Modelling dependence in finance using copulas*

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*These slides may be downloaded from http://gro.creditlyonnais.fr (the direct link is http://gro.creditlyonnais.fr/content/wp/copula-stat2001-canada.pdf).

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Agenda

1. The Gaussian assumption in finance
2. Copulas and multivariate financial models
3. An open field for risk management
    - Market risk
    - Operational risk
    - Credit risk
4. New pricing methods with copulas
    - Multi-asset options
    - Credit derivatives

1 The Gaussian assumption in finance

We consider the 'universal' financial model. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space. The asset prices processes $S_1(t)$ and $S_2(t)$ are given by the SDE representation

$$\begin{cases} \mathrm{d}S_1(t) &= \mu_1 S_1(t)\,\mathrm{d}t + \sigma_1 S_1(t)\,\mathrm{d}W_1(t) \\ \mathrm{d}S_2(t) &= \mu_2 S_2(t)\,\mathrm{d}t + \sigma_2 S_2(t)\,\mathrm{d}W_2(t) \end{cases}$$

where $W_1(t)$ and $W_2(t)$ are two $\mathcal{F}_t$–brownian motions with

$$\mathbb{E}\left[W_1(t)\,W_2(t) \mid \mathcal{F}_{t_0}\right] = \rho\,(t - t_0)$$

It comes that the logarithm return of assets is gaussian (= Gaussian assumption in finance).

Empirical facts: **Asset returns are not gaussian** (see the financial econometric literature on ARCH, long-memory, Lévy processes, etc.).

Problem: Univariate financial models are not gaussian, but multivariate financial models are gaussian!

2 Copulas and multivariate financial models

1. How to define multivariate financial models compatible with univariate non-Gaussian financial models?

$$\boxed{\text{Copula} = \text{a powerful tool}}$$

2. How to obtain tractable multivariate financial models (in terms of computational time)?
3. How to specify multivariate financial models which may be understood/used by the finance industry?

$$\boxed{\text{Copula} = \text{a promising tool}}$$

Copulas have been already incorporated in some software solutions:

- SAS Risk Dimensions
- Palisade @Risk

2.1 Pearson correlation and dependence

Pearson correlation $\rho =$ linear dependence measure.

For two given asset prices processes $S_1(t)$ and $S_2(t)$ which are GBM, the range of the correlation is

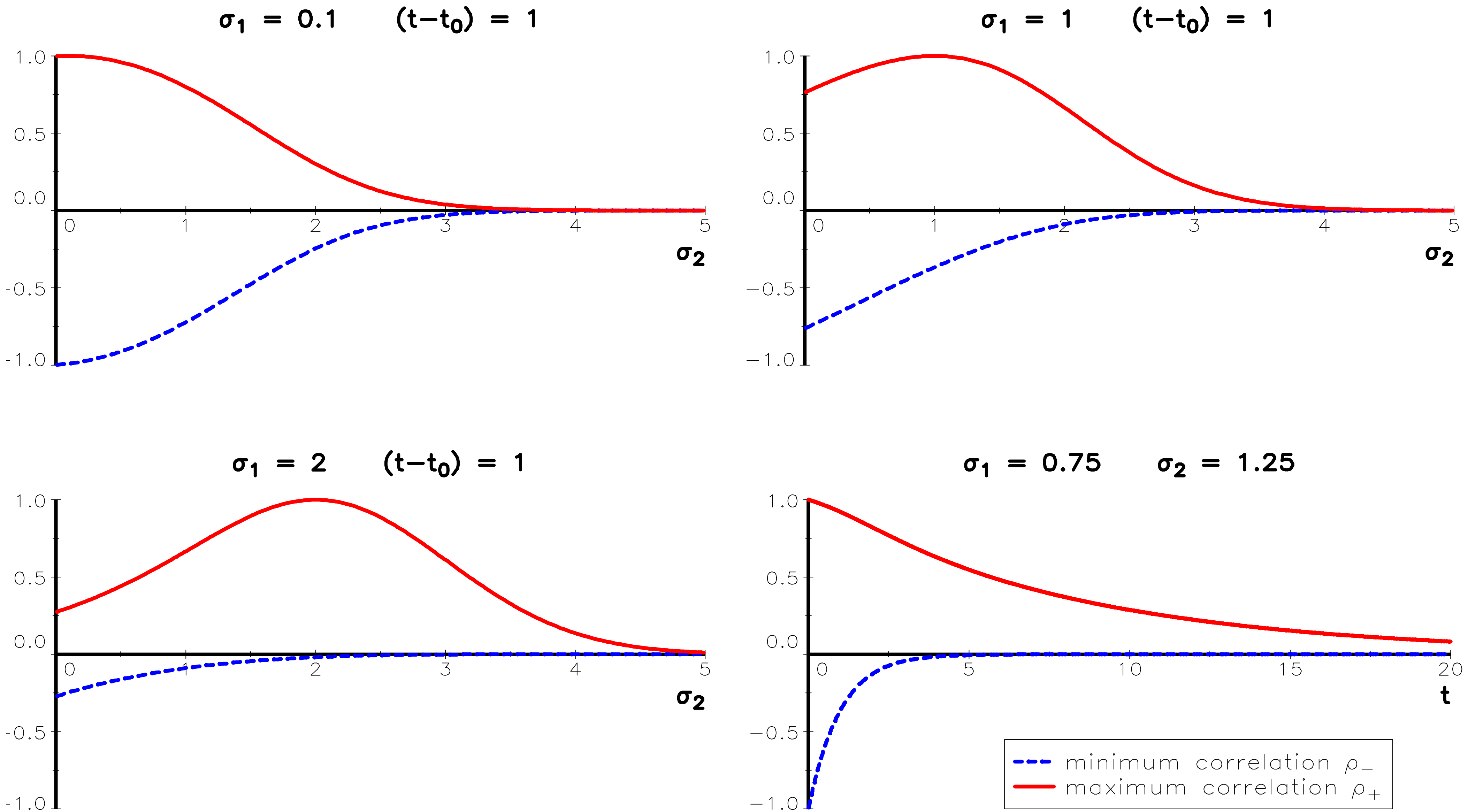
$$\rho^- \leq \rho(S_1(t), S_2(t)) \leq \rho^+$$

with

$$\rho^\pm = \frac{\exp(\pm\sigma_1\sigma_2(t-t_0)) - 1}{\sqrt{\exp\left(\sigma_1^2(t-t_0)\right) - 1} \cdot \sqrt{\exp\left(\sigma_2^2(t-t_0)\right) - 1}}$$

$\rho(S_1(t), S_2(t)) = \rho^-$ (resp. ρ^+) $\Leftrightarrow \mathbf{C}\langle S_1(t), S_2(t)\rangle = \mathbf{C}^-$ (resp. $\mathbf{C}^+$) $\Leftrightarrow S_2(t) = f(S_1(t))$ with f a decreasing (resp. increasing) function

$$\boxed{\text{Perfect dependence} \neq |\rho| = 1}$$



Permissible range of $\rho(S_1(t), S_2(t))$
when $S_1(t)$ and $S_2(t)$ are two GBM processes

2.2 Copula: a new tool in finance

- introduced by Embrechts et al. [14].
- Market risk: Bouyé et al. [2], Cherubini et al. [8], Durrleman et al. [13], Embrechts et al. [15], Luciano et al. [24], Tibiletti [33].
- Credit risk: Coutant et al. [11], Frey et al. [17], Georges et al. [18], Giesecke [19], Hamilton et al. [20], Lindskog et al. [23], Maccarinelli et al. [25].
- Operational risk: Ceske et al. [5] [6], Frachot et al. [16].
- Asset prices modelling: Bouyé et al. [4], Malavergne et al. [26], Patton [27], Rockinger et al. [28], Scaillet [31].
- Credit derivatives pricing: Li [21], Georges et al. [18], Schönbucher et al. [32].
- Multi-asset options pricing: Bikos [1], Cherubini et al. [7], Coutant et al. [10], Durrleman [12], Rosenberg [29] [30].

2.3 Copulas in a nutshell

A copula function $\mathbf{C}$ is a multivariate probability distribution with uniform $[0,1]$ margins.

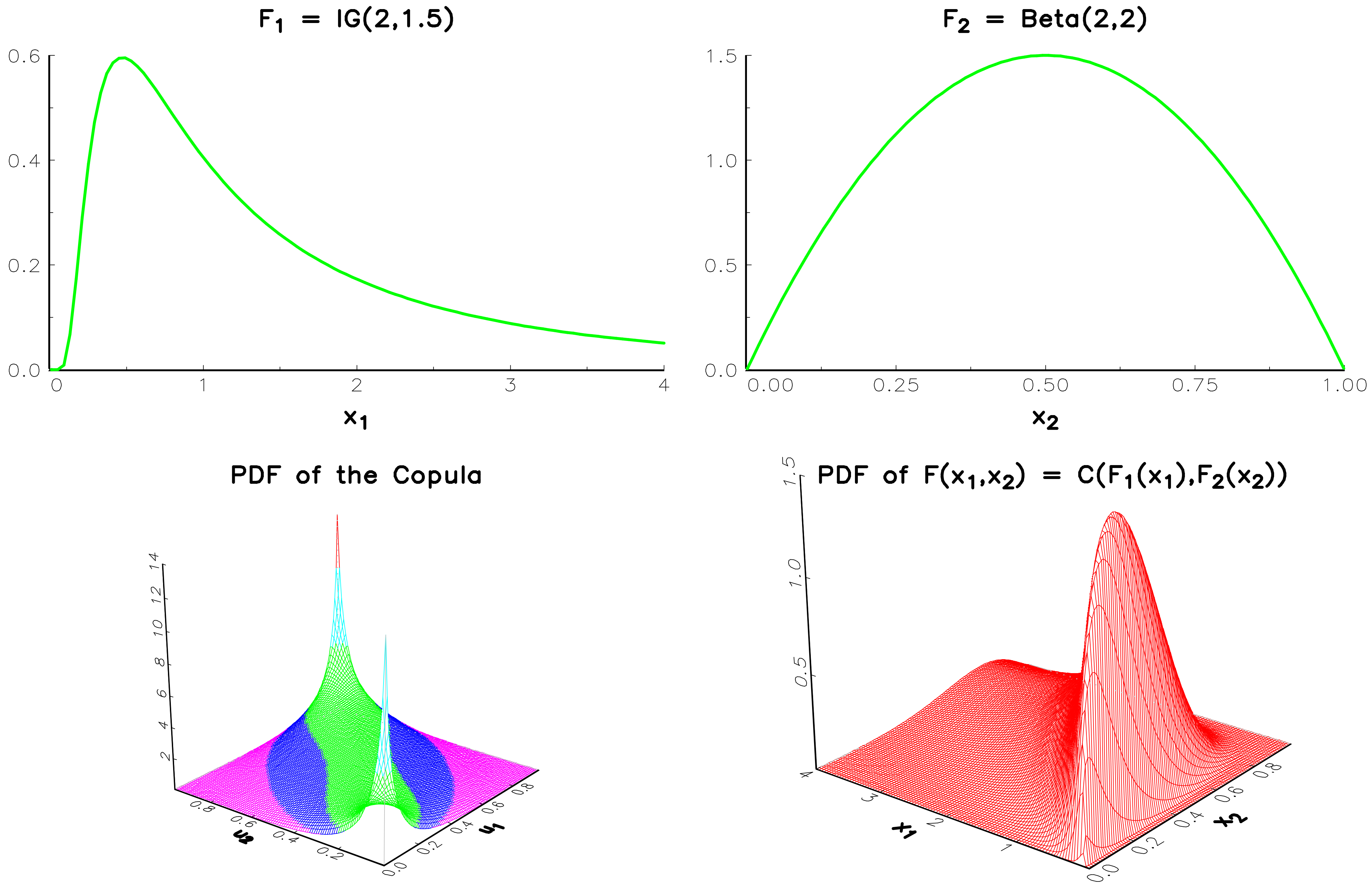
$\mathbf{C}\left(\mathbf{F}_1\left(x_1\right),\ldots,\mathbf{F}_N\left(x_N\right)\right)$ defines a multivariate cdf $\mathbf{F}$ with margins $\mathbf{F}_1,\ldots,\mathbf{F}_N \Rightarrow \mathbf{F}$ is a probability distribution with given marginals.

The copula function of the random variables $(X_1,\ldots,X_N)$ is **invariant** under strictly increasing transformations ($\partial_x h_n\left(x\right) > 0$):

$$\mathbf{C}\left\langle X_1,\ldots,X_N\right\rangle = \mathbf{C}\left\langle h_1\left(X_1\right),\ldots,h_N\left(X_N\right)\right\rangle$$

... the copula is invariant while the margins may be changed at will, it follows that is precisely the copula which captures those properties of the joint distribution which are invariant under a.s. strickly increasing transformations (Schweizer and Wolff [1981]).

