How to Design Target-Date Funds?*

Benjamin Bruder
Research & Development
Lyxor Asset Management, Paris
benjamin.bruder@lyxor.com

Léo Culerier
Research & Development
Lyxor Asset Management, Paris
leo.culerier@lyxor.com

Thierry Roncalli
Research & Development
Lyxor Asset Management, Paris
thierry.roncalli@lyxor.com

September 2012

Abstract

Several years ago, the concept of target-date funds emerged to complement traditional balanced funds in defined-contribution pension plans. The main idea is to delegate the dynamic allocation with respect to the retirement date of individuals to the portfolio manager. Owing to its long-term horizon, a target-date fund is unique and cannot be compared to a mutual fund. Moreover, the objective of the individual is to contribute throughout their working life by investing a part of their income in order to maximise their pension benefits. The main purpose of this article is to analyse and understand dynamic allocation in a target-date fund framework. We show that the optimal exposure in the risky portfolio varies over time and is very sensitive to the parameters of both the market and the investor's. We then deduce some practical guidelines to better design target-date funds for the asset management industry.

Keywords: target-date fund, retirement system, dynamic asset allocation, stochastic optimal control, market portfolio, risk aversion, stock/bond asset mix policy.

JEL classification: C61, D91, G11, J26.

1 Introduction

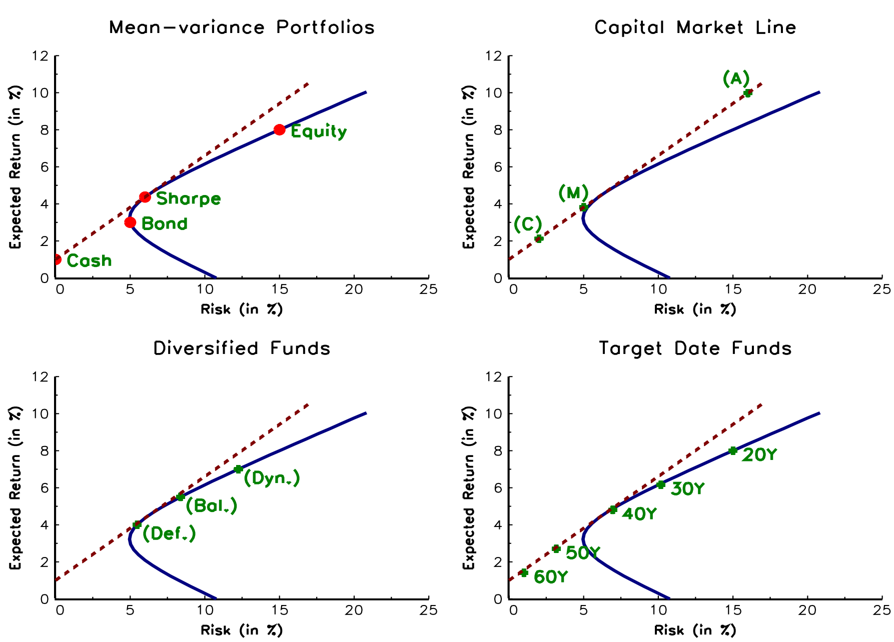
In 1952, Markowitz introduced the first mathematical formulation for portfolio optimisation. For Markowitz, "*the investor does (or should) consider expected return a desirable thing and variance of return an undesirable thing*". Indeed, Markowitz shows that an efficient portfolio is the portfolio that maximises the expected return for a given level of risk (corresponding to the variance of return). Markowitz concludes that there is not only one optimal portfolio, but a set of optimal portfolios which is called the 'efficient frontier'. By studying the liquidity preference, Tobin (1958) shows that the efficient frontier becomes a straight line in the presence of a risk-free asset. In this case, optimal portfolios correspond to a combination of the risk-free asset and one particular efficient portfolio named the tangency portfolio. Sharpe (1964) summarises Markowitz and Tobin's results as follows: "*the*

*We are grateful to Karl Eychenne and Nicolas Gaussel for their helpful comments.

process of investment choice can be broken down into two phases: first, the choice of a unique optimum combination of risky assets[1]*; and second, a separate choice concerning the allocation of funds between such a combination and a single riskless asset*". This two-step procedure is today known as the *separation theorem* (Lintner, 1965).

In Figure 1, we have illustrated these results. The first panel represents the efficient frontier of Markowitz between equities and bonds. By combining the mean-variance optimised portfolio with the higher Sharpe ratio and the risk-free rate, we obtain the capital market line. In the second panel, we have reported the risk/return profile of optimal portfolios corresponding to different values for the risk aversion parameter. The aggressive (A) investor leverages the tangency portfolio in order to take more risk and to expect better performance. For the moderate (M) investor, the optimal portfolio is closed to the tangency portfolio. The conservative (C) investor will take less risk, hence the reason why they allocate their wealth between the tangency portfolio (risky assets) and the risk-free asset. This theoretical view of asset allocation is however far from the 'popular advice' on portfolio allocation. In the third panel, we have shown typical diversified funds designed by the asset management industry. Generally, we distinguish three fund profiles: defensive, balanced and dynamic. The difference between them comes from the relative proportion of stocks and bonds[2]. It means that in practice, the composition of the risky portfolio varies according to the investor's risk aversion. This portfolio construction contradicts the separation theorem, which states that the composition of the risky portfolio should be the same for all investors. This paradox, known as the *asset allocation puzzle* (Canner *et al.*, 1997), has resulted in wealth of literature being written in an attempt to find some explanations (see Campbell (2000) for a survey).

Figure 1: The asset allocation puzzle



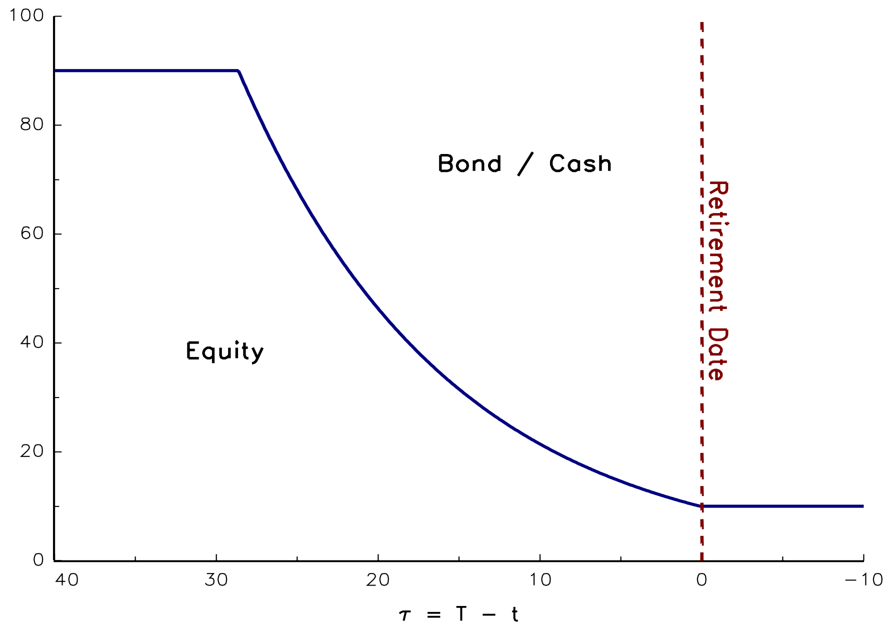
---

[1]It is precisely the tangency portfolio.

[2]For a defensive fund, the proportion of stocks and bonds is 20% and 80% respectively, whereas they are the reverse for the dynamic fund. In the case of the balanced fund, we have the same proportions.

Another puzzle concerns lifecycle funds or target-date funds. In a lifestyle fund like diversified funds, the asset mix policy depends on the risk aversion (or the lifestyle) of the investor. In a target-date fund, the asset mix policy depends on the time to retirement of the investor. In Figure 2, we have reported the dynamic asset allocation of a typical target-date fund[3]. When the individual is young, they invest principally in equities whereas the allocation will be more heavily weighted toward bonds (or cash) as they approach retirement. The speed with which a target-date fund changes its asset allocation is known as the 'glide path'.

Figure 2: An example of glide path



These funds have been gaining popularity since early 21$^{\text{st}}$ century. One of the reasons is the major shift from defined benefit (DB) toward defined contribution (DC) pension plans and the transfer of investment risk from the corporate sector to households. Another reason concerns certain regulatory reform and tax benefits. For example, the Pension Protection Act of 2006 in the US requires companies who have underfunded their pension plans to pay higher premiums to the Pension Benefit Guaranty Corporation. DB liabilities may induce large pension costs for the sponsor and incite it to promote DC pension plans. In this context, it is not surprising that target-date funds encounter high growth:

> "*Target-date funds are fast becoming a fixed feature of the defined-contribution landscape. Over the past half dozen years, assets in target-date funds have grown more than fivefold from* \$71 *billion at the end of 2005 to approximately* \$378 *billion at year-end 2011. In its most recent study, Vanguard reported that* 82% *of its retirement plans offered target-date funds, and nearly one fourth of participants invested only in a target-date fund. The consultant Casey Quirk estimates that target-date funds will consume more than half of all defined-contribution assets by 2020*" (Morningstar [39], 2012).

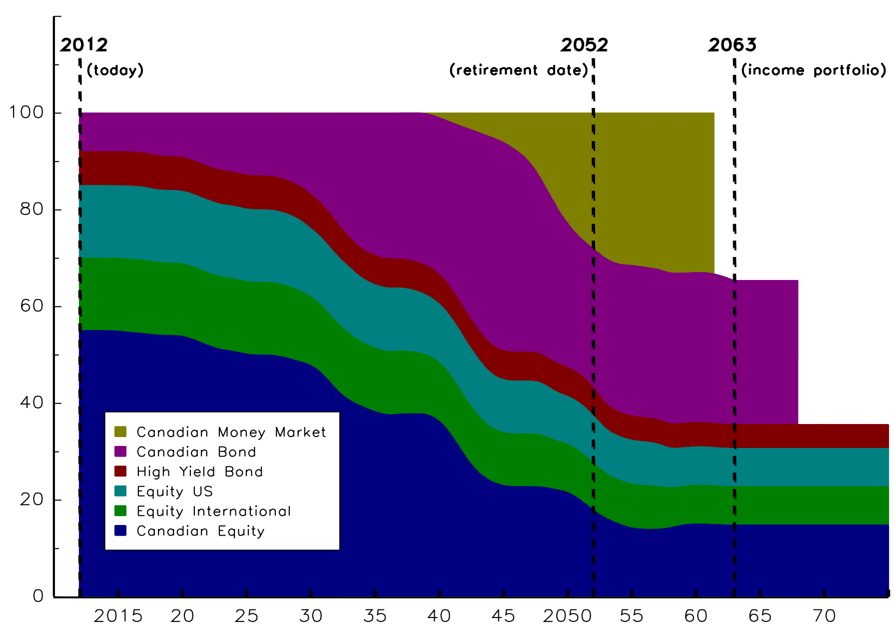
[3] $\tau = T - t$ is the remaining time until the retirement.

It is also a concentrated industry, because, according to the Morningstar report, three asset managers (Fidelity, Vanguard and T. Rowe Price) hold 78% of the target-date mutual funds. The previous figures only concern open target-date assets and not the overall target-date assets managed in retirement systems. The Investment Company Institute (2012) reports that U.S. retirement assets were $17.9 trillion at year-end 2011. The largest components are IRAs ($4.9), DC plans ($4.5) and private-sector DB pension funds ($2.4). For DC plans, the asset allocation depends on the age of participants:

> "*On average, younger participants allocate a larger portion of their portfolio to equities [...] According to research conducted by ICI and the Employee Benefit Research Institute (EBRI), at year-end 2010, individuals in their twenties invested* 44% *of their assets in equity funds and company stock,* 37% *in target-date funds and non-target-date balanced funds [...] All told, participants in their twenties had 74 percent of their 401(k) assets in equities. By comparison, at year-end 2010, individuals in their sixties invested* 41% *of their assets in equity funds and company stock,* 16% *in target-date funds and non-target-date balanced funds [...] All told, participants in their sixties had* 49% *of their 401(k) assets in equities*" (Investment Company Institute [27], 2012).

Indeed, target-date funds represent 27% and 9% respectively of the DC plan assets for individuals in their twenties and in their sixties. This therefore confirms the appeal of target-date funds for younger participants.

Figure 3: Allocation of the Fidelity ClearPath® 2045 Retirement Portfolio



Source: `www.fidelity.ca/cs/Satellite/en/public/products/managed_solutions/clearpath`.

One of the big challenges of target-date funds is the design of glide paths. Let us consider for example the Fidelity ClearPath® Retirement Portfolios. In Figure 3, we have reported the dynamic allocation for Canadian individuals who will retire in 40 years using the simulation tool provided on Fidelity's web site. We notice how the split between stocks, bonds and cash evolves in relation to time. The allocation becomes more conservative when we approach the retirement date: equities represent more than 80% of the portfolio today whereas their weight is only 31% in 50 years' time; the bond allocation increases during 25 years and then stabilises at around 30%; the weight of the high yield asset class varies between 7% and 5% and therefore is very stable; the allocation in cash begins 12 years before the retirement date and represents 35% at the end. If we consider other target-date funds, glide paths present similar patterns on average. But differences among target-date funds may be very large. For example, Morningstar (2012) reports that equity allocation varies between 20% (38% and 85% respectively) and 78% (86% and 100% respectively) if the target year is 2015 (2025 and 2055 respectively). It is rather surprising to obtain such broad allocation range particularly for shorter target dates.

