Equally-weighted Risk contributions: a new method to build risk balanced diversified portfolios

Séminaire ???

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Agenda

- The asset allocation problem
- Heuristic solutions: minimum variance and 1/n portfolios
- A new method: the Equal Risk Contributions (ERC) portfolio
- Properties of the ERC portfolio
- Some practical applications

1 The asset allocation problem

- The classical mean-variance model
- Two heuristic solutions
  - The minimum variance portfolio
  - The 1/n portfolio

1.1 The limitations of the classical mean-variance model

The classical asset-allocation problem is:

$$\begin{aligned} x^{\star}(\phi) &= \arg\min_{x} x^{\top}\Sigma x - \phi x^{\top}\mu \\ &\text{u.c. } \mathbf{1}^{\top}x = 1 \text{ and } \mathbf{0} \leq x \leq \mathbf{1} \end{aligned}$$

$\Rightarrow$ Main difficulty is the sensitivity of the solution to small changes in the inputs.

An illustrative example

(0) (μ_i, σ_i) are respectively equal to $(8\%, 12\%)$, $(7\%, 10\%)$, $(7.5\%, 11\%)$, $(8.5\%, 13\%)$ and $(8\%, 12\%)$. The correlation matrix is $C_5(\rho)$ with $\rho = 60\%$.

(a) $\rho = 50\%$

(b) $\mu_2 = 8\%$

(c) $\rho = 50\%$ and $\mu_2 = 8\%$

(d) $\mu_3 = 10\%$

$\Rightarrow$ We compute optimal portfolios x^* such that $\sigma(x^*) = 10\%$:

Case	(0)	(a)	(b)	(c)	(d)
x_1	23.96	26.46	6.87	6.88	21.64
x_2	6.43	0.00	44.47	33.61	0.00
x_3	16.92	6.97	0.00	0.00	22.77
x_4	28.73	40.11	41.79	52.63	33.95
x_5	23.96	26.46	6.87	6.88	21.64

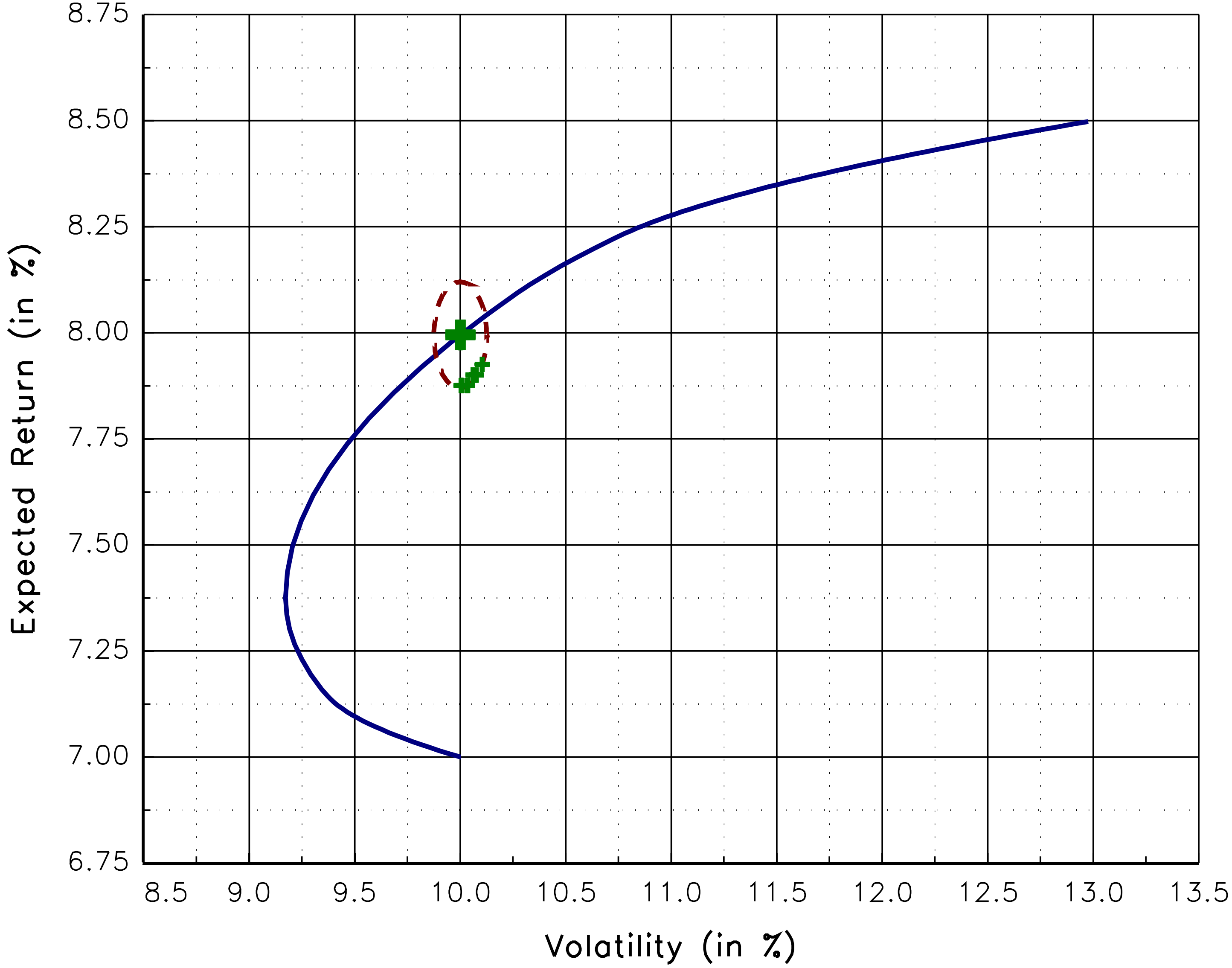
Some equivalent portfolios

Let us consider the case (0) and the optimal portfolio x^* with $(\mu(x^*), \sigma(x^*)) = (7.99\%, 10\%)$. Here are some portfolios which are closed to the optimal portfolio:

x_1	23.96		5	5	35	35	50	5	5	10
x_2	6.43	25		25	10	25	10	30		25
x_3	16.92	5	40		10	5	15		45	10
x_4	28.73	35	20	30	5	35	10	35	20	45
x_5	23.96	35	35	40	40		15	30	30	10
$\mu(x)$	7.99	7.90	7.90	7.90	7.88	7.90	7.88	7.88	7.88	7.93
$\sigma(x)$	10.00	10.07	10.06	10.07	10.01	10.07	10.03	10.00	10.03	10.10

$\Rightarrow$ These portfolios have very different compositions, but lead to very close mean-variance features.

$\Rightarrow$ **Some of these portfolios appear more balanced and, in some sense, more diversified than the optimal potfolio.**


8.75
8.50
8.25
8.00
7.75
7.50
7.25
7.00
6.75
Expected Return (in %)
8.5
9.0
9.5
10.0
10.5
11.0
11.5
12.0
12.5
13.0
13.5
Volatility (in %)

1.2 Some solutions

- Portfolio resampling (Michaud, 1989)
- Robust asset allocation (Tütüncü and Koenig, 2004)

⇒ Market practice: many investors prefer more heuristic solutions, which are computationally simple to implement and appear robust as they are not dependent on expected returns.

- **The minimum variance (mv) portfolio**
  It is obtained for $\phi = 0$ in the mean-variance problem and does not depend on the expected returns.
- **The equally-weighted (ew or 1/n) portfolio**
  Another simple way is to attribute the same weight to all the assets of the portfolio (Bernartzi and Thaler, 2001).

The minimum variance portfolio

In the previous example, the mv portfolio is $x_1 = 11.7\%$, $x_2 = 49.1\%$, $x_3 = 27.2\%$, $x_4 = 0.0\%$ and $x_5 = 11.7\%$.

$\Rightarrow$ This portfolio suffers from large concentration in the second asset whose volatility is the lowest.

The 1/n portfolio

In the previous example, the ew portfolio is
$x_1 = x_2 = x_3 = x_4 = x_5 = 20\%$.

$\Rightarrow$ This portfolio does not take into account volatilities and cross-correlations!

How to build a 1/n portfolio which takes into account risk?

2 Theoretical aspects of the ERC portfolio

- Definition
- Mathematical link between the ew, mv and erc portfolios
- Some analytics
- $\sigma_{\text{mv}} \leq \sigma_{\text{erc}} \leq \sigma_{1/\text{n}}$
- Numerical algorithm to compute the ERC portfolio
- The existence of the ERC portfolio

2.1 Definition

Let $\sigma(x) = \sqrt{x^\top \Sigma x}$ be the risk of the portfolio x. The Euler decomposition gives us:

$$\sigma(x) = \sum_{i=1}^{n} \sigma_i(x) = \sum_{i=1}^{n} x_i \times \frac{\partial\, \sigma(x)}{\partial\, x_i}.$$

with $\partial_{x_i} \sigma(x)$ the marginal risk contribution and $\sigma_i(x) = x_i \times \partial_{x_i} \sigma(x)$ the risk contribution of the i^{th} asset.

