Understanding the dependence in financial models with copulas

Stochastic Models in Finance

Ecole Polytechnique, Paris, 04/23/2001

Thierry Roncalli

Groupe de Recherche Opérationnelle

Crédit Lyonnais

http://gro.creditlyonnais.fr

Joint work with V. Durrleman, A. Nikeghbali, G. Riboulet, E. Bouyé, J. Messines, etc.

1 What is a copula function?

Definition 1 (Schweizer and Sklar [1974]) *A two-dimensional copula (or 2-copula) is a function* $\mathbf{C}$ *with the following properties:*

1. $\operatorname{Dom}\mathbf{C} = [0,1] \times [0,1]$;
2. $\mathbf{C}(0,u) = \mathbf{C}(u,0) = 0$ *and* $\mathbf{C}(u,1) = \mathbf{C}(1,u) = u$ *for all* u *in* $[0,1]$;
3. $\mathbf{C}$ *is 2-increasing:*

$$\mathbf{C}(v_1,v_2) - \mathbf{C}(v_1,u_2) - \mathbf{C}(u_1,v_2) + \mathbf{C}(u_1,u_2) \geq 0$$

whenever $(u_1,u_2) \in [0,1]^2$, $(v_1,v_2) \in [0,1]^2$ *such* $0 \leq u_1 \leq v_1 \leq 1$ *and* $0 \leq u_2 \leq v_2 \leq 1$.

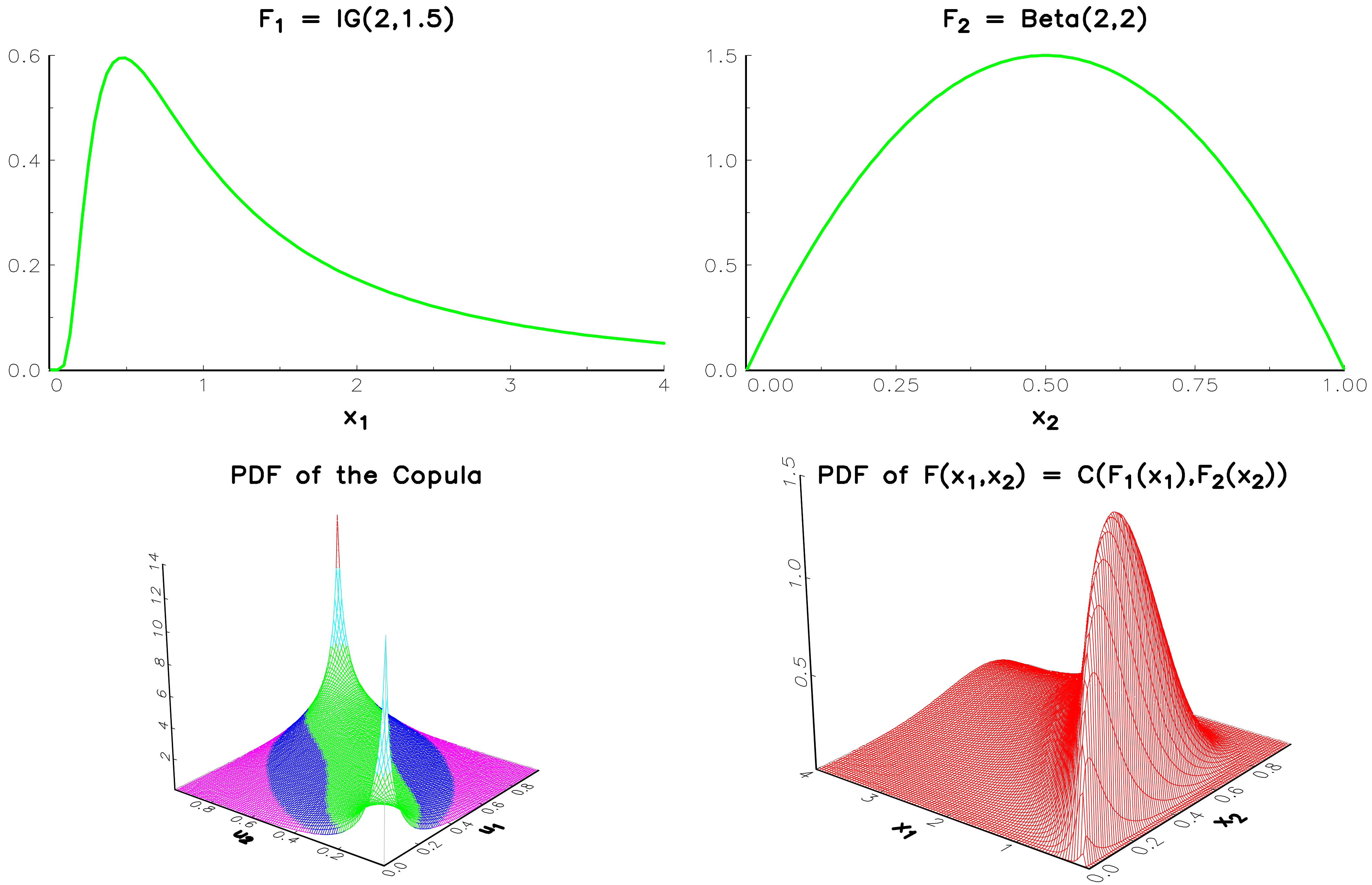
$\Rightarrow$ 2-Copulas are also doubly stochastic measures on the unit square.

Theorem 1 *Let* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *be 2 univariate distributions. It comes that* $\mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2))$ *defines a bivariate probability distribution with margins* $\mathbf{F}_1$ *and* $\mathbf{F}_2$ *(because the integral transforms are uniform distributions).*

Theorem 2 *Let* $\mathbf{F}$ *be a 2-dimensional distribution function with margins* $\mathbf{F}_1$ *and* $\mathbf{F}_2$*. Then* $\mathbf{F}$ *has a copula representation:*

$$\mathbf{F}(x_1, x_2) = \mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2))$$

The copula $\mathbf{C}$ *is unique if the margins are continuous. Otherwise, only the subcopula is uniquely determined on* $\operatorname{Ran}\mathbf{F}_1 \times \operatorname{Ran}\mathbf{F}_2$*.*



Bivariate distribution with given marginals

2 What is a dependence function?

The copula function of **random variables** (X_1, X_2) is **invariant** under strictly increasing transformations ($\partial_x h_n(x) > 0$):

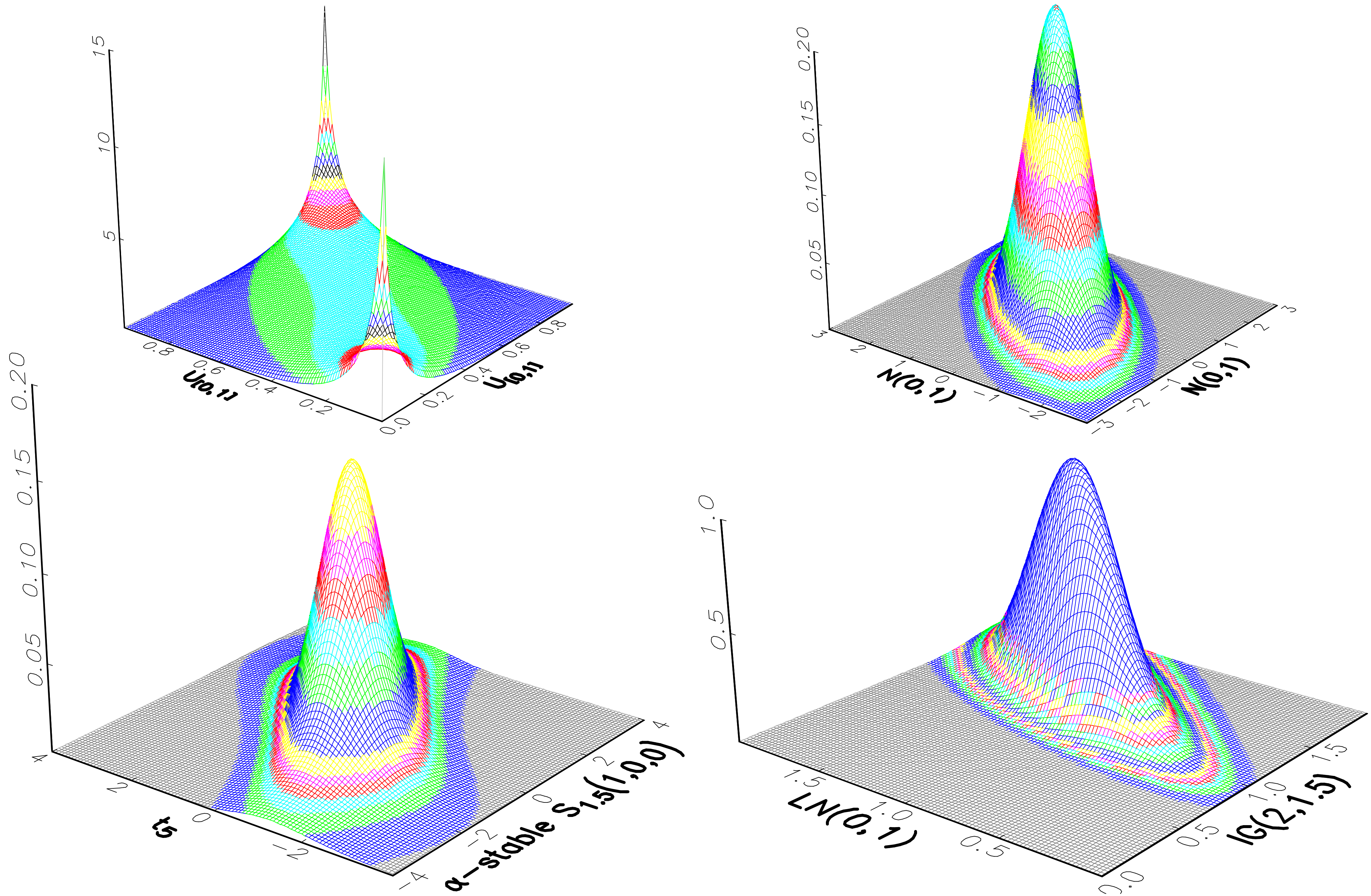
$$\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}\langle h_1(X_1), h_2(X_2)\rangle$$