$\Rightarrow$ **Copula = dependence function of r.v.** (Deheuvels [1978]).



Bivariate distribution with given marginals

2.4 The Normal copula

Two caracteristics in finance: High dimensional problems (e.g. a portfolio with 1000 securities) and probabilistic properties of the models (e.g. markovian property).

All the copula functions are not good candidates for financial application in an industry point of view.

The Normal copula has not yet been extensively studied (see however Song [2000]). Nevertheless, it may be an 'industrial' copula.

Remark 1 *The multivariate normal distribution is very tractable. It is very easy to estimate the parameters and simulation is straightforward. Moreover, this distribution has nice properties.*

$\boxed{\textbf{Is it also the case for the Normal copula?}}$

The copula function

$$\mathbf{C}\left(\mathbf{u};\rho\right)=\Phi\left(\Phi^{-1}\left(u_1\right),\ldots,\Phi^{-1}\left(u_N\right);\rho\right)$$

The density is

$$c\left(\mathbf{u};\rho\right)=\left|\rho\right|^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\varsigma^{\top}\left(\rho^{-1}-\mathbb{I}\right)\varsigma\right)$$

with $\varsigma=\left(\Phi^{-1}\left(u_1\right),\ldots,\Phi^{-1}\left(u_N\right)\right)$.

The Ψ transform We define the operator Ψ as follows

$$\begin{aligned}\Psi\left[\mathbf{F}\right] \;:\; \mathbb{R} &\longrightarrow \mathbb{R}\\ x &\longmapsto \Psi\left[\mathbf{F}\right](x)=\Phi^{-1}\left(\mathbf{F}\left(x\right)\right)\end{aligned}$$

We note also Ψ^{-1} the (left) inverse operator ($\Psi^{-1}\circ\Psi=1$), i.e. $\Psi^{-1}\left[\mathbf{F}\right](x)=\mathbf{F}^{[-1]}\left(\Phi\left(x\right)\right)$.

Estimation The log-likelihood function is

$$\ell(\mathbf{u};\rho) = -\frac{T}{2}\ln|\rho| - \frac{1}{2}\sum_{t=1}^{T}\varsigma_t^{\top}\left(\rho^{-1}-\mathbb{I}\right)\varsigma_t$$

and the ML estimate of ρ is also $\hat{\rho}_{\text{ML}} = \frac{1}{T}\sum_{t=1}^{T}\varsigma_t^{\top}\varsigma_t$.

Two-stage (Joe and Xu [1996]) and omnibus (Genest, Ghoudi and Rivest [1995]) estimators can then be obtained with $\varsigma_t = \left(\Psi\left[\mathbf{F}_1\right]\left(x_1^t\right),\ldots,\Psi\left[\mathbf{F}_N\right]\left(x_N^t\right)\right)$:

1. IFM estimate: $\mathbf{F}_n$ = MLE of the n^{th} marginal distribution.
2. Omnibus estimate : $\mathbf{F}_n = n^{\text{th}}$ empirical distribution.

$\Rightarrow$ The data are mapped to uniforms and transformed with the inverse gaussian distribution. The correlation parameter ρ of the Normal copula is then equal to the Pearson product moment of the transformed data.

Simulation

- Generate a gaussian vector $\mathbf{v}$ of random variables with correlation ρ.
- To simulate a vector $\mathbf{x}$ of random variables with marginals $\mathbf{F}_1, \ldots, \mathbf{F}_N$ and a Normal copula with parameters ρ, we use the following transformation

$$\mathbf{x} = \left(\Psi^{-1} [\mathbf{F}_1] (v_1), \ldots, \Psi^{-1} [\mathbf{F}_N] (v_N) \right)$$

Application to marketing Segmentation is a useful tool for marketing (and scoring). Let Y be a random variable which corresponds to the target. The main idea is to define classes

Class	Definition
1	$Y \leq y_1$
$\vdots$	
M	$y_{M-1} \leq Y \leq y_M$

For example, let Y be a (potential) rentability index. The bank would like to capture the most profitable customers. It can define the targets y_m by a quantile rule $\Pr\{Y \leq y_m\} = \tau_m$. For each class, it will define a specific customer relationship policy. For example, it will decide how much to spend on capture customers for a specific class.

One statistical tool which are used is the linear quantile regression

$$Y_n = X_n^\top \beta + u_n$$

where X_n are the characteristics of the individual n. If we reformulate the linear regression with only positive terms:

$$Y_n = X_n^\top \beta + u_n = \sum_{k=1}^{K} x_{n,k} \left(\beta_n^+ - \beta_n^- \right) + u_n^+ - u_n^-$$

We can show that the solution of the quantile regression $\Pr\{Y \leq y_m\} = \tau_m$ may be found using linear programming:

$$\begin{aligned} z &= \arg\min c^\top z \\ & \text{u.c.} \quad \begin{cases} Az = y \\ z \geq 0 \end{cases} \end{aligned}$$

where $X = (X_1, \ldots, X_N)^\top$, $A = (X, -X, I_N, -I_N)$, $y = (Y_1, \ldots, Y_N)^\top$, $z = \left(\beta^+, \beta^-, u^+, u^-\right)^\top \in \mathbb{R}^{2K+2N}$ and $c = (\mathbf{0}, \mathbf{0}, \tau_m \mathbf{1}, (1 - \tau_m)\, \mathbf{1})^\top$.

Computational issues ($\dim A \simeq N \times 2N$) = very large-scale problem. Portnoy and Koenker [1997] suggest then to use an interior-point method.

Problem: how to proceed when (Y, X) are not gaussian? One solution is to assume that only the copula of (Y, X) is Normal. In this case, we can use the Portnoy-Koenker algorithm with the transformed variables $Y_i = \Psi\left[\mathbf{F}_i\right](X_i)$.

Let consider the bivariate case*. We have $\partial_1 \mathbf{C}(u_1,u_2) = \Phi(\varsigma)$ where $\varsigma = \left(\mathbf{1}-\rho^{\mathbf{2}}\right)^{-\frac{1}{2}}\left[\Phi^{-1}(u_2) - \rho\Phi^{-1}(u_1)\right]$. The relationship between u_2 and u_1 in the expression $\Pr\{U_2 \le u_2 \mid U_1 = u_1\} = \tau$ is also given by

$$u_2 = \Phi\left(\rho\Phi^{-1}(u_1) + \sqrt{1-\rho^2}\Phi^{-1}(\tau)\right)$$

If the margins are gaussian, we obtain the well-known curve

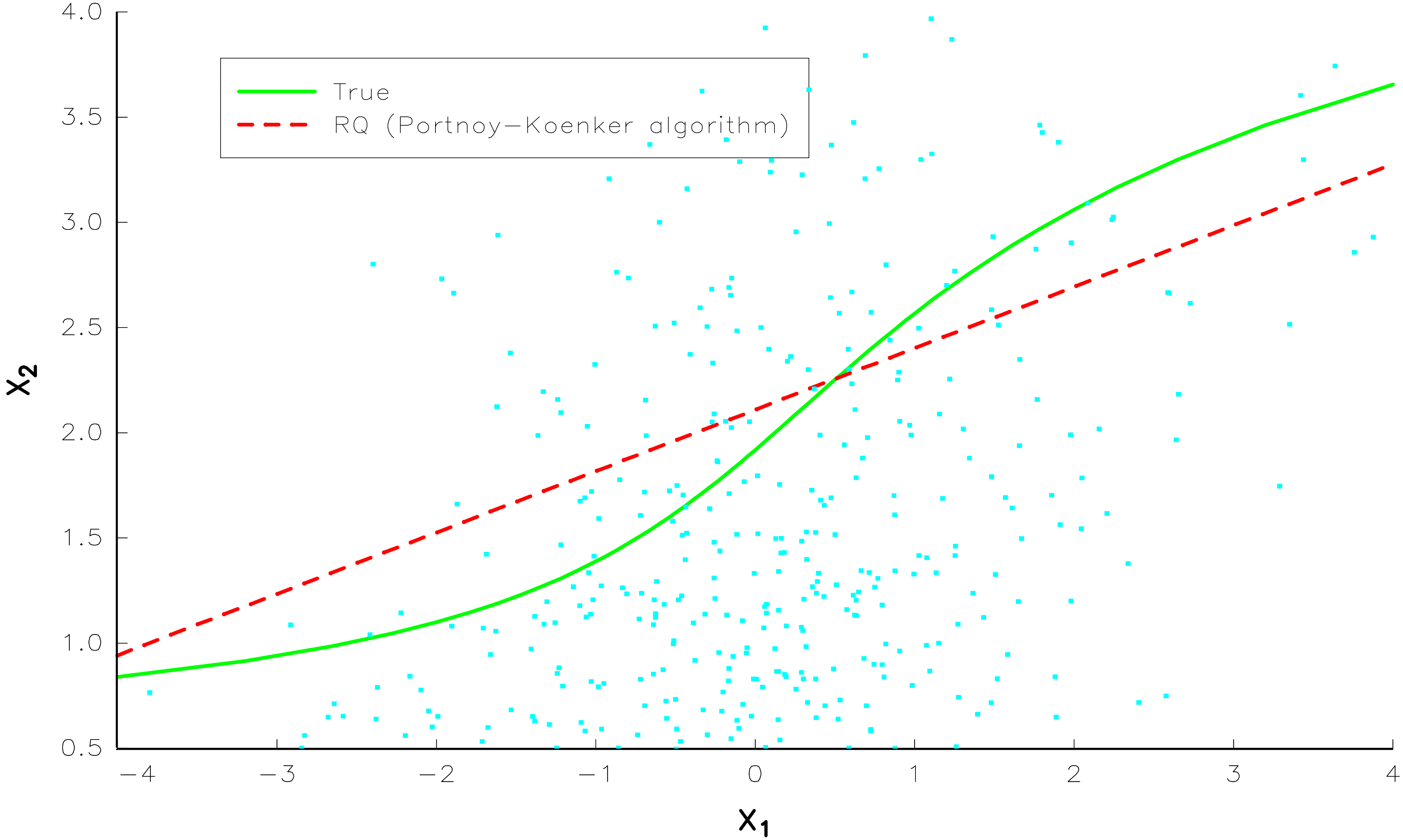
$$X_2 = \left[\mu_2 - \rho\frac{\sigma_2}{\sigma_1}\mu_1 + \sqrt{1-\rho^2}\Phi^{-1}(\tau)\right] + \rho\frac{\sigma_2}{\sigma_1}X_1$$

We remark that the relationship is **linear**. When the margins are not gaussian, the relationship is linear in the Ψ projection space:

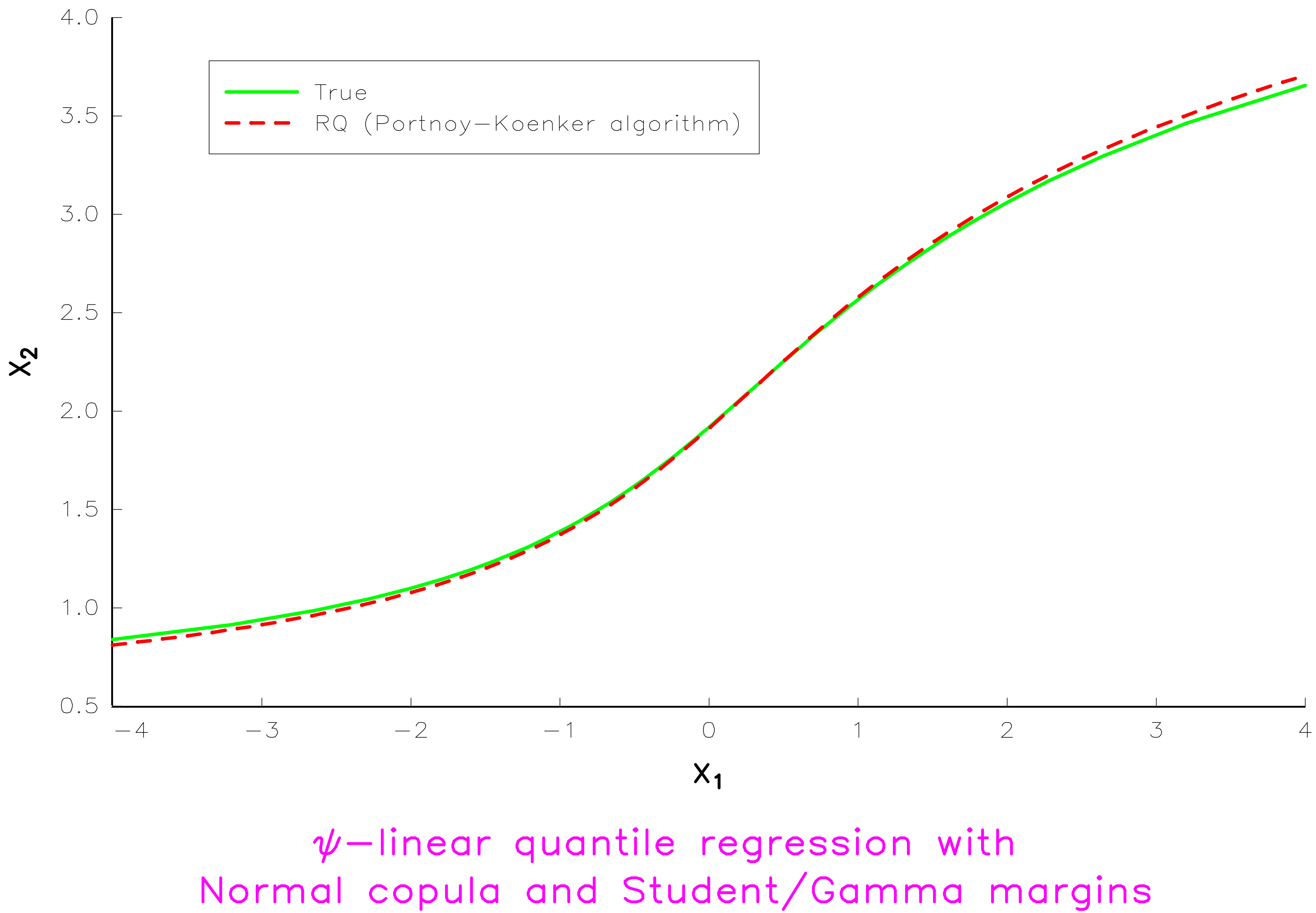
$$\Psi[\mathbf{F}_2](X_2) = a + b\Psi[\mathbf{F}_1](X_1)$$

where $a = \sqrt{1-\rho^2}\Phi^{-1}(\tau)$ and $b = \rho$.