Starting from the definition of the risk contribution $\sigma_i(x)$, the idea of the ERC strategy is to find a risk-balanced portfolio such that the risk contribution is the same for all assets of the portfolio:

$$\sigma_i(x) = \sigma_j(x)$$

Remark: We restrict ourseleves to cases without short selling.

An example

Let us consider the previous case (0). We have:

$\sigma(x) = 9.5\%$	x_i	$\partial_{x_i}\,\sigma(x)$	$x_i \times \partial_{x_i}\,\sigma(x)$	$c_i(x)$
1	19.2%	0.099	0.019	20%
2	23.0%	0.082	0.019	20%
3	20.8%	0.091	0.019	20%
4	17.7%	0.101	0.019	20%
5	19.2%	0.099	0.019	20%

$c_i(x)$ is the relative contribution of the asset i to the risk of the portfolio:

$$\begin{aligned} c_i(x) &= \sigma_i(x)/\sigma(x) \\ &= \frac{x_i \times \partial_{x_i}\,\sigma(x)}{\sigma(x)} \end{aligned}$$

2.2 Mathematical comparison of the ew, mv and erc portfolios

- ew

$$x_i = x_j$$

- mv

$$\frac{\partial\,\sigma\,(x)}{\partial\,x_i} = \frac{\partial\,\sigma\,(x)}{\partial\,x_j}$$

- erc

$$x_i \frac{\partial\,\sigma\,(x)}{\partial\,x_i} = x_j \frac{\partial\,\sigma\,(x)}{\partial\,x_j}$$

$\Rightarrow$ **The ERC portfolio may be viewed as a portfolio "between" the 1/n portfolio and the minimum variance portfolio.**

2.3 Some analytics

The correlations are the same The solution is:

$$x_i = \frac{\sigma_i^{-1}}{\sum_{j=1}^n \sigma_j^{-1}}$$

The weight allocated to each component i is given by the ratio of the inverse of its volatility with the harmonic average of the volatilities.

The volatilities are the same We have:

$$x_i \propto \left(\sum_{k=1}^n x_k \rho_{ik} \right)^{-1}$$

The weight of the asset i is proportional to the inverse of the weighted average of correlations of component i with other components.

The general case We have:

$$x_i \propto \beta_i^{-1}$$

The weight of the asset i is proportional to the inverse of its beta.

Remark 1 *The solution for the two previous cases is endogeneous since x_i is a function of itself directly and through the constraint that $\sum x_i = 1$.*

2.4 The main theorem ($\sigma_{\text{mv}} \leq \sigma_{\text{erc}} \leq \sigma_{1/\text{n}}$)

Consider the following optimization problem:

$$\begin{aligned} x^{\star}(c) &= \arg\min \sqrt{x^{\top}\Sigma x} \\ &\text{u.c.} \begin{cases} \sum_{i=1}^{n} \ln x_i \geq c \\ \mathbf{1}^{\top} x = 1 \\ \mathbf{0} \leq x \leq \mathbf{1} \end{cases} \end{aligned}$$

Notice that if $c_1 \leq c_2$, we have $\sigma(x^{\star}(c_1)) \leq \sigma(x^{\star}(c_2))$. Moreover, we have $x^{\star}(-\infty) = x_{\text{mv}}$ and $x^{\star}(-n \ln n) = x_{1/\text{n}}$. The ERC portfolio corresponds to a particular value of c such that $-\infty \leq c \leq -n \ln n$. We also obtain the following inequality:

$$\sigma_{\text{mv}} \leq \sigma_{\text{erc}} \leq \sigma_{1/\text{n}}$$

$\Rightarrow$ **The ERC portfolio may be viewed as a form of variance-minimizing portfolio subject to a constraint of sufficient diversification in terms of component weights.**

2.5 Another result

Let us consider the minimum variance portfolio with a constant correlation matrix $C_n(\rho)$. The solution is:

$$x_i = \frac{-((n-1)\rho + 1)\sigma_i^{-2} + \rho \sum_{j=1}^{n} \left(\sigma_i \sigma_j\right)^{-1}}{\sum_{k=1}^{n} \left(-((n-1)\rho + 1)\sigma_k^{-2} + \rho \sum_{j=1}^{n} \left(\sigma_k \sigma_j\right)^{-1}\right)}$$

The lower bound of $C_n(\rho)$ is achieved for $\rho = -(n-1)^{-1}$. and we have:

$$x_i = \frac{\sum_{j=1}^{n} \left(\sigma_i \sigma_j\right)^{-1}}{\sum_{k=1}^{n} \sum_{j=1}^{n} \left(\sigma_k \sigma_j\right)^{-1}} = \frac{\sigma_i^{-1}}{\sum_{k=1}^{n} \sigma_k^{-1}} \longrightarrow \text{erc}$$

2.6 Numerical algorithm to compute the ERC portfolio

The ERC portfolio is the solution of the following problem:

$$x^{\star} = \left\{ x \in [0,1]^n : x^\top \mathbf{1} = 1, \sigma_i(x) = \sigma_j(x) \text{ for all } i, j \right\}$$

If the ERC portfolio exists, there exists a constant $c > 0$ such that:

$$x \odot \Sigma x = c\mathbf{1} \text{ u.c. } \mathbf{1}^\top x = 1 \text{ and } \mathbf{0} \leq x \leq \mathbf{1}$$

We may show that solving this problem is equivalent to solving the following non-linear optimisation problem:

$$\begin{aligned} x^{\star} &= \arg\min f(x) \\ & \text{u.c. } \mathbf{1}^\top x = 1 \text{ and } \mathbf{0} \leq x \leq \mathbf{1} \end{aligned}$$

with:

$$f(x) = n \sum_{i=1}^{n} x_i^2 (\Sigma x)_i^2 - \sum_{i,j=1}^{n} x_i x_j (\Sigma x)_i (\Sigma x)_j$$

$\Rightarrow$ numerical solution may be obtained with a SQP algorithm.

2.7 An example

We consider a universe of 4 risky assets. Volatilities are respectively 10%, 20%, 30% and 40%.

The case of the constant correlation matrix

$$\rho = C_4 (50\%)$$

The $1/n$ portfolio

$\sigma(x) = 20.2\%$	x_i	$\partial_{x_i} \sigma(x)$	$x_i \times \partial_{x_i} \sigma(x)$	$c_i(x)$
1	25%	0.068	0.017	8.5%
2	25%	0.149	0.037	18.5%
3	25%	0.242	0.060	30.0%
4	25%	0.347	0.087	43.1%

The minimum variance portfolio

$\sigma(x) = 10.0\%$	x_i	$\partial_{x_i}\,\sigma(x)$	$x_i \times \partial_{x_i}\,\sigma(x)$	$c_i(x)$
1	100%	0.100	0.100	100%
2	0%	0.100	0.000	0%
3	0%	0.150	0.000	0%
4	0%	0.200	0.000	0%

The ERC portfolio

$\sigma(x) = 15.2\%$	x_i	$\partial_{x_i}\,\sigma(x)$	$x_i \times \partial_{x_i}\,\sigma(x)$	$c_i(x)$
1	48%	0.079	0.038	25%
2	24%	0.158	0.038	25%
3	16%	0.237	0.038	25%
4	12%	0.316	0.038	25%

The case of a general correlation matrix

$$\rho = \begin{pmatrix} 1.00 & & & \\ 0.80 & 1.00 & & \\ 0.00 & 0.00 & 1.00 & \\ 0.00 & 0.00 & -0.50 & 1.00 \end{pmatrix}$$

The $1/n$ portfolio

$\sigma(x) = 11.5\%$	x_i	$\partial_{x_i} \sigma(x)$	$x_i \times \partial_{x_i} \sigma(x)$	$c_i(x)$
1	25%	0.056	0.014	12.3%
2	25%	0.122	0.030	26.4%
3	25%	0.065	0.016	14.1%
4	25%	0.217	0.054	47.2%

The minimum variance portfolio

$\sigma(x) = 8.6\%$	x_i	$\partial_{x_i}\,\sigma(x)$	$x_i \times \partial_{x_i}\,\sigma(x)$	$c_i(x)$
1	74.5%	0.086	0.064	74.5%
2	0%	0.138	0.000	0%
3	15.2%	0.086	0.013	15.2%
4	10.3%	0.086	0.009	10.3%

The ERC portfolio

$\sigma(x) = 10.3\%$	x_i	$\partial_{x_i}\,\sigma(x)$	$x_i \times \partial_{x_i}\,\sigma(x)$	$c_i(x)$
1	38.4%	0.067	0.026	25%
2	19.2%	0.134	0.026	25%
3	24.3%	0.106	0.026	25%
4	18.2%	0.141	0.026	25%

2.8 Existence of the ERC portfolio

$\Rightarrow$ In the first version of the working paper, we found examples without numerical solution. These examples suggest us that the ERC portfolio may not exist in some cases.

$\Rightarrow$ In the last version of the paper, we prove that the ERC portfolio always exists !