Here are some examples :

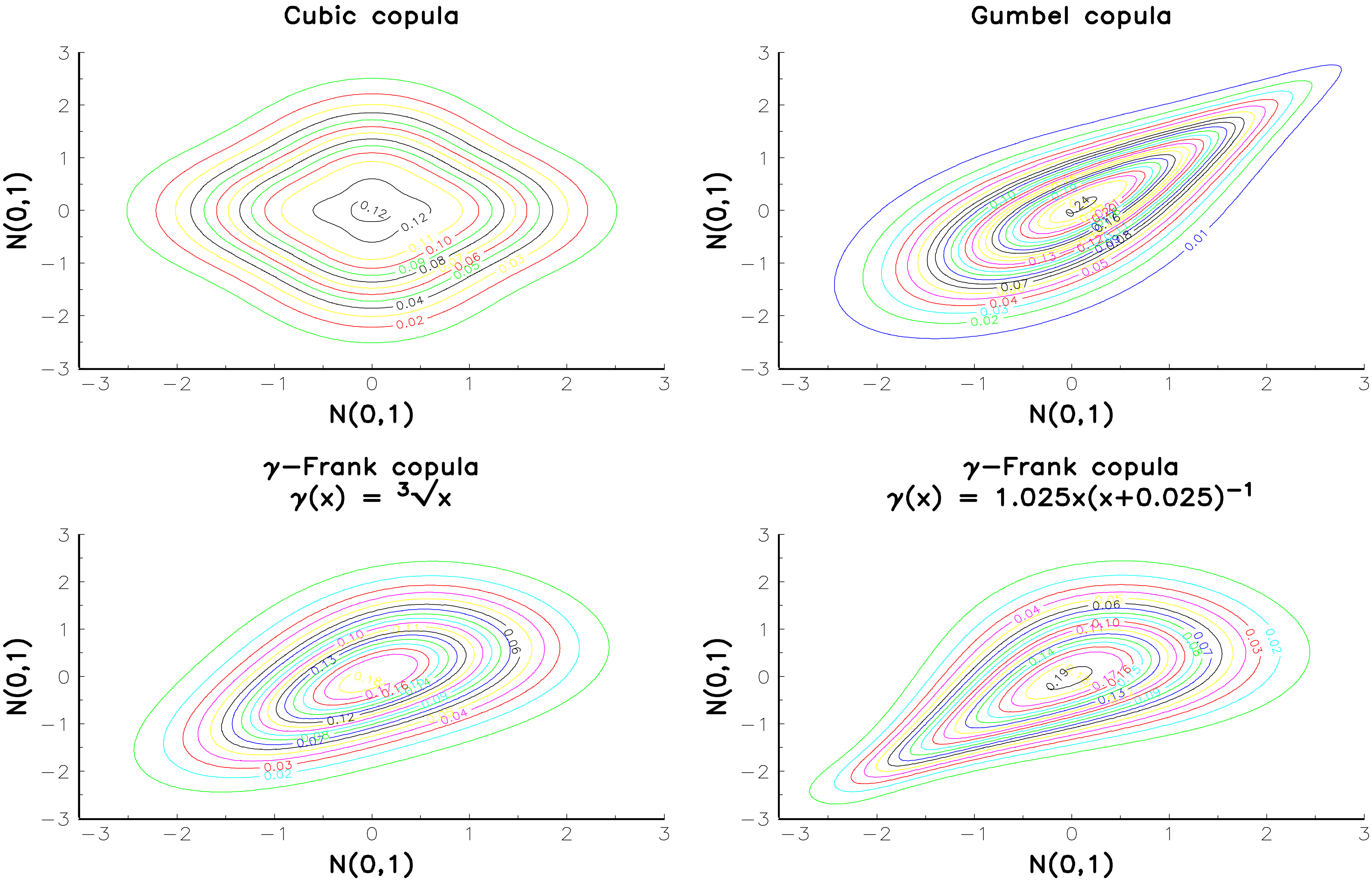
$$\begin{aligned}\mathbf{C}\langle X_1, X_2\rangle &= \mathbf{C}\langle \ln X_1, X_2\rangle \\ &= \mathbf{C}\langle \ln X_1, \exp X_2\rangle \\ &= \mathbf{C}\left\langle (X_1 - K_1)^+, (X_2 - K_2)^+\right\rangle\end{aligned}$$

> *... the copula is invariant while the margins may be changed at will, it follows that is precisely the copula which captures those properties of the joint distribution which are invariant under a.s. strickly increasing transformations* (Schweizer and Wolff [1981]).

⇒ **Copula = dependence function of random variables.**



Normal Copula with $\tau = 0.5$



4 Distributions with Gaussian Margins

2.1 The Normal copula

Remark 1 *The multivariate normal distribution is very tractable. It is very easy to estimate the parameters and simulation is straightforward. Moreover, this distribution has nice properties and most of tractable statistical methods (linear regression, factor analysis, etc.) assume the normality.*

Is it always the case for the Normal copula?

Statistical model	Statistical problem	Algorithm
Quantile regression	$x_2 = \mathbf{q}(x_1;\alpha)$ $\Pr\{X_2 \le x_2 \mid X_1 = x_1\} = \alpha$	PK
Mean regression	$x_2 = \mathbf{e}(x_1)$ $x_2 = \mathbb{E}[X_2 \mid X_1 = x_1]$	LS
PCA	Find the best combinations of X_1 and X_2 to explain $\operatorname{cov}(X_1, X_2)$	EIG

2.1.1 The Ψ transform

We define the operator Ψ as follows

$$\begin{aligned} \Psi\left[\mathbf{F}\right] \ : \ & \mathbb{R} \longrightarrow \mathbb{R} \\ & x \longmapsto \Psi\left[\mathbf{F}\right](x) = \Phi^{-1}\left(\mathbf{F}(x)\right) \end{aligned}$$

We note also Ψ^{-1} the (left) inverse operator ($\Psi^{-1} \circ \Psi = 1$), i.e. $\Psi^{-1}\left[\mathbf{F}\right](x) = \mathbf{F}^{[-1]}\left(\Phi(x)\right)$.

2.1.2 Quantile regression

Costinot, Roncalli and Teïletche [2000] show that

$$\frac{\partial}{\partial u_1}\mathbf{C}\left(u_1,u_2\right)=\Phi\left(\varsigma\right)$$

with

$$\varsigma=\frac{\Phi^{-1}\left(u_2\right)-\beta\Phi^{-1}\left(u_1\right)}{\sqrt{1-\beta^2}}$$

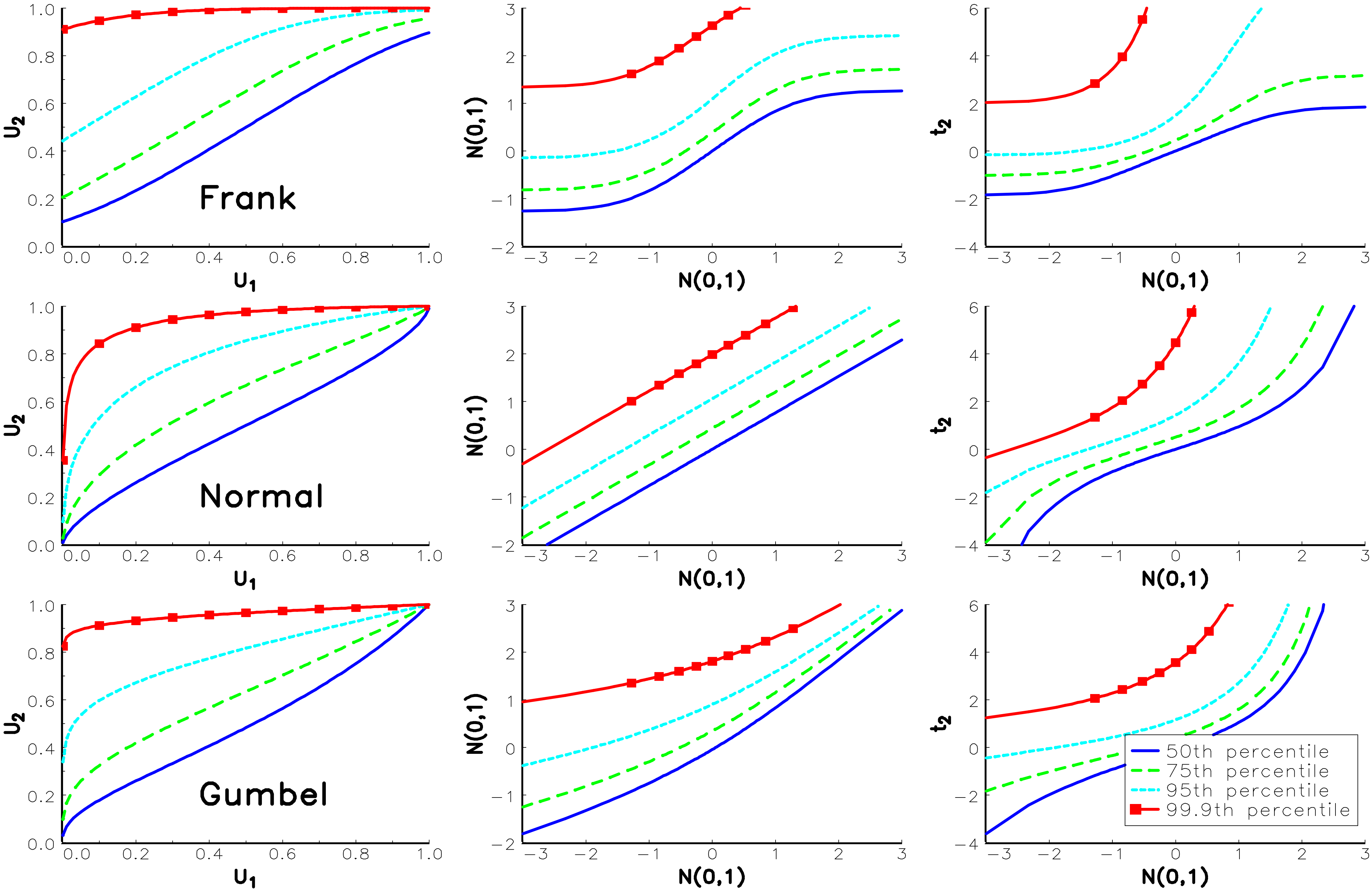
The expression of the function $u_2=\mathbf{q}^{\star}\left(u_1;\alpha\right)$ is also