There are many popular justifications given for these allocations (Quinn, 1991), but most of them does not make economic sense (Jagannathan and Kocherlakota, 1996). Hence the reason why academic research may help to understand this asset allocation puzzle. The seminal work of Merton (1969, 1971) is the usual framework used to analyse dynamic asset allocation. Most research introduces some clarifications of the Merton model. For example, Munk *et al.* (2004) consider the mean-reverting property of equity returns and the uncertainty of inflation risk. By calibrating their model with historical US data from the period 1951-2003, they found results consistent with the investment recommendation "more stocks for longer-term investors". This research confirms the conclusion made by Brennan and Xia (2002) that the optimal stock-bond mix depends on the investor's horizon in the presence of inflation risk and highlights the influence of mean reversion[4]. Other clarifications concern the behaviour of optimal allocation in the latter years before the retirement date (Basu and Drew, 2009), the effect of the stock-bond correlation on hedging demand (Henderson, 2005), the labour supply flexibility (Bodie *et al.*, 1992), the longevity risk (Cocco and Gomez, 2012), the effect of real estate assets (Martellini and Milhau, 2010) and the return predictability (Larsen and Munk, 2012).

Although the various clarifications are interesting to understand the allocation of target-date funds, they are not enough to explain the dynamics of glide paths. Labour income is generally the key factor to understanding their behaviour. It explains the breadth of literature on the relationship between stock-bond asset mix and labour income[5]. Our paper adopts this approach by considering stochastic permanent contribution to the target-date fund. In this case, we show that the behaviour of optimal exposure is similar to many glide paths used in the investment industry. Nevertheless, we also notice that the solution is very sensitive to input parameters like the equity risk premium, the contribution uncertainty or the stock-bond correlation. But the most important factor concerns the personal profile of the individual. The degree of risk aversion, retirement date and income perspective of the investor are certainly the key elements to design target-date funds.

Our paper is organised as follows. Section two presents the theoretical model based on the Merton framework. Since we assume that interest rates and permanent contribution are stochastic, we have to solve the optimisation problem by considering numerical analysis. In Section three, we analyse the dynamics of risky exposure and define the glide path. Then, we highlight some patterns for the design of target-date funds with respect to our theoretical model. Section five offers some concluding remarks.

[4]See also Barberis (2000), Campbell *et al.* (2001) or Wachter (2002) for related results.

[5]We could cite for example the works of Bodie *et al.* (2004), Cocco *et al.* (2005), Gomes and Michaelides (2005), Henderson (2005), Munk and Sørensen (2010) or Viceira (2001).

2 A theoretical model

We consider the intertemporal model of Merton (1971) in which we introduce a stochastic permanent contribution π_t. It is the main difference between the design of a mutual fund and a target-date fund:

- In a mutual fund, the individual makes an initial investment and seeks to maximise the net asset value for a given horizon. Generally, the investment horizon for such funds is three or five years;
- In a target-date fund, the individual makes an initial investment and continues to contribute throughout their working life. The investor's objective is then to maximise their pension benefits during retirement.

Another difference between a mutual fund and a target-date fund concerns then the investment horizon. The latter is typically equal to forty years.

Generally, retirement system models do not use permanent contribution π_t as a state variable, but prefer to consider labour income L_t. Of course, most of the permanent contribution comes from a savings component related to labour income and we can link these two state variables as follows:

$$\pi_t = \varpi_t L_t$$

where ϖ_t is the savings ratio of the individual. But this equation ignores the possibility that the permanent contribution could come from family heritage, from inheritance or even from the employer's funding contribution system[6]. Hence why we prefer to consider the permanent contribution, and not the labour income, as a (exogenous) state variable.

2.1 The framework

Let us first introduce some notations. We consider a target-date fund with maturity T owned by only one investor. Its value at time t is denoted by X_t. It could be invested in a risky portfolio S_t with a proportion α_t and in a zero-coupon bond $B_{t,T}$ with a proportion $1-\alpha_t$. Since the investor has some income or private wealth and would like to use the target-date fund for retirement pension benefits, they regularly contribute to the fund. We note their contribution as π_t. The dynamics of the target-date fund is then:

$$\frac{\mathrm{d}X_t}{X_t} = \alpha_t \frac{\mathrm{d}S_t}{S_t} + (1-\alpha_t)\frac{\mathrm{d}B_{t,T}}{B_{t,T}} + \frac{\pi_t \,\mathrm{d}t}{X_t}$$

Let us now precise the dynamics of S_t, $B_{t,T}$ and π_t. For this last one, we assume that $\pi_t = p_t Q_t$ where p_t is the average contribution behaviour for the representative agent and Q_t is a random factor related to contribution uncertainty. The state variable Q_t is important, because the investor does not know exactly what their contribution will be in the future. We then have:

$$\begin{cases} \mathrm{d}S_t = \mu_t S_t \,\mathrm{d}t + \sigma_t S_t \,\mathrm{d}W_t^S \\ \mathrm{d}Q_t = \theta_t Q_t \,\mathrm{d}t + \zeta_t Q_t \,\mathrm{d}W_t^Q \\ \mathrm{d}B_{t,T} = r_t B_{t,T} \,\mathrm{d}t + \Gamma_{t,T} B_{t,T} \,\mathrm{d}W_t^B \end{cases}$$

[6] This last case may be illustrated by the French PERCO retirement system. The individual contribution is completed by a (capped) employer contribution π_t^E. It implies that the permanent contribution becomes:

$$\pi_t = \varpi_t L_t + \pi_t^E$$

The objective of individuals is generally to obtain the maximum contribution $\pi_{\max}^E$ of the employer meaning that the permanent contribution is not a proportion of the labour income.

Let $W_t = \left(W_t^S, W_t^Q, W_t^B\right)$ be the vector of brownian motions. We assume that:

$$\mathbb{E}\left[W_t W_t^\top\right] = \begin{pmatrix} 1 & \rho_{S,Q} & \rho_{S,B} \\ \rho_{S,Q} & 1 & \rho_{Q,B} \\ \rho_{S,B} & \rho_{Q,B} & 1 \end{pmatrix} \mathrm{d}t$$

2.2 Optimal solution

Let $\mathcal{U}(x)$ be the investor's utility function which depends on the value x reached by the fund. We assume that their objective is to maximise their utility at time T. We then have:

$$\alpha_t^* = \arg\max \mathbb{E}_t\left[\mathcal{U}(X_T)\right]$$

The investor's problem is then to find the optimal level of exposure α_t^* to the risky portfolio depending on the behaviour of the assets and the investor's contribution policy. Since we cannot obtain a closed formula for α_t^*, we use stochastic optimal control.

We use the zero-coupon bond $B_{t,T}$ of maturity T as the numéraire. We introduce the notation $X_{t,T} = X_t / B_{t,T}$. As the self-financing condition is invariant by change of numéraire (El Karoui *et al.*, 1995), we obtain:

$$\frac{\mathrm{d}X_{t,T}}{X_{t,T}} = \alpha_t \frac{\mathrm{d}S_{t,T}}{S_{t,T}} + \frac{\pi_t \,\mathrm{d}t}{B_{t,T} X_{t,T}}$$

In Appendix A.1, we show that the forward dynamics of (X_t, Q_t) is:

$$\begin{cases} \mathrm{d}X_{t,T} = \left(\alpha_t \mu_{t,T} X_{t,T} + p_t Q_{t,T}\right) \mathrm{d}t + \alpha_t \sigma_{t,T} X_{t,T} \,\mathrm{d}B_t^S \\ \mathrm{d}Q_{t,T} = \theta_{t,T} Q_{t,T} \,\mathrm{d}t + \zeta_{t,T} Q_{t,T} \,\mathrm{d}B_t^Q \end{cases}$$

with[7]:

$$\begin{cases} \mu_{t,T} = \mu_t - r_t + \Gamma_{t,T}^2 - \rho_{S,B}\sigma_t\Gamma_{t,T} \\ \sigma_{t,T} = \sqrt{\sigma_t^2 + \Gamma_{t,T}^2 - 2\rho_{S,B}\sigma_t\Gamma_{t,T}} \\ \theta_{t,T} = \theta_t - r_t + \Gamma_{t,T}^2 - \rho_{Q,B}\zeta_t\Gamma_{t,T} \\ \zeta_{t,T} = \sqrt{\zeta_t^2 + \Gamma_{t,T}^2 - 2\rho_{Q,B}\zeta_t\Gamma_{t,T}} \end{cases}$$

B_t^S and B_t^Q are two brownian motions such that $\mathbb{E}\left[B_t^S B_t^Q\right] = \rho^\star \,\mathrm{d}t$ with:

$$\rho^\star = \frac{\rho_{S,Q}\sigma_t\zeta_t - \rho_{S,B}\sigma_t\Gamma_{t,T} - \rho_{Q,B}\zeta_t\Gamma_{t,T} + \Gamma_{t,T}^2}{\sigma_{t,T}\zeta_{t,T}} \tag{1}$$

Let $\mathcal{J}$ be the function defined by:

$$\mathcal{J}(t,x,q) = \sup_\alpha \mathbb{E}_t\left[\mathcal{U}(X_T) \mid Q_{t,T} = q, X_{t,T} = x\right]$$

The optimal exposure is then given by this relationship[8]:

$$\alpha_t^* = -\frac{\mu_{t,T}\partial_x\mathcal{J}(t,x,q) + \rho^\star\sigma_{t,T}\zeta_{t,T} q \partial_{x,q}^2\mathcal{J}(t,x,q)}{x\sigma_{t,T}^2\partial_x^2\mathcal{J}(t,x,q)} \tag{2}$$

[7]To obtain this result and to solve our model, we have to introduce some technical restrictions to the process $\mu_t - r_t$ and $\theta_t - r_t$ so that $X_{t,T}$ and $Q_{t,T}$ are two Markov processes. For example, our analysis is valid if $\mu_t - r_t$ and $\theta_t - r_t$ are deterministic.

[8]See Appendix A.2 for more details.

2.3 Some specific cases

Let us first consider that the investor does not contribute to the fund. In this case, $\pi_t = 0$ and the solution (2) reduces to:

$$\alpha_t^* = -\frac{\mu_{t,T}\partial_x \mathcal{J}(t,x)}{x\sigma_{t,T}^2 \partial_x^2 \mathcal{J}(t,x)} \tag{3}$$

Moreover, if we assume that the interest rate is constant and the parameters μ_t and σ_t do not depend on time t, we obtain Merton's well-known result (1969):

$$\alpha_t^* = -\frac{(\mu - r)}{\sigma^2} \cdot \frac{\partial_x \mathcal{J}(t,x)}{x\partial_x^2 \mathcal{J}(t,x)}$$

In the case of the CRRA utility function $\mathcal{U}(x) = x^\gamma/\gamma$ with $\gamma < 1$, the proportion of the risky portfolio remains constant for the entire period $[0, T]$ and does not depend on the value x of the fund:

$$\alpha_t^* = \frac{(\mu - r)}{(1-\gamma)\sigma^2} = \bar{\alpha} \tag{4}$$

The constant mix strategy is then the optimal dynamic allocation strategy.

If $Q_t = 1$, it means that there is no uncertainty on the future contribution of the investor. Moreover, if we assume that the interest rate is nul, Merton's results (4) become:

$$\alpha_t^* = \bar{\alpha} + \frac{\mu \int_t^T \pi_u \, \mathrm{d}u}{(1-\gamma)\sigma^2 x} \tag{5}$$

Therefore, the optimal exposure depends on the future contributions to be made by the investor. This result was already found by Merton (1971) when he introduced noncapital gain income:

> "*[...] one finds that, in computing the optimal decision rules, the individual capitalizes the lifetime flow of wage income as the market (risk-free) rate of interest and then treats the capitalized value of an addition to the current stock of wealth* " (Merton (1971), page 395).

3 Dynamics of the risky exposure

In this section, we study the dynamics of the risky exposure α_t^* with respect to the different parameters. First, we analyse the impact of permanent contribution. Second, we define precisely what the glide path of a target-date fund is and how it is related to the optimal allocation. Then, we measure the impact of the different input parameters (stochastic contribution, stochastic interest rates and correlations) on the solution.

3.1 Comparison with the optimal solution of Merton

Let us consider the equation (5). We notice that α_t^* is an increasing function of the expected return μ and the risk aversion parameter γ and a decreasing function of the volatility σ. We retrieve Merton's stylised fact. We also notice that α_t^* is larger than $\bar{\alpha}$ if $\pi_t \geq 0$. It means that the investor is more exposed in a target-date fund than in the constant mix strategy, because of the contribution effect. The difference between α_t^* and $\bar{\alpha}$ depends on the behaviour of the contribution function π_t and the value of the wealth x of the investor.