$\Rightarrow$ When the optimization problem is tricky, we have to replace the constrained problem with another optimization problem without the constraint $\sum x_i = 1$ and rescale the solution.

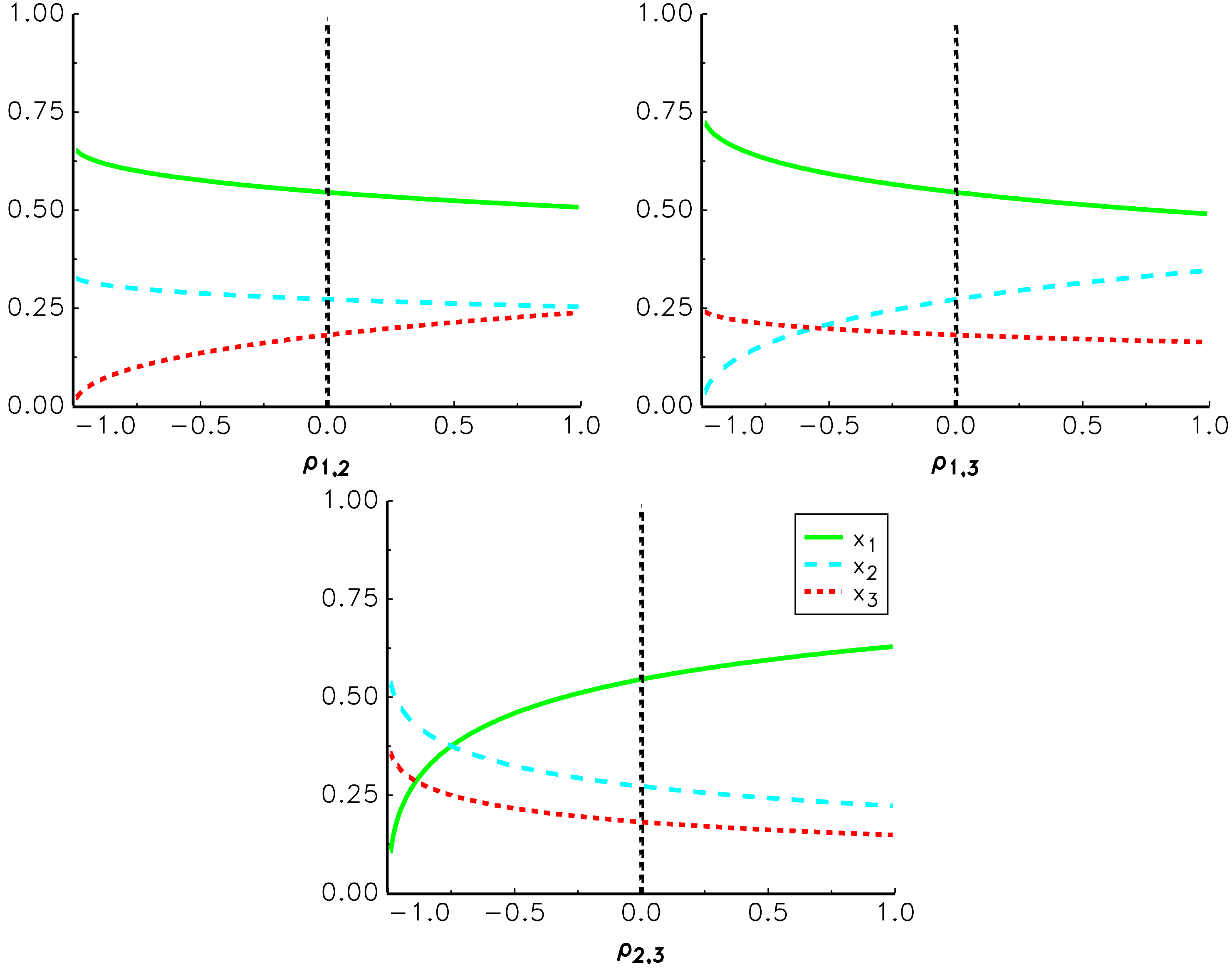
2.9 The impact of the cross-correlations

In the case where the cross-correlations are the same, the ERC portfolio does not depend on the correlation matrix and may be computed analytically. The question is then the following*:

How much the general ERC solution differ from the ERC solution in the constant correlations case?

Let us consider an example with $(\sigma_1, \sigma_2, \sigma_3) = (10\%, 20\%, 30\%)$ and $\rho = C_3(r)$. The solution is $x_1 = 54.54\%$, $x_2 = 27.27\%$ and $x_3 = 18.18\%$. In the next Figure, we represent the ERC solutions when one cross-correlation $\rho_{i,j}$ changes. In particular, x_1 may be very small if the cross-correlation between the second and third assets is negative.

*It was suggested by A. Steiner (http://www.andreassteiner.net/performanceanalysis/).



1.00
0.75
0.50
0.25
0.00
−1.0
−0.5
0.0
0.5
1.0
$\rho_{1,2}$
$\rho_{1,3}$
$\rho_{2,3}$
x_1
x_2
x_3

We consider now different examples with 3 or 4 assets and a general correlation matrix:

$$\rho = \begin{pmatrix} 1 & & & \\ \rho_{1,2} & 1 & & \\ \rho_{1,3} & \rho_{2,3} & 1 & \\ \rho_{1,4} & \rho_{2,4} & \rho_{3,4} & 1 \end{pmatrix}$$

(a) 3 assets with same volatilities $\sigma_i = 20\%$.

(b) 3 assets with $(\sigma_1, \sigma_2, \sigma_3) = (10\%, 20\%, 30\%)$.

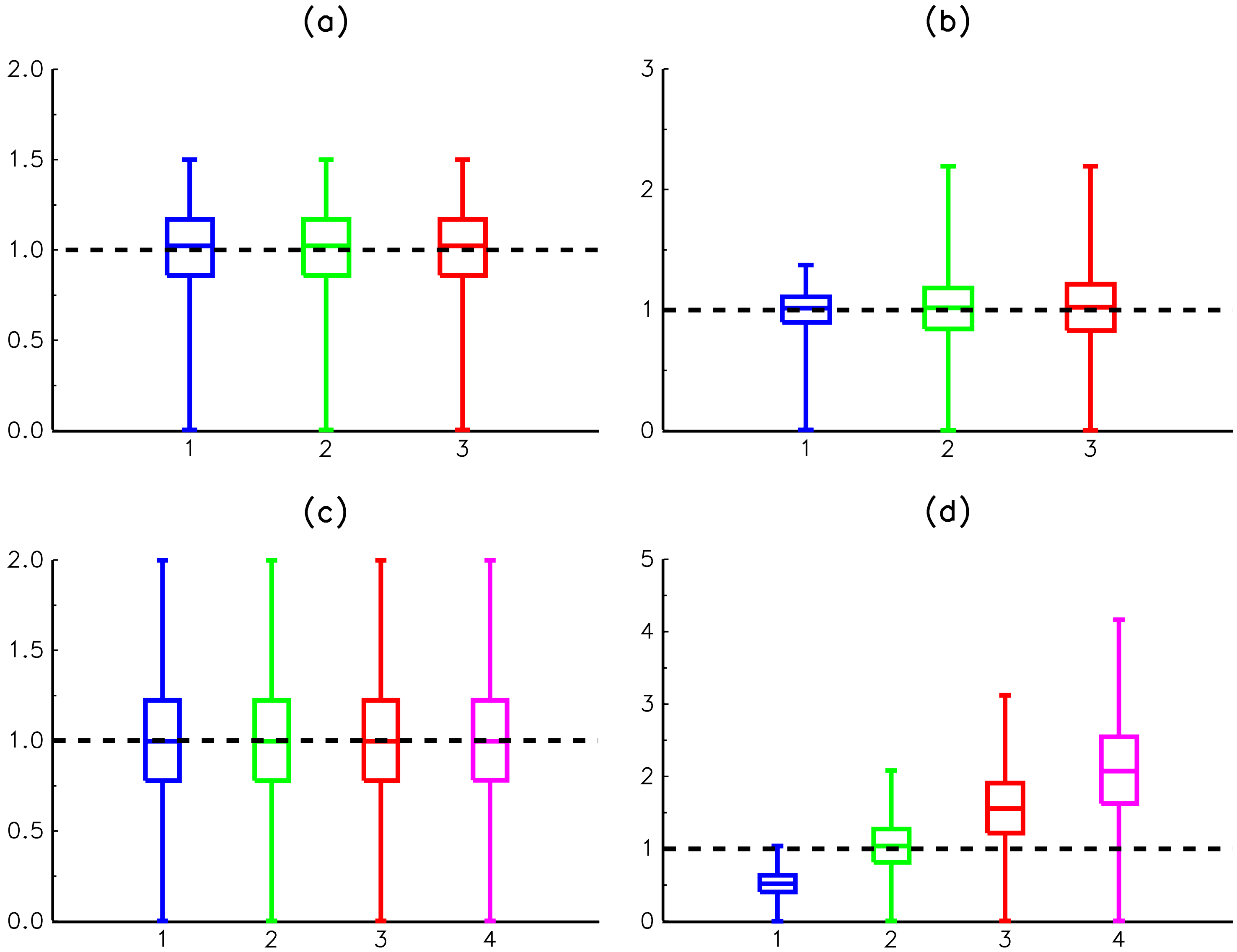
(c) 4 assets with same volatilities $\sigma_i = 20\%$.

