A Gauss Implementation of Skew Normal/Student distributions (SN, ST, MSN and MST)
The Skew library

Thierry Roncalli
University of Evry

Thibault Lagache
GRO, Crédit Agricole SA

This version: November 2004

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Chapter 1

Introduction

1.1 Installation

1. The file *skew.zip* is a zipped archive file. Copy this file under the root directory of Gauss, for example **D:\GAUSS60**.
2. Unzip the file. Directories will then be created and files will be copied over them:

target_path	*readme.txt*
*target_path***dlib**	DLLs
*target_path***lib**	library file
*target_path***skew****examples**	example and tutorial files
*target_path***skew****src**	source code files
*target_path***src**	source code files

3. If your root of Gauss is **D:\GAUSS60**, the installation is finished, otherwise you have to modify the paths of the library using notepad or the LibTool. Another way to update the library is to run Gauss, l**og on to the *skew**src* directory**, delete the path with the command **lib skew -n** and add the path to the library with the command **lib skew -a**.

1.2 Getting started

Gauss 6.0.57+ for Windows is required to use the **SKEW** routines.

1.2.1 readme.txt file

The file *readme.txt* contains last minute information on the **SKEW** procedures. Please read it before using them.

1.2.2 Setup

In order to use these procedures, the **SKEW** library must be active. This is done by including **SKEW** in the LIBRARY statement at the top of your program:

library skew;

1.3 What is SKEW?

SKEW is a Gauss library for computing skew distribution functions. **SKEW** contains the procedures whose list is given below:

- SN
  - PDF
    `pdf = pdfSN(x,mu,sigma,alpha);`
  - CDF
    `cdf = cdfSN(x,mu,sigma,alpha);`
  - INV
    `q = cdfSNi(x,mu,sigma,alpha);`
  - RND
    `u = rndSN(r,c,mu,sigma,alpha);`
- MSN
  - PDF
    `pdf = pdfMSN(x,mu,sigma,alpha);`
  - CDF
    `cdf = cdfMSN(x,mu,sigma,alpha);`
  - RND
    `u = rndMSN(r,mu,sigma,alpha);`
- ST
  - PDF
    `pdf = pdfST(x,mu,sigma,alpha,nu);`
  - CDF
    `cdf = cdfST(x,mu,sigma,alpha,nu);`
  - INV
    `q = cdfSTi(x,mu,sigma,alpha,nu);`
  - RND
    `u = rndST(r,c,mu,sigma,alpha,nu);`
- MST
  - PDF
    `pdf = pdfMST(x,mu,sigma,alpha,nu);`
  - CDF
    `cdf = cdfMST(x,mu,sigma,alpha,nu);`
  - RND
    `u = rndMST(r,mu,sigma,alpha,nu);`
- ML estimation of the parameters
  - `{mu,sigma,alpha,nu} = ml_skew(data,sv,model);`

1.4 Using Online Help

SKEW library supports Windows Online Help. Before using the browser, you have to verify that the **SKEW** library is activated by the `library` command.

Chapter 2

The Skew probability distribution functions

The following presentation is based on Azzalini *et al.* (2003).

2.1 The *Skew Normal* distribution function

2.1.1 The multivariate case

We consider the Gaussian random vector $X \sim \mathcal{N}(0, \Sigma)$. Let $\phi_d(x; \Sigma)$ be the associated density function. We have:

$$\phi_d(x; \Sigma) = (2\pi)^{-d/2} |\Sigma|^{-1/2} \exp\left(-\frac{1}{2} x^\top \Sigma^{-1} x\right)$$

with d the dimension of the random vector. We also note $\Phi_d(x; \Sigma)$ the cumulative density function., and $\phi(x) = \phi_1(x; 1)$ and $\Phi(x) = \Phi_1(x; 1)$.

The density function of the SN distribution is defined by:

$$2\phi_d(x - \mu; \Sigma) \Phi\left(\alpha^\top \omega^{-1}(x - \mu)\right)$$

with $\omega = \operatorname{diag}^{1/2}(\Sigma)$. We say that X follows a *Skew Normal* distribution function with parameters μ, Σ and α and we write $X \sim \mathcal{MSN}(\mu, \Sigma, \alpha)$. We remark that the distribution function of $X \sim \mathcal{MSN}(\mu, \Sigma, 0)$ is the standard Normal distribution $\mathcal{N}(\mu, \Sigma)$. We verify the property: $X = \mu + \omega X^\star$ with $X^\star \sim \mathcal{MSN}(0, \Omega, \alpha)$ and $\Omega = \omega^{-1}\Sigma\omega^{-1}$ the correlation matrix of Σ .

Let us introduce the δ vector:

$$\delta = \frac{\Omega\alpha}{\sqrt{1 + \alpha^\top \Omega \alpha}}$$

Then $X^\star \sim \mathcal{MSN}(0, \Omega, \alpha)$ has the following stochastic representation:

$$X^\star = \begin{cases} U & \text{if } U_0 > 0 \\ -U & \text{if } U_0 \leq 0 \end{cases}$$

with:

$$\begin{pmatrix} U_0 \\ U \end{pmatrix} \sim \mathcal{N}\left(0, \Omega_+(\delta) = \begin{pmatrix} 1 & \delta^\top \\ \delta & \Omega \end{pmatrix}\right)$$

We deduce that:

$$\begin{aligned}
\Pr\{X \leq x\} &= \Pr\left\{X^\star \leq \omega^{-1}(x-\mu)\right\} \\
&= \Pr\left\{U \leq \omega^{-1}(x-\mu) \mid U_0 > 0\right\} \\
&= \frac{\Pr\left\{U \leq \omega^{-1}(x-\mu), U_0 > 0\right\}}{\Pr\{U_0 > 0\}} \\
&= 2\left(\Pr\left\{U \leq \omega^{-1}(x-\mu)\right\} - \Pr\left\{U \leq \omega^{-1}(x-\mu), U_0 \leq 0\right\}\right) \\
&= 2\left(\Phi_d\left(\omega^{-1}(x-\mu);\Omega\right) - \Phi_{d+1}\left(u_+;\Omega_+(\delta)\right)\right) \\
&= 2\Phi_{d+1}\left(u_+;\Omega_+(-\delta)\right)
\end{aligned}$$

with:

$$u_+ = \begin{pmatrix} 0 \\ \omega^{-1}(x-\mu) \end{pmatrix}$$

We use this representation to compute the multidimensional cdf and to simulate the random vector $X \sim \mathcal{MSN}(\mu, \Sigma, \alpha)$.