It is an increasing function of the ratio between the future contributions and the actual value of the target-date fund. In Table 1, we have shown the numerical values taken by α_t^* when $\mu = 3\%$, $\sigma = 15\%$, $\gamma = -3$ and $\pi_t = \pi$. In this case, the optimal exposure of the constant mix strategy is 33.3%. If we suppose that the investor plans to have a future cumulative contribution $\Pi_t = \int_t^T \pi_u \, \mathrm{d}u$ equal to their actual wealth, the optimal exposure is then twice the exposure $\bar{\alpha}$ obtained by Merton[9]. If the future contribution is relatively small compared to the actual wealth, the exposure is close to the exposure of the constant mix strategy. We could then deduce the following economic behaviour of the investor:

- the exposure is an increasing function of the maturity because the future contribution Π_t is an increasing function of the remaining time $T - t$; It implies, for example, that a young worker has greater exposure than an individual close to retirement;

- the exposure is smaller for an investor who does not expect to have significant future income with respect to their initial wealth than for an investor for whom future growth is expected to be significant; this implies that individuals take less risk if they have more initial wealth.

These different properties are illustrated in Figure 4 with the previous values of the parameters μ, σ and γ. We assume that $\pi_t = 5\%$ while T is fixed to 40 years. In the first panel, we confirm that exposure decreases with the value of the wealth x and the time[10] t. In the top-right panel, we illustrate the impact of the wealth x. The smaller the wealth, the greater the risk exposure.

Table 1: Optimal exposure α_t^* (in %) with respect to x and π

	$T - t = 1$ year				$T - t = 10$ years			
x/π	0	100	1000	10000	0	100	1000	10000
1000	33.3	36.7	66.7	366.7	33.3	66.7	366.7	3366.7
2000	33.3	35.0	50.0	200.0	33.3	50.0	200.0	1700.0
5000	33.3	34.0	40.0	100.0	33.3	40.0	100.0	700.0
10000	33.3	33.7	36.7	66.7	33.3	36.7	66.7	366.7
100000	33.3	33.4	33.7	36.7	33.3	33.7	36.7	66.7

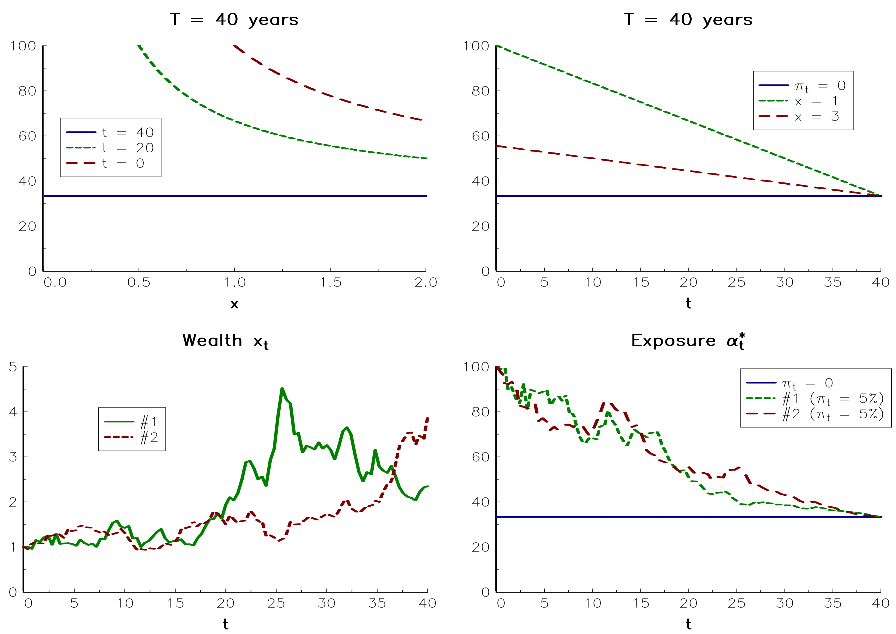
The initial wealth x is reported in rows whereas the future contribution π corresponds to columns. Their values are in dollars. For example, if the initial wealth is 10000 dollars and the future contribution is 1000 dollars per year, the optimal exposure α_t^* is 36.7% (respectively 66.7%) if the time to retirement $T - t$ is 1 year (respectively 10 years).

Remark 1 *We notice that if we multiply the initial wealth x and the future contribution π_t by the same factor, the optimal exposure does not change. We could then normalize the results, because they only depend on the relative ratio between the future contribution and the current wealth. For example, we verify in Table 1 that the optimal exposure for $x = 1000$ dollars and $\pi = 100$ dollars is the same for $x = 10000$ dollars and $\pi = 1000$ dollars. This is why we normalize x and set it to 1.*

[9] It is the case when the actual wealth x is equal to 10000 dollars, the future contribution π is 1000 per year and the time to retirement $T - t$ is 10 years.

[10] meaning that it increases with the remaining time $T - t$.

Figure 4: Computation of the risky exposure α_t^*



3.2 Defining the glide path

The glide path of the target-date fund is generally defined as dynamic asset allocation $\{\alpha_t^*, t \leq T\}$. In Figure 4, the bottom-left panel corresponds to two simulations of the wealth x whereas the bottom-right panel indicates the corresponding optimal exposure α_t^*. We first notice that the allocation depends on the path of the wealth. It implies that optimal exposure and wealth are both endogenous:

$$X_t \longrightarrow \alpha_t^* \longrightarrow \mathrm{d}X_t \longrightarrow X_{t+\mathrm{d}t}$$

Therefore, there is not one dynamic asset allocation path as illustrated in bottom-left panel. Hence why it is more pertinent to define the glide path as the expected dynamic asset allocation:

$$g_t = \mathbb{E}_0 \left[\alpha_t^*\right] = \mathbb{E}\left[\alpha_t^* \mid X_0 = x_0\right]$$

The glide path can be calculated using Monte Carlo simulations. In the case where the interest rate is nul and the contribution function is linear, we obtain the approximated formula given in Appendix A.4. In Figure 5, we have reported the glide path computed by Monte Carlo and by approximation[11]. We notice that the approximated formula gives a good result[12].

The glide path depends on the shape of the contribution function π_t. We give two examples in Figure 6. In the first example (top-left panel), the investor does not contribute when they are relatively young, whereas upon retirement their contribution is at its maximum. In this case, we obtain the glide path given in the bottom-left panel. During 10 years, the glide

[11] We use the same parameters values as before. For the contribution function, a and b are set to 0.002 and 0.01.

[12] The small difference comes from the convexity bias due to Jensen's inequality.

Figure 5: Computation of the glide path

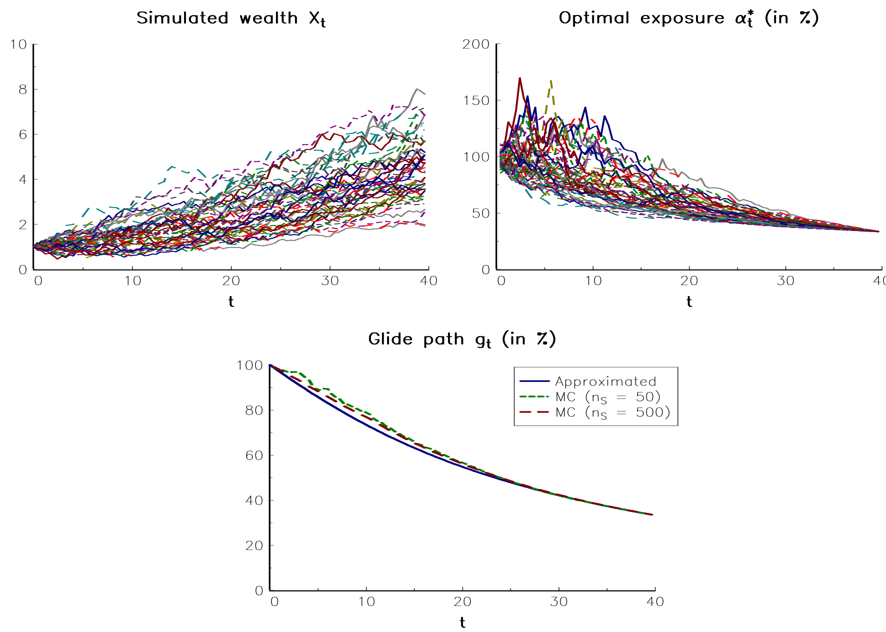
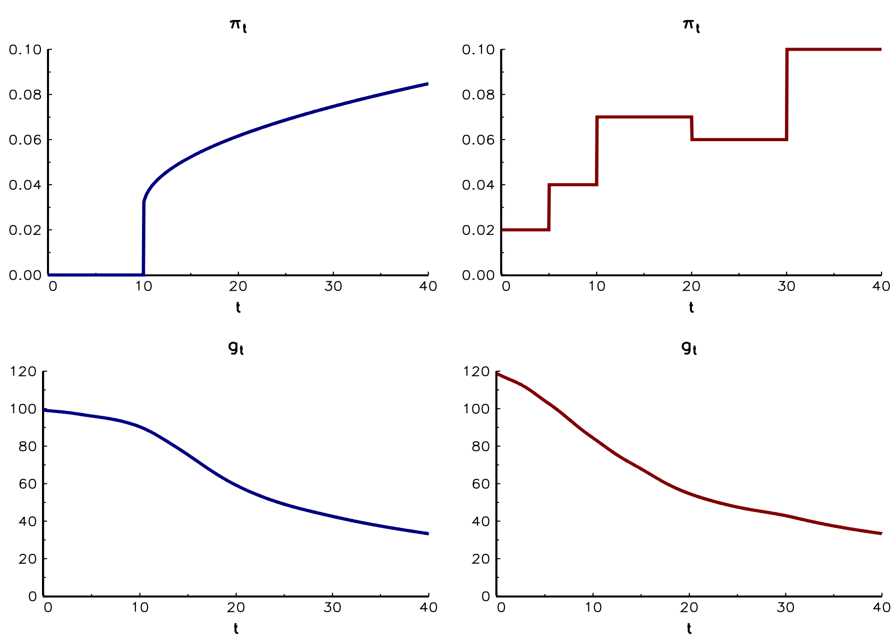


Figure 6: Some examples of glide path

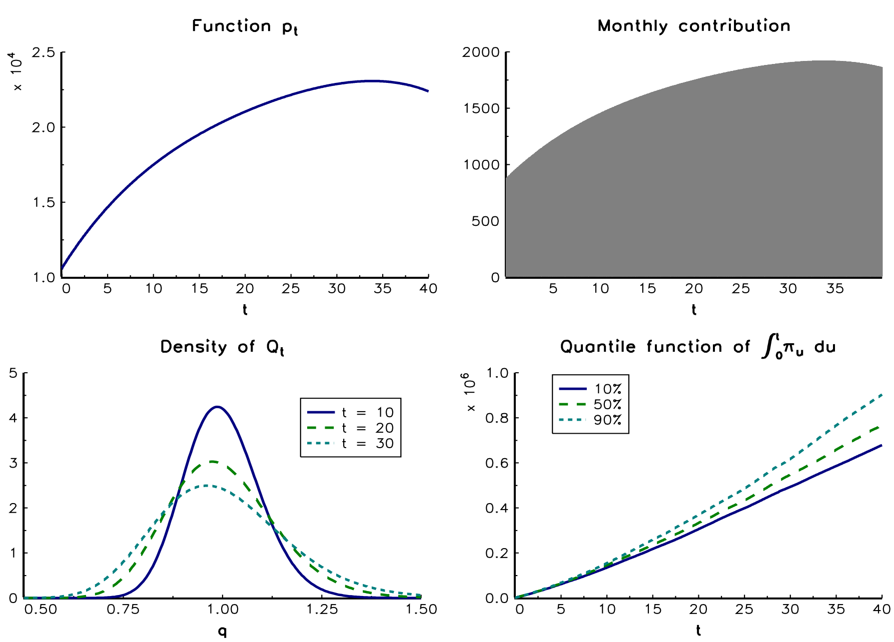


path is approximatively constant because their future contribution remains constant. We could also consider an example where the contribution function is not a monotone function (top-right panel). In this case, the shape of the glide path is more complex as illustrated in the bottom-right panel. We notice in this example that the exposure of the investor to the risky asset is leveraged at the beginning of their working life.

3.3 Introducing uncertainty on the contribution function

In the sequel, we assume that $\mu_t = 3\%$, $\sigma_t = 15\%$, $\theta_t = 0$, $\zeta_t = 3\%$, $r_t = 1\%$ and $\Gamma_{t,T} = 0$. Moreover, all the correlations are set to zero. For the utility function, we continue to use the CRRA function with $\gamma = -3$. The contribution π_t is calibrated using the French saving report done by INSEE (ref. [26], Figure 6, page 114). We obtain the function p_t reported in the first panel in Figure 7. In order to better understand these data, we have indicated the corresponding monthly contribution in the second panel. The average monthly contribution is equal to 1650 dollars whereas the minimum and the maximum are respectively 880 and 1900 dollars. In the third panel, we have represented the probability density function of Q_t. We verify that the variance of Q_t increases with the time t due to the economic uncertainty. Finally, the fourth panel corresponds to the quantile distribution of the cumulative contribution $\int_0^t \pi_u \,\mathrm{d}u$.