$$u_2=\Phi\left(\beta\Phi^{-1}\left(u_1\right)+\sqrt{1-\beta^2}\Phi^{-1}\left(\alpha\right)\right)$$

If the margins are gaussians, we obtain the well-known curve

$$X_2=\left[\mu_2-\beta\frac{\sigma_2}{\sigma_1}\mu_1+\sqrt{1-\beta^2}\Phi^{-1}\left(\alpha\right)\right]+\beta\frac{\sigma_2}{\sigma_1}X_1$$

We remark that the relationship is **linear**. When the margins are not gaussians, the relationship is linear in the Ψ projection space.



Regression quantiles with different copula functions

Remark 2 *If we assume that the dependence function is Normal, we can use the Portnoy-Koenker algorithm with the transformed variables* $Y_i = \Psi[\mathbf{F}_i](X_i)$. *Let* $\hat{a}$ *and* $\hat{b}$ *be the estimates of the linear quantile regression*

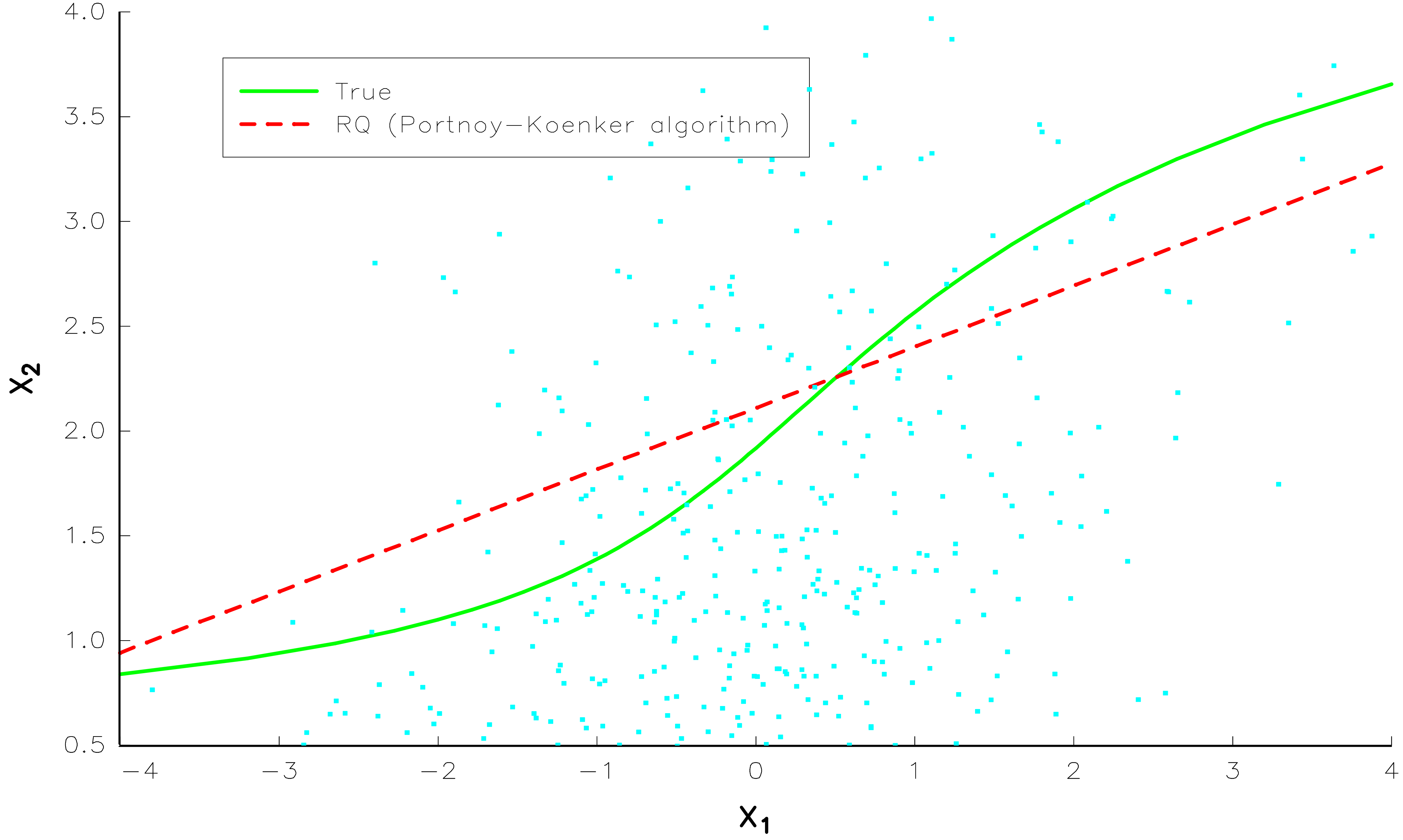
$$\begin{cases} Y_2 = a + bY_1 + U \\ \Pr\{Y_2 \leq y_2 \mid Y_1 = y_1\} = \alpha \end{cases}$$

The quantile regression curve of X_2 *on* X_1 *is then obtained as follows*

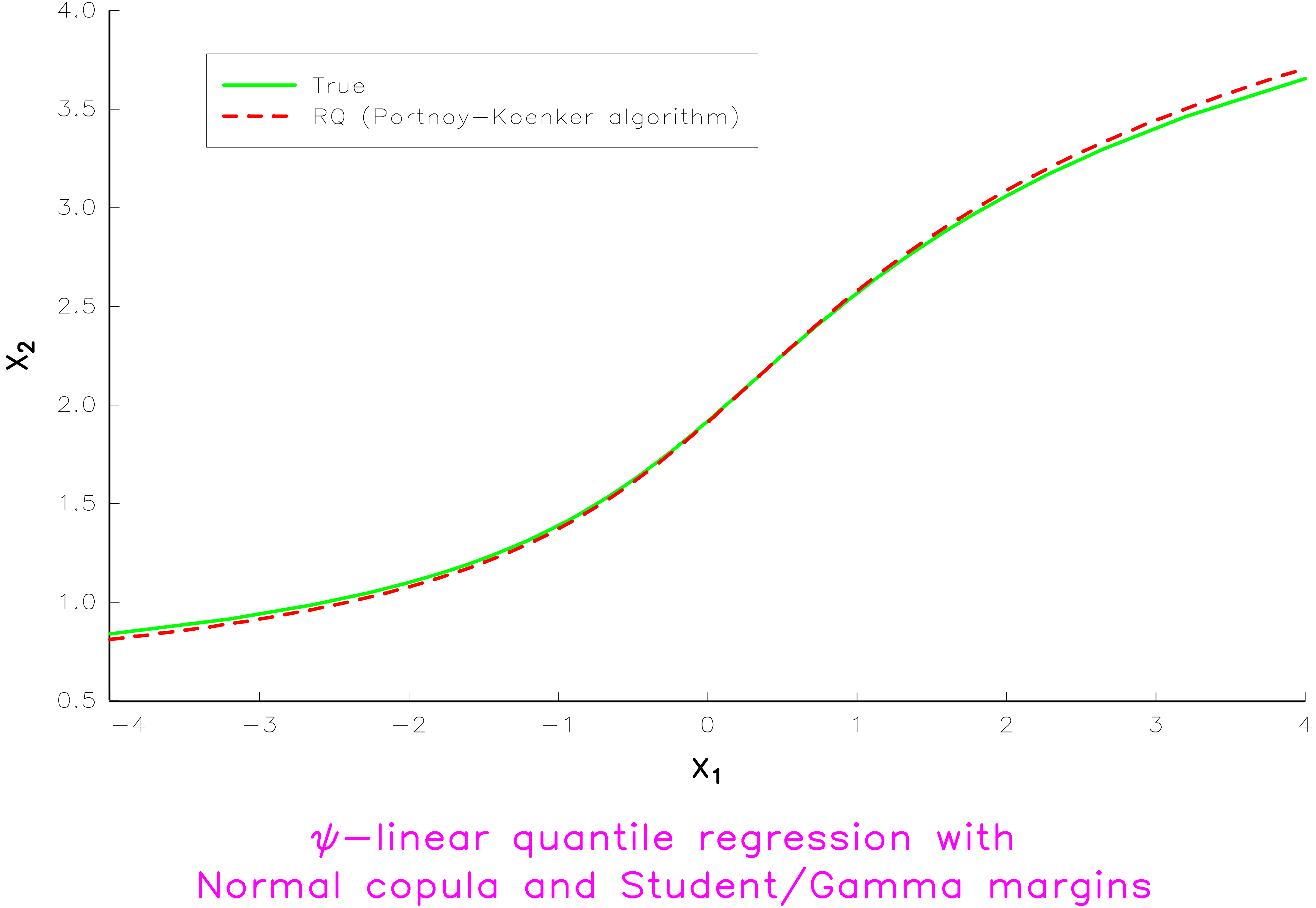
$$X_2 = \Psi^{-1}[\mathbf{F}_2]\left(\hat{a} + \hat{b}\Psi[\mathbf{F}_1](X_1)\right)$$

Linearity = Normality

Can we extend the previous analysis to other statistical models (linear regression, factor analysis, etc.)?