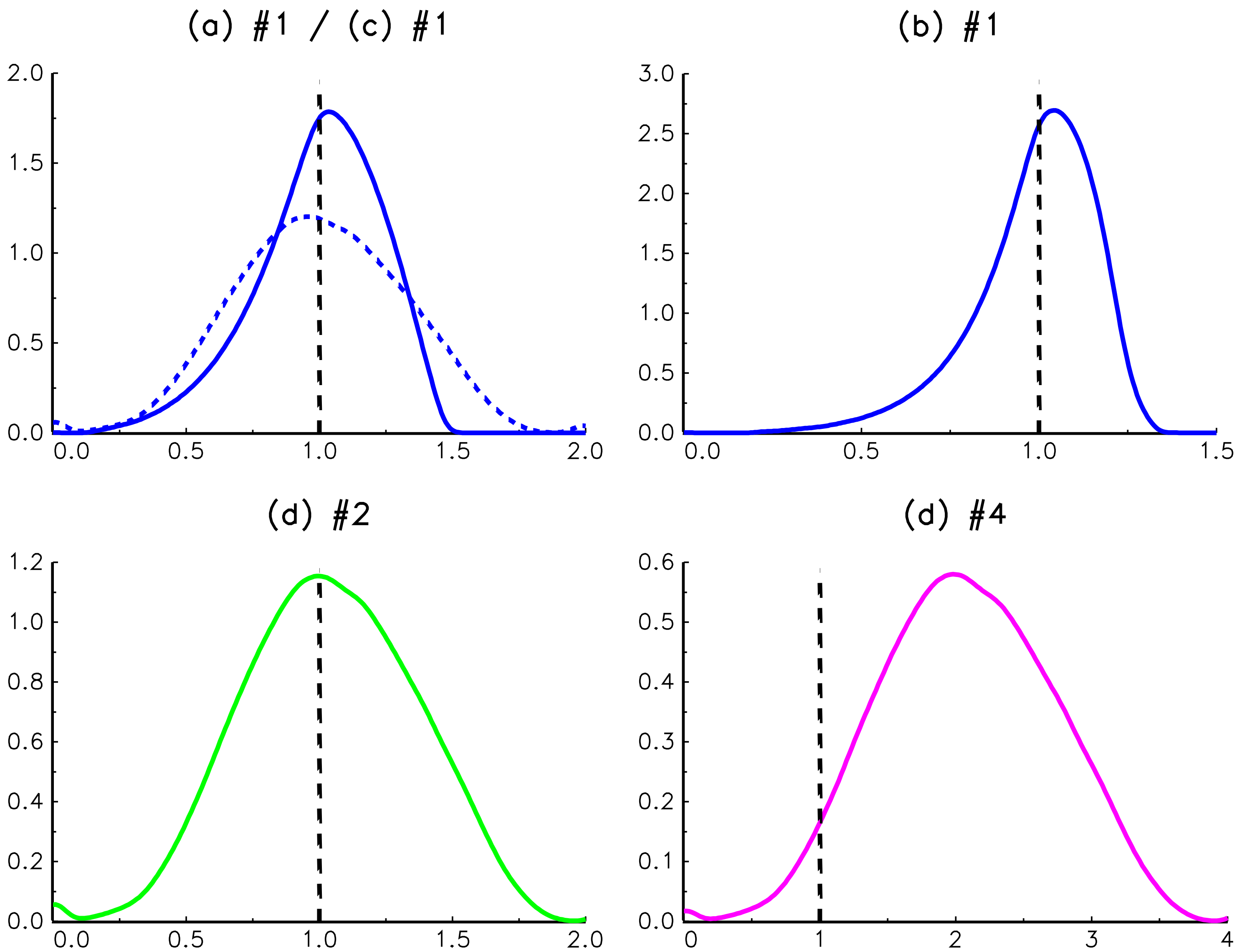
(d) 4 assets with $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (10\%, 20\%, 30\%, 40\%)$.

We simulate correlation matrices with $\rho_{i,j} \sim \mathcal{U}_{[-1,1]}$ and $\min \lambda(\rho) > 0$. Then, we compute the ratios between the weights of the general ERC solution and the weights of the ERC solution with constant correlations (the ratio is equal to 1 if the two solutions are the same).

In the next Figures, we report the box plots (min/25%/Median/75%/max) of the ratios for the four cases and we estimate the corresponding density.

$\Rightarrow$ The difference between the two solutions may be very large (see in particular the ratio for the 4th asset in case (d)).


(a)
2.0
1.5
1.0
0.5
0.0
1
2
3
(b)
3
2
1
0
1
2
3
(c)
2.0
1.5
1.0
0.5
0.0
1
2
3
4
(d)
5
4
3
2
1
0
1
2
3
4


(a) #1 / (c) #1
(b) #1
(d) #2
(d) #4

In the case of four assets, the ERC solution with constant correlations is:

$$x^{*}=\begin{pmatrix}48\%\\24\%\\16\%\\12\%\end{pmatrix}$$

Now, with the following correlation matrix:

$$\rho=\begin{pmatrix}1 & & & \\ 0.20 & 1 & & \\ 0.20 & -0.20 & 1 & \\ 0.00 & -0.60 & -0.60 & 1\end{pmatrix}$$

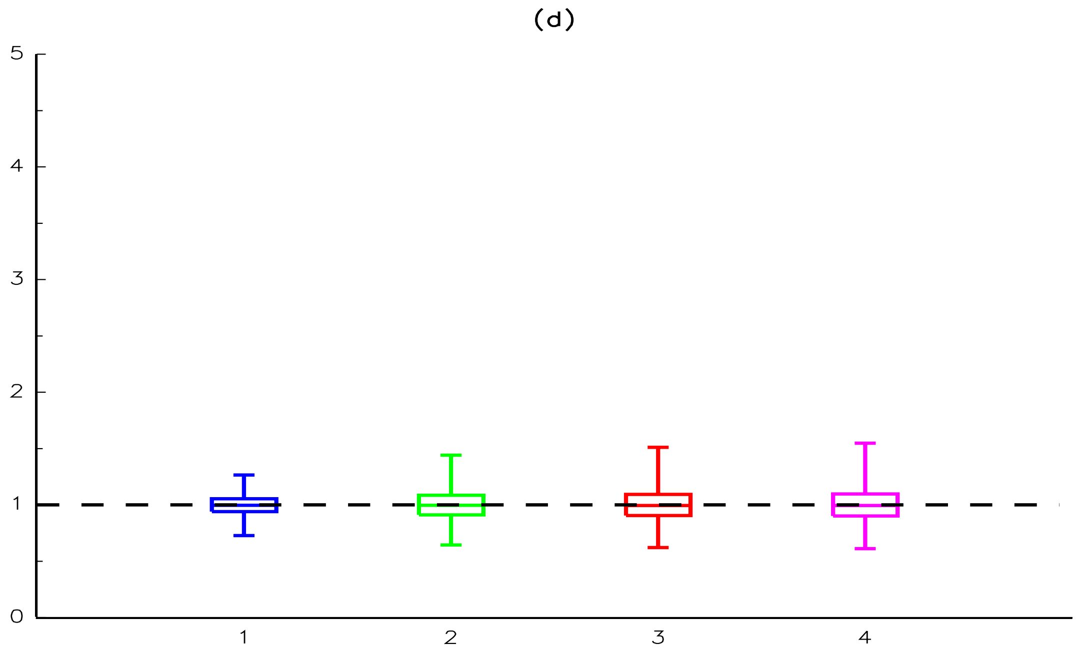
we have:

$$x^{*}=\begin{pmatrix}10.55\%\\38.97\%\\25.98\%\\24.50\%\end{pmatrix}$$

$\Rightarrow$ The two solutions are very different!

What becomes of the previous result if we only consider correlation matrices that make sense in finance?

$\Rightarrow$ The differences are reduced, but remain large (especially for high dimensions).



Box plot for the case (d) when all $\rho_{i,j}$ are positive

3 Applications

- Sector indices
- Agriculture commodity
- Global diversified portfolio
- BRIC

3.1 Sector indices

- FTSE-Datastream
- 10 sector indices.
- January 1973 / May 2008
- Rolling estimation of the covariance matrix with a 1Y window.