*see [9].



Linear quantile regression with
Normal copula and Student/Gamma margins


4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
−4
−3
−2
−1
0
1
2
3
4
X2
X1
True
RQ (Portnoy−Koenker algorithm)
ψ−linear quantile regression with
Normal copula and Student/Gamma margins

3 An open field for risk management

The bank must compute the capital needed to support the risk exposure of an operation (market, credit, operational, etc.). In general, the capital charge is determined so that the estimated probability of unexpected loss exhausting capital is less than some target insolvency rate.

3.1 General framework of capital allocation

Quantile notion of the risk Let $\mathbf{F}$ be the (**potential**) loss probability distribution (we denote ϑ the corresponding r.v.) and $1-\alpha$ be the target insolvency rate. The capital charge VaR (or Capital-at-Risk/Value-at-Risk) is defined by

$$\Pr\{\vartheta > \mathrm{VaR}\} = 1 - \alpha$$

or by

$$\mathrm{VaR} = \inf\{x \mid \mathbf{F}(x) \geq 1-\alpha\}$$

In general, we distinguish Economic Capital (computed with internal models) and Regulatory Capital (computed according to methods given by the Basel Commitee on Banking Supervision).

Let consider an example of equity portfolio with a long position in the security S_t. We define the loss variable as follows: $\vartheta = S_{t+1} - S_t$. We assume two distributions: $\vartheta \sim \mathcal{N}(0,1)$ and $\vartheta \sim \mathbf{t}_4$.

Rating	*Regulatory*	BBB	A	AA	AAA
α	99%	99.75%	99.9%	99.95%	99.97%
Return time	100 days	400 days	4 years	8 years	13 years
$\Phi^{-1}(\alpha)$	2.33	2.81	3.09	3.29	3.43
$t_4^{-1}(\alpha)$	3.75	5.60	7.17	8.61	9.83

Let consider now a portfolio with different securities. The Capital-at-Risk will be influenced by

- the assumption on the distributions of individual risk factors;
- and by the assumption on the **dependence** between the different risk factors.

An example (market risk) Three portfolios with five commodities of the London Metal Exchange (see [3]):

	AL	AL-15	CU	NI	PB
P_1	1	1	1	1	1
P_2	-1	-1	-1	1	1
P_3	2	1	-3	4	5

- Gaussian margins and Normal copula

	90%	95%	99%	99.5%	99.9%
P_1	7.26	9.33	13.14	14.55	17.45
P_2	4.04	5.17	7.32	8.09	9.81
P_3	13.90	17.82	25.14	27.83	33.43

- Student margins ($\nu = 4$) and Normal copula

	90%	95%	99%	99.5%	99.9%
P_1	6.51	8.82	14.26	16.94	24.09
P_2	3.77	5.00	7.90	9.31	13.56
P_3	12.76	17.05	27.51	32.84	49.15

3.2 Operational risk

Industry definition = "*the risk of direct or indirect loss resulting from inadequate or failed internal processs, people and systems or from external events*" (thefts, desasters, etc.).

Operational risk is now explicitly concerned by the New Basel Capital Accord (banks have to allocate capital for operational risk since 2005).

Loss Distribution Approach (LDA) Under this approach, the bank estimates, for each business line/risk type cell, the probability distributions of the severity (single event impact) and of the one year event frequency using its internal data. With these two distributions, the bank then computes the probability distribution of the aggregate operational loss. The total required capital is the sum of the *Value-at-Risk* of each business line and event type combination.

Let i and j denote a given business line and a given event type.

- $\zeta(i,j)$ is the random variable which represents the **amount of one loss event** for the business line i and the event type j. The **loss severity distribution** of $\zeta(i,j)$ is denoted by $\mathbf{F}_{i,j}$.
- $N(i,j)$ is the random variable which represents the **number of one year events** for the business line i and the event type j. The **loss frequency distribution** of $\zeta(i,j)$ is denoted by $\mathbf{P}_{i,j}$.

In **LDA**, the loss for the business line i and the event type j is

$$\vartheta(i,j) = \sum_{n=0}^{N(i,j)} \zeta_n(i,j)$$

The distribution $\mathbf{G}_{i,j}$ of $\vartheta(i,j)$ is then a compound distribution

$$\mathbf{G}_{i,j}(x) = \begin{cases} \sum_{n=1}^{\infty} p_{i,j}(n)\,\mathbf{F}_{i,j}^{n\star}(x) & x > 0 \\ p_{i,j}(0) & x = 0 \end{cases}$$

For a given target insolvency rate $1-\alpha$, the capital charge corresponds to

$$\mathrm{CaR}\left(i,j;\alpha\right)=\mathbf{G}_{i,j}^{-1}\left(\alpha\right)$$

Computing the total capital charge The total capital charge for the bank will be then the simple summation of the capital charges accross each of the business lines and event types:

$$\mathrm{CaR}\left(\alpha\right)=\sum_{i=1}^{I}\sum_{j=1}^{J}\mathrm{CaR}\left(i,j;\alpha\right)$$

Problem: The Basel Commitee on Banking Supervision assumes implicitely that the different losses are perfectly dependent.

Let ϑ_1 and ϑ_2 be two losses with distributions $\mathbf{G}_1$ and $\mathbf{G}_2$. We denote ϑ the total loss with distribution $\mathbf{G}$. We have

$$\begin{aligned}
\mathrm{CaR}(\alpha) &= \mathbf{G}^{-1}(\alpha) \\
&= \mathrm{CaR}_1(\alpha) + \mathrm{CaR}_2(\alpha) \\
&= \mathbf{G}_1^{-1}(\alpha) + \mathbf{G}_2^{-1}(\alpha)
\end{aligned}$$

It is equivalent to assume that $\mathbf{C}\langle\vartheta_1,\vartheta_2\rangle = \mathbf{C}^+$. In this case, we have $\vartheta_2 = \mathbf{G}_2^{(-1)}(\mathbf{G}_1(\vartheta_1))$. Let us denote ϖ the function $x \mapsto x + \mathbf{G}_2^{(-1)}(\mathbf{G}_1(x))$. We have

$$\begin{aligned}
\alpha &= \Pr\{\vartheta_1 + \vartheta_2 \leq \mathrm{CaR}(\alpha)\} \\
&= \mathbb{E}\left[\mathbf{1}_{[\varpi(\vartheta_1) \leq \mathrm{CaR}(\alpha)]}\right] \\
&= \mathbf{G}_1\left(\varpi^{-1}(\mathrm{CaR}(\alpha))\right)
\end{aligned}$$

It comes that $\mathrm{CaR}(\alpha) = \varpi\left(\mathbf{G}_1^{(-1)}(\alpha)\right)$ and we obtain the relationship

$$\mathrm{CaR}(\alpha) = \mathbf{G}_1^{(-1)}(\alpha) + \mathbf{G}_2^{(-1)}\left(\mathbf{G}_1\left(\mathbf{G}_1^{(-1)}(\alpha)\right)\right) = \mathrm{CaR}_1(\alpha) + \mathrm{CaR}_2(\alpha)$$

Correlated aggregate loss distributions or correlated frequencies? The total loss distribution ϑ for the bank as whole is defined by

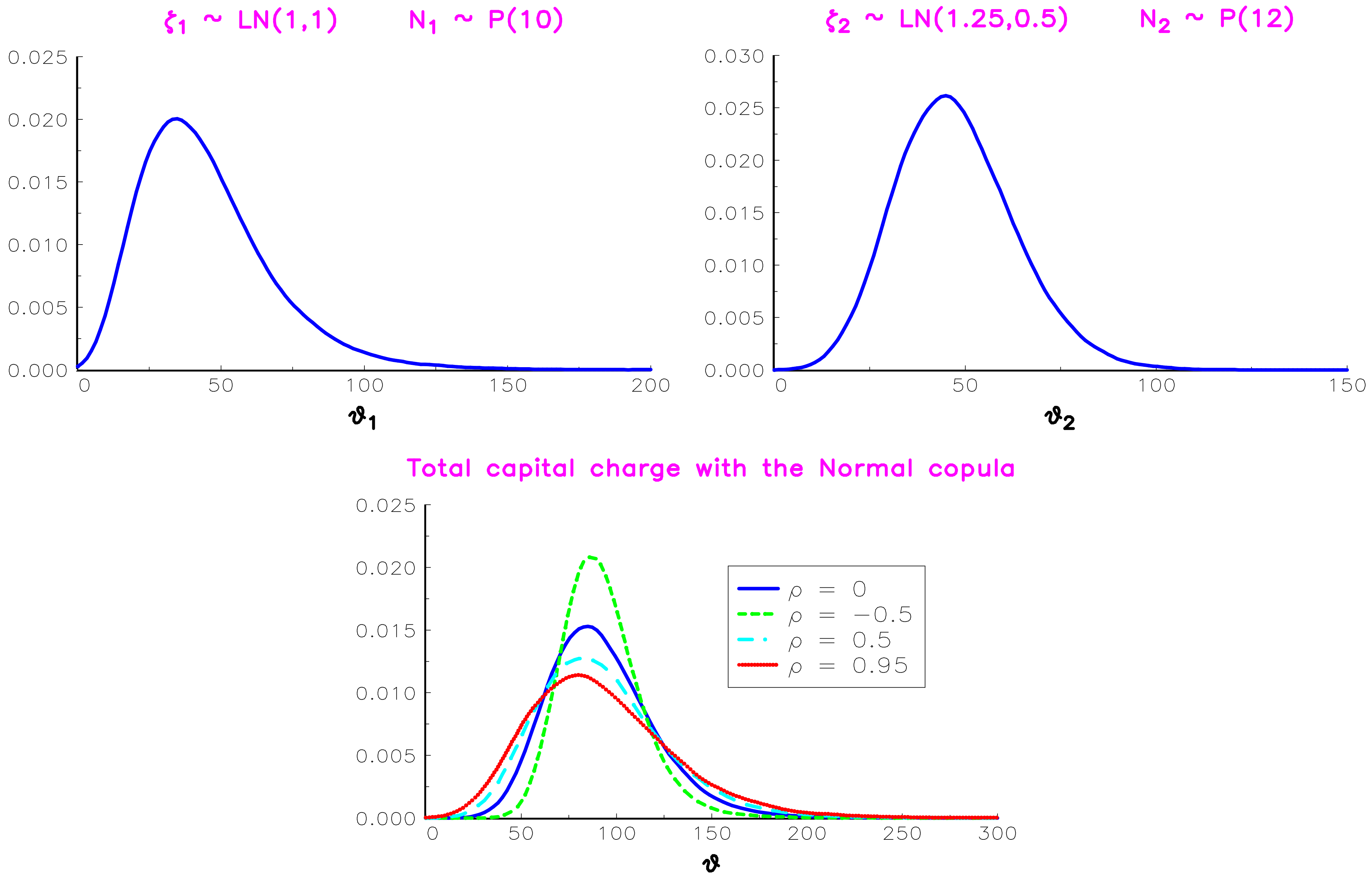
$$\vartheta = \sum_{i=1}^{I} \sum_{j=1}^{J} \vartheta(i,j)$$

In this case, we could introduce the dependence directly between the aggregate loss distributions.

