Course 2024–2025 in Sustainable Finance
Lecture 6. Exercise — Calculating the Prevalence of Undernourishment

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[1] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Agenda

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Let X and R be the random variables representing energy intake and energy requirement, respectively.

Question 1

We assume that the random vector (X, R) follows a bivariate log-normal distribution: $(X, R) \sim \mathcal{LN}\left(\mu_x, \sigma_x^2, \mu_r, \sigma_r^2, \rho\right)$.

Question (a)

Find the probability distribution of $D = X/R$. Then, calculate the prevalence of undernourishment, denoted by $\mathrm{PoU}^{\star} = \Pr\{X < R\}$.

We have:

$$\ln D = \ln X - \ln R \sim \mathcal{N}\left(\mu_d, \sigma_d^2\right)$$

where:

$$\mu_d = \mathbb{E}\left[\ln X - \ln R\right] = \mathbb{E}\left[\ln X\right] - \mathbb{E}\left[\ln R\right] = \mu_x - \mu_r$$

and:

$$\begin{aligned}
\sigma_d^2 &= \operatorname{var}\left(\ln X - \ln R\right) \\
&= \operatorname{var}\left(\ln X\right) + \operatorname{var}\left(\ln R\right) - 2\operatorname{cov}\left(\ln X, \ln R\right) \\
&= \sigma_x^2 + \sigma_r^2 - 2\rho\sigma_x\sigma_r
\end{aligned}$$

We deduce that $D = X/R$ is a log-normal random variable: $D \sim \mathcal{LN}\left(\mu_d, \sigma_d^2\right)$.

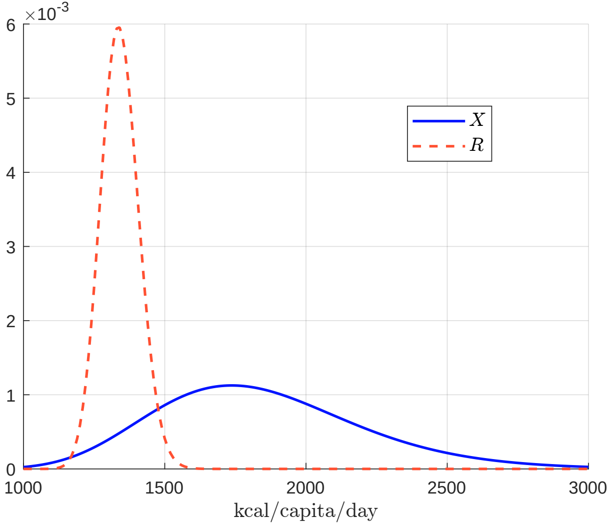
It follows that:

$$
\begin{aligned}
\mathrm{PoU}^{\star} &= \Pr\left\{X < R\right\} \\
&= \Pr\left\{D < 1\right\} \\
&= \Pr\left\{\ln D < 0\right\} \\
&= \Phi\left(-\frac{\mu_d}{\sigma_d}\right) \\
&= \Phi\left(-\frac{\mu_x - \mu_r}{\sqrt{\sigma_x^2 + \sigma_r^2 - 2\rho\sigma_x\sigma_r}}\right)
\end{aligned}
$$

Question (b)

Assume that $\mu_x = 7.50$, $\sigma_x = 0.20$, $\mu_r = 7.20$, and $\sigma_r = 0.05$. Plot the density functions of X and R.

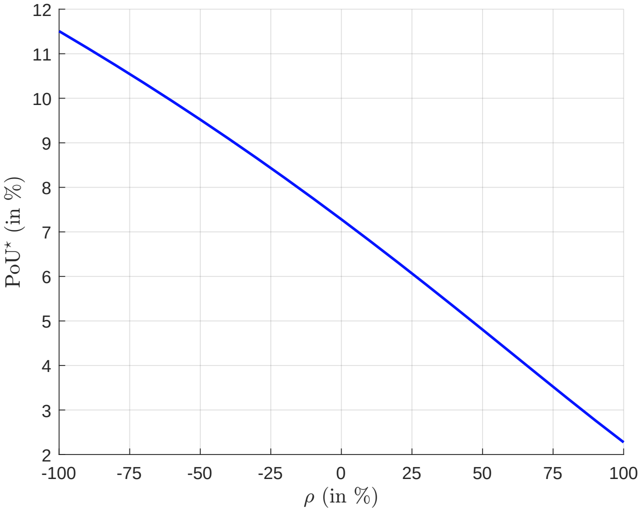
Figure 1: Probability density functions of X and R



Question (c)

Plot the function of $\mathrm{PoU}^{\star}$ as ρ varies in the interval $[-1, 1]$. Comment on the results.