Figure 7: Dynamics of the contribution function π_t

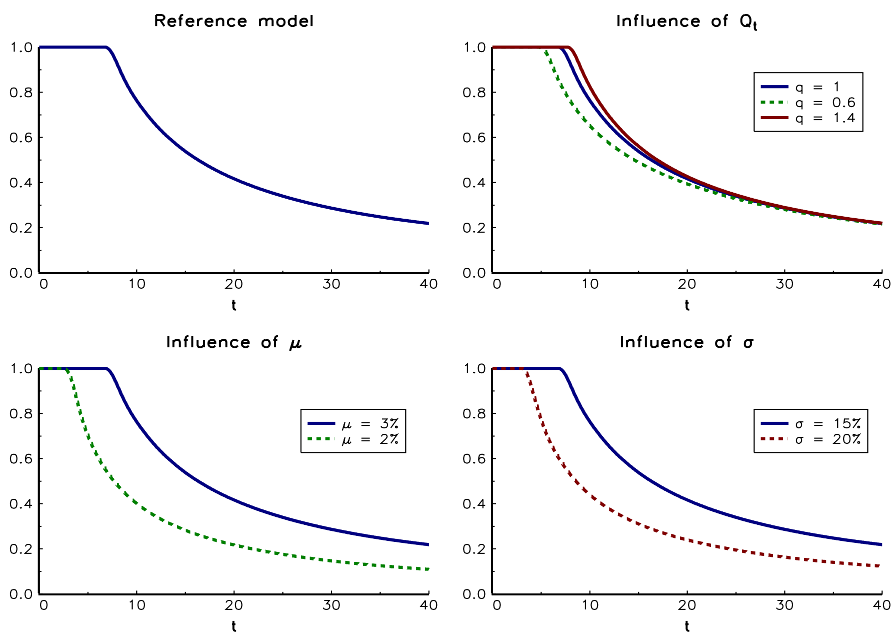


The representation of the optimal exposure α_t^* is complicated because it depends on three state variables: the time t, the wealth X_t and the degree of economic uncertainty Q_t. Hence why we prefer to compute the glide path g_t:

$$g_t = \mathbb{E}_0 \left[\alpha_t^*\right] = \mathbb{E}\left[\alpha_t^* \mid X_0 = x_0, Q_0 = q_0\right]$$

For the illustration, x_0 is set to USD 10 000 and q_0 is equal to 1. We have reported the glide path g_t in Figure 8. If we consider the first panel, we retrieve the typical behaviour of glide paths used in the investment industry. But this glide path is valid for an investor with an initial wealth of USD 10 000 and an average total contribution of USD 800 000 (see Figure 7). If the initial wealth is equal to USD 500 000, the glide path is completely different. So, the glide paths used in the investment industry correspond to investors whose initial wealth is very small compared with their future revenues. In the second panel, we show the influence of economic uncertainty of future contributions. We notice that the influence of Q_t is similar to the ratio between the actual wealth and the future contributions. Of course, the level of the glide path is highly dependent of the risk premium $\mu - r$ and the volatility σ of the risky asset (Panels 3 and 4 in Figure 8). More precisely, we observe that the most important factor is the Sharpe ratio divided by the volatility. This result has been already exhibited by Merton (1969).

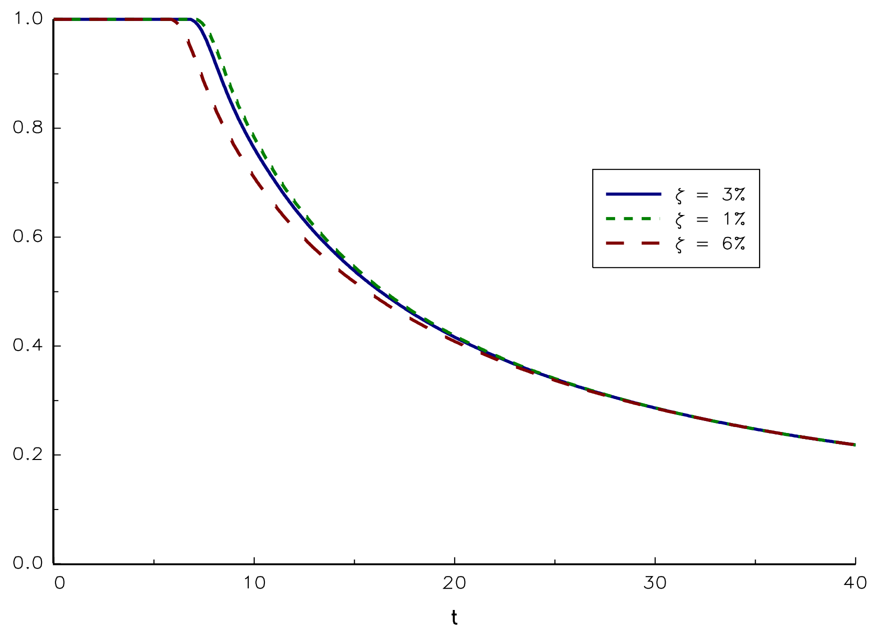
Figure 8: Glide path with random economic factor



The first panel corresponds to the glide path using default values for parameters. The second panel represents the influence of Q_t which is the state variable of the contribution uncertainty. In average, Q_t is equal to 1. If $Q_t < 1$, the realized future contribution is then below the expected future contribution. The third panel shows the impact of the expected return μ of the risky asset whereas the fourth panel indicates how the glide path changes with the volatility of the risky asset.

Remark 2 *More surprising is the influence of the volatility ζ_t, which measures the standard deviation of the future contribution. As expected, it reduces the risky exposure, but the impact is small (see Figure 9). One explanation is the leptokurtic behavior of the log-normal distribution of Q_t. Increasing ζ_t implies a more dispersed probability distribution, but with occurrences of high values of Q_t. This second effect partially compensates the effect of dispersion. Nevertheless, the impact of ζ_t implies that stable future contribution induces more risky exposures than random future contributions (the typical example concerns individuals working in the public sector compared to individuals working in the private sector).*

Figure 9: Influence of the volatility ζ_t on the glide path



3.4 Implications of stochastic interest rates

In the previous paragraph, we assumed that the volatility $\Gamma_{t,T}$ of the zero-coupon bond is equal to zero. We now consider that interest rates are stochastic. For that, we use the Ho and Lee (1995) specification:

$$\Gamma_{t,T} = \varphi \cdot (T - t)$$

When $\varphi = 0.1\%$, we obtain the results of the first panel in Figure 10. Introducing the volatility of interest rates reduces the optimal exposure, particularly at the beginning of the time period. To understand this result, we notice that the interest rates volatility $\Gamma_{t,T}$ has three effects:

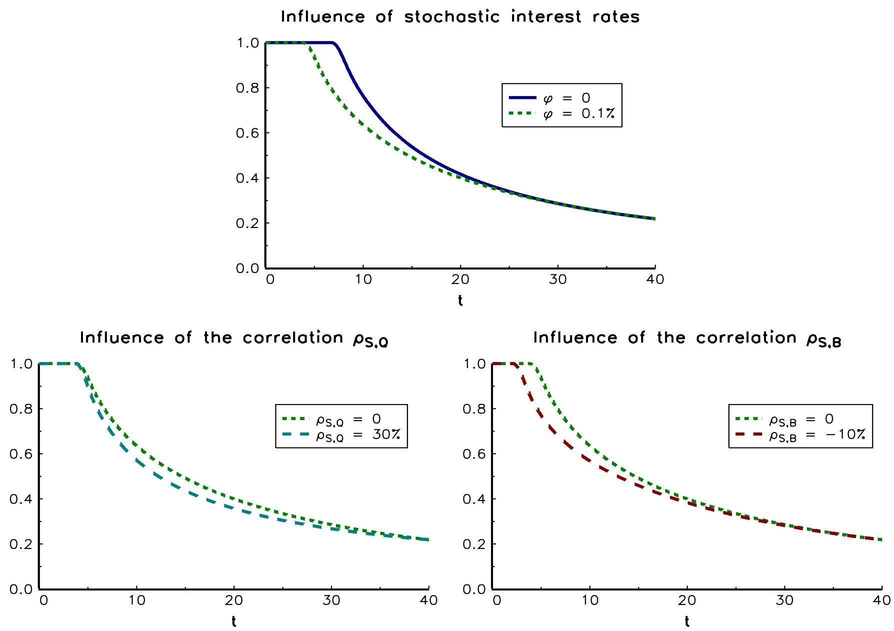
1. it increases the forward volatility $\sigma_{t,T}$ of the risky asset S_t;
2. it increases the forward volatility $\zeta_{t,T}$ of the economic factor Q_t;
3. it introduces a positive forward correlation ρ^* between the risky asset and the economic factor.

All these three effects have a negative impact on the optimal exposure $\alpha_t^\star$. From an economic point of view, we could interpret this effect because the zero-coupon bond is the only asset to hedge the liabilities (which corresponds to the terminal wealth) in our model. The hedging exposure increases then with interest rate volatilities. Hence why this effect is likely to disappear if we introduce the cash or short-term bonds in the asset universe.

3.5 What is the impact of the correlations?

We continue our example by studying the influence of the correlations. The parameter φ of the interest rates volatility is always set to 0.1%. In Figure 10, we observe that a positive

Figure 10: Impact of stochastic interest rates and correlations



In the first panel, we compare the glide path with and without interest rates volatility φ. The impact of the correlation $\rho_{S,Q}$ between the risky asset returns and the contribution π_t is represented in the second panel. The third panel illustrates the impact of the stock-bond correlation $\rho_{S,B}$.

correlation between S_t and Q_t or a negative correlation between S_t and $B_{t,T}$ decreases the optimal exposure.

The parameter $\rho_{S,Q}$ measures the correlation between the return of the risky asset and the contribution to the fund. It is generally admitted that when the equity market performs well, the contribution is improved because the investor is encouraged to participate in the growth of the equity market. Another reason is that the economy is more likely to be in a period of expansion because of the positive correlation between equity performance and the business cycle. In the Lucas (1978) model, the payoff of the risky asset has a positive covariance with consumption, which implies that the correlation with savings is negative. For these reasons, we could admit than when S_t increases, Q_t also increases. This explains why we assume that the correlation $\rho_{S,Q}$ is positive.

For the correlation $\rho_{S,B}$ between the risky asset and the zero-coupon bond, it is more complicated. Empirical works show that the stock-bond correlation is time-varying. In the long-run, this correlation is generally positive because of the present value model. However, it is negative in periods of flight-to-quality (Ilmanen, 2003) or periods of low inflation (Eychenne *et al.*, 2011). This has been the case for the past 15 years and is the reason why we prefer to impose negative correlation $\rho_{S,B}$.

Indeed, the impact of the correlations $\rho_{S,Q}$ and $\rho_{S,B}$ is more complex than presented here because Equation (1) shows that the results depend on the volatility of Q_t, $B_{t,T}$ and

S_t. Appendix A.5 presents an in-depth analysis. For the single effects, we could summarise the results as follows:

Parameter	$\rho_{S,Q}$	$\rho_{S,B}$	$\rho_{Q,B}$	$\Gamma_{t,T}$	σ_t ($\Gamma_{t,T} \simeq 0$)	ζ_t
Impact	+	−	−	?	sign of $\rho_{S,Q}$	?

For the multi-effects, it depends on the numerical values of the parameters.

Remark 3 *$\rho_{S,Q}$ measures the correlation between the risky asset and the permanent contribution. If $\rho_{S,Q}$ is close to one, the exposure to the risky asset tends to* 0*. In this case, it is preferable that the individual invest a large part of their savings component into bonds. This is the typical case for employee savings plan. By investing a large part of their money into the stocks of their company, individuals face the simultaneous risk of losing their job and experiencing large losses for their financial assets. But when financial income is highly correlated with labour income (or the economic cycle), individuals demand a high risk premium. It is one of the lessons of the Lucas model and is certainly more rational to diversify financial income and labour income. This explains why the exposure to the risky asset falls to zero when the correlation tends to one.*

4 Patterns of target-date funds

In this section, we exhibit the four patterns that characterise a target-date fund. First, it is very sensitive to the personal profile of the investor. Second, a target-date fund exhibits a contrarian allocation strategy. Third, the risk budget must be controlled. And finally, tactical asset allocation has to be considered. These four patterns help us to better design target-date funds as shown in the last paragraph of this section.

4.1 The personal profile of the investor

We may divide the parameters of our model into two families:

- some parameters concern the financial assets;
- other parameters relate to the individual.

These two types of parameters influence the design of the target-date fund in a different way. For example, asset parameters, like the expected return or the volatility of the risky asset, are defined by the portfolio manager. Indeed, the fund manager has the mandate to change the allocation according to their short-term or long-term views on the different asset classes. In some sense, these parameters are endogenous and change from one portfolio manager to another. The investors' parameters are more exogenous and concern:

1. the retirement date of the investor;
2. the risk aversion of the investor;
3. the actual and future wealth of the investor.

These three parameters are very important for designing the target-date fund.