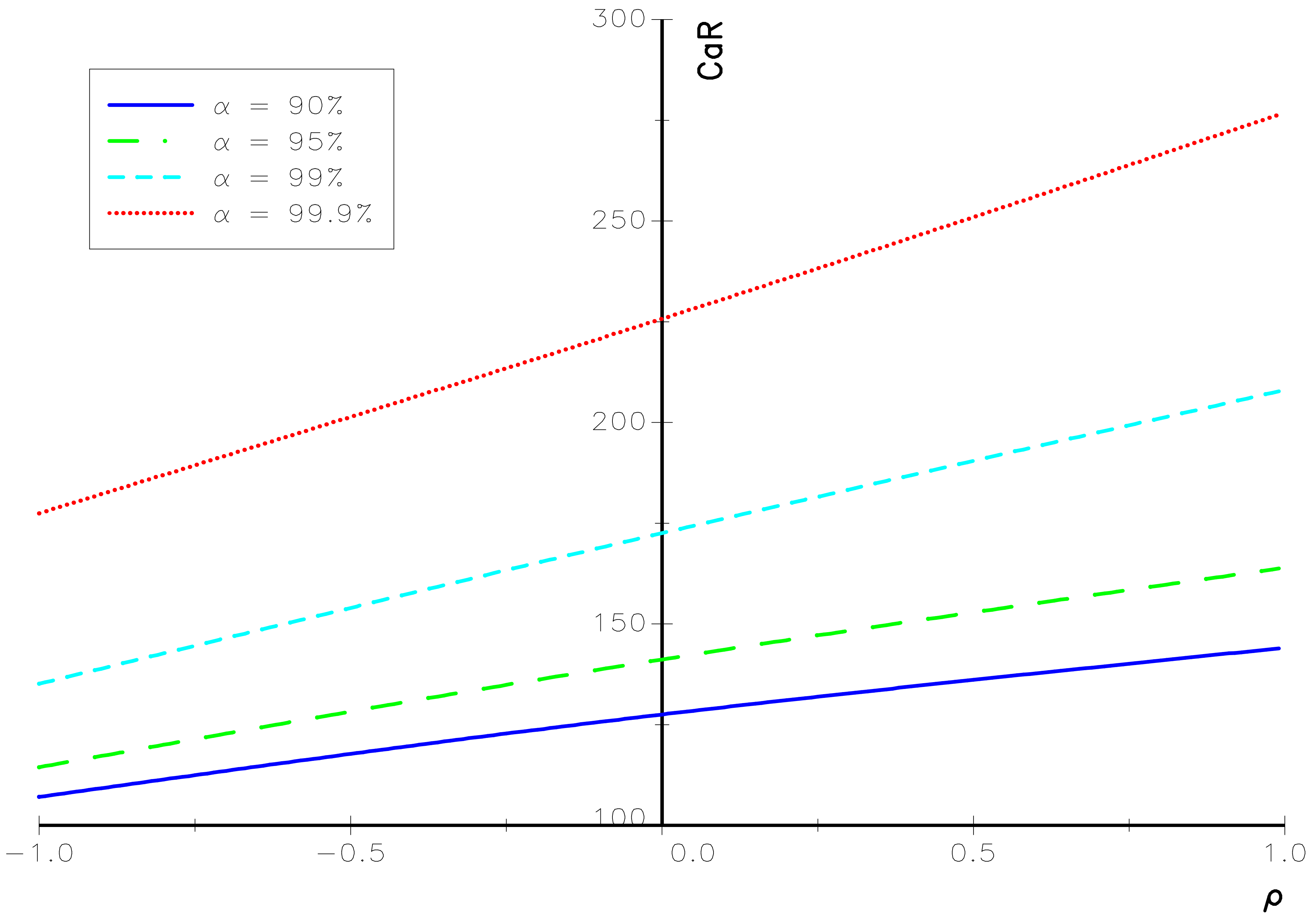
Or, we could introduce the dependence indirectly between the frequency distributions:

$$\vartheta = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{n=0}^{N(i,j)} \zeta_n(i,j)$$

For example, we could use multivarariate Poisson distributions generated from the Normal copula (Song [2000]).



Impact of the copula on the total loss distribution



Impact of the parameter ρ on the Capital-at-Risk

3.3 The choice of the copula function

Let consider two markets (for example the equity market and the bond market). We are interested in the probability that the loss in one market is greater than its value-at-risk given that the loss in the second market is already greater than its value-at-risk:

$$\begin{aligned}\lambda(\alpha) &= \Pr\{\vartheta_2 > \mathrm{VaR}_2(\alpha) \mid \vartheta_1 > \mathrm{VaR}_1(\alpha)\} \\ &= \Pr\left\{\vartheta_2 > \mathbf{F}_2^{-1}(\alpha) \mid \vartheta_1 > \mathbf{F}_1^{-1}(\alpha)\right\} \\ &= \frac{\Pr\{\mathbf{F}_2(\vartheta_2) > \alpha, \mathbf{F}_1(\vartheta_1) > \alpha\}}{\Pr\{\mathbf{F}_1(\vartheta_1) > \alpha\}} \\ &= \frac{1 - 2\alpha + \mathbf{C}(\alpha, \alpha)}{1 - \alpha}\end{aligned}$$

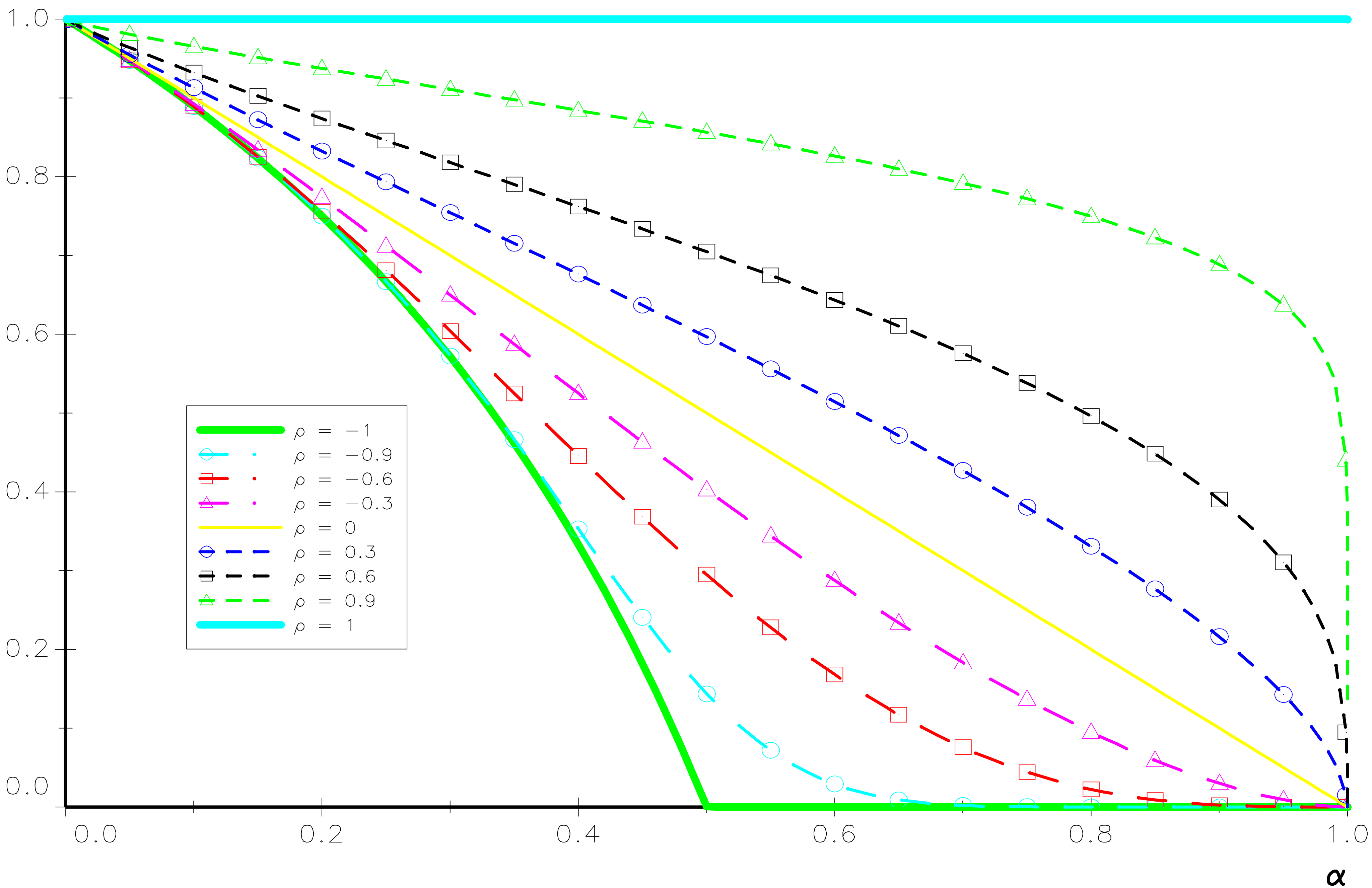
$\lambda(\alpha)$ depends on the copula, but not on the margins.

The limit case $\lambda = \lim_{\alpha \to 1} \lambda(\alpha)$ is called the tail dependence coefficient.

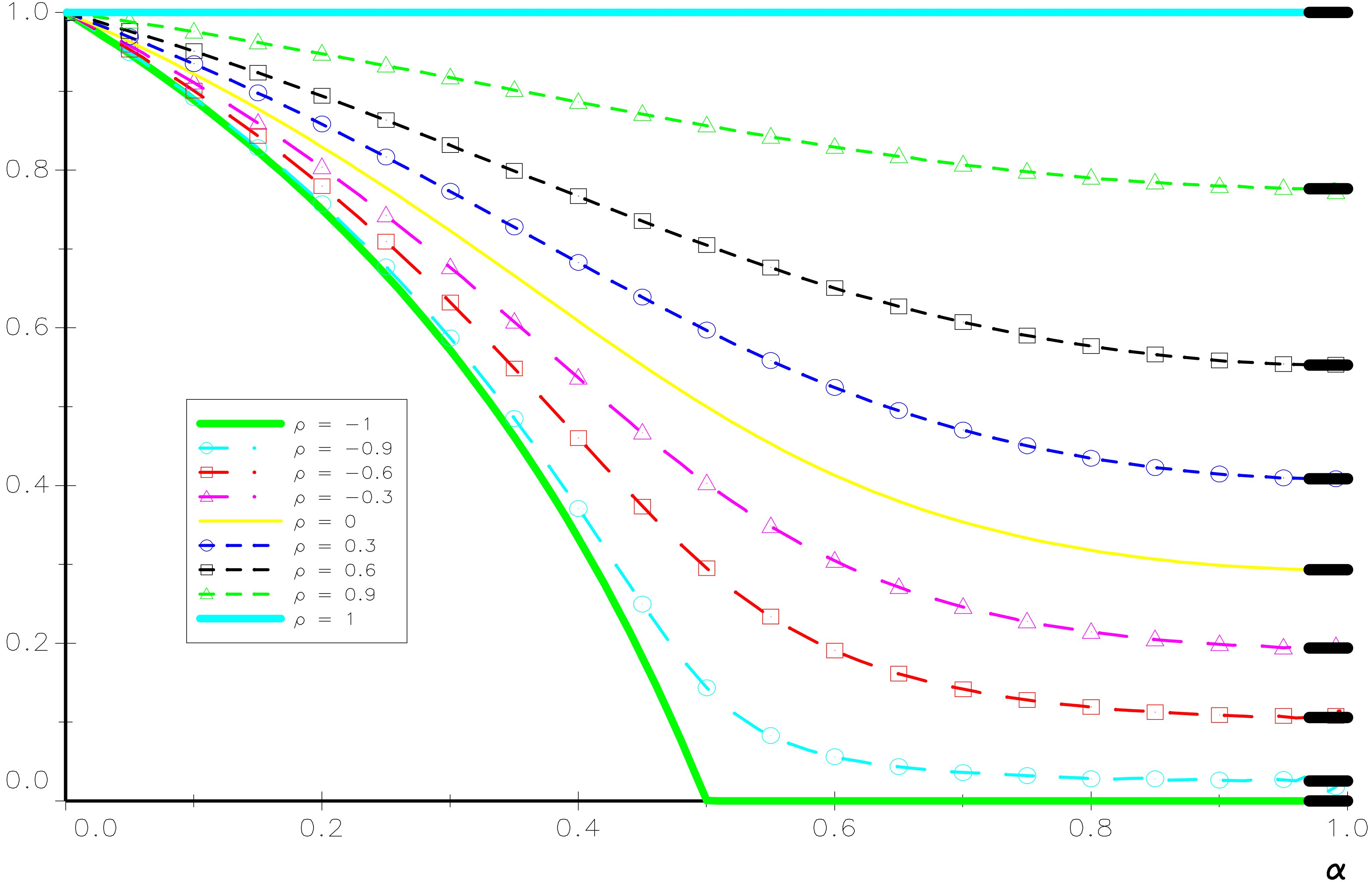
Remark 2 *The measure λ is the probability that one variable is extreme given that the other is extreme.*

1. Normal copula $\Rightarrow$ extremes are asymptotically independent for $\rho \neq 1$, i.e $\lambda = 0$ for $\rho < 1$.
2. Student copula $\Rightarrow$ extremes are asymptotically dependent for $\rho \neq -1$.