2.1.2 The univariate case

With $d = 1$ and $\Sigma = \sigma^2$, the SN density function becomes:

$$\frac{2}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\Phi\left(\alpha\frac{x-\mu}{\sigma}\right)$$

To compute the cdf, we use the stochastic representation. We have:

$$\begin{aligned}
\Pr\{X \leq x\} &= 2\left(\Phi\left(\frac{x-\mu}{\sigma}\right) - \Phi_2\left(0, \frac{x-\mu}{\sigma};\delta\right)\right) \\
&= 2\Phi_2\left(0, \frac{x-\mu}{\sigma};-\delta\right)
\end{aligned}$$

with:

$$\delta = \frac{\alpha}{\sqrt{1+\alpha^2}}$$

We may obtain another expression by using another stochastic representation: if U_1 and U_2 are two standard Gaussian random variates with correlation ρ, then the distribution function of the random variate $\max(U_1, U_2)$ is $\mathcal{SN}(0, 1, \alpha)$ with:

$$\alpha = \sqrt{\frac{1-\rho}{1+\rho}} \geq 0$$

We deduce that:

$$\begin{aligned}
\Pr\{X \leq x\} &= \Pr\left\{X^\star \leq \frac{x-\mu}{\sigma}\right\} \\
&= \Pr\left\{\max(U_1, U_2) \leq \frac{x-\mu}{\sigma}\right\} \\
&= \Phi_2\left(\frac{x-\mu}{\sigma}, \frac{x-\mu}{\sigma};\rho\right)
\end{aligned}$$

with:

$$\rho = \frac{1-\alpha^2}{1+\alpha^2}$$

If $\alpha \leq 0$, we use the property $-X^\star \sim \mathcal{SN}(0,1,-\alpha)$ and we obtain:

$$\begin{aligned}\Pr\{X \leq x\} &= 1 - \Pr\left\{-X^\star \leq -\frac{x-\mu}{\sigma}\right\} \\ &= 1 - \Phi_2\left(-\frac{x-\mu}{\sigma}, -\frac{x-\mu}{\sigma}; \rho\right)\end{aligned}$$

2.2 The *Skew Student* distribution function

2.2.1 The multivariate case

Let $X \sim \mathcal{MSN}(0,\Sigma,\alpha)$. We consider the random vector Y such that:

$$Y = \mu + \frac{X}{\sqrt{\chi^2_\nu/\nu}}$$

The distribution function of Y is *Skew Student* and we note$\mathcal{MST}(\mu,\Sigma,\alpha,\nu)$. Let $t_d(z;\Sigma,\nu)$ and $T_d(z;\Sigma,\nu)$ be the pdf and cdf functionsof the t distribution with ν degrees of freedom. The density function of Y is:

$$2t_d(y-\mu;\Sigma,\nu)\,T_1\left(\alpha^\top\omega^{-1}(y-\mu)\sqrt{\frac{\nu+d}{Q+\nu}};1,\nu+d\right)$$

with $Q = (y-\mu)^\top \Sigma^{-1}(y-\mu)$. We remark that we have:

$$\begin{aligned}\Pr\{Y \leq y\} &= \Pr\left\{\frac{X^\star}{\sqrt{\chi^2_\nu/\nu}} \leq \omega^{-1}(y-\mu)\right\} \\ &= \Pr\left\{\frac{U}{\sqrt{\chi^2_\nu/\nu}} \leq \omega^{-1}(y-\mu) \mid U_0 > 0\right\} \\ &= 2\Pr\left\{\frac{1}{\sqrt{\chi^2_\nu/\nu}}\begin{pmatrix}-U_0 \\ U\end{pmatrix} \leq \begin{pmatrix}0 \\ \omega^{-1}(y-\mu)\end{pmatrix}\right\} \\ &= 2T_{d+1}(u_+;\Omega_+(-\delta),\nu)\end{aligned}$$

To simulate Y, we use the relashionship with the $\mathcal{MSN}$ distribution function.

2.2.2 The univariate case

The density function becomes:

$$\frac{2}{\sigma}t_1\left(\frac{y-\mu}{\sigma};1,\nu\right)T_1\left(\alpha\frac{y-\mu}{\sigma}\sqrt{\frac{\nu+1}{\left(\frac{y-\mu}{\sigma}\right)^2+\nu}};1,\nu+1\right)$$

To compute the cdf, we use the following result:

$$\Pr\{Y \leq y\} = 2T_2\left(0,\frac{y-\mu}{\sigma};-\delta;\nu\right)$$

If $\alpha \geq 0$, we may show that:

$$\Pr\{Y \leq y\} = T_2\left(\frac{x-\mu}{\sigma}, \frac{x-\mu}{\sigma}; \rho\right)$$

Bibliography

[1] Azzalini, A. and Capitanio A., Distributions generated by perturbation of symmetry with emphasis on a multivariate skew t distribution, *JRSS B*, **65**, pp. 367-389, 2003.