# Tracking Problems, Hedge Fund Replication, and Alternative Beta

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## Abstract

As hedge fund replication based on factor models has encountered growing interest among professionals and academics, and despite the launch of numerous products (indexes and mutual funds) in the past year, it has faced many critics. In this paper, we consider two of the main critiques, namely the lack of reactivity of hedge fund replication, its deficiency in capturing tactical allocations, and the lack of access to the alpha of hedge funds. To address these problems, we consider hedge fund replication as a general tracking problem which may be solved by means of Bayesian filters. Using the example provided by Roncalli and Teiletche (2008), we detail how the Kalman filter tracks changes in exposures, and show that it provides a replication methodology with a satisfying economic interpretation. Finally, we address the problem of accessing the pure alpha by proposing a core/satellite approach of alternative investments between high-liquid alternative beta and less liquid investments. Non-normality and non-linearities documented on hedge fund returns are investigated using the same framework in a companion paper [Roncalli and Weisang (2009)].

1 The views expressed in this paper are those of the authors and do not necessarily represent those of Lyxor Alternative Investments.

Over the past decade, hedge fund replication has encountered a growing interest both from an academic and a practitioner perspective. Recently, Della Casa et al. (2008) reported the results of an industry survey showing that, even though only 7% of the surveyed institutions had invested in hedge fund replication products in 2007, three times as many were considering investing in 2008. Despite this surge in interest, the practice still faces many critics. If the launch of numerous products (indexes and mutual funds) by several investment banks in the past year can be taken as proof of the attraction of the 'clones' of hedge funds (HF) as investment vehicles, there remain nonetheless several shortcomings which need to be addressed. For instance, according to the same survey, 13% of the potential investors do not invest because they do not believe that replicating hedge funds' returns was possible; 16% deplore the lack of track record of the products; another 16% consider the products as black boxes. Finally, 25% of the same investors do not invest for a lack of understanding of the methodologies employed, while 31% of them were not interested for they see the practice as only replicating an average performance, thus failing to give access to one of the main attractive features of investing in one hedge fund, namely its strategy of management.

As a whole, the reasons put forward by these institutions compound different fundamental questions left unanswered by the literature. Since the seminal work of Fung and Hsieh (1997), most of the literature [Agarwal and Naik (2000), Amenc et al. (2003, 2007), Fung and Hsieh (2001), inter alia] has focused on assessing and explaining the characteristics of HF returns in terms of their (possibly time-varying) exposures to some underlying factors. Using linear factor models, these authors report the incremental progress in the explanatory power of the different models proposed. Yet, for now, the standard rolling-windows OLS regression methodology, used to capture the dynamic exposures of the underlying HF's portfolio, has failed to show consistent out-of-sample results, stressing the difficulty of capturing the tactical asset allocation (TAA) of HF's managers. More recently, more advanced methodologies, in particular Markov-Switching models and Kalman Filter (KF), have been introduced [Amenc et al. (2008), Roncalli and Teiletche (2008)] and show superior results to the standard rolling-windows OLS approach. From the point of view of investors, however, the complexity of these algorithms certainly does not alleviate the lack of understanding in the replication procedure. Furthermore, despite superior dynamic procedures and an ever expanding set of explanatory factors, some nonlinear features of HF returns [Diez de los Rios and Garcia (2008)] as well as a substantial part of their performance remain unexplainable, unless surmising ultrahigh frequency trading and investments in illiquid assets or in derivative instruments by HF managers. To our knowledge, while commonly accepted by most authors, because of practical difficulties, these explanations have not led to a systematic assessment nor have they been subject to systematic replication procedures. In this paper, we address two of the main critiques formulated on hedge fund replication. First, using the notion of tracking problems and Bayesian filters and their associated algorithms, we address the alleged failure of HF replication to capture the tactical allocations of the HF industry. Using the linear Gaussian model as a basis for the discussion, we provide the readers with an intuition for the inner tenets of the Kalman Filter. We illustrate how one can obtain sensible results, in terms of alternative betas. Second, we address the problem of accessing the part of the HF performances attributed to uncaptured dynamic strategies or investments in illiquid assets, i.e., the alpha of HF.

## Framework

Although HF replication is at the core of this paper, we would like to inscribe our contribution in a larger framework, albeit limited to a few financial perspectives. Thus, after a description of HF replication, this section introduces the notion of tracking problems. After a brief and succinct formal definition, we show how this construct indeed underpins many different practices in finance, including some hedge fund replication techniques and some investment strategies such as, for example, Global Tactical Asset Allocation (henceforth, GTAA). It is armed with this construct and the tools associated to it that we tackle three of the main critiques heard in the context of hedge fund replication in subsequent sections.

## Hedge fund replication

### Rationale behind HF replication

Even though HF returns' characteristics make them an attractive investment, investing in hedge funds is limited for many investors due to regulatory or minimum size constraints, in particular for retail and institutional investors. Hedge funds as an investment vehicle have also suffered from several criticisms: lack of transparency of the management's strategy, making it difficult to conduct risk assessment for investors; poor liquidity, particularly relevant in periods of stress; and the problem of a fair pricing of their management fees. It is probably the declining average performance of the hedge fund industry coupled with a number of interrogations into the levels of fees [Fung and Hsieh (2007)] which led many major investors to seek means of capturing hedge fund investments strategies and performance without investing directly into these alternative investment vehicles [Amenc et al. (2007)]. Hence, the idea of replicating hedge funds' portfolios, already common in the context of equity portfolios, gained momentum.

### Factor models[2]

Starting with the work of Fung and Hsieh (1997) as an extension of Sharpe's style regression analysis [Sharpe (1992)] to the world of hedge funds, factor-based models were first introduced as tools for performance analysis.

2 With the growing interest in hedge fund replication over the last decade, it is not surprising to find that there exists a rich literature which is almost impossible to cover extensively. A comparison of the factor and the pay-off distribution approaches can be found in Amenc et al. (2007). We also refer the interested reader to Amin and Kat (2003), Kat (2007), or Kat and Palaro (2006) for a more detailed account of the pay-off distribution approach

The underlying assumption of Sharpe's style regression is that there exists, as in standard Arbitrage Pricing Theory (APT), a return-based style (RBS) factor structure for the returns of all the assets that compose the investment world of the fund's manager [Fung and Hsieh (1997), Sharpe (1992)]. Factor-based models for hedge fund replication make a similar assumption but use asset-based style (ABS) factors. While RBS factors describe risk factors and are used to assess performance, ABS factors are directly selected with the purpose of being directly transposable into investment strategies. ABS factors have been used to take into account dynamic trading strategies with possibly nonlinear pay-off profiles [Agarwal and Naik (2000), Fung and Hsieh (2001)]. The idea of replicating a hedge fund's portfolio is therefore to take long and short positions in a set of ABS factors suitably selected so as to minimize the error with respect to the individual hedge fund or the hedge fund index.