The copula function has then a great influence of the aggregation of risks (in particular, stress-testing is very sensitive to the choice of the copula — see [2]).



Quantile-dependent measure $\lambda(\alpha)$ for the Normal copula



$\lambda(\alpha)$ for the Student copula and $\nu = 1$

4 New pricing methods with copulas

Let consider an European Call option. The payoff is $G(T) = (S(T) - K)^+$. Under some conditions, the price $P(t_0)$ of this contingent claim is given by

$$P(t_0) = e^{-r(T-t_0)} \mathbb{E}^{\mathbb{Q}} \left[G(T) \middle| \mathcal{F}_{t_0} \right]$$

with $\mathbb{Q}$ the *martingale* probability measure.

For a spread option, we have $G(T) = (S_2(T) - S_1(T) - K)^+$ and we obtain a similar expression for the price.

Spread option is a special case of two-asset options. Since some years, multi-asset options are traded very frequently. The main difference with option with only one underlying is that the *martingale* probability measure is **multidimensional**.

4.1 Coherent valuation of multi-asset options

The Black-Scholes model In the BS model, we have

$$\mathrm{d}S(t) = rS(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t)$$

under $\mathbb{Q}$. The price of an European option is then a function of the volatility σ. However, when we compute the implied volatility from the option prices for different values of the strike K, it is not constant. This is the volatility smile effect.

Option models in banks Banks have then developped sophisticated models (e.g. stochastic volatility models) to take into account the smile effect.

To this day, therefore, the BS model continues to be used, out of analytical and computational convenience, for contingent claims based on different assets.

Problem: the margins of the multivariate Risk-Neutral Distribution (RND) are not the univariate RND.

$\Rightarrow$ In this case, we may show that there exists arbitrage opportunities inside the same bank (see [10]).

$\Rightarrow$ Moreover, one-asset options could be viewed as limits of multi-asset options — see e.g. the Basket option $G(T) = (\alpha_1 S_1(T) + \alpha_2 S_2(T) - K)^{+}$.

The copula construction Let $\mathbb{Q}$ be the multivariate RND of the random vector $\mathbf{S}(T) \mid \mathcal{F}_{t_0}$. In [10], we show that the margins of $\mathbb{Q}$ are necessarily univariate RND*. Using Sklar's theorem, it comes that $\mathbb{Q}$ admits the following canonical decomposition

$$\mathbb{Q}(S_1(T), \ldots, S_N(T)) = \mathbf{C}^{\mathbb{Q}}(\mathbb{Q}_1(S_1(T)), \ldots, \mathbb{Q}_N(S_N(T)))$$

$\mathbf{C}^{\mathbb{Q}}$ is called the risk-neutral copula (RNC).

*We prove this by using properties of the Girsanov theorem applied to multivariate probability measure.

Relationships between $\mathbf{C}^{\mathbb{Q}}$ and $\mathbf{C}^{\mathbb{P}}$

Let $\mathbb{P}$ be the objective (or historical) distribution. We denote by $\mathbf{C}^{\mathbb{P}}$ the objective copula. We can prove the following proposition:

Proposition 1 *If the drift and the diffusion of the asset prices vector* $\mathbf{S}(t)$ *are of the form* $\mu(t) \odot \mathbf{S}(t)$ *and* $\sigma(t) \odot \mathbf{S}(t)$ *and if risk premiums are non stochastic, then the risk-neutral copula* $\mathbf{C}^{\mathbb{Q}}$ *is equal to the objective copula* $\mathbf{C}^{\mathbb{P}}$*.*

Implication of this proposition: in this case, the univariate RND can be estimated using the options market whereas the RNC can be estimated using the spot market. So, the spot market contains useful information to price multi-asset options.

The case of the spread option In [12], Valdo Durrleman shows that the price $P(t_0)$ is

$$\begin{aligned} P(t_0) &= S_2(t_0) - S_1(t_0) - Ke^{-r(T-t_0)} + \\ & e^{-r(T-t_0)} \int_{-\infty}^{K} \int_{0}^{+\infty} f_1(x) \cdot \partial_1 \mathbf{C}^{\mathbb{Q}}(\mathbf{F}_1(x), \mathbf{F}_2(x+y)) \, \mathrm{d}x \, \mathrm{d}y \end{aligned}$$

A remark The copula construction implies that we can associate a risk-neutral copula to a multivariate risk-neutral distribution. But it does not mean that the combination of univariate RND with a copula define necessarily a multivariate risk-neutral distribution (see [10] for further details).

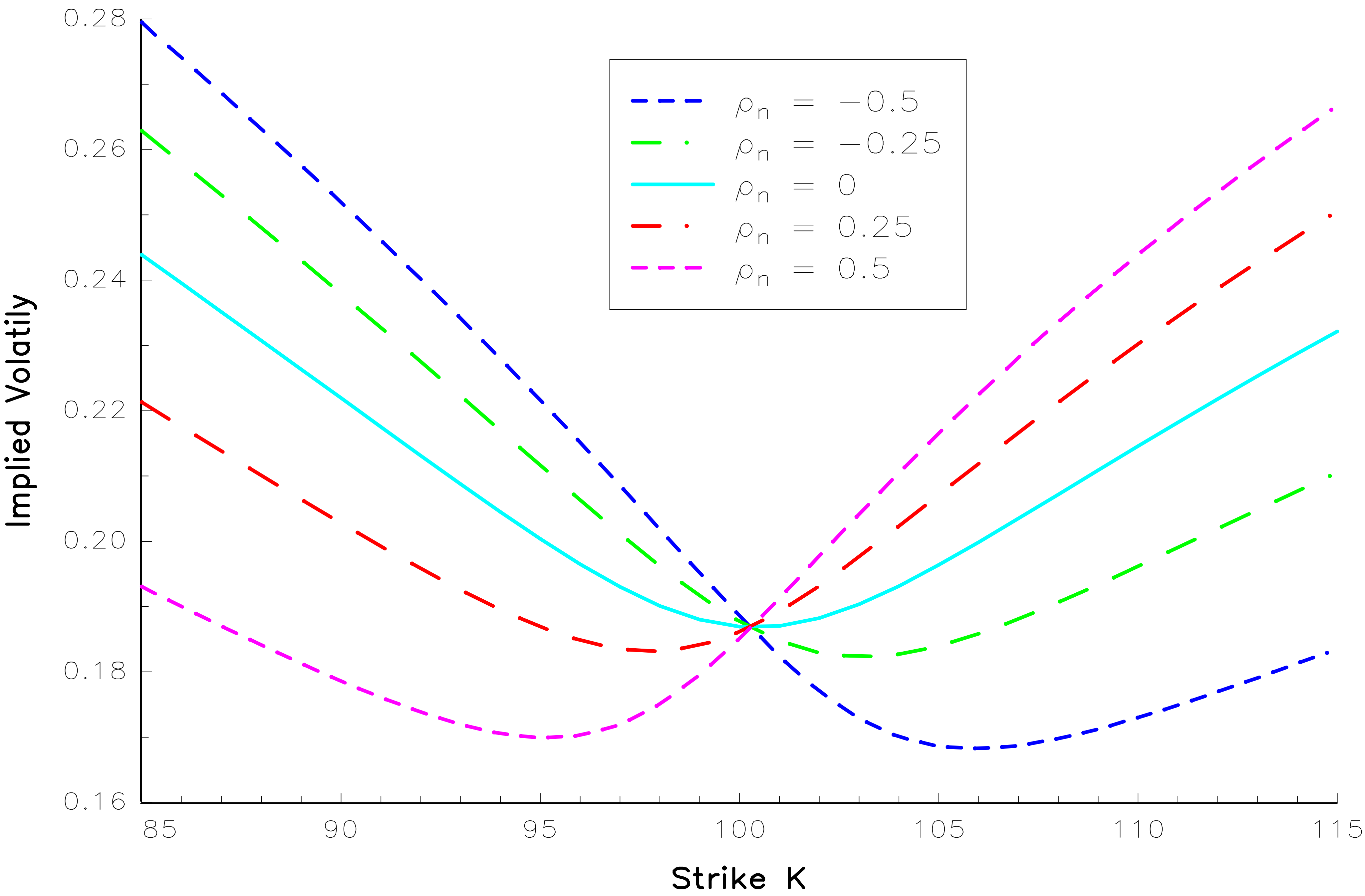
BS pricing in stochastic volatility environment We assume that the asset prices $S_n(t)$ are given by the Heston model

$$\begin{cases} \mathrm{d}S_n(t) &= \mu_n S_n(t)\,\mathrm{d}t + \sqrt{V_n(t)}S_n(t)\,\mathrm{d}W_n^1(t) \\ \mathrm{d}V_n(t) &= \kappa_n(V_n(\infty) - V_n(t))\,\mathrm{d}t + \sigma_n\sqrt{V_n(t)}\,\mathrm{d}W_n^2(t) \end{cases}$$

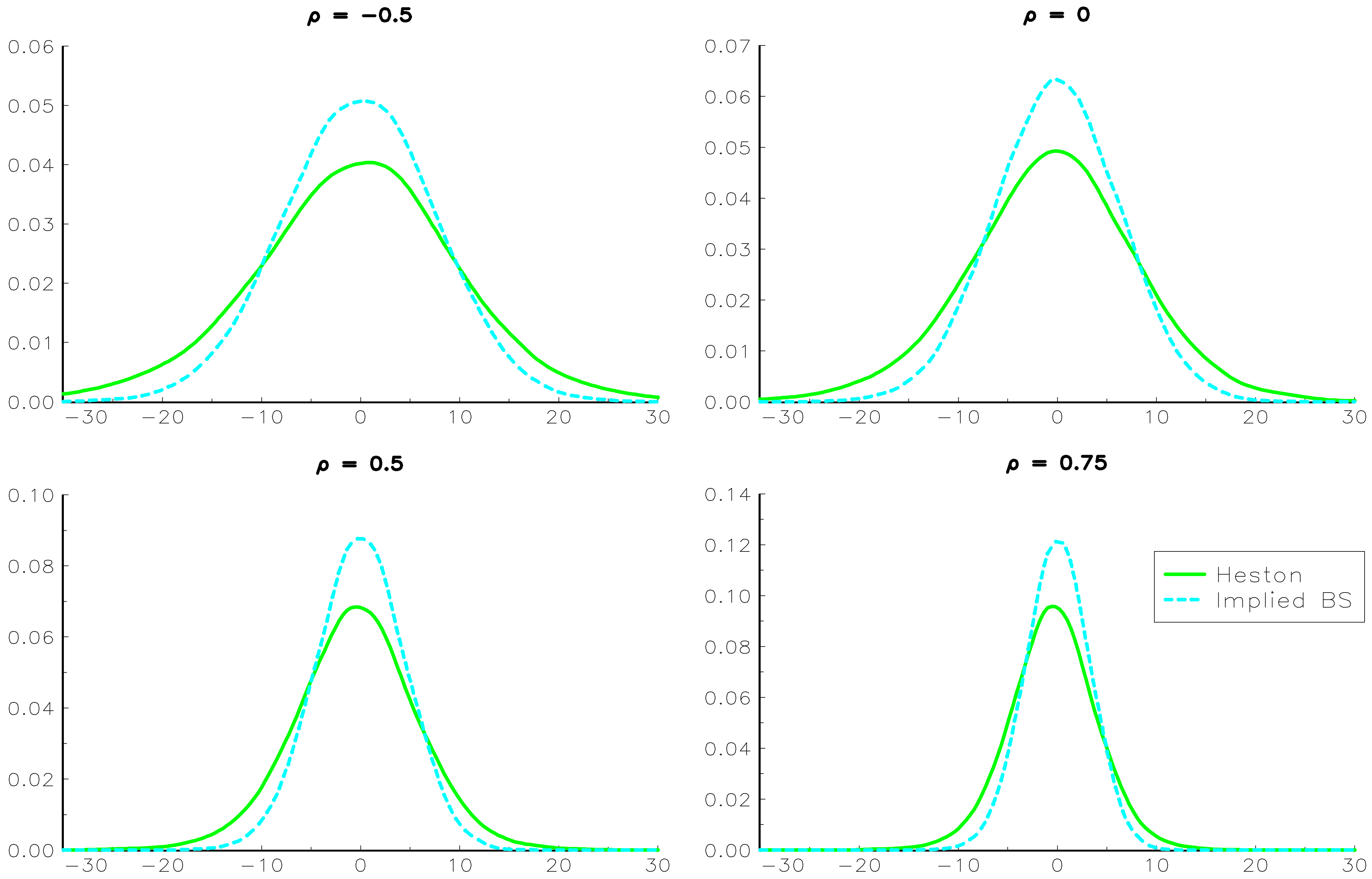
with $\mathbb{E}\left[W_n^1(t)\,W_n^2(t) \mid \mathcal{F}_{t_0}\right] = \rho_n(t - t_0)$, $\kappa_n > 0$, $V_n(\infty) > 0$ and $\sigma_n > 0$. The market prices of risk processes are $\lambda_n^1(t) = (\mu_n - r)/\sqrt{V_n(t)}$ and $\lambda_n^2(t) = \lambda_n\sigma_n^{-1}\sqrt{V_n(t)}$.