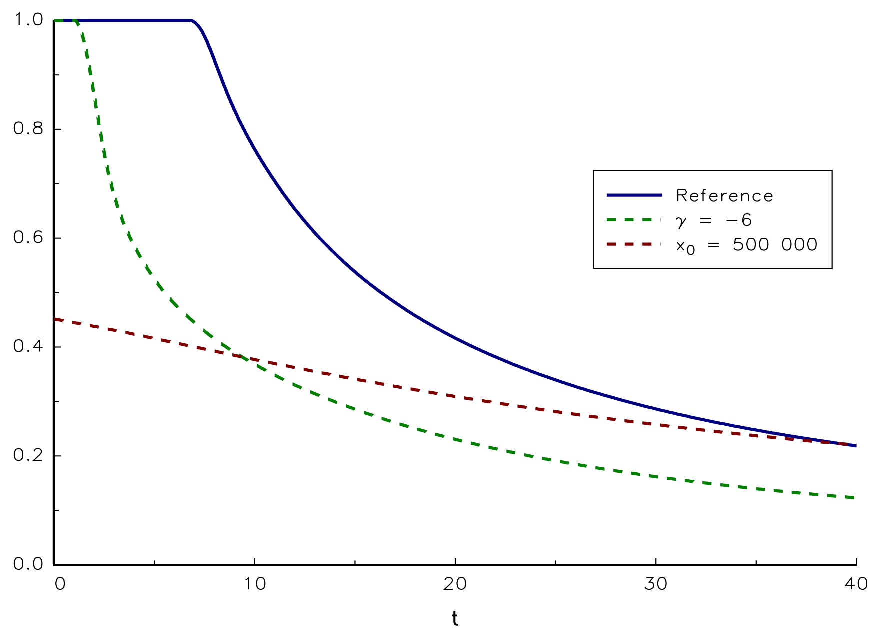
Most of the time in the industry, target-date funds are only structured according to retirement date. For example, Fidelity manages nine target-date funds called Fidelity ClearPath® Retirement Portfolios, each of them corresponding to a specific retirement date[13]. Thereby,

[13] The retirement dates begin in 2005 and end in 2045 on a constant scale of 5 years.

the Fidelity ClearPath® Retirement Portfolio 2045 is designed for investors who will retire in or around the year 2045.

The risk aversion of the investor is not often used to design target-date funds. However, the asset management industry has created diversified funds by mapping investor risk profiles to fund profiles. Defensive, balanced and aggressive funds correspond generally to the three investor risk profiles: conservative (low risk), moderate (medium risk) and aggressive (high risk). In Figure 11, we compare the previous solution obtained in section 3.3 with the solution if $\gamma = -6$. We verify that if the investor is more sensitive to risk, the exposure to the risky asset decreases largely.

Figure 11: Impact of investor parameters



The solid blue line corresponds to the reference exposure. It has been obtained for an initial wealth x_0 of 10000 dollars and a risk aversion parameter γ equal to -3. For the short dashed green line, we assume that the investor has a higher risk aversion ($\gamma = -6$), implying that the risky exposure is reduced in a quasi homogeneous way. The dashed red line shows the impact of the initial wealth ($x_0 = 500000$ dollars).

Another important parameter which is never used to build a target-date fund is the ratio $\mathcal{R}$ between the expected future contribution of the investor and their current wealth. In section 3.3, we assume that their current wealth is USD 10 000 whereas their future cumulative contribution is close to USD 800 000 dollars (third panel in Figure 7). So, the results we have obtained are for an investor with a very high ratio ($\mathcal{R}$ is equal to 80). This is the typical profile of an individual with a high level of education and who expects to be a high earner in the future. However, if this ratio is small, we obtain a different profile. In Figure 11, we illustrated the glide path when the individual's current wealth is USD 500 000 meaning that $\mathcal{R}$ is equal to 1.6. This result is not intuitive, because we might think that the individual will invest more in risky assets if their future contribution is smaller in order to maintain their pension benefits. But this argument means that the individual has a lower level of risk aversion.

Remark 4 *The previous figures of the initial wealth and the future contribution may appear to be high. Nevertheless, the glide path remains the same if we divide these two quantities by the same factor. For example, if x_0 is equal to USD $1\,000$ and the future contribution is USD $80\,000$, the solution always corresponds to the same reference curve.*

4.2 The contrarian nature of target-date funds

Let us consider the optimal exposure $\alpha_t^*(x)$ at time t. If we consider the deterministic case, we obtain:

$$\frac{\partial\,\alpha_t^*(x)}{\partial\,x} \leq 0 \tag{6}$$

This result remains valid if we assume that[14] $\partial_x \mathcal{J}(t,x,q) \geq 0$, $\partial_x^2 \mathcal{J}(t,x,q) \leq 0$ and $\rho^\star = 0$. It means that the investment strategy is contrarian.

By defining the dynamic asset allocation by the glide path, the effect of the wealth dynamics vanishes. In particular, if the risky asset has performed very well during a period, the wealth of the investor increases and is certainly larger than the wealth expected:

$$X_t \geq \mathbb{E}[X_t] \Rightarrow \alpha_t^* \leq g_t$$

In this case, the target-date fund takes more risk if the dynamic asset allocation is given by the glide path g_t. We have illustrated this difference of behaviour[15] in Figure 12. $X(T;g_t)$ (resp. $X(T;\alpha_t^\star)$) is the terminal value of the target-date fund when the dynamic asset allocation corresponds to the glide path (resp. the optimal exposure). We verify that for large values of $X(T;\alpha_t^\star)$, $X(T;g_t)$ is bigger. If the performance of the fund is not very good, the difference is not significant.

These results are biased however, because we assume that the risky asset is a geometric brownian motion. If the risky asset is mean-reverting, the glide path will give a smaller performance than the optimised exposure. The reason for this is that the risky exposure of the fund will be the highest at the top of the market and the smallest at the bottom of the market in the case of the glide path. For the optimised exposure, the opposite applies.

Remark 5 *The previous result (6) remains valid if we assume that the contribution is stochastic and if $\rho^\star \geq 0$ (see Appendix A.5 for the proof).*

4.3 Managing the risk budget

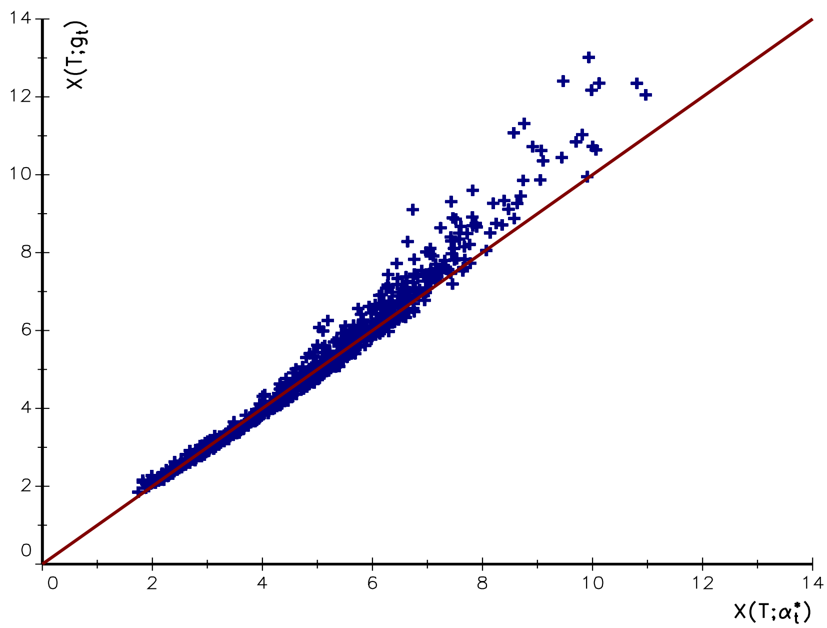
In the original Merton's model, the volatility of the strategy is constant. Indeed, we have:

$$\begin{aligned}\sigma\left(\frac{\mathrm{d}X_t}{X_t}\right) &= \sigma\left(\alpha_t^\star \frac{\mathrm{d}S_t}{S_t}\right) \\ &= \frac{(\mu - r)}{(1-\gamma)\,\sigma^2}\sigma\sqrt{\mathrm{d}t} \\ &= \frac{\mathrm{SR}}{(1-\gamma)}\sqrt{\mathrm{d}t}\end{aligned}$$

[14] The first two hypotheses are generally verified because the investor is risk averse meaning that the utility function is concave.

[15] We use the same parameters as those in Figure 5.

Figure 12: Glide path versus optimal exposure



Moreover, we do not make the distinction between the instantaneous volatility $\sigma_t\left(\mathrm{d}X_t/X_t\right)$ and the unconditional volatility $\sigma_0\left(\mathrm{d}X_t/X_t\right)$. In the target-date fund, this result does not hold. The instantaneous volatility depends on the level of the wealth X_t:

$$\begin{aligned}\sigma_t\left(\frac{\mathrm{d}X_t}{X_t}\right) &= \sigma\left(\alpha_t^\star\frac{\mathrm{d}S_t}{S_t}\right)\\ &= \bar{\alpha}\left(1+\frac{\mu\int_t^T\pi_u\,\mathrm{d}u}{X_t}\right)\sigma\sqrt{\mathrm{d}t}\\ &= \frac{\mathrm{SR}}{(1-\gamma)}\left(1+\frac{\mu\int_t^T\pi_u\,\mathrm{d}u}{X_t}\right)\sqrt{\mathrm{d}t}\end{aligned}$$

For the unconditional volatility, we obtain:

$$\sigma_0^2\left(\frac{\mathrm{d}X_t}{X_t}\right)=\sigma_0^2\left(\alpha_t^\star\right)\sigma^2\,\mathrm{d}t+g_t^2\sigma^2\,\mathrm{d}t$$

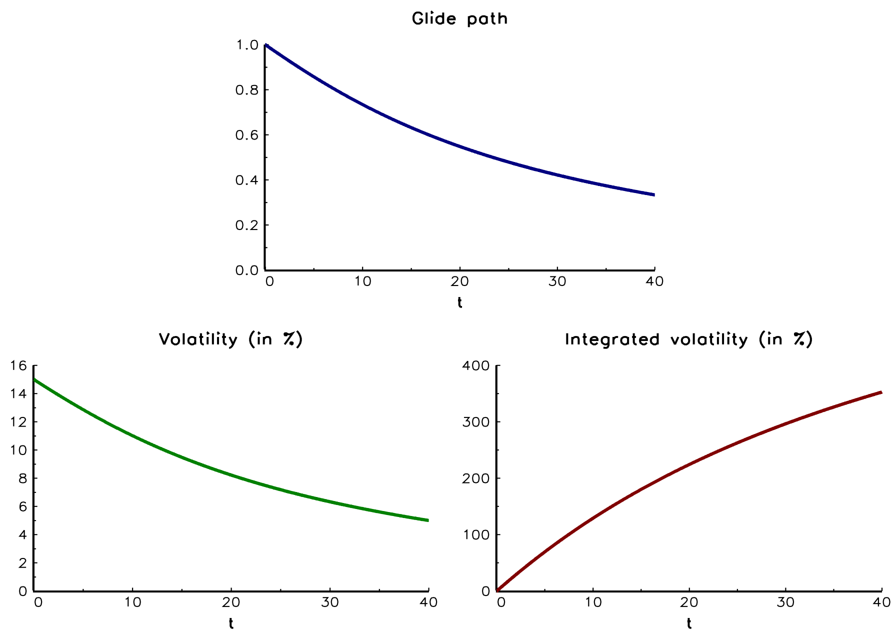
The two measures $\sigma_t\left(\mathrm{d}X_t/X_t\right)$ and $\sigma_0\left(\mathrm{d}X_t/X_t\right)$ are then different. They are equal if the dynamic asset allocation is based on the glide path. In this case, we obtain:

$$\sigma\left(\frac{\mathrm{d}X_t}{X_t}\right)=\sigma\left(g_t\frac{\mathrm{d}S_t}{S_t}\right)=g_t\sigma\sqrt{\mathrm{d}t}$$

Using the glide path in a target-date fund is then equivalent to targeting a volatility budget which is a decreasing function of the time t. Figure 13 illustrates this property using our previous deterministic example.

Remark 6 *This result is similar to another one when we introduce stochastic volatility in the Merton model. In this last case, the optimal solution is to consider the volatility target strategy.*

Figure 13: Volatility budget of the target-date fund



4.4 The importance of tactical asset allocation

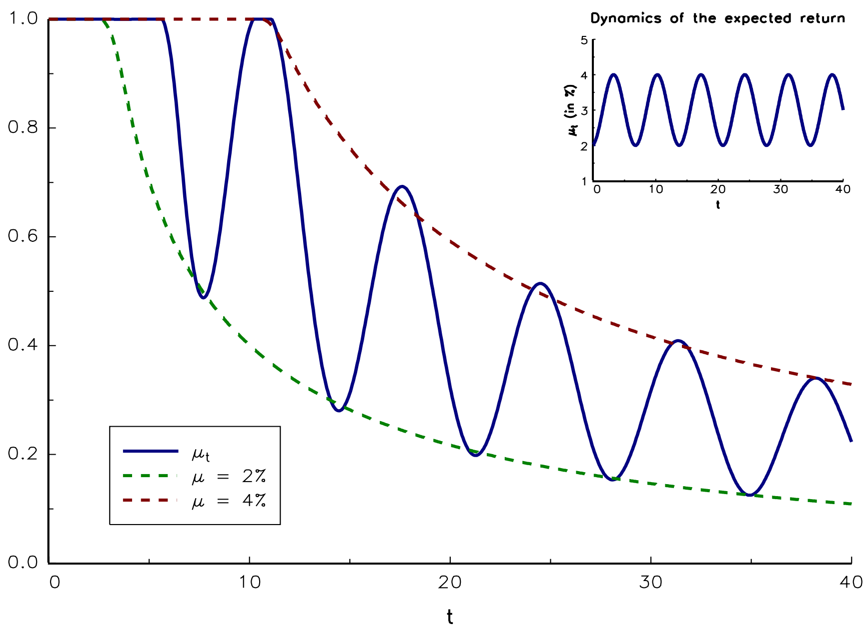
In the previous paragraphs, we assume that the risk premium and the volatility of the risky asset are constant. In real life, however, this is not the case. For example, the volatility of the equity market exhibits some regimes or may be viewed as a mean-reverting process. Since the work of Lucas (1978), we know that risk premia are time-varying. In particular, we observe a covariance effect with the consumption behaviour of investors due to the fact that output exhibits some cycle (Darolles *et al.*, 2010). In this context, defining a glide path with a constant risk premium and a constant volatility is suboptimal.

