Course 2023-2024 in Financial Risk Management
Lecture 2. Market Risk

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[1] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

General information

1. **Overview**
   The objective of this course is to understand the theoretical and practical aspects of risk management
2. **Prerequisites**
   M1 Finance or equivalent
3. **ECTS**
   4
4. **Keywords**
   Finance, Risk Management, Applied Mathematics, Statistics
5. **Hours**
   Lectures: 36h, Training sessions: 15h, HomeWork: 30h
6. **Evaluation**
   There will be a final three-hour exam, which is made up of questions and exercises
7. **Course website**
   `http://www.thierry-roncalli.com/RiskManagement.html`

Objective of the course

The objective of the course is twofold:

1. knowing and understanding the financial regulation (banking and others) and the international standards (especially the Basel Accords)
2. being proficient in risk measurement, including the mathematical tools and risk models

Class schedule

Course sessions

- September 15 (6 hours, AM+PM)
- September 22 (6 hours, AM+PM)
- September 19 (6 hours, AM+PM)
- October 6 (6 hours, AM+PM)
- October 13 (6 hours, AM+PM)
- October 27 (6 hours, AM+PM)

Tutorial sessions

- October 20 (3 hours, AM)
- October 20 (3 hours, PM)
- November 10 (3 hours, AM)
- November 10 (3 hours, PM)
- November 17 (3 hours, PM)

Class times: Fridays 9:00am-12:00pm, 1:00pm–4:00pm, University of Evry, Room 209 IDF

Agenda

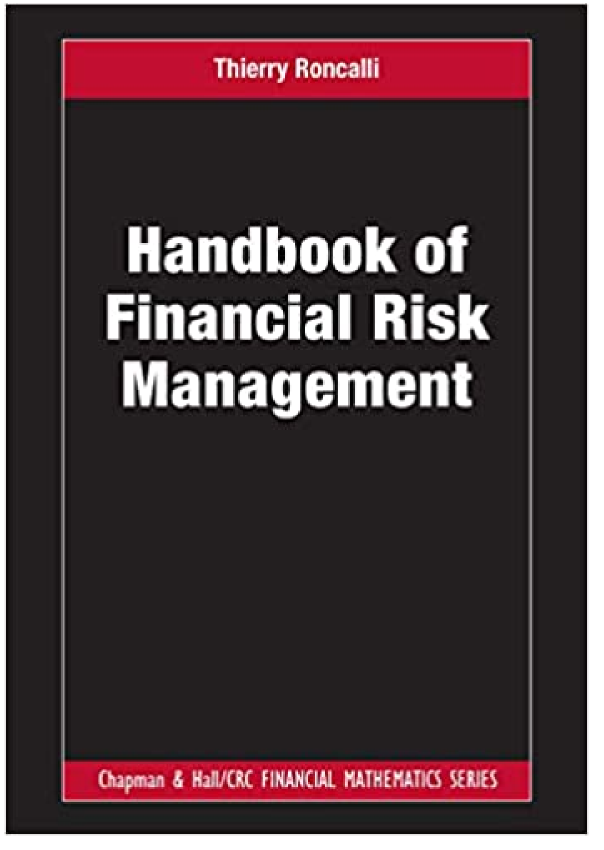
- Lecture 1: Introduction to Financial Risk Management
- Lecture 2: Market Risk
- Lecture 3: Credit Risk
- Lecture 4: Counterparty Credit Risk and Collateral Risk
- Lecture 5: Operational Risk
- Lecture 6: Liquidity Risk
- Lecture 7: Asset Liability Management Risk
- Lecture 8: Model Risk
- Lecture 9: Copulas and Extreme Value Theory
- Lecture 10: Monte Carlo Simulation Methods
- Lecture 11: Stress Testing and Scenario Analysis
- Lecture 12: Credit Scoring Models

Agenda

- Tutorial Session 1: Market Risk
- Tutorial Session 2: Credit Risk
- Tutorial Session 3: Counterparty Credit Risk and Collateral Risk
- Tutorial Session 4: Operational Risk & Asset Liability Management Risk
- Tutorial Session 5: Copulas, EVT & Stress Testing

Textbook

- Roncalli, T. (2020), *Handbook of Financial Risk Management*, Chapman & Hall/CRC Financial Mathematics Series.



Additional materials

- Slides, tutorial exercises and past exams can be downloaded at the following address:

  `http://www.thierry-roncalli.com/RiskManagement.html`

- Solutions of exercises can be found in the companion book, which can be downloaded at the following address:

  `http://www.thierry-roncalli.com/RiskManagementBook.html`

Agenda

- Lecture 1: Introduction to Financial Risk Management
- **Lecture 2: Market Risk**
- Lecture 3: Credit Risk
- Lecture 4: Counterparty Credit Risk and Collateral Risk
- Lecture 5: Operational Risk
- Lecture 6: Liquidity Risk
- Lecture 7: Asset Liability Management Risk
- Lecture 8: Model Risk
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- Lecture 12: Credit Scoring Models

Most important dates

- 19 October 1987: Stock markets crashed and the Dow Jones Industrial Average index dropped by more than 20% in the day
- 1988: Publication of the Basel I Accord
- 1990s: Japanese asset price bubble
- 1994: Bond market massacre
- October 1994: Publication of *RiskMetrics* by J.P. Morgan
- January 1996: Amendment to incorporate market risks (Basel I)
- 2004: Measuring market risks is the same in Basel II
- 2008: Global Financial Crisis (GFC)
- 2009: Basel 2.5
- January 2019: Revision of market risk in Basel III (also known as the **fundamental review of the trading book** or FRTB)

Definition

According to the Basel Committee, market risk is defined as "*the risk of losses (in on- and off-balance sheet positions) arising from movements in market prices. The risks subject to market risk capital requirements include but are not limited to:*

- *default risk, interest rate risk, credit spread risk, equity risk, foreign exchange (FX) risk and commodities risk for trading book instruments*;
- *FX risk and commodities risk for banking book instruments*."

Portfolio	Fixed Income	Equity	Currency	Commodity	Credit
Trading	✓	✓	✓	✓	✓
Banking			✓	✓	

$\Rightarrow$ trading book $\neq$ banking book

The Basel I/II framework

To compute the capital charge, banks have the choice between two approaches:

1. the standardized measurement method (SMM)
2. the internal model-based approach (IMA)

$\Rightarrow$ Banks quickly realized that they can sharply reduce their capital requirements by adopting internal models

Standardized measurement method (SMM)

Five main risk categories:

1. Interest rate risk
2. Equity risk
3. Currency risk
4. Commodity risk
5. Price risk on options and derivatives

For each category, a capital charge is computed to cover:

- the general market risk
- the specific risk

Standardized measurement method (SMM)

The capital charge $\mathcal{K}$ is equal to the risk exposure E times the capital charge weight K:

$$\mathcal{K} = E \cdot K$$

- For the specific risk, the risk exposure corresponds to the notional of the instrument, whether it is a long or a short position
- For the general market risk, long and short positions on different instruments can be offset

The case of equity risk

- The capital charge for specific risk is 4% if the portfolio is liquid and well-diversified and 8% otherwise
- For the general market risk, the risk weight is equal to 8% and applies to the net exposure

Remark

Under Basel 2.5, the capital charge for specific risk is set to 8% whatever the liquidity of the portfolio

The case of equity risk

Example

We consider a \$100 mn short exposure on the S&P 500 index futures contract and a \$60 mn long exposure on the Apple stock.

The capital charge for specific risk is[2]:

$$\mathcal{K}^{\mathrm{Specific}} = 100 \times 4\% + 60 \times 8\% = 4 + 4.8 = 8.8$$

The net exposure is $-\$40$ mn. We deduce that the capital charge for the general market risk is:

$$\mathcal{K}^{\mathrm{General}} = |-40| \times 8\% = 3.2$$

It follows that the total capital charge for this equity portfolio is \$12 mn.

[2] We assume that the S&P 500 index is liquid and well-diversified, whereas the exposure on the Apple stock is not diversified.

The case of interest rate risk (specific risk)

- For government instruments, the capital charge weights are:

Rating	AAA to AA−	A+ to BBB−			BB+ to B−	Below B−	NR
Maturity		0−6M	6M−2Y	2Y+			
K	0%	0.25%	1.00%	1.60%	8%	12%	8%

- In the case of other instruments (PSE, banks and corporates), the capital charge weights are:

Rating	AAA to BBB−			BB+ to BB−	Below BB−	NR
Maturity	0−6M	6M−2Y	2Y+			
K	0.25%	1.00%	1.60%	8%	12%	8%

The case of interest rate risk (specific risk)

Example

We consider a trading portfolio with the following exposures: a long position of \$50 mn on Euro-Bund futures, a short position of \$100 mn on three-month T-Bills and a long position of \$10 mn on an investment grade (IG) corporate bond with a three-year residual maturity.

$\Rightarrow$ Why the capital charge for specific risk is equal to \$0, \$0 and \$160 000?

The case of interest rate risk (general market risk)

Two methods:

- Maturity approach
- Duration approach (price sensitivity with respect to a change in yield)

Internal model-based approach

The use of an internal model is conditional upon the approval of the supervisory authority:

- **Qualitative criteria**
  - Independent risk control unit
  - Daily reports
  - Daily risk management
  - Etc.
- **Quantitative criteria**
  - The value-at-risk (VaR) is computed on a daily basis with a 99% confidence level. The minimum holding period of the VaR is 10 trading days. If the bank computes a VaR with a shorter holding period, it can use the square-root-of-time rule
  - Relevant risk factors
  - Sample period: at least one year
  - The value of the multiplication factor depends on the quality of the internal model with a range between 3 and 4. The quality of the internal model is related to its ex-post performance measured by the backtesting procedure
  - **Stress testing & Backtesting**

The square-root-of-time rule

The holding period to define the capital is 10 trading days. For that, banks can compute the one-day VaR and converts it to a ten-day VaR:

$$\mathrm{VaR}_{\alpha}\left(w;\text{ten days}\right)=\sqrt{10}\times\mathrm{VaR}_{\alpha}\left(w;\text{one day}\right)$$

Required capital

The required capital at time t is equal to:

$$\mathcal{K}_t = \max\left(\mathrm{VaR}_{t-1}, (3+\xi)\cdot\frac{1}{60}\sum_{i=1}^{60}\mathrm{VaR}_{t-i}\right)$$

where VaR_t is the 10-day value-at-risk calculated at time t and ξ is the penalty coefficient ($0 \leq \xi \leq 1$)

Required capital

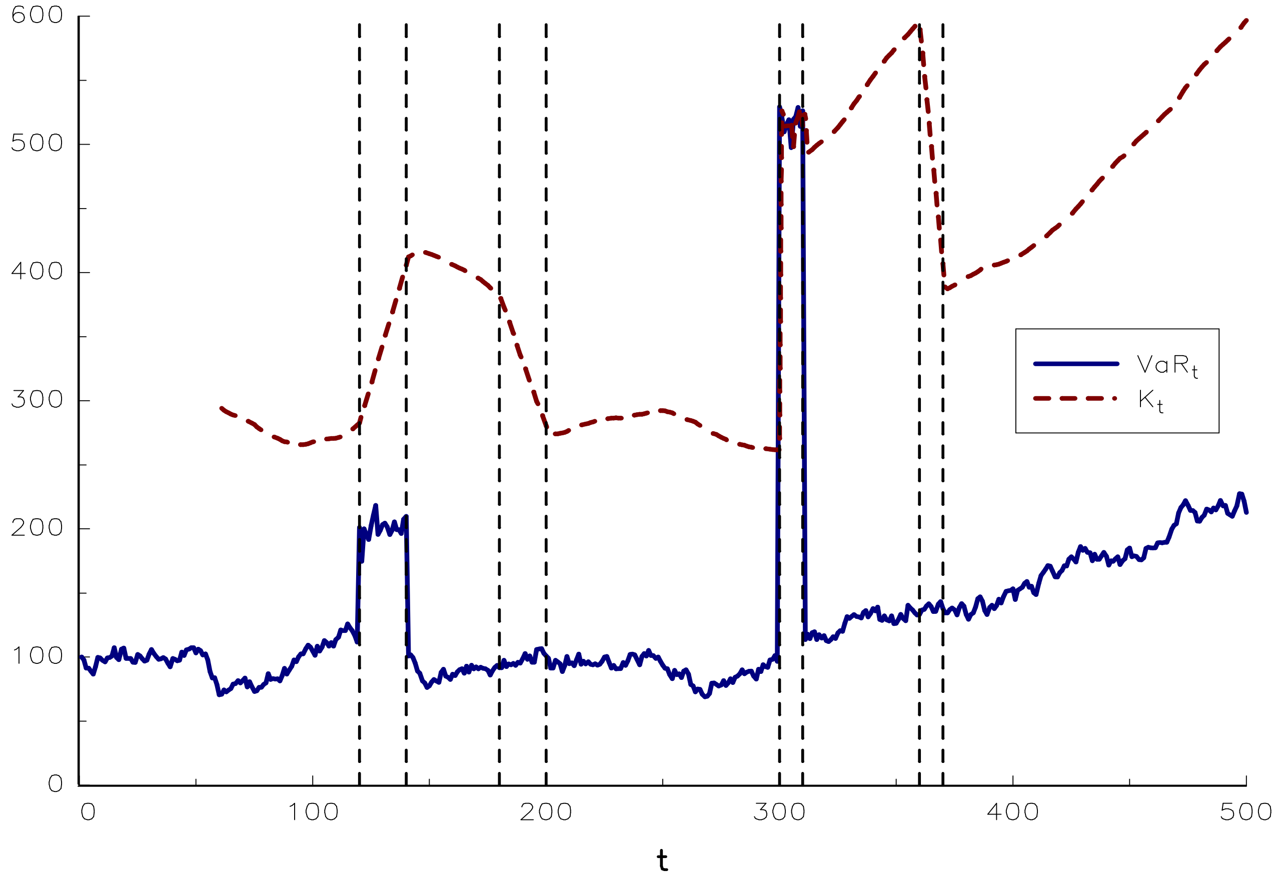


Figure: Calculation of the required capital with the VaR

Backtesting

Definition

Backtesting consists of verifying that the internal model is consistent with a 99% confidence level