A generic procedure for HF replication using factor models [Agarwal and Naik (2000), Fung and Hsieh (2001), Sharpe (1992)] can be decomposed in two steps. In step 1, one estimates a model of the HF returns as $r_k^{HF} = \sum_{i=1}^{m} w^{(i)} r_k^{(i)} + \varepsilon_k$. Given the estimated positions $\hat{w}^{(i)}$ (on the ABS factor $r^{(i)}$ resulting from step 1, step 2 simply constructs the 'clone' of the hedge fund by $r_k^{Clone} \sum_{i=1}^{m} = r_k^{(i)}$. The factor-based approach is thus very intuitive and natural. There are, however, several caveats to this exercise. Contrary to the passive replication of equity indices, the replication of hedge funds returns must take into account key unobservable determinants of hedge fund investment strategies such as the returns from the assets in the manager's portfolio; dynamic trading strategies; or the use of leverage [Fung and Hsieh (1997, 2001)].

Fairly recently, attempts to capture the dynamic nature of the HF portfolio allocation have been explored in the literature in order to improve the in-sample explanatory powers and the quality of the out-of-sample replication. One method, used extensively [Fung and Hsieh (2004), Hasanhodzic and Lo (2007), Jaeger (2008), Lo (2008), inter alia], is to use rolling-windows OLS where the coefficients $\{w_k^{(i)}\}$ $t_k$ are estimated by running the OLS regressions of $\{r_l^{HF}\}_{l=k-L}^{k-1}$ on the set of factors $\{r_l^{(i)}\}_{l=k-L}^{k-1}$ for l =1,…,m. A common choice for the window length L is 24 months, even though one could consider a longer time-span trading-off the dynamic character of the coefficients for more stable and more robust estimates. By means of an example, Roncalli and Teiletche (2008) have demonstrated that such a methodology poorly captures the dynamic allocation in comparison with the Kalman filter (KF). The use of KF estimation, however, requires caution in its implementation, making the estimation of the positions $\{w_k^{(i)}\}$ a non-trivial matter. Markov regime-switching models have also been considered [Amenc et al. (2008)]. The idea therein is that HF managers switch from one type of portfolio exposure to another depending on some state of the world, assumed to be discrete in nature. One possible interpretation is to consider that the active management consists of changing the asset allocation depending on two states of the economy (high and low). Justifying the number of states or their interpretation is, however, tricky.

## Definition of the tracking problem

We follow Arulampalam et al. (2002) and Ristic et al. (2004) in their definition of the general tracking problem. We note $x_k \in \mathbb{R}^{n_x}$ the vector of states and $z_k \in \mathbb{R}^{n_x}$ the measurement vector at time index k. In our setting, we assume that the evolution of $x_k$ is given by a first-order Markov model $x_k = f(t_k, x_{k-1}, \upsilon_k)$, where $f$ is a (non-)linear function and $\upsilon_k$ a noise process. In general, the state $x_k$ is not observed directly, but partially through the measurement vector $z_k$. It is further assumed that the measurement vector is linked to the target state vector through the following measurement equation $z_k = h(t_k, x_k, \eta_k)$, where h is a (non-)linear function, and $\eta_k$ is a second noise process independent from $\upsilon_k$. Our goal is thus to estimate $x_k$ from the set of all available measurements $z_{1:k} = \{z_i, i=1,\ldots,k\}$.

**Remark 1 –** In the rest of the paper, a system in the following format will be referred to as a tracking problem (henceforth TP) $\{x_k = f(t_k, x_{k-1}, \upsilon_k); z_k = h(t_k, x_k, \eta_k)\}$ (1)

## Link between GTAA, HF replication, and tracking problems

The two problems of replicating a global tactical asset allocation (GTAA) strategy and HF replication can be seen as belonging to the same class of approaches. For the clarity of our exposé, we decompose the return of a hedge fund into two components

$$r_k^{(\mathrm{HF})} = \underbrace{\sum_{i=1}^{m} w_k^{(i)} r_k^{(i)}}_{\text{GTAA ABS factors}} + \underbrace{\sum_{i=m+1}^{p} w_k^{(i)} r_k^{(i)}}_{\text{HF ABS factors}}. \quad (2)$$

GTAA is an investment strategy that attempts to exploit short-term market inefficiencies by establishing positions in an assortment of markets with a goal to profit from relative movements across those markets. This top-down strategy focuses on general movements in the market rather than on performance of individual securities. Beside GTAA, hedge fund managers may invest in a larger universe. A part of the universe is composed of the asset classes found in GTAA strategy and another part of the universe is composed of other alternative asset classes and strategies, such as stock picking strategies (which may be found in equity market neutral, long/short event driven hedge funds), high frequency trading, non-linear exposures using derivatives, and illiquid assets (corresponding to distressed securities, real estate or private equity).

and to Agarwal and Naik (2000, 2004), Fung and Hsieh (1997, 1999, 2001, 2007), and Hasanhodzic and Lo (2007) for the systematic quantitative replication of strategies using factor models as proposed by many investment banks as hedge funds' clones products. Also, some excellent popularizing books on hedge funds and their replication can be found [Jaeger (2008), Lo (2008)].

The idea of HF replication, in particular to create investment vehicles, is to replicate the first term on the RHS of (2). If we note $\eta_k = \sum_{i=m+1}^{p} w_k^{(i)} r_k^{(i)}$, then HF replication can be described as a TP $\{w_k = w_{k-1} + \upsilon_k; r_k^{(HF)} = r_k^T w_k + \eta_k\}$ (3). We must, however, stress two points before continuing. First, HF replication will work best at the industry level using aggregates of hedge funds' performances as the replication benchmark. Diez de los Rios and Garcia (2008) report a large proportion of the HF industry to be following long/short equity strategies (about 30%)[3]. The performance of a single HF following an L/S equity strategy is explained by its proprietary model of stock picking and its proprietary model to choose its beta, such that its portfolio will be long of a 100% of the selected stocks, and short of x% of its benchmark index. It is almost impossible to determine without inside information the portfolio of stocks picked by the HF manager as it depends on its targeted risk profile and the private views of the managers. However, because of the efficiency of liquid markets, as an aggregate, the performance of all the L/S equity HF will be proportional to $1-\bar{x}$, where $\bar{x}$ is the average taken over all L/S equity funds of their exposure. In other words, the performance of the aggregate will be proportional to the beta of the entire industry, and the idiosyncratic decisions of each manager are averaged out. It is worth noting that in this case, as the underlying asset classes are standard, replicating an aggregate of L/S equity HF is about the similar to replicating a GTAA strategy. This point is all the more salient since other HF strategies are not represented in a proportion equivalent to the L/S equity HF [Fung and Hsieh (2004)].

Seemingly, one weakness of the approach we propose is that only the beta of HF strategies seems to matters. One could rightly argue, however, that an attractive feature of investing in single HF is the promise of absolute performance. Even in the case of L/S equity strategies, Fung and Hsieh (2004) further argued that they produce 'portable' absolute overperformances, which they termed 'alternative alphas,' that are not sensitive to traditional asset classes. We contend nonetheless, as our decomposition above between GTAA ABS factors and HF ABS factors hinted at, that one must be realistic between what can and cannot be replicated. If HF performances can be divided between a beta component and a non-replicable alpha component, it is because HF managers engage in trading at high-frequencies or in illiquid assets, thus benefiting from local and transient market inefficiencies or illiquidity premia. Moreover, if considering these typical HF ABS factors is very useful in explaining the performance of the HF industry, these items cannot in good measure be replicated from an investment perspective. Thus, we already need to point out that not all of the HF strategies can be successfully replicated using the method we advocate in this paper. This is perhaps the one good news for the HF industry. Even though we will demonstrate one can truly capture a substantial part of the performance of the industry as a whole, still they individually retain some edge, particularly those practicing true alternative strategies. The next sections expose and provide the tools to capture the tactical allocation of a manager's portfolio.