ICB Classification

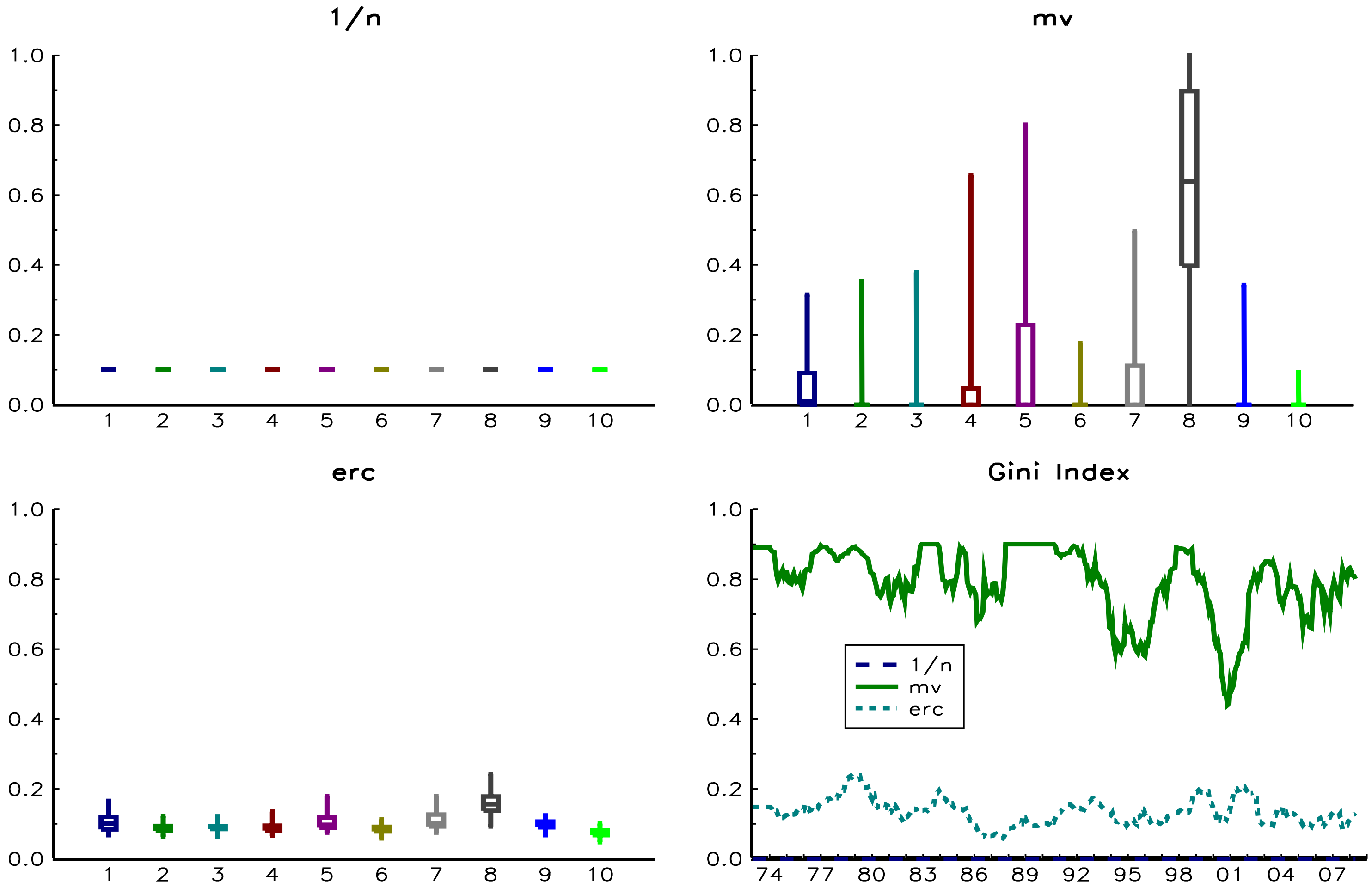
	ICB Code	Industry name	Sector mnemonic
1	0001	Oil & Gas	OILGS
2	1000	Basic Materials	BMATR
3	2000	Industrials	INDUS
4	3000	Consumer Goods	CNSMG
5	4000	Healthcare	HLTHC
6	5000	Consumer Services	CNSMS
7	6000	Telecommunications	TELCM
8	7000	Utilities	UTILS
9	8000	Financials	FINAN
10	9000	Technology	TECNO
		All	TOTMK

Results

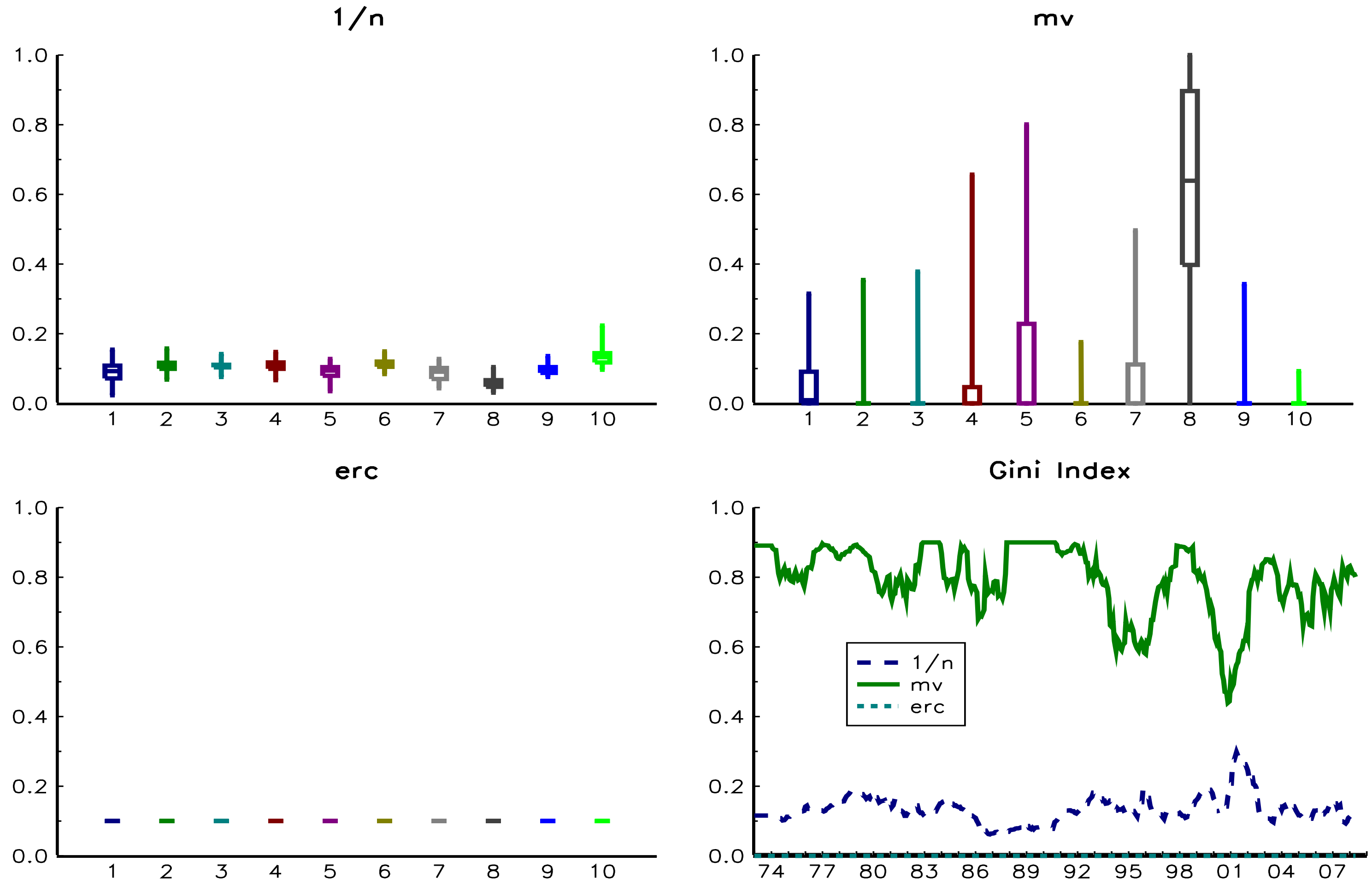
	1/n	mv	erc
Return	11.52%	10.37%	11.39%
Volatility	15.05%	11.54%	14.23%
Sharpe	0.77	0.90	0.80
VaR 1D 1%	−2.41%	−1.90%	−2.28%
VaR 1W 1%	−5.45%	−4.47%	−5.16%
VaR 1M 1%	−11.18%	−9.37%	−10.42%
DD 1D	−18.63%	−14.71%	−18.40%
DD 1W	−25.19%	−17.71%	−24.73%
DD 1M	−27.09%	−21.13%	−26.24%
DD Max	−45.29%	−46.15%	−44.52%
$\bar{H}_w$	0.00%	53.81%	0.89%
$\bar{G}_w$	0.00%	79.35%	13.50%
$\bar{T}_w$	0.00%	5.13%	1.00%
$\bar{H}_{rc}$	0.73%	53.81%	0.00%
$\bar{G}_{rc}$	13.37%	79.35%	0.00%

Remark: G is the Gini Index, H is the Herfindahl index and T is the turnover statistic. In the previous table, we present the average values of these statistics for both the weights ($\bar{H}_w$, $\bar{G}_w$ and $\bar{T}_w$) and the risk contributions ($\bar{H}_{rc}$ and $\bar{G}_{rc}$).

