

The Correlation Problem in Operational Risk

Antoine Frachot, Thierry Roncalli and Eric Salomon
Groupe de Recherche Opérationnelle
Group Risk Management, Crédit Agricole SA, France*

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1 Introduction

Diversification effects in operational risk modelling are crucial, in particular when capital computations are performed. Though the Basel II consultative paper CP3 is voluntarily vague about the correlation which should be expected between (say) internal fraud and damage to physical assets, common sense suggests that operational risk events might be, at least partially, decorrelated. Indeed, that all severe operational risk losses occur simultaneously and systematically in the same year is rather dubious and is hardly supported by empirical evidence.

Besides if perfect correlation across risk types or across business lines was to be accepted, capital charges by risk types or business lines should have to be summed, leading in some cases to much higher a capital charge than in the *Standard Approach* as proposed by Basel II. As regulators intend to promote the use of the most advanced methodology (*Advanced Measurement Approach* or *AMA*), at least for internationally active banks, the perfect correlation hypothesis would be thus clearly disincentive and would go against regulators' stated goals.

The vagueness of the consultative paper is quite understandable as it is a difficult task to assess the level of correlation to be expected between different risk types and/or business units. So far banks are quite short of historical data, which implies that reliable estimations of correlation can hardly be extracted from historical data. Nevertheless, as this paper aims to demonstrate, this difficulty can be overcome by using data-independent theoretical calculations and secondly by validating these theoretical results with empirical simulations. This is actually the main goal of this paper. Moreover, this paper proves that *Loss Distribution Approach*-like models (LDA) cannot, by construction, show high levels of correlation between aggregate losses corresponding to various risk type \times business line classes. By this, we do not suggest in any sense that LDA models are ill-suited for operational risk modelling. On the contrary we firmly believe that LDA models should be at the root of the *AMA* method. We only stress that there are **strong arguments in favor of low levels of correlation between aggregate losses**, suggesting that capital charge summation is exagerately conservative. As a result we are confident that low correlations will be justifiable vis-à-vis of our supervisors, which will eventually provide banks with strong incentives to use advanced methods.

The paper is organized as follows. The first section reminds the basic roots of the standard LDA model as it is currently understood in operational risk modelling. In the second section we show how correlation can be introduced in the standard LDA model. We then derive simple calculations to assess the level of correlation which can be obtained in the framework of standard LDA models. Finally we provide empirical simulations partly based on Crédit Lyonnais historical loss data and we give an evaluation of the level of correlation.

* *address*: Groupe de Recherche Opérationnelle, Le Centorial, 18, rue du Quatre Septembre – 75002 Paris – France; *E-mail*: antoine.frachot@creditylyonnais.fr or thierry.roncalli@creditylyonnais.fr

2 The standard LDA model

This section aims at reminding the basic features of the standard LDA model. By “standard LDA model”, we mean the LDA model as it is currently understood by operational risk community and as it is planned to be used in practice. Conversely appendix B provides an example of a non-standard LDA model.

2.1 Main assumptions

Following the usual LDA methodology, the aggregate loss is naturally defined as a random sum of individual losses:

$$L = \sum_{n=1}^N X_n = X_1 + \dots + X_N$$

where L is the aggregate loss, N is the annual number of losses (i.e. frequency of events) and X_n are loss amounts (i.e. severity of event). In the course of this paper, “aggregate loss” will refer to the loss incurred in a “class of risk”, where a class designates one cell among the 7 risk types \times 8 business lines cells defined by the consultative paper.

Accordingly aggregate losses result from two distinct sources of randomness (i.e. frequency and severity) which both have to be modelled. In essence the LDA model as used in operational risk or in actuarial sciences assumes the three following assumptions within each class of risk:

- (i) N and (X_1, X_2, \dots) are independent random variables,
- (ii) X_1, X_2, \dots is a set of independent random variables,
- (iii) and X_1, X_2, \dots follow the same marginal distribution.

The first assumption means that frequency and severity are two independent sources of randomness. It is obviously debatable as we may imagine that the number of losses *and* their corresponding amounts both move together, i.e. are partly correlated. As an example, Appendix B proposes a tractable extension of the LDA model allowing for frequency and severity correlation. Assumptions 2 and 3 mean that two different losses within the same homogeneous class are independent and identically distributed. Provided that data collection processes are reliable and that homogeneity can be assumed, the two last assumptions are quite reasonable. In particular the granularity imposed by the consultative paper (i.e. 7 risk types \times 8 business lines, see Annexes 6 and 7 in [3]) aims to ensure that assumption (iii) is satisfied.

However even if some extensions are proposed in Appendix B, we prefer to focus on the standard LDA framework, i.e. a model which satisfies the three previous assumptions, because of the following reasons:

- first the standard LDA as just defined is the usual framework which has been adopted by the insurance industry for a long time at their satisfaction¹;
- secondly standard LDA models are the kind of models which are used in practice and which are actually implemented in existing commercial softwares;
- third, any extension of the standard LDA model may degenerate into an over-sophisticated model in comparison with the paucity of historical data which are actually available for its calibration.

¹The previous assumptions are standard in Actuarial Sciences. They define the collective risk model (see Definition 1.14 and Chapter 4 in KLUGMAN et al. [10]).

2.2 Capital charge computations

Let us now turn to capital charge computations under the previous assumptions. We denote \mathbf{G} the distribution of the aggregate loss L . This distribution is a compounded distribution since it is a mixture of the severity and frequency distributions. In the sequel it will be referred to as the **aggregate loss distribution**. How compounding is done is now common practice and does not need to be detailed any further at this stage of exposition.

The last consultative paper CP3 defines the regulatory capital requirement as follows:

Definition 1 (Basel Committee on Banking Supervision [3]) *The regulatory capital requirement (or Capital-at-Risk) is the sum of expected loss (EL) and unexpected loss (UL) for a one year holding period and a 99.9 percent confidence interval.*

This definition implies that frequency distribution must be understood on a yearly basis. In the spirit of a Value-at-Risk-like measure, the regulatory capital requirement K is the 99.9% percentile of distribution of the aggregate loss:

$$K = \mathbf{G}^{-1}(99.9\%)$$

meaning that one expects to incur a loss higher than K (“in average”) once every 1000 years².