Linear quantile regression with
Normal copula and Student/Gamma margins



True
RQ (Portnoy−Koenker algorithm)
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
−4
−3
−2
−1
0
1
2
3
4
X2
X1
ψ−linear quantile regression with
Normal copula and Student/Gamma margins

2.2 Copulas and markov processes

The multiplication product of copulas have been defined by Darsow, Nguyen and Olsen [1992] in the following manner

$$\begin{array}{rcl} \mathbf{I}^2 & \longrightarrow & \mathbf{I} \\ (x,y) & \longmapsto & (\mathbf{C}_1 * \mathbf{C}_2)(x,y) = \int_0^1 \partial_2 \mathbf{C}_1(x,s)\, \partial_1 \mathbf{C}_2(s,y)\, \mathrm{d}s \end{array}$$

2.2.1 Markov processes and $*$ product of 2-copulas

Darsow, Nguyen and Olsen [1992] prove the following theorem:

Theorem 3 *Let $X = \{X_t, \mathcal{F}_t; t \geq 0\}$ be a stochastic process and let $\mathbf{C}_{s,t}$ denote the copula of the random variables X_s and X_t. Then the following are equivalent*

(i) The transition probabilities $P_{s,t}(x, \mathcal{A}) = \Pr\{X_t \in \mathcal{A} \mid X_s = x\}$ satisfy the Chapman-Kolmogorov equations

$$P_{s,t}(x, \mathcal{A}) = \int_{-\infty}^{\infty} P_{s,\theta}(x, \mathrm{d}y)\, P_{\theta,t}(y, \mathcal{A})$$

for all $s < \theta < t$ and almost all $x \in \mathbb{R}$.

(ii) For all $s < \theta < t$,

$$\mathbf{C}_{s,t} = \mathbf{C}_{s,\theta} * \mathbf{C}_{\theta,t} \tag{1}$$

In the conventional approach, one specifies a Markov process by giving the initial distribution μ *and a family of transition probabilities* $P_{s,t}(x, \mathcal{A})$ *satisfying the Chapman-Kolmogorov equations. In our approach, one specifies a Markov process by giving all of the marginal distributions and a family of 2-copulas satisfying (1). Ours is accordingly an alternative approach to the study of Markov processes which is different in principle from the conventional one. Holding the transition probabilities of a Markov process fixed and varying the initial distribution necessarily varies all of the marginal distributions, but holding the copulas of the process fixed and varying the initial distribution does not affect any other marginal distribution* (Darsow, Nguyen and Olsen [1992]).

2.2.2 Understanding the dependence structure of diffusion processes

The Brownian copula is

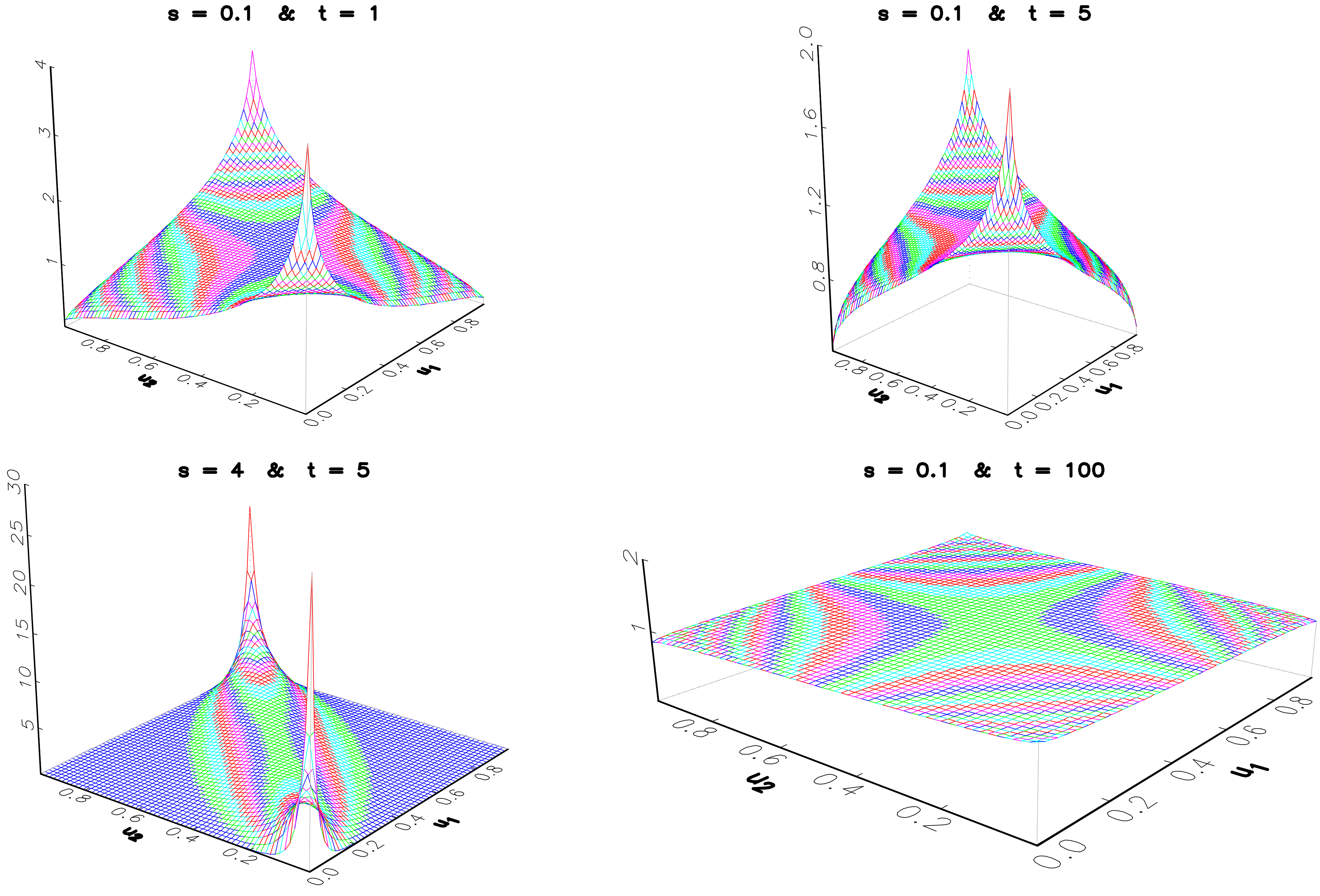
$$\mathbf{C}_{s,t}\left(u_1,u_2\right) = \int_0^{u_1} \Phi\left(\frac{\sqrt{t}\Phi^{-1}\left(u_2\right) - \sqrt{s}\Phi^{-1}\left(u\right)}{\sqrt{t-s}}\right) \mathrm{d}u$$

The copula of a Geometric Brownian motion is the Brownian copula.
The Ornstein-Uhlenbeck copula is

$$\mathbf{C}_{s,t}\left(u_1,u_2\right) = \int_0^{u_1} \Phi\left(\frac{\hbar\left(t_0,s,t\right)\Phi^{-1}\left(u_2\right) - \hbar\left(t_0,s,s\right)\Phi^{-1}\left(u\right)}{\hbar\left(s,s,t\right)}\right) \mathrm{d}u$$

with

$$\hbar\left(t_0,s,t\right) = \sqrt{e^{2a(t-s)} - e^{-2a(s-t_0)}}$$



PDF of the Brownian copula

$\Rightarrow$ Interpretation of the parameters in term of the time dependence.