## Capturing tactical allocation with Bayesian filters

The prior density of the state vector at time k is given by the following equation $p(x_k \mid z_{1:k-1}) = \int p(x_k \mid x_{1:k-1}) p(x_{k-1} \mid z_{1:k-1})\, dx_{k-1}$ (4), where we used the fact that our model is a first-order Markov model to write $p(x_k \mid x_{1:k-1}, z_{1:k-1}) = p(x_k \mid x_{k-1})$. This equation is known as the Bayes prediction step. It gives an estimate of the probability density function of $x_k$ given all available information until $t_{k-1}$. At time $t_k$, as a new measurement value zk becomes available, one can update the probability density of $x_k$: $p(x_k \mid z_{1:k}) \propto p(z_k \mid x_k) p(x_k \mid z_{1:k-1})$ (5). This equation is known as the Bayes update step. The Bayesian filter corresponds to the system of the two recursive equations (4) and (5). In order to initialize the recurrence algorithm, we assume the probability distribution of the initial state vector $p(x_0)$ to be known.

Using Bayesian filters, we do not only derive the probability distributions $p(x_k \mid z_{1:k-1})$ and $p(x_k \mid z_{1:k})$, but we may also compute the best estimates $\hat{x}_{k|k-1}$ and $\hat{x}_{k|k}$ which are given by $\hat{x}_{k|k-1} = \mathbb{E}[x_k \mid z_{1:k-1}] = \int x_k p(x_k \mid z_{1:k-1}) dx_k$ and $\hat{x}_{k|k} = \mathbb{E}[x_k \mid z_{1:k}] = \int x_k p(x_k \mid z_{1:k}) dx_k$. To gain better understanding of the advantages of using the tracking problem's formalization as well as Bayesian filters to answer the problem at hand, we examine here HF replications in a Gaussian linear framework using KF. In a companion paper, we also considered the use of particle filters to allow for more flexible specification of the density function [Roncalli and Weisang (2009)].

## Hedge fund replication: the Gaussian linear case

In this section, in order to substantiate our claim that the tactical asset allocation of a portfolio is retrievable, we start by providing an intuition of the inner workings of the KF algorithm. We also test, with the aid of an example, the capacity of KF to determine plausible weights for a replicating portfolio of a standard HF index. Furthermore, we show that the replicating portfolio provides a qualitatively sensible explanation for the behavior of the HFRI index over the period 1994-2008, while enabling us to capture a significant part of its performance.[4] Finally, we look into the types of strategies that one could consider when replicating in the HF industry.

### Understanding linear Gaussian approach and Bayesian filtering to replication strategies

While the KF algorithm described in appendix is well known to many engineers and econometricians, the classic contemporaneous representation (A1) provides little insight into how KF dynamically modifies the estimated weights to track the exposures of the portfolio as described

3 Fung and Hsieh (2004) report further that in March 2003 about 40% of the HFs reported in the TASS database list long/short equity as their primary investment style. There are historical reasons for that. L/S equity strategy was the strategy used by the first HF on record, created in 1949 by A.W. Jones.

4 To be more precise, the study period for all the computations done in the rest of this paper begins in January 1994 and ends in September 2008.

in TP (3). In the following, using the innovations representation of the KF algorithm, we explain with finer details the dynamic adjustments of the recursion.

### Innovation representation of linear state-space models

The dynamic described by the equations (A1) can be re-written in terms of the tracking error $\hat{e}_k$. It suffices to recombine (A1) into $\hat{x}_{k+1|k} = c_{k+1} + F_{k+1}(\hat{x}_{k|k-1} + \hat{P}_{k|k-1} H_k^T \hat{V}_k^{-1} \hat{e}_k)$; $\square = c_{k+1} + F_{k+1}\hat{x}_{k|k-1} + K_k\hat{e}_k$, where $K_k = F_{k+1} \hat{P}_{k|k-1} H_k^T \hat{V}_k^{-1}$ is called the gain matrix. The state-space is then represented as $\{Z_k = d_k + H_k\hat{x}_{k|k-1} + \hat{e}_k;\ \hat{x}_{k+1|k} = c_{k+1} + F_{k+1} \hat{x}_{k|k-1} + K_k\hat{e}_k\}$ (6) , where the two noise processes $\upsilon_k$ and $\eta_k$ have been replaced by the process $\hat{e}_k$, and the transition equation is defined on the estimate of the state vector $\hat{x}_{k|k-1}$, and not directly on the state vector $x_k$.

In the case of the tracking problem (3), the innovation representation yields $\{r_k^{(HF)} = r_k^T \hat{w}_{k|k-1} + \hat{e}_k;\ \hat{w}_{k+1|k} = \hat{w}_{k|k-1} + K_k\hat{e}_k\}$. It can be shown that the gain matrix $K_k$ can be construed as the matrix of 'regression' coefficients of $\hat{w}_{k+1|k}$ on $\hat{e}_k$ the innovation at time tk (cf. appendix).

### Interpretation of the correction mechanism of the Kalman filter

At time $t_k$, KF performs an update of the previous weights estimates $\hat{w}_{k|k-1}$ by applying the correction term $K_k\hat{e}_k = \hat{P}_{k|k-1}r_k\hat{e}_k^*$, where $\hat{e}_k^* = \hat{e}_k/\hat{V}_k$ is the normalized tracking error. Recall also that $\hat{P}_{k|k-1} = \mathbb{E}[w_k - \hat{w}_{k|k-1})(w_k - \hat{w}_{k|k-1})^T \mid r^{(HF)}_{1:k-1}]$ is the variance matrix of the state estimation error $w_k - \hat{w}_{k|k-1}$.

We are now in a position to explain how KF adjusts the weights between two rebalancing dates. Here are some facts to understand the statistical prediction-correction system behind KF.

1. First, notice that the larger the normalized tracking error $\hat{e}_k^*$, the larger the change in the allocation $\hat{e}_k^* \nearrow \Rightarrow |\Delta\, \hat{w}^{(i)}_{k+1|k}| \searrow$

This remark compounds three smaller ones.

a) The size of $\hat{e}_k^*$ takes into account the relative size of $\hat{e}_k$ with respect to its covariance $\hat{V}_k$.

b) Note that $\hat{V}_k = r_k^T \hat{P}_{k|k-1}r_k + S_k$. Thus, the variance of the tracking error depends on the covariance matrix $\hat{P}_{k|k-1}$ of the past state estimation error and the variance of the current observation noise $\eta_k$. Hence, the larger the recent past errors of the Kalman Filter, the smaller the normalized tracking error $\hat{e}_k^*$ will be. In other words, ceteris paribus, the smaller the recent past errors, the 'stronger' is the algorithm's reaction to the last observed tracking error. We have
$\hat{V}_k \nearrow \Rightarrow |\Delta\hat{w}^{(i)}_{k+1|k}| \searrow$

c) $\hat{e}_k^*$ is a relative measure of the correction on $|\Delta\, \hat{w}^{(i)}_{k+1|k}|$, but it does not indicate the direction of change
$\hat{e}_k^* > 0 \Rightarrow \Delta\hat{w}^{(i)}_{k+1|k} > 0$ or $\Delta\hat{w}^{(i)}_{k+1|k} < 0$

2. Second, assume that $\hat{P}_{k|k-1}$ is a diagonal matrix. The errors on the estimated weights are not correlated. The direction of change for the asset class I will then be given by the sign of $r_k^{(i)} \times \hat{e}_k$
$r_k^{(i)} \times \hat{e}_k > 0 \Rightarrow \Delta\hat{w}^{(i)}_{k+1|k} > 0$

   The directions are then adjusted to take into account the volatility of the Kalman filter errors on the estimated weights. For the i[th] factor, we have
$\Delta\hat{w}^{(i)}_{k+1|k} = (\hat{P}_{k|k-1})_{i,i}\, r_k^{(i)}\, \hat{e}_k^*$ .