The total loss L of the bank is then the sum of aggregate losses for each business line \times loss type class. Let H be the number of classes (where $H = 7 \times 8$ in the Basel II context). Therefore:

$$L = \sum_{h=1}^H L_h$$

where L_h is the aggregate loss corresponding to the h^{th} class. With natural notations the regulatory capital requirement for each business line/loss type class is then:

$$K_h = \mathbf{G}_h^{-1}(99.9\%)$$

2.3 Diversification effects according to CP3

At this point the main issue concerns the aggregation of the K capital charges in order to obtain the regulatory capital requirement of the bank as a whole. The Basel Committee statement is the following:

“Risk measures for different operational risk estimates must be added for purposes of calculating the regulatory minimum capital requirement. However, the bank may be permitted to use internally determined correlations in operational risk losses across individual operational risk estimates, provided it can demonstrate to a high degree of confidence and to the satisfaction of the national supervisor that its systems for determining correlations are sound, implemented with integrity, and take into account the uncertainty surrounding any such correlation estimates (particularly in periods of stress). The bank must validate its correlation assumptions.” (§ 629 (d) page 126 in [3]).

Following the lines of the consultative paper and using our notations, it is thus first suggested to sum the H capital charges, i.e.:

$$K = \sum_{h=1}^H K_h \quad (1)$$

²The return period is the mean of the duration between two consecutive exceeding of the Capital-at-Risk. Because the distribution of the excess event $\{L > K\}$ is geometric with probability $\Pr\{L > K\}$, it comes that the return time is

$$\dot{t}(K) = \frac{1}{\Pr\{L > K\}} = \frac{1}{1 - \mathbf{G}(\mathbf{G}^{-1}(0.999))} = 1000$$

As it is now well-known this methodology assumes implicitly that aggregate losses L_h are perfectly correlated (see proof in Appendix A). In some sense it would be exactly as if everything went wrong simultaneously for all business line \times risk type classes. In mathematical terms, it would imply that both frequency and severity of losses are driven by one single source of randomness instead of possibly $H = 7 \times 8$ independent sources. Even if we consider that the “real world” is anywhere in between 1 and 56 sources of randomness, we strongly believe that the one-single-source case is too much a conservative way of capturing losses dependency. The following technical note provides a numerical illustration of this point.

Remark 1 We note $\check{t}_1, \dots, \check{t}_H$ the return times of the different Capital-at-Risk. Suppose that the different aggregate losses L_h are perfectly correlated. The return time of the Capital-at-Risk (1) for the whole bank is

$$\begin{aligned}\check{t} &= \frac{1}{\Pr\{L_1 > K_1, \dots, L_H > K_H\}} \\ &= \min(\check{t}_1, \dots, \check{t}_H)\end{aligned}$$

Accordingly, the return time is effectively equal to 1000 years. If instead L_k are actually independent, \check{t} is now equal to:

$$\check{t} = \prod_{h=1}^H \check{t}_h$$

leading in this case to a return time of 1000^H years instead of 1000. With only two independent classes, we already obtain a return time corresponding to a 99,9999% confidence level! Finally the general case is somewhere in between, that is:

$$\min(\check{t}_1, \dots, \check{t}_H) < \check{t} < \prod_{h=1}^H \check{t}_h$$

*It is worth stressing that the method of aggregation (1) is therefore disincentive when aggregated losses are actually not perfectly correlated (which is the most likely case in practice): the more granular the bank risk measurement system is, the bigger the global capital charge. As a result **banks would have no incentive to refine their risk measurement granularity.***

3 Dependence and diversification effects in the standard LDA model

In the previous section we suggested that reality was somewhere in between perfect correlation and full-independence (between two distinct classes of losses). In practice one has to be more precise since capital charge computations require to use some correlation value whose soundness (following the words of the consultative paper) has to be demonstrated with a high degree of confidence. This issue has been adressed in FRACHOT et al. [8] whose lines of discussion are reminded in the sequel.

Before that, it is necessary to clarify which correlation we are talking about since each class of losses is driven by two elementary sources of randomness, i.e. frequency and severity. In this respect correlation between aggregate losses may result from correlation between frequencies or between severities or between both. For ease of exposition, we restrict our discussion to only two classes of risk as the lines of reasoning can be extended easily to more than two classes. Let us note L_1 and L_2 the two aggregate losses with the following additional notations:

$$\begin{aligned}L &= L_1 + L_2 \\ &= \underbrace{\sum_{n=1}^{N_1} X_n}_{L_1} + \underbrace{\sum_{m=1}^{N_2} Y_m}_{L_2}\end{aligned}$$

where L is the global aggregate loss (i.e. at the bank’s level).

3.1 Frequency correlation

We refer to **frequency correlation** when annual frequencies N_1 and N_2 are not independent variables. In practice it means that we may observe that, historically, the number of (say) external fraud events is high (respectively low) when the number of (say) internal fraud events is also high (respectively low). It likely happens when both frequencies N_1 and N_2 share common dependence with respect to some variables such as gross income, economic cycle, size of business etc. Empirically frequency correlation could be evidenced and measured by computing the historical correlation between past frequencies of events, provided of course that data are recorded for a sufficiently long period of time.

3.2 Severity correlation

On the other hand **severity correlation** is more difficult to tackle in the context of the LDA model. Mathematically it would mean that loss X_n - randomly drawn from the first class of events - and Y_m - drawn from the second one - are not independent with one another. It may be observed when (say) internal fraud loss amounts are high (respectively low) when (say) external fraud loss amounts are high (respectively low). Empirically it could be evidenced if for example mean loss amounts of two classes are correlated over time.

