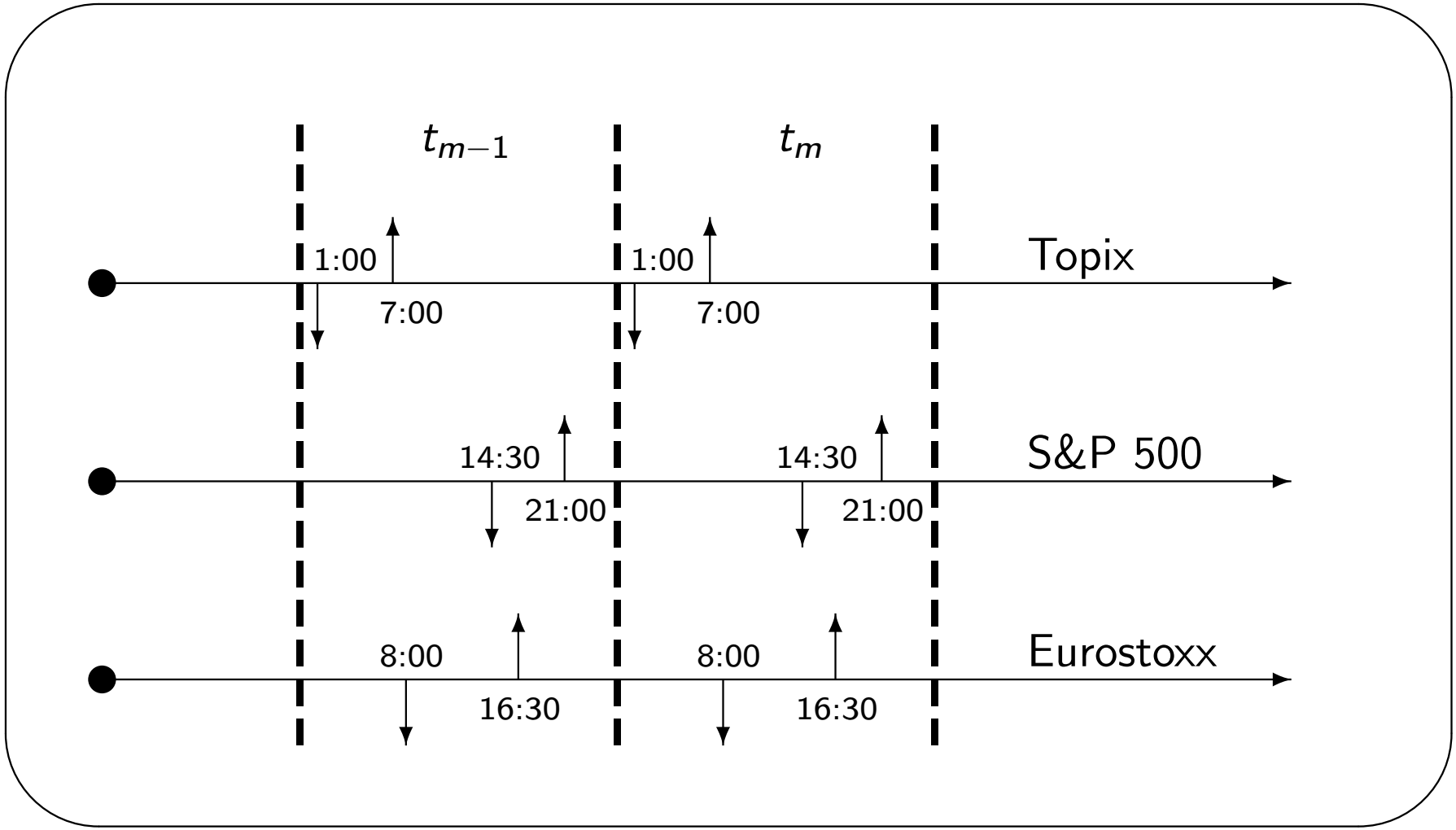


Figure 1.9, Page 31

Figure: Density of the estimator $\hat{\rho}$ with asynchronous returns

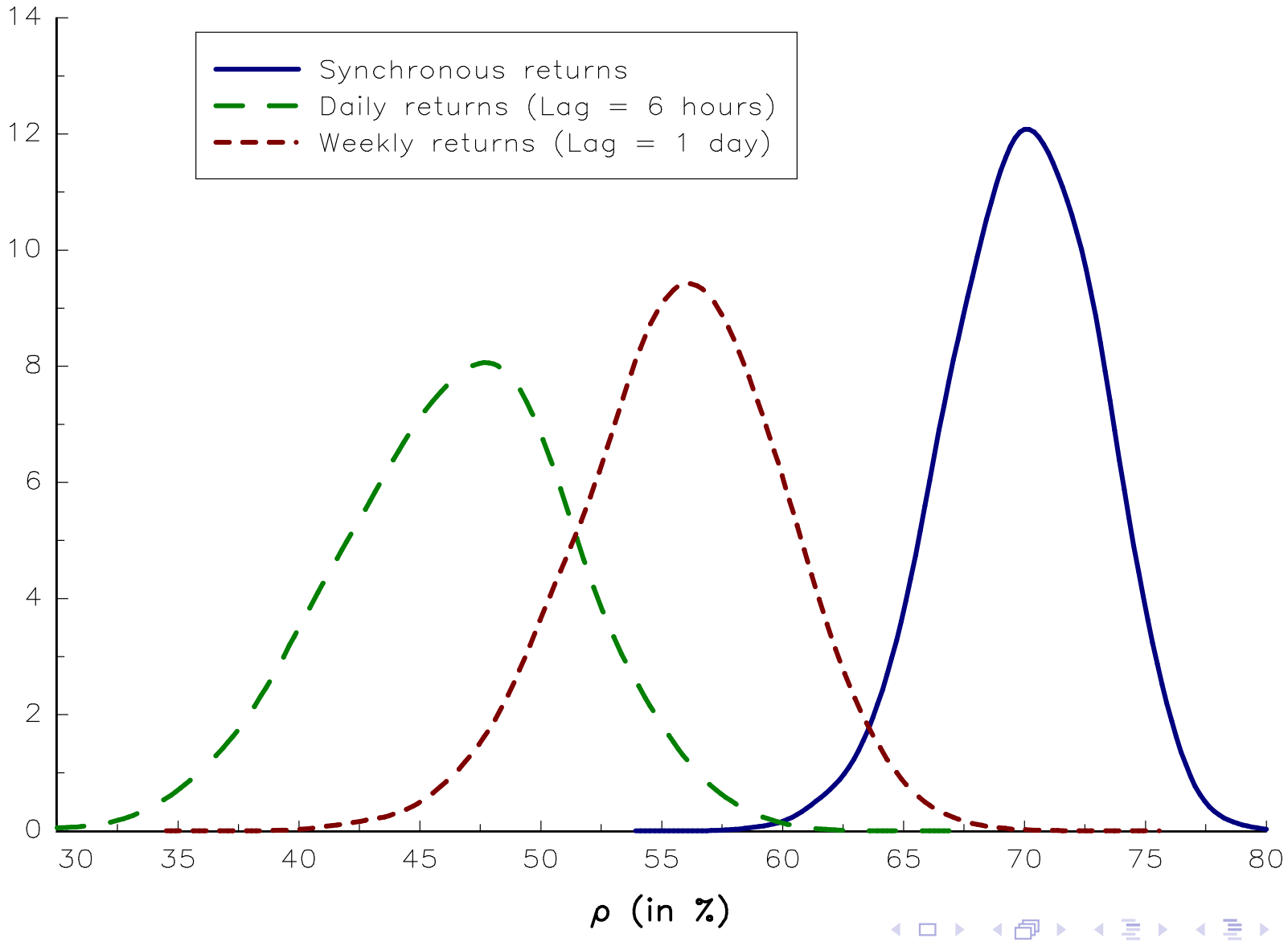