   If KF has made a lot of errors on the weight of one factor (which means that the weights have highly changed in the past), it will perform a large correction $(\hat{P}_{k|k-1})_{i,i} \nearrow \Rightarrow |\Delta\hat{w}^{(i)}_{k+1|k}| \nearrow$

3. Third, assume that $\hat{P}_{k|k-1}$ is a not diagonal matrix. The correction done by KF takes into account of the correlations between the errors on the estimated weights $\Delta\hat{w}^{(i)}_{k+1|k} = \hat{w}^{(i)}_{k+1|k} = \hat{w}^{(i)}_{k|k-1} = \hat{e}_k^* \sum_{j=1}^{m} (\hat{P}_{k|k-1})_{i,j}\, r_k^{(j)}$ .

   Suppose that $\hat{e}_k < 0$ and $r_k^{(1)} > 0$. According to point 2 above, the weight of the first factor should be reduced. However, because of the correlations between the errors on the estimated weights, there may be an opposite correction $\Delta\hat{w}^{(1)}_{k+1|k}$, because for instance the errors on the other factors are negatively correlated with the error on the first factor and the performances of the other factors are negative.

4. Finally, notice that when, at time tk, the replication strategy has the same performance as the fund's strategy, KF does not change the estimated weights $\hat{e}_k = 0 \Rightarrow \hat{w}^{(i)}_{k+1|k} = \hat{w}^{(i)}_{k|k-1}$ .

### An example with a well-diversified Hedge Fund index

As in Roncalli and Teiletche (2008), we consider replicating the HFRI Fund Weighted Composite index as an example. The model considered (6F) is

$$\begin{cases} r_k^{(HF)} &=& \sum_{i=1}^{6} w_k^{(i)}\, r_k^{(i)} + \eta_k \\ \mathbf{w}_k &=& \mathbf{w}_{k-1} + \boldsymbol{\nu}_k \\ Q_k &=& \operatorname{diag}(\sigma_1^2, \ldots, \sigma_m^2) \end{cases} \tag{2}$$

where the set of factors that served as a basis for this exercise is: an equity exposure in the S&P 500 index (SPX), a long/short position between Russell 2000 and S&P 500 indexes (RTY/SPX), a long/short position between DJ Eurostoxx 50 and S&P 500 indexes (SX5E/SPX), a long/short position between Topix and S&P 500 indexes (TPX/SPX), a bond position in the 10-year U.S. Treasury (UST), and an FX position in the EUR/USD.

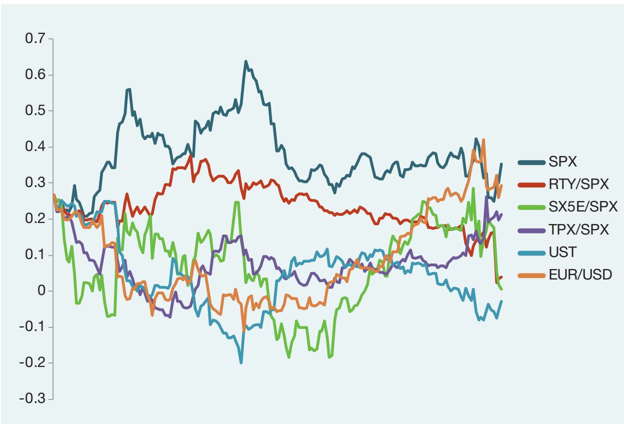



**Figure 1 – Estimated weights of the 6F model (Jan 1994 - Sep 2008)**

### Results

To present realistic results, we assumed that replication of the exposures to each factor was done using futures[5] (hedged in USD) and that the sampling period is one month. The study period begins in January 1994 and ends in September 2008. We estimated the model described in (8)[6]. The estimates of the parameters are (in %) $\hat{\sigma}^2_\eta = 5.48\ 10^{-5}$, $\hat{\sigma}^2_1 = 7.34\ 10^{-4}$, $\hat{\sigma}^2_2 = 2.83\ 10^{-4}$, $\hat{\sigma}^2_3 = 2.09\ 10^{-3}$, $\hat{\sigma}^2_4 = 4.26\ 10^{-4}$, $\hat{\sigma}^2_5 = 5.25\ 10^{-4}$ and $\hat{\sigma}^2_6 = 6.26\ 10^{-4}$. The resulting estimated exposures are presented in Figure 1.

## Interpretation of the results

A closer look at the results of the previous estimation demonstrates, as we show below, that replication using KF provides better replicators than traditional methods in the sense that it captures a better part of the performance of the HF benchmark while providing estimated weights that possess a sensible explanation of the dynamic investment strategy of the underlying index. To do so, we first introduce the alternative beta concept, before moving to an attribution of performance (AP) of the replicating strategy.

### The alternative beta concept

As mentioned in Hasanhodzic and Lo (2007), Lo (2008), and Roncalli and Teiletche (2008), we may compute the attribution of performance of the return $r_k^{(HF)}$ of hedge funds indices in several ways. In practice, the attribution of performance is often done directly on the absolute returns.[7] First, rewrite the return of the hedge fund portfolio using the following decomposition

$$r_k^{(\mathrm{HF})} = \sum_{i=1}^{m}\left(w_k^{(i)} - \bar{w}^{(i)}\right)\left(r_k^{(i)} - r_k^{(0)}\right) + \sum_{i=1}^{m}\bar{w}^{(i)} r_k^{(i)} + \left(1 - \sum_{i=1}^{m}\bar{w}^{(i)} r_k^{(0)}\right) + \eta_k. \qquad (3)$$

where $\bar{w}^{(i)}$ are the fixed weights on the different asset classes, e.g. $\bar{w}^{(i)} = \mathbb{E}(w_k^{(i)})$[8] and $r_k^{(0)}$ is the return of the risk-free asset.

| | Traditional | | Alternative | | Total |
|---|---|---|---|---|---|
| Period | Alpha | Beta | Alpha | Beta | |
| 1994-2008 | 3.80 | 5.92 | 2.22 | 7.55 | 9.94 |
| 1997-2008 | 3.14 | 5.46 | 1.14 | 7.55 | 8.77 |
| 2000-2008 | 2.20 | 4.02 | 1.48 | 4.75 | 6.30 |