For the brownian copula, we have

$$\mathbf{C}_{s,\infty} = \mathbf{C}^{\perp}$$

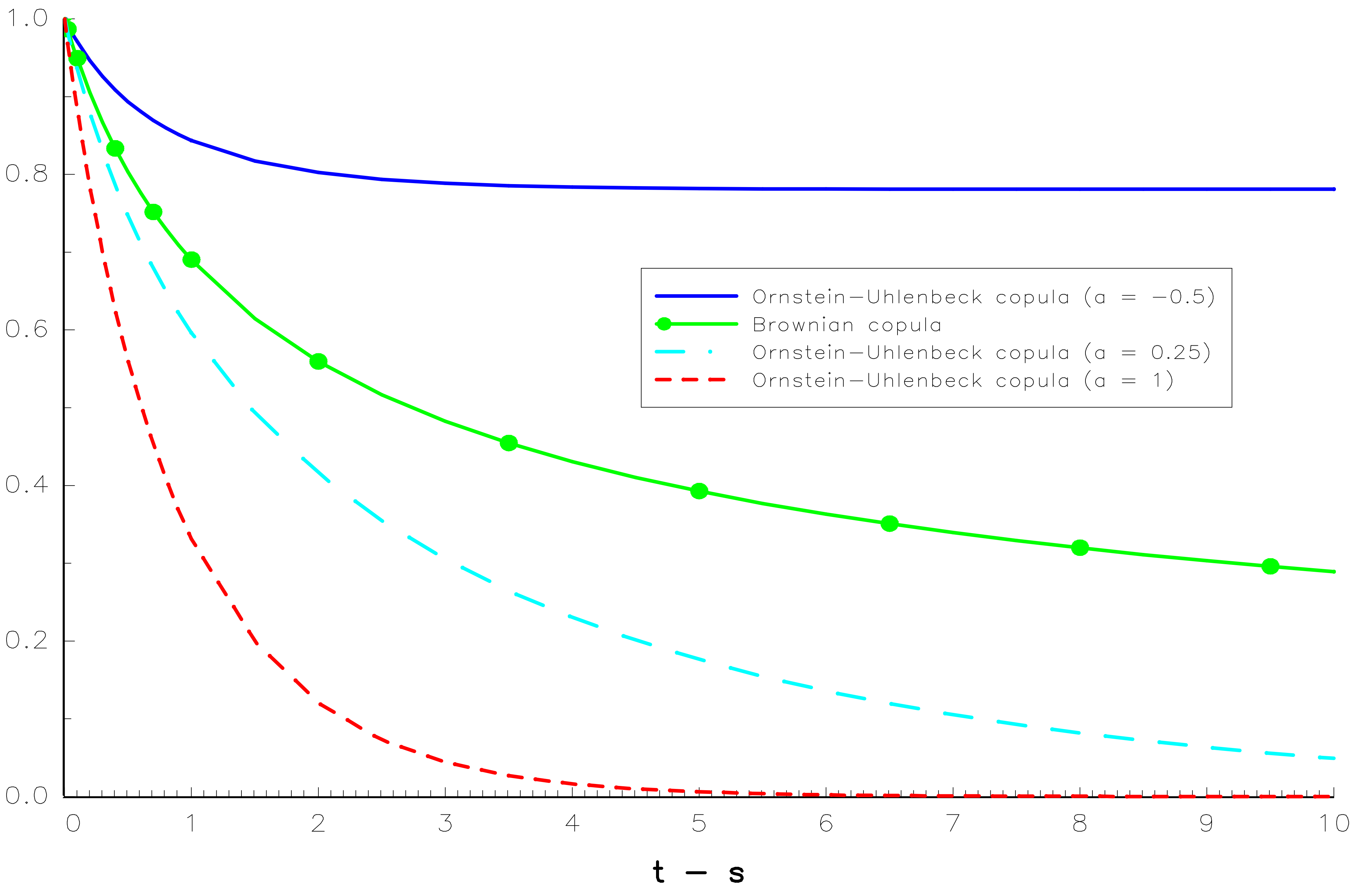
For the Ornstein-Uhlenbeck copula, we verify that

$$\lim_{a \longrightarrow \infty} \mathbf{C}_{s,t}(u_1, u_2) = \mathbf{C}^{\perp}$$

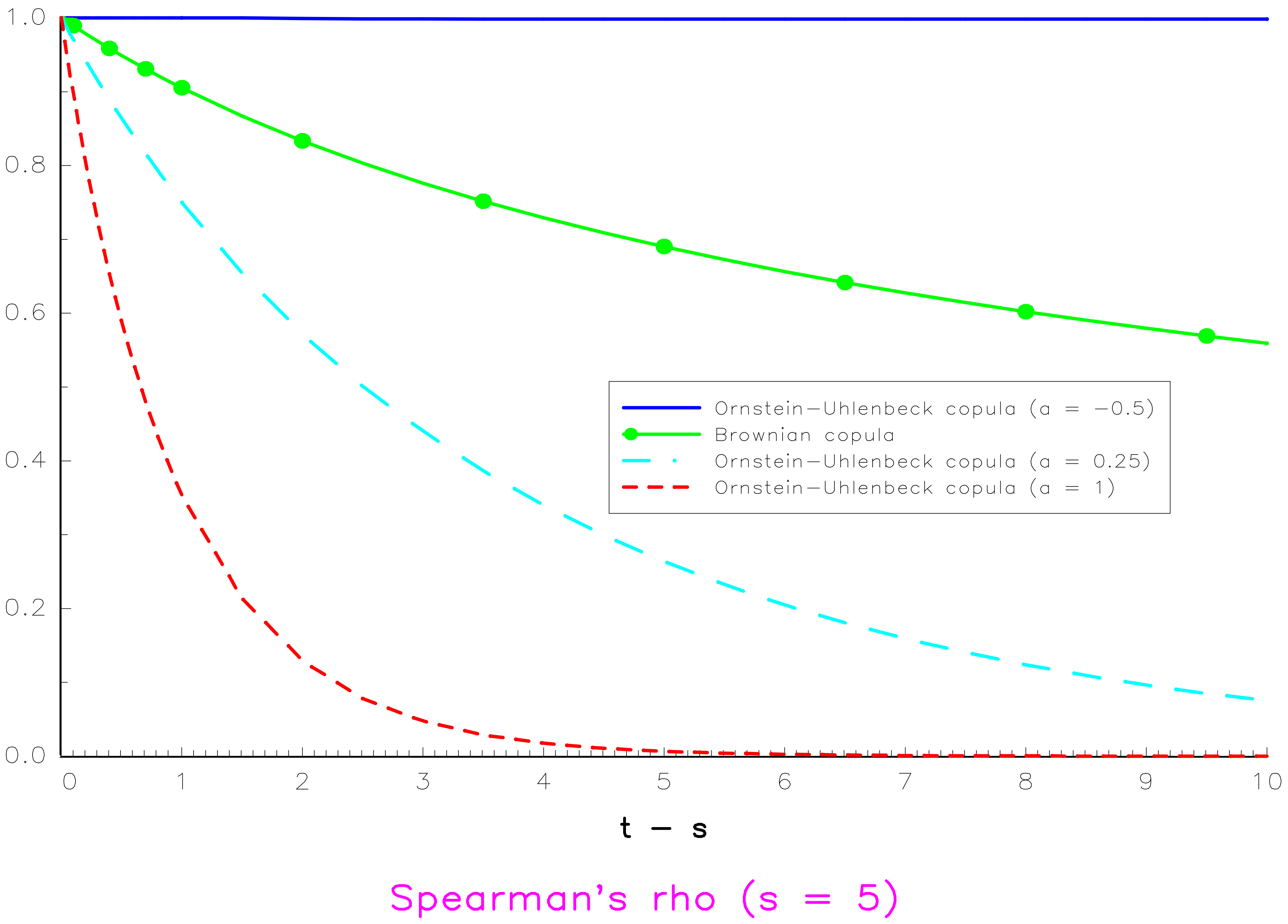
but we have

$$\lim_{a \longrightarrow -\infty} \mathbf{C}_{s,t}(u_1, u_2) = \mathbf{C}^{+}$$

Question: What are the copulas such that $\mathbf{C}_{s,\infty} \neq \mathbf{C}^{\perp}$?



Spearman's rho (s = 1)


1.0
0.8
0.6
0.4
0.2
0.0
0
1
2
3
4
5
6
7
8
9
10
Ornstein−Uhlenbeck copula (a = −0.5)
Brownian copula
Ornstein−Uhlenbeck copula (a = 0.25)
Ornstein−Uhlenbeck copula (a = 1)
t − s
Spearman's rho (s = 5)

2.2.3 CKLS revisited

Chan, Karolyi, Longstaff and Sanders [1992] consider the following process for interest rates

$$\begin{cases} \mathrm{d}r(t) = (\alpha + \beta r(t))\,dt + \sigma r(t)^{\gamma}\,\mathrm{d}W(t) \\ r(t_0) = r_0 \end{cases}$$

Special cases are the following models:

$\mathcal{P}_i$	Process	Model
$\mathcal{P}_1$	$dr(t) = \mu\,dt + \sigma\,dW(t)$	Merton [1973]
$\mathcal{P}_2$	$dr(t) = a(b - r(t))\,dt + \sigma\,dW(t)$	Vasicek [1977]
$\mathcal{P}_3$	$dr(t) = \sigma r(t)\,dW(t)$	Dothan [1978]
$\mathcal{P}_4$	$dr(t) = a(b - r(t))\,dt + \sigma r(t)\,dW(t)$	Brennan et Schwartz [1980]
$\mathcal{P}_5$	$dr(t) = \sigma r(t)^{\frac{3}{2}}\,dW(t)$	CIR [1980]
$\mathcal{P}_6$	$dr(t) = \kappa(\theta - r(t))\,dt + \sigma\sqrt{r(t)}\,dW(t)$	CIR [1985]
$\mathcal{P}_7$	$dr(t) = \kappa\left(\theta - \sqrt{r(t)}\right)dt + \sigma\sqrt{r(t)}\,dW(t)$	Longstaff [1989]

Main result: $\beta \simeq 0$ **and** $\gamma \geq 1$

Problem: different margins and different time dependence.