On the other hand, a basic feature of actuarial models requires to assume that individual losses are jointly independent *within* one specific business line/risk type class. **Therefore it is conceptually difficult to assume simultaneously severity independence within each class and severity correlation between two classes.** It is also mathematically impossible in general but by chance.

Remark 2 *Indeed the covariance between two losses belonging to the same class, i.e. $\text{cov}(X_n, X_m)$ can be decomposed as:*

$$\text{cov}(X_n, X_m) = \mathbb{E}[\text{cov}(X_n, X_m | Y_p)] + \text{cov}(\mathbb{E}[X_n | Y_p], \mathbb{E}[X_m | Y_p])$$

for any loss Y_p coming from another class. Both terms are different from zero under the assumption of non-zero correlation between losses from two distinct classes, the last term being always positive. As a result, zero-correlation between two losses within the same class, i.e. $\text{cov}(X_n, X_m)$, occurs only by chance and therefore never holds in practice.

If one wishes to draw some operational conclusions from the previous discussion, we should say that adding correlation between frequencies of events is quite an easy task and does not destroy the very nature of the LDA model. In particular it does not change the way capital charges are computed for each class. Therefore this type of correlation can be taken into account at minimal cost. On the contrary correlation between loss amounts necessarily alters, in a substantial extent, the basic foundations of the standard LDA model and requires to build an entirely new family of models, whose an example is given in the appendix. Unfortunately such an extension is far away from what is reasonably feasible considering the data we currently dispose of.

3.3 Aggregate loss correlation

According to previous subsections, we now suppose that **aggregate loss correlation** is fundamentally conveyed by the underlying correlation between frequencies:

$$\left. \begin{array}{l} \text{cor}(N_1, N_2) \neq 0 \\ \text{cor}(X, Y) = 0 \end{array} \right\} \Rightarrow \text{cor}(L_1, L_2) \neq 0$$

As said before, it is the simplest and cheapest way to add correlation between aggregate losses. We now address the issue of the extent by which frequency correlation translates into aggregate losses correlation.

As a first trivial result, it is clear that:

$$\text{cor}(L_1, L_2) \leq \text{cor}(N_1, N_2)$$

Furthermore we expect that, even with strong frequency-correlation (i.e. $\text{cor}(N_1, N_2)$ close to one), aggregate losses may show low level of correlation. In addition one may also guess that it is particularly true for high severity events since severity-independence likely dominates frequency-correlation. This point will be confirmed in the next section.

This remark is worth being stressed as the correlation considered by the Basle Committee is unambiguously the aggregate loss correlation since it is this correlation which is used when capital charges are to be aggregated. Therefore there is a risk of misunderstandings between banks and national supervisors: the fact that a bank uses a low aggregate loss correlation does not mean that the dependence in the LDA model (as measured here by the frequency correlation) is low.

4 Empirical findings about correlation between aggregate losses

This section aims to investigate how and in what extent frequency correlation generates aggregate loss correlation. Empirical simulations, partly based on Credit Lyonnais data, will then be provided.

Here we restrict ourselves to the case where frequencies are correlated while severities are not. Accordingly any correlation between two aggregate losses is conveyed through frequency correlation, consistently with the arguments developed in the previous section. In this context, we can easily compute the correlation between two aggregate losses associated to two classes. Straightforward calculations give the following formula for the covariance:

$$\begin{aligned} \text{cov}(L_1, L_2) &= \mathbb{E}[L_1 L_2] - \mathbb{E}[L_1] \mathbb{E}[L_2] \\ &= \mathbb{E}\left[\sum_{n=1}^{N_1} X_n \sum_{m=1}^{N_2} Y_m\right] - \mathbb{E}\left[\sum_{n=1}^{N_1} X_n\right] \mathbb{E}\left[\sum_{m=1}^{N_2} Y_m\right] \\ &= (\mathbb{E}[N_1 N_2] - \lambda_1 \lambda_2) \mathbb{E}[X] \mathbb{E}[Y] \end{aligned}$$

and the correlation:

$$\text{cor}(L_1, L_2) = \frac{(\mathbb{E}[N_1 N_2] - \lambda_1 \lambda_2)}{\sqrt{\lambda_1 \lambda_2}} \times \frac{\mathbb{E}[X] \mathbb{E}[Y]}{\sqrt{\mathbb{E}[X^2] \mathbb{E}[Y^2]}} \quad (2)$$

$$= \text{cor}(N_1, N_2) \times \eta(X) \times \eta(Y) \quad (3)$$

by noting η the function:

$$\eta(X) = \frac{\mathbb{E}[X]}{\sqrt{\mathbb{E}[X^2]}}$$

Simple calculations show that:

$$\begin{aligned} \eta(X) &= \frac{\mathbb{E}[X]}{\sqrt{\sigma^2[X] + \mathbb{E}^2[X]}} \\ &= \frac{1}{\sqrt{1 + c^2(X)}} \leq 1 \end{aligned}$$

with:

$$c^2(X) = \frac{\sigma^2[X]}{\mathbb{E}^2[X]}$$

provided that X is a positive random variable, which is assumed here as X is homogeneous to a loss amount. As a result, aggregate loss correlation is always lower than frequency correlation:

$$0 \leq \text{cor}(L_1, L_2) \leq \text{cor}(N_1, N_2) \leq 1$$

In particular, should frequency correlations be equal to 1, i.e. perfect correlation between N_1 and N_2 , the resulting correlation between aggregate losses would be significantly lower than 1. It is especially true

when severity distributions are fat-tailed, i.e. high-severity risk. Thus for high severity risk types ($c \gg 1$), aggregate loss correlations may be very small even if the frequency correlation is high. Conversely aggregate loss correlations are closed to frequency correlation only when the variance of losses is small in comparison with the mean of losses. This is typically what we have in mind for high frequency - low severity risk. In Figure 1, we report the upper bound (when $\text{cor}(N_1, N_2)$ is equal to one) of the aggregate loss correlation (in %) with respect to the coefficient of variation $c(X)$.