Figure 1.10, Page 33

Figure: Hayashi-Yoshida estimator

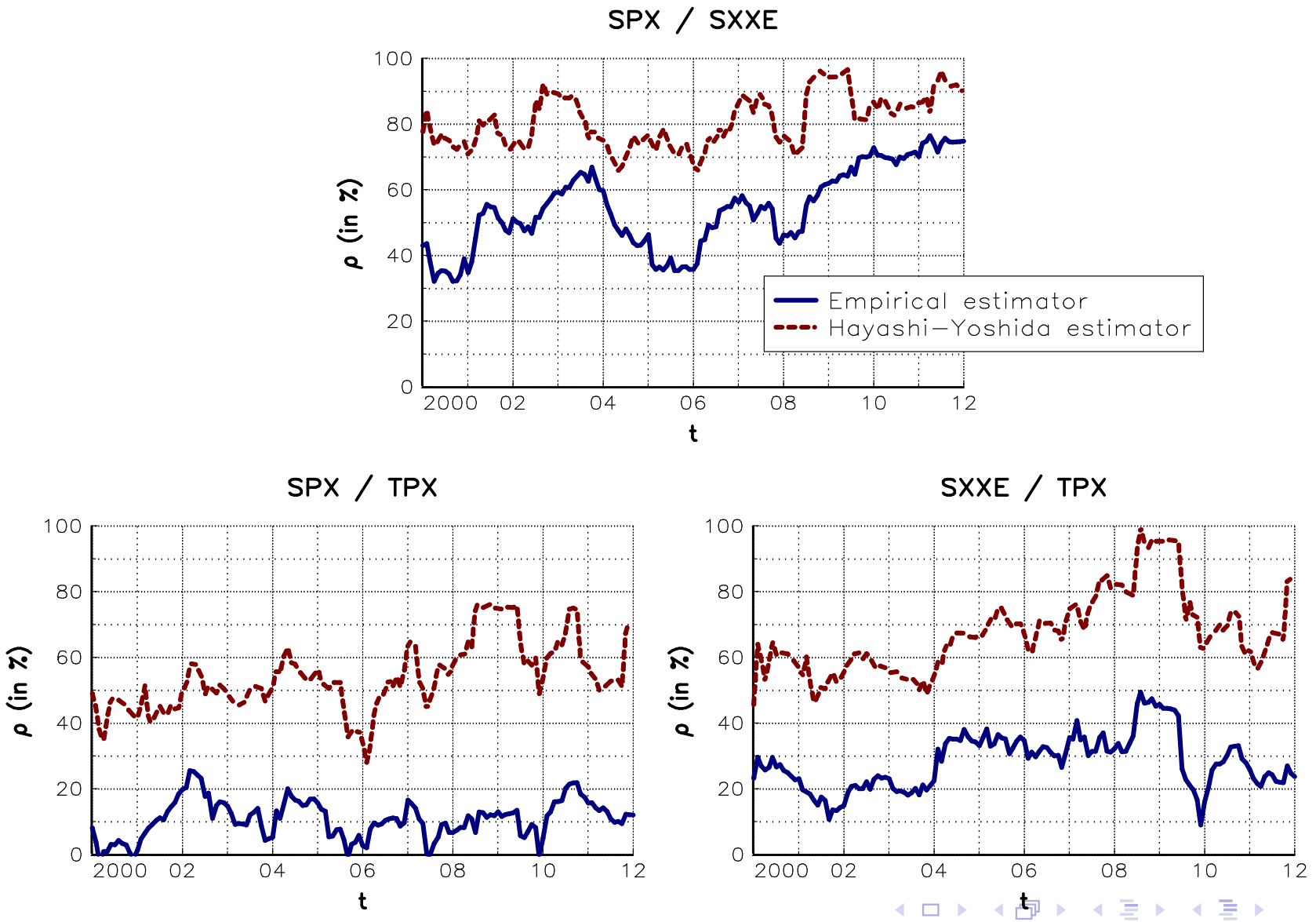


Figure 1.11, Page 35

Figure: Cumulative weight W_m of the IGARCH model