⇒ For instance, we expect that the realized loss exceeds the VaR figure once every 100 observations on average

Table: Value of the penalty coefficient ξ for a sample of 250 observations

Zone	Number of exceptions	ξ
Green	0 – 4	0.00
Yellow	5	0.40
	6	0.50
	7	0.65
	8	0.75
	9	0.85
Red	10+	1.00

Statistical approach of backtesting

We note w the portfolio, $\mathrm{VaR}_\alpha(w;h)$ the value-at-risk calculated at time $t-1$, and $L_t(w)$ the daily loss at time t:

$$L_t(w) = -\Pi_t(w) = \mathrm{MtM}_{t-1} - \mathrm{MtM}_t$$

By definition, we have:

$$\Pr\{L_t(w) \geq \mathrm{VaR}_\alpha(w;h)\} = 1-\alpha$$

Let e_t be the random variable which is equal to 1 if there is an exception and 0 otherwise. e_t is a Bernoulli random variable with parameter p:

$$p = \Pr\{e_t = 1\} = \Pr\{L_t(w) \geq \mathrm{VaR}_\alpha(w;h)\} = 1-\alpha$$

Let $N_e(t_1;t_2) = \sum_{t=t_1}^{t_2} e_t$ be the number of exceptions for the period $[t_1, t_2]$. We assume that the exceptions are independent across time.

Main result

$N_e(t_1;t_2)$ is a binomial random variable $\mathcal{B}(n;1-\alpha)$

Statistical approach of backtesting

Table: Probability distribution (in %) of the number of exceptions ($n = 250$ trading days)

	$\alpha = 99\%$		$\alpha = 98\%$	
m	$\Pr\{N_e = m\}$	$\Pr\{N_e \leq m\}$	$\Pr\{N_e = m\}$	$\Pr\{N_e \leq m\}$
0	8.106	8.106	0.640	0.640
1	20.469	28.575	3.268	3.908
2	25.742	54.317	8.303	12.211
3	21.495	75.812	14.008	26.219
4	13.407	89.219	17.653	43.872
5	6.663	95.882	17.725	61.597
6	2.748	98.630	14.771	76.367
7	0.968	99.597	10.507	86.875
8	0.297	99.894	6.514	93.388
9	0.081	99.975	3.574	96.963
10	0.020	99.995	1.758	98.720

Statistical approach of backtesting

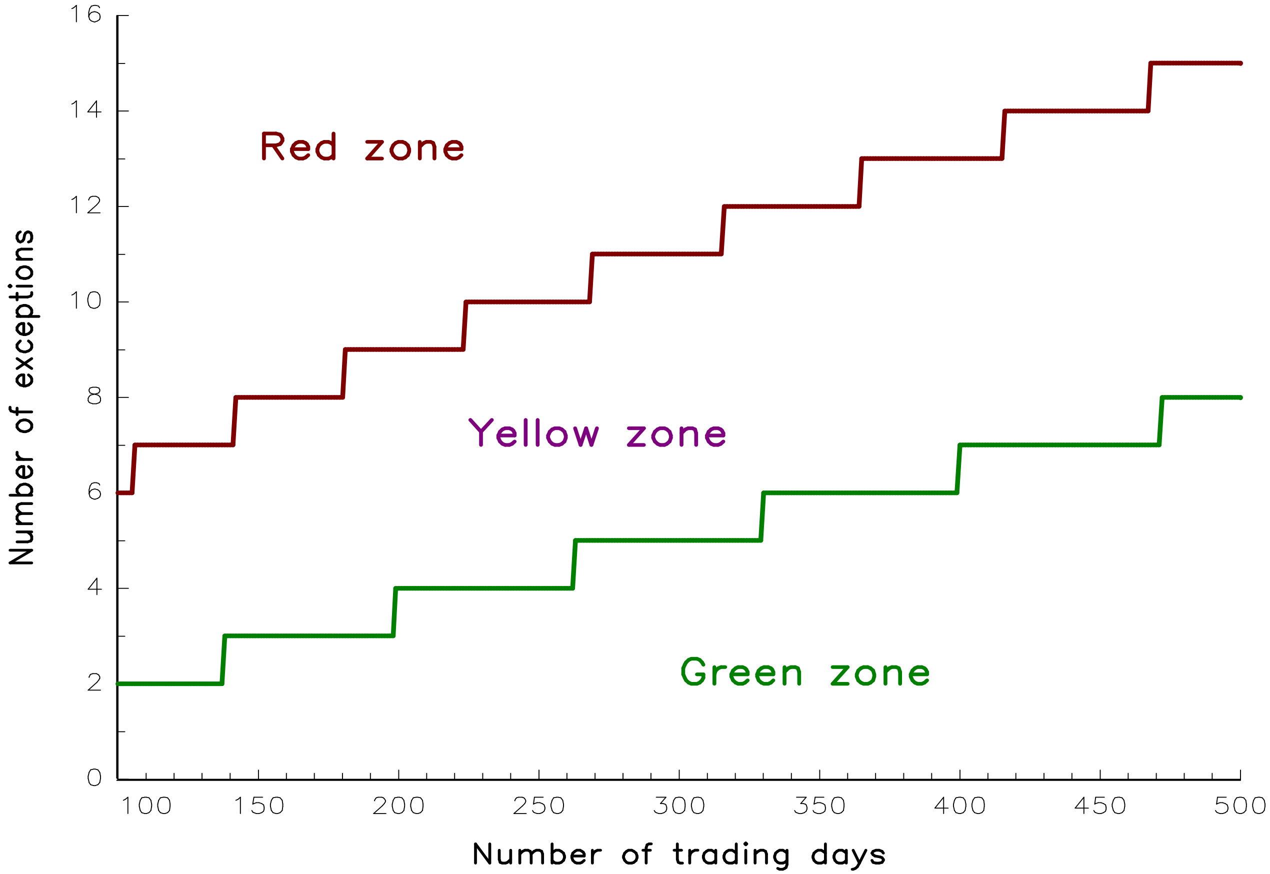


Figure: Color zones of the backtesting procedure ($\alpha = 99\%$)

The Basel 2.5 framework

The required capital becomes:

$$\mathcal{K}_t = \mathcal{K}_t^{\mathrm{VaR}} + \mathcal{K}_t^{\mathrm{SVaR}} + \mathcal{K}_t^{\mathrm{SRC}} + \mathcal{K}_t^{\mathrm{IRC}} + \mathcal{K}_t^{\mathrm{CRM}}$$

where $\mathcal{K}_t^{\mathrm{VaR}}$ is the VaR capital and $\mathcal{K}_t^{\mathrm{SRC}}$ (Basel II), and:

- $\mathcal{K}_t^{\mathrm{SVaR}}$ is the **Stressed VaR**
- $\mathcal{K}_t^{\mathrm{IRC}}$ is the **incremental risk charge** (IRC), which measures the impact of rating migrations and defaults
- $\mathcal{K}_t^{\mathrm{CRM}}$ is the **comprehensive risk measure** (CRM), which corresponds to a supplementary capital charge for credit exotic trading portfolios

The stressed VaR

Definition

The stressed VaR has the same characteristics than the traditional VaR (99% confidence level and 10-day holing period), but the model inputs are *"calibrated to historical data from a continuous 12-month period of significant financial stress relevant to the bank's portfolio"*.

⇒ This implies that the historical period to compute the SVaR is completely different than the historical period to compute the VaR[3]

[3] For instance, a typical period is the 2008 year which both combines the subprime mortgage crisis and the Lehman Brothers bankruptcy

The Basel III framework

Banks have the choice between two approaches for computing the capital charge:

1. a standardized method (SA-TB[4])
2. an internal model-based approach (IMA)

$\Rightarrow$ SMM is replaced by SA-TB and IMA is revisited

Remark

Contrary to the previous framework, the SA-TB method is very important even if banks calculate the capital charge with the IMA method. Indeed, the bank must implement SA-TB in order to meet the output (or capital) floor requirement, which is set at 72.5% in January 2027:

$$\mathcal{K}_t = \max\left(\mathcal{K}_t^{IMA}, 72.5\% \times \mathcal{K}_t^{SA\text{-}TB}\right)$$

[4] TB means trading book

SA-TB

The standardized capital charge is the sum of three components:

1. sensitivity-based capital requirement
2. the default risk capital (DRC)
3. the residual risk add-on (RRAO)

Some comments:

- The first component must be viewed as the pure market risk and is the equivalent of the capital charge for the general market risk
- The second component captures the jump-to-default risk (JTD) and replaces the specific risk
- The last component captures specific risks that are difficult to measure in practice

Sensitivity-based capital requirement

We have:

$$\mathcal{K} = \mathcal{K}^{\text{Delta}} + \mathcal{K}^{\text{Vega}} + \mathcal{K}^{\text{Curvature}}$$

$\Rightarrow$ a capital charge for delta, vega and curvature risks

7 risk classes:

1. General interest rate risk (GIRR)
2. Credit spread risk(CSR) on non-securitization products
3. Credit spread risk(CSR) on non-correlation trading portfolio (non-CTP)
4. Credit spread risk(CSR) on correlation trading portfolio (CTP)
5. Equity risk
6. Commodity risk
7. Foreign exchange risk

Delta and vega risk components

- We first begin to calculate the weighted sensitivity of each risk factor $\mathcal{F}_j$:
$$\mathrm{WS}_j = S_j \cdot \mathrm{RW}_j$$
where S_j and RW_j are the net sensitivity of the portfolio with respect to the risk factor and the risk weight of $\mathcal{F}_j$
- Second, we calculate the capital requirement for the risk bucket $\mathcal{B}_k$:
$$\mathcal{K}_{\mathcal{B}_k} = \sqrt{\max\left(\sum_j \mathrm{WS}_j^2 + \sum_{j' \neq j} \rho_{j,j'} \mathrm{WS}_j \mathrm{WS}_{j'}, 0\right)}$$
where $\mathcal{F}_j \in \mathcal{B}_k$.
- Finally, we aggregate the different buckets for a given risk class:
$$\mathcal{K}^{\mathrm{Delta/Vega}} = \sqrt{\sum_k \mathcal{K}_{\mathcal{B}_k}^2 + \sum_{k' \neq k} \gamma_{k,k'} \mathrm{WS}_{\mathcal{B}_k} \mathrm{WS}_{\mathcal{B}_{k'}}}$$
where $\mathrm{WS}_{\mathcal{B}_k} = \sum_{j \in \mathcal{B}_k} \mathrm{WS}_j$ is the weighted sensitivity of the bucket $\mathcal{B}_k$.

Delta and vega risk components

The capital requirement for delta and vega risks can be viewed as a Gaussian risk measure with the following parameters:

1. the sensitivities S_j of the risk factors that are calculated by the bank;
2. the risk weights RW_j of the risk factors;
3. the correlation $\rho_{j,j'}$ between risk factors within a bucket;
4. the correlation $\gamma_{k,k'}$ between the risk buckets.

Curvature risk component

The curvature risk uses a similar methodology, but it is based on two adverse scenarios: (1) the risk factor is shocked upward and (2) the risk factor is shocked downward

The curvature risk is close to the gamma risk that we encounter in the theory of options

Practical computation of dela, vega and curvature risks

Three steps:

1. defining the risk factors
2. calculating the sensitivities
3. calculating the risk-weighted sensitivities WS_j

Defining the risk factors

$\Rightarrow$ The Basel Committee gives a very precise list of risk factors by asset classes

For instance, the equity delta risk factors are the equity spot prices and the equity repo rates, the equity vega risk factors are the implied volatilities of options, and the equity curvature risk factors are the equity spot prices

In the case of the interest rate risk class (GIRR), the risk factors include the yield curve[5], a flat curve of market-implied inflation rates for each currency and some cross-currency basis risks

[5]The risk factors correspond to the following tenors of the yield curve: 3M, 6M, 1Y, 2Y, 3Y, 5Y, 10Y, 15Y, 20Y and 30Y.