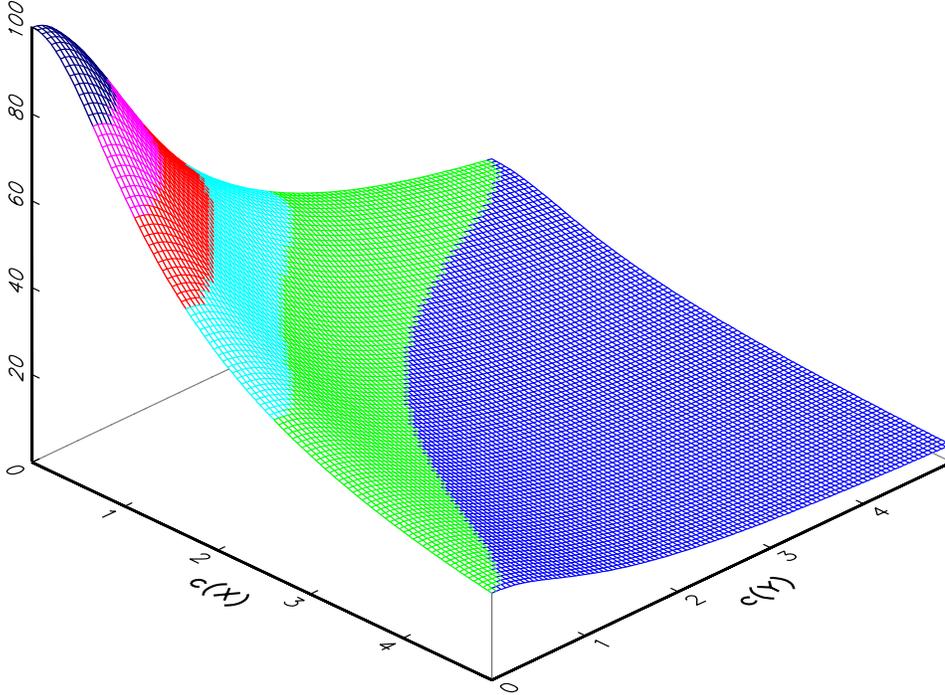


Figure 1: Upper bound of the aggregate loss correlation (in %)

Let us now consider the case where severity distributions are lognormal: $X \sim \mathcal{LN}(\mu_X, \sigma_X)$ and $Y \sim \mathcal{LN}(\mu_Y, \sigma_Y)$. We have:

$$\eta(X) = e^{-\frac{1}{2}\sigma_X^2}$$

and:

$$\text{cor}(L_1, L_2) = \text{cor}(N_1, N_2) \times \exp\left(-\frac{1}{2}\sigma_X^2 - \frac{1}{2}\sigma_Y^2\right)$$

The correlation between the aggregate losses is then a very simple formula which depends only on the frequency correlation and the standard deviation of the logarithm of the losses. According to our previous discussion, aggregate loss correlations are non-increasing functions of parameter σ . The aggregate loss correlation is then a decreasing function of the kurtosis or the magnitude of the tail of the distribution. In Figure 2, we report the upper bound (when $\text{cor}(N_1, N_2)$ is equal to one) of the aggregate loss correlation (in %) with respect to the parameters σ_X and σ_Y . To give a clearer idea of where we are in practice, we have indicated in Figure 2 the area corresponding to Credit Lyonnais data. **Because $\sigma > 1.5$ for all severity distributions, the maximum correlation is less than 4%.**

5 Conclusion

This paper has demonstrated that aggregate losses are necessarily low as long as we remain under the standard assumptions of LDA models. Moreover empirical findings show that the correlation between two

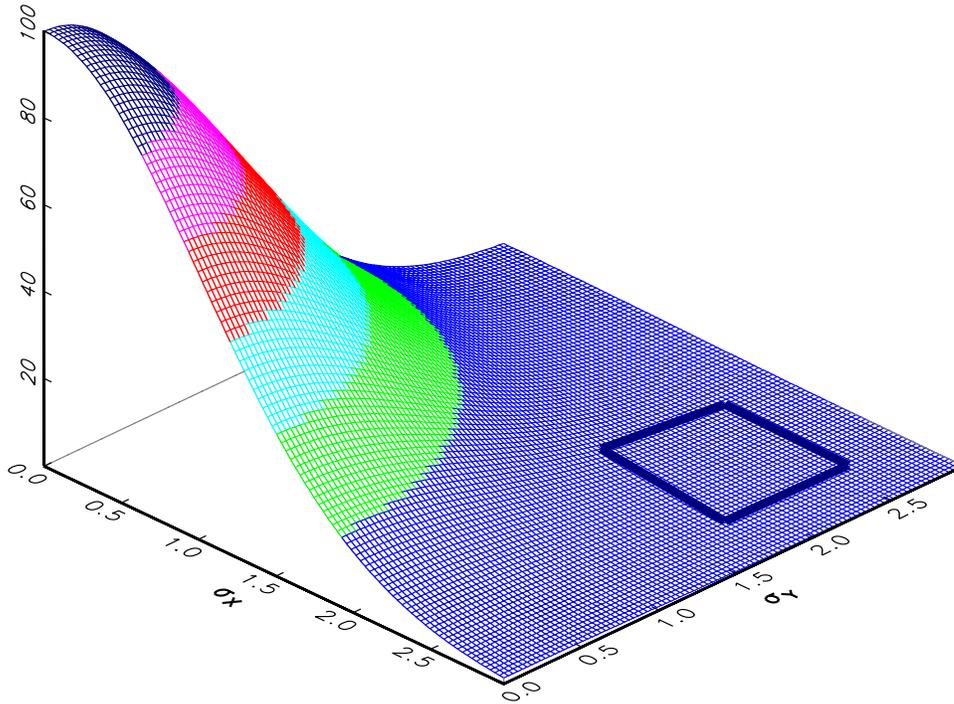


Figure 2: Upper bound of the correlation (in %) between two aggregate losses when individual losses are lognormal distributed

aggregate losses is typically below 5%, which opens a wide scope for large diversification effects, much larger than those the Basel Committee seems to have in mind. In other words, summing up capital charges is in substantial contradiction with the type of correlation consistent with the standard LDA model.