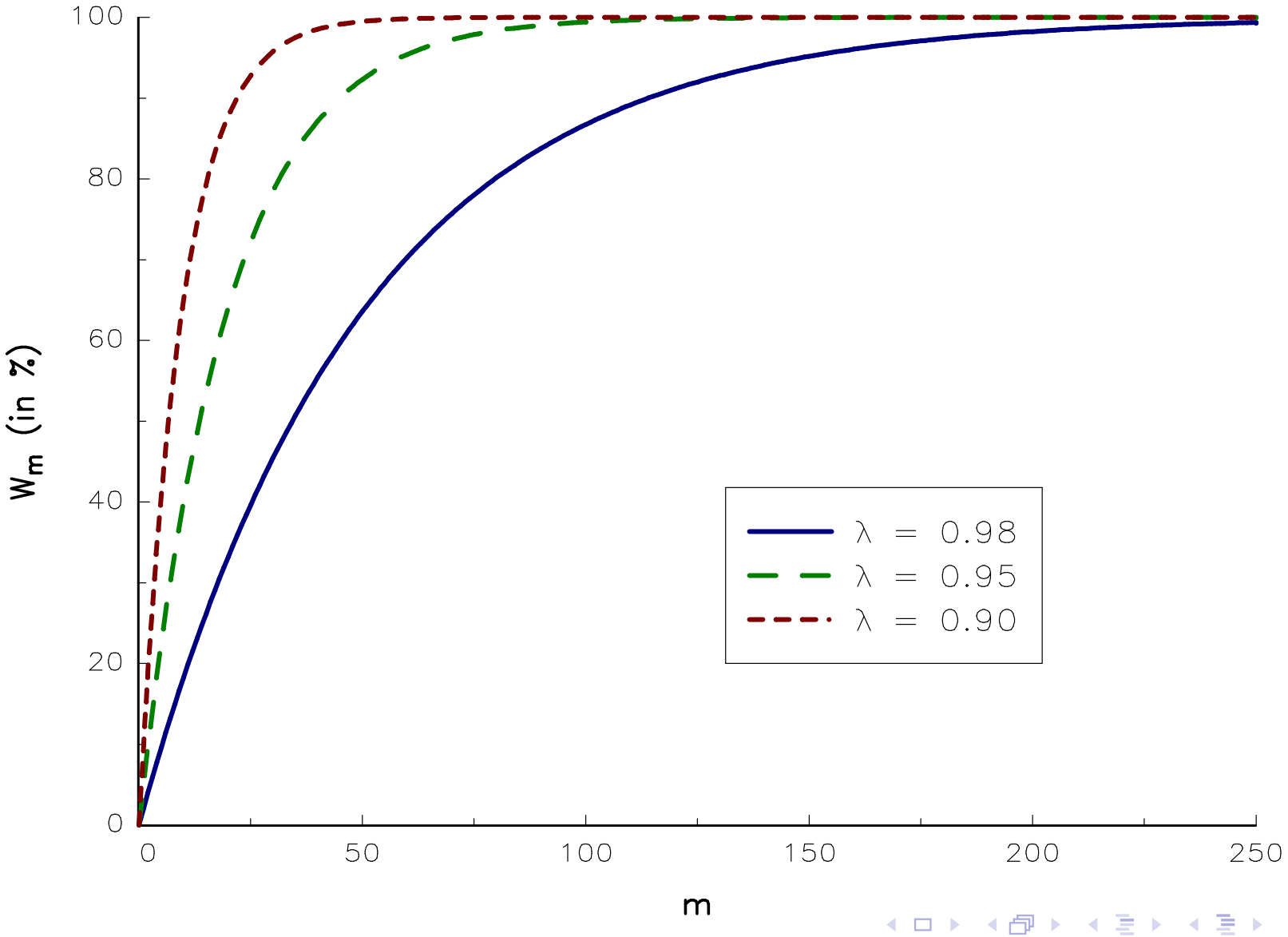


Figure 1.12, Page 36

Figure: Estimation of the S&P 500 volatility

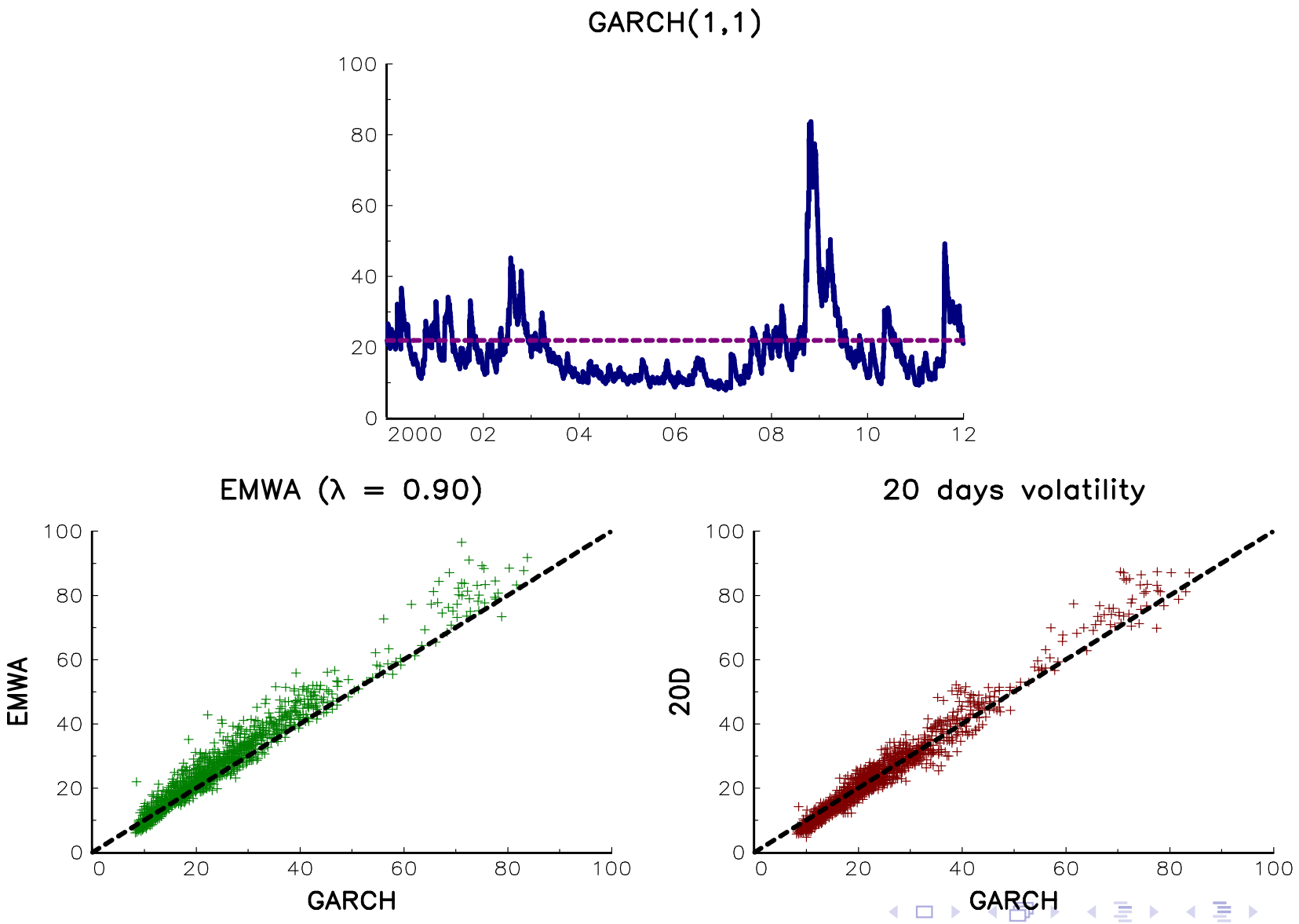
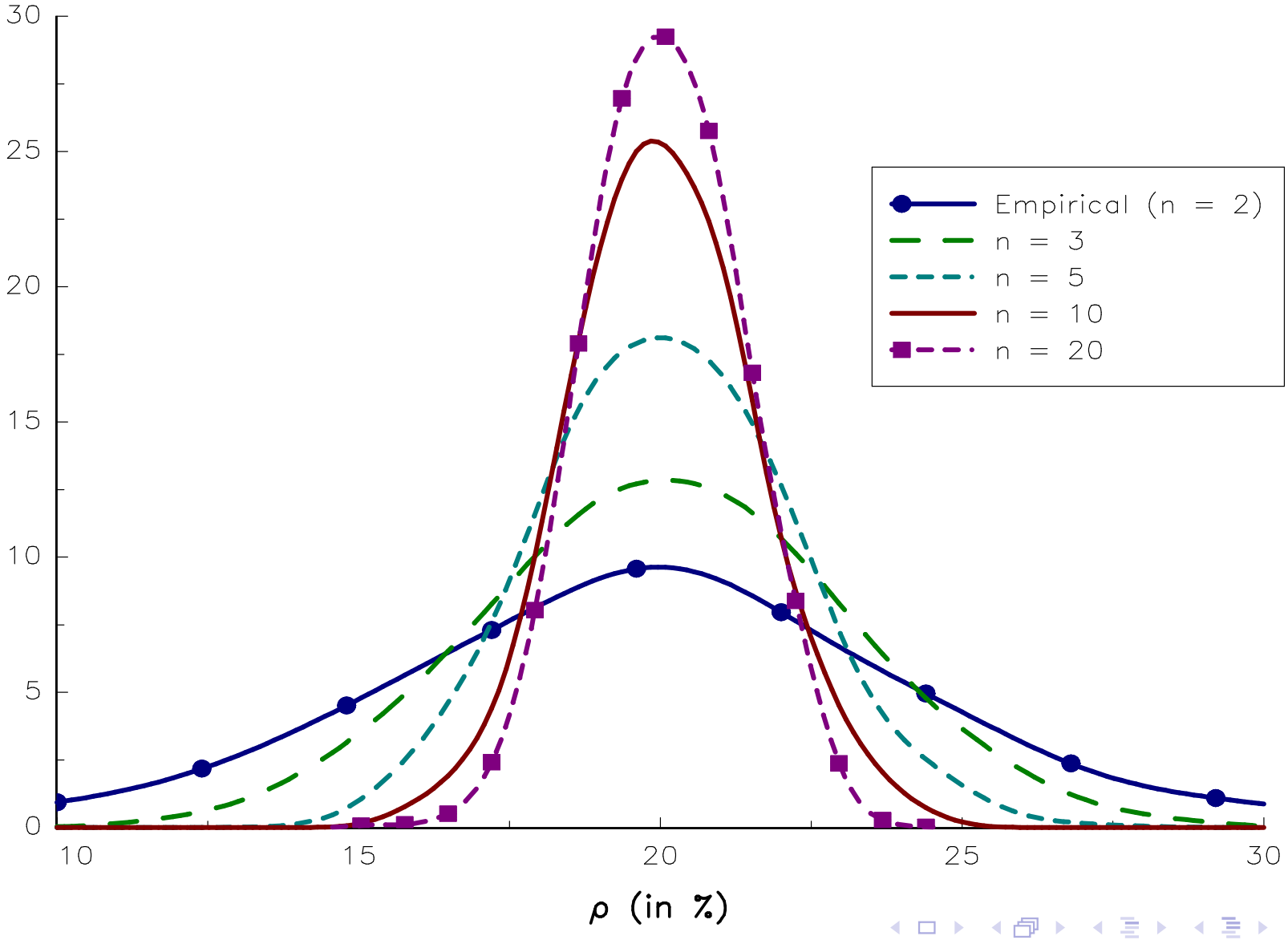