Calculating the sensitivities

The equity delta sensitivity of the instrument i with respect to the equity risk factor $\mathcal{F}_j$ is given by:

$$S_{i,j} = \boldsymbol{\Delta}_i \left(\mathcal{F}_j\right) \cdot \mathcal{F}_j$$

where $\boldsymbol{\Delta}_i \left(\mathcal{F}_j\right)$ measures the (discrete) delta of the instrument i by shocking the equity risk factor $\mathcal{F}_j$ by 1%:

$$S_{i,j} = \frac{P_i \left(1.01 \cdot \mathcal{F}_j\right) - P_i \left(\mathcal{F}_j\right)}{1.01 \cdot \mathcal{F}_j - \mathcal{F}_j} \cdot \mathcal{F}_j = \frac{P_i \left(1.01 \cdot \mathcal{F}_j\right) - P_i \left(\mathcal{F}_j\right)}{0.01}$$

Remark

- *If the instrument corresponds to a stock, the sensitivity is exactly the price of this stock when the risk factor is the stock price, and zero otherwise*
- *If the instrument corresponds to an European option on this stock, the sensitivity is the traditional delta of the option times the stock price*

Calculating the sensitivities

For the vega sensitivity, we have:

$$S_{i,j} = \boldsymbol{\upsilon}_i \left(\mathcal{F}_j \right) \cdot \mathcal{F}_j$$

where $\mathcal{F}_j$ is the implied volatility and $\boldsymbol{\upsilon}_i \left(\mathcal{F}_j \right)$ is the vega of the instrument

Calculating the risk-weighted sensitivities

We use the figures given in BCBS (2019) for the risk weight RW_j, the correlation $\rho_{j,j'}$ and the correlation $\gamma_{k,k'}$

Internal model-based approach

A trading desk is "*an unambiguously defined group of traders or trading accounts that implements a well-defined business strategy operating within a clear risk management structure*".

$\Rightarrow$ Internal models are implemented at the **trading desk** level, meaning that some trading desks are approved for the use of internal models, while other trading desks must use the SA-TB approach

Capital requirement for modellable risk factors

Main differences with Basel I/II

The value-at-risk at the 99% confidence level is replaced by the expected shortfall at the 97.5% confidence level. Moreover, the 10-day holding period is not valid for all instruments

Expected shortfall

The expected shortfall is the average loss beyond the value-at-risk

Capital requirement for modellable risk factors

Impact of the liquidity

$$\mathrm{ES}_\alpha(w) = \sqrt{\sum_{k=1}^{5}\left(\mathrm{ES}_\alpha(w; h_k)\sqrt{\frac{h_k - h_{k-1}}{h_1}}\right)^2}$$

- $\mathrm{ES}_\alpha(w; h_1)$ is the expected shortfall of the portfolio w at horizon 10 days by considering all risk factors
- $\mathrm{ES}_\alpha(w; h_k)$ is the expected shortfall of the portfolio w at horizon h_k days by considering the risk factors $\mathcal{F}_j$ that belongs to the liquidity class k
- h_k is the horizon of the liquidity class k, which is given below:

Liquidity class k	1	2	3	4	5
Liquidity horizon h_k	10	20	40	60	120

Capital requirement for modellable risk factors

Liquidity buckets

1. Interest rates (specified currencies and domestic currency of the bank), equity prices (large caps), FX rates (specified currency pairs).
2. Interest rates (unspecified currencies), equity prices (small caps) and volatilities (large caps), FX rates (currency pairs), credit spreads (IG sovereigns), commodity prices (energy, carbon emissions, precious metals, non-ferrous metals).
3. FX rates (other types), FX volatilities, credit spreads (IG corporates and HY sovereigns).
4. Interest rates (other types), IR volatility, equity prices (other types) and volatilities (small caps), credit spreads (HY corporates), commodity prices (other types) and volatilities (energy, carbon emissions, precious metals, non-ferrous metals).
5. Credit spreads (other types) and credit spread volatilities, commodity volatilities and prices (other types).

Capital requirement for modellable risk factors

How to calculate the expected shortfall for a period of stress?

$$\mathrm{ES}_{\alpha}\left(w;h\right) = \mathrm{ES}_{\alpha}^{(\mathrm{reduced,stress})}\left(w;h\right) \cdot \min\left(\frac{\mathrm{ES}_{\alpha}^{(\mathrm{full,current})}\left(w;h\right)}{\mathrm{ES}_{\alpha}^{(\mathrm{reduced,current})}\left(w;h\right)}, 1\right)$$

where $\mathrm{ES}_{\alpha}^{(\mathrm{full,current})}$ is the expected shortfall based on the current period with the full set of risk factors, $\mathrm{ES}_{\alpha}^{(\mathrm{reduced,current})}$ is the expected shortfall based on the current period with a restricted set of risk factors and $\mathrm{ES}_{\alpha}^{(\mathrm{reduced,stress})}$ is the expected shortfall based on the stress period with the restricted set of risk factors

Remark

The previous formula assumes that there is a proportionality factor between the full set and the restricted set of risk factors:

$$\frac{\mathrm{ES}_{\alpha}^{(\mathrm{full,stress})}\left(w;h\right)}{\mathrm{ES}_{\alpha}^{(\mathrm{full,current})}\left(w;h\right)} \approx \frac{\mathrm{ES}_{\alpha}^{(\mathrm{reduced,stress})}\left(w;h\right)}{\mathrm{ES}_{\alpha}^{(\mathrm{reduced,current})}\left(w;h\right)}$$

Capital requirement for modellable risk factors

Example

In the table below, we have calculated the 10-day expected shortfall for a given portfolio:

Set of risk factors	Period	Liquidity class				
		1	2	3	4	5
Full	Current	100	75	34	12	6
Reduced	Current	88	63	30	7	5
Reduced	Stress	112	83	47	9	7

Capital requirement for modellable risk factors

Table: Scaled expected shortfall

k	Sc_k	Full Current	Reduced Current	Reduced Stress	Full/Stress (not scaled)	Full Stress
1	1	100.00	88.00	112.00	127.27	127.27
2	1	75.00	63.00	83.00	98.81	98.81
3	$\sqrt{2}$	48.08	42.43	66.47	53.27	75.33
4	$\sqrt{2}$	16.97	9.90	12.73	15.43	21.82
5	$\sqrt{6}$	14.70	12.25	17.15	8.40	20.58
Total		135.80	117.31	155.91		180.38

The scaling factor is equal to $Sc_k = \sqrt{\left(h_k - h_{k-1}\right)/h_1}$, the scaled expected shortfall is equal to $\mathrm{ES}_\alpha^\star\left(w; h_k\right) = Sc_k \cdot \mathrm{ES}_\alpha\left(w; h_k\right)$ and the total expected shortfall is given by $\mathrm{ES}_\alpha\left(w\right) = \sqrt{\sum_{k=1}^5 \left(\mathrm{ES}_\alpha^\star\left(w; h_k\right)\right)^2}$

Capital requirement for modellable risk factors

The final step for computing the capital requirement (also known as the 'internally modelled capital charge') is to apply this formula:

$$\mathrm{IMCC} = \varrho \cdot \mathrm{IMCC}_{global} + (1 - \varrho) \cdot \sum_{k=1}^{5} \mathrm{IMCC}_k$$

where:

- ϱ is equal to 50%
- IMCC_{global} is the stressed ES calculated with the internal model and cross-correlations between risk classes
- IMCC_k is the stressed ES calculated at the risk class level (interest rate, equity, foreign exchange, commodity and credit spread)

Other capital requirements

- Concerning non-modellable risk factors, the capital requirement is based on stress scenarios, that are equivalent to a stressed expected shortfall SES
- The default risk capital (DRC) is calculated using a value-at-risk model with a 99.9% confidence level with the same default probabilities that are used for the IRB approach

Capital requirement for the market risk

For eligible trading desks, we have:

$$\mathcal{K}_t^{\mathrm{IMA}} = \max\left(\mathrm{IMCC}_{t-1} + \mathrm{SES}_{t-1}, \frac{m_c \sum_{i=1}^{60} \mathrm{IMCC}_{t-i} + \sum_{i=1}^{60} \mathrm{SES}_{t-i}}{60}\right) + \mathrm{DRC}$$

where $m_c = 1.5 + \xi$ and $0 \leq \xi \leq 0.5$

Table: Value of the penalty coefficient ξ in Basel III

Zone	Number of exceptions	ξ
Green	0 – 4	0.00
Amber	5	0.20
	6	0.26
	7	0.33
	8	0.38
	9	0.42
Red	10+	0.50

Coherent risk measures

We note $\mathcal{R}(w)$ as the risk measure of portfolio w

Coherent risk measure

1. Subadditivity
$$\mathcal{R}(w_1 + w_2) \leq \mathcal{R}(w_1) + \mathcal{R}(w_2)$$
2. Homogeneity
$$\mathcal{R}(\lambda w) = \lambda \mathcal{R}(w) \quad \text{if } \lambda \geq 0$$
3. Monotonicity
$$\text{if } w_1 \prec w_2, \text{ then } \mathcal{R}(w_1) \geq \mathcal{R}(w_2)$$
4. Translation invariance
$$\text{if } m \in \mathbb{R}, \text{ then } \mathcal{R}(w + m) = \mathcal{R}(w) - m$$

$\Rightarrow$ Translation invariance implies that:

$$\mathcal{R}(w + \mathcal{R}(w)) = \mathcal{R}(w) - \mathcal{R}(w) = 0$$

Some risk measures

The portfolio's loss is equal to $L(w) = -P_t(w) R_{t+h}(w)$

- Volatility of the loss

$$\mathcal{R}(w) = \sigma(L(w)) = \sigma(w)$$

- Standard deviation-based risk measure

$$\mathcal{R}(w) = \mathrm{SD}_c(w) = \mathbb{E}[L(w)] + c \cdot \sigma(L(w)) = -\mu(w) + c \cdot \sigma(w)$$

- Value-at-risk

$$\mathcal{R}(w) = \mathrm{VaR}_\alpha(w) = \inf\{\ell : \Pr\{L(w) \leq \ell\} \geq \alpha\}$$

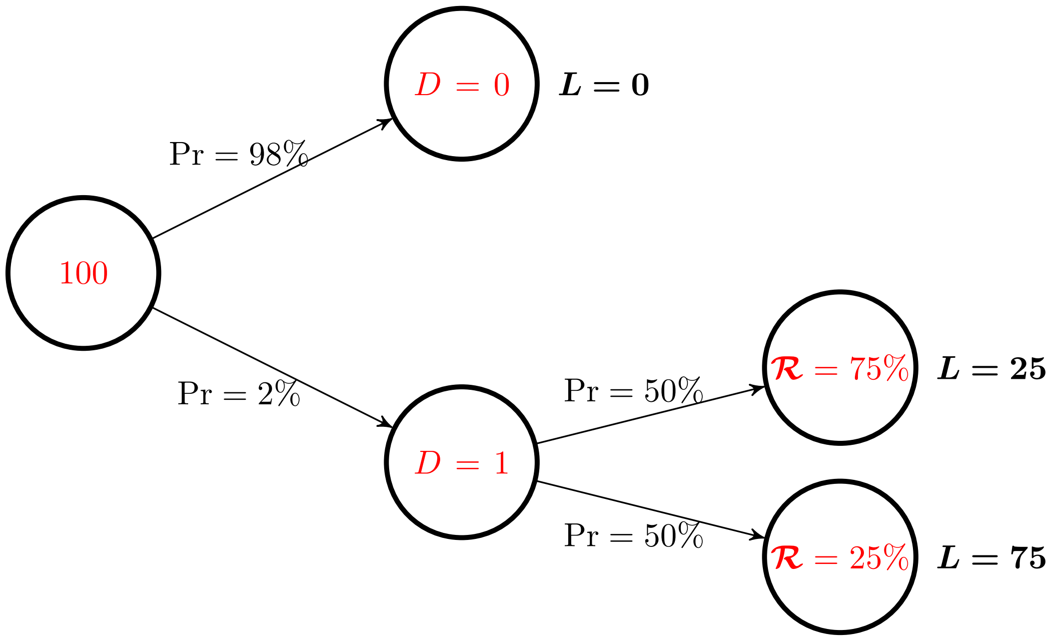
- Expected shortfall

$$\mathcal{R}(w) = \mathrm{ES}_\alpha(w) = \mathbb{E}[L(w) \mid L(w) \geq \mathrm{VaR}_\alpha(w)] = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(w)\,\mathrm{d}u$$

The value-at-risk is not always subadditive

Example

We consider a \$100 defaultable zero-coupon bond, whose default probability is equal to 200 bps. We assume that the recovery rate $\mathcal{R}$ is a binary random variable with $\Pr\{\mathcal{R} = 0.25\} = \Pr\{\mathcal{R} = 0.75\} = 50\%$.