Obviously nothing prevents from extending the standard LDA model to more general a model where resulting correlations would be higher. It would require to allow frequency and severity to be correlated with one another and within a risk type \times business line class, which is a clear departure from the standard LDA model. Roughly speaking, it would mean that when losses come at a high frequency, their severities are also (in average) higher, and reciprocally. In Appendix B, we explore this kind of model in the simplest possible way. However, even though this kind of correlation between frequency and severity can make sense in practice, this cannot be done but at the expense of model tractability, and the extended model thus obtained is far out of reach of what current databases and state-of-the-art technology can cope with. Above all, our preliminary computations show that, even in this highly-correlated model, the resulting correlation between aggregate losses is probably not higher than 10%.

We finally propose the following simplified formula for the global capital charge:

$$K = EL + \sqrt{\sum_{i,j=1}^H \rho_{i,j} (K_i - EL_i) \times (K_j - EL_j)}$$

where EL_h is the expected loss of the h^{th} class. According to our previous results, we suggest to take $\rho_{i,j}$ ($i \neq j$) as a constant not higher than 5%-10%. Obviously this formula should be computed with the constraints imposed by the Basel Committee concerning the way subsidiaries should be taken into account³.

³

“Since experience has shown that capital is generally not freely transferable within a banking group, especially during times of stress, each banking subsidiary within the group must be adequately capitalised on a stand-alone basis.” (Principle 3 page 3 in [4]).

References

- [1] Operational Risk — Consultative Document, Supporting document to the New Basel Capital Accord, *Basel Committee on Banking Supervision*, January 2001.
- [2] Working Paper on the Regulatory Treatment of Operational Risk, *Basel Committee on Banking Supervision*, September 2001
- [3] The New Basel Capital Accord – Third Consultative Paper, *Basel Committee on Banking Supervision*, April 2003
- [4] Principles for the Home-host Recognition of AMA Operational Risk Capital, *Basel Committee on Banking Supervision*, January 2004
- [5] EVANS, M., N. HASTINGS and B. PEACOCK [1993], *Statistical Distributions*, John Wiley & Sons
- [6] FELLER, W. [1971], *An Introduction to Probability Theory and Its Applications*, Volume II, *second edition*, John Wiley & Sons
- [7] FRACHOT, A., P. GEORGES and T. RONCALLI [2001], Loss distribution approach for operational risk, *Working Paper*, Crédit Lyonnais, Groupe de Recherche Opérationnelle
- [8] FRACHOT, A., O. MOUDOULAUD and T. RONCALLI [2003], Loss distribution approach in practice, in M. Ong (ed.), *The Basel Handbook: A Guide for Financial Practitioners*, Risk Books
- [9] JOHNSON, N.L., S. KOTZ and N. BALAKRISHNAN [1997], *Discrete Multivariate Distributions*, Wiley Series in Probability and Statistics, John Wiley & Sons
- [10] KLUGMAN, S.A., H.H. PANJER and G.E. WILLMOT [1998], *Loss Models: From Data to Decisions*, *Wiley Series in Probability and Mathematical Statistics*, John Wiley & Sons

A Aggregation of quantiles and perfect correlation

We consider two random variables L_1 et L_2 . Let \mathbf{G}_1 and \mathbf{G}_2 be the distributions of L_1 and L_2 . The capital charges for a confidence level α are respectively $K_1(\alpha) = \mathbf{G}_1^{-1}(\alpha)$ and $K_2(\alpha) = \mathbf{G}_2^{-1}(\alpha)$. We would like to aggregate these two capital charges, that is we would like to compute the capital charge of $L = L_1 + L_2$. Let $K_{1+2}(\alpha)$ be the capital charge defined by $\inf \{x : \Pr \{L_1 + L_2 \leq x\} \geq \alpha\}$. Let \mathbf{G}_{1+2} be the distribution of $L = L_1 + L_2$. We have

$$\mathbf{G}_{1+2}(x) = \iint_{x_1+x_2 \leq x} d\mathbf{G}_{1+2}(x_1, x_2) = \iint_{x_1+x_2 \leq x} d\mathbf{C}(\mathbf{G}_1(x_1), \mathbf{G}_2(x_2))$$

The capital charge $K_{1+2}(\alpha)$ does not depend only on the margins, but on the copula \mathbf{C} between L_1 and L_2 too.

We consider the case of the upper Fréchet copula $\mathbf{C} = \mathbf{C}^+$. We suppose that \mathbf{G}_1 and \mathbf{G}_2 are continuous. Following the comonotony definition, we have $L_2 = \mathbf{G}_2^{-1}(\mathbf{G}_1(L_1))$. Let ϖ the function defined by $x \mapsto x + \mathbf{G}_2^{-1}(\mathbf{G}_1(x))$. We have

$$\begin{aligned} \alpha &= \Pr \{L_1 + L_2 \leq K_{1+2}(\alpha)\} \\ &= \mathbb{E}[\mathbf{1} \{\varpi(L_1) \leq K_{1+2}(\alpha)\}] \\ &= \mathbf{G}_1(\varpi^{-1}(K_{1+2}(\alpha))) \end{aligned}$$

We deduce that $K_{1+2}(\alpha) = \varpi(\mathbf{G}_1^{-1}(\alpha))$ and we obtain the following result:

$$\begin{aligned} K_{1+2}(\alpha) &= \mathbf{G}_1^{-1}(\alpha) + \mathbf{G}_2^{-1}(\mathbf{G}_1(\mathbf{G}_1^{-1}(\alpha))) \\ &= \mathbf{G}_1^{-1}(\alpha) + \mathbf{G}_2^{-1}(\alpha) \\ &= K_1(\alpha) + K_2(\alpha) \end{aligned}$$