Box plot of the monthly weights



Box plot of the monthly risk contributions



3.2 Agriculture commodity

- Datastream
- Basket of light agricultural commodities
- January 1979 / December 2007
- Rolling estimation of the covariance matrix with a 1Y window.

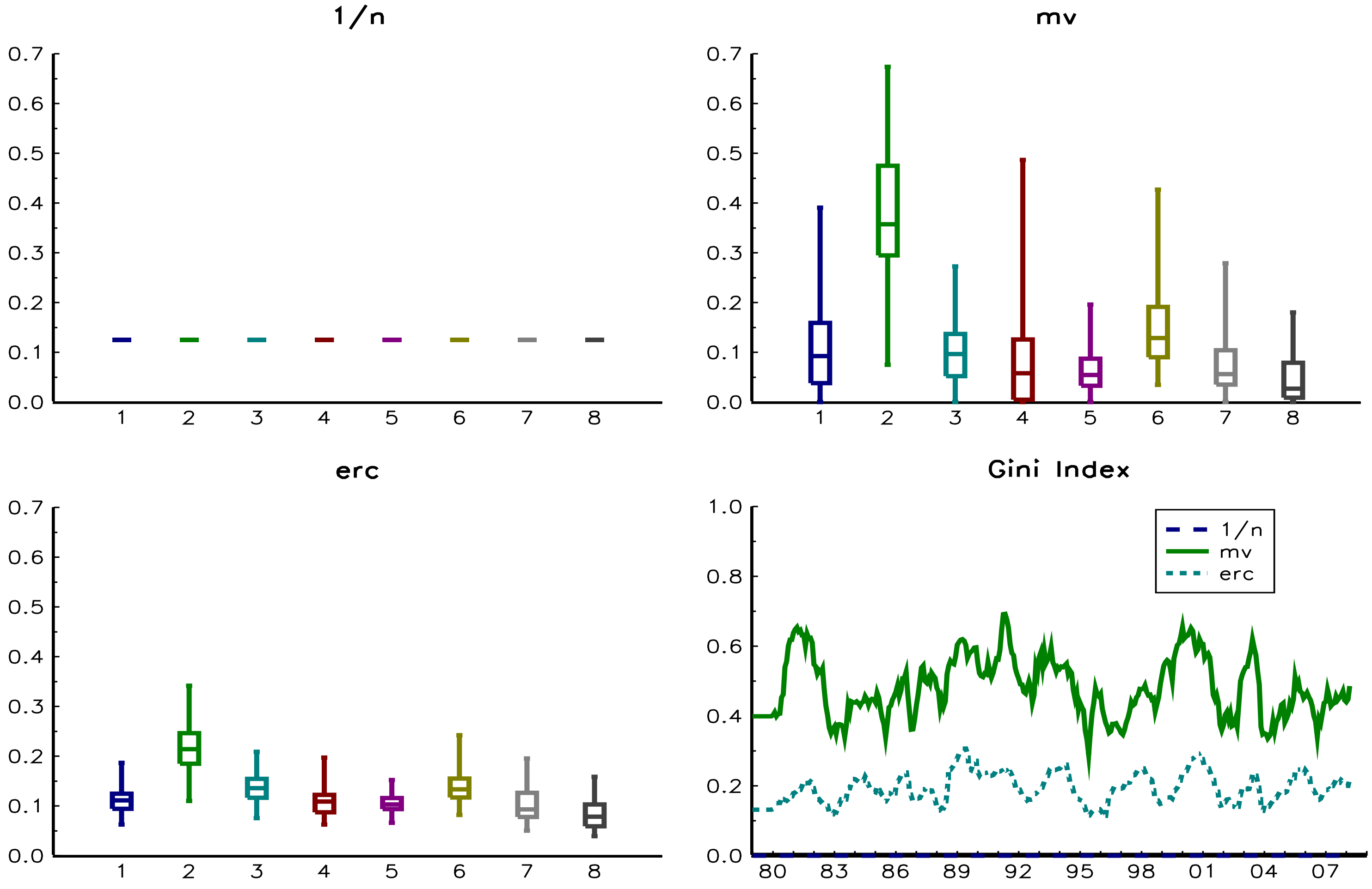
The basket

	Commodity Name	DS Code	Exchange Market
1	Corn	CC.	CBOT
2	Live Cattle	CLC	CME
3	Lean Hogs	CLH	CME
4	Soybeans	CS.	CBOT
5	Wheat	CW.	CBOT
6	Cotton	NCT	NYBOT
7	Coffee	NKC	NYBOT
8	Sugar	NSB	NYBOT

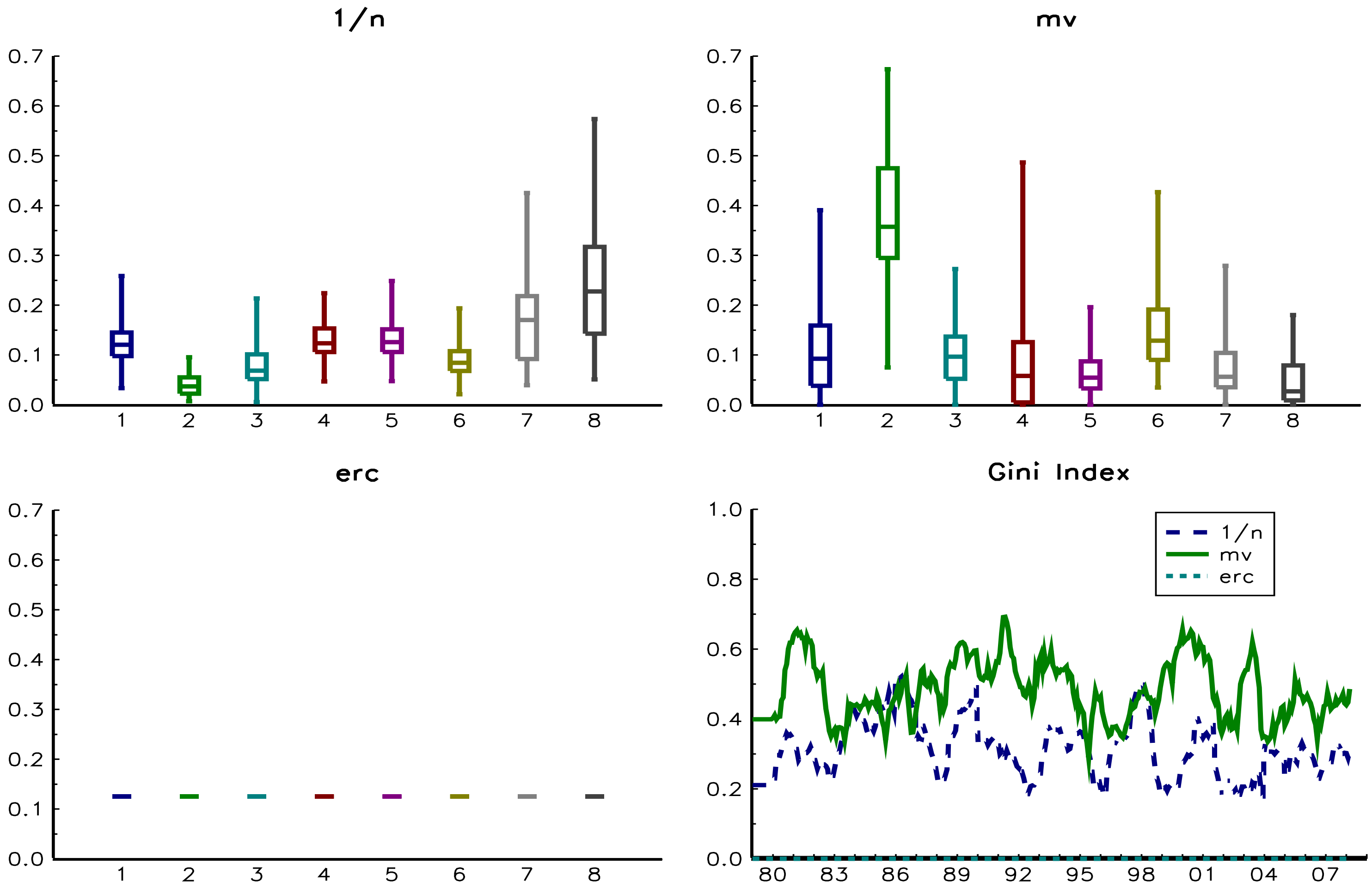
Results

	1/n	mv	erc
Return	10.2%	14.3%	12.1%
Volatility	12.4%	10.0%	10.7%
Sharpe	0.27	0.74	0.49
VaR 1D 1%	-1.97%	-1.58%	-1.64%
VaR 1W 1%	-4.05%	-3.53%	-3.72%
VaR 1M 1%	-7.93%	-6.73%	-7.41%
DD 1D	-5.02%	-4.40%	-3.93%
DD 1W	-8.52%	-8.71%	-7.38%
DD 1M	-11.8%	-15.1%	-12.3%
DD Max	-44.1%	-30.8%	-36.9%
$\bar{H}_w$	0.00%	14.7%	2.17%
$\bar{G}_w$	0.00%	48.1%	19.4%
$\bar{T}_w$	0.00%	4.90%	1.86%
$\bar{H}_{rc}$	6.32%	14.7%	0.00%
$\bar{G}_{rc}$	31.3%	48.1%	0.00%

Box plot of the monthly weights



Box plot of the monthly risk contributions



3.3 Global diversified portfolio

- Datastream
- Basket of major asset classes
- January 1995 / December 2007
- Rolling estimation of the covariance matrix with a 1Y window.