**Table 1 – Estimated yearly alpha (in %)**

The first approach to consider is the traditional alpha/beta decomposition derived from the CAPM $r_k^{(HF)} = \alpha_k + \beta_k$ where $\beta_k$ is the component of return attributed to the benchmarks and where the sensitivities of the fund's portfolio to the benchmarks are considered constant. In this alpha/beta decomposition, we thus have

$$\begin{cases} \alpha_k &= r_k^{(\mathrm{HF})} - \beta_k \\ \beta_k &= \left(1 - \sum_{i=1}^{m}\bar{w}^{(i)}\right) r_k^{(0)} + \sum_{i=1}^{m}\bar{w}^{(i)} r_k^{(i)} \end{cases} \qquad (4)$$

When taking the mathematical expectations in (10), one finds that the traditional alpha/beta decomposition will always underestimate the systematic part of the performance – the beta – and overestimate the idiosyncratic part – the alpha – as the contribution of the covariance between the factors and the exposures is lumped into the idiosyncratic part. Instead, one may consider another decomposition $r_k^{(HF)} = \alpha_k^{AB} + \beta_k^{TB} + \beta_k^{AB}$ where $\beta_k^{TB}$ is the traditional beta and + $\beta_k^{AB}$ is called alternative beta. We have

$$\begin{aligned} \beta_k^{AB} &= \left(1 - \sum_{i=1}^{m} w_k^{(i)}\right) r_k^{(0)} + \sum_{i=1}^{m} w_k^{(i)} r_k^{(i)} - \beta_k^{TB} \\ \square &= \sum_{i=1}^{m}\left(w_k^{(i)} - \bar{w}^{(i)}\right)\left(r_k^{(i)} - r_k^{(0)}\right). \end{aligned} \qquad (5)$$

The alternative beta $\beta_k^{AB}$ thus captures the part of the performance of the fund due to an active management of the portfolio's expositions to the different benchmarks. For a discussion of active versus passive management, we refer the reader to Roncalli and Teiletche (2008) and Lo (2008). After approximating $w_k$ with $\hat{w}_{k|k-1}$, the clone gives access to the sum of the traditional beta and the alternative beta $r_k^{Clone} = (1 - \sum_{i=1}^m \hat{w}_{k|k-1}^{(i)}) r_k^{(0)} + \sum_{i=1}^m \hat{w}_{k|k-1}^{(i)} r_k^{(i)}$. The term $\alpha_k^{AB}$ is called the alternative alpha. It is computed as $\alpha_k^{AB} = r_k^{(HF)} - r_k^{Clone}$. We have reported the performance attribution of $\alpha/\beta$ components in Figure 2. Notice that a large part of the HF returns are not explained by the traditional alpha but by the alternative beta. For the

5 When the future does not exist, we approximate the monthly performance by the monthly return of the corresponding TR index minus the one-month domestic Libor and the hedging cost.

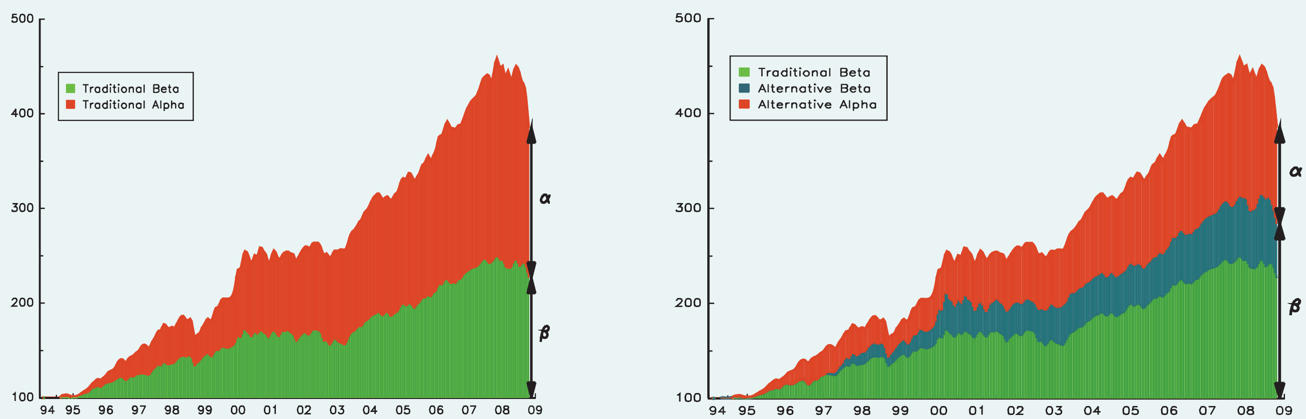
6 The parameters $w_0$ and $P_0$ are initialized at $w_0 = 0$; $P_0 = I_{6x6}$.

7 In this case, we assume that the cash investment is part of the beta component.

8 There are several ways to compute the fixed weights. One approach is to consider the mean of the dynamic weights $\bar{w}^{(i)} = \frac{1}{n}\sum_{k=1}^{n}\hat{w}_{k|k-1}^{(i)}$. Another approach is to compute the OLS regression on the entire period $t_0, \ldots t_n$ $r_k^{(HF)} = \sum_{i=1}^{m}\bar{w}^{(i)} r_k^{(i)} + \eta_k$. Finally, we may estimate the weights using the Kalman filter by imposing that $Q_k = 0_{(m\times m)}$. In this case, the weights $\bar{w}_k^{(i)}$ correspond to the recursive OLS estimates.

| Factor | Cash | SPX | RTY/SPX | SX5E/SPX | TPX/SPX | UST | EUR/USD | Total (Clone) |
|---|---|---|---|---|---|---|---|---|
| Performance (in %) | 4% | 51% | 14% | 22% | 6% | 2% | 11% | 164% |

Table 2 – Attribution of performance of the replicated strategy




(a) Traditional α/β

(b) Alternative α/β

Figure 2 – Attribution of performance

entire period, the alternative alpha explains about 23% of the HF returns whereas the alternative beta explains about 77%. The decomposition between alpha and beta over several periods is reported in Table 1. Note that the alpha is overestimated using traditional beta.

### Performance attribution of the replicated strategy

In Table 2, we report the performance attribution of the ABS factors' exposures for our example. The main contributor to the replicated strategy is the long equity exposure. It is interesting to note that three other strategies have a significant contribution. They are the two L/S equity strategies on small caps and Eurozone and the FX position EUR/USD. Finally, the last two positions have a small absolute contribution to the performance: the L/S equity on Japan and the 10-year U.S. bond position. In a first approach, one may consider the elimination of these factors. However, they may help track the volatility of the HF index, therefore contributing to the performance as well.

Interestingly, using the KF estimates, we are now able to explain the success of the HF industry between 2000 and 2003. Notice in Figure 3 that the highest exposure of the HF industry to the directional equity market was in March 2000 and represented more than 60% of the overall exposure. After March 2000, the HF industry decreased the leverage on equity and modified the bets on L/S equity. In the right graph, we compare the performance of the alternative beta strategy with respect to two other strategies. The first one uses the fixed allocation of March 2000 for all the asset classes and the second corresponds to the alternative beta, except for the directional equity exposure which is fixed and equal to the equity beta of March 2000. It appears that the relative good performance of the HF industry may be explained by two components: equity deleverage and good bets on L/S equity on RTY/SPX and SX5E/SPX. We estimate that with respect to the allocation of March 2000, the equity deleverage explains 40% of the outperformance whereas the reallocation of the L/S equity explains about 60% of the outperformance.