Theorem 1 *The aggregation principle which consists in summing up all the individual capital charges corresponds to the case where the dependence between the random losses is the upper Fréchet copula \mathbf{C}^+ (or in other words when the random losses are perfectly correlated). In this case, we do not face to two source of randomness but only one:*

$$L = \varpi(L_1)$$

B Correlation between frequency and severity

We suppose now that X_1, X_2, \dots are conditionnally independent to N . We have $X | N \sim \mathcal{LN}(\mu(N), \sigma^2(N))$ with $\mu(N) = \mu + \alpha N$ and $\sigma^2(N) = \sigma^2 + \beta \times N$. Because $\mathbb{E}[e^{cN}] = e^{\lambda(e^c - 1)}$, we have $\mathbb{E}[X] = e^{\mu + \frac{1}{2}\sigma^2} e^{\lambda(e^{\alpha + \frac{\beta}{2}} - 1)}$ and $\mathbb{E}[X^2] = e^{2\mu + 2\sigma^2} e^{\lambda(e^{2\alpha + 2\beta} - 1)}$. We deduce that $\sigma[X^2] = e^{2\mu + \sigma^2} \left(e^{\sigma^2} e^{\lambda(e^{2\alpha + 2\beta} - 1)} - e^{2\lambda(e^{\alpha + \frac{\beta}{2}} - 1)} \right)$. For the aggregate loss, we need some intermediare results. We have $\mathbb{E}[Ne^{cN}] = \lambda e^c e^{\lambda(e^c - 1)}$ and $\mathbb{E}[N^2 e^{cN}] = (\lambda e^c + \lambda^2 e^{2c}) e^{\lambda(e^c - 1)}$. It comes that

$$\mathbb{E}[L] = \lambda e^{\mu + \frac{1}{2}\sigma^2} e^{\alpha + \frac{\beta}{2}} e^{\lambda(e^{\alpha + \frac{\beta}{2}} - 1)}$$

and

$$\begin{aligned} \mathbb{E}[L^2] &= e^{2\mu + \sigma^2} (\lambda e^{2\alpha + \beta} + \lambda^2 e^{4\alpha + 2\beta}) e^{\lambda(e^{2\alpha + \beta} - 1)} - \lambda e^{2\mu + \sigma^2} e^{2\alpha + \beta} e^{\lambda(e^{2\alpha + \beta} - 1)} + \\ &\quad \lambda e^{2\mu + 2\sigma^2} e^{2\alpha + 2\beta} e^{\lambda(e^{2\alpha + 2\beta} - 1)} \end{aligned}$$

We deduce that

$$\sigma[L^2] = \lambda e^{2\mu + \sigma^2} e^{2\alpha + \beta} \left(\lambda e^{2\alpha + \beta} e^{\lambda(e^{2\alpha + \beta} - 1)} + e^{\sigma^2 + \beta} e^{\lambda(e^{2\alpha + 2\beta} - 1)} - \lambda e^{2\lambda(e^{\alpha + \frac{\beta}{2}} - 1)} \right)$$

For the covariance, we have $\mathbb{E}[L_1 L_2] = e^{\mu_1 + \frac{1}{2}\sigma_1^2} e^{\mu_2 + \frac{1}{2}\sigma_2^2} \mathbb{E}[N_1 N_2 e^{\alpha_1 N_1 + \frac{\beta_1}{2} \times N_1} e^{\alpha_2 N_2 + \frac{\beta_2}{2} \times N_2}]$. We suppose that (N_1, N_2) have jointly a bivariate Poisson distribution (JOHNSON et al. [9]). It means that $N_1 = N_{11} + N_{12}$ and $N_2 = N_{22} + N_{12}$ where N_{11}, N_{12} and N_{22} are mutually independent Poisson random variables with parameters $\lambda_{11}, \lambda_{12}$ and λ_{22} . The correlation between N_1 and N_2 is $\frac{\lambda_{12}}{\sqrt{(\lambda_{11} + \lambda_{12})(\lambda_{22} + \lambda_{12})}}$. Given values for λ_1, λ_2 and $\text{cor}(N_1, N_2)$, it comes that $\lambda_{11} = \lambda_1 - \text{cor}(N_1, N_2) \sqrt{\lambda_1 \lambda_2}$, $\lambda_{22} = \lambda_2 - \text{cor}(N_1, N_2) \sqrt{\lambda_1 \lambda_2}$ and $\lambda_{12} = \text{cor}(N_1, N_2) \sqrt{\lambda_1 \lambda_2}$. It comes that

$$\begin{aligned} \mathbb{E}[L_1 L_2] &= e^{\mu_1 + \frac{1}{2}\sigma_1^2} e^{\mu_2 + \frac{1}{2}\sigma_2^2} \times e^{\lambda_{11} \left(e^{\alpha_1 + \frac{\beta_1}{2}} - 1 \right) + \lambda_{22} \left(e^{\alpha_2 + \frac{\beta_2}{2}} - 1 \right) + \lambda_{12} \left(e^{\alpha_1 + \alpha_2 + \frac{\beta_1 + \beta_2}{2}} - 1 \right)} \times \\ &\quad \left[\lambda_{11} \lambda_{22} e^{\alpha_1 + \alpha_2 + \frac{\beta_1 + \beta_2}{2}} + \lambda_{22} \lambda_{12} e^{\alpha_1 + 2\alpha_2 + \frac{\beta_1}{2} + \beta_2} + \lambda_{11} \lambda_{12} e^{2\alpha_1 + \alpha_2 + \beta_1 + \frac{\beta_2}{2}} + \right. \\ &\quad \left. \lambda_{12} e^{\alpha_1 + \alpha_2 + \frac{\beta_1 + \beta_2}{2}} + \lambda_{12}^2 e^{2\alpha_1 + 2\alpha_2 + \beta_1 + \beta_2} \right] \end{aligned}$$

