Course 2023-2024 in Financial Risk Management
Lecture 5. Operational Risk

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[1]The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

General information

1. **Overview**
   The objective of this course is to understand the theoretical and practical aspects of risk management
2. **Prerequisites**
   M1 Finance or equivalent
3. **ECTS**
   4
4. **Keywords**
   Finance, Risk Management, Applied Mathematics, Statistics
5. **Hours**
   Lectures: 36h, Training sessions: 15h, HomeWork: 30h
6. **Evaluation**
   There will be a final three-hour exam, which is made up of questions and exercises
7. **Course website**
   `http://www.thierry-roncalli.com/RiskManagement.html`

Objective of the course

The objective of the course is twofold:

1. knowing and understanding the financial regulation (banking and others) and the international standards (especially the Basel Accords)
2. being proficient in risk measurement, including the mathematical tools and risk models

Class schedule

Course sessions

- September 15 (6 hours, AM+PM)
- September 22 (6 hours, AM+PM)
- September 19 (6 hours, AM+PM)
- October 6 (6 hours, AM+PM)
- October 13 (6 hours, AM+PM)
- October 27 (6 hours, AM+PM)

Tutorial sessions

- October 20 (3 hours, AM)
- October 20 (3 hours, PM)
- November 10 (3 hours, AM)
- November 10 (3 hours, PM)
- November 17 (3 hours, PM)

Class times: Fridays 9:00am-12:00pm, 1:00pm–4:00pm, University of Evry, Room 209 IDF

Agenda

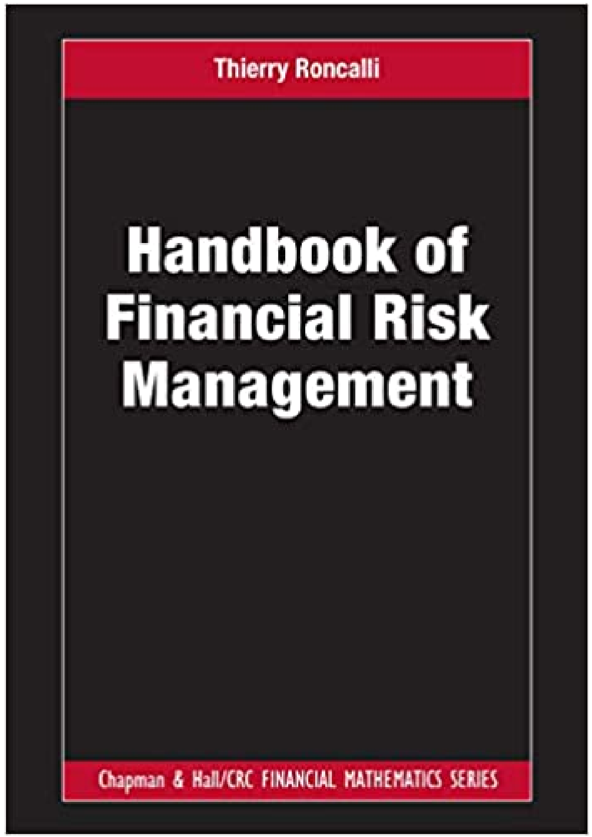
- Lecture 1: Introduction to Financial Risk Management
- Lecture 2: Market Risk
- Lecture 3: Credit Risk
- Lecture 4: Counterparty Credit Risk and Collateral Risk
- Lecture 5: Operational Risk
- Lecture 6: Liquidity Risk
- Lecture 7: Asset Liability Management Risk
- Lecture 8: Model Risk
- Lecture 9: Copulas and Extreme Value Theory
- Lecture 10: Monte Carlo Simulation Methods
- Lecture 11: Stress Testing and Scenario Analysis
- Lecture 12: Credit Scoring Models

Agenda

- Tutorial Session 1: Market Risk
- Tutorial Session 2: Credit Risk
- Tutorial Session 3: Counterparty Credit Risk and Collateral Risk
- Tutorial Session 4: Operational Risk & Asset Liability Management Risk
- Tutorial Session 5: Copulas, EVT & Stress Testing

Textbook

- Roncalli, T. (2020), *Handbook of Financial Risk Management*, Chapman & Hall/CRC Financial Mathematics Series.



Additional materials

- Slides, tutorial exercises and past exams can be downloaded at the following address:

  `http://www.thierry-roncalli.com/RiskManagement.html`

- Solutions of exercises can be found in the companion book, which can be downloaded at the following address:

  `http://www.thierry-roncalli.com/RiskManagementBook.html`

Agenda

A long list of operational risk losses:

- 1983: Banco Ambrosiano Vatican Bank (money laundering)
- 1995: Barings (rogue trading)
- 1996: Summitomo Bank (rogue trading)
- 1996: Crédit Lyonnais (headquarter fire)
- Etc.

Since the end of the nineties, new themes: operational risk, legal risk, compliance, money laundering, etc.

Definition

Definition

The Basel Committee defines the operational risk in the following way:

> *"Operational risk is defined as the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events. This definition includes legal risk, but excludes strategic and reputational risk"*

Loss event type classification

1. Internal fraud ("*losses due to acts of a type intended to defraud, misappropriate property or circumvent regulations, the law or company policy, excluding diversity/discrimination events, which involves at least one internal party*")
    1. Unauthorized activity
    2. Theft and fraud
2. External fraud ("*losses due to acts of a type intended to defraud, misappropriate property or circumvent the law, by a third party*")
    1. Theft and fraud
    2. Systems security
3. Employment practices and workplace safety ("*losses arising from acts inconsistent with employment, health or safety laws or agreements, from payment of personal injury claims, or from diversity/discrimination events*")
    1. Employee relations
    2. Safe environment
    3. Diversity & discrimination

Loss event type classification

4. Clients, products & business practices (“*losses arising from an unintentional or negligent failure to meet a professional obligation to specific clients (including fiduciary and suitability requirements), or from the nature or design of a product*”)
   1. Suitability, disclosure & fiduciary
   2. Improper business or market practices
   3. Product flaws
   4. Selection, sponsorship & exposure
   5. Advisory activities
5. Damage to physical assets (“*losses arising from loss or damage to physical assets from natural disaster or other events*”)
   1. Disasters and other events

Loss event type classification

6. Business disruption and system failures ("*losses arising from disruption of business or system failures*")
    1. Systems
7. Execution, delivery & process management ("*losses from failed transaction processing or process management, from relations with trade counterparties and vendors*")
    1. Transaction capture, execution & maintenance
    2. Monitoring and reporting
    3. Customer intake and documentation
    4. Customer/client account management
    5. Trade counterparties
    6. Vendors & suppliers

Loss data collection exercise (LDCE)

Table: Internal losses larger than €20 000 per year

Year	pre 2002	2002	2003	2004	2005	2006	2007
n_L	14 017	10 216	13 691	22 152	33 216	36 386	36 622
L (in € bn)	3.8	12.1	4.6	7.2	9.7	7.4	7.9
n_B	24	35	55	68	108	115	117

More and more operational risk losses:

- Société Générale in 2008 ($7.2 bn), Morgan Stanley in 2008 ($9.0 bn), BPCE in 2008 ($1.1 bn), UBS in 2011 ($2 bn), JPMorgan Chase in 2012 ($5.8 bn), etc.
- Libor scandal: $2.5 bn for Deutsche Bank, $1 bn for Rabobank, $545 mn for UBS, etc.
- Forex scandal: six banks (BoA, Barclays, Citi, JPM, UBS and RBS) agreed to pay fines totaling $5.6 bn in May 2015
- BNP Paribas payed a fine of $8.9 bn in June 2014 (anti-money laundering control)
- Etc.

Basel II versus Basel III

Basel II

- Basic indicator approach (BIA)
- The standardized approach (TSA)
- Advanced measurement approaches (AMA)

Basel III

- Standardized approach (SA-OR)
- Pillar II

Basic indicator approach (BIA)

The capital charge is a fixed percentage of annual gross income:

$$\mathcal{K} = \alpha \cdot \overline{\mathrm{GI}}$$

where $\alpha = 15\%$ and $\overline{\mathrm{GI}}$ is the average of the positive gross income over the previous three years:

$$\overline{\mathrm{GI}} = \frac{\max\left(\mathrm{GI}_{t-1}, 0\right) + \max\left(\mathrm{GI}_{t-2}, 0\right) + \max\left(\mathrm{GI}_{t-3}, 0\right)}{\sum_{k=1}^{3} \mathbb{1}\left\{\mathrm{GI}_{t-k} > 0\right\}}$$

The standardized approach (TSA)

TSA is an extended version of BIA:

$$\mathcal{K}_{j,t} = \beta_j \cdot \mathrm{GI}_{j,t}$$

where β_j and $\mathrm{GI}_{j,t}$ are a fixed percentage and the gross income corresponding to the j^{th} business line. The total capital charge is the three-year average of the sum of all the capital charges:

$$\mathcal{K} = \frac{1}{3} \sum_{k=1}^{3} \max \left(\sum_{j=1}^{8} \mathcal{K}_{j,t-k}, 0 \right)$$

If the values of gross income are all positive, the total capital charge becomes:

$$\mathcal{K} = \frac{1}{3} \sum_{k=1}^{3} \sum_{j=1}^{8} \beta_j \cdot \mathrm{GI}_{j,t-k} = \sum_{j=1}^{8} \beta_j \cdot \overline{\mathrm{GI}}_j$$

The standardized approach (TSA)

Table: Mapping of business lines for operational risk

Level 1	Level 2	β_j
Corporate Finance	Corporate Finance Municipal/Government Finance Merchant Banking Advisory Services	18%
Trading & Sales	Sales Market Making Proprietary Positions Treasury	18%
Retail Banking	Retail Banking Private Banking Card Services	12%
Commercial Banking$^\natural$	Commercial Banking	12%
Payment & Settlement	External Clients	18%
Agency Services	Custody Corporate Agency Corporate Trust	15%
Asset Management	Discretionary Fund Management Non-Discretionary Fund Management	12%
Retail Brokerage	Retail Brokerage	12%

The standardized approach (TSA)

What is the difference between corporate finance, trading & sales and commercial banking?