To compute prices of spread options, we consider that the RNC is the Normal copula with parameter ρ. We compare then the Heston prices with these given by the BS model using ATM implied volatilities.

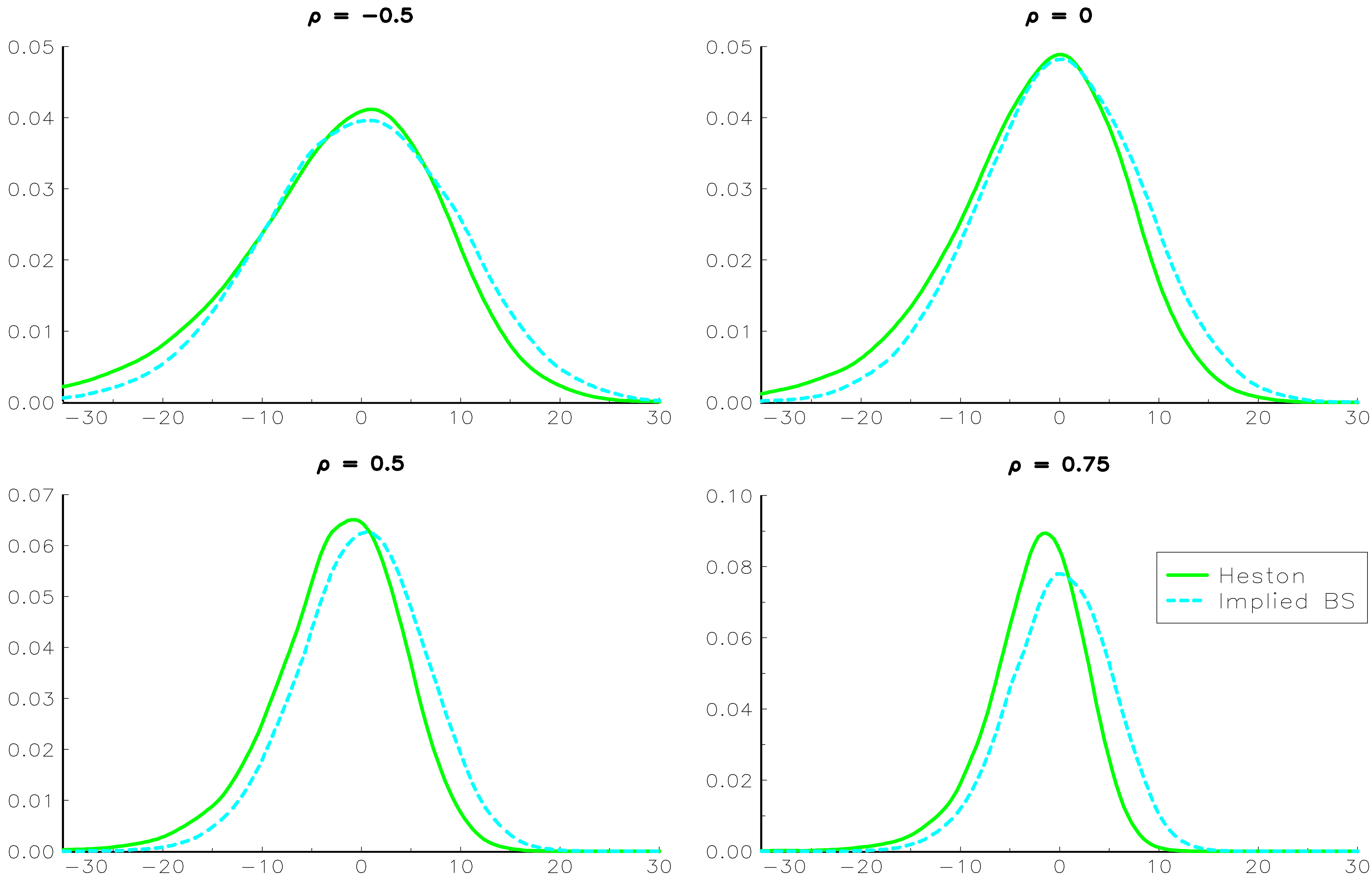
Numerical values (two assets with same characteristics except ρ_n): $S_n(t_0) = 100$, $\tau = 1/12$, $r = 5\%$, $V_n(t_0) = V_n(\infty) = \sqrt{20\%}$, $\kappa_n = 0.5$, $\sigma_n = 90\%$ and $\lambda_n = 0$.



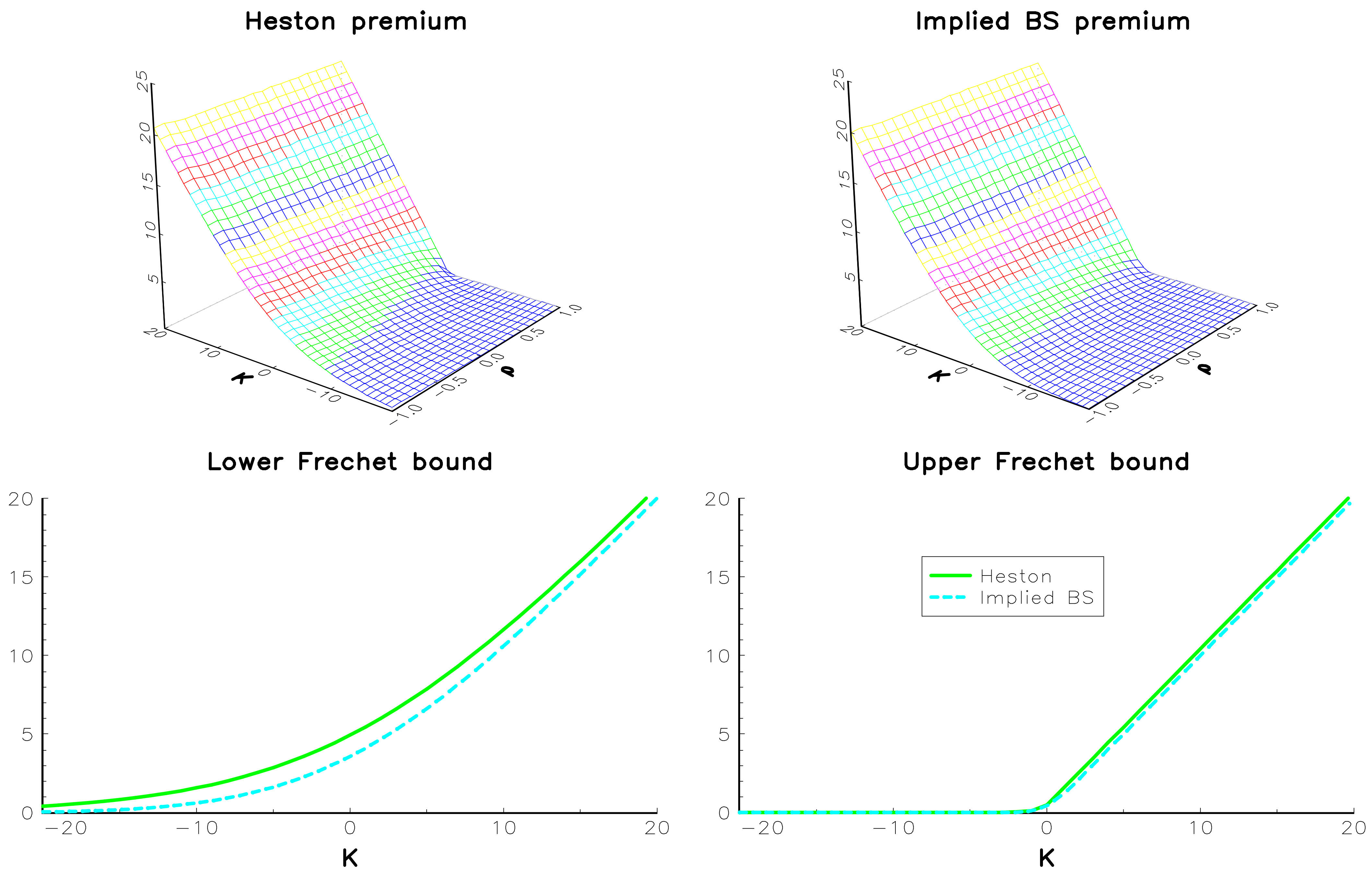
Volatility smile of the Heston model (one-month maturity)



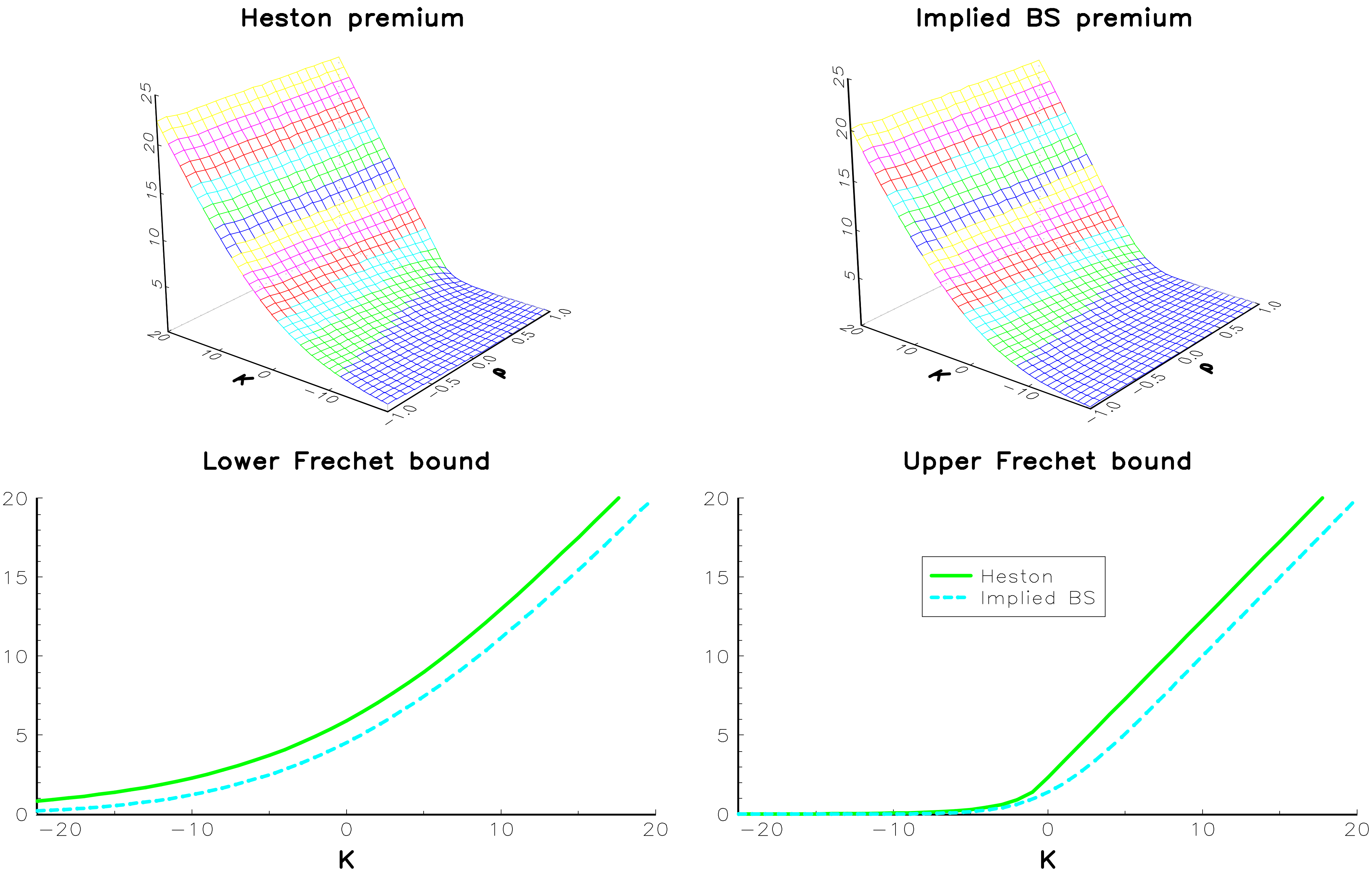
Density of the RND of the spread ($\rho_1 = -0.75$, $\rho_2 = -0.50$)



Density of the RND of the spread (ρ_1=−0.75, ρ_2=0.50)



European Spread Put premium ($\rho_1=-0.75$, $\rho_2=-0.50$)



European Spread Put premium ($\rho_1=-0.75$, $\rho_2=0.50$)

How to build 'forward-looking' indicators for the dependence function?