Let us consider a Markov process with Student margins and an Ornstein-Uhlenbeck copula.

$$\frac{r(t) - r_0 e^{-a(t-t_0)} + b\left(1 - e^{-a(t-t_0)}\right)}{\sigma\sqrt{\frac{1-e^{-2a(t-t_0)}}{2a}}} \sim \mathbf{t}_\nu$$

$\nu = \infty \Rightarrow$ Ornstein-Uhlenbeck process.

$$\nu \to 1 \quad \overset{?}{\Longrightarrow} \quad \beta = 0 \text{ and } \gamma \geq 1$$

2.2.4 Characterization of Markov copulas

Markov property $\Leftrightarrow$ Markov copula (Markov property does not depend on margins specifications).

Markov copulas may be characterized using the $\star$ product — $\mathbf{C}_{s,t} = \mathbf{C}_{s,\theta} * \mathbf{C}_{\theta,t}$ is not a sufficient condition.

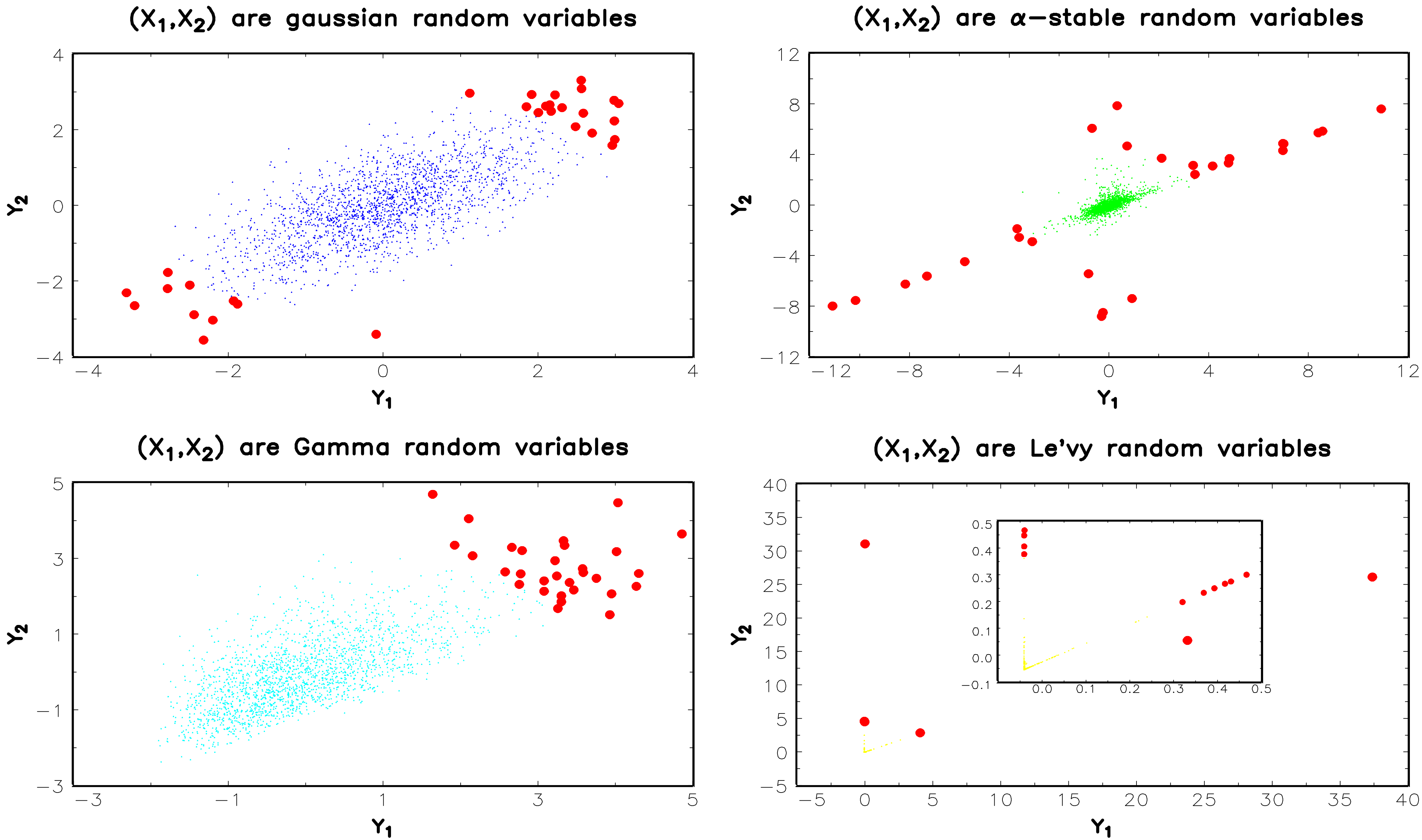
Problem: not very tractable.

Other solution: Markov sub-algebras (Partitions of unity example).

3 An open field for risk management

With copulas, it appears that the risk can be splitted into two parts: the individual risks and the dependence structure between them.

- **Coherent multivariate statistical model = Coherent model for individual risks + coherent dependence function**
- **Coherent model for individual risks** = taking into account fat-tailed distributions, etc.
- **coherent dependence function** = understanding the aggregation of quantiles of the individual risks.



Bivariate distributions with same first and second moments

= Gaussian VaRs are equal

3.1 Value-at-Risk

⇒ The influence of margins

Rating	*VaR*	BBB	A	AA	AAA
α	99%	99.75%	99.9%	99.95%	99.97%
Return time	100 days	400 days	4 years	8 years	13 years
$\frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(0.99)}$	1	1.20	1.33	1.41	1.48
$\frac{t_4^{-1}(\alpha)}{t_4^{-1}(0.99)}$	1	1.49	1.91	2.30	2.62

⇒ The influence of the dependence function: If a bivariate copula $\mathbf{C}$ is such that*

$$\lim_{u \to 1} \frac{\bar{\mathbf{C}}(u,u)}{1-u} = \lambda$$

exists, then $\mathbf{C}$ has **upper tail dependence** for $\lambda \in (0,1]$ and no upper tail dependence for $\lambda = 0$.

*$\bar{\mathbf{C}}$ is the joint survival function, that is $\bar{\mathbf{C}}(u_1,u_2) = 1 - u_1 - u_2 + \mathbf{C}(u_1,u_2)$.

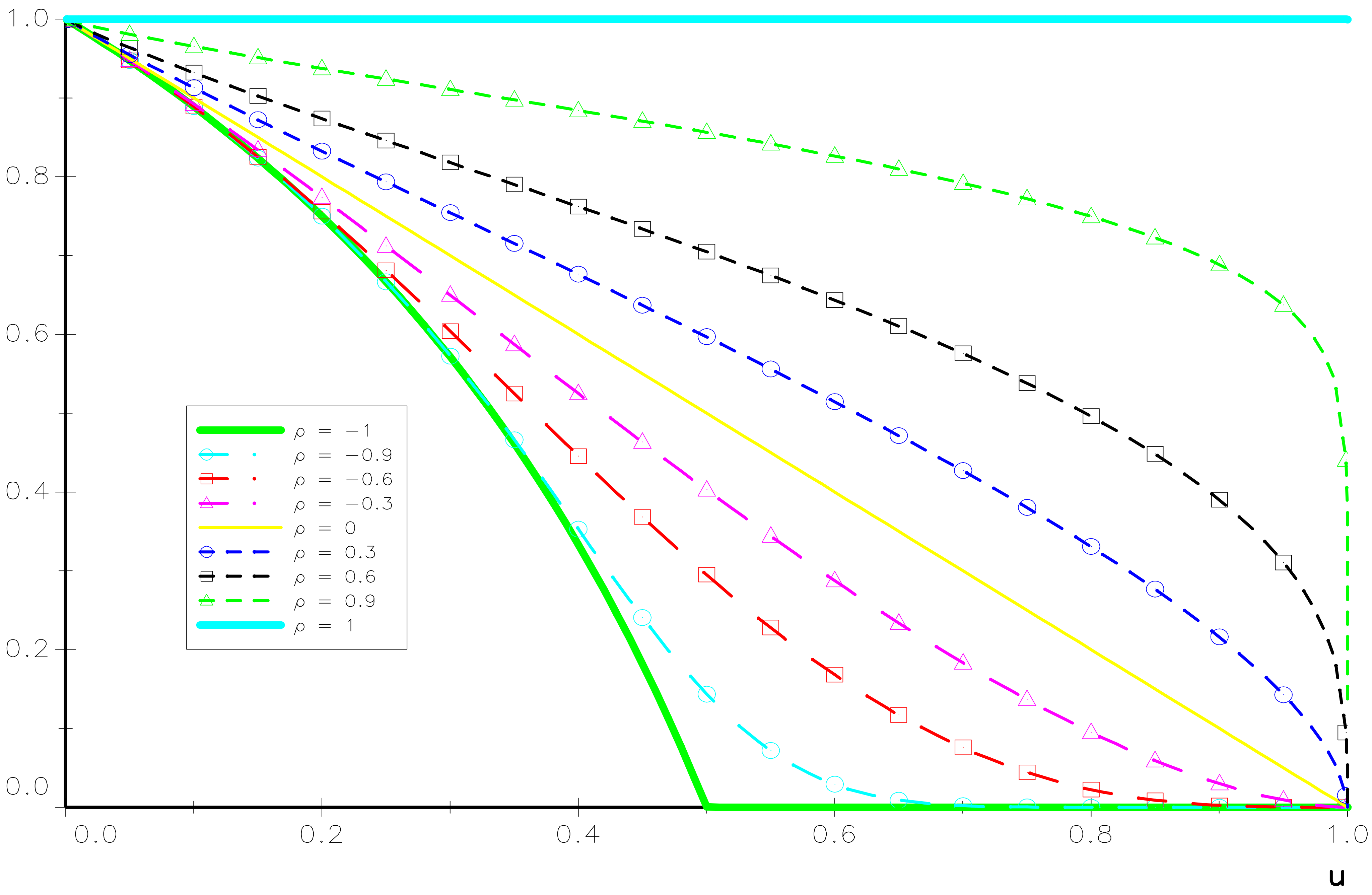
Remark 3 *The measure λ is the probability that one variable is extreme given that the other is extreme.*