$\Rightarrow$ $\mathbf{F}(0) = \Pr\{L \leq 0\} = 98\%$, $\mathbf{F}(25) = \Pr\{L_i \leq 25\} = 99\%$ and $\mathbf{F}(75) = \Pr\{L_i \leq 75\} = 100\%$

The value-at-risk is not always subadditive

The 99% value-at-risk is equal to \$25, and we have:

$$\mathrm{ES}_{99\%}(L) = \mathbb{E}[L \mid L \geq 25] = \frac{25+75}{2} = \$50$$

We now consider two zero-coupon bonds with iid default times:

	$L_1 = 0$	$L_1 = 25$	$L_1 = 75$	
$L_2 = 0$	96.04%	0.98%	0.98%	98.00%
$L_2 = 25$	0.98%	0.01%	0.01%	1.00%
$L_2 = 75$	0.98%	0.01%	0.01%	1.00%
	98.00%	1.00%	1.00%	

We deduce that the probability distribution function of $L = L_1 + L_2$ is:

ℓ	0	25	50	75	100	150
$\Pr\{L = \ell\}$	96.04%	1.96%	0.01%	1.96%	0.02%	0.01%
$\Pr\{L \leq \ell\}$	96.04%	98%	98.01%	99.97%	99.99%	100%

It follows that $\mathrm{VaR}_{99\%}(L) = 75$ and:

$$\mathrm{ES}_{99\%}(L) = \frac{75 \times 1.96\% + 100 \times 0.02\% + 150 * 0.01\%}{1.96\% + 0.02\% + 0.01\%} = \$75.63$$

Value-at-risk

Definition

The value-at-risk $\mathrm{VaR}_\alpha(w;h)$ is defined as the potential loss which the portfolio w can suffer for a given confidence level α and a fixed holding period h:

$$\Pr\{L(w) \leq \mathrm{VaR}_\alpha(w;h)\} = \alpha \Leftrightarrow \mathrm{VaR}_\alpha(w;h) = \mathbf{F}_L^{-1}(\alpha)$$

Three parameters are necessary to compute this risk measure:

- the holding period h, which indicates the time period to calculate the loss;
- the confidence level α, which gives the probability that the loss is lower than the value-at-risk;
- the portfolio w, which gives the allocation in terms of risky assets and is related to the risk factors.

Expected shortfall

Definition

The expected shortfall $\mathrm{ES}_\alpha(w;h)$ is defined as the expected loss beyond the value-at-risk of the portfolio:

$$\mathrm{ES}_\alpha(w;h) = \mathbb{E}\left[L(w) \mid L(w) \geq \mathrm{VaR}_\alpha(w;h)\right]$$

We notice that $\mathrm{ES}_\alpha(w;h) \geq \mathrm{VaR}_\alpha(w;h)$

Three methods

Let $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ be the vector of risk factors. We assume that there is a pricing function g such that:

$$P_t(w) = g(\mathcal{F}_{1,t}, \ldots, \mathcal{F}_{m,t}; w)$$

We deduce that the expression of the random loss is equal to:

$$L(w) = P_t(w) - g(\mathcal{F}_{1,t+h}, \ldots, \mathcal{F}_{m,t+h}; w) = \ell(\mathcal{F}_{1,t+h}, \ldots, \mathcal{F}_{m,t+h}; w)$$

where ℓ is the loss function. We have:

$$\widehat{\mathrm{VaR}}_\alpha(w; h) = \hat{\mathbf{F}}_L^{-1}(\alpha) = -\hat{\mathbf{F}}_\Pi^{-1}(1-\alpha)$$

and:

$$\widehat{\mathrm{ES}}_\alpha(w; h) = \frac{1}{1-\alpha}\int_\alpha^1 \hat{\mathbf{F}}_L^{-1}(u)\,\mathrm{d}u$$

1. the historical (or empirical or non-parametric) VaR/ES
2. the analytical (or parametric or Gaussian) VaR/ES
3. the Monte Carlo (or simulated) VaR/ES

Historical methods

Two approaches:

- order statistic approach
- kernel approach

Let $(\mathcal{F}_{1,s}, \ldots, \mathcal{F}_{m,s})$ be the vector of risk factors observed at time $s < t$. If we calculate the future P&L with this historical scenario, we obtain:

$$\Pi_s(w) = g(\mathcal{F}_{1,s}, \ldots, \mathcal{F}_{m,s}; w) - P_t(w)$$

If we consider n_S historical scenarios ($s = 1, \ldots, n_S$), the empirical distribution $\hat{\mathbf{F}}_\Pi$ is described by the following probability distribution:

$\Pi(w)$	$\Pi_1(w)$	$\Pi_2(w)$	$\cdots$	$\Pi_{n_S}(w)$
p_s	$1/n_S$	$1/n_S$		$1/n_S$

Order statistic approach

Theorem (HFRM, page 67)

Let $X_1, \ldots, X_n$ be a sample from a continuous distribution $\mathbf{F}$. Suppose that for a given scalar $\alpha \in \left]0,1\right[$, there exists a sequence $\{a_n\}$ such that $\sqrt{n}\left(a_n - n\alpha\right) \rightarrow 0$. We can show that:

$$\sqrt{n}\left(X_{(a_n:n)} - \mathbf{F}^{-1}(\alpha)\right) \rightarrow \mathcal{N}\left(0, \frac{\alpha(1-\alpha)}{f^2\left(\mathbf{F}^{-1}(\alpha)\right)}\right)$$

$\Rightarrow \hat{\mathbf{F}}^{-1}(\alpha) = X_{(n\alpha:n)}$

- If $n_s = 1\,000$, $\hat{\mathbf{F}}^{-1}(90\%)$ is the 900^{th} order statistic
- If $n_s = 2\,00$, $\hat{\mathbf{F}}^{-1}(90.5\%)$ is the 181^{th} order statistic

Order statistic approach

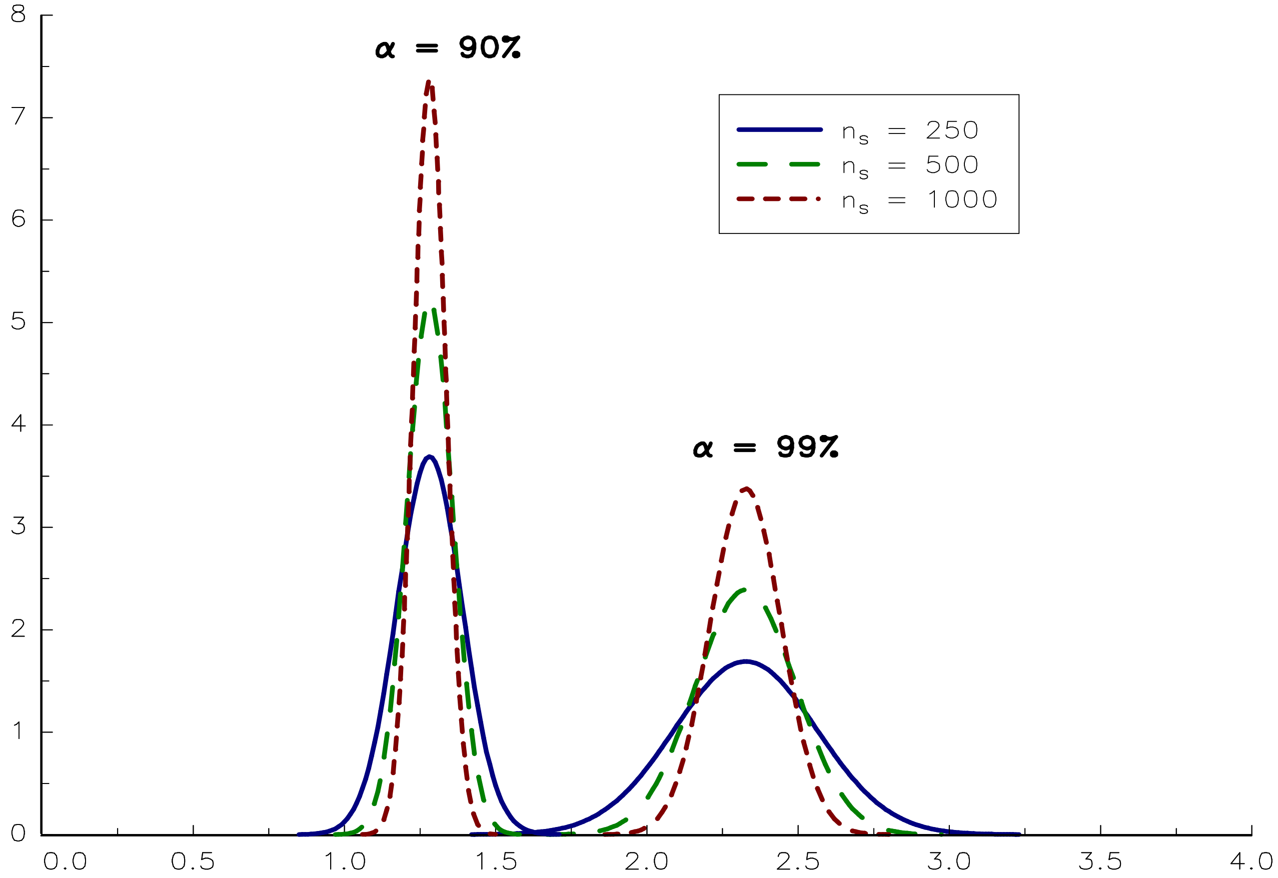


Figure: Density of the quantile estimator (Gaussian case)

Application to the value-at-risk

We calculate the order statistics associated to the P&L sample $\{\Pi_1(w), \ldots, \Pi_{n_S}(w)\}$:

$$\min_s \Pi_s(w) = \Pi_{(1:n_S)} \leq \Pi_{(2:n_S)} \leq \cdots \leq \Pi_{(n_S-1:n_S)} \leq \Pi_{(n_S:n_S)} = \max_s \Pi_s(w)$$

It follows that:

$$\mathrm{VaR}_\alpha(w;h) = -\Pi_{(n_S(1-\alpha):n_S)}$$

Application to the value-at-risk

Remark

If $n_S(1-\alpha)$ is not an integer, we consider the interpolation scheme:

$$\mathrm{VaR}_\alpha(w;h) = -\left(\Pi_{(q:n_S)} + (n_S(1-\alpha) - q)\left(\Pi_{(q+1:n_S)} - \Pi_{(q:n_S)}\right)\right)$$

where $q = q_\alpha(n_S) = \lfloor n_S(1-\alpha)\rfloor$ is the integer part of $n_S(1-\alpha)$.

In the case where we use 250 historical scenarios, the 99% value-at-risk is the mean between the second and third largest losses:

$$\begin{aligned}
\mathrm{VaR}_{99\%}(w;h) &= -\left(\Pi_{(2:250)} + (2.5-2)\left(\Pi_{(3:250)} - \Pi_{(2:250)}\right)\right) \\
&= -\frac{1}{2}\left(\Pi_{(2:250)} + \Pi_{(3:250)}\right) \\
&= \frac{1}{2}\left(L_{(249:250)} + L_{(248:250)}\right)
\end{aligned}$$

Application to the value-at-risk

Example

We consider a portfolio composed of 10 stocks Apple and 20 stocks Coca-Cola. The current date is 2 January 2015.

Remark

Data are available at
http://www.thierry-roncalli.com/download/frm-data1.xlsx

Application to the value-at-risk

The mark-to-market of the portfolio is:

$$P_t(w) = 10 \times P_{1,t} + 20 \times P_{2,t}$$

where $P_{1,t}$ and $P_{2,t}$ are the stock prices of Apple and Coca-Cola. We assume that the market risk factors corresponds to the daily stock returns $R_{1,t}$ and $R_{2,t}$. We deduce that the P&L for the scenario s is equal to:

$$\Pi_s(w) = \underbrace{10 \times P_{1,s} + 20 \times P_{2,s}}_{g(R_{1,s},R_{2,s};w)} - P_t(w)$$

where $P_{i,s} = P_{i,t} \times (1 + R_{i,s})$ is the simulated price of stock i for the scenario s.