## Which strategies may be replicated?

The example provided above is of course no proof that the methodology we have exposed so far is the panacea to the replication problem. Rather, the preceding example could almost be taken as a teaching case used to demonstrate the aptitudes of this formulation of the replication problem to provide satisfying answers. It is, however, important to better understand what types of strategies followed by the HF industry may subject themselves well to this replication process. To try to provide an answer to this problem, we thus estimated the 6F[9] on a series of HF

9 We also estimated a factor model using seven factors (7F) including some nonreplicable factors traditionally used in the literature [Hasanhodzic and Lo (2007)]. If this (7F) model performs better on a number of accounts, providing better performances, lower volatility, lower volatility of the tracking error, better correlation of the returns of the tracker with its benchmark, one must however make note of three facts. First, any gain is in general small and parsimony considerations suggest a smaller model. Second, from an investment point of view some of the factors in (7F) are not easily implementable, and any gain in performance may be offset by additional implementation costs these factors could involve. Third, the gain in the tracking performance is reflected, even if only slightly, by higher drawdowns. Detailed results are available from the authors on demand.

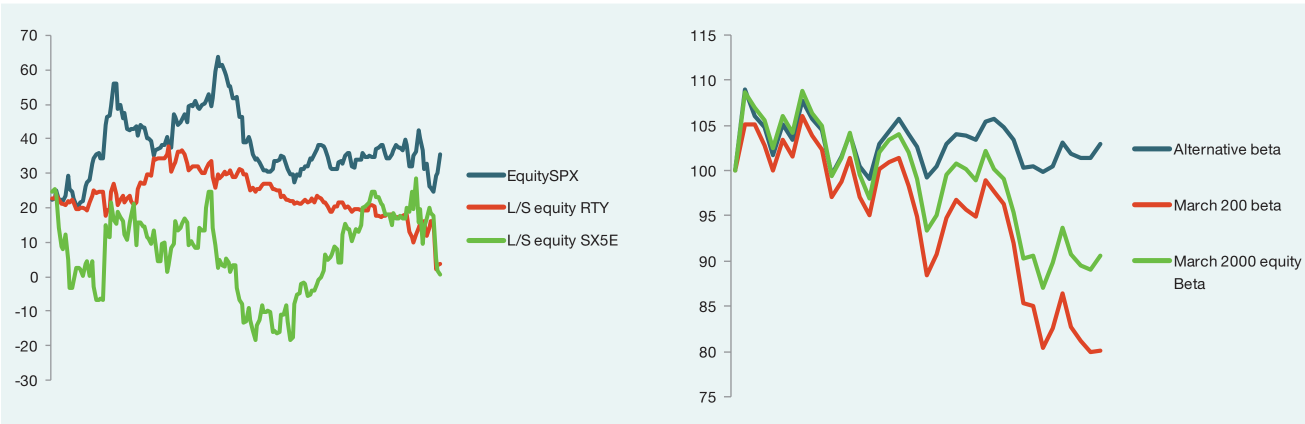



**Figure 3 – Replication during the equity bear market**

indexes representing general categories of strategies. The HFRI index trackers using the 6F model were compared to their benchmarks (detailed results are available from the authors). The key points of an analysis of our results can be summarized in the following way. Overall, HF trackers have smaller Sharpe ratios than their respective indexes, even though they generally exhibit lower volatilities. However, they also present a smaller risk if one measures risk as the maximum drawdown or as excess kurtosis of the returns. Some strategies present low correlation with their respective trackers and one can thus conclude that they are difficult to replicate by the method employed here. This concerns mainly illiquid strategies (i.e., distressed securities), strategies with small betas (i.e., relative value), and strategies based on stock picking (like merger arbitrage or equity market neutral). Also of note, some tracker may not have a high correlation with their respective index, but may still exhibit similar performance. This is, for example, the case of funds of funds (FOF in the tables). One reason for this may be that part of the alternative betas of the underlying funds is captured by the fee structure of the FOF and thus do not appear in their performance, while the replicating process provides a direct access to this part of the performance.

Finally, on a more particular note, it is worth taking a look at two particular strategies. First, on the “emerging market: Russia/E. Europe” HFRI index, it is worth noting that the model performs particularly poorly, pointing at the fact that in our pool of factors, none had a strong relation with the economy of that region of the world. Second, the “macro: syst. diversified” is the one case where the model produces a clone with higher drawdowns than the actual HFRI macro: syst. diversified. One reason behind these poor results is probably the set of factors used. Another reason could be the inadequacy of factor models in this case, but one could ask why, if the concept of factor model is the underlying problem, our results do not show more results similar to these. This illustrates that the better results obtained with our replication methodology cannot replace a careful choice of the set of factors. It is also a sign that if a better selection methodology is found, it would still have to rely on some economic insight, echoing results found in the literature [Amenc et al. (2007)].

## Alpha considerations

In the previous sections, we have developed and demonstrated the use of Bayesian filters to answer the question of HF replication. In this section, we focus on the part of the HF performance left unexplained by the methods presented above. We thus look into the alternative alpha component, and look for possible explanations of its origin. In the previous sections, we suggested possible sources including high frequency trading and investments in illiquid assets. To these two, we add here another component which stems not from specific strategies but from the fact that, by construction, a replicating portfolio implements its exposure with a time lag with respect to the replicated HF profile. We focus here on the impacts of the implementation lag and the illiquid investments, in this respective order. Nonlinearities are addressed in a companion paper [Roncalli and Weisang (2009)].

Starting with the impact of the implementation lag, note first that replication clones are obtained using lagged exposures with a lag d = -1. If one uses d = 0, one assumes that one can implement at time $t_k$ the true exposures of the period $[t_k, t_{k+1}]$ and ford > 0, the implemented exposures are those estimated for the period [k + d, k + d +1]. Putting to test our claim that the implementation lag contributes to the alpha, we computed backtests of the portfolios obtained for d = 0, 1, 2 using the 6F model presented above and the HFRI Fund Weighted Composite Index and compared them with the case d = -1. The results obtained are provided in Table 3. Unsurprisingly, with the added information, the results are substantially better, with the best results for the contemporaneous implementation (d = 0). The part of the HF performance explained by the alternative beta clone jumps by about 10% to 85%, reducing the alpha

| d | $\hat{\mu}^{HF}_{1Y}$ | $\hat{\mu}^{Clone}_{1Y}$ | $\pi_{AB}$ | $\sigma_{TE}$ | $\rho$ | $\tau$ | $\varrho$ |
|---|---|---|---|---|---|---|---|
| -1 | 9.94 | 7.55 | 75.93 | 3.52 | 87.35 | 67.10 | 84.96 |
| 0 | 9.94 | 8.39 | 84.45 | 1.94 | 96.17 | 80.18 | 94.55 |
| 1 | 9.94 | 8.42 | 84.77 | 2.05 | 95.71 | 80.09 | 94.42 |
| 2 | 9.94 | 8.26 | 83.11 | 2.22 | 94.96 | 78.42 | 93.59 |

$\hat{\mu}_{1Y}$ is the annualized performance; $\pi_{AB}$ the proportion of the HF index performance explained by the tracker and $\sigma_{TE}$ the yearly tracking error. $\rho$, $\tau$ and $\varrho$ are respectively the linear correlation, the Kendall tau and the Spearman rho between the monthly returns of the HF index and the tracker. All statistics are expressed in percents.