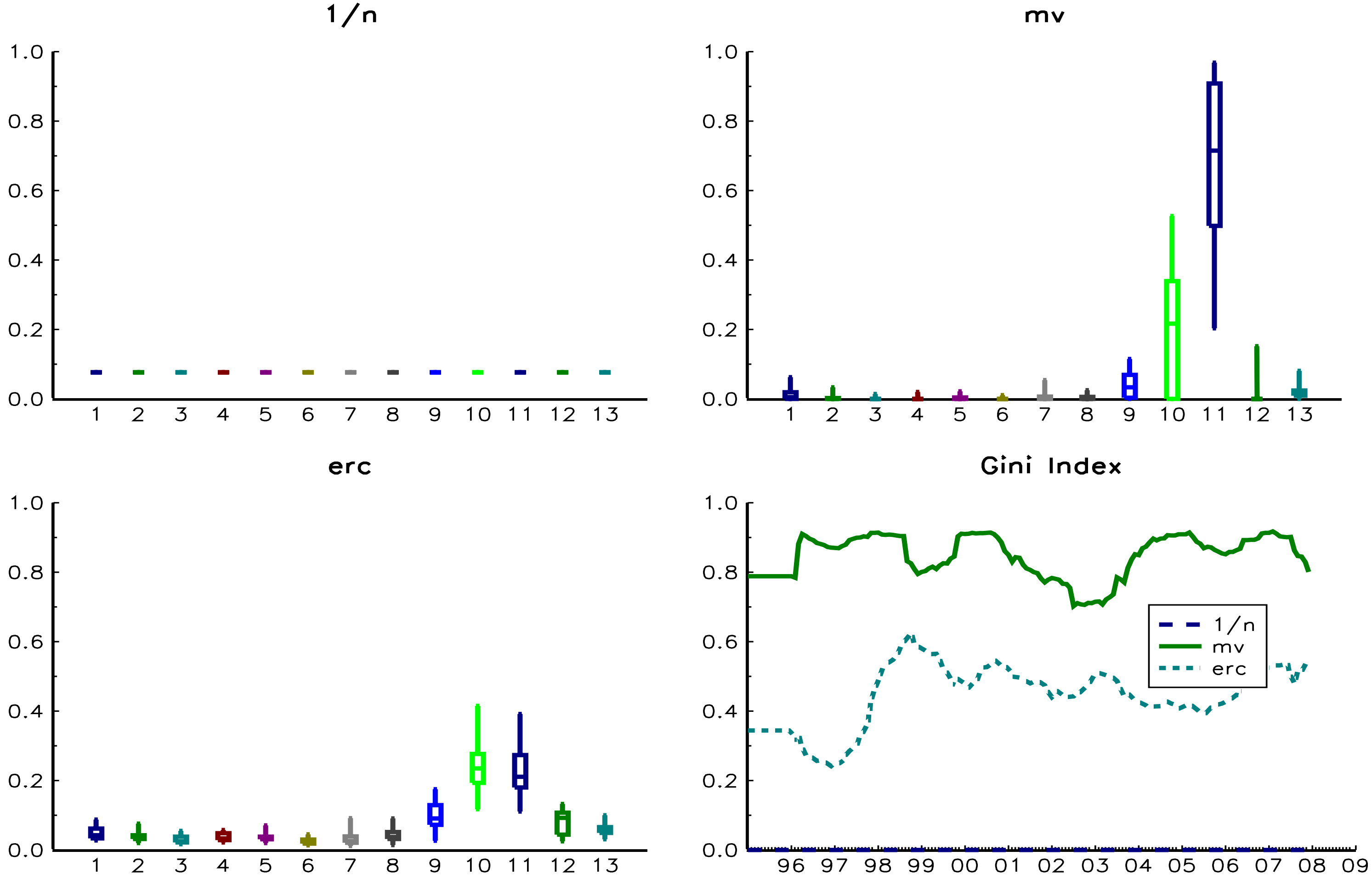
The basket

	Code	Name
1	SPX	S&P 500
2	RTY	Russell 2000
3	EUR	DJ Euro Stoxx 50 Index
4	GBP	FTSE 100
5	JPY	Topix
6	MSCI-LA	MSCI Latin America
7	MSCI-EME	MSCI Emerging Markets Europe
8	ASIA	MSCI AC Asia ex Japan
9	EUR-BND	JP Morgan Global Govt Bond Euro
10	USD-BND	JP Morgan Govt Bond US
11	USD-HY	ML US High Yield Master II
12	EM-BND	JP Morgan EMBI Diversified
13	GSCI	S&P GSCI

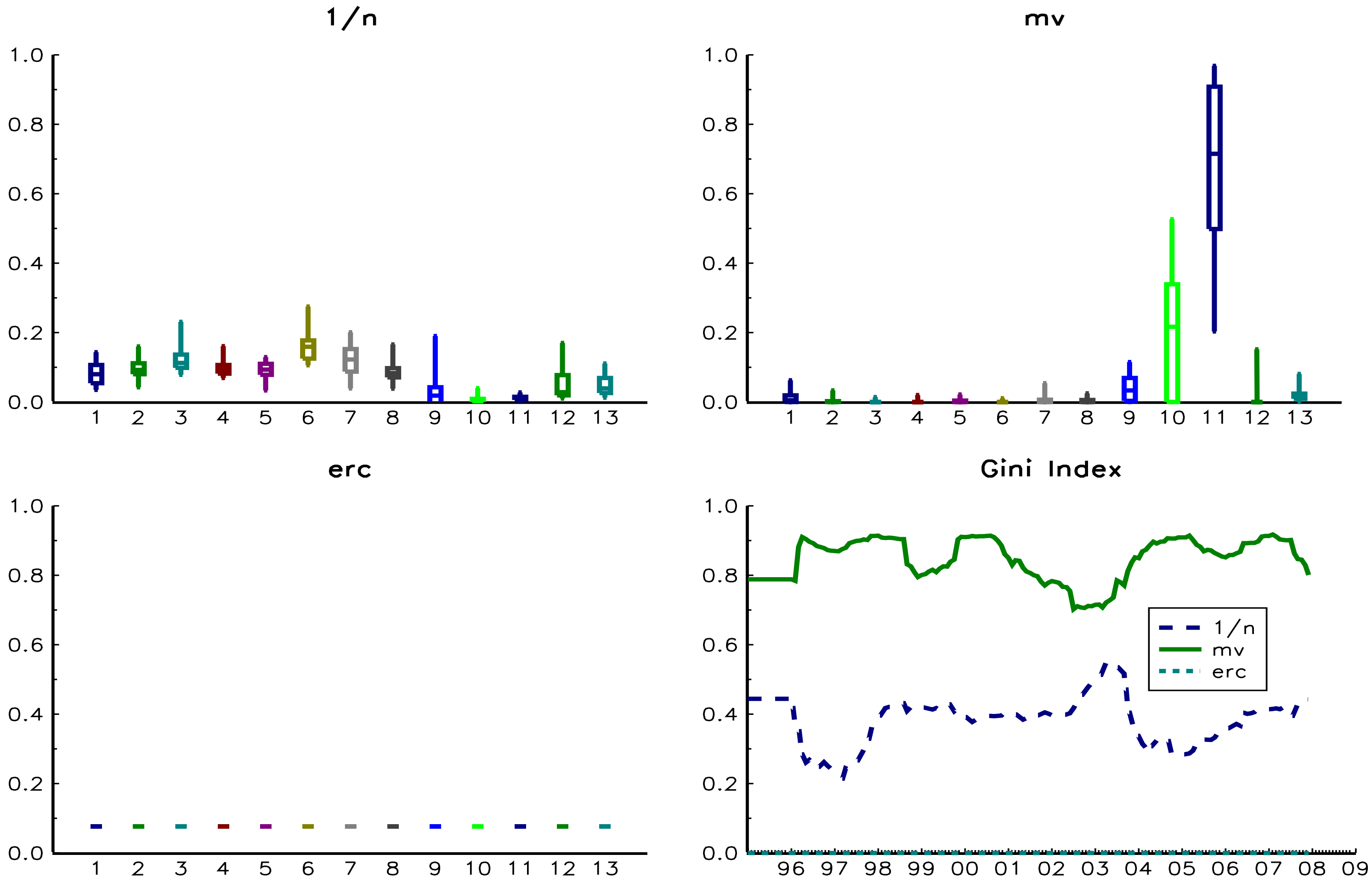
Results

	1/n	mv	erc
Return	9.99%	7.08%	8.70%
Volatility	9.09%	2.81%	4.67%
Sharpe	0.62	0.97	0.93
VaR 1D 1%	−1.61%	−0.51%	−0.74%
VaR 1W 1%	−4.37%	−1.48%	−1.94%
VaR 1M 1%	−8.96%	−3.20%	−3.87%
DD 1D	−3.24%	−1.64%	−3.20%
DD 1W	−6.44%	−3.03%	−3.64%
DD 1M	−14.53%	−5.06%	−6.48%
DD Max	−29.86%	−7.55%	−9.01%
$\bar{H}_w$	0.00%	56.52%	8.44%
$\bar{G}_w$	0.00%	84.74%	44.65%
$\bar{T}_w$	0.00%	3.93%	2.28%
$\bar{H}_{rc}$	4.19%	56.52%	0.00%
$\bar{G}_{rc}$	38.36%	84.74%	0.00%

Box plot of the monthly weights



Box plot of the monthly risk contributions



3.4 Equity market neutral hedge funds

$\Rightarrow$ EMN HF exhibits a beta against equity indices closed to zero.