Application to the value-at-risk

Table: Computation of the market risk factors $R_{1,s}$ and $R_{2,s}$

s	Date	Apple		Coca-Cola	
		Price	$R_{1,s}$	Price	$R_{2,s}$
1	2015-01-02	109.33	-0.95%	42.14	-0.19%
2	2014-12-31	110.38	-1.90%	42.22	-1.26%
3	2014-12-30	112.52	-1.22%	42.76	-0.23%
4	2014-12-29	113.91	-0.07%	42.86	-0.23%
5	2014-12-26	113.99	1.77%	42.96	0.05%
6	2014-12-24	112.01	-0.47%	42.94	-0.07%
7	2014-12-23	112.54	-0.35%	42.97	1.46%
8	2014-12-22	112.94	1.04%	42.35	0.95%
9	2014-12-19	111.78	-0.77%	41.95	-1.04%
10	2014-12-18	112.65	2.96%	42.39	2.02%

Application to the value-at-risk

- We calculate the historical risk factors. For instance, we have:

$$R_{1,1} = \frac{109.33}{110.38} - 1 = -0.95\%$$

- We calculate the simulated prices. For instance, in the case of the 9^{th} scenario, we obtain:

$$\begin{aligned} P_{1,s} &= 109.33 \times (1 - 0.77\%) = \$108.49 \\ P_{2,s} &= 42.14 \times (1 - 1.04\%) = \$41.70 \end{aligned}$$

- We then deduce the simulated mark-to-market $\mathrm{MtM}_s(w) = g(R_{1,s}, R_{2,s}; w)$

Application to the value-at-risk

Table: Computation of the simulated P&L $\Pi_s(w)$

s	Date	Apple $R_{1,s}$	Apple $P_{1,s}$	Coca-Cola $R_{2,s}$	Coca-Cola $P_{2,s}$	$\mathrm{MtM}_s(w)$	$\Pi_s(w)$
1	2015-01-02	−0.95%	108.29	−0.19%	42.06	1 924.10	−12.00
2	2014-12-31	−1.90%	107.25	−1.26%	41.61	1 904.66	−31.44
3	2014-12-30	−1.22%	108.00	−0.23%	42.04	1 920.79	−15.31
4	2014-12-29	−0.07%	109.25	−0.23%	42.04	1 933.37	−2.73
5	2014-12-26	1.77%	111.26	0.05%	42.16	1 955.82	19.72
23	2014-12-01	−3.25%	105.78	−0.62%	41.88	1 895.35	−40.75
69	2014-09-25	−3.81%	105.16	−1.16%	41.65	1 884.64	−51.46
85	2014-09-03	−4.22%	104.72	0.34%	42.28	1 892.79	−43.31
108	2014-07-31	−2.60%	106.49	−0.83%	41.79	1 900.68	−35.42
236	2014-01-28	−7.99%	100.59	0.36%	42.29	1 851.76	−84.34
242	2014-01-17	−2.45%	106.65	−1.08%	41.68	1 900.19	−35.91
250	2014-01-07	−0.72%	108.55	0.30%	42.27	1 930.79	−5.31

Application to the value-at-risk

If we rank the scenarios, the worst P&Ls are -84.34, -51.46, -43.31, -40.75, -35.91 and -35.42. We deduce that the daily historical VaR is equal to:

$$\mathrm{VaR}_{99\%}\left(w; \text{one day}\right) = \frac{1}{2}\left(51.46 + 43.31\right) = \$47.39$$

If we assume that $m_c = 3$, the corresponding capital charge represents 23.22% of the portfolio value:

$$\mathcal{K}_t^{\mathrm{VaR}} = 3 \times \sqrt{10} \times 47.39 = \$449.54$$

Application to the expected shortfall

Since the expected shortfall is the expected loss beyond the value-at-risk, it follows that the historical expected shortfall is given by:

$$\mathrm{ES}_\alpha(w;h) = \frac{1}{q_\alpha(n_S)} \sum_{s=1}^{n_S} \mathbb{1}\{L_s \geq \mathrm{VaR}_\alpha(w;h)\} \cdot L_s$$

or:

$$\mathrm{ES}_\alpha(w;h) = -\frac{1}{q_\alpha(n_S)} \sum_{s=1}^{n_S} \mathbb{1}\{\Pi_s \leq -\mathrm{VaR}_\alpha(w;h)\} \cdot \Pi_s$$

where $q_\alpha(n_S) = \lfloor n_s(1-\alpha) \rfloor$ is the integer part of $n_s(1-\alpha)$.

Computation of the ES

We have:

$$\mathrm{ES}_\alpha(w;h) = -\frac{1}{q_\alpha(n_S)} \sum_{i=1}^{q_\alpha(n_S)} \Pi_{(i:n_S)}$$

Application to the expected shortfall

We have:

$$\mathrm{ES}_{99\%}(w;\text{one day}) = \frac{84.34 + 51.46}{2} = \$67.90$$

and:

$$\mathrm{ES}_{97.5\%}(w;\text{one day}) = \frac{84.34 + 51.46 + 43.31 + 40.75 + 35.91 + 35.42}{6} = \$48.53$$

We remind that $\mathrm{VaR}_{99\%}(w;\text{one day}) = \$47.39$.

Analytical methods

We speak about analytical value-at-risk when we are able to find a closed-form formula of $\mathbf{F}_L^{-1}(\alpha)$

Gaussian value-at-risk

Suppose that $L(w) \sim \mathcal{N}\left(\mu(L), \sigma^2(L)\right)$. In this case, we have $\Pr\left\{L(w) \leq \mathbf{F}_L^{-1}(\alpha)\right\} = \alpha$ or:

$$\Pr\left\{\frac{L(w) - \mu(L)}{\sigma(L)} \leq \frac{\mathbf{F}_L^{-1}(\alpha) - \mu(L)}{\sigma(L)}\right\} = \alpha \Leftrightarrow \Phi\left(\frac{\mathbf{F}_L^{-1}(\alpha) - \mu(L)}{\sigma(L)}\right) = \alpha$$

We deduce that:

$$\frac{\mathbf{F}_L^{-1}(\alpha) - \mu(L)}{\sigma(L)} = \Phi^{-1}(\alpha) \Leftrightarrow \mathbf{F}_L^{-1}(\alpha) = \mu(L) + \Phi^{-1}(\alpha)\,\sigma(L)$$

The expression of the value-at-risk is then:

$$\mathrm{VaR}_\alpha(w; h) = \mu(L) + \Phi^{-1}(\alpha)\,\sigma(L)$$

if $\alpha = 99\%$, we obtain:

$$\mathrm{VaR}_{99\%}(w; h) = \mu(L) + 2.33 \times \sigma(L)$$

Gaussian value-at-risk

Example

We consider a short position of \$1 mn on the S&P 500 futures contract. We estimate that the annualized volatility $\hat{\sigma}_{\mathrm{SPX}}$ is equal to 35%

The portfolio loss is equal to $L(w) = N \times R_{\mathrm{SPX}}$ where N is the exposure amount ($-\$1$ mn) and R_{SPX} is the (Gaussian) return of the S&P 500 index. We deduce that the annualized loss volatility is $\hat{\sigma}(L) = |N| \times \hat{\sigma}_{\mathrm{SPX}}$. The value-at-risk for a one-year holding period is:

$$\mathrm{VaR}_{99\%}(w; \text{one year}) = 2.33 \times 10^6 \times 0.35 = \$815\,500$$

By using the square-root-of-time rule, we deduce that:

$$\mathrm{VaR}_{99\%}(w; \text{one day}) = \frac{815\,500}{\sqrt{260}} = \$50\,575$$

Gaussian expected shortfall

By definition, we have:

$$\begin{aligned}\mathrm{ES}_{\alpha}(w) &= \mathbb{E}\left[L(w) \mid L(w) \geq \mathrm{VaR}_{\alpha}(w)\right] \\ &= \frac{1}{1-\alpha}\int_{\mathbf{F}_{L}^{-1}(\alpha)}^{\infty} x f_{L}(x)\,\mathrm{d}x\end{aligned}$$

where f_L and $\mathbf{F}_L$ are the density and distribution functions of the loss $L(w)$

The Gaussian expected shortfall of the portfolio w is equal to:

$$\mathrm{ES}_{\alpha}(w) = \mu(L) + \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{(1-\alpha)}\sigma(L)$$

Proof

$$\mathrm{ES}_\alpha(w) = \frac{1}{1-\alpha}\int_{\mu(L)+\Phi^{-1}(\alpha)\sigma(L)}^{\infty} \frac{x}{\sigma(L)\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu(L)}{\sigma(L)}\right)^2\right) \mathrm{d}x$$

With the variable change $t = \sigma(L)^{-1}(x-\mu(L))$, we obtain:

$$\begin{aligned}
\mathrm{ES}_\alpha(w) &= \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^{\infty} (\mu(L)+\sigma(L)\,t)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t^2\right)\mathrm{d}t \\
&= \frac{\mu(L)}{1-\alpha}\left[\Phi(t)\right]_{\Phi^{-1}(\alpha)}^{\infty} + \frac{\sigma(L)}{(1-\alpha)\sqrt{2\pi}}\int_{\Phi^{-1}(\alpha)}^{\infty} t\exp\left(-\frac{1}{2}t^2\right)\mathrm{d}t \\
&= \mu(L) + \frac{\sigma(L)}{(1-\alpha)\sqrt{2\pi}}\left[-\exp\left(-\frac{1}{2}t^2\right)\right]_{\Phi^{-1}(\alpha)}^{\infty} \\
&= \mu(L) + \frac{\sigma(L)}{(1-\alpha)\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left[\Phi^{-1}(\alpha)\right]^2\right)
\end{aligned}$$

Gaussian VaR vs Gaussian ES

The value-at-risk and the expected shortfall are both a standard deviation-based risk measure. They coincide when the scaling parameters $c_{\mathrm{VaR}} = \Phi^{-1}\left(\alpha_{\mathrm{VaR}}\right)$ and $c_{\mathrm{ES}} = \phi\left(\Phi^{-1}\left(\alpha_{\mathrm{ES}}\right)\right) / \left(1 - \alpha_{\mathrm{ES}}\right)$ are equal.

Table: Scaling factors c_{VaR} and c_{ES}

α (in %)	95.0	96.0	97.0	97.5	98.0	98.5	99.0	99.5
c_{VaR}	1.64	1.75	1.88	1.96	2.05	2.17	**2.33**	2.58
c_{ES}	2.06	2.15	2.27	**2.34**	2.42	2.52	2.67	2.89

Linear factor models

When $g(\mathcal{F}_t; w) = \sum_{i=1}^{n} w_i P_{i,t}$, the random P&L is equal to:

$$\begin{aligned}
\Pi(w) &= P_{t+h}(w) - P_t(w) \\
&= \sum_{i=1}^{n} w_i P_{i,t+h} - \sum_{i=1}^{n} w_i P_{i,t} \\
&= \sum_{i=1}^{n} w_i \left(P_{i,t+h} - P_{i,t}\right)
\end{aligned}$$

We assume that the asset returns are the risk factors :

$$P_{i,t+h} = P_{i,t}\left(1 + R_{i,t+h}\right)$$

where $R_{i,t+h}$ is the asset return between t and $t+h$. In this case, we obtain:

$$\Pi(w) = \sum_{i=1}^{n} w_i P_{i,t} R_{i,t+h}$$

The covariance model

Let R_t be the vector of asset returns. We note $W_{i,t} = w_i P_{i,t}$ the wealth invested (or the nominal exposure) in asset i and $W_t = (W_{1,t}, \ldots, W_{n,t})$. It follows that:

$$\Pi(w) = \sum_{i=1}^{n} W_{i,t} R_{i,t+h} = W_t^\top R_{t+h}$$

If we assume that $R_{t+h} \sim \mathcal{N}(\mu, \Sigma)$, we deduce that $\mu(\Pi) = W_t^\top \mu$ and $\sigma^2(\Pi) = W_t^\top \Sigma W_t$. Therefore, the expression of the value-at-risk is:

$$\mathrm{VaR}_\alpha(w; h) = -W_t^\top \mu + \Phi^{-1}(\alpha) \sqrt{W_t^\top \Sigma W_t}$$

Example

We consider the Apple/Coca-Cola example. The nominal exposures are \$1 093.3 (Apple) and \$842.8 (Coca-Cola). The estimated standard deviation of daily returns is equal to 1.3611% for Apple and 0.9468% for Coca-Cola, whereas the cross-correlation is equal to 12.0787%. It follows that:

$$\begin{aligned}
\sigma^2(\Pi) &= W_t^\top \Sigma W_t \\
&= 1\,093.3^2 \times \left(\frac{1.3611}{100}\right)^2 + 842.8^2 \times \left(\frac{0.9468}{100}\right)^2 + \\
&\quad 2 \times \frac{12.0787}{100} \times 1\,093.3 \times 842.8 \times \frac{1.3611}{100} \times \frac{0.9468}{100} \\
&= 313.80
\end{aligned}$$

We deduce that the 99% daily value-at-risk is equal to:

$$\mathrm{VaR}_{99\%}(w; \text{one day}) = \Phi^{-1}(0.99)\sqrt{313.80} = \$41.21$$

The factor model

- CAPM (HFRM, pages 76-77)
- APT (HFRM, page 77 and Exercise 2.4.5 page 119)
- Application to a bond portfolio (HFRM, pages 77-80)

Some other topics

- Volatility forecasting EWMA, GARCH and SV models (HFRM, pages 80-83 and Section 10.2.4 page 664)
- Other probability distributions (HFRM, pages 84-90)
- Cornish-Fisher approximation (HFRM, pages 85-87)

$$\mathrm{VaR}_\alpha(w;h) = \mu(L) + Z(\alpha;\gamma_1(L),\gamma_2(L)) \times \sigma(L)$$

where:

$$Z(\alpha;\gamma_1,\gamma_2) = z_\alpha + \frac{1}{6}\left(z_\alpha^2 - 1\right)\gamma_1 + \frac{1}{24}\left(z_\alpha^3 - 3z_\alpha\right)\gamma_2 - \frac{1}{36}\left(2z_\alpha^3 - 5z_\alpha\right)\gamma_1^2$$

and $z_\alpha = \Phi^{-1}(\alpha)$

Monte Carlo methods

- We assume a given probability distribution $\mathbf{H}$ for the risk factors:

$$(\mathcal{F}_{1,t+h}, \ldots, \mathcal{F}_{m,t+h}) \sim \mathbf{H}$$

- We simulate n_S scenarios of risk factors and calculate the simulated P&L $\Pi_s(w)$ for each scenario s
- We calculate the empirical quantile using the order statistic approach

$\Rightarrow$ The **Monte Carlo VaR/ES** is a **historical VaR/ES** with simulated scenarios or the Monte Carlo VaR/ES is a **parametric VaR/ES** for which it is difficult to find an analytical formula

Identification of risk factors

We consider a portfolio containing w_S stocks and w_C call options on this stock. We note S_t and $\mathcal{C}_t$ the stock and option prices at time t. We have:

$$\Pi(w) = w_S(S_{t+h} - S_t) + w_C(\mathcal{C}_{t+h} - \mathcal{C}_t)$$

If we use asset returns as risk factors, we get:

$$\Pi(w) = w_S S_t R_{S,t+h} + w_C \mathcal{C}_t R_{C,t+h}$$

where $R_{S,t+h}$ and $R_{C,t+h}$ are the returns of the stock and the option for the period $[t, t+h]$

$\Rightarrow$ **Two risk factors**: $R_{S,t+h}$ and $R_{C,t+h}$?