Figure 1.13, Page 38

Figure: Density of the uniform correlation estimator



Result (1.19), Page 39

$$\hat{C} = \left(\begin{array}{cccc|cccc} 1.00 & & & & & & & \\ 0.77 & 1.00 & & & & & & \\ 0.77 & 0.77 & 1.00 & & & & & \\ 0.77 & 0.77 & 0.77 & 1.00 & & & & \\ \hline 0.50 & 0.50 & 0.50 & 0.50 & 1.00 & & & \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.59 & 1.00 & & \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.59 & 0.59 & 1.00 & \\ 0.50 & 0.50 & 0.50 & 0.50 & 0.59 & 0.59 & 0.59 & 1.00 \end{array}\right)$$

Result (1.21), Page 39

$$\hat{C} = \left(\begin{array}{cccc|ccccc} 1.00 & & & & & & & & \\ 0.88 & 1.00 & & & & & & & \\ 0.88 & 0.94 & 1.00 & & & & & & \\ 0.63 & 0.67 & 0.66 & 1.00 & & & & & \\ \hline 0.73 & 0.78 & 0.78 & 0.63 & 1.00 & & & & \\ 0.58 & 0.62 & 0.60 & 0.54 & 0.59 & 1.00 & & & \\ 0.56 & 0.59 & 0.58 & 0.56 & 0.60 & 0.54 & 1.00 & & \\ 0.64 & 0.68 & 0.66 & 0.65 & 0.69 & 0.62 & 0.67 & 1.00 & \end{array}\right)$$

Table 1.8, Page 45

Table: Sensitivity of the MVO portfolio to input parameters

ρ		70%	90%		90%	
σ_2				18%	18%	
μ_1						9%
x_1	38.3	38.3	44.6	13.7	−8.0	60.6
x_2	20.2	25.9	8.9	56.1	74.1	−5.4
x_3	41.5	35.8	46.5	30.2	34.0	44.8