**Table 3 – Results of time lags implementation on the replicating portfolios**

component from around 25% to about 15%. In other words, in our example, 40% of the alternative alpha is explained by the implementation delay. In this particular case, we can therefore propose a new breakdown on the HF performance.

75% of the performance corresponds to alternative beta which may be reproduced by the tracker and 25% is the alternative alpha of which 10% corresponds in fact to alternative beta which may not be implemented and are lost due to the dynamic allocation and 15% makes up a component that we call the pure alternative alpha. It is also interesting to note that the volatility of the pure alpha component ($\sigma_{TE}$ for d = 0, 1, 2) is lower and is half of the volatility of the alternative alpha. We represent in Figure 4 the evolution of the two components of the alternative alpha, with $\alpha_1$ representing the contribution of the implementation lag to the alternative alpha and $\alpha_2$ the pure alternative alpha.

We now turn to our second claim that the alternative alpha stems from the illiquidity premia associated with investment in illiquid assets. Using the results of our previous experiment on implementation delay, we focus on explaining the pure component of the alternative alpha. One possible way to substantiate this claim would be, for example, to extract the pure alpha component and run an analysis in the same fashion as it was done at first for HF replication using regressions to determine whether factors representing different illiquid assets, such as distressed securities or private equity, are able to explain the returns of the pure alternative alpha. We proceed differently here by keeping in mind the idea to demonstrate that it is possible to access the performance of this pure component from an investment perspective. One idea then is to build a core/satellite portfolio where the core is the alternative beta and the satellite is a basket of illiquid or optional strategies. The previous construction of alternative investments has some important advantages. For example, one could consider a portfolio with 70% of alternative beta, 10% of optional or quantitative strategies, 10% of real estate, and 10% of private equity. The core/satellite approach permits us to distinguish clearly liquid and illiquid investments, small term and long term investments. In our example, these three satellite strategies are respectively proxied by equally weighted portfolios of the SGI volatility premium index and JP Morgan

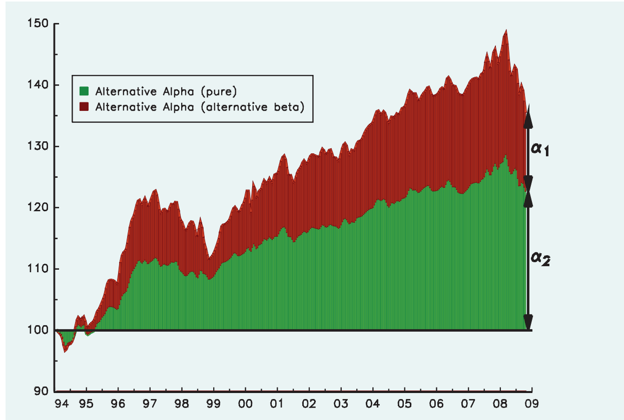



**Figure 4 – The decomposition of the alternative alpha**

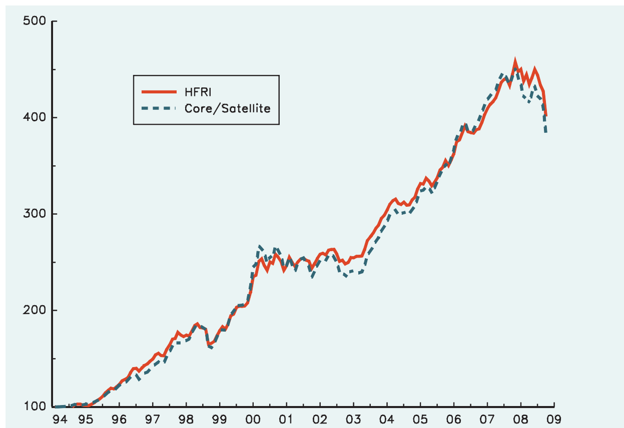



**Figure 5 – The core/satellite approach to alternative investments**

carry max index, UK IPD TR all property index and NCREIF property index, and LPX buyout index and LPX venture index. The results of this approach are displayed in Figure 5.

After obtaining these results, there is no doubt in our mind that, in this case at least, the pure alternative alpha component can be replicated by means of this core/satellite strategy. One may wonder, however, why there is apparently no need to take into account a high frequency factor. Beside the fact that it is rather good news from a practitioner point-of-view, one must point out that in our example, we replicated the HFRI Fund Weighted Composite Index, which is the most general industry aggregate provided by Hedge Fund Research, Inc. As such, in light of the results presented by Diez de los Rios and Garcia (2008), we surmise that the effect of high frequency trading, which would appear as nonlinear, is negligible.

## Discussion

In the sections above, we demonstrated the efficiency of Bayesian filters – in particular the Kalman filter – in capturing the tactical asset allocation. Furthermore, completed by a core/satellite portfolio strategy, we showed this approach would be enough to replicate a general HF index like HFRI. Nonetheless, one could legitimately ask 'so what?' question. Hedge returns are renowned to be generated using complex financial instruments generating highly nonlinear returns. Obviously the exercise above does not include any complex product, and tactical asset allocation is more the realm of 'traditional' managers than hedge funds. Thus, it seems, it falls short of answering the question at hand: replicating any hedge fund track records. Our answer is threefold.

First, the philosophy of replication that we pursued here is a readily available methodology that directly translates into implementable investments. One could take issue and point out that it does not take into account the risk management perspective of hedge fund replication, i.e., using the methodology for risks assessment. But, the issue then is: who is the end user of clones and hedge fund replicates? In what we presented above, nothing forbids the inclusion of 'rule-based' factors mechanically reproducing an alternative strategy to represent a certain type of risk. Unfortunately, as we demonstrate in a companion paper [Roncalli and Weisang (2009)], these types of factors are often difficult to implement as they are extremely dependent on the data available, which are themselves not necessarily representative of investable opportunities. Second, although it has been documented from the beginning of hedge fund replication [Fung and Hsieh (1997)], the existence and presence of nonlinearities in hedge fund returns seem persistent in only a handful of strategies [Diez de los Rios and Garcia (2008)]. Thus, a core/satellite approach capturing on one hand the tactical allocation between different asset classes, combined on the other hand with buy-and-hold strategies to capture risk premia of illiquid investments presents clear advantages, transparency not being last on the list. Again, nothing prevents the inclusion of rule-based factors in the tactical allocation part if the goal is risk assessment. Finally, the framework of tracking problems and their solvability using Bayesian filters provides readily available extensions. For example, using particle filters, one can try to integrate some nonlinearity in the replication methods [Roncalli and Weisang (2009)]. From the academics' point of view, introducing particle filters opens a door for a better understanding of HF returns and the underlying risks of the HF strategies. If it already has direct implications from a risk management perspective, we also surmise that particles filters are one of the main avenues toward a better monitoring of, for now, unaccounted risks, as they are contained in the higher moments of the returns' distribution.