Identification of risk factors

The problem is that the option price $\mathcal{C}_t$ is a non-linear function of the underlying price S_t:

$$\mathcal{C}_t = f_C(S_t)$$

This implies that:

$$\begin{aligned} \Pi(w) &= w_S S_t R_{S,t+h} + w_C \left(f_C(S_{t+h}) - \mathcal{C}_t\right) \\ &= w_S S_t R_{S,t+h} + w_C \left(f_C(S_t(1 + R_{S,t+h})) - \mathcal{C}_t\right) \end{aligned}$$

$\Rightarrow$ **One risk factor**: $R_{S,t+h}$?

The Black-Scholes formula

The price of the call option is equal to:

$$C_{\mathrm{BS}}\left(S_t, K, \Sigma_t, T, b_t, r_t\right) = S_t e^{(b_t - r_t)\tau} \Phi\left(d_1\right) - K e^{-r_t \tau} \Phi\left(d_2\right)$$

where:

- S_t is the current price of the underlying asset
- K is the option strike
- Σ_t is the volatility parameter,
- T is the maturity date
- b_t is the cost-of-carry[6]
- r_t is the interest rate
- the parameter $\tau = T - t$ is the time to maturity
- The coefficients d_1 and d_2 are defined as follows:

$$d_1 = \frac{1}{\Sigma_t \sqrt{\tau}} \left(\ln \frac{S_t}{K} + b_t \tau \right) + \frac{1}{2} \Sigma_t \sqrt{\tau} \quad \text{and} \quad d_2 = d_1 - \Sigma_t \sqrt{\tau}$$

[6]The cost-of-carry depends on the underlying asset. We have $b_t = r_t$ for non-dividend stocks and total return indices, $b_t = r_t - d_t$ for stocks paying a continuous dividend yield d_t, $b_t = 0$ for forward and futures contracts and $b_t = r_t - r_t^\star$ for foreign exchange options where $r_t^\star$ is the foreign interest rate.

Identification of risk factors

We can write the option price as follows:

$$\mathcal{C}_t = f_{\mathrm{BS}}\left(\theta_{\mathrm{contract}}; \theta\right)$$

where $\theta_{\mathrm{contract}}$ are the parameters of the contract (strike K and maturity T) and θ are the other parameters

- S_t is obviously a risk factor
- If Σ_t is not constant, the option price may be sensitive to the volatility risk
- The option may be impacted by changes in the interest rate or the cost-of-carry

$\Rightarrow$ The choice of risk factors depends on the derivative contract (volatility risk, dividend risk, yield curve risk, correlation risk, etc.)

Methods to calculate VAR and ES risk measures

1. The method of full pricing (option repricing)
2. The method of sensitivities (delta-gamma-vega approximation)
3. The hybrid method

The method of full pricing

We recall that the P&L of the s^{th} scenario has the following expression:

$$\Pi_s(w) = g(\mathcal{F}_{1,s}, \ldots, \mathcal{F}_{m,s}; w) - P_t(w)$$

In the case of the previous example, the P&L becomes then:

$$\Pi_s(w) = \begin{cases} w_S S_t R_s + w_C \left(f_C\left(S_t\left(1+R_s\right); \Sigma_t\right) - \mathcal{C}_t\right) & \text{with one risk factor} \\ w_S S_t R_s + w_C \left(f_C\left(S_t\left(1+R_s\right), \Sigma_s\right) - \mathcal{C}_t\right) & \text{with two risk factors} \end{cases}$$

where the pricing function is:

$$f_C(S; \Sigma) = C_{\mathrm{BS}}(S, K, \Sigma, T-h, b_t, r_t)$$

Remark

In the model with two risk factors, we have to simulate the underlying price and the implied volatility. For the single factor model, we use the current implied volatility Σ_t instead of the simulated value Σ_s.

Application to the VaR and ES

Example

We consider a long position on 100 call options with strike $K = 100$. The value of the call option is \$4.14, the residual maturity is 52 days and the current price of the underlying asset is \$100. We assume that $\Sigma_t = 20\%$ and $b_t = r_t = 5\%$. The objective is to calculate the daily 99% VaR and the daily 97.5% ES with 250 historical scenarios, whose first nine values are the following:

s	1	2	3	4	5	6	7	8	9
R_s	−1.93	−0.69	−0.71	−0.73	1.22	1.01	1.04	1.08	−1.61
$\Delta\Sigma_s$	−4.42	−1.32	−3.04	2.88	−0.13	−0.08	1.29	2.93	0.85

Remark

Data are available at
http://www.thierry-roncalli.com/download/frm-data1.xlsx

Application to the VaR and ES

$\Rightarrow$ The implied volatility is equal to 20%

For the first scenario, R_s is equal to -1.93% and S_{t+h} is equal to $100 \times (1 - 1.93\%) = 98.07$. The residual maturity τ is equal to $51/252$ years. It follows that:

$$\begin{aligned}
d_1 &= \frac{1}{20\% \times \sqrt{51/252}} \left(\ln \frac{98.07}{100} + 5\% \times \frac{51}{252} \right) + \frac{1}{2} \times 20\% \times \sqrt{\frac{51}{252}} = -0.0592 \\
d_2 &= -0.0592 - 20\% \times \sqrt{\frac{51}{252}} = -0.1491
\end{aligned}$$

We deduce that:

$$\begin{aligned}
\mathcal{C}_{t+h} &= 98.07 \times e^{(5\%-5\%)\frac{51}{252}} \times \Phi(-0.0592) - 100 \times e^{5\% \times \frac{51}{252}} \times \Phi(-0.1491) \\
&= 98.07 \times 1.00 \times 0.4764 - 100 \times 1.01 \times 0.4407 \\
&= 3.093
\end{aligned}$$

The simulated P&L for the first historical scenario is then equal to:

$$\Pi_s = 100 \times (3.093 - 4.14) = -104.69$$

Application to the VaR and ES

Table: Daily P&L of the long position on the call option when the risk factor is the underlying price

s	R_s (in %)	S_{t+h}	$\mathcal{C}_{t+h}$	Π_s
1	−1.93	98.07	3.09	−104.69
2	−0.69	99.31	3.72	−42.16
3	−0.71	99.29	3.71	−43.22
4	−0.73	99.27	3.70	−44.28
5	1.22	101.22	4.81	67.46
6	1.01	101.01	4.68	54.64
7	1.04	101.04	4.70	56.46
8	1.08	101.08	4.73	58.89
9	−1.61	98.39	3.25	−89.22

$\Rightarrow$ With the 250 historical scenarios, the 99% value-at-risk is equal to \$154.79, whereas the 97.5% expected shortfall is equal to \$150.04

The option return R_C is not a new risk factor

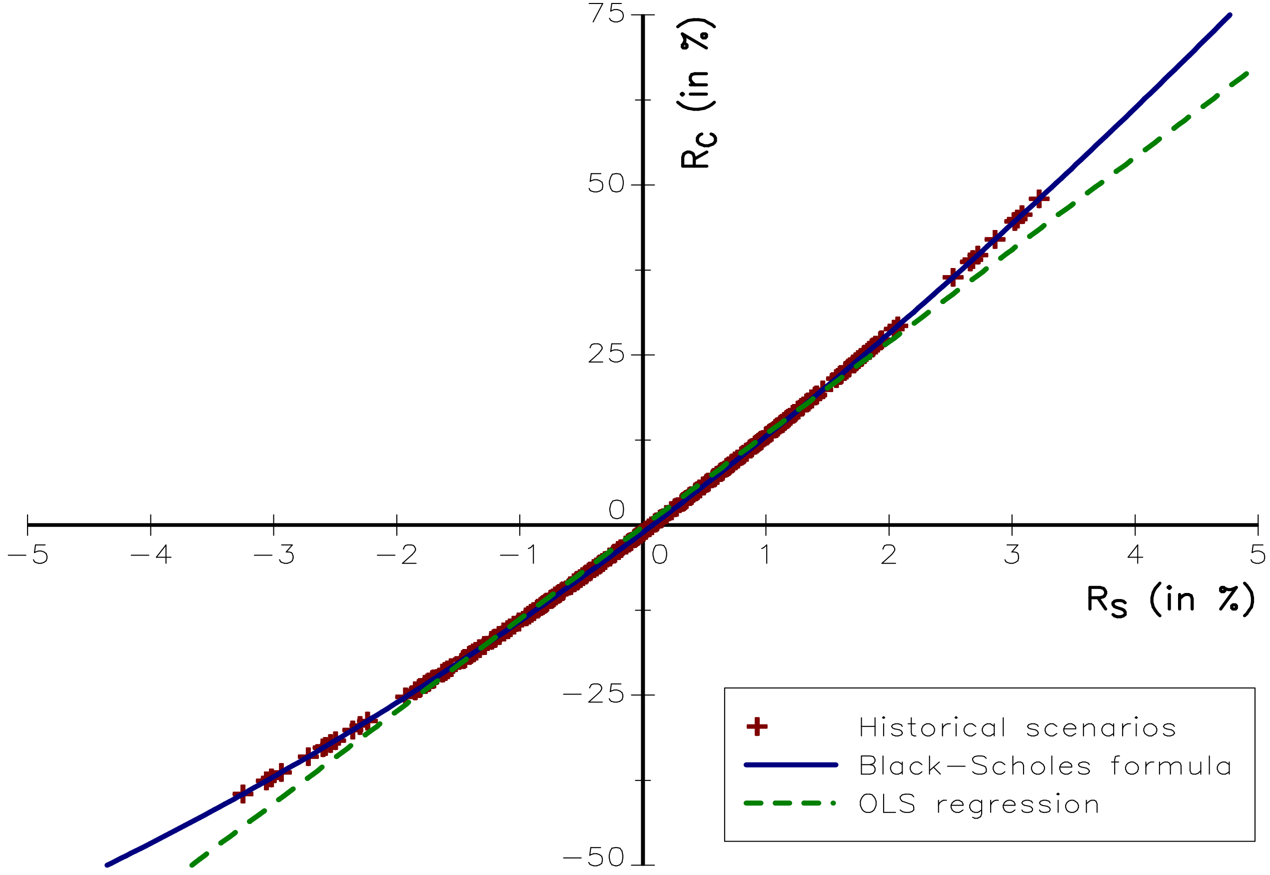


Figure: Relationship between the asset return R_S and the option return R_C

Adding the risk factor Σ_t

$$\Sigma_{t+h} = \Sigma_t + \Delta\Sigma_s$$

Table: Daily P&L of the long position on the call option when the risk factors are the underlying price and the implied volatility

s	R_s (in %)	S_{t+h}	$\Delta\Sigma_s$ (in %)	Σ_{t+h}	$\mathcal{C}_{t+h}$	Π_s
1	−1.93	98.07	−4.42	15.58	2.32	−182.25
2	−0.69	99.31	−1.32	18.68	3.48	−65.61
3	−0.71	99.29	−3.04	16.96	3.17	−97.23
4	−0.73	99.27	2.88	22.88	4.21	6.87
5	1.22	101.22	−0.13	19.87	4.79	65.20
6	1.01	101.01	−0.08	19.92	4.67	53.24
7	1.04	101.04	1.29	21.29	4.93	79.03
8	1.08	101.08	2.93	22.93	5.24	110.21
9	−1.61	98.39	0.85	20.85	3.40	−74.21

$\Rightarrow \mathrm{VaR}_{99\%}(w;\text{one day}) = \$181.70$ and $\mathrm{ES}_{97.5\%}(w;\text{one day}) = \$172.09$

The method of sensitivities

The previous approach is called *full pricing*, because it consists in re-pricing the option

In the method based on the Greek coefficients, the idea is to approximate the change in the option price by a Taylor expansion:

- Delta approach
- Delta-gamma approach
- Delta-gamma-theta approach
- Delta-gamma-theta-vega approach
- Etc.