We finally obtain

$$\begin{aligned} \text{cor}(L_1, L_2) &= e^{\lambda_1 \left(e^{\alpha_1 + \frac{\beta_1}{2}} - 1 \right) + \lambda_2 \left(e^{\alpha_2 + \frac{\beta_2}{2}} - 1 \right)} \left[e^{\text{cor}(N_1, N_2) \sqrt{\lambda_1 \lambda_2} \left(e^{\alpha_1 + \alpha_2 + \frac{\beta_1 + \beta_2}{2}} - e^{\alpha_1 + \frac{\beta_1}{2}} - e^{\alpha_2 + \frac{\beta_2}{2}} + 1 \right)} \times \right. \\ &\quad \left\{ \sqrt{\lambda_1 \lambda_2} + \text{cor}(N_1, N_2) \left(1 + \lambda_1 \left(e^{\alpha_1 + \frac{\beta_1}{2}} - 1 \right) + \lambda_2 \left(e^{\alpha_2 + \frac{\beta_2}{2}} - 1 \right) \right) \right. \\ &\quad \left. \left. \sqrt{\lambda_1 \lambda_2} \text{cor}^2(N_1, N_2) \left(1 - e^{\alpha_1 + \frac{\beta_1}{2}} - e^{\alpha_2 + \frac{\beta_2}{2}} + e^{\alpha_1 + \alpha_2 + \frac{\beta_1 + \beta_2}{2}} \right) \right\} - \sqrt{\lambda_1 \lambda_2} \right] \times \\ &\quad \prod_{i=1}^2 \left(\lambda_i e^{2\alpha_i + \beta_i} e^{\lambda(e^{2\alpha_i + \beta_i} - 1)} + e^{\sigma_i^2 + \beta_i} e^{\lambda(e^{2\alpha_i + 2\beta_i} - 1)} - \lambda_i e^{2\lambda \left(e^{\alpha_i + \frac{\beta_i}{2}} - 1 \right)} \right)^{-1/2} \end{aligned}$$

The correlations obtained with this framework are larger than the previous ones. However, they remain small. To give an idea of the level of correlations, we have computed the correlation (in %) with Crédit Lyonnais risk measurement system. For that, we have introduced a parameter c which represents the proportion of the mean and the variance of the logarithm of X explained by the Poisson random variable⁴. Figure 3 represents the maximum correlation, Figure 4 corresponds to the second largest correlation whereas we have reported the minimum correlation in Figure 5. We remark that for acceptable values of c and $\text{cor}(N_1, N_2)$ – that is $c \leq 50\%$ and $\text{cor}(N_1, N_2) \leq 50\%$ – the maximum correlation is 18% ! And if we are interested in the mean correlation, it is equal to 8% in the worst case ($c = 100\%$ and $\text{cor}(N_1, N_2) = 100\%$).

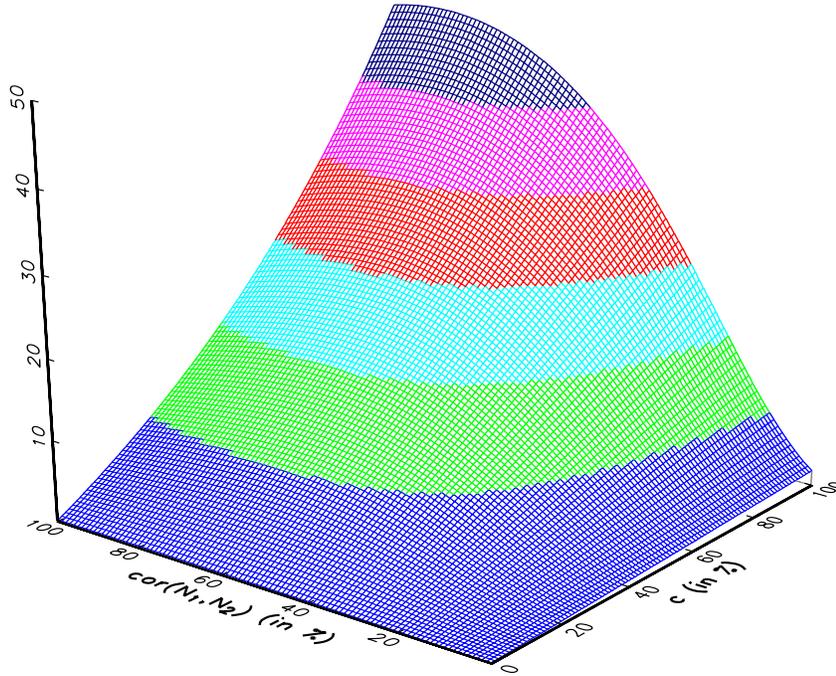


Figure 3: Maximum correlation (in %) with Crédit Lyonnais risk measurement system

⁴To be more explicit, let us introduce λ_{LDA} , μ_{LDA} and σ_{LDA} the estimates obtained with the LDA compound Poisson lognormal model. The parameters λ , μ , α , σ and β are such that

$$\begin{aligned}\lambda &= \lambda_{\text{LDA}} \\ \mu &= (1 - c) \mu_{\text{LDA}} \\ \alpha &= c \frac{\mu_{\text{LDA}}}{\lambda_{\text{LDA}}} \\ \sigma &= \sqrt{(1 - c) \sigma_{\text{LDA}}^2} \\ \beta &= c \frac{\sigma_{\text{LDA}}^2}{\lambda_{\text{LDA}}}\end{aligned}$$

We verify that $\mathbb{E}[\mu(N)] = \mu_{\text{LDA}}$ and $\mathbb{E}[\sigma^2(N)] = \sigma_{\text{LDA}}^2$.