$\Rightarrow$ Coles, Currie and Tawn [1999] define the quantile-dependent measure of dependence as follows

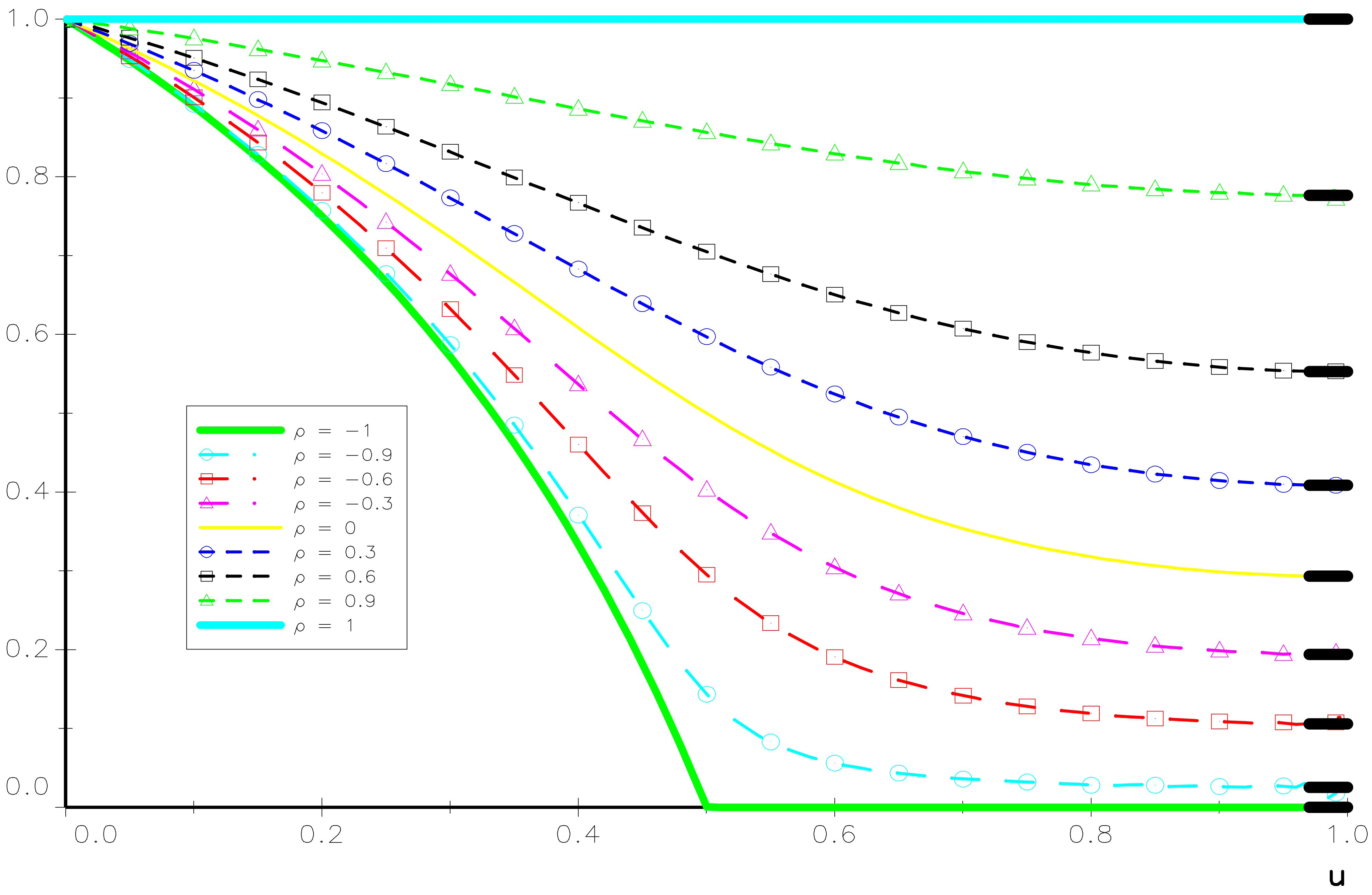
$$\lambda(u) = \Pr\left\{X_2 > F_2^{-1}(u) \mid X_1 > F_1^{-1}(u)\right\} = \Pr\{U_2 > u \mid U_1 > u\} = \frac{\bar{\mathbf{C}}(u,u)}{1-u}$$

$u = \alpha \Rightarrow$ *VaR* interpretation.

1. Normal copula $\Rightarrow$ extremes are asymptotically independent for $\rho \neq 1$, i.e $\lambda = 0$ for $\rho < 1$.
2. Student copula $\Rightarrow$ extremes are asymptotically dependent for $\rho \neq -1$.



Quantile-dependent measure λ(u) for the Normal copula



λ(u) for the Student copula and $\nu = 1$

Let (X_1, X_2) be a random vector with copula $\mathbf{C}$. The law of the maximum of (X_1, X_2) in a sample of size n has density

$$\begin{aligned} f_{\max}(x_1, x_2) &= n\mathbf{C}^{n-1}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)) f_1(x_1) f_2(x_2) \times \\ & c(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)) + \\ & n(n-1)\mathbf{C}^{n-2}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)) f_1(x_1) f_2(x_2) \times \\ & \partial_1\mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)) \partial_2\mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)) \end{aligned}$$

$\Rightarrow$ Illustration with the Normal copula and different values of ρ.

LME example:

	AL	AL-15	CU	NI	PB
P_1	1	1	1	1	1
P_2	-1	-1	-1	1	1
P_3	2	1	-3	4	5

- Gaussian margins and Normal copula

	90%	95%	99%	99.5%	99.9%
P_1	7.26	9.33	13.14	14.55	17.45
P_2	4.04	5.17	7.32	8.09	9.81
P_3	13.90	17.82	25.14	27.83	33.43

- Student margins ($\nu = 4$) and Normal copula

	90%	95%	99%	99.5%	99.9%
P_1	6.51	8.82	14.26	16.94	24.09
P_2	3.77	5.00	7.90	9.31	13.56
P_3	12.76	17.05	27.51	32.84	49.15

- Gaussian margins and Student copula ($\nu = 1$)

	90%	95%	99%	99.5%	99.9%
P_1	5.69	7.95	13.19	15.38	20.06
P_2	3.82	5.55	9.75	11.65	16.41
P_3	13.41	19.36	34.16	40.55	54.48

Value-at-risk based on Student margins and a Normal copula (Gauss software, Pentium III 550 Mhz, 100000 simulations)

Number of assets	Computational time
2	0.1 sc
10	24.5 sc
100	4 mn 7 sc
500	33 mn 22 sc
1000	1 hr 44 mn 45 sc

3.2 Stress testing

Stress testing program = what are the larger risks in the portfolio?
⇒ **Extreme value theory** allows to model the maxima or minima of a distribution and to apply stress scenarios to a portfolio.

Problem: **multivariate** stress scenarios.

3.2.1 Multivariate extreme value theory

An extreme value copula satisfy the following condition

$$\mathbf{C}\left(u_1^t, \ldots, u_N^t\right) = \mathbf{C}^t\left(u_1, \ldots, u_N\right) \quad \forall\, t > 0$$

For example, the Gumbel copula is an extreme value copula:

$$\begin{aligned}\mathbf{C}\left(u_1^t, u_2^t\right) &= \exp\left(-\left[\left(-\ln u_1^t\right)^{\alpha} + \left(-\ln u_2^t\right)^{\alpha}\right]^{\frac{1}{\alpha}}\right) \\ &= \left[\exp\left(-\left[(-\ln u_1)^{\alpha} + (-\ln u_1)^{\alpha}\right]^{\frac{1}{\alpha}}\right)\right]^t = \mathbf{C}^t\left(u_1, u_2\right)\end{aligned}$$

What is the link between extreme value copulas and the multivariate extreme value theory? **The joint limit distribution G of multivariate extremes is of the type**

$$\mathbf{G}\left(\chi_1^+,\ldots,\chi_N^+\right)=\mathbf{C}_\star\left(\mathbf{G}_1\left(\chi_1^+\right),\ldots,\mathbf{G}_N\left(\chi_N^+\right)\right)$$

where $\mathrm{C}_\star$ is an extreme value copula and G_n a non-degenerate univariate extreme value distribution.

Univariate theory $\Rightarrow$ Fisher-Tippet theorem.

Multivariate theory $\Rightarrow$ the class of multivariate extreme value distribution is the class of **extreme value copulas** with nondegenerate marginals.

Let $\mathbf{D}$ be a multivariate distribution with unit exponential survival margins and $\mathbf{C}$ an extreme value copula. Using the relation

$$\mathbf{C}\left(u_1,\ldots,u_N\right)=\mathbf{C}\left(\mathrm{e}^{-\tilde{u}_1},\ldots,\mathrm{e}^{-\tilde{u}_N}\right)=\mathbf{D}\left(\tilde{u}_1,\ldots,\tilde{u}_N\right)$$

we have $\mathbf{D}^t\left(\tilde{\mathbf{u}}\right)=\mathbf{D}\left(t\tilde{\mathbf{u}}\right)$ and then $\mathbf{D}$ is a *min-stable multivariate exponential* (MSMVE) distribution.