- **Corporate finance**: mergers and acquisitions, underwriting, securitization, syndications, IPO, debt placements
- **Trading & sales**: buying and selling of securities and derivatives, own position securities, lending and repos, brokerage
- **Commercial banking**: project finance, real estate, export finance, trade finance, factoring, leasing, lending, guarantees, bills of exchange

Advanced measurement approaches (AMA)

The AMA method is defined by certain criteria without refereing to a specific statistical model:

- The capital charge should cover the one-year operational loss at the 99.9% confidence level $(\mathrm{UL} + \mathrm{EL})$
- A minimum five-year observation period of internal loss data
- The model can incorporate the risk mitigation impact of insurance, which is limited to 20% of the total operational risk capital charge

Advanced measurement approaches (AMA)

Table: Distribution of annualized operational losses (in %)

Business line	Event type							All
	1	2	3	4	5	6	7	
Corporate Finance	0.2	0.1	0.6	93.7	0.0	0.0	5.4	28.0
Trading & Sales	11.0	0.3	2.3	29.0	0.2	1.8	55.3	13.6
Retail Banking	6.3	19.4	9.8	40.4	1.1	1.5	21.4	32.0
Commercial Banking	11.4	15.2	3.1	35.5	0.4	1.7	32.6	7.6
Payment & Settlement	2.8	7.1	0.9	7.3	3.2	2.3	76.4	2.6
Agency Services	1.0	3.2	0.7	36.0	18.2	6.0	35.0	2.6
Asset Management	11.1	1.0	2.5	30.8	0.3	1.5	52.8	2.5
Retail Brokerage	18.1	1.4	6.3	59.5	0.1	0.2	14.4	5.1
Unallocated	6.5	2.8	28.4	28.3	6.5	1.3	26.2	6.0
All	6.1	8.0	6.0	52.4	1.4	1.2	24.9	100.0

Basel III (SA-OR or SMA)

Remark

The standardized measurement approach (SMA) will replace the three approaches of the Basel II framework in 2022. AMA may be used for Pillar 2

The SMA is based on three components:

1. Business indicator (BI)
2. Business indicator component (BIC)
3. Internal loss multiplier (ILM)

Basel III (SA-OR or SMA)

- The business indicator is a proxy of the operational risk:

$$\mathrm{BI} = \mathrm{ILDC} + \mathrm{SC} + \mathrm{FC}$$

where ILDC is the interest, leases and dividends component, SC is the services component and FC is the financial component. The underlying idea is to list the main activities that generate operational risk:

$$\left\{\begin{array}{l} \mathrm{ILDC} = \min\left(\left|\mathrm{INC} - \mathrm{EXP}\right|, 2.25\% \cdot \mathrm{IRE}\right) + \mathrm{DIV} \\ \mathrm{SC} = \max\left(\mathrm{OI}, \mathrm{OE}\right) + \max\left(\mathrm{FI}, \mathrm{FE}\right) \\ \mathrm{FC} = \left|\Pi_{\mathrm{TB}}\right| + \left|\Pi_{\mathrm{BB}}\right| \end{array}\right.$$

where INC represents the interest income, EXP the interest expense, IRE the interest earning assets, DIV the dividend income, OI the other operating income, OE the other operating expense, FI the fee income, FE the fee expense, Π_{TB} the net P&L of the trading book and Π_{BB} the net P&L of the banking book

Basel III (SA-OR or SMA)

- The business indicator component is given by:

$$\mathrm{BIC} = 12\% \cdot \min\left(\mathrm{BI}, \$1 \text{ bn}\right) + 15\% \cdot \min\left(\mathrm{BI} - 1, \$30 \text{ bn}\right) + 18\% \cdot \min\left(\mathrm{BI} - 30\right)^{+}$$

- The internal loss multiplier is equal to:

$$\mathrm{ILM} = \ln\left(e^{1} - 1 + \left(\frac{15 \cdot \bar{L}}{\mathrm{BIC}}\right)^{0.8}\right)$$

where $\bar{L}$ is the average annual operational risk losses over the last 10 years

- The capital charge for the operational risk is then equal to:

$$\mathcal{K} = \mathrm{ILM} \cdot \mathrm{BIC}$$

LDA and operational risk

The operational risk loss L of the bank is divided into a matrix of homogenous losses:

$$L = \sum_{k=1}^{K} S_k$$

where S_k is the sum of losses of the k^{th} cell and K is the number of cells in the matrix (Basel II $= 7 \times 8 = 56$ cells)

Definition

Definition

LDA is a method to model the random loss S_k of a particular cell. It assumes that S_k is the random sum of homogeneous individual losses:

$$S_k = \sum_{n=1}^{N_k(t)} X_n^{(k)}$$

where $N_k(t)$ is the random number of individual losses for the period $[0, t]$ and $X_n^{(k)}$ is the n^{th} individual loss

Two sources of uncertainty:

1. We don't know what will be the magnitude of each loss event (severity risk)
2. We don't know how many losses will occur in the next year (frequency risk)

Assumptions

We consider the random sum:

$$S = \sum_{n=1}^{N(t)} X_n$$

The loss distribution approach is based on the following assumptions:

- The number of losses $N(t)$ follows the loss frequency distribution $\mathbf{P}$
- The sequence of individual losses X_n is independent and identically distributed (iid)
- The corresponding probability distribution $\mathbf{F}$ is called the loss severity distribution
- The number of events is independent from the amount of loss events

The probability distribution $\mathbf{G}$ of S is the compound distribution $(\mathbf{P}, \mathbf{F})$

Exercise I

Exercise

We assume that the number of losses is distributed as follows:

n	0	1	2	3
$p(n)$	50%	30%	17%	3%

The loss amount can take the values \$100 and \$200 with probabilities 70% and 30%

Show that:

s	0	100	200	300	400	500	600
$\Pr\{S = s\}$	50%	21%	17.33%	8.169%	2.853%	0.567%	0.081%

Compound distribution

The cumulative distribution function of S can be written as:

$$\mathbf{G}(s) = \begin{cases} \sum_{n=1}^{\infty} p(n)\,\mathbf{F}^{n\star}(s) & \text{for} \quad s > 0 \\ p(0) & \text{for} \quad s = 0 \end{cases}$$

where $\mathbf{F}^{n\star}$ is the n-fold convolution of $\mathbf{F}$ with itself:

$$\mathbf{F}^{n\star}(s) = \Pr\left\{\sum_{i=1}^{n} X_i \leq s\right\}$$

Compound distribution

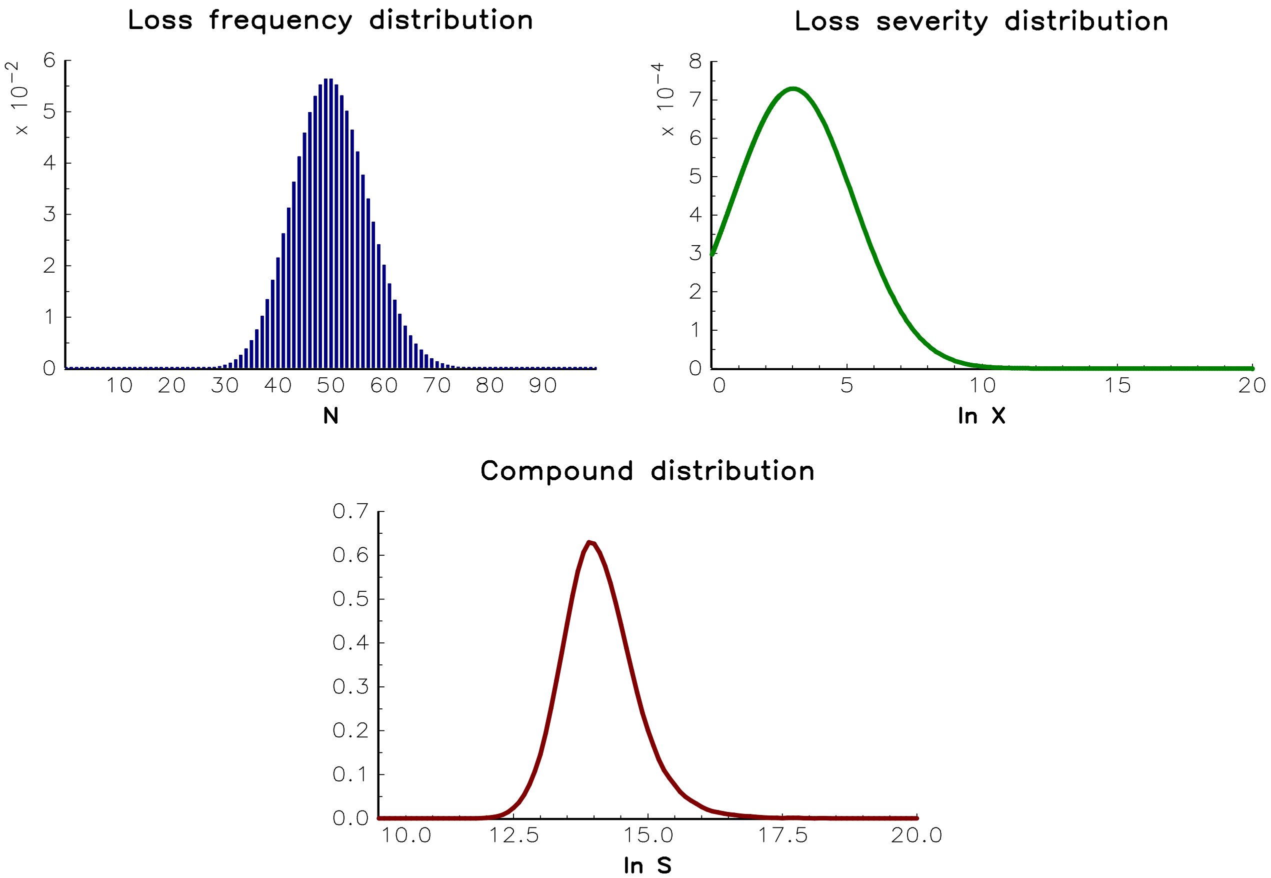


Figure: Compound distribution when $N \sim \mathcal{P}(50)$ and $X \sim \mathcal{LN}(8,5)$