Figure 2: Relationship between the correlation ρ and the prevalence of undernourishment $\mathrm{PoU}^{\star}$



Remark

We observe a decreasing function between ρ and $\mathrm{PoU}^{\star}$, because the standard deviation of D is a decreasing function of the correlation between X and R and $\mu_x > \mu_r$. Conversely, if $\mu_x < \mu_r$, the relationship between ρ and $\mathrm{PoU}^{\star}$ becomes increasing.

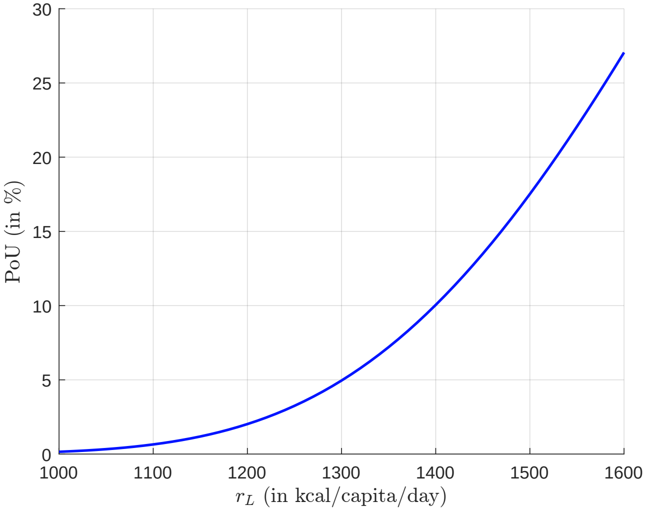
Question (d)

Compute the prevalence of undernourishment defined by $\mathrm{PoU} = \Pr\{X < r_L\}$. Plot the relationship between r_L and PoU when $r_L \in [1000, 1600]$.

We have:

$$\mathrm{PoU} = \Pr\left\{X < r_L\right\} = \Pr\left\{\ln X < \ln r_L\right\} = \Phi\left(\frac{\ln r_L - \mu_x}{\sigma_x}\right)$$

Figure 3: Prevalence of undernourishment PoU



Question (e)

Find the value of $r_L^\star$ such as $\mathrm{PoU} = \mathrm{PoU}^\star$. Calibrate the parameter $r_L^\star$ for the prevalence of undernourishment calculated in Question 1.(c).

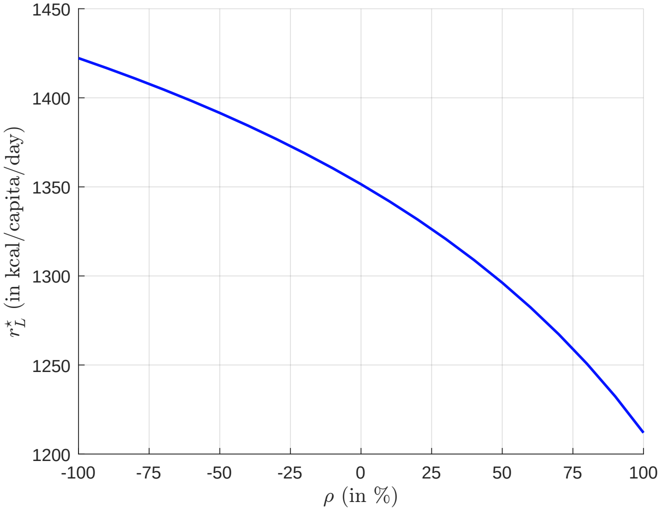
We deduce that:

$$\begin{aligned}\mathrm{PoU} = \mathrm{PoU}^{\star} \quad &\Leftrightarrow \quad \Phi\left(\frac{\ln r_L^{\star} - \mu_x}{\sigma_x}\right) = \Phi\left(-\frac{\mu_x - \mu_r}{\sqrt{\sigma_x^2 + \sigma_r^2 - 2\rho_{x,r}\sigma_x\sigma_r}}\right) \\ &\Leftrightarrow \quad \frac{\ln r_L^{\star} - \mu_x}{\sigma_x} = -\frac{\mu_x - \mu_r}{\sqrt{\sigma_x^2 + \sigma_r^2 - 2\rho_{x,r}\sigma_x\sigma_r}}\end{aligned}$$

which implies:

$$r_L^{\star} = \exp\left(\mu_x - \frac{(\mu_x - \mu_r)\,\sigma_x}{\sqrt{\sigma_x^2 + \sigma_r^2 - 2\rho_{x,r}\sigma_x\sigma_r}}\right)$$

Figure 4: Calibration of $r_L^\star$



Question 2

We want to calibrate the probability distribution function of X. We assume that $X \sim \mathcal{LN}\left(\mu_x, \sigma_x^2\right)$.