In general, EMN is a pure stock picking process: the strategy holds Long / short equity positions, with long positions hedged with short positions in the same and in related sectors, so that the equity market neutral investor should be little affected by sector-wide events.

How to calibrate the positions of the long / short bets ?

3.4.1 Market practice

Let us denote by x_i and y_i the long and short positions of the i^{th} bet. Let $\sigma(x_i + y_i)$ be the volatility of this long / short position.

$\Rightarrow$ One of the most popular practice is to assign a weight w_i for the i^{th} bet that is inversely proportional to its volatility:

$$w_i \propto \frac{1}{\sigma(x_i + y_i)}$$

The underlying idea is to have long /short positions that have the same risk contributions.

3.4.2 ERC applied to L/S strategy

Let $x = (x_1, \ldots, x_n)^\top$ and $y = (y_1, \ldots, y_n)^\top$ be the vectors of weights for the long and short positions. The covariance matrix of the augmented vector of the returns is defined by the matrix
$\Sigma = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{pmatrix}$.

If we define $\sigma(x, y) = \sqrt{x^\top \Sigma_{xx} x + y^\top \Sigma_{yy} y + 2x^\top \Sigma_{xy} y}$ as the volatility of the EMN portfolio, the risk contribution of the i^{th} bet is:

$$\sigma_i(x, y) = x_i \frac{\partial\, \sigma(x, y)}{\partial\, x_i} + y_i \frac{\partial\, \sigma(x; y)}{\partial\, y_i}$$

(a) The first idea to calibrate the weights x and y is to solve the following problem:

$$\left\{\begin{array}{l} \sigma_i\left(x,y\right)=\sigma_j\left(x,y\right) \\ x_i>0 \\ y_i<0 \end{array}\right.$$

However, this problem is not well defined because it has several solutions.

(b) We may impose the exact matching $y_i=-x_i$ between the long and short positions:

$$\left\{\begin{array}{l} \sigma_i\left(x,y\right)=\sigma_j\left(x,y\right) \\ x_i>0 \\ x_i+y_i=0 \\ \sigma\left(x,y\right)=\sigma^* \end{array}\right.$$

Moreover, in order to have a unique solution, we have to impose a volatility target σ^* for the strategy. **This is the right problem.**

(c) Let $w_i = x_i = -y_i$ be the weight of the i^{th} bet. The previous problem is equivalent to solving:

$$\begin{cases} \sigma_i(w) = \sigma_j(w) \\ w_i > 0 \\ \sigma(w) = \sigma^* \end{cases}$$

Note that the covariance matrix of the long /short bets is:

$$\begin{aligned} \Sigma_{ww} &= A \Sigma A^\top \\ &= \Sigma_{xx} + \Sigma_{yy} - \Sigma_{xy} - \Sigma_{yx} \end{aligned}$$

with $A = \begin{pmatrix} I_n & -I_n \end{pmatrix}$.

3.4.3 The right way to solve the ERC problem

Since problems (b) and (c) are conceptually the same, (c) is preferred because it is numerically more easy to solve. Since the constraint $\sigma(w) = \sigma^*$ is difficult to manage, a slight modification in introduced:

$$\begin{cases} \sigma_i(w) = \sigma_j(w) \\ w_i > 0 \\ w_1 = \alpha \end{cases}$$

Let us denote w^α the solution of this problem when the first weight w_1 is equal to α. Since $\sigma(c \times w^\alpha) = c \times \sigma(w^\alpha)$, if we multiply all the weights by a constant c, the relationships $\sigma_i(c \times w^\alpha) = \sigma_j(c \times w^\alpha)$ remain valid. Hence, if one would like to impose a target value, one can just scale the solution w^α:

$$w = \frac{\sigma^*}{\sigma(w^\alpha)} w^\alpha$$

3.4.4 Market practice = ERC solution?

We know that $w_i \propto \frac{1}{\sigma(w_i)}$ when the cross-correlations $\rho_{i,j}^{w,w}$ between the L/S positions are the same. We have:

$$\rho_{i,j}^{w,w} = \frac{\rho_{i,j}^{x,x}\sigma_i^x\sigma_j^x + \rho_{i,j}^{y,y}\sigma_i^y\sigma_j^y - \rho_{i,j}^{x,y}\sigma_i^x\sigma_j^y - \rho_{i,j}^{y,x}\sigma_i^y\sigma_j^x}{\sqrt{\left(\sigma_i^x\right)^2 + \left(\sigma_i^y\right)^2 - 2\rho_{i,i}^{x,y}\sigma_i^x\sigma_i^y}\sqrt{\left(\sigma_j^x\right)^2 + \left(\sigma_j^y\right)^2 - 2\rho_{j,j}^{x,y}\sigma_j^x\sigma_j^y}}$$

We assume that the assets have the same volatilities. We have:

$$\rho_{i,j}^{w,w} \propto \rho_{i,j}^{x,x} + \rho_{i,j}^{y,y} - \rho_{i,j}^{x,y} - \rho_{i,j}^{y,x}$$

Moreover, assuming a risk model of $S+1$ factors, with a general risk factor and S sectorial factors, $\rho_{i,j}^{x,x}$ is equal to ρ_s if the two assets belong to the same factor s, otherwise it is equal to ρ^*. It follows that:

$$\rho_{i,j}^{w,w} \propto \begin{cases} \rho_s + \rho_s - \rho_s - \rho_s & \text{if } i,j \in s \\ \rho^* + \rho^* - \rho^* - \rho^* & \text{otherwise} \end{cases}$$

We verify that $\rho_{i,j}^{w,w} = 0$.

The market practice corresponds to the ERC solution when:

1. The long and short positions of each bet are taken in the same sector.
2. The volatilities of the assets are equal.
3. We assume a risk model of $S+1$ factors with one factor by sector and a general risk factor.

$\Rightarrow$ From a practical point of view, only the first and third conditions are important because even if the volatilities are not equal, we have $\rho_{i,j}^{w,w} \simeq 0$ because the covariance is very small compared to the product of the volatilities.

3.4.5 An example

We have 4 sectors.

The intra-sector correlations are respectively 50%, 30%, 20% and 60% whereas the inter-sector correlation is 10%.

We consider a portfolio with 5 L/S positions.

The volatilities are respectively 10%, 40%, 30%, 10% and 30% for the long positions, and 20%, 20%, 20%, 50% and 20% for the short positions.

The bets are done respectively in the 1st sector, 2nd, 3rd, 4th and 4th sectors.

The volatility target σ^* is 10%.

Market Practice

$\sigma(x) = 10\%$	x_i	$\partial_{x_i}\sigma(x)$	$x_i \times \partial_{x_i}\sigma(x)$	$c_i(x)$
1	27.3%	0.080	0.022	21.9%
2	12.2%	0.177	0.022	21.5%
3	14.5%	0.151	0.022	22.0%
4	10.6%	0.160	0.017	16.9%
5	19.6%	0.090	0.018	17.6%

ERC

$\sigma(x) = 10\%$	x_i	$\partial_{x_i}\sigma(x)$	$x_i \times \partial_{x_i}\sigma(x)$	$c_i(x)$
1	26.0%	0.077	0.020	20.0%
2	11.7%	0.171	0.020	20.0%
3	13.9%	0.144	0.020	20.0%
4	11.5%	0.174	0.020	20.0%
5	20.9%	0.095	0.020	20.0%

4 Conclusion

Traditional asset allocation $\Rightarrow$ concentrated portfolios

ERC $\Rightarrow$ Equal risk contributions of the various components
$\Rightarrow$ a good compromise between minimum variance and $1/n$ portfolios

The main application is the construction of indices.
$\Rightarrow$ Risk indexation (like fundamental indexation) is an alternative to market-cap indexations.

5 References

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