Regulatory capital

The capital charge (or the capital-at-risk) corresponds to the percentile α:

$$\mathrm{CaR}(\alpha) = \mathbf{G}^{-1}(\alpha)$$

The regulatory capital is obtained by setting $\alpha = 99.9\%$:

$$\mathcal{K} = \mathrm{CaR}(99.9\%)$$

Here are the different steps to implement the loss distribution approach:

- for each cell of the operational risk matrix, we estimate the loss frequency distribution and the loss severity distribution
- we calculate the capital-at-risk
- we define the copula function between the different cells of the operational risk matrix, and deduce the aggregate capital-at-risk

Estimation of the loss severity distribution

Let $\{x_1, \ldots, x_T\}$ the sample collected for a given cell of the operational risk matrix. We consider that the individual losses follow a given parametric distribution **F**:

$$X \sim \mathbf{F}(x; \theta)$$

where θ is the vector of parameters

The goal is to estimate θ (and **F**)

Two issues:

- The choice of **F**
- The choice of the estimation method

Some candidates for the loss severity distribution

- Gamma $X \sim \mathcal{G}(\alpha, \beta)$ where $\alpha > 0$ and $\beta > 0$

$$\mathbf{F}(x;\theta) = \frac{\gamma(\alpha, \beta x)}{\Gamma(\alpha)}$$

- Log-gamma $X \sim \mathcal{LG}(\alpha, \beta)$ where $\alpha > 0$ and $\beta > 0$

$$\mathbf{F}(x;\theta) = \frac{\gamma(\alpha, \beta \ln x)}{\Gamma(\alpha)}$$

- Log-logistic $X \sim \mathcal{LL}(\alpha, \beta)$ where $\alpha > 0$ and $\beta > 0$

$$\mathbf{F}(x;\theta) = \frac{1}{1 + (x/\alpha)^{-\beta}} = \frac{x^{\beta}}{\alpha^{\beta} + x^{\beta}}$$

- Log-normal $X \sim \mathcal{LN}(\mu, \sigma^2)$ where $x > 0$ and $\sigma > 0$

$$\mathbf{F}(x;\theta) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$$

- Generalized extreme value $X \sim \mathcal{GEV}(\mu, \sigma, \xi)$ where $x > \mu - \sigma/\xi$, $\sigma > 0$ and $\xi > 0$

$$\mathbf{F}(x;\theta) = \exp\left\{-\left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/\xi}\right\}$$

- Pareto $X \sim \mathcal{P}(\alpha, x_{-})$ where $x \geq x_{-}$, $\alpha > 0$ and $x_{-} > 0$

$$\mathbf{F}(x;\theta) = 1 - \left({}^{x}/_{x_{-}}\right)^{-\alpha}$$

Some candidates for the loss severity distribution

Table: Density function, mean and variance of parametric probability distribution

Distribution	$f(x;\theta)$	$\mathbb{E}[X]$	$\operatorname{var}(X)$
$\mathcal{G}(\alpha,\beta)$	$\dfrac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$	$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^2}$
$\mathcal{LG}(\alpha,\beta)$	$\dfrac{\beta^\alpha (\ln x)^{\alpha-1}}{x^{\beta+1}\Gamma(\alpha)}$	$\left(\dfrac{\beta}{\beta-1}\right)^\alpha$ if $\beta>1$	$\left(\dfrac{\beta}{\beta-2}\right)^\alpha - \left(\dfrac{\beta}{\beta-1}\right)^{2\alpha}$ if $\beta>2$
$\mathcal{LL}(\alpha,\beta)$	$\dfrac{\beta(x/\alpha)^{\beta-1}}{\alpha\left(1+(x/\alpha)^\beta\right)^2}$	$\dfrac{\alpha\pi}{\beta\sin(\pi/\beta)}$ if $\beta>1$	$\alpha^2\left(\dfrac{2\pi}{\beta\sin(2\pi/\beta)} - \dfrac{\pi^2}{\beta^2\sin^2(\pi/\beta)}\right)$ if $\beta>2$
$\mathcal{LN}(\mu,\sigma^2)$	$\dfrac{1}{x\sigma\sqrt{2\pi}}\exp\left(-\dfrac{1}{2}\left(\dfrac{\ln x-\mu}{\sigma}\right)^2\right)$	$\exp\left(\mu+\dfrac{1}{2}\sigma^2\right)$	$\exp\left(2\mu+\sigma^2\right)\left(\exp\left(\sigma^2\right)-1\right)$
$\mathcal{GEV}(\mu,\sigma,\xi)$	$\dfrac{1}{\sigma}\left[1+\xi\left(\dfrac{x-\mu}{\sigma}\right)\right]^{-(1+1/\xi)} \exp\left\{-\left[1+\xi\left(\dfrac{x-\mu}{\sigma}\right)\right]^{-1/\xi}\right\}$	$\mu+\dfrac{\sigma}{\xi}\left(\Gamma(1-\xi)-1\right)$ if $\xi<1$	$\dfrac{\sigma^2}{\xi^2}\left(\Gamma(1-2\xi)-\Gamma^2(1-\xi)\right)$ if $\xi<\frac{1}{2}$
$\mathcal{P}(\alpha,x_-)$	$\dfrac{\alpha x_-^\alpha}{x^{\alpha+1}}$	$\dfrac{\alpha x_-}{\alpha-1}$ if $\alpha>1$	$\dfrac{\alpha x_-^2}{(\alpha-1)^2(\alpha-2)}$ if $\alpha>2$

Estimation methods

Method of maximum likelihood (HFRM, Section 10.1.2, page 614)

The log-likelihood function associated to the sample is:

$$\ell(\theta) = \sum_{i=1}^{T} \ln f(x_i; \theta)$$

where $f(x; \theta)$ is the density function

Generalized method of moments (HFRM, Section 10.1.3, page 628)

The empirical moments are:

$$\begin{cases} h_{i,1}(\theta) = x_i - \mathbb{E}[X] \\ h_{i,2}(\theta) = (x_i - \mathbb{E}[X])^2 - \mathrm{var}(X) \end{cases}$$

Estimation methods

If we consider that $X \sim \mathcal{LN}\left(\mu, \sigma^2\right)$, the log-likelihood function is:

$$\ell(\theta) = -\sum_{i=1}^{T} \ln x_i - \frac{T}{2} \ln \sigma^2 - \frac{T}{2} \ln 2\pi - \frac{1}{2} \sum_{i=1}^{T} \left(\frac{\ln x_i - \mu}{\sigma} \right)^2$$

whereas the empirical moments are:

$$\left\{ \begin{array}{l} h_{i,1}(\theta) = x_i - e^{\mu + \frac{1}{2}\sigma^2} \\ h_{i,2}(\theta) = \left(x_i - e^{\mu + \frac{1}{2}\sigma^2} \right)^2 - e^{2\mu + \sigma^2} \left(e^{\sigma^2} - 1 \right) \end{array} \right.$$

Estimation methods

Example

We assume that the individual losses take the following values expressed in thousand dollars: 10.1, 12.5, 14, 25, 317.3, 353, 1 200, 1 254, 52 000 and 251 000

We find that:

- $\hat{\alpha}_{\text{ML}} = 15.70$ and $\hat{\beta}_{\text{ML}} = 1.22$ for the log-gamma distribution
- $\hat{\alpha}_{\text{ML}} = 293\,721$ and $\hat{\beta}_{\text{ML}} = 0.51$ for the log-logistic distribution
- $\hat{\mu}_{\text{ML}} = 12.89$ and $\hat{\sigma}_{\text{ML}} = 3.35$ for the log-normal distribution
- $\hat{\mu}_{\text{GMM}} = 16.26$ and $\hat{\sigma}_{\text{GMM}} = 1.40$ for the log-normal distribution

An important bias

The truncation process of loss data collection

- Data are recorded only when their amounts are higher than some thresholds
- Loss thresholds vary across banks, time, business lines, etc.