Question (a)

Give the first two moments of X. We will denote them by $\mu(X)$ and $\sigma^2(X)$.

We have:

$$\mu(X) = e^{\mu_x + \frac{1}{2}\sigma_x^2}$$

and:

$$\sigma^2(X) = e^{2\mu_x + \sigma_x^2}\left(e^{\sigma_x^2} - 1\right) = \mu^2(X)\left(e^{\sigma_x^2} - 1\right)$$

Question (b)

Deduce the coefficient of variation $\mathrm{CV}(X)$.

We deduce that:

$$\mathrm{CV}(X) = \frac{\sigma(X)}{\mu(X)} = \sqrt{e^{\sigma_x^2} - 1}$$

Question (c)

Find the moment estimators of μ_x and σ_x from $\mu(X)$ and $\mathrm{CV}(X)$.

We have:

$$\begin{aligned} \mathrm{CV}(X) = \sqrt{e^{\sigma_x^2} - 1} \quad & \Leftrightarrow \quad e^{\sigma_x^2} = \mathrm{CV}^2(X) + 1 \\ & \Leftrightarrow \quad \sigma_x = \sqrt{\ln\left(\mathrm{CV}^2(X) + 1\right)} \end{aligned}$$

and:

$$\begin{aligned} \mu(X) = e^{\mu_x + \frac{1}{2}\sigma_x^2} \quad & \Leftrightarrow \quad \mu_x = \ln \mu(X) - \frac{1}{2}\sigma_x^2 \\ & \Leftrightarrow \quad \mu_x = \ln \mu(X) - \frac{1}{2}\ln\left(\mathrm{CV}^2(X) + 1\right) \\ & \Leftrightarrow \quad \mu_x = \ln \mu(X) - \ln \sqrt{\mathrm{CV}^2(X) + 1} \\ & \Leftrightarrow \quad \mu_x = \ln \frac{\mu(X)}{\sqrt{\mathrm{CV}^2(X) + 1}} \end{aligned}$$

Question (d)

We consider an hypothetical country with a population of 1 million and two food components (cereals and fruits/vegetables), whose the food balance sheet is as follows:

	Production	Imports	Exports	ΔStocks	Feed	Waste
#1	349 000	1 025	40 000	−1 000	45 000	9 000
#2	50 000	9 010	1 000	6 000	500	500

All the figures are expressed in tonnes. Calculate the food available for human consumption. Assuming that the average energy density is 3 500 and 500 Calories/kg for cereals and fruits/vegetables respectively, find the average dietary energy consumption $\mu(X)$ expressed in kcal/capita/day.

The food available for human consumption is equal to:

$$\begin{aligned} Q_1 &= 349\,000 + 1\,025 - 40\,000 - (-1\,000) - 45\,000 - 9\,000 \\ &= 257\,025 \text{ tonnes} \end{aligned}$$

and

$$\begin{aligned} Q_2 &= 50\,000 + 9\,010 - 1\,000 - 6\,000 - 500 - 500 \\ &= 51\,010 \text{ tonnes} \end{aligned}$$

We deduce that the ADEC value is:

$$\begin{aligned} \mu(X) &= \frac{257\,025 \times 10^3 \times 3\,500 + 51\,010 \times 10^3 \times 500}{365 \times 10^6} \\ &= 2\,534.50 \text{ kcal/capita/day} \end{aligned}$$

Question (e)

The average dietary energy consumptiongeneral]Dietary energy consumption (DEC) by household expenditure deciles are 1 650 (first decile), 1 985, 2 150, 2 350, 2 530, 2 550, 2 650, 2 750, 3 100 and 3 630 (last decile). Calculate the coefficient of variation $\mathrm{CV}(X \mid Y)$. Check that $\mu(X \mid Y) = \mu(X)$. Assuming that $\mathrm{CV}(X \mid R) = 0.20$, calculate $\mathrm{CV}(X)$.