The delta approach

We define the delta approach as follows:

$$\mathcal{C}_{t+h} - \mathcal{C}_t \simeq \boldsymbol{\Delta}_t \left(S_{t+h} - S_t\right)$$

where $\boldsymbol{\Delta}_t$ is the option delta:

$$\boldsymbol{\Delta}_t = \frac{\partial\, C_{\mathrm{BS}}\left(S_t, \Sigma_t, T\right)}{\partial\, S_t}$$

The delta approach applied to delta neutral portfolios

If we consider the introductory example, we have:

$$\begin{aligned}
\Pi(w) &= w_S(S_{t+h} - S_t) + w_C(\mathcal{C}_{t+h} - \mathcal{C}_t) \\
&\simeq (w_S + w_C\boldsymbol{\Delta}_t)(S_{t+h} - S_t) \\
&= (w_S + w_C\boldsymbol{\Delta}_t) S_t R_{S,t+h}
\end{aligned}$$

With the delta approach, we aggregate the risk by netting the different delta exposures[7]. In particular, the portfolio is delta neutral if the net exposure is zero:

$$w_S + w_C\boldsymbol{\Delta}_t = 0 \Leftrightarrow w_S = -w_C\boldsymbol{\Delta}_t$$

With the delta approach, the VaR/ES of delta neutral portfolios is then equal to zero

[7] A long (or short) position on the underlying asset is equivalent to $\boldsymbol{\Delta}_t = 1$ (or $\boldsymbol{\Delta}_t = -1$).

The delta-gamma approach

We can use the second-order approximation or the delta-gamma approach:

$$\mathcal{C}_{t+h} - \mathcal{C}_t \simeq \boldsymbol{\Delta}_t \left(S_{t+h} - S_t\right) + \frac{1}{2}\boldsymbol{\Gamma}_t \left(S_{t+h} - S_t\right)^2$$

where $\boldsymbol{\Gamma}_t$ is the option gamma:

$$\boldsymbol{\Gamma}_t = \frac{\partial^2 \, C_{\mathrm{BS}}\left(S_t, \Sigma_t, T\right)}{\partial \, S_t^2}$$

Comparison between delta and delta-gamma approaches

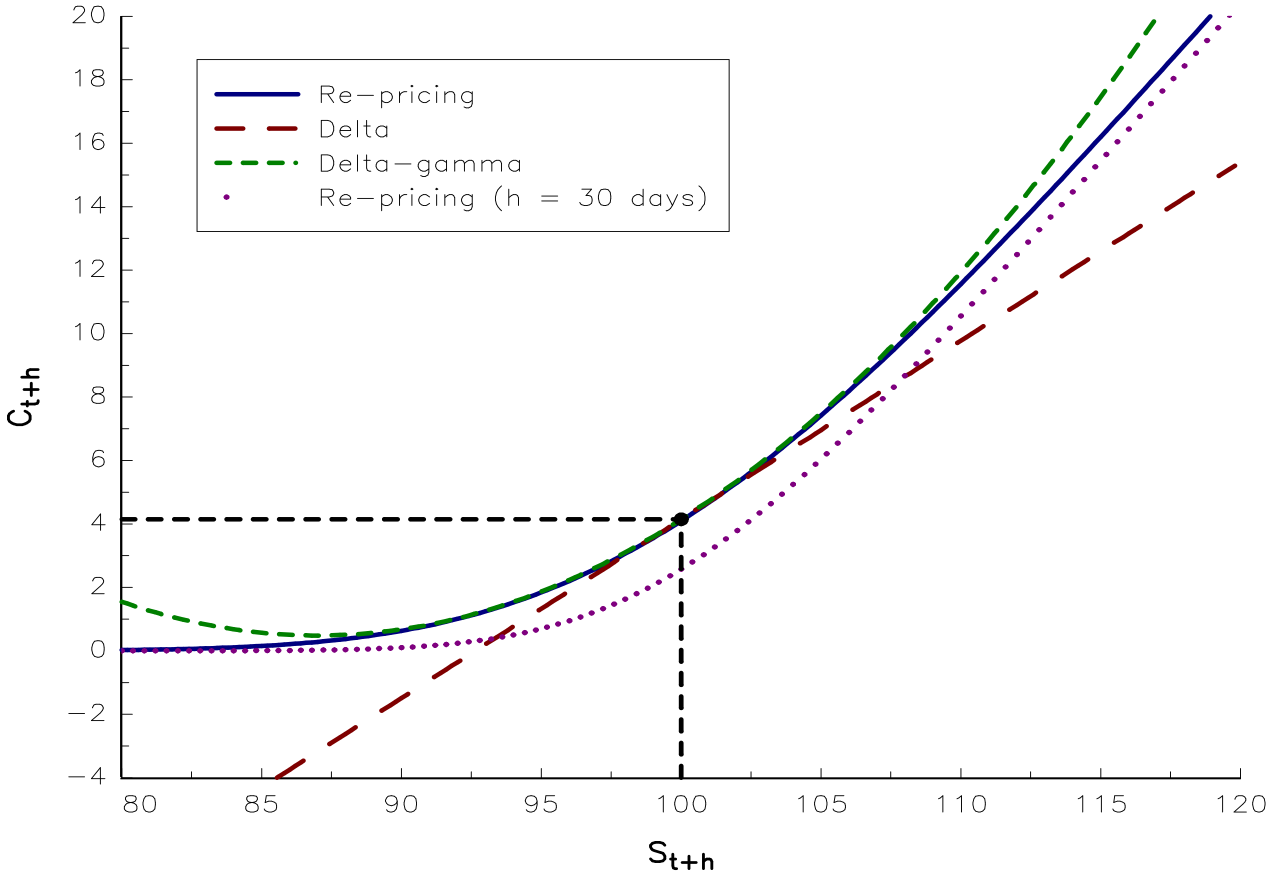


Figure: Approximation of the option price with the Greek coefficients

Extension to other risk factors

The Taylor expansion can be generalized to a set of risk factors $\mathcal{F}_t = (\mathcal{F}_{1,t}, \ldots, \mathcal{F}_{m,t})$:

$$\begin{aligned} \boldsymbol{\mathcal{C}}_{t+h} - \boldsymbol{\mathcal{C}}_t \quad \simeq \quad & \sum_{j=1}^{m} \frac{\partial\, \boldsymbol{\mathcal{C}}_t}{\partial\, \mathcal{F}_{j,t}} \left(\mathcal{F}_{j,t+h} - \mathcal{F}_{j,t}\right) + \\ & \frac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{m} \frac{\partial^2\, \boldsymbol{\mathcal{C}}_t}{\partial\, \mathcal{F}_{j,t}\, \partial\, \mathcal{F}_{k,t}} \left(\mathcal{F}_{j,t+h} - \mathcal{F}_{j,t}\right) \left(\mathcal{F}_{k,t+h} - \mathcal{F}_{k,t}\right) \end{aligned}$$

The delta-gamma-theta-vega approach is defined as follows:

$$\boldsymbol{\mathcal{C}}_{t+h} - \boldsymbol{\mathcal{C}}_t \simeq \boldsymbol{\Delta}_t \left(S_{t+h} - S_t\right) + \frac{1}{2} \boldsymbol{\Gamma}_t \left(S_{t+h} - S_t\right)^2 + \boldsymbol{\Theta}_t h + \boldsymbol{\upsilon}_t \left(\Sigma_{t+h} - \Sigma_t\right)$$

where $\boldsymbol{\Theta}_t = \partial_t\, C_{\mathrm{BS}}\left(S_t, \Sigma_t, T\right)$ is the option theta and $\boldsymbol{\upsilon}_t = \partial_{\Sigma_t}\, C_{\mathrm{BS}}\left(S_t, \Sigma_t, T\right)$ is the option vega

$\Rightarrow$ We can also include vanna and volga effects

The Black-Scholes Greek coefficients

$$
\begin{aligned}
\boldsymbol{\Delta}_t &= e^{(b_t - r_t)\tau} \Phi(d_1) \\
\boldsymbol{\Gamma}_t &= \frac{e^{(b_t - r_t)\tau} \phi(d_1)}{S_t \Sigma_t \sqrt{\tau}} \\
\boldsymbol{\Theta}_t &= -r_t K e^{-r_t \tau} \Phi(d_2) - \frac{1}{2\sqrt{\tau}} S_t \Sigma_t e^{(b_t - r_t)\tau} \phi(d_1) - \\
&\quad (b_t - r_t) S_t e^{(b_t - r_t)\tau} \Phi(d_1) \\
\boldsymbol{\upsilon}_t &= e^{(b_t - r_t)\tau} S_t \sqrt{\tau} \phi(d_1)
\end{aligned}
$$

(HFRM, Exercise 2.4.7 page 121)

Application to the VaR and ES

In the case of our previous example (Slide 82), we obtain $\boldsymbol{\Delta}_t = 0.5632$, $\boldsymbol{\Gamma}_t = 0.0434$, $\boldsymbol{\Theta}_t = -11.2808$ and $\boldsymbol{\upsilon}_t = 17.8946$

We have:

- $\Pi_1^{\boldsymbol{\Delta}}(w) = 100 \times 0.5632 \times (98.07 - 100) = -108.69$
- $\Pi_1^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}}(w) = -108.69 + 100 \times \frac{1}{2} \times 0.0434 \times (98.07 - 100)^2 = -100.61$
- $\Pi_1^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}+\boldsymbol{\Theta}}(w) = -100.61 - 11.2808 \times 1/252 = -105.09$
- $\Pi_1^{\boldsymbol{\upsilon}}(w) = 100 \times 17.8946 \times (15.58\% - 20\%) = -79.09$
- $\Pi_1^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}+\boldsymbol{\Theta}+\boldsymbol{\upsilon}}(w) = -105.90 - 79.09 = -184.99$

Application to the VaR and ES

Table: Calculation of the P&L based on the Greek sensitivities

s	R_s (in %)	S_{t+h}	Π_s	Π_s^{Δ}	$\Pi_s^{\Delta+\Gamma}$	$\Pi_s^{\Delta+\Gamma+\Theta}$
1	-1.93	98.07	-104.69	-108.69	-100.61	-105.09
2	-0.69	99.31	-42.16	-38.86	-37.83	42.30
3	-0.71	99.29	-43.22	-39.98	-38.89	-43.37
4	-0.73	99.27	-44.28	-41.11	-39.96	-44.43
5	1.22	101.22	67.46	68.71	71.93	67.46
6	1.01	101.01	54.64	56.88	59.09	54.61
7	1.04	101.04	56.46	58.57	60.91	56.44
8	1.08	101.08	58.89	60.82	63.35	58.87
9	-1.61	98.39	-89.22	-90.67	-85.05	-89.53
$\mathrm{VaR}_{99\%}(w;\text{one day})$			154.79	171.20	151.16	155.64
$\mathrm{ES}_{97.5\%}(w;\text{one day})$			150.04	165.10	146.37	150.84

Application to the VaR and ES

Table: Calculation of the P&L using the vega coefficient

s	S_{t+h}	Σ_{t+h}	Π_s	Π_s^{υ}	$\Pi_s^{\Delta+\upsilon}$	$\Pi_s^{\Delta+\Gamma+\upsilon}$	$\Pi_s^{\Delta+\Gamma+\Theta+\upsilon}$
1	98.07	15.58	−182.25	−79.09	−187.78	−179.71	−184.19
2	99.31	18.68	−65.61	−23.62	−62.48	−61.45	−65.92
3	99.29	16.96	−97.23	−54.40	−94.38	−93.29	−97.77
4	99.27	22.88	6.87	51.54	10.43	11.58	7.10
5	101.22	19.87	65.20	−2.33	66.38	69.61	65.13
6	101.01	19.92	53.24	−1.43	55.45	57.66	53.18
7	101.04	21.29	79.03	23.08	81.65	84.00	79.52
8	101.08	22.93	110.21	52.43	113.25	115.78	111.30
9	98.39	20.85	−74.21	15.21	−75.46	−69.84	−74.32
$\mathrm{VaR}_{99\%}\,(w;\text{one day})$			181.70	77.57	190.77	179.29	183.76
$\mathrm{ES}_{97.5\%}\,(w;\text{one day})$			172.09	73.90	184.90	169.34	173.81

The hybrid method

The hybrid method consists of combining the two approaches:

1. we first calculate the P&L for each (historical or simulated) scenario with the method based on the sensitivities;
2. we then identify the worst scenarios;
3. we finally revalue these worst scenarios by using the full pricing method.