"A bank must have an appropriate de minimis gross loss threshold for internal loss data collection, for example €10 000. The appropriate threshold may vary somewhat between banks, and within a bank across business lines and/or event types. However, particular thresholds should be broadly consistent with those used by peer banks" (BCBS, 2006, page 153)

Operational risk loss data

Remark

- *Operational risk loss data cannot be reduced to the sample of individual losses, but also requires specifying the threshold H_i for each individual loss x_i*
- *The form of operational loss data is then $\{(x_i, H_i), i = 1, \ldots, T\}$, where x_i is the observed value of X knowing that X is larger than the threshold H_i*

From a statistical point of view, we have:

- The true distribution is the probability distribution of X
- The sample distribution is the probability distribution of $X \mid X \geq H_i$

Dealing with loss thresholds

Analytics of the sample distribution

The sample distribution is equal to:

$$
\begin{aligned}
\mathbf{F}^{\star}(x;\theta \mid H) &= \Pr\{X \leq x \mid X \geq H\} \\
&= \frac{\Pr\{X \leq x, X \geq H\}}{\Pr\{X \geq H\}} \\
&= \frac{\Pr\{X \leq x\} - \Pr\{X \leq \min(x, H)\}}{\Pr\{X \geq H\}} \\
&= \mathbb{1}\{x \geq H\} \cdot \frac{\mathbf{F}(x;\theta) - \mathbf{F}(H;\theta)}{1 - \mathbf{F}(H;\theta)}
\end{aligned}
$$

It follows that the density function is:

$$
f^{\star}(x;\theta \mid H) = \mathbb{1}\{x \geq H\} \cdot \frac{f(x;\theta)}{1 - \mathbf{F}(H;\theta)}
$$

Dealing with loss thresholds

Analytics of the sample distribution

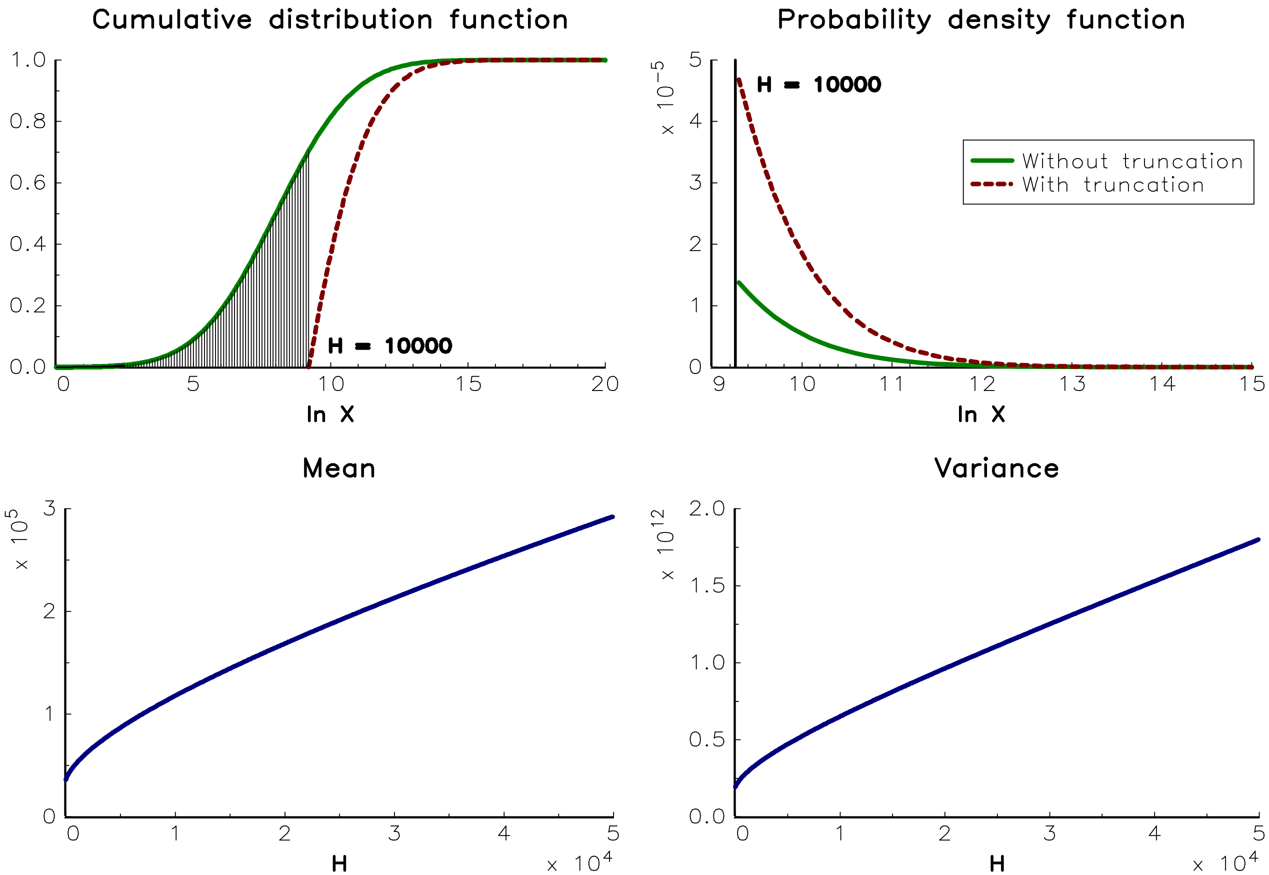


Figure: Impact of the loss threshold H on the sample distribution ($X \sim \mathcal{LN}(8,5)$)

Dealing with loss thresholds

Application to the method of maximum likelihood

We have:

$$\begin{aligned}\ell(\theta) &= \sum_{i=1}^{T} \ln f^{\star}(x_i;\theta \mid H_i) \\ &= \sum_{i=1}^{T} \ln f(x_i;\theta) + \sum_{i=1}^{T} \ln \mathbb{1}\{x_i \geq H_i\} - \sum_{i=1}^{T} \ln(1 - \mathbf{F}(H_i;\theta))\end{aligned}$$

where H_i is the threshold associated to the i^{th} observation

The correction term is $-\sum_{i=1}^{T} \ln(1 - \mathbf{F}(H_i;\theta))$

Dealing with loss thresholds

Application to the method of maximum likelihood

In the case of the log-normal model, the log likelihood function is:

$$\ell(\theta) = -\frac{T}{2}\ln 2\pi - \frac{T}{2}\ln\sigma^2 - \sum_{i=1}^{T}\ln x_i - \frac{1}{2}\sum_{i=1}^{T}\left(\frac{\ln x_i - \mu}{\sigma}\right)^2 - \sum_{i=1}^{T}\ln\left(1 - \Phi\left(\frac{\ln H_i - \mu}{\sigma}\right)\right)$$

Dealing with loss thresholds

Application to the generalized method of moments

The empirical moments become:

$$\left\{ \begin{array}{l} h_{i,1}(\theta) = x_i - \mathbb{E}\left[X \mid X \geq H_i\right] \\ h_{i,2}(\theta) = \left(x_i - \mathbb{E}\left[X \mid X \geq H_i\right]\right)^2 - \operatorname{var}\left(X \mid X \geq H_i\right) \end{array} \right.$$

There is no reason that the conditional moment $\mathbb{E}\left[X^m \mid X \geq H_i\right]$ is equal to the unconditional moment $\mathbb{E}\left[X^m\right]$

Dealing with loss thresholds

Application to the generalized method of moments

In the case of the log-normal model, the empirical moments are:

$$\left\{\begin{array}{l} h_{i,1}(\theta) = x_i - a_1(\theta, H_i)\, e^{\mu + \frac{1}{2}\sigma^2} \\ h_{i,2}(\theta) = x_i^2 - 2x_i a_1(\theta, H_i)\, e^{\mu + \frac{1}{2}\sigma^2} + 2a_1^2(\theta, H_i)\, e^{2\mu + \sigma^2} - a_2(\theta, H_i)\, e^{2\mu + 2\sigma^2} \end{array}\right.$$

where:

$$a_k(\theta, H) = \frac{1 - \Phi\left(\frac{\ln H - \mu - k\sigma^2}{\sigma}\right)}{1 - \Phi\left(\frac{\ln H - \mu}{\sigma}\right)}$$

Dealing with loss thresholds

Illustration

Example

We assume that the individual losses take the following values expressed in thousand dollars: 10.1, 12.5, 14, 25, 317.3, 353, 1 200, 1 254, 52 000 and 251 000

The ML estimates are $\hat{\mu}_{\mathrm{ML}} = 12.89$ and $\hat{\sigma}_{\mathrm{ML}} = 3.35$ for the log-normal distribution

Example

The previous losses have been collected using a unique threshold that is equal to \$5 000

The ML estimates become $\hat{\mu}_{\mathrm{ML}} = 8.00$ and $\hat{\sigma}_{\mathrm{ML}} = 5.71$ for the log-normal distribution