In [1], Aris Bikos suggests the following method:

1. estimate the univariate RND $\widehat{\mathbb{Q}}_n$ using Vanilla options;
2. estimate the copula $\hat{\mathbf{C}}$ using multi-asset options by imposing that $\mathbb{Q}_n = \widehat{\mathbb{Q}}_n$;
3. derive "forward-looking" indicators directly from $\hat{\mathbf{C}}$.

An example of the computation of the implied parameter $\widehat{\rho}$

- BS model: LN distribution calibrated with ATM options; Pricing kernel = LN distributions + Normal copula

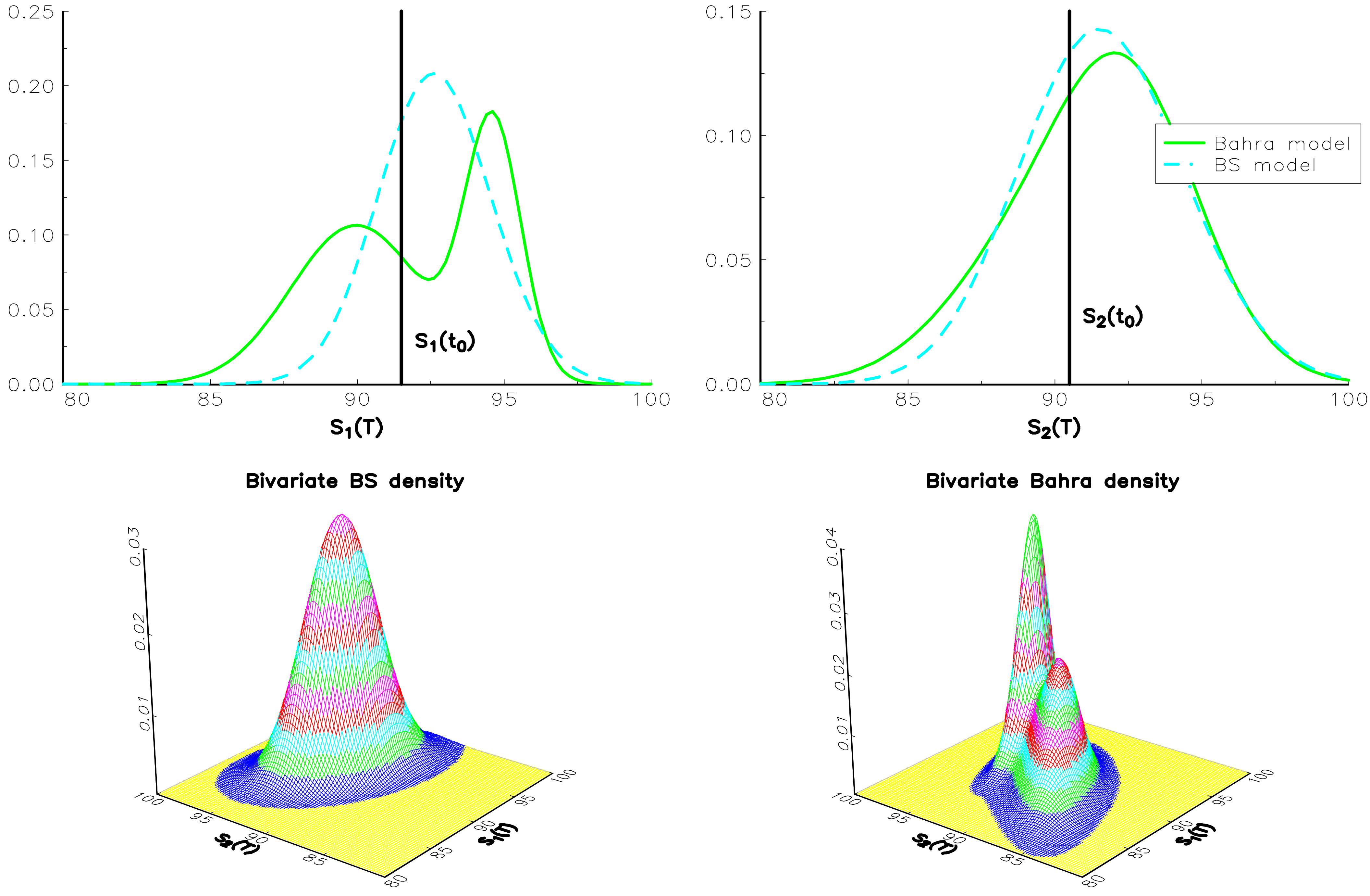
$$\widehat{\rho}_1 = -0.341$$

- Bahra model: mixture of LN distributions calibrated with eight European prices; Pricing kernel = MLN distributions + Normal copula

$$\widehat{\rho}_2 = 0.767$$

Remark 3 *$\widehat{\rho}_1$ and $\widehat{\rho}_2$ are parameters of the Normal Copula. $\widehat{\rho}_1$ is a Pearson correlation, not $\widehat{\rho}_2$.*

$\Rightarrow$ BS model: negative dependence / Bahra model: positive dependence.



A spread option example

4.2 The pricing of credit derivatives

In multi-asset options, the risk is a <u>market</u> risk (because of the **volatility** of the asset prices). In credit derivatives, the risk is a <u>credit</u> risk (because of the **default** of the counterparties).

A default is generally described by a *survival* function $\mathbf{S}(t) = \Pr\{T > t\}$. Let $\breve{\mathbf{C}}$ be a *survival* copula. A multivariate survival function $\mathbf{S}$ can be defined as follows

$$\mathbf{S}(t_1, \ldots, t_N) = \breve{\mathbf{C}}(\mathbf{S}_1(t_1), \ldots, \mathbf{S}_N(t_N))$$

where $(\mathbf{S}_1, \ldots, \mathbf{S}_N)$ are the marginal survival functions. Nelsen [1999] notices that "$\breve{\mathbf{C}}$ *couples the joint survival function to its univariate margins in a manner completely analogous to the way in which a copula connects the joint distribution function to its margins*".

$\Rightarrow$ Introducing dependence between defaultable securities can then be done using the copula framework (see [21] and [25]).

Some examples*

The Default Digital Put (DDP) option

The European DDP "*pays off 1 at t iff there has been a default at some time before (or including) t*".

If we assume that the interest rate and the first default $\tau = \bigwedge_{n=1}^{N} T_n$ are independent, the price at time t_0 of the DDP of maturity t is then†

$$\begin{aligned} \mathrm{DDP}(t_0, t) &= \mathbb{E}\left[e^{-\int_{t_0}^{t} r(s)\,\mathrm{d}s} \mathbf{1}_{[\tau < t]} \right] \\ &= (1 - S_\tau(t))\, P(t_0, t) \\ &= \Big(1 - \breve{\mathbf{C}}(\mathbf{S}_1(t), \ldots, \mathbf{S}_N(t))\Big)\, P(t_0, t) \end{aligned}$$

*These examples are taken from [18].

†$P(t_0, t)$ is the default-free bond price.

The First-to-Default (FtoD) option – A first-to-default is a contingent claim that pays at the first of N credit events an amount $\varpi(\tau)$.

$$\begin{aligned}
\mathrm{FtoD}(t_0,t) &= \mathbb{E}\left[\varpi(\tau)\,e^{-r(\tau)}\mathbf{1}_{[\tau<t]}\right] \\
&= \int_{t_0}^{t}\varpi(s)\,e^{-r(s)}f_\tau(s)\,\mathrm{d}s \\
&= \sum_{n=1}^{N}\int_{t_0}^{t}\varpi(s)\,e^{-r(s)}f_n(s)\,\partial_n\breve{\mathbf{C}}\left(\mathbf{S}_1(s),\ldots,\mathbf{S}_N(s)\right)\mathrm{d}s
\end{aligned}$$

The n^{th}-to-Default ($P_{n:N}$) option

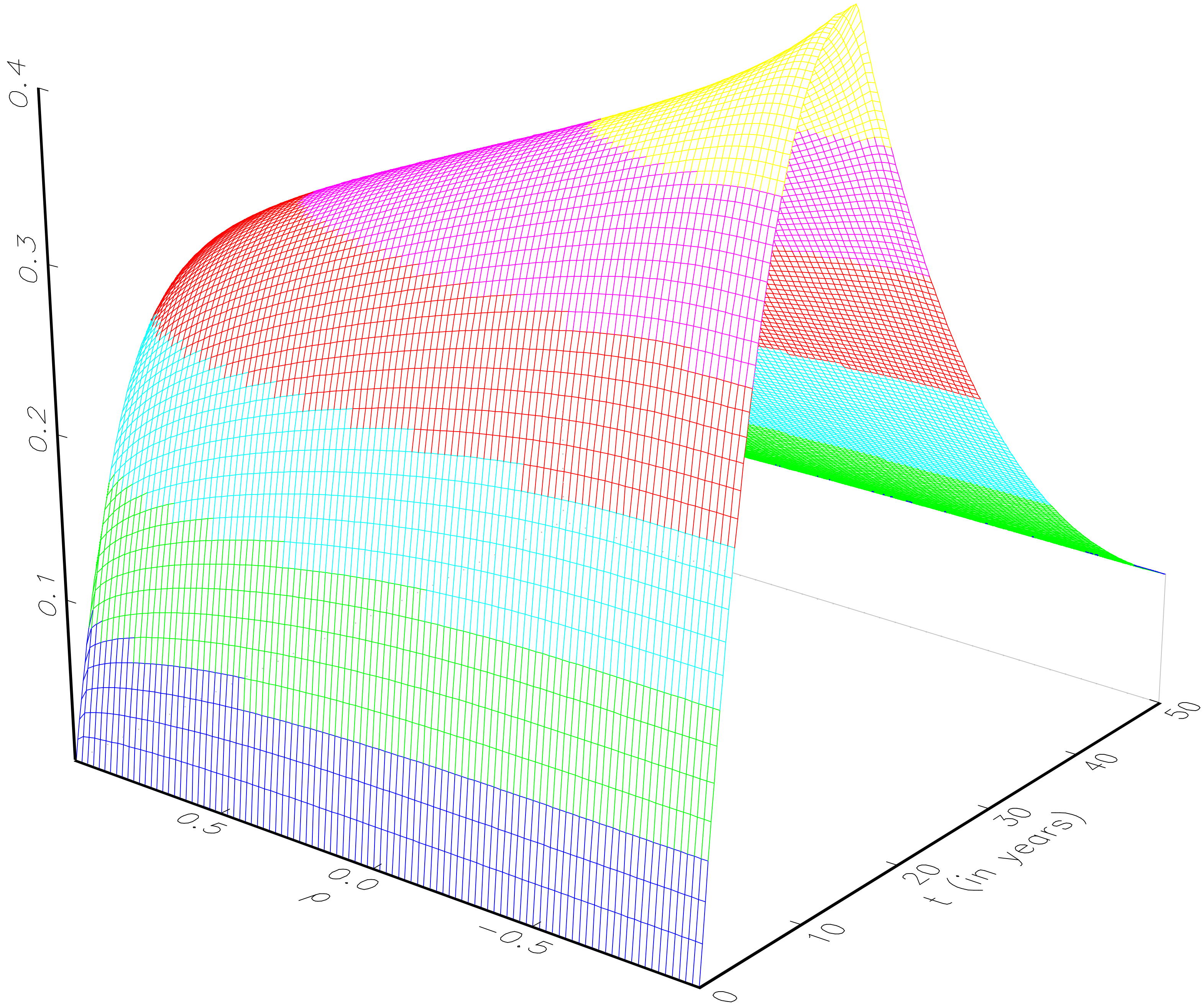
$$\begin{aligned}
P_{n:N}(t_0,t) &= \mathbb{E}\left[\varpi(T_{n:N})\,e^{-r(T_{n:N})}\mathbf{1}_{[T_{n:N}<t]}\right] \\
&= \int_{t_0}^{t}\varpi(s)\,e^{-r(s)}f_{n:N}(s)\,\mathrm{d}s
\end{aligned}$$

We have of course $P_{1:N} = \mathrm{FtoD}$ and $P_{N:N} = \mathrm{LtoD}$ (*last-to-default*).