Figure 1.16, Page 46

Figure: Uncertainty of the efficient frontier

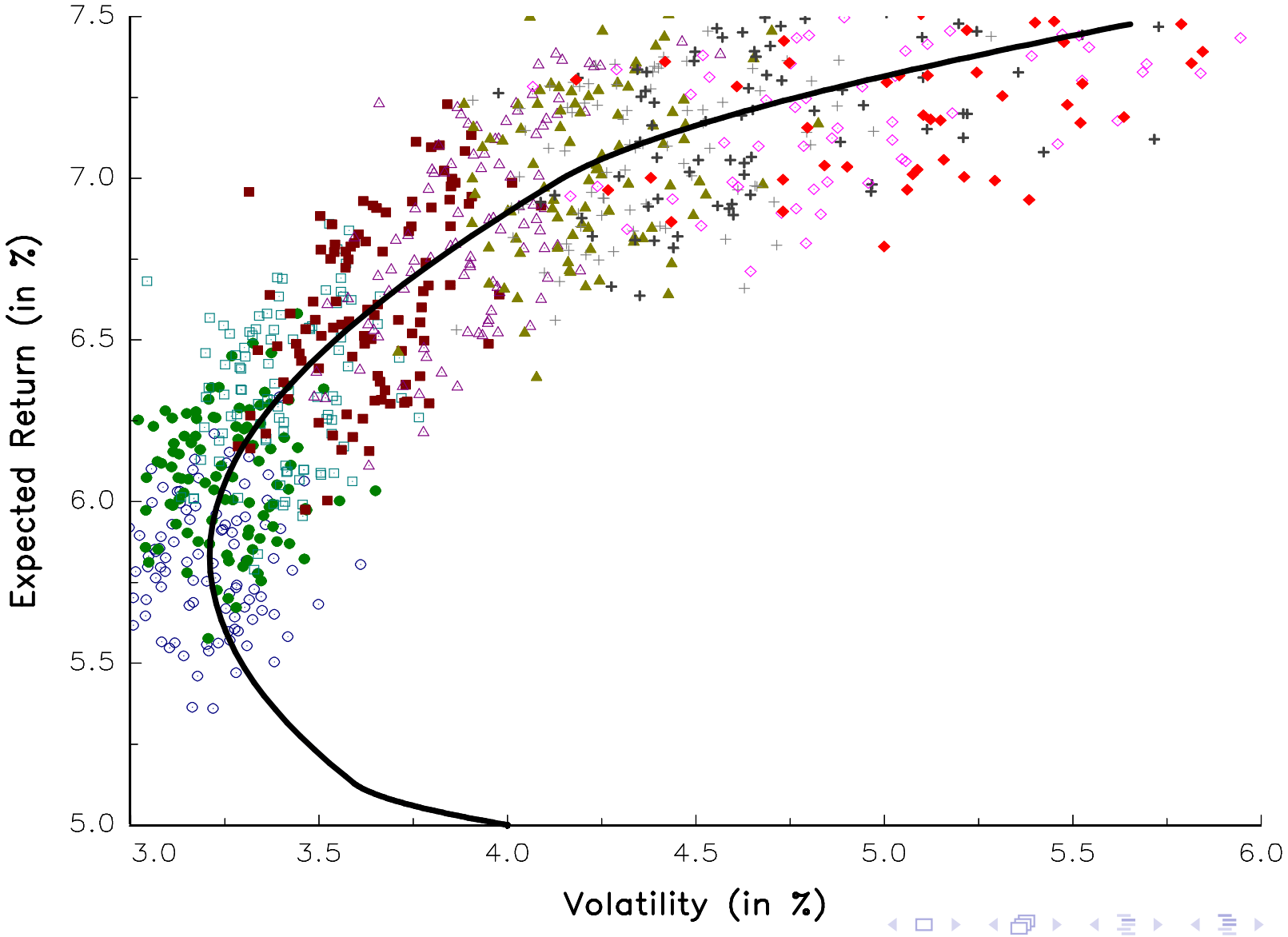
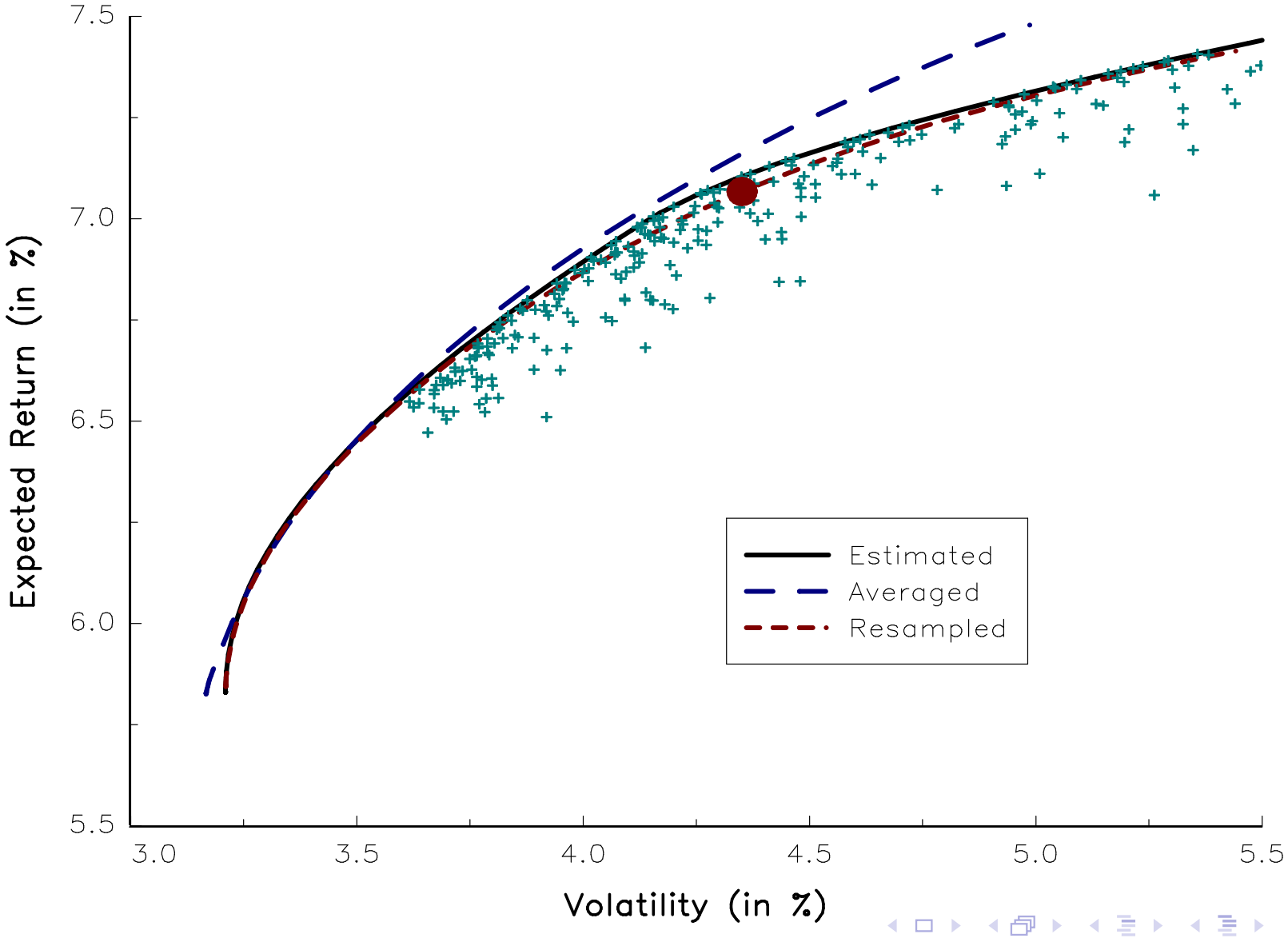


Figure 1.17, Page 48

Figure: Resampled efficient frontier



Result (1.23), Page 49

$$\hat{C} = \left(\begin{array}{cccc:cccc} 1.00 & & & & & & & \\ 0.73 & 1.00 & & & & & & \\ 0.72 & 0.76 & 1.00 & & & & & \\ 0.61 & 0.64 & 0.64 & 1.00 & & & & \\ \hdashline 0.72 & 0.76 & 0.75 & 0.64 & 1.00 & & & \\ 0.71 & 0.75 & 0.74 & 0.63 & 0.74 & 1.00 & & \\ 0.63 & 0.66 & 0.65 & 0.56 & 0.66 & 0.65 & 1.00 & \\ 0.68 & 0.72 & 0.71 & 0.60 & 0.71 & 0.70 & 0.62 & 1.00 \end{array}\right)$$