Dealing with loss thresholds

Illustration

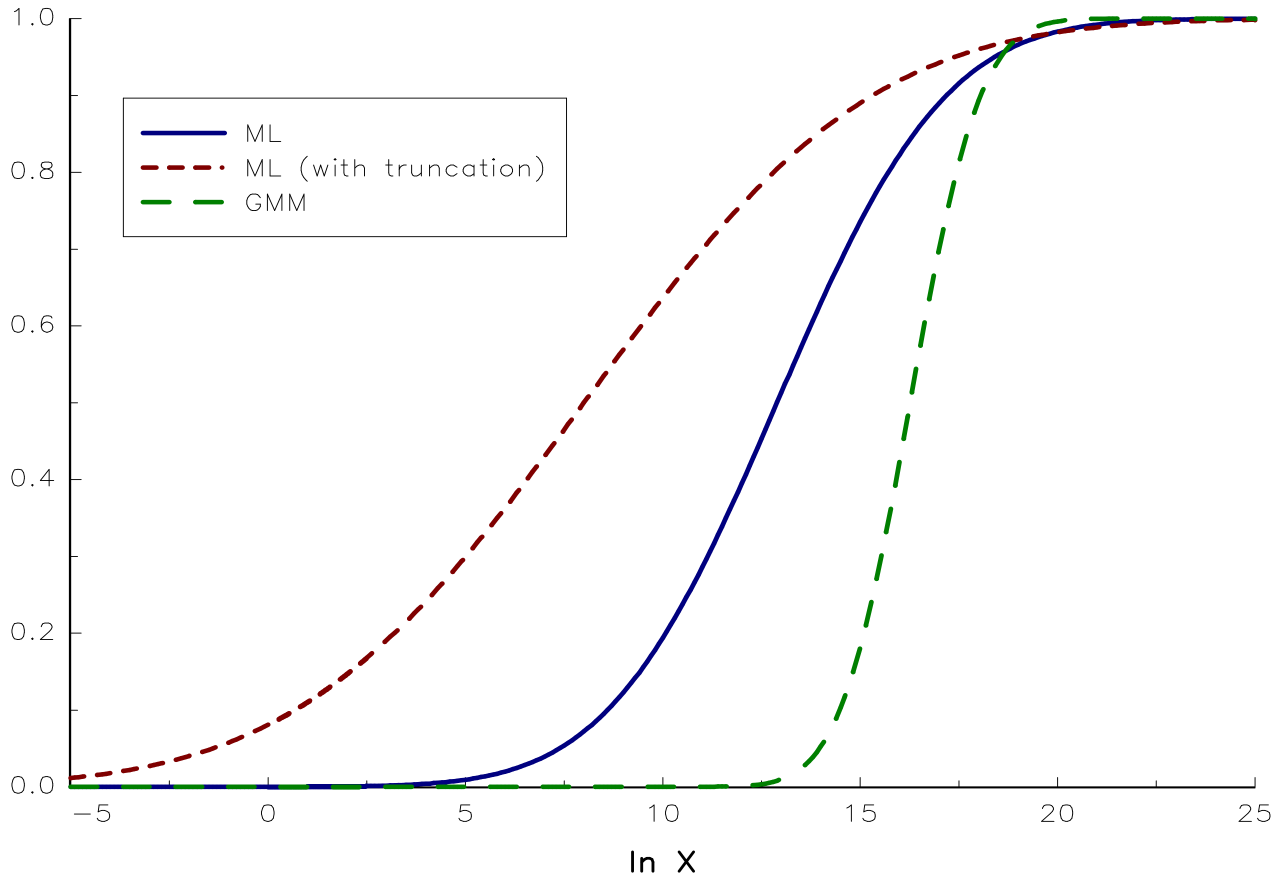


Figure: Comparison of the estimated severity distributions

Choice of the severity distribution

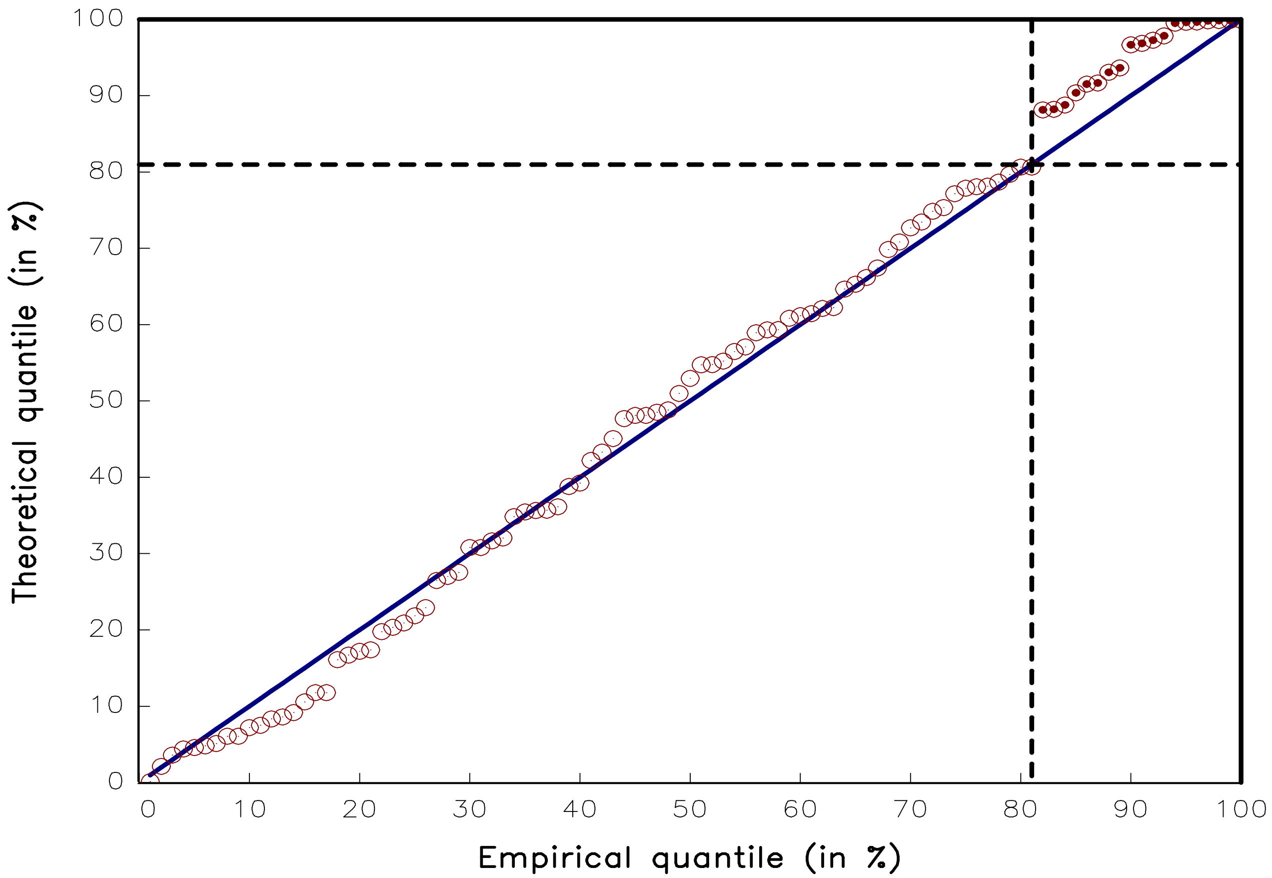


Figure: An example of QQ plot where extreme events are underestimated

Estimation of the loss frequency distribution

The goal is now to estimate $\mathbf{P}$

Counting process

Let $N(t)$ be the number of losses occurring during the time period $[0, t]$. The number of losses for the time period $[t_1, t_2]$ is then equal to:

$$N(t_1; t_2) = N(t_2) - N(t_1)$$

We generally made the following statements about the process $N(t)$:

- The distribution of the number of losses $N(t; t+h)$ for each $h > 0$ is independent of t; moreover, $N(t; t+h)$ is stationary and depends only on the time interval h
- The random variables $N(t_1; t_2)$ and $N(t_3; t_4)$ are independent if the time intervals $[t_1, t_2]$ and $[t_3, t_4]$ are disjoint
- No more than one loss may occur at time t

Poisson process

These simple assumptions define a Poisson process:

1. There exists a scalar $\lambda > 0$ such that the distribution of $N(t)$ has a Poisson distribution with parameter λt
2. The duration between two successive losses is *iid* and follows the exponential distribution distribution $\mathcal{E}(\lambda)$
3. The probability mass function of the Poisson process is:

$$p(n) = \Pr\{N(t) = n\} = \frac{e^{-\lambda t} \cdot (\lambda t)^n}{n!}$$

Remark

If $N_1 \sim \mathcal{P}(\lambda_1)$ *and* $N_2 \sim \mathcal{P}(\lambda_2)$, *then* $N_1 + N_2 \sim \mathbf{P}(\lambda_1 + \lambda_2)$. *We deduce that:*

$$\sum_{k=1}^{K} N\left(\frac{k-1}{K}; \frac{k}{K}\right) = N(1)$$

where $N((k-1)/K; k/K) \sim \mathcal{P}(\lambda/K)$

Estimation of λ

The ML estimator is the mean of the annual number of losses:

$$\hat{\lambda}_{\mathrm{ML}} = \frac{1}{n_y} \sum_{y=1}^{n_y} N_y$$

where N_y is the number of losses occurring at year y

Since we have $\lambda = \mathbb{E}\left[N(1)\right] = \mathrm{var}\left(N(1)\right)$, the MM estimator based on the first moment is equal to:

$$\hat{\lambda}_{\mathrm{MM}} = \hat{\lambda}_{\mathrm{ML}} = \frac{1}{n_y} \sum_{y=1}^{n_y} N_y$$

whereas the MM estimator based on the first moment is equal to:

$$\hat{\lambda}_{\mathrm{MM}} = \frac{1}{n_y} \sum_{y=1}^{n_y} \left(N_y - \bar{N}\right)^2$$

where $\bar{N}$ is the average number of losses

Estimation of λ

Example

The annual number of losses from 2006 to 2015 is the following: 57, 62, 45, 24, 82, 36, 98, 75, 76 and 45. The mean is equal to 60 whereas the variance is equal to 474.40

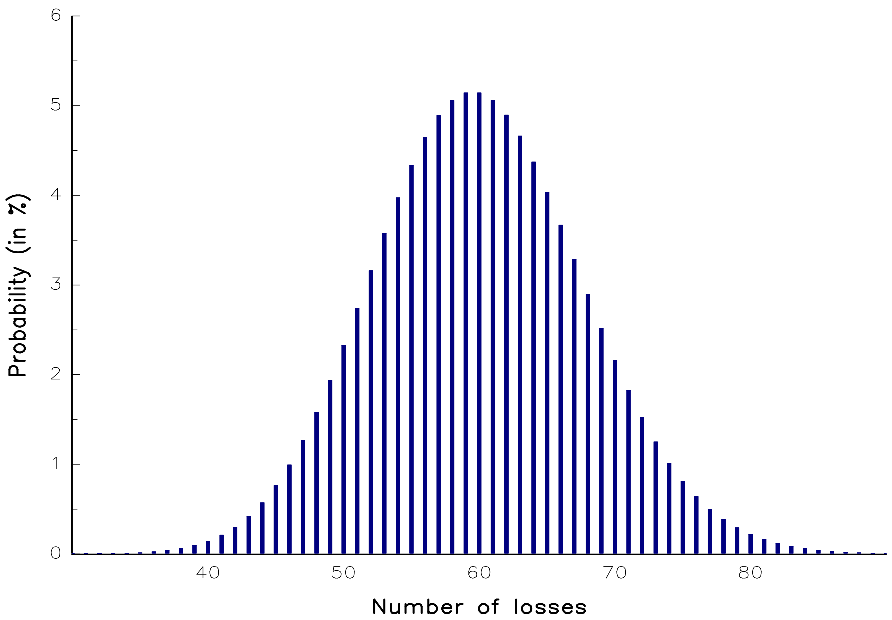


Figure: PMF of the Poisson distribution $\mathcal{P}(60)$

Not possible to observe an annual number of losses equal to 24 and 98!