Suppose for example that the expected return of the risky asset is given by a sine function:

$$\mu_t = 3\% + 1\% \cdot \sin\left(\frac{2\pi}{7}(T-t)\right)$$

It implies that the expected return varies between 2% and 4% and that the cycle period is 7 years. We have illustrated the corresponding glide path in Figure 14. We observe that the glide path is located between two curves, which are the glide paths for the lower and upper bounds of μ_t. The variations may be very large. For example, if t is equal to 11 years, the optimal exposure is 100% while it becomes 30% three years later. If we assume that the volatility is time-varying, we obtain similar conclusions. In finance, we generally observe that performance is negatively correlated to risk. It is especially true for equities. In this case, we will have a double effect: a negative outlook on expected return will decrease the optimal exposure, and this jump will be amplified because of the increase in volatility. Taking into account the short-run behaviour of asset classes is also necessarily in order to optimise the glide path. Using strategic asset allocation to estimate the long-run path of financial assets in terms of performance and risk is insufficient and therefore must be supplemented by tactical asset allocation.

Figure 14: Impact of a time-varying risk premium



4.5 Target-date funds in practice

The previous patterns help to design target-date funds in a practical way. In particular, most academic models including our framework assume that the investor's choices are driven by a utility function. This utility function describes the behaviour of the investor with respect to risks and potential returns. But such a utility function is not directly observable and nobody is able to explicitly state its utility function. In practice, we have seen that asset management offers balanced funds, each corresponding to a particular attitude toward risk. They are then described by their exposure to risky assets. However, this characterises the risk aversion backwards (from the optimal strategy to the risk aversion) and may appear unsatisfactory.

Here, we develop an alternative approach. Instead of modelling the risk aversion of the investor, we introduce the maximum cumulated quantity of risk that they are willing to take throughout the investment horizon. This risk budget is directly related to the standard deviation of the wealth at maturity. Therefore, the attitude toward risk is expressed in terms of the confidence interval of the wealth at maturity. This constraint involves the cumulated variance of the fund's net asset value over the course of the investment horizon. The instantaneous variance of the fund value is given by[16]:

$$\sigma_t^2\left(\mathrm{d}X_{t,T}\right) = \alpha_t^2 \sigma_t^2 X_{t,T}^2 \,\mathrm{d}t$$

The constraint holds on the cumulated variance of the fund value:

$$\int_0^T \alpha_t^2 \sigma_t^2 X_{t,T}^2 \,\mathrm{d}t \leq V^2 \tag{7}$$

[16] We consider the net asset value of the fund in the forward framework, as the variations of the current net asset value due to interest rates do not change the expected value of the terminal wealth.

where V^2 is the total variance budget of the strategy from inception date to maturity. Once the risk has been constrained, the investor can focus on achieving the highest possible average return. This is a translation of Markowitz problem in a dynamical framework. The objective is to maximise the expected wealth at maturity, under the constraint (7):

$$\alpha_t^* = \arg\max \mathbb{E}_0 \left[X_T\right] \text{ such that } \int_0^T \alpha_t^2 \sigma_t^2 X_{t,T}^2 \,\mathrm{d}t \leq V^2$$

We suppose that the Sharpe ratio of the forward price $S_{t,T}$ of the risky asset is constant, and that the interest rates are deterministic. In this case, the optimal allocation policy is given by:

$$\alpha_t^* = \frac{V}{\sigma_t \sqrt{T} X_{t,T}}$$

The optimal proportion of risky assets is a linear function of the total risk budget V, and it is inversely proportional to the forward wealth and the volatility of the risky asset. Let $v^\star = V / \mathbb{E}\left[X_t\right]$ be the risk budget expressed in terms of the terminal wealth. The glide path may be approximated by the following system if we assume that the sharpe ratio $\mathrm{SR} = \left(\mu_t - r_t\right)/\sigma_t$ is constant:

$$\begin{aligned} g_t &= \frac{V}{\sigma_t \sqrt{T} x_t} \\ \mathrm{d}x_t &= \frac{V}{x_t} \mathrm{SR} \,\mathrm{d}t + \pi_t \,\mathrm{d}t \\ x_t &= v^\star x_T \end{aligned} \tag{8}$$

We could then use a classical optimisation algorithm and the Runge-Kutta method to determine the glide path g_t, the associated expected wealth x_t and the risk budget V.

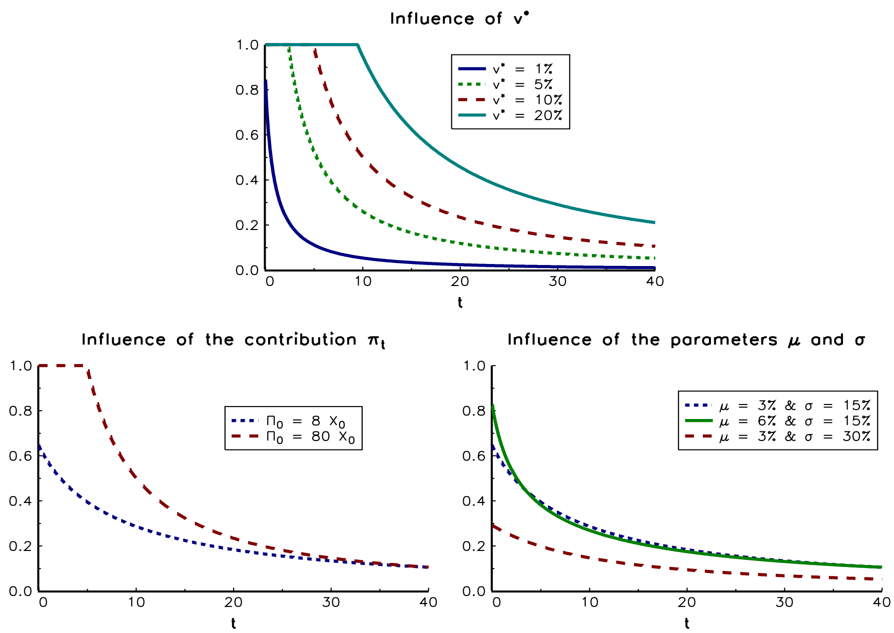
Let us consider our previous example defined in Section 3.3 page 12. We assume that $\mu = 3\%$, $\sigma = 15\%$ and $r = 1\%$. In Figure 15, we have reported in the first panel the glide path[17] corresponding to different values of the risk budgeting ratio $v^\star$. For example, $v^\star = 10\%$ corresponds to a volatility budget V equal to USD 82 600. In this case, X_T takes the value USD 826 000[18]. If the investor finds this volatility budget V too high, they could reduce it by choosing a smaller value for $v^\star$. For example, he could fix $v^\star$ to 1% implying that the risky exposure is largely reduced. In the second panel, we show the influence of the contribution function. Please note that our results are based on an initial wealth X_0 equal to USD 10 000 and a cumulative contribution Π_0 close to USD 800 000. If the contribution is divided by a factor of 10, we observe a significant impact on the risky exposure. Finally, the last panel illustrates the relationship between the glide path and the parameters of the risky asset. In the previous theoretical model, an increase of μ is equivalent to a decrease in σ. In this practical model, it is not the case, because of the risk budget constraint. This model then has an appealing property, because it contains a volatility target mechanism.

From a professional point of view, the model (8) is more powerful than the theoretical model based on the Merton approach, because the results are quite similar to the glide paths obtained via the utility framework although it is more tractable. Moreover, instead of relying on an abstract risk aversion parameter, it uses an explicit standard deviation parameter V, which is the risk budget of the investor. Finally, we could easily extend the model by considering tactical asset allocation.

[17] with a maximum exposure of 100%

[18] We verify that the ratio between V and X_T is 10%.

Figure 15: Glide path with the risk budgeting approach



5 Conclusion

This paper proposes a model to understand the design of target-date funds. Our model is based on the seminal work of Merton (1969, 1971) in which we introduce stochastic interest rates and stochastic contribution. We show that the optimal exposure to risky assets could be obtained by solving a two-dimensional PDE. We illustrate the impact of the different parameters such as the risk premium or the volatility of the risky asset, the correlations between the risky asset, the interest rate and the savings component, etc. We notice that the most important factors for designing a target-date fund relate to the investors' parameters: retirement date, level of risk aversion and income perspective.

This paper highlights the role of the glide path. In the investment industry, these glide paths are defined on creation of the target-date fund and are hardly revisited. By analysing the theoretical behaviour of the solution, we find that defining the optimal exposure is somewhere equivalent to defining a risk budget path. This is why we propose to design target-date funds by using a glide path, not in terms of weighting but rather in terms of risk[19]. This practical solution has the advantage of encompassing the risk aversion of the individual and verifying the target risk pattern.

Our model may be improved in three ways. First, we assume that the retirement date is constant. Introducing a stochastic retirement date is more realistic for two reasons: ageing population in developed countries forces governments to delay retirement[20] and the jobless of individuals in their fifties or sixties imply that they will use perhaps their pension savings

[19] This approach is related to risk parity portfolios defined in Maillard *et al.* (2010) and Bruder and Roncalli (2012).

[20] See Lachance (2008) for an example of a lifecycle model with a stochastic retirement date.

before their retirement date. Second, our model assumes that retirement is the maturity of the optimization problem. By assuming that, we ignore how many years the individual will live after retirement and if the pension benefits will be sufficient, especially if the individual lives a long time. A more realistic model consists then in considering the financial life of the individual after the retirement date. Last, but not least, the investment policy of the individual does not take the housing problem (Kraft and Munk, 2011) into consideration. This last issue leads to a fundamental question for individuals: how will renting or owning residential real estate modify optimal exposure to risky assets?

A Mathematical results

A.1 Forward dynamics

We have:

$$\begin{aligned} S_{t,T} &= \frac{S_t}{B_{t,T}} \\ &= \frac{S_0}{B_{0,T}} \exp\left(\int_0^t \left(\mu_u - r_u - \frac{1}{2}\left(\sigma_u^2 - \Gamma_{u,T}^2\right)\right) \mathrm{d}u + \int_0^t \sigma_u \,\mathrm{d}W_u^S - \Gamma_{u,T}\,\mathrm{d}W_u^B\right) \end{aligned}$$

$S_{t,T}$ is then a log-normal process and could be written as follows:

$$S_{t,T} = S_{0,T} \exp\left(\int_0^t \left(\mu_{u,T} - \frac{1}{2}\sigma_{u,T}^2\right) \mathrm{d}u + \int_0^t \sigma_{u,T}\,\mathrm{d}B_u^S\right)$$

By construction, B_t^S is a brownian motion and we have:

$$\begin{aligned} \sigma_{t,T}^2 \,\mathrm{d}t &= \operatorname{var}\left(\sigma_t W_t^S - \Gamma_{t,T} W_t^B\right) \\ &= \left(\sigma_t^2 + \Gamma_{t,T}^2 - 2\rho_{S,B}\sigma_t\Gamma_{t,T}\right) \mathrm{d}t \end{aligned}$$

It implies that:

$$\begin{aligned} \mu_{t,T} &= \mu_t - r_t - \frac{1}{2}\left(\sigma_t^2 - \Gamma_{t,T}^2\right) + \frac{1}{2}\sigma_{t,T}^2 \\ &= \mu_t - r_t + \Gamma_{t,T}^2 - \rho_{S,B}\sigma_t\Gamma_{t,T} \end{aligned}$$

In a similar way, we deduce that $Q_{t,T}$ is a log-normal process:

$$Q_{t,T} = Q_{0,T} \exp\left(\left(\theta_{t,T} - \frac{1}{2}\zeta_{t,T}^2\right) t + \zeta_{t,T} B_t^Q\right)$$

with:

$$\left\{\begin{array}{l} \theta_{t,T} = \theta_{t,T} - r_t + \Gamma_{t,T}^2 - \rho_{Q,B}\zeta_t\Gamma_{t,T} \\ \zeta_{t,T} = \sqrt{\zeta_t^2 + \Gamma_{t,T}^2 - 2\rho_{Q,B}\zeta_t\Gamma_{t,T}} \end{array}\right.$$