Result (1.24), Page 51

$$\hat{C} = \left(\begin{array}{cccc|ccccc} 1.00 & & & & & & & & \\ 0.77 & 1.00 & & & & & & & \\ 0.77 & 0.80 & 1.00 & & & & & & \\ 0.65 & 0.67 & 0.65 & 1.00 & & & & & \\ \hline 0.72 & 0.71 & 0.72 & 0.63 & 1.00 & & & & \\ 0.61 & 0.64 & 0.63 & 0.58 & 0.65 & 1.00 & & & \\ 0.60 & 0.64 & 0.62 & 0.60 & 0.63 & 0.62 & 1.00 & & \\ 0.65 & 0.67 & 0.67 & 0.67 & 0.67 & 0.63 & 0.66 & 1.00 & \end{array}\right)$$

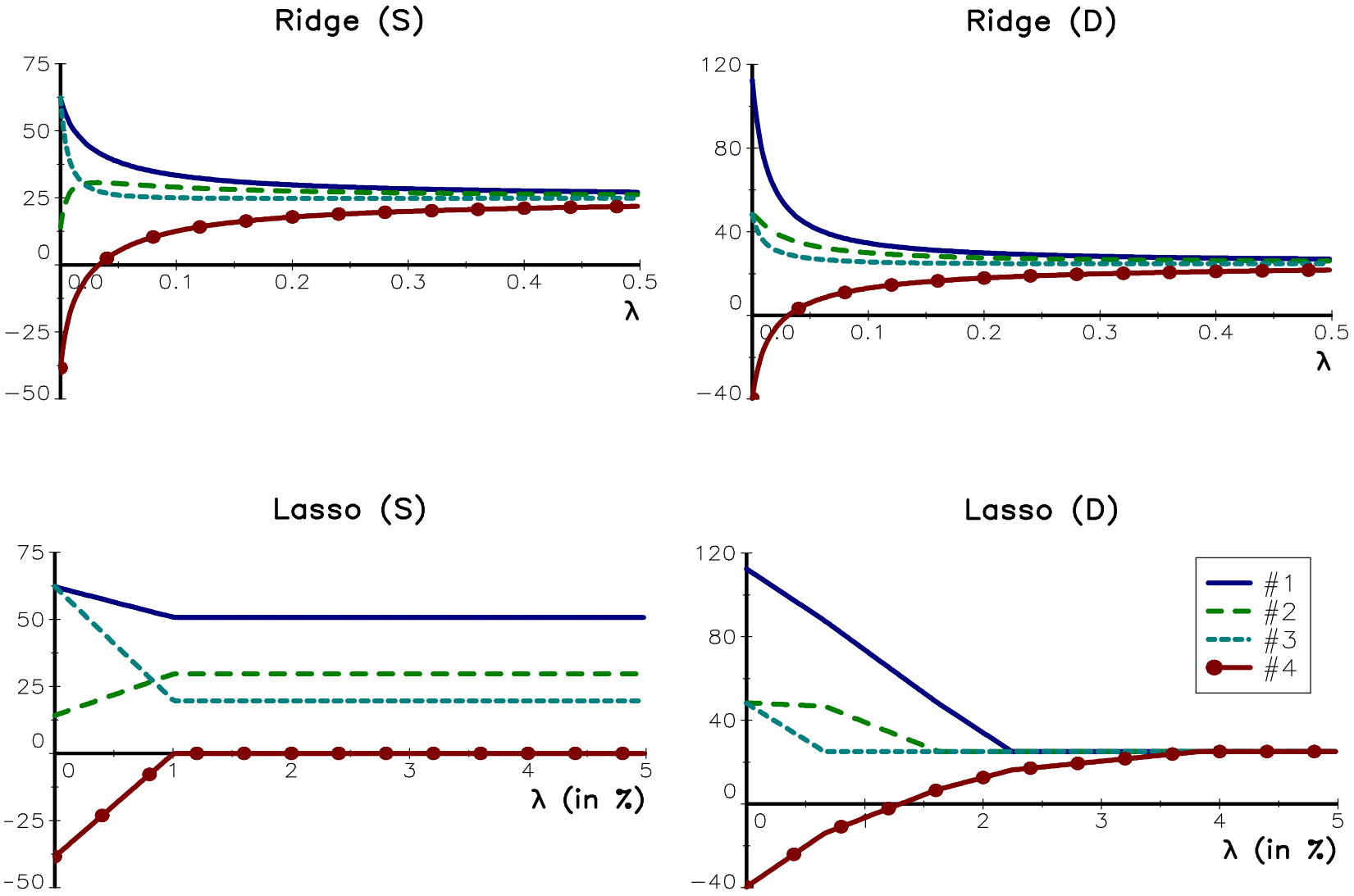
Table 1.9, Page 53

Table: Solutions of penalized mean-variance optimization

	MVO		Ridge		Lasso	
	(NC)	(C)	(S)	(D)	(S)	(D)
$x_1^\star$	112.29	62.09	38.88	51.62	24.41	25.00
$x_2^\star$	48.30	14.17	28.06	36.85	11.36	25.00
$x_3^\star$	48.10	62.21	27.34	29.34	27.78	25.00
$x_4^\star$	−39.69	−38.48	−1.57	−0.47	0.00	20.42

Figure 1.18, Page 54

Figure: Weights of penalized MVO portfolios (in %)



Tables 1.10 & 1.11, Page 55

Table: Principal component analysis of the covariance matrix Σ

Asset / Factor	1	2	3
1	65.35%	−72.29%	−22.43%
2	69.38%	69.06%	−20.43%
3	30.26%	−2.21%	95.29%
Eigenvalue	8.31%	0.84%	0.26%
% cumulated	88.29%	97.20%	100.00%

Table: Principal component analysis of the information matrix $\mathscr{I}$

Asset / Factor	1	2	3
1	−22.43%	−72.29%	65.35%
2	−20.43%	69.06%	69.38%
3	95.29%	−2.21%	30.26%
Eigenvalue	379.97	119.18	12.04
% cumulated	74.33%	97.65%	100.00%

Figure 1.19, Page 56

Figure: PCA applied to the stocks of the FTSE index (June 2012)