Let x_j^y be the average dietary energy consumptiongeneral]Dietary energy consumption (DEC) for the j^{th} household expenditure decile. We have:

$$\sigma(X \mid Y) = \sqrt{\sum_{j=1}^{10} f_j \left(x_i^y - \bar{x}^y\right)^2} = 533.4906$$

where $f_j = 10\%$ is the frequency of each decile group and $\bar{x}^y = \sum_{j=1}^{10} f_j x_i^y$ is the mean. We verify that $\bar{x}^y = 2\,534.50 = \mu(X)$. It follows that:

$$\mathrm{CV}(X \mid Y) = \frac{\sigma(X \mid Y)}{\mu(X \mid Y)} = \frac{533.4906}{2\,534.50} = 0.2105$$

We deduce that:

$$\mathrm{CV}(X) = \sqrt{\mathrm{CV}^2(X \mid Y) + \mathrm{CV}^2(X \mid R)} = \sqrt{0.2105^2 + 0.20^2} = 0.2904$$

Question (f)

Calibrate the parameters μ_x and σ_x using the previous figures. What is the prevalence of undernourishment if we assume that $r_L = 1\,850$? Give an estimate of the number of undernourished people.

We have:

$$\sigma_x = \sqrt{\ln\left(0.2904^2 + 1\right)} = 0.2845$$

and:

$$\mu_x = \ln \frac{2\,534.50}{\sqrt{0.2904^2 + 1}} = 7.7973$$

The prevalence of undernourishment is equal to:

$$\mathrm{PoU} = \Phi\left(\frac{1\,850 - 7.7973}{0.2845}\right) = 16.75\%$$

Finally, the number of undernourished people is around 167 500:

$$\mathrm{NoU} = N \cdot \mathrm{PoU} = 10^6 \times 16.75\% = 167\,451$$

Question 3

We seek to estimate the minimum dietary energy requirement (MDER).

Question (a)

We recall that the body mass index (BMI) is expressed in kg/m^2 and is defined as the ratio of the weight (in kg) to the square of the height (in meter):

$$\mathrm{BMI} = \frac{\mathrm{weight}}{\mathrm{height}^2}$$

The ideal value of BMI is 22. To define undernourished people, we assume that their weight is below a reference value:

$$\mathrm{weight} \leq \mathrm{weight}^{\star}$$

where:

$$\mathrm{weight}^{\star} = \mathrm{BMI}\left(p\right) \cdot \mathrm{height}^2$$

where p is a percentile value that depends on age.

Question (a) (Cont'd)

If the age is less than ten years, p is set to 50%, while for individuals 10 years and older, it is set to 5%. Below, we give the values of $\mathrm{BMI}\,(p)$ and the average height per age and sex:

	Age	-3	3–10	10–18	18–30	30–60	60+
$\mathrm{BMI}\,(p)$	Female	15.5	15.5	17.0	17.5	17.5	17.5
	Male	15.5	15.5	17.0	18.5	18.5	18.5
Height	Female	0.80	1.20	1.55	1.60	1.60	1.60
	Male	0.88	1.24	1.58	1.72	1.72	1.72

Calculate the reference value $\mathrm{weight}^{\star}$ per age and sex to determine the undernourishment.

We have:

$$\text{weight}^{\star}_{age,sex} = \text{BMI}_{age,sex}\,(p) \cdot \text{height}^{2}_{age,sex}$$

For example, we have:

$$\begin{aligned}\text{weight}^{\star}_{-3yr,female} &= \text{BMI}_{-3yr,female}\,(5\%) \cdot \text{height}^{2}_{-3yr,female} \\ &= 15.5 \times 0.80^{2} \\ &= 9.92\,\text{kg}\end{aligned}$$

We obtain the following results:

Age	−3	3–10	10–18	18–30	30–60	60+
Female	9.92	22.32	40.84	44.80	44.80	44.80
Male	12.00	23.83	42.44	54.73	54.73	54.73

This means that a 40-year-old woman is considered undernourished if she weighs less than 44.8 kg.

Question (b)

We assume that the basal metabolic rate (BMR) is given by the Schofield equation:

$$\mathrm{BMR} = \alpha + \beta \cdot \mathrm{weight}^{\star}$$

where α_j and β_j are the estimates of the linear regression between weight and BMR:

	Age	-3	3–10	10–18	18–30	30–60	60+
α	Female	-31.1	485.9	692.6	486.6	845.6	658.5
	Male	-30.4	504.3	658.2	692.2	873.1	587.7
β	Female	58.317	20.315	13.384	14.818	8.126	9.082
	Male	59.512	22.706	17.686	15.057	11.472	11.711

Calculate the basal metabolic rate for each group.