Negative binomial distribution

When the variance exceeds the mean, we use the negative binomial distribution $\mathcal{NB}(r,p)$:

$$p(n) = \binom{r+n-1}{n}(1-p)^r p^n = \frac{\Gamma(r+n)}{n!\,\Gamma(r)}(1-p)^r p^n$$

where $r > 0$ and $p \in [0,1]$. We have:

$$\mathbb{E}[\mathcal{NB}(r,p)] = \frac{p \cdot r}{1-p}$$

and:

$$\operatorname{var}(\mathcal{NB}(r,p)) = \frac{p \cdot r}{(1-p)^2}$$

Remark

We verify that:

$$\operatorname{var}(\mathcal{NB}(r,p)) = \frac{1}{1-p} \cdot \mathbb{E}[\mathcal{NB}(r,p)] > \mathbb{E}[\mathcal{NB}(r,p)]$$

Negative binomial distribution and Poisson process

The negative binomial distribution corresponds to a Poisson process where the intensity parameter is random and follows a gamma distribution:

$$\begin{aligned} \mathcal{NB}(r, p) &\sim \mathcal{P}(\Lambda) \\ \Lambda &\sim \mathcal{G}(\alpha, \beta) \end{aligned}$$

where $\alpha = r$ and $\beta = (1 - p) / p$

$\Rightarrow$ See HFRM, Exercise 5.4.6, page 346 and HFRM-CB, Section 5.4.6, pages 113-116

Application to the example

- Using the previous example, we obtain:

$$\hat{r}_{\mathrm{MM}} = \frac{m^2}{v - m} = \frac{60^2}{474.40 - 60} = 8.6873$$

and

$$\hat{p}_{\mathrm{MM}} = \frac{v - m}{v} = \frac{474.40 - 60}{474.40} = 0.8735$$

where m is the mean and v is the variance of the sample

- If we use the method of maximum likelihood, we obtain $\hat{r}_{\mathrm{ML}} = 7.7788$ and $\hat{p}_{\mathrm{ML}} = 0.8852$
- We deduce that:

$$\mathcal{NB}\left(\hat{r}_{\mathrm{ML}}, \hat{p}_{\mathrm{ML}}\right) \sim \mathcal{P}\left(\Lambda\right)$$

where:

$$\Lambda \sim \mathcal{G}\left(7.7788, 0.1296\right)$$

- $\mathcal{P}\left(60\right)$ and $\mathcal{NB}\left(\hat{r}_{\mathrm{ML}}, \hat{p}_{\mathrm{ML}}\right)$ have the same mean, but not the same variance

Application to the example

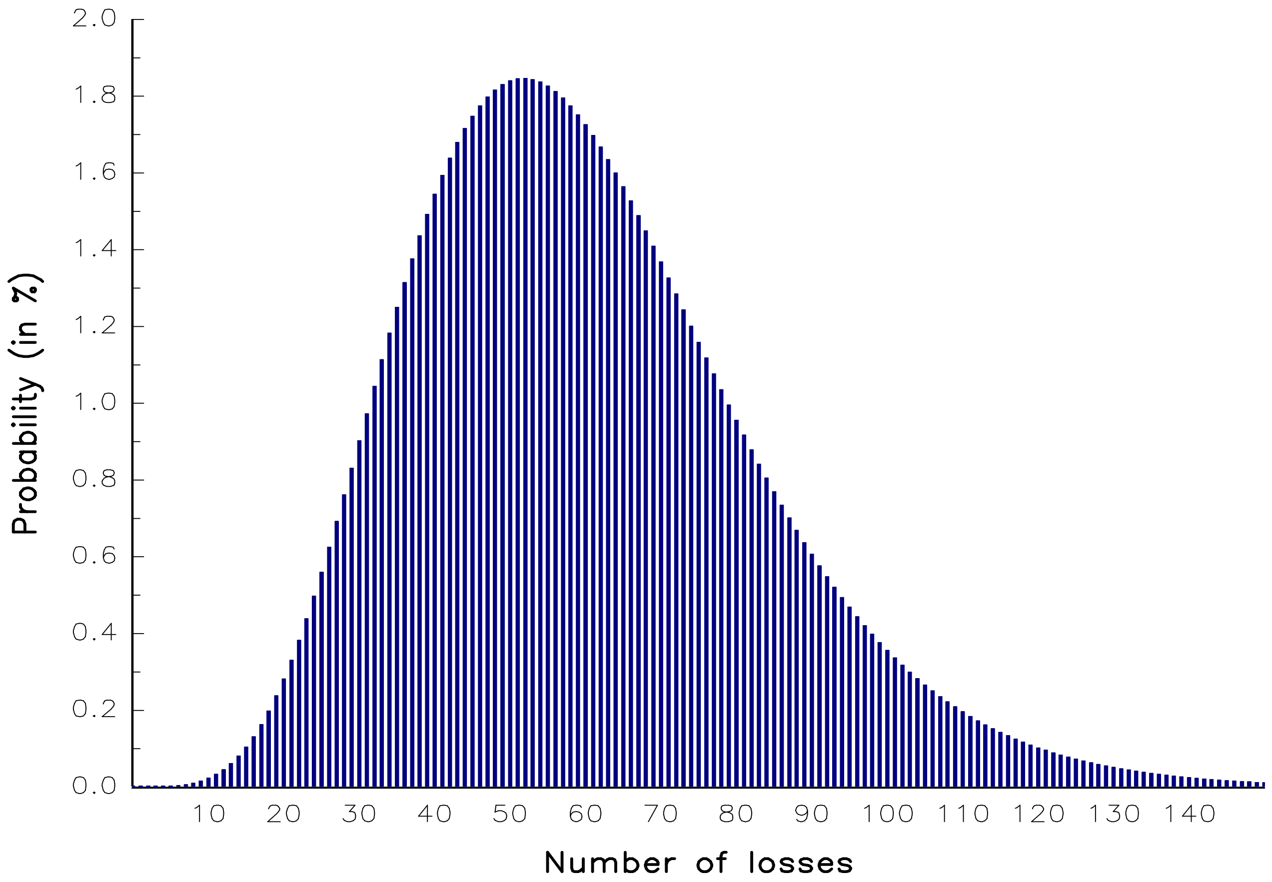


Figure: PMF of the negative binomial distribution

Application to the example

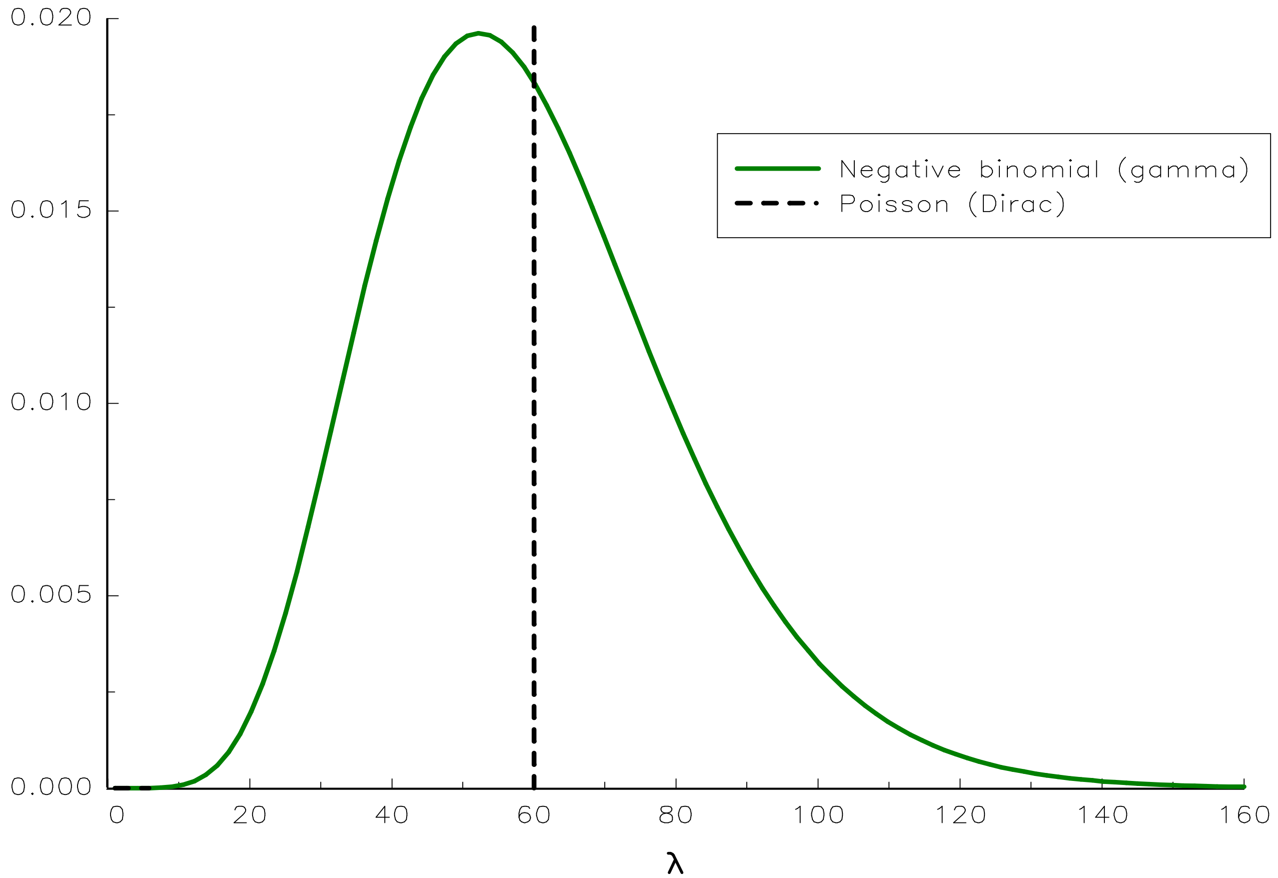


Figure: Probability density function of the parameter λ

Dealing with a loss threshold

- The loss threshold has an impact on the sample frequency distribution
- For instance, if the threshold H is set at a high level, then the average number of reported losses is low
- Let $N_H(t)$ be the number of events that are larger than the threshold H:

$$N_H(t) = \sum_{i=1}^{N(t)} \mathbb{1}\{X_i > H\}$$

- We can show that (HFRM, page 326):

$$\mathbb{E}[N_H(t)] = \mathbb{E}[N(t)] \cdot \Pr\{X_i > H\} = \mathbb{E}[N(t)] \cdot (1 - \mathbf{F}(H; \theta))$$