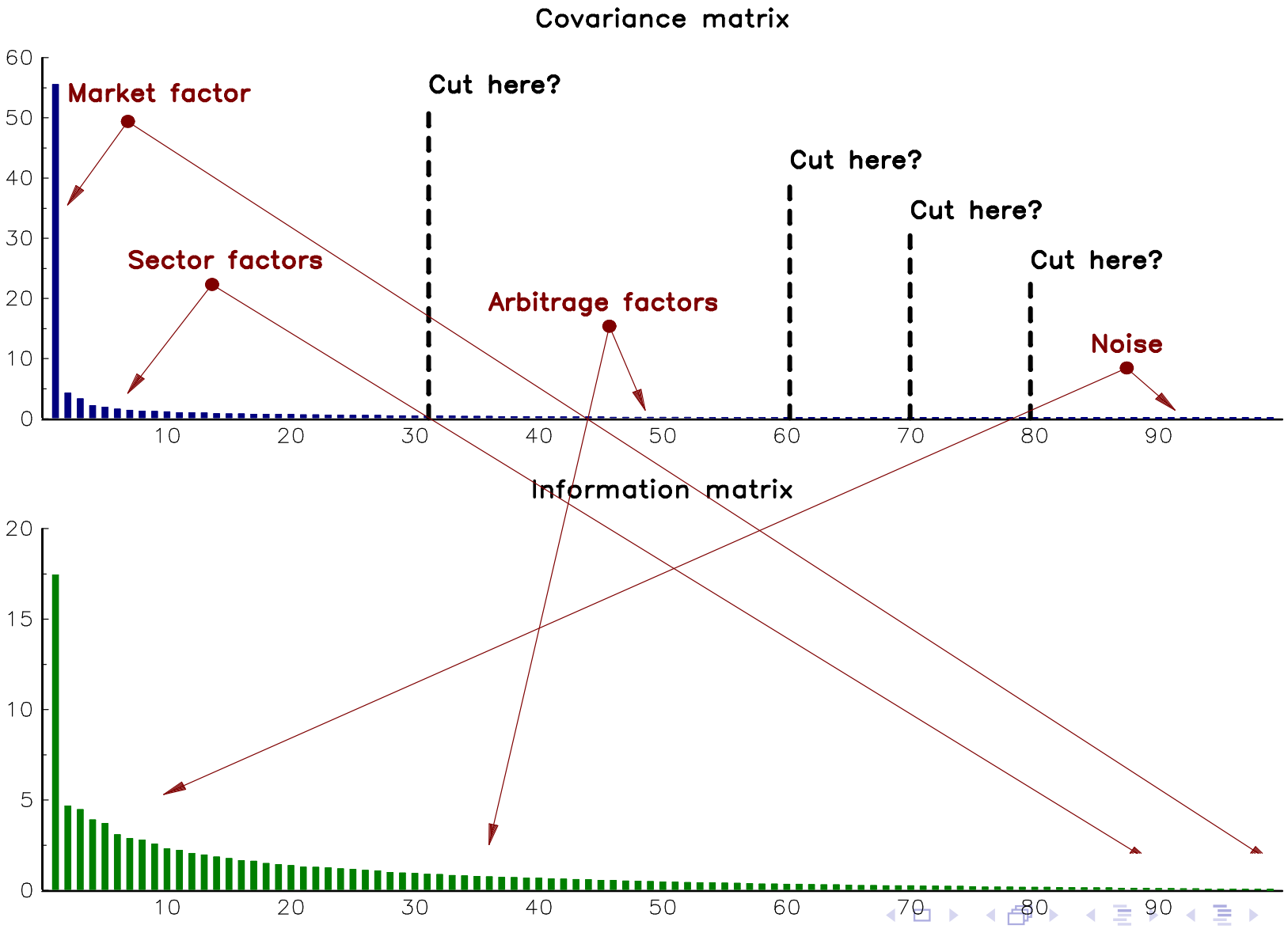


Table 1.12, Page 57

Table: Effect of deleting a PCA factor

$x^\star$	MV	$\lambda_1 = 0$	$\lambda_2 = 0$	$\lambda_3 = 0$	$\lambda_4 = 0$	$\lambda_5 = 0$	$\lambda_6 = 0$
$x_1^\star$	15.29	15.77	20.79	27.98	0.00	13.40	0.00
$x_2^\star$	10.98	16.92	1.46	12.31	0.00	8.86	0.00
$x_3^\star$	34.40	12.68	35.76	28.24	52.73	53.38	2.58
$x_4^\star$	0.00	22.88	0.00	0.00	0.00	0.00	0.00
$x_5^\star$	1.01	17.99	2.42	0.00	15.93	0.00	0.00
$x_6^\star$	38.32	13.76	39.57	31.48	31.34	24.36	97.42

Table 1.13, Page 60

Table: Limiting the turnover of MVO portfolios

τ^+	5.00	10.00	25.00	50.00	75.00	x^0
$x_1^\star$		35.00	36.40	42.34	45.59	30.00
$x_2^\star$		45.00	42.50	30.00	24.74	45.00
$x_3^\star$		15.00	21.10	27.66	29.67	15.00
$x_4^\star$		5.00	0.00	0.00	0.00	10.00
$\mu(x^\star)$		5.95	6.06	6.13	6.14	6.00
$\sigma(x^\star)$		15.00	15.00	15.00	15.00	15.69

n_b	n_x	Number of solved QP problems Heuristic	Backward	Forward
50	10	40	1 220	455
	40	10	455	1 220
500	50	450	123 975	23 775
	450	50	23 775	123 975
1 500	100	1 400	1 120 700	145 050
	1 000	500	625 250	1 000 500

Table 1.14, Page 63

Table: Sampling the SX5E index with the heuristic algorithm

k	Stock	b_i	$\sigma\left(x_{(k)} \mid b\right)$
1	Nokia	0.45	0.18
2	Carrefour	0.60	0.23
3	Repsol	0.71	0.28
4	Unibail-Rodamco	0.99	0.30
5	Muenchener Rueckver	1.34	0.32
6	RWE	1.18	0.36
7	Koninklijke Philips	1.07	0.41
8	Generali	1.06	0.45
9	CRH	0.82	0.51
10	Volkswagen	1.34	0.55
42	LVMH	2.39	3.67
43	Telefonica	3.08	3.81
44	Bayer	3.51	4.33
45	Vinci	1.46	5.02
46	BBVA	2.13	6.53
47	Sanofi	5.38	7.26
48	Allianz	2.67	10.76
49	Total	5.89	12.83
50	Siemens	4.36	30.33

Table 1.15, Page 63

Table: Sampling the SX5E index with the backward elimination algorithm

k	Stock	b_i	$\sigma\left(x_{(k)} \mid b\right)$
1	Iberdrola	1.05	0.11
2	France Telecom	1.48	0.18
3	Carrefour	0.60	0.22
4	Muenchener Rueckver	1.34	0.26
5	Repsol	0.71	0.30
6	BMW	1.37	0.34
7	Generali	1.06	0.37
8	RWE	1.18	0.41
9	Koninklijke Philips	1.07	0.44
10	Air Liquide	2.10	0.48
42	GDF Suez	1.92	3.49
43	Bayer	3.51	3.88
44	BNP Paribas	2.26	4.42
45	Total	5.89	4.99
46	LVMH	2.39	5.74
47	Allianz	2.67	7.15
48	Sanofi	5.38	8.90
49	BBVA	2.13	12.83
50	Siemens	4.36	30.33