The basal metabolic rate (BMR) is given by the Schofield equation:

$$\mathrm{BMR}_{age,sex} = \alpha_{age,sex} + \beta_{age,sex} \cdot \mathrm{weight}^{\star}_{age,sex}$$

For example, we have:

$$\begin{aligned} \mathrm{BMR}_{-3yr,female} &= \alpha_{-3yr,female} + \beta_{-3yr,female} \cdot \mathrm{weight}^{\star}_{-3yr,female} \\ &= -31.1 + 58.317 \times 9.92 \\ &= 547.40\,\text{kcal/capita/day} \end{aligned}$$

We obtain the following results:

Age	−3	3–10	10–18	18–30	30–60	60+
Female	547	939	1 239	1 150	1 210	1 065
Male	684	1 045	1 409	1 516	1 501	1 229

Question (c)

The minimum dietary energy requirement is equal to the physical activity level (PAL) times the basal metabolic rate:

$$\mathrm{MDER} = \mathrm{PAL} \cdot \mathrm{BMR}$$

We assume that the population is on average lightly active, implying that $\mathrm{PAL} = 1.55$. Calculate the minium dietary energy requirement for the different groups.

The minimum dietary energy requirement is equal to the physical activity level times the basal metabolic rate:

$$\mathrm{MDER}_{age,sex} = \mathrm{PAL}_{age,sex} \cdot \mathrm{BMR}_{age,sex}$$

We obtain the following results:

Age	−3	3–10	10–18	18–30	30–60	60+
Female	848	1 456	1 921	1 783	1 875	1 651
Male	1 060	1 620	2 184	2 350	2 326	1 904

This means that a 40-year-old woman is considered undernourished if her dietary energy consumption is less than 1 875 Calories per day.

Question (d)

We assume that the proportion of females and males is the same, while the distribution of the population by age is as follows:

Age	-3	3–10	10–18	18–30	30–60	60+
Frequency	9.9%	15%	16%	18%	29%	12.1%

Calculate the minium dietary energy requirement of the population.

The minimum dietary energy requirement of the population is the weighted average of the different MDER values:

$$\mathrm{MDER} = \sum_{sex} \sum_{age} f_{age,sex} \cdot \mathrm{MDER}_{age,sex}$$

where $f_{age,sex}$ is the frequency of the group in the population. Finally, we obtain:

$$\mathrm{MDER} = 1\,849.90 \text{ kcal/capita/day}$$

Question 4

We want to calculate the depth of the food deficitgeneral]Depth of the food deficit:

$$\mathrm{FD} = \int_{x<r_L} (\bar{r} - x)\, f_x(x)\, \mathrm{d}x$$

where r_L is the minimum dietary energy requirement (MDER), $\bar{r}$ is the average dietary energy requirement (ADER), and $f_x(x)$ is the probability density function of the dietary energy consumption X.

Question (a)

What is the interpretation of the indicator FD?

FD indicates how many calories would be needed to ensure that undernourished would be eliminated if properly distributed.

Question (b)

Find the probabilistic expression of the indicator FD.

We have:

$$\begin{aligned}
\mathrm{FD} &= \int_{x<r_L} (\bar{r} - x)\, f_x(x)\, \mathrm{d}x \\
&= \int_0^{r_L} \bar{r} f_x(x)\, \mathrm{d}x - \int_0^{r_L} x f_x(x)\, \mathrm{d}x \\
&= \bar{r} \int_0^{r_L} f_x(x)\, \mathrm{d}x - \frac{\int_0^{r_L} x f_x(x)\, \mathrm{d}x}{\int_0^{r_L} f_x(x)\, \mathrm{d}x} \int_0^{r_L} f_x(x)\, \mathrm{d}x \\
&= (\bar{r} - \mathbb{E}[X \mid X < r_L]) \cdot \Pr\{X < r_L\} \\
&= \mathrm{PoU} \cdot (\bar{r} - \mathbb{E}[X \mid X < r_L])
\end{aligned}$$

The depth of the food deficit is the product of the prevalence of undernourishment and the difference between the average dietary energy requirement and the average dietary energy consumption, conditional on consumption being below the minimum requirement. Another expression is:

$$\mathrm{FD} = \mathrm{PoU} \cdot \mathbb{E}\left[(\bar{r} - X)_+ \mid X < r_L\right]$$

because $\bar{r} \geq r_L$. $\mathbb{E}\left[(\bar{r} - X)_+ \mid X < r_L\right]$ is the expected shortfall of food security.