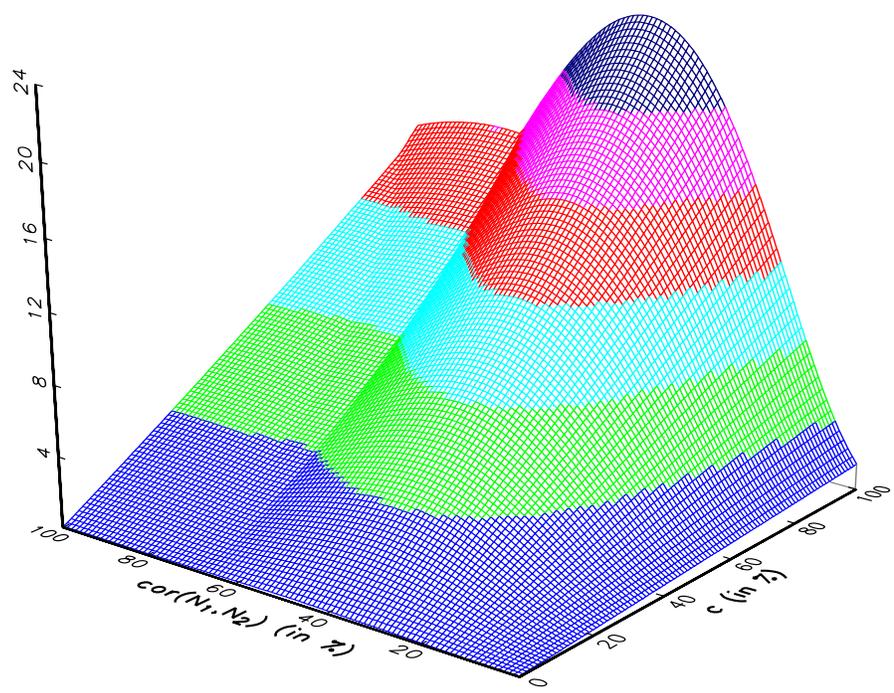


Figure 4: Second largest correlation (in %) with Crédit Lyonnais risk measurement system

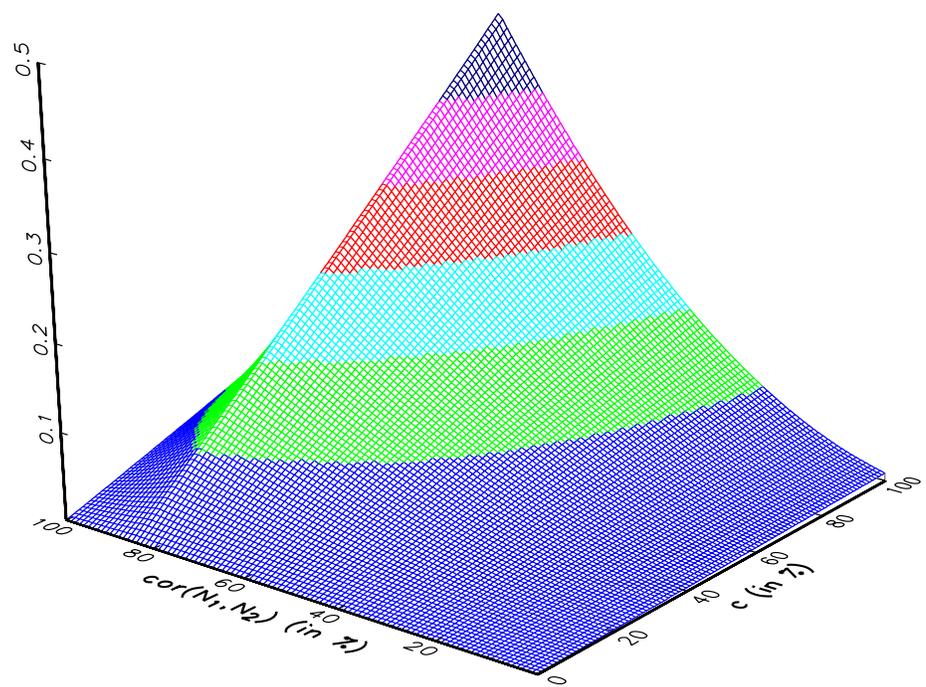


Figure 5: Minimum correlation (in %) with Crédit Lyonnais risk measurement system

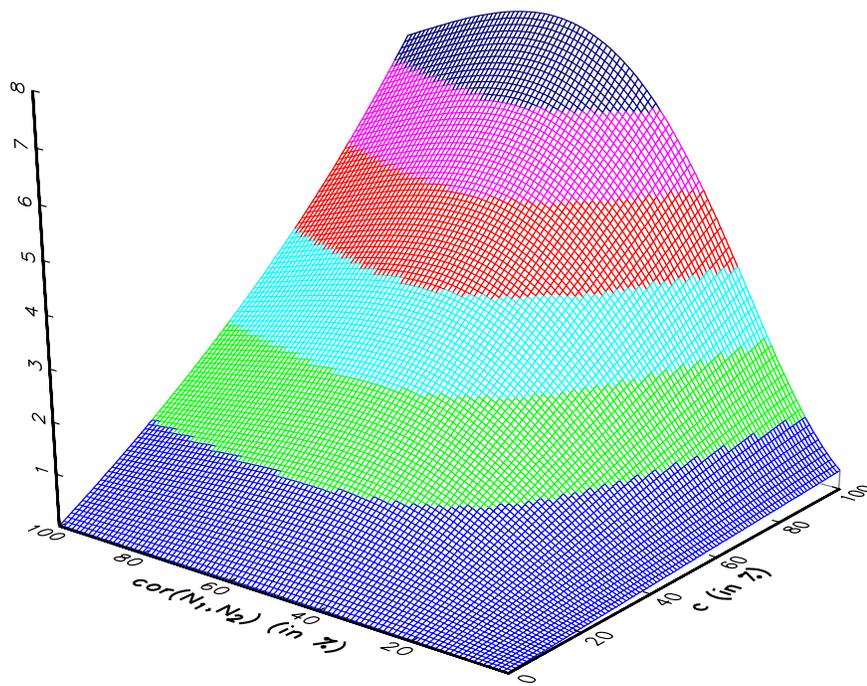


Figure 6: Mean correlation (in %) with Crédit Lyonnais risk measurement system