## Conclusion

In this paper, after providing a formal statistical framework to hedge fund replication, we limited ourselves to demonstrate that linear factor models can efficiently recover the tactical allocation using an adequate methodology. Furthermore, we considered how hedge fund replicates could reproduce the alpha. For sake of space, and because they present completely different challenges, we left the study of the replication of nonlinearities in hedge fund returns to another paper [Roncalli and Weisang (2009)]. Nevertheless, we believe the results presented in here to be very interesting both for the practitioners and the academics. From the practitioners' point of view, by grounding all of our approaches into a general and coherent framework, and by meticulously adding complexity to the methodology, we demonstrated that a robust replication process can be obtained by means of mainstream statistical methods, such as the Kalman filter, provided that careful thought is given to the specification of the model and the type of instruments used in the replication process (particularly with respect to liquidity or other trading considerations). It is perhaps necessary to remind the reader again that as an investment toolbox to manage HF exposures (both long and short) and liquidity, the first quality of a HF clone should not be to be a hedge fund in itself. As such, and in line with this HF replication philosophy, our core/satellite approach showed that this robust approach (Kalman filter and liquid instruments) can still be supplemented by other illiquid investments to capture and reproduce more efficiently the risk profile of the hedge fund industry. Incidentally, it also hints at the efficiency of the 'core' method to capture the HF betas to classic asset classes. From the academic's point of view, the new framework provided allows for readily available extensions, with similar problems having been already studied in other disciplines like engineering and signal processing.

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## Appendix 1

### Contemporaneous representation of the Kalman Filter

If one assumes the tracking problem to be linear and Gaussian, one may prove that the optimal algorithm to estimate the state vector is the Kalman filter. The state space model is then given by

$x_k = c_k + F_k x_{k-1} + \nu_k$

$z_k = d_k + H_k x_{k-1} + \eta_k$

with $\nu_k \sim N(0,Q_k)$ and $\eta_k \sim N(0,S_k)$. Moreover, the initial distribution of the state vector is $p(x_0) = \phi(x_0, \hat{x}_0, \hat{P}_0)$, where $\phi(x, m, P)$ is the Gaussian pdf with argument x, mean m and covariance matrix P. The Bayes filter is then described by the following recursive equations

$$\begin{aligned} p(\mathbf{x}_k \mid \mathbf{z}_{1:k-1}) &= \phi\left(\mathbf{x}_{k-1}, \hat{\mathbf{x}}_{k|k-1}, \hat{\mathrm{P}}_{k|k-1}\right) \\ p(\mathbf{x}_k \mid \mathbf{z}_{1:k}) &= \phi\left(\mathbf{x}_k, \hat{\mathbf{x}}_{k|k}, \hat{\mathrm{P}}_{k|k}\right) \end{aligned}$$

with

$$\begin{cases} \hat{\mathbf{x}}_{k|k-1} = \mathbf{c}_k + F_k\,\hat{\mathbf{x}}_{k-1|k-1} \\ \hat{\mathrm{P}}_{k|k-1} = F_k\,\hat{\mathrm{P}}_{k-1|k-1}\,F_k^\top + Q_k \\ \hat{\mathbf{z}}_{k|k-1} = \mathbf{d}_k + H_k\,\hat{\mathbf{x}}_{k|k-1} \\ \hat{\mathbf{e}}_k = \mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1} \\ \hat{V}_k = H_k\,\hat{\mathrm{P}}_{k|k-1} H_k^\top + S_k \\ \hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \hat{\mathrm{P}}_{k|k-1} H_k^\top \hat{V}_k^{-1}\,\hat{\mathbf{e}}_k \\ \hat{\mathrm{P}}_{k|k} = \hat{\mathrm{P}}_{k|k-1} - \hat{\mathrm{P}}_{k|k-1}\,H_k^\top \hat{V}_k^{-1} H_k\,\hat{\mathrm{P}}_{k|k-1} \end{cases} \tag{A1}$$

The set of equations (A1) describes the Kalman filter algorithm. The previous quantities can be interpreted as follows

- $\hat{x}_{k|k-1} = \mathbb{E}[x_k \mid z_{1:k-1}]$ is the estimate of $x_k$ based on all available information until time index $t_{k-1}$;
- $\hat{P}_{k|k-1}$ is the covariance matrix of the estimator $\hat{x}_{k|k-1}$: $\hat{P}_{k|k-1} = \mathbb{E}[(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})^T \mid z_{1:k-1}]$;
- $\hat{z}_{k|k-1} = \mathbb{E}[z_k \mid z_{1:k-1}]$ is the estimate of $z_k$ based on all available information until time index $t_{k-1}$;
- $\hat{e}_k = z_k \mid \hat{z}_{k|k-1}$ is the estimated tracking error;
- $\hat{V}_k$ is the covariance matrix of the tracking error $\hat{V}_k = \mathbb{E}[e_k e_k^T]$;
- $\hat{x}_{k|k} = \mathbb{E}[x_k \mid z_{1:k}]$ is the estimate of $x_k$ based on all available information until time index $t_k$;
- Finally, $\hat{P}_{k|k}$ is the covariance matrix of $\hat{x}_{k|k}$ : $\hat{P}_{k|k} = \mathbb{E}[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^T \mid z_{1:k}]$.

### Interpretation of KF estimates updates

The joint density of the observational vectors $z_1,\ldots,z_k$ can be written as $p(z_1,\ldots,z_k) = p(z_1) \prod^k_{l=2} p(z_l \mid z_{l-1})$. Transforming from $z_l$ to $\hat{e}_l = z_k - \hat{z}_{l|l-1}$,[10] we have $p(\hat{e}_1,\ldots,\hat{e}_k) = p(\hat{e}_1) \prod^k_{l=2} p(\hat{e}_l)$ since $p(z_1) = p(\hat{e})$ and the Jacobian of the transformation is unity because each $\hat{e}_l$ is $z_l$ minus a linear function of $z_1,\ldots,z_{l-1}$ for $l = 2,\ldots,k$. We deduce then that $\hat{e}_1,\ldots,\hat{e}_k$ are independent from each other and that $\hat{e}_l,\ldots,\hat{e}_k$ are independent from $z_{1:l-1}$. This last property, combined with some well known results of multivariate regression, provides us with an interpretation of the gain matrix and the dynamical adjustment of the weights. Noticing that

$$\begin{aligned} \mathbb{E}(\mathbf{x}_{k+1} \mid \mathbf{z}_1, \ldots, \mathbf{z}_k) &= \mathbb{E}(\mathbf{x}_{k+1} \mid \mathbf{z}_1, \ldots, \mathbf{z}_{k-1}, \hat{\mathbf{e}}_k) \\ &= \mathbb{E}(\mathbf{x}_{k+1} \mid \mathbf{z}_1, \ldots, \mathbf{z}_k) + \Sigma_{\hat{\mathbf{z}}_{1:k-1}\,\hat{\mathbf{e}}_k} \Sigma^{-1}_{\hat{\mathbf{e}}_k\,\hat{\mathbf{e}}_k} \hat{\mathbf{e}}_k, \end{aligned}$$

where the second equality is a useful result from multivariate regression. Hence, we see that in equation (6), the gain matrix $K_k$ can be construed as the matrix of 'regression' coefficients of $x_{k+1}$ on $\hat{e}_k$ the innovation at time $t_k$ [cf. e.g., Durbin and Koopman (2001) or Hamilton (1994)].

10 By definition, $\hat{e}_l = z_l - \mathbb{E}(z_l \mid z_{1:l-1})$, i.e. $\hat{e}_l$ is the part of $z_l$ that cannot be predicted from the past. For this reason, the process $\hat{e}_k$ is sometimes called the innovation process.