Question (c)

Find the analytical value of FD when $X \sim \mathcal{LN}\left(\mu_x, \sigma_x^2\right)$.

Following Roncalli (2021, page 319), we introduce the notation $\Phi_c(x) = \Phi((x - \mu_x)/\sigma_x)$, and we calculate the conditional moment $\mu'_m(X) = \mathbb{E}[X^m \mid X < r_L]$ for $m \geq 1$ by using the change of variable $y = \ln x$:

$$\begin{aligned}
\mu'_m(X) &= \frac{1}{\Phi_c(\ln r_L)} \int_0^{r_L} \frac{x^m}{x\sigma_x\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x - \mu_x}{\sigma_x}\right)^2\right) \mathrm{d}x \\
&= \frac{1}{\Phi_c(\ln r_L)} \int_{-\infty}^{\ln r_L} \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y - \mu_x}{\sigma_x}\right)^2 + my\right) \mathrm{d}y
\end{aligned}$$

We have:

$$\begin{aligned}
-\frac{1}{2}\left(\frac{y - \mu_x}{\sigma_x}\right)^2 + my &= -\frac{1}{2}\left(\frac{y^2 - 2y\left(\mu_x + m\sigma_x^2\right) + \mu_x^2}{\sigma_x^2}\right) \\
&= -\frac{1}{2}\left(\frac{y - \left(\mu_x + m\sigma_x^2\right)}{\sigma_x}\right)^2 + \left(m\mu_x + \frac{1}{2}m^2\sigma_x^2\right)
\end{aligned}$$

We deduce that:

$$\mu_m'(X) = \frac{\exp\left(m\mu_x + m^2\sigma_x^2/2\right)}{\Phi_c(\ln r_L)} \int_{-\infty}^{\ln r_L} \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y - \left(\mu_x + m\sigma_x^2\right)}{\sigma_x}\right)^2\right)$$

Using the change of variable $z = \dfrac{y - \left(\mu_x + m\sigma_x^2\right)}{\sigma_x}$, it follows that:

$$\mu_m'(X) = \frac{\exp\left(m\mu_x + m^2\sigma_x^2/2\right)}{\Phi_c(\ln r_L)} \int_{-\infty}^{z_L} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) \mathrm{d}z$$

where:

$$z_L = \frac{\ln r_L - \left(\mu_x + m\sigma_x^2\right)}{\sigma_x}$$

Finally, we obtain:

$$\mu'_m(X) = \frac{\Phi_c\left(\ln r_L - m\sigma_x^2\right)}{\Phi_c\left(\ln r_L\right)} \exp\left(m\mu_x + \frac{1}{2}m^2\sigma_x^2\right)$$

and:

$$\mathbb{E}\left[X \mid X < r_L\right] = \mu'_1(X) = \frac{\Phi\left(\dfrac{\ln r_L - \mu_x - \sigma_x^2}{\sigma_x}\right)}{\Phi\left(\dfrac{\ln r_L - \mu_x}{\sigma_x}\right)} \exp\left(\mu_x + \frac{1}{2}\sigma_x^2\right)$$

The analytical expression of the depth of the food deficit is:

$$\mathrm{FD} = \bar{r}\Phi\left(\frac{\ln r_L - \mu_x}{\sigma_x}\right) - e^{\mu_x + \frac{1}{2}\sigma_x^2}\Phi\left(\frac{\ln r_L - \mu_x - \sigma_x^2}{\sigma_x}\right)$$

because:

$$\mathrm{PoU} = \Pr\left\{X < r_L\right\} = \Phi\left(\frac{\ln r_L - \mu_x}{\sigma_x}\right)$$

Question (d)

Calculate the depth of the food deficit in the case of Question 2 if we assume that the average dietary energy requirement is equal to 2 500 Calories per person per day.

The depth of the food deficit is equal to:

$$\begin{aligned}
\mathrm{FD} &= 2\,500 \times \Phi\left(\frac{\ln 1\,850 - 7.7973}{0.2845}\right) - \\
&\quad \exp\left(7.7973 + \frac{1}{2}0.2845^2\right) \times \Phi\left(\frac{\ln 1\,850 - 7.7973 - 0.2845^2}{0.2845}\right) \\
&= 150.2968 \text{ kcal/capita/year}
\end{aligned}$$