Let $\rho^\star$ be the correlation between the two brownian motions B_t^S and B_t^Q. It satisfies the following relationship:

$$\langle \mathrm{d}S_{t,T}, \mathrm{d}Q_{t,T} \rangle = \rho^\star \sigma_{t,T}\zeta_{t,T}\,\mathrm{d}t$$

We have also:

$$\begin{aligned} \rho^\star \sigma_{t,T}\zeta_{t,T}\,\mathrm{d}t &= \left\langle \sigma_t \mathrm{d}W_t^S - \Gamma_{t,T}\mathrm{d}W_t^B, \zeta_t \mathrm{d}W_t^Q - \Gamma_{t,T}\mathrm{d}W_t^B \right\rangle \\ &= \left(\rho_{S,Q}\sigma_t\zeta_t - \rho_{Q,B}\zeta_t\Gamma_{t,T} - \rho_{S,B}\sigma_t\Gamma_{t,T} + \Gamma_{t,T}^2\right) \mathrm{d}t \end{aligned}$$

We deduce that:

$$\rho^\star = \frac{\rho_{S,Q}\sigma_t\zeta_t - \rho_{Q,B}\zeta_t\Gamma_{t,T} - \rho_{S,B}\sigma_t\Gamma_{t,T} + \Gamma_{t,T}^2}{\sqrt{\sigma_t^2 + \Gamma_{t,T}^2 - 2\rho_{S,B}\sigma_t\Gamma_{t,T}}\sqrt{\zeta_t^2 + \Gamma_{t,T}^2 - 2\rho_{Q,B}\zeta_t\Gamma_{t,T}}}$$

A.2 Solving the HJB equation

Let $\mathcal{J}$ be the function defined by:

$$\mathcal{J}(t,x,q)=\sup_{\alpha}\mathbb{E}_t\left[\mathcal{U}(X_T)\mid Q_{t,T}=q,X_{t,T}=x\right]$$

We may show that $\mathcal{J}$ verifies the following HJB equation (Pham, 2009, Prigent, 2007):

$$\partial_t\mathcal{J}(t,x,q)+\max_{\alpha}\mathcal{H}(t,x,q,\alpha)=0$$

where $\mathcal{H}$ is the associated Hamiltonian:

$$\begin{aligned}\mathcal{H}(t,x,q,\alpha) &= (\alpha\mu_{t,T}x+p_tq)\,\partial_x\mathcal{J}(t,x,q)+\theta_{t,T}q\partial_q\mathcal{J}(t,x,q)+\\ &\quad \frac{1}{2}\alpha^2\sigma_{t,T}^2x^2\partial_x^2\mathcal{J}(t,x,q)+\alpha\rho^{\star}\sigma_{t,T}\zeta_{t,T}xq\partial_{x,q}^2\mathcal{J}(t,x,q)+\frac{1}{2}\zeta_{t,T}^2q^2\partial_q^2\mathcal{J}(t,x,q)\end{aligned}$$

At the optimum, we have:

$$\begin{aligned}\frac{\partial\,\mathcal{H}(t,x,q,\alpha)}{\partial\,\alpha} &= \mu_{t,T}x\mathcal{J}(t,x,q)+\alpha\sigma_{t,T}^2x^2\partial_x^2\mathcal{J}(t,x,q)+\\ &\quad \rho^{\star}\sigma_{t,T}\zeta_{t,T}xq\partial_{x,q}^2\mathcal{J}(t,x,q)\\ &= 0\end{aligned}$$

We deduce that the optimal exposure is:

$$\alpha_t^*=-\frac{\mu_{t,T}\partial_x\mathcal{J}(t,x,q)+\rho^{\star}\sigma_{t,T}\zeta_{t,T}q\partial_{x,q}^2\mathcal{J}(t,x,q)}{x\sigma_{t,T}^2\partial_x^2\mathcal{J}(t,x,q)}$$

The HJB equation becomes a two-dimensional PDE:

$$\partial_t\mathcal{J}(t,x,q)+\mathcal{H}(t,x,q,\alpha_t^*)=0 \tag{9}$$

From a numerical point of view, the HJB equation (9) is solved on the space $[x_-,x_+]\times[q_-,q_+]$ with $t\in[0,T]$. We impose boundary on the allocation such that $0\leq\alpha_t\leq 1$. For the boundary conditions, we assume that $\mathcal{J}(T,x,q)=U(x)$, $\partial_q\mathcal{J}(t,x,q_-)=\partial_q\mathcal{J}(t,x,q_-+\mathrm{d}q)$, $\partial_q\mathcal{J}(t,x,q_+)=\partial_q\mathcal{J}(t,x,q_+-\mathrm{d}q)$, $\partial_x\mathcal{J}(t,x_-,q)=\frac{x_-}{\gamma-1}\partial_x^2\mathcal{J}(t,x_-,q)$ and $\partial_x\mathcal{J}(t,x_+,q)=\frac{x_+}{\gamma-1}\partial_x^2\mathcal{J}(t,x_+,q)$. We consider that $q_-=0$ and $q_+=2$ whereas the bounds x_- and x_+ depend on the initial wealth and the expected future contribution of the investor. For our example, we have taken $x_-\simeq 0$ and $x_+=2\times 10^6$. We then use the Hopscotch algorithm to solve the HJB equation numerically (Kurpiel and Roncalli, 1999).

A.3 Derivation of the relationship (5)

Without interest rates and uncertainty of savings ($\pi_t=p_t$), the dynamics of the wealth becomes:

$$\frac{\mathrm{d}X_t^{\pi}}{X_t^{\pi}}=\tilde{\alpha}_t\frac{\mathrm{d}S_t}{S_t}$$

with $X_t^{\pi}=X_t+\int_t^T\pi_u\,\mathrm{d}u$ and $\tilde{\alpha}_t=\alpha_tX_t/X_t^{\pi}$. It comes that:

$$X_t^{\pi}=X_0^{\pi}\exp\left(\int_0^t\left(\tilde{\alpha}_u\mu_u-\frac{1}{2}\tilde{\alpha}_u^2\sigma_u^2\right)\mathrm{d}u+\int_0^t\tilde{\alpha}_u\sigma_u\,\mathrm{d}B_u\right)$$

With $\mathcal{U}(x) = x^\gamma/\gamma$, we have:

$$\max_{\tilde{\alpha}_t} \mathbb{E}_t \left[\mathcal{U}\left(X_T^\pi\right)\right] \Leftrightarrow \max_{\tilde{\alpha}_t} \gamma \left(\tilde{\alpha}_t \mu_t - \frac{1}{2}\tilde{\alpha}_t^2 \sigma_t^2\right) + \frac{1}{2}\gamma^2 \tilde{\alpha}_t^2 \sigma_t^2$$

The optimal solution is then:

$$\tilde{\alpha}_t^\star = \frac{\mu_t}{(1-\gamma)\,\sigma_t^2}$$

or:

$$\alpha_t^\star = \frac{\mu_t}{(1-\gamma)\,\sigma_t^2}\left(1 + \frac{\int_t^T \pi_u \,\mathrm{d}u}{x}\right)$$

A.4 Computation of the glide path when the contribution function is linear

Let $m_t = \mathbb{E}[X_t]$. If π_t is deterministic, we have:

$$\begin{aligned} \mathrm{d}m_t &= \alpha_t \mu m_t \,\mathrm{d}t + \pi_t \,\mathrm{d}t \\ &= \bar{\alpha}\mu m_t \,\mathrm{d}t + \left(\bar{\alpha}\mu \int_t^T \pi_u \,\mathrm{d}u + \pi_t\right)\mathrm{d}t \end{aligned}$$

In the case where $\pi_t = at + b$, It comes that:

$$\begin{aligned} \mathrm{d}m_t &= \bar{\alpha}\mu m_t \,\mathrm{d}t + \left(\bar{\alpha}\mu \left[\frac{1}{2}au^2 + bu\right]_t^T + at + b\right)\mathrm{d}t \\ &= \bar{\alpha}\mu m_t \,\mathrm{d}t + \left(\bar{\alpha}\mu \left(\frac{1}{2}a\left(T^2 - t^2\right) + b\,(T-t)\right) + at + b\right)\mathrm{d}t \end{aligned}$$

and:

$$m_t = x_0 e^{\bar{\alpha}\mu t} + \bar{\alpha}\mu\left(\frac{1}{2}aT^2 + bT\right)t - \bar{\alpha}\mu\left(\frac{1}{6}at^2 + \frac{1}{2}bt\right)t + \left(\frac{1}{2}at + b\right)t$$

If the wealth is sufficient high[21], we have :

$$g_t \simeq \bar{\alpha} + \frac{\mu\left(\frac{1}{2}a\left(T^2 - t^2\right) + b\,(T-t)\right)}{(1-\gamma)\,\sigma^2 m_t}$$

A.5 Impact of the correlations on optimal exposure

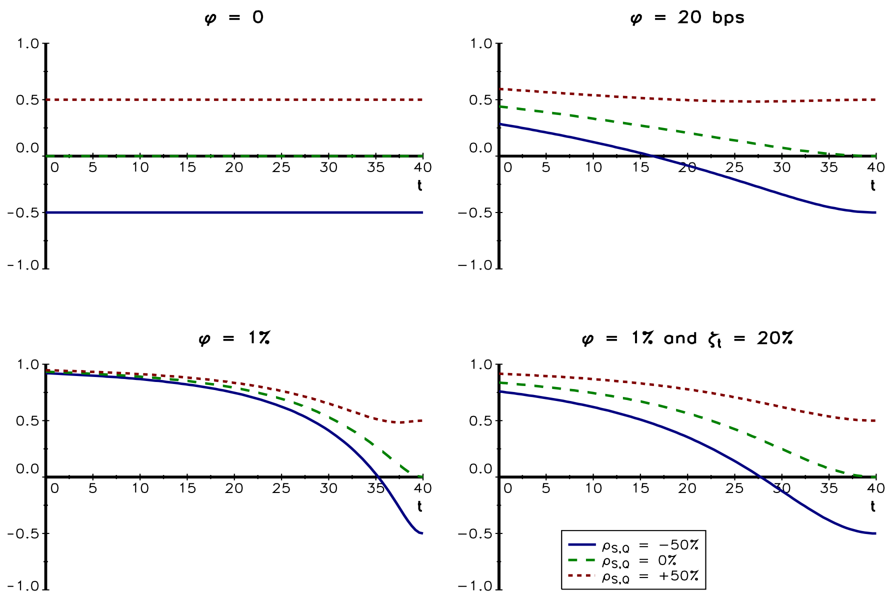
We notice that the correlation parameters $\rho_{S,Q}$, $\rho_{S,B}$ and $\rho_{Q,B}$ influence the optimal solution $\alpha_t^\star$ because they modify the forward correlation $\rho^\star$. So, if we are able to quantify the impact of $\rho^\star$, we are able to analyse the impact of these correlation parameters.

Let us first consider the relationship (1) between the correlation parameters and the forward correlation. We consider the previous values of parameters: $\sigma_t = 15\%$, $\zeta_t = 3\%$ and $T = 40$ years. In Figure 16, we have reported the value of $\rho^\star$ with respect to time t for three values of $\rho_{S,Q}$. If φ is set to zero and if there is not correlation between the risky asset S_t and the economic factor Q_t or the bond $B_{t,T}$, $\rho^\star$ is exactly equal to $\rho_{S,Q}$. If φ

[21] in order to diminish the convexity effect in Jensen's inequality.

increases, the correlation $\rho^\star$ increases too. In particular[22], we have $\rho^\star \to \rho_{S,Q}$ when $t \to T$, $\rho^\star \geq \rho^\star$ when $t \to 0$ and $\rho^\star \to 1$ when $t \to 0$ and $\varphi \to \infty$. These behaviors are illustrated in the second and third panels in Figure 16, whereas the impact of ζ_t is reported in the fourth panel. Figures 17 and 18 shows how the forward correlation changes with respect to the correlations $\rho_{S,B}$ and $\rho_{Q,B}$. The forward correlation is a decreasing function of these two parameters, but the impact depend on the respective magnitude of the volatility of Q_t, B_t, T and S_t. It explains that the impact of $\rho_{Q,B}$ is small in Figure 18.