Dealing with a loss threshold

In the case of the Poisson distribution, we also prove that:

$$\mathbf{P}_H(\lambda) = \mathbf{P}(\lambda_H)$$

We deduce that the estimator $\hat{\lambda}$ has the following expression:

$$\hat{\lambda} = \frac{\hat{\lambda}_H}{1 - \mathbf{F}\left(H; \hat{\theta}\right)}$$

where:

- $\hat{\lambda}_H$ is the average number of losses that are collected above the threshold H
- $\mathbf{F}\left(x; \hat{\theta}\right)$ is the parametric estimate of the severity distribution.

Remark

This approach is only valid if the loss threshold is unique

Dealing with a loss threshold

Example

We consider that the bank has collected the loss data from 2006 to 2015 with a threshold of \$20 000. For a given event type, the calibrated severity distribution corresponds to a log-normal distribution with parameters $\hat{\mu} = 7.3$ and $\hat{\sigma} = 2.1$, whereas the annual number of losses is the following: 23, 13, 50, 12, 25, 36, 48, 27, 18 and 35

Using the Poisson distribution, we obtain $\hat{\lambda}_H = 28.70$. The probability that the loss exceeds the threshold H is equal to:

$$\Pr\{X > 20\,000\} = 1 - \Phi\left(\frac{\ln(20\,000) - 7.3}{2.1}\right) = 10.75\%$$

This means that only 10.75% of losses can be observed when we apply a threshold of \$20 000. We deduce that the estimate of the Poisson parameter is equal to:

$$\hat{\lambda} = \frac{28.70}{10.75\%} = 266.90$$

Calculating the capital charge

Several approaches:

- Monte Carlo approach
- Method of characteristic functions
- Panjer recursive approach
- Single loss approximation

Monte Carlo approach

Algorithm

Compute the capital-at-risk for an operational risk cell

Initialize the number of simulations n_S
for $j = 1 : n_S$ **do**
  Simulate an annual number n of losses from the frequency distribution $\mathbf{P}$
  $S_j \leftarrow 0$
  for $i = 1 : n$ **do**
    Simulate a loss X_i from the severity distribution $\mathbf{F}$
    $S_j = S_j + X_i$
  end for
end for
Calculate the order statistics $S_{1:n_S}, \ldots, S_{n_S:n_S}$
Deduce the capital-at-risk $\mathrm{CaR} = S_{\alpha n_S : n_S}$ with $\alpha = 99.9\%$
return CaR

Monte Carlo approach

Illustration

- We assume that $N(1) \sim \mathcal{P}(4)$ and $X_i \sim \mathcal{LN}(8, 4)$
- The simulated values of $N(1)$ are 3, 4, 1, 2, 3, etc.
- The simulated values of X_i are 3388.6, 259.8, 13328.3, 39.7, 1220.8, 1486.4, 15197.1, 3205.3, 5070.4, 84704.1, 64.9, 1237.5, 187073.6, 4757.8, 50.3, 2805.7, etc.

For the first simulation, we have three losses and we obtain:

$$S_1 = 3388.6 + 259.8 + 13328.3 = \$16\,976.7$$

For the second simulation, the number of losses is equal to four and the compound loss is equal to:

$$S_2 = 39.7 + 1220.8 + 1486.4 + 15197.1 = \$17\,944.0$$

For the third simulation, we obtain:

$$S_3 = \$3\,205.3$$

Monte Carlo approach

The Monte Carlo method is powerful and the most used approach for computing the capital charge for operational risk

But be careful about the convergence!

Panjer recursion

Theorem

Panjer (1981) showed that if the pmf of the counting process $N(t)$ satisfies:

$$p(n) = \left(a + \frac{b}{n}\right) p(n-1)$$

where a and b are two scalars, then the following recursion holds:

$$g(x) = p(1) f(x) + \int_0^x \left(a + b\frac{y}{x}\right) f(y) g(x-y) \, \mathrm{d}y$$

where $x > 0$

Panjer recursion

For discrete severity distributions satisfying $f_n = \Pr\{X_i = n\delta\}$ where δ is the monetary unit (e.g. \$10 000), the Panjer recursion becomes:

$$g_n = \Pr\{S = n\delta\} = \frac{1}{1 - af_0} \sum_{j=1}^{n} \left(a + \frac{bj}{n}\right) f_j g_{n-j}$$

where:

$$g_0 = \sum_{n=0}^{\infty} p(n)(f_0)^n = \begin{cases} p(0)\, e^{bf_0} & \text{if } a = 0 \\ p(0)(1 - af_0)^{-1-b/a} & \text{otherwise} \end{cases}$$

The capital-at-risk is then equal to:

$$\mathrm{CaR}(\alpha) = n^\star \delta$$

where:

$$n^\star = \inf \left\{ n : \sum_{j=0}^{n} g_j \geq \alpha \right\}$$

Panjer recursion

Example

We consider the compound Poisson distribution with log-normal losses and different sets of parameters:

(a) $\lambda = 5$, $\mu = 5$, $\sigma = 1.0$

(b) $\lambda = 5$, $\mu = 5$, $\sigma = 1.5$

(c) $\lambda = 5$, $\mu = 5$, $\sigma = 2.0$

(d) $\lambda = 50$, $\mu = 5$, $\sigma = 2.0$

We perform a discretization of the severity distribution:

$$f_n = \Pr\left\{n\delta - \frac{\delta}{2} \leq X_i \leq n\delta + \frac{\delta}{2}\right\} = \mathbf{F}\left(n\delta + \frac{\delta}{2}\right) - \mathbf{F}\left(n\delta - \frac{\delta}{2}\right)$$

with the convention $f_0 = \mathbf{F}(\delta/2)$

Panjer recursion

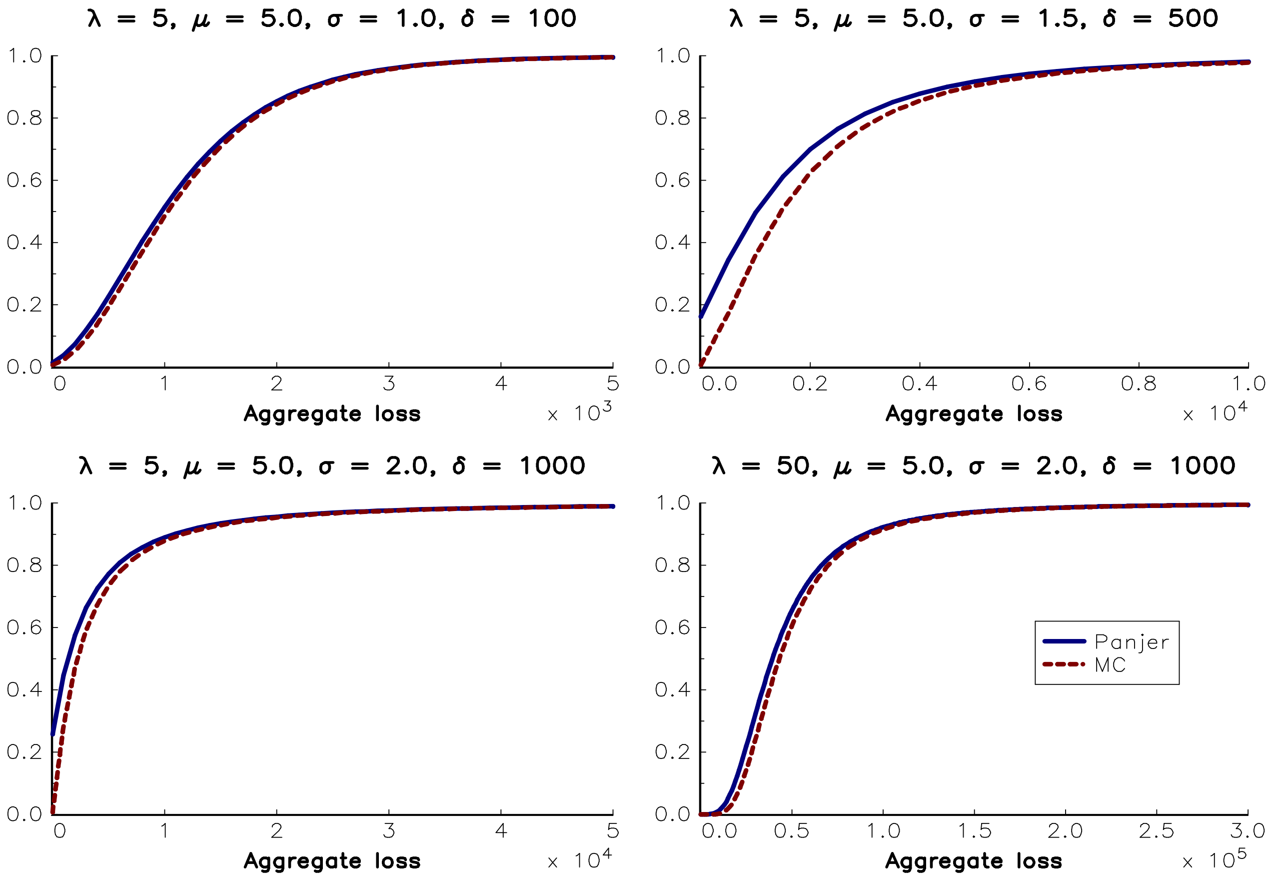


Figure: Comparison between the Panjer and MC compound distributions

Panjer recursion

Table: Comparison of the capital-at-risk calculated with Panjer recursion and Monte Carlo simulations