$\Rightarrow$ The underlying idea is to consider the faster approach to locate the value-at-risk, and then to use the most accurate approach to calculate the right value

The hybrid method

Table: The 10 worst scenarios identified by the hybrid method

i	Full pricing		Greeks					
			$\Delta - \Gamma - \Theta - \upsilon$		$\Delta - \Theta$		$\Delta - \Theta - \upsilon$	
	s	Π_s	s	Π_s	s	Π_s	s	Π_s
1	100	-183.86	100	-186.15	182	-187.50	134	-202.08
2	1	-182.25	1	-184.19	169	-176.80	100	-198.22
3	134	-181.15	134	-183.34	27	-174.55	1	-192.26
4	27	-163.01	27	-164.26	134	-170.05	169	-184.32
5	169	-162.82	169	-164.02	69	-157.66	27	-184.04
6	194	-159.46	194	-160.93	108	-150.90	194	-175.36
7	49	-150.25	49	-151.43	194	-149.77	49	-165.41
8	245	-145.43	245	-146.57	49	-147.52	182	-164.96
9	182	-142.21	182	-142.06	186	-145.27	245	-153.37
10	79	-135.55	79	-136.52	100	-137.38	69	-150.68

Backtesting

mark-to-model $\neq$ mark-to-market

For on-exchange products, the simulated P&L is equal to:

$$\Pi_s(w) = \underbrace{P_{t+1}(w)}_{\text{mark-to-model}} - \underbrace{P_t(w)}_{\text{mark-to-market}}$$

whereas the realized P&L is equal to:

$$\Pi(w) = \underbrace{P_{t+1}(w)}_{\text{mark-to-market}} - \underbrace{P_t(w)}_{\text{mark-to-market}}$$

Backtesting

For exotic options and OTC derivatives, the simulated P&L is the difference between two mark-to-model values:

$$\Pi_s(w) = \underbrace{P_{t+1}(w)}_{\text{mark-to-model}} - \underbrace{P_t(w)}_{\text{mark-to-model}}$$

and the realized P&L is also the difference between two mark-to-model values:

$$\Pi(w) = \underbrace{P_{t+1}(w)}_{\text{mark-to-model}} - \underbrace{P_t(w)}_{\text{mark-to-model}}$$

$\Rightarrow$ **Model risk**

Model risk

4 types of model risk:

1. Operational risk
2. Parameter risk
3. Mis-specification risk
4. Hedging risk

(HFRM, Chapter 9, Page 491)

On the importance of risk allocation

Let us consider two trading desks A and B, whose risk measure is respectively $\mathcal{R}(w_A)$ and $\mathcal{R}(w_B)$. At the global level, the risk measure is equal to $\mathcal{R}(w_{A+B})$. The question is then how to allocate $\mathcal{R}(w_{A+B})$ to the trading desks A and B:

$$\mathcal{R}(w_{A+B}) = \mathcal{RC}_A(w_{A+B}) + \mathcal{RC}_B(w_{A+B})$$

Remark

This question is an important issue for the bank because risk allocation means capital allocation:

$$\mathcal{K}(w_{A+B}) = \mathcal{K}_A(w_{A+B}) + \mathcal{K}_B(w_{A+B})$$

Capital allocation is not neutral, because **it will impact the profitability of business units that compose the bank**

Euler allocation principle

- We decompose the P&L as follows:

$$\Pi = \sum_{i=1}^{n} \Pi_i$$

  where Π_i is the P&L of the i^{th} sub-portfolio

- We note $\mathcal{R}(\Pi)$ the risk measure associated with the P&L
- We consider the risk-adjusted performance measure (RAPM) defined by:

$$\mathrm{RAPM}(\Pi) = \frac{\mathbb{E}[\Pi]}{\mathcal{R}(\Pi)}$$

- We consider the portfolio-related RAPM of the i^{th} sub-portfolio defined by:

$$\mathrm{RAPM}(\Pi_i \mid \Pi) = \frac{\mathbb{E}[\Pi_i]}{\mathcal{R}(\Pi_i \mid \Pi)}$$

Euler allocation principle

Based on the notion of RAPM, Tasche (2008) states two properties of **risk contributions** that are desirable from an economic point of view:

1. Risk contributions $\mathcal{R}(\Pi_i \mid \Pi)$ to portfolio-wide risk $\mathcal{R}(\Pi)$ satisfy the full allocation property if:

$$\sum_{i=1}^{n} \mathcal{R}(\Pi_i \mid \Pi) = \mathcal{R}(\Pi)$$

2. Risk contributions $\mathcal{R}(\Pi_i \mid \Pi)$ are RAPM compatible if there are some $\varepsilon_i > 0$ such that:

$$\mathrm{RAPM}(\Pi_i \mid \Pi) > \mathrm{RAPM}(\Pi) \Rightarrow \mathrm{RAPM}(\Pi + h\Pi_i) > \mathrm{RAPM}(\Pi)$$

for all $0 < h < \varepsilon_i$

$\Rightarrow$ This property means that assets with a better risk-adjusted performance than the portfolio continue to have a better RAPM if their allocation increases in a small proportion

Euler allocation principle

Tasche (2008) shows that if there are risk contributions that are RAPM compatible, then $\mathcal{R}(\Pi_i \mid \Pi)$ is uniquely determined as:

$$\mathcal{R}(\Pi_i \mid \Pi) = \left. \frac{\mathrm{d}}{\mathrm{d}h} \mathcal{R}(\Pi + h\Pi_i) \right|_{h=0}$$

and the risk measure is homogeneous of degree 1

If we consider the risk measure $\mathcal{R}(w)$ defined in terms of weights, the risk contribution of sub-portfolio i is uniquely defined as:

$$\mathcal{RC}_i = w_i \frac{\partial\, \mathcal{R}(w)}{\partial\, w_i}$$

and the risk measure satisfies the Euler decomposition (or the Euler allocation principle):

$$\mathcal{R}(w) = \sum_{i=1}^{n} w_i \frac{\partial\, \mathcal{R}(w)}{\partial\, w_i} = \sum_{i=1}^{n} \mathcal{RC}_i$$

Application to Gaussian risk measures

If we assume that the portfolio return $R(w)$ is a linear function of the weights w, the expression of the standard deviation-based risk measure becomes:

$$\mathcal{R}(w) = -\mu(w) + c \cdot \sigma(w) = -w^\top \mu + c \cdot \sqrt{w^\top \Sigma w}$$

where μ and Σ are the mean vector and the covariance matrix of sub-portfolios

We have:

$$\frac{\partial \mathcal{R}(w)}{\partial w} = -\mu + c \cdot \frac{1}{2} \left(w^\top \Sigma w\right)^{-1/2} (2\Sigma w) = -\mu + c \cdot \frac{\Sigma w}{\sqrt{w^\top \Sigma w}}$$

The risk contribution of the i^{th} sub-portfolio is then:

$$\mathcal{RC}_i = w_i \cdot \left(-\mu_i + c \cdot \frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}}\right)$$

Application to Gaussian risk measures

We verify that the standard deviation-based risk measure satisfies the full allocation property:

$$\begin{aligned}
\sum_{i=1}^{n} \mathcal{RC}_i &= \sum_{i=1}^{n} w_i \cdot \left(-\mu_i + c \cdot \frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}}\right) \\
&= w^\top \left(-\mu + c \cdot \frac{\Sigma w}{\sqrt{w^\top \Sigma w}}\right) \\
&= -w^\top \mu + c \cdot \sqrt{w^\top \Sigma w} \\
&= \mathcal{R}(w)
\end{aligned}$$

Application to Gaussian risk measures

- Gaussian VaR risk contribution:

$$\mathcal{RC}_i = w_i \cdot \left(-\mu_i + \Phi^{-1}(\alpha) \cdot \frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}} \right)$$

- Gaussian ES risk contribution:

$$\mathcal{RC}_i = w_i \cdot \left(-\mu_i + \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{(1-\alpha)} \cdot \frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}} \right)$$

Application to Gaussian risk measures

block

We consider the Apple/Coca-Cola portfolio that has been used for calculating the Gaussian VaR. We recall that the nominal exposures were \$1 093.3 (Apple) and \$842.8 (Coca-Cola), the estimated standard deviation of daily returns was equal to 1.3611% for Apple and 0.9468% for Coca-Cola and the cross-correlation of stock returns was equal to 12.0787%.

Application to Gaussian risk measures

Table: Risk decomposition of the 99% Gaussian value-at-risk

Asset	w_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
Apple	1093.3	2.83%	30.96	75.14%
Coca-Cola	842.8	1.22%	10.25	24.86%
$\mathcal{R}(w)$			41.21	

Table: Risk decomposition of the 99% Gaussian expected shortfall

Asset	w_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^{\star}$
Apple	1093.3	3.24%	35.47	75.14%
Coca-Cola	842.8	1.39%	11.74	24.86%
$\mathcal{R}(w)$			47.21	

Application to non-normal risk measures

Generalized formulas

- The risk contribution for the value-at-risk is equal to:

$$\mathcal{RC}_i = \mathbb{E}\left[L_i \mid L(w) = \mathrm{VaR}_\alpha(L)\right]$$

- The risk contribution for the expected shortfall is equal to:

$$\mathcal{RC}_i = \mathbb{E}\left[L_i \mid L(w) \geq \mathrm{VaR}_\alpha(L)\right]$$

$\Rightarrow$ These formulas can easily be applied to historical and Monte Carlo risk measures (HFRM, pages 109-116)

Calculating the Gaussian VaR risk contribution

Asset returns are assumed to be Gaussian:

$$R \sim \mathcal{N}(\mu, \Sigma)$$

The portfolio's loss is equal to:

$$L(w) = -R(w) = -\sum_{i=1}^{n} w_i R_i = -w^\top R$$

We notice that:

$$L_i = -w_i R_i$$

and:

$$\mathbb{E}[L_i \mid L(w) = \mathrm{VaR}_\alpha(w; h)] = -w_i \mathbb{E}[R_i \mid L(w) = \mathrm{VaR}_\alpha(w; h)]$$

Calculating the Gaussian VaR risk contribution

We have:

$$\left(\begin{array}{c} R \\ L(w) \end{array} \right) = \left(\begin{array}{c} I_n \\ -w^\top \end{array} \right) R$$

and:

$$\left(\begin{array}{c} R \\ L(w) \end{array} \right) \sim \mathcal{N}\left(\left(\begin{array}{c} \mu \\ -w^\top \mu \end{array} \right), \left(\begin{array}{cc} \Sigma & -\Sigma w \\ -w^\top \Sigma & w^\top \Sigma w \end{array} \right) \right)$$

We would like to calculate:

$$\mathcal{RC}_i = -w_i \mathbb{E}\left[R_i \mid L(w) = \mathrm{VaR}_\alpha(w; h) \right]$$

Conditional distribution in the case of the normal distribution

Let us consider a Gaussian random vector defined as follows:

$$\left(\begin{array}{c} X \\ Y \end{array}\right) \sim \mathcal{N}\left(\left(\begin{array}{c} \mu_x \\ \mu_y \end{array}\right), \left(\begin{array}{cc} \Sigma_{x,x} & \Sigma_{x,y} \\ \Sigma_{y,x} & \Sigma_{y,y} \end{array}\right)\right)$$

We have:

$$Y \mid X = x \sim \mathcal{N}\left(\mu_{y|x}, \Sigma_{y,y|x}\right)$$

where:

$$\mu_{y|x} = \mathbb{E}\left[Y \mid X = x\right] = \mu_y + \Sigma_{y,x}\Sigma_{x,x}^{-1}\left(x - \mu_x\right)$$

and:

$$\Sigma_{y,y|x} = \operatorname{cov}\left(Y \mid X = x\right) = \Sigma_{y,y} - \Sigma_{y,x}\Sigma_{x,x}^{-1}\Sigma_{x,y}$$

Calculating the Gaussian VaR risk contribution

Since $\mathrm{VaR}_\alpha(w;h) = -w^\top \mu + \Phi^{-1}(\alpha)\sqrt{w^\top \Sigma w}$, we have:

$$\begin{array}{rcl}
\mathbb{E}\left[R \mid L(w) = \mathrm{VaR}_\alpha(w;h)\right] & = & \mathbb{E}\left[R \mid L(w) = -w^\top \mu + \Phi^{-1}(\alpha)\sqrt{w^\top \Sigma w}\right] \\
& = & \mu - \Sigma w \left(w^\top \Sigma w\right)^{-1} \cdot \\
& & \left(-w^\top \mu + \Phi^{-1}(\alpha)\sqrt{w^\top \Sigma w} - \left(-w^\top \mu\right)\right) \\
& = & \mu - \Phi^{-1}(\alpha)\,\Sigma w \dfrac{\sqrt{w^\top \Sigma w}}{\left(w^\top \Sigma w\right)^{-1}} \\
& = & \mu - \Phi^{-1}(\alpha)\dfrac{\Sigma w}{\sqrt{w^\top \Sigma w}}
\end{array}$$

and:

$$\mathcal{RC}_i = -w_i \left(\mu - \Phi^{-1}(\alpha)\frac{\Sigma w}{\sqrt{w^\top \Sigma w}}\right)_i = -w_i \mu_i + \Phi^{-1}(\alpha)\frac{w_i \cdot (\Sigma w)_i}{\sqrt{w^\top \Sigma w}}$$

Exercises

- Value-at-risk
  - Exercise 2.4.2 – Covariance matrix
  - Exercise 2.4.4 – Value-at-risk of a long/short portfolio
  - Exercise 2.4.4 – Value-at-risk of an equity portfolio hedged with put options
- Expected shortfall
  - Exercise 2.4.10 – Expected shortfall of an equity portfolio
  - Exercise 2.4.11 – Risk measure of a long/short portfolio
- Options and derivatives
  - Exercise 2.4.6 – Risk management of exotic options
  - Exercise 2.4.7 – P&L approximation with Greek sensitivities

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