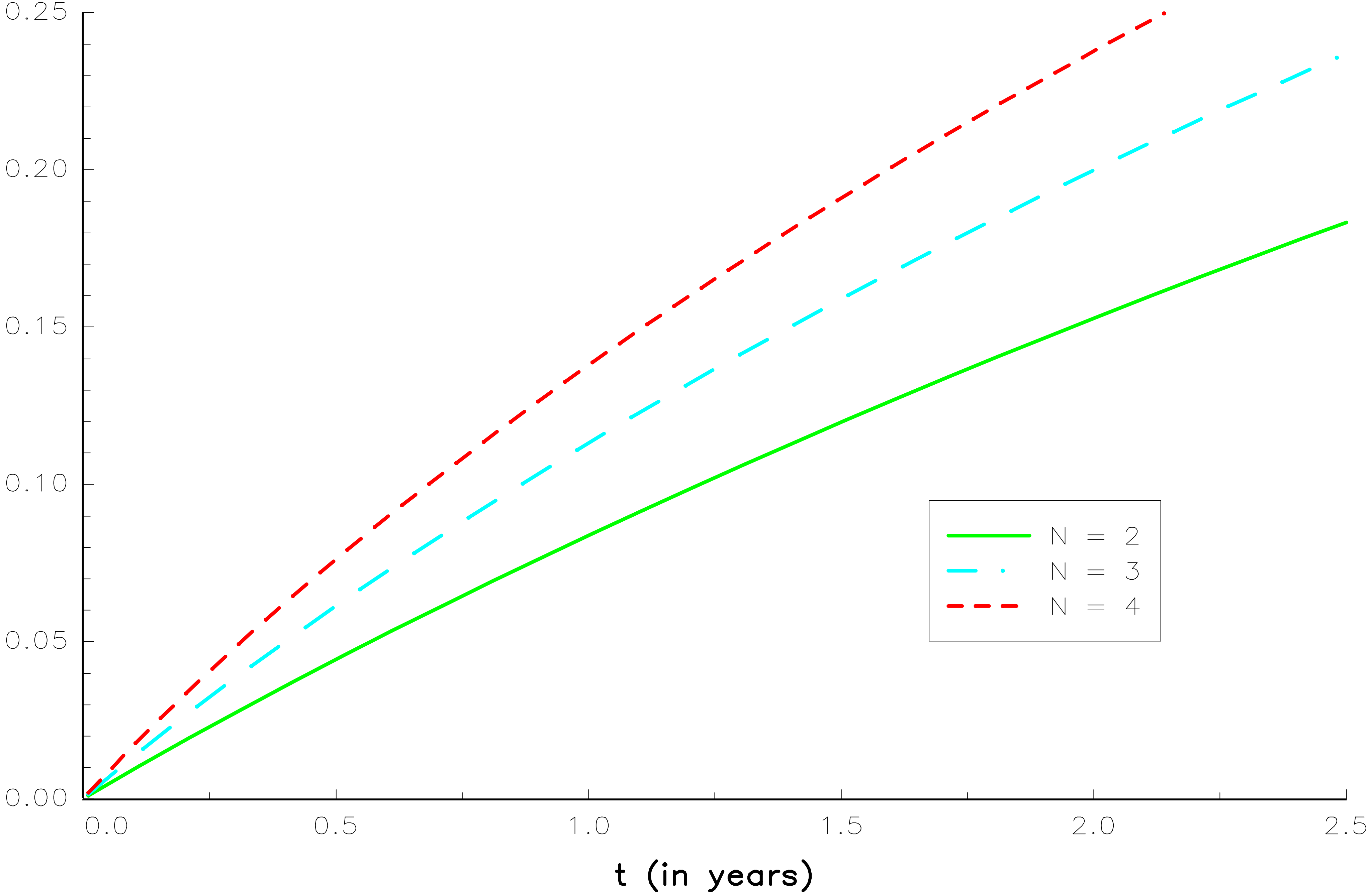
Numerical illustrations

$\varpi = 1$ — $r = 5\%$ — Exponential survival times with hasard rates λ_n — the survival copula is a Normal copula with a matrix of parameters of the form

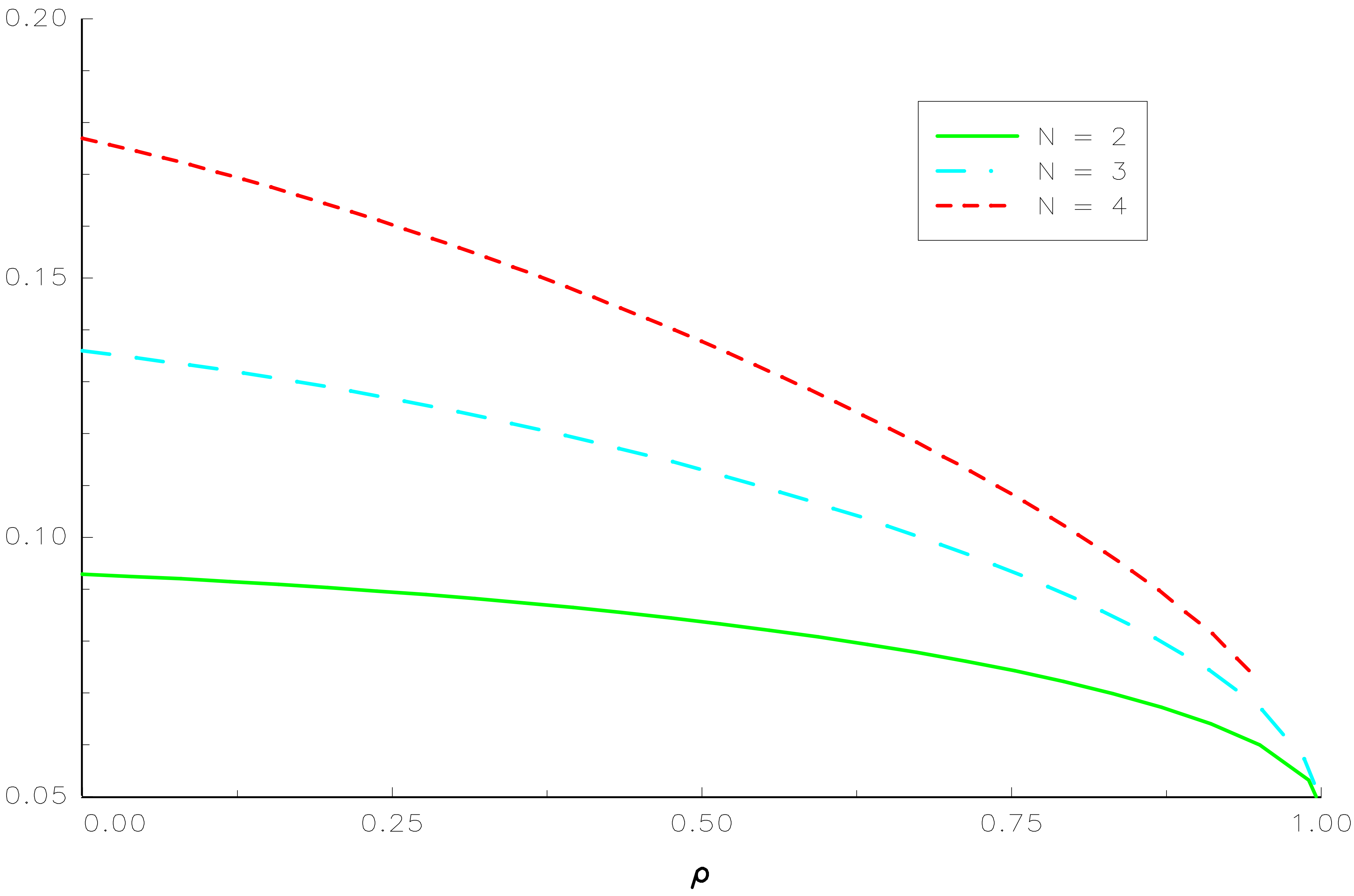
$$\begin{bmatrix} 1 & \rho & \cdots & \rho \\ & 1 & \ddots & \vdots \\ & & 1 & \rho \\ & & & 1 \end{bmatrix}$$



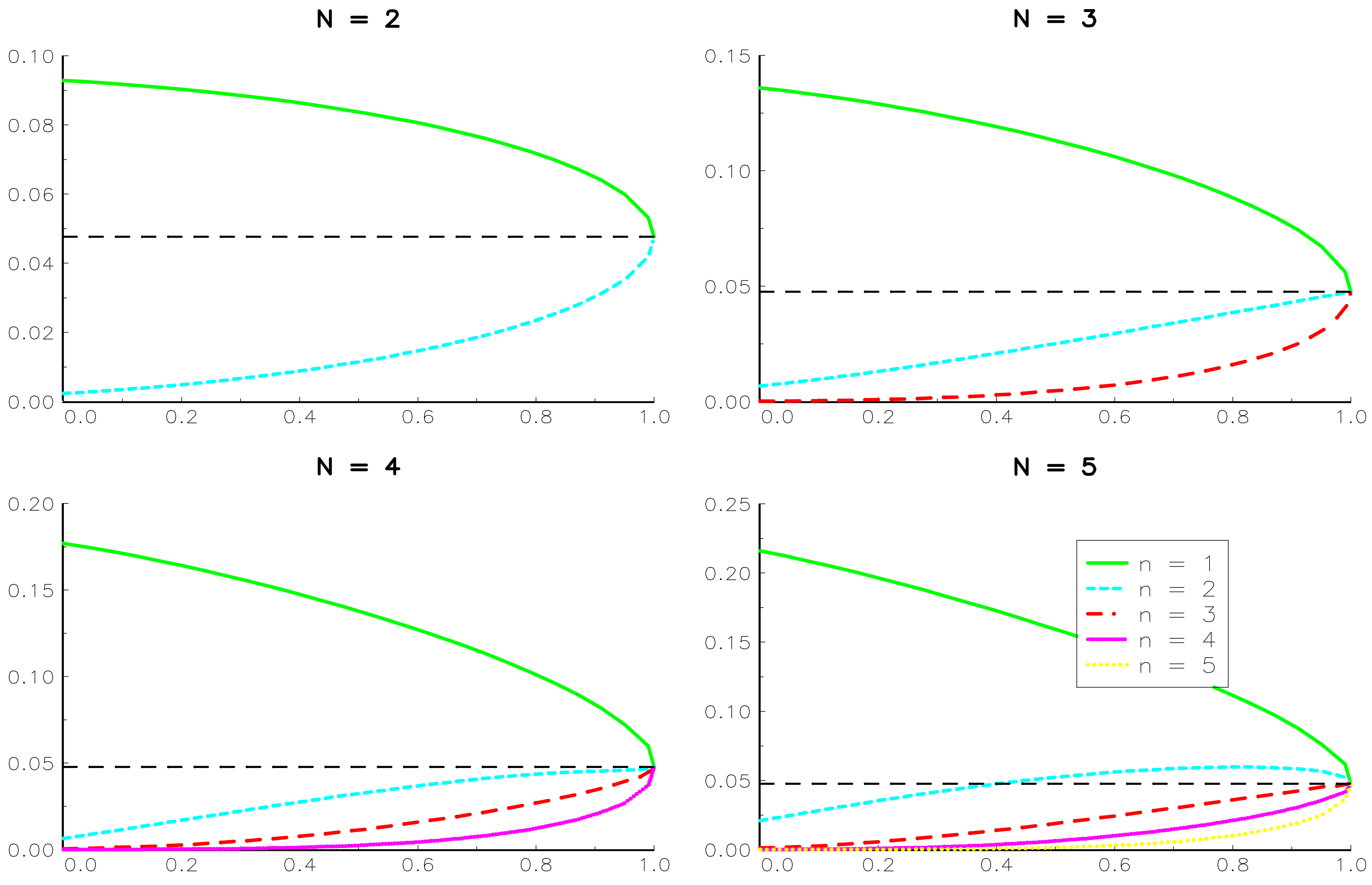
Influence of the parameter ρ on the DDP premium ($N = 2$)



Value of the FtoD premium for ρ = 0.5



Value of the FtoD premium for t = 1



Value of the premium $P_{n:N}(t_0,t)$ for $t = 1$

5 Conclusion

The use of copulas in finance is very recent.

However, private communications with professionals of other banks indicate that copulas are **largely** studied (and used) in most banks.

And professionals expect a lot from copulas to solve (and understand) many financial problems.

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