α	Panjer recursion				Monte Carlo simulations			
	(a)	(b)	(c)	(d)	(a)	(b)	(c)	(d)
90%	2400	4500	11000	91000	2350	4908	11648	93677
95%	2900	6500	19000	120000	2896	6913	19063	123569
99%	4300	13500	52000	231000	4274	13711	51908	233567
99.5%	4900	18000	77000	308000	4958	17844	77754	310172
99.9%	6800	32500	182000	604000	6773	32574	185950	604756

Single loss approximation

If the severity belongs to the family of subexponential distributions (HFRM, pages 333-336), Böcker and Klüppelberg (2005) showed that:

$$\begin{aligned}
\mathbf{G}^{-1}(\alpha) &= \mathrm{EL} + \mathrm{UL}(\alpha) \\
&\approx \mathbb{E}[N(1)] \cdot \mathbb{E}[X_i] + \mathbf{F}^{-1}\left(1 - \frac{1-\alpha}{N(1)}\right) - \mathbb{E}[X_i] \\
&\approx (\mathbb{E}[N(1)] - 1) \cdot \mathbb{E}[X_i] + \mathbf{F}^{-1}\left(1 - \frac{1-\alpha}{\mathbb{E}[N(1)]}\right)
\end{aligned}$$

If $N(1) \sim \mathcal{P}(\lambda)$ and $X_i \sim \mathcal{LN}\left(\mu, \sigma^2\right)$, we obtain:

$$\mathrm{EL} = \lambda \exp\left(\mu + \frac{1}{2}\sigma^2\right)$$

and:

$$\mathrm{UL}(\alpha) \approx \exp\left(\mu + \sigma\Phi^{-1}\left(1 - \frac{1-\alpha}{\lambda}\right)\right) - \exp\left(\mu + \frac{1}{2}\sigma^2\right)$$

Single loss approximation

Remark

A better approximation of the capital-at-risk is:

$$\mathbf{G}^{-1}(\alpha) \approx \left(\mathbf{P}^{-1}(\alpha) - 1\right) \mathbb{E}[X_i] + \mathbf{F}^{-1}\left(1 - \frac{1-\alpha}{\mathbb{E}[N(1)]}\right)$$

where $\mathbf{P}$ *is the cumulative distribution function of the counting process* $N(1)$

How to compute the total capital charge?

The operational risk loss L of the bank is divided into a matrix of homogenous losses:

$$L = \sum_{k=1}^{K} S_k$$

where S_k is the sum of losses of the k^{th} cell and K is the number of cells in the matrix (Basel II $= 7 \times 8 = 56$ cells)

Using LDA, we know how to compute S_k. **But how to compute the total loss L?**

The solution is given by the copula approach

It only works with the Monte Carlo approach and uses the method of the empirical quantile function (HFRM, Section 13.1.3.2, pages 805-808)

Probability distribution of a given scenario

We assume that $N(t) \sim \mathcal{P}(\lambda)$. Let τ_n be the arrival time of the n^{th} loss:

$$\tau_n = \inf\{t \geq 0 : N(t) = n\}$$

- We know that $T_n = \tau_n - \tau_{n-1} \sim \mathcal{E}(\lambda)$
- We recall that the losses $X_n \sim \mathbf{F}$
- We note $T_n(x)$ the duration between two losses exceeding x
- We have $T_n(x) \equiv T_1(x)$

Theorem

We have:

$$T_n(x) \sim \mathcal{E}(\lambda(1 - \mathbf{F}(x)))$$

and:

$$\mathbb{E}[T_n(x)] = \frac{1}{\lambda(1 - \mathbf{F}(x))}$$

Probability distribution of a given scenario

Proof

By using the fact that a finite sum of exponential times is an Erlang distribution, we have:

$$\begin{aligned}
\Pr\left\{T_1(x) > t\right\} &= \sum_{n\geq 1} \Pr\left\{\tau_n > t; X_1 < x, \ldots, X_{n-1} < x; X_n \geq x\right\} \\
&= \sum_{n\geq 1} \Pr\left\{\tau_n > t\right\} \cdot \mathbf{F}(x)^{n-1} \cdot (1 - \mathbf{F}(x)) \\
&= \sum_{n\geq 1} \mathbf{F}(x)^{n-1} \cdot (1 - \mathbf{F}(x)) \cdot \left(\sum_{k=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!}\right) \\
&= (1 - \mathbf{F}(x)) \cdot \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \left(\sum_{n=k}^{\infty} \mathbf{F}(x)^n\right) \\
&= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \mathbf{F}(x)^k \\
&= e^{-\lambda (1-\mathbf{F}(x)) t}
\end{aligned}$$

Calibration of a set of scenarios

- We define a scenario as "a loss of x or higher occurs once every d years"
- We assume that the severity distribution is $\mathbf{F}(x;\theta)$ and the frequency distribution is $\mathcal{P}(\lambda)$
- Suppose that we face different scenarios $\{(x_s, d_s), s = 1, \ldots, n_S\}$. We may estimate the implied parameters underlying the expert judgements using the method of moments:

$$\left(\hat{\lambda}_{\mathrm{MM}}, \hat{\theta}_{\mathrm{MM}}\right) = \arg\min \sum_{s=1}^{n_S} w_s \cdot \left(d_s - \frac{1}{\lambda \left(1 - \mathbf{F}(x_s;\theta)\right)}\right)^2$$

  where w_s is the weight of the s^{th} scenario
- We can show that the optimal weights w_s correspond to the inverse of the variance of d_s:

$$w_s = \frac{1}{\mathrm{var}(d_s)} = \lambda \left(1 - \mathbf{F}(x_s;\theta)\right)$$

Calibration of a set of scenarios

Numerical solution

To solve the previous optimization program, we proceed by iterations:

- Let $\left(\hat{\lambda}_m, \hat{\theta}_m\right)$ be the solution of the minimization program:

$$\left(\hat{\lambda}_m, \hat{\theta}_m\right) = \arg\min \sum_{j=1}^{p} \hat{\lambda}_{m-1} \cdot \left(1 - \mathbf{F}\left(x_s; \hat{\theta}_{m-1}\right)\right) \cdot \left(d_s - \frac{1}{\lambda\left(1 - \mathbf{F}\left(x_s; \theta\right)\right)}\right)^2$$

- Under some conditions, the estimator $\left(\hat{\lambda}_m, \hat{\theta}_m\right)$ converge to the optimal solution
- We can simplify the optimization program by using the following approximation:

$$w_s = \frac{1}{\operatorname{var}\left(d_s\right)} = \frac{1}{\mathbb{E}\left[d_s\right]} \simeq \frac{1}{d_s}$$

Calibration of a set of scenarios

Example

We assume that the severity distribution is log-normal and consider the following set of expert's scenarios:

x_s (in \$ mn)	1	2.5	5	7.5	10	20
d_s (in years)	1/4	1	3	6	10	40

Calibration of a set of scenarios

#1 If $w_s = 1$, we obtain $\hat{\lambda} = 43.400$, $\hat{\mu} = 11.389$ and $\hat{\sigma} = 1.668$

#2 Using the approximation $w_s \simeq 1/d_s$, the estimates become $\hat{\lambda} = 154.988$, $\hat{\mu} = 10.141$ and $\hat{\sigma} = 1.855$

#3 The optimal estimates are $\hat{\lambda} = 148.756$, $\hat{\mu} = 10.181$ and $\hat{\sigma} = 1.849$

Here are the estimated values of the duration:

x_s (in \$ mn)	1	2.5	5	7.5	10	20
#1	0.316	1.022	2.964	5.941	10.054	39.997
#2	0.271	0.968	2.939	5.973	10.149	39.943
#3	0.272	0.970	2.941	5.974	10.149	39.944

Exercises

- Severity distribution
  - Exercise 5.4.1 – Estimating the loss severity distribution
  - Exercise 5.4.5 – Parametric estimation of the loss severity distribution
- Frequency distribution
  - Exercise 5.4.2 – Estimation of the loss frequency distribution
- Other topics
  - Exercise 5.4.3 – Using the method of moments in operational risk models
  - Exercise 5.4.6 – Mixed Poisson process

References

- Basel Committee on Banking Supervision (2006)
  International Convergence of Capital Measurement and Capital Standards — A Revised Framework — Comprehensive version, June 2006.
- Basel Committee on Banking Supervision (2017)
  Basel III: Finalising Post-crisis Reforms, December 2017.
- Roncalli, T. (2020)
  Handbook of Financial Risk Management, Chapman and Hall/CRC Financial Mathematics Series, Chapter 5.
- Roncalli, T. (2020)
  Handbook of Financial Risk Management – Companion Book, Chapter 5.