Figure 16: The case $\rho_{S,B} = \rho_{S,Q} = 0$



Because $\mathcal{J}(t,x,q)$ is a utility function, we could assume that $\partial_x \mathcal{J}(t,x,q) \geq 0$ and $\partial_x^2 \mathcal{J}(t,x,q) \leq 0$. Another expression of the equation (2) is:

$$\begin{aligned}\alpha^*_{t,x} &= \frac{\mu_{t,T}}{\sigma^2_{t,T}}\left(-\frac{\partial_x \mathcal{J}(t,x,q)}{x\partial_x^2 \mathcal{J}(t,x,q)}\right) - \rho^\star \frac{\zeta_{t,T}}{\sigma_{t,T}}\left(\frac{q\partial^2_{x,q}\mathcal{J}(t,x,q)}{x\partial_x^2 \mathcal{J}(t,x,q)}\right) \\ &= \frac{1}{\mathcal{R}(t,x,q)}\frac{\mu_{t,T}}{\sigma^2_{t,T}} - \rho^\star \frac{1}{\mathcal{E}_x(t,x,q)}\frac{\zeta_{t,T}}{\sigma_{t,T}}\end{aligned}$$

where $\mathcal{R}(t,x,q) \geq 0$ is the relative risk aversion and $\mathcal{E}_x(t,x,q) \geq 0$ is a function which depends on the derivatives of the marginal utility. It comes that:

$$\frac{\partial\, \alpha^*_{t,x}}{\partial\, \rho^\star} \leq 0$$

[22]In the case $\rho_{S,B} = \rho_{Q,B} = 0$, the formula (1) reduces to:

$$\rho^\star = \frac{\rho_{S,Q}\sigma_t\zeta_t + \varphi^2 T^2}{\sqrt{\sigma_t^2 + \varphi^2 T^2}\sqrt{\zeta_t^2 + \varphi^2 T}}$$

Figure 17: The case $\varphi = 20$ bps

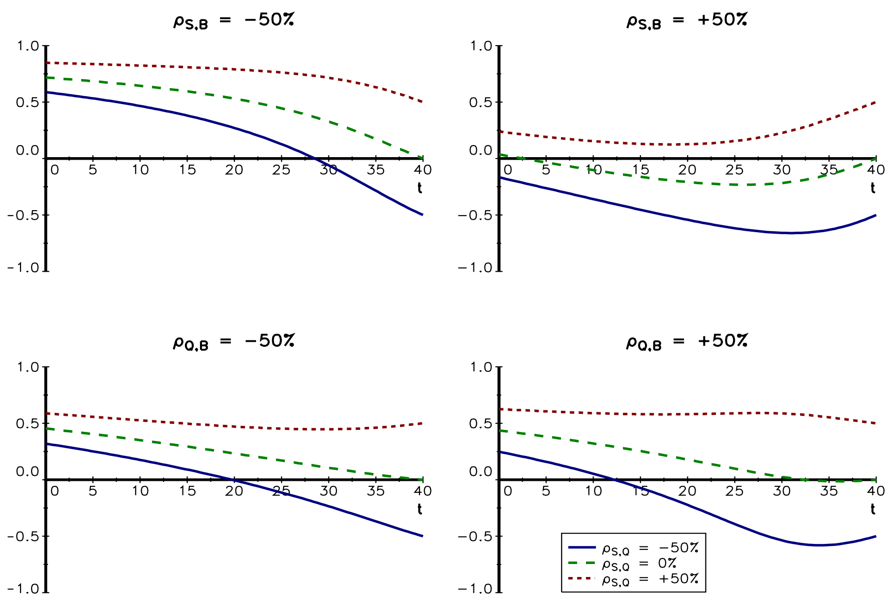
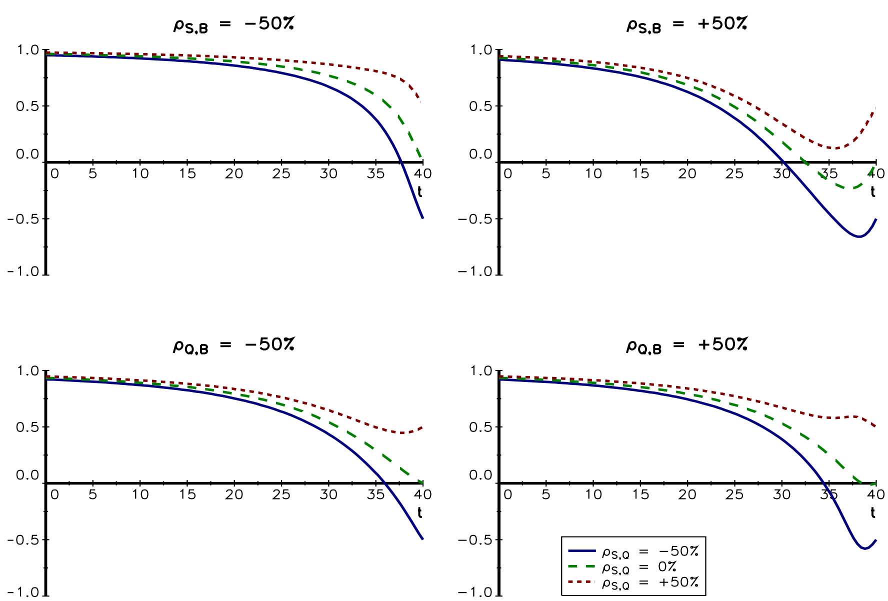


Figure 18: The case $\varphi = 1\%$



References

[1] Bajeux-Besnainou I., Jordan J.V. and Portait R. (2003), Dynamic Asset Allocation for Stocks, Bonds, and Cash, *Journal of Business*, 76(2), pp. 263-287.

[2] Barberis N. (2000), Investing for the Long Run When Returns Are Predictable, *Journal of Finance*, 55(1), pp. 225-264.

[3] Basu A. and Drew M.E. (2009), Portfolio Size Effect in Retirement Accounts: What does it imply for Lifecycle Asset Allocation Funds?, *Journal of Portfolio Management*, 35(3), pp. 61-72.

[4] Bodie Z. (2003), Life-Cycle Investing in Theory and Practice, *Financial Analysts Journal*, 59, pp. 24-29.

[5] Bodie Z., Merton R. and Samuelson W.F. (1992), Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model, *Journal of Economic Dynamics and Control*, 16(3-4), pp. 427-449.

[6] Bodie Z., Detemple J.B., Otruba S. and Walter S. (2004), Optimal Consumption-Portfolio Choices and Retirement Planning, *Journal of Economic Dynamics and Control*, 28(6), pp. 1115-1148.

[7] Bodie Z., Detemple J. and Rindisbacher M. (2009), Life Cycle Finance and the Design of Pension Plans, `www.ssrn.com/abstract=1396835`.

[8] Booth P. and Yakoubov Y. (2000), Investment Policy for Defined-Contribution Pension Scheme Members close to Retirement: An Analysis of the Lifestyle Concept, *North American Actuarial Journal*, 4(2), pp. 1-19.

[9] Brennan M.J. and Xia Y. (2002), Dynamic Asset Allocation under Inflation, *Journal of Finance*, 57(3), pp. 1201-1238.

[10] Bruder B., Jamet G. and Lasserre G. (2010), Beyond Liability-Driven Investment: New Perspectives on Defined Benefit Pension Fund Management, *Lyxor White Paper Series*, 2, `www.lyxor.com`.

[11] Bruder B. and Roncalli T. (2012), Managing Risk Exposures using the Risk Budgeting Approach, *Working Paper*, `www.ssrn.com/abstract=2009778`.

[12] Campbell J.Y. (2000), Asset Pricing at the Millennium, *Journal of Finance*, 55(4), pp. 1515-1567.

[13] Campbell J.Y., Cocco J., Gomes F., Maenhout P.J. and Viciera L.M. (2001), Stock Market Mean Reversion and the Optimal Equity Allocation of a Long-Lived Investor, *Review of Finance*, 5(3), pp. 269-292.

[14] Campbell J.Y. and Viciera L.M. (2002), *Strategic Asset Allocation*, Oxford University Press.

[15] Canner N., Mankiw N.G. and Weil D.N. (1997), An Asset Allocation Puzzle, *American Economic Review*, 87(1), pp. 181-191.

[16] Cochrane J.H. (2001), *Asset Pricing*, Princeton University Press.

[17] Cocco J.F. and Gomes F.J. (2012), Longevity Risk, Retirement Savings, and Financial Innovation, *Journal of Financial Economics*, 103(3), pp. 507-529.

[18] Cocco J.F., Gomes F.J. and Maenhout P.J. (2005), Consumption and Portfolio Choice over the Life Cycle, *Review of Financial Studies*, 18(2), pp. 491-533.

[19] Darolles S., Eychenne K. and Martinetti S. (2010), Time Varing Risk Premiums & Business Cycles: A Survey, *Lyxor White Paper Series*, 4, `www.lyxor.com`.

[20] Geman H., El Karoui N. and Rochet J-C. (1995), Changes of Probability Measure and Option Pricing, *Journal of Applied Probability*, 32(2), pp. 443-458.

[21] Eychenne K., Martinetti S. and Roncalli T. (2011), Strategic Asset Allocation, *Lyxor White Paper Series*, 6, `www.lyxor.com`.

[22] Gomes F. and Michaelides A. (2005), Optimal Life-Cycle Asset Allocation: Understanding the Empirical Evidence, *Journal of Finance*, 60(2), pp. 869-904.

[23] Henderson V. (2005), Explicit Solutions to an Optimal Portfolio Choice Problem with Stochastic Income, *Journal of Economic Dynamics and Control*, 29(7), pp. 1237-1266.

[24] Ho T.S.Y. and Lee S-B. (1986), Term Structure Movements and Pricing Interest Rate Contingent Claims, *Journal of Finance*, 41(5), pp. 1011-1029.

[25] Ilmanen A. (2003), Stock-Bond Correlations, *Journal of Fixed Income*, 13(2), pp. 55-66.

[26] INSEE (2006), *Épargne et Patrimoine des Ménages*, L'Économie Française, `www.insee.fr/fr/ffc/docs_ffc/ecofrac06c.pdf`.

[27] Investment Company Institute (2012), Investment Company Fact Book – A Review of Trends and Activity in the U.S. Investment Company Industry, 52nd edition, `www.icifactbook.org`.

[28] Jagannathan R. and Kocherlakota N.R. (1996), Why Should Older People Invest Less in Stocks Than Younger People?, *Quarterly Review, Federal Reserve Bank of Minneapols*, 20(3), pp. 11-23.

[29] Kraft H. and Munk C. (2011), Optimal Housing, Consumption, and Investment Decisions over the Life Cycle, *Management Science*, 57(6), pp. 1025-1041.

[30] Kurpiel A. and Roncalli T. (2000), Hopscotch Methods for Two State Financial Models, *Journal of Computational Finance*, 3/2, pp. 53-89.

[31] Lachance M-E. (2008), Pension Reductions: Can Welfare be Preserved by Delaying Retirement?, *Journal of Pension Economics and Finance*, 7(2), pp. 157-177.

[32] Larsen L.S. and Munk C. (2012), The Costs of Suboptimal Dynamic Asset Allocation: General Results and Applications to Interest Rate Risk, Stock Volatility Risk, and Growth/Value Tilts, *Journal of Economic Dynamics and Control*, 36(2), pp. 266-293.

[33] Lucas R.E. (1978), Asset Prices in an Exchange Economy, *Econometrica*, 46(6), pp. 1429-1445.

[34] Maillard S., Roncalli T. and Teiletche J. (2010), The Properties of Equally Weighted Risk Contributions Portfolios, *Journal of Portfolio Management*, 36(4), pp. 60-70.

[35] Markowitz H. (1952), Portfolio Selection, *Journal of Finance*, 7(1), pp. 77-91.

[36] MARTELLINI L. and MILHAU V. (2010), From Deterministic to Stochastic Life-Cycle Investing: Implications for the Design of Improved Forms of Target Date Funds, *EDHEC-Risk Institute Publication*.

[37] MERTON R.C. (1969), Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case, *Review of Economics and Statistics*, 51(3), pp. 247-257.

[38] MERTON R.C. (1971), Optimum Consumption and Portfolio Rules in a Continuous-Time Model, *Journal of Economic Theory*, 3(4), pp. 373-413.

[39] Morningstar Fund Research (2012), Target-Date Series Research Paper: 2012 Industry Survey, `corporate.morningstar.com`.

[40] MUNK C. and SØRENSEN C. (2010), Dynamic Asset Allocation with Stochastic Income and Interest Rates, *Journal of Financial Economics*, 96(3), pp. 433-462.

[41] MUNK C., SØRENSEN C. and VINTHER T.N. (2004), Dynamic Asset Allocation under Mean-reverting Returns, Stochastic Interest Rates and Inflation Uncertainty: Are Popular Recommendations Consistent with Rational Behavior?, *International Review of Economics and Finance*, 13(2), pp. 141-166.

[42] PHAM H. (2009), *Continuous-time Stochastic Control and Optimization With Financial Applications*, Stochastic Modeling and Applied Probability Series, 61, Springer.

[43] PRIGENT J.-L. (2007), *Portfolio Optimization and Performance Analysis*, Chapman & Hall.

[44] QUINN J.B. (1991), *Making the Most of Your Money*, Simon and Schuster.

[45] POTERBA J., RAUH J., VENTI S. and WISE D. (2009), Lifecycle Asset Allocation Strategies and the Distribution of 401(k) Retirement Wealth, in D. Wise (ed.), *Developments in the Economics of Aging* , University of Chicago Press, pp. 15-50.

[46] SHILLER R.J. (2006), Life-Cycle Personal Accounts Proposal for Social Security: An Evaluation of President Bush's Proposal, *Journal of Policy Modeling*, 28, pp. 427-444.

[47] VICEIRA L.M. (2001), Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income, *Journal of Finance*, 56(2), pp. 433-470.

[48] WACHTER, J.A. (2002), Portfolio and Consumption Decisions under Mean-Reverting Returns: An Exact Solution for Complete Markets, *Journal of Financial and Quantitative Analysis*, 37(1), pp. 63-91.