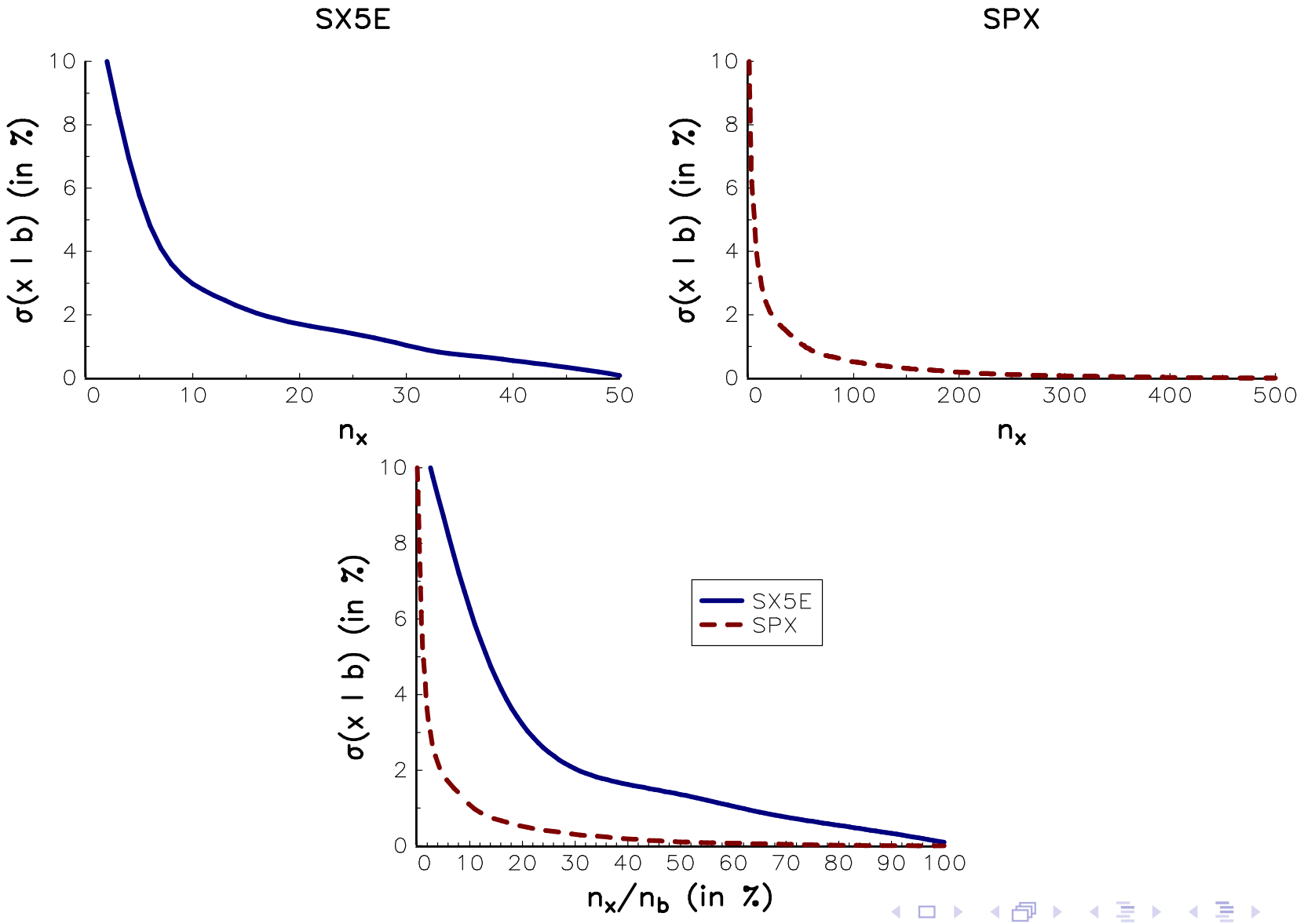
Table 1.16, Page 64

Table: Sampling the SX5E index with the forward selection algorithm

k	Stock	b_i	$\sigma\left(x_{(k)} \mid b\right)$
1	Siemens	4.36	12.83
2	Banco Santander	3.65	8.86
3	Bayer	3.51	6.92
4	Eni	3.32	5.98
5	Allianz	2.67	5.11
6	LVMH	2.39	4.55
7	France Telecom	1.48	3.93
8	Carrefour	0.60	3.62
9	BMW	1.37	3.35
41	Société Générale	1.07	0.50
42	CRH	0.82	0.45
43	Air Liquide	2.10	0.41
44	RWE	1.18	0.37
45	Nokia	0.45	0.33
46	Unibail-Rodamco	0.99	0.28
47	Repsol	0.71	0.24
48	Essilor	1.17	0.18
49	Muenchener Rueckver	1.34	0.11
50	Iberdrola	1.05	0.00

Figure 1.20, Page 65

Figure: Sampling the SX5E and SPX indices



Tables 1.17 & 1.18, Pages 68 & 69

Table: Minimum variance portfolio when $x_i \geq 10\%$

$\tilde{x}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
56.195	0.000	0.000	15.00	100.00			
23.805	0.000	0.000	20.00	10.00	100.00		
10.000	1.190	0.000	19.67	10.50	58.71	100.00	
10.000	1.625	0.000	23.98	17.38	16.16	67.52	100.00

Table: Minimum variance portfolio when $10\% \leq x_i \leq 40\%$

$\tilde{x}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
40.000	0.000	0.915	20.20	100.00			
40.000	0.000	0.000	20.00	30.08	100.00		
10.000	0.915	0.000	21.02	35.32	61.48	100.00	
10.000	1.050	0.000	26.27	39.86	25.70	73.06	100.00

Tables 1.19 & 1.20, Pages 69 & 70

Table: Mean-variance portfolio when $10\% \leq x_i \leq 40\%$ and $\mu^\star = 6\%$

$\tilde{x}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
40.000	0.000	0.125	15.81	100.00			
30.000	0.000	0.000	20.00	13.44	100.00		
20.000	0.000	0.000	25.00	41.11	70.00	100.00	
10.000	1.460	0.000	24.66	23.47	19.06	73.65	100.00

Table: MSR portfolio when $10\% \leq x_i \leq 40\%$

$\tilde{x}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
40.000	0.000	0.342	17.13	100.00			
39.377	0.000	0.000	20.00	18.75	100.00		
10.000	0.390	0.000	23.39	36.25	66.49	100.00	
10.623	0.000	0.000	30.00	50.44	40.00	79.96	100.00