Theorem 4 (Pickands representation of MSMVE distributions) *Let $\mathbf{D}\left(\tilde{\mathbf{u}}\right)$ be a survival function with exponential margins. $\mathbf{D}$ satisfies*

$$\begin{aligned}\mathbf{D}\left(\tilde{\mathbf{u}}\right) &= \exp\left[-\left(\sum_{n=1}^{N}\tilde{u}_n\right)B\left(w_1,\ldots,w_N\right)\right] \\ B\left(\mathbf{w}\right) &= \int\cdots\int_{\mathcal{S}_N}\max_{1\leq n\leq N}\left(q_n w_n\right)\,\mathrm{d}S\left(\mathbf{q}\right)\end{aligned}$$

with $w_n=\tilde{u}_n/\sum_1^N\tilde{u}_n$ and where $\mathcal{S}_N$ is the N-dimensional unit simplex and S a finite measure on $\mathcal{S}_N$. B is a convex function and $\max\left(w_1,\ldots,w_N\right)\leq B\left(\mathbf{w}\right)\leq 1$.

It comes necessarily that an extreme value copula verifies

$$\mathbf{C}^{\perp} \prec \mathbf{C} \prec \mathbf{C}^{+}$$

Maximum domain of attraction: $\mathbf{F} \in \mathsf{MDA}(\mathbf{G})$ iff

1. $\mathbf{F}_n \in \mathsf{MDA}(\mathbf{G}_n)$ for all $n = 1\ldots,N$;
2. $\mathbf{C} \in \mathsf{MDA}(\mathbf{C}_\star)$.

3.2.2 Bivariate stress testing

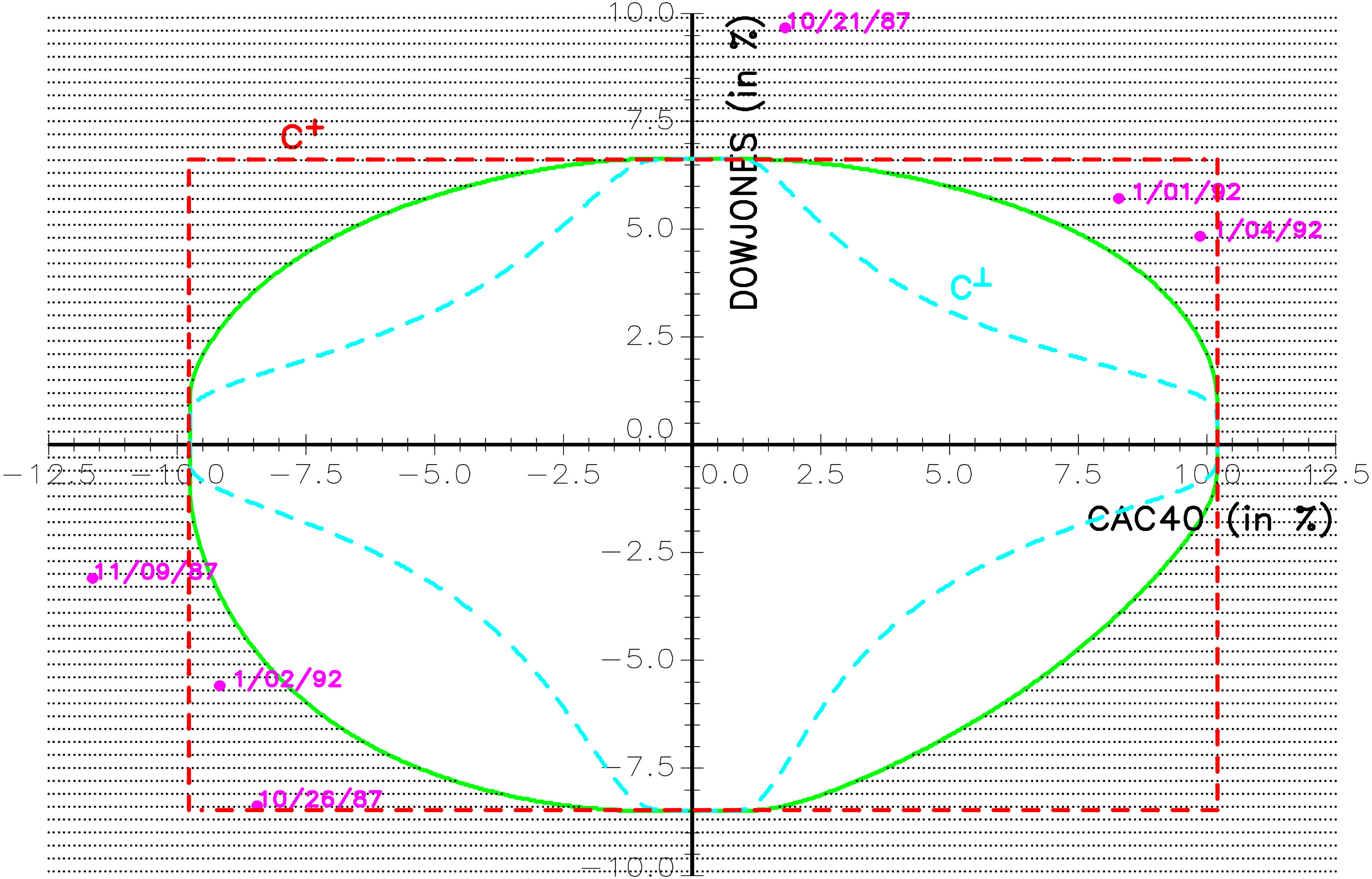
A *failure area* = set of values $\left(\chi_1^+, \chi_2^+\right)$ such that

$$\Pr\left\{\chi_1^+ > \chi_1, \chi_2^+ > \chi_2\right\} = 1 - \mathbf{G}_1(\chi_1) - \mathbf{G}_2(\chi_2) + \mathbf{C}\left(\mathbf{G}_1(\chi_1), \mathbf{G}_2(\chi_2)\right)$$

equals a given level of probability.

Return time of the CAC40/DowJones example of Costinot, Roncalli and Teïletche [2000]:

Date	CAC40	DowJones	EVT	Gaussian hyp.
10/19/1987	−10.14%	−25.63%	105.79	1.44×10^{14}
10/21/1987	+1.80%	+9.67%	18.14	2.88×10^{14}
10/26/1987	−8.45%	−8.38%	9.18	1.80×10^{13}
11/09/1987	−11.65%	−3.10%	8.12	2.30×10^{9}
01/01/1992	+8.28%	+5.71%	6.85	1.66×10^{8}
01/02/1992	−9.18%	−5.59%	6.39	2.96×10^{9}
01/04/1992	+9.87%	+4.83%	7.06	2.05×10^{9}



Failure area for a 5 years waiting time

3.2.3 Multivariate stress testing

see BDNRR [2000].

3.3 Dependence in credit risk models

Coutant, S., P. Martineu, J. Messines, G. Riboulet and T. Roncalli [2001], Revisiting the dependence in credit risk models, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*

Portfolio with liquid credits $\neq$ Portfolio with no liquid credits.

$\Rightarrow$ downgrading risk $\neq$ default risk.

What is the influence of introducing a dependence function?

1. impact on the joint migration probability distribution;
2. impact on the joint survival distribution.

3.3.1 The credit migration approach (CreditMetrics)

Dependence = Normal copula

3.3.2 The actuarial approach (CreditRisk+)

The model

The defaults at time horizon T are given by a set of Bernoulli random variables B_n

$$B_n = \begin{cases} 1 & \text{if the firm } n \text{ has defaulted at time } T \\ 0 & \text{otherwise} \end{cases}$$

The parameters p_n of the B_n's are stochastic. We have

$$p_n = P_n \sum_{k=1}^{K} \theta_{n,k} X_k$$

where $\{X_k\}$ are K independent $\mathbf{H}$-distributed factors. Moreover, given those factors, the defaults are conditionally independent.

One common risk factor

We note μ and σ the mean and the standard deviation of X. Let $\mathbf{F}_n(b_n \mid X = x)$ be the conditional marginal distribution function:

$$\mathbf{F}_n(b_n \mid X = x) = \begin{cases} 0 & \text{if} \quad b_n < 0 \\ 1 - p_n & \text{if} \quad 0 \leq b_n < 1 \\ 1 & \text{if} \quad b_n \geq 1 \end{cases}$$

We introduce a mapping random variable G_n with $G_n = g\left(B_n\right)$ defined on the real line. There exists then a value $g_n^\star$ such that

$$\Pr\left\{G_n \leq g_n^\star\right\} = \Pr\left\{B_n \leq 0\right\}$$

$$\mathbf{F}_n(g_n \mid X = x) = \begin{cases} 1 - P_n \frac{x}{\mu} & \text{if} \quad g_n \leq g_n^\star \\ 1 & \text{if} \quad g_n > g_n^\star \end{cases}$$

Since the default events are held as independent, we have

$$\mathbf{F}(g_1, \ldots, g_N) = \int_0^\infty \prod_{n=1}^N \mathbf{F}_n(g_n \mid X = x) h\left(x\right) \, \mathrm{d}x$$

- General case
  - No approximation

$$\mathbf{C}'(u_1, u_2) = \left(\frac{\sigma^2 + \mu^2}{\mu^2}\right) u_1 u_2 - \frac{\sigma^2}{\mu^2}(u_1 + u_2 - 1)$$

  - Bernoulli-Poisson approximation

$$\mathbf{C}'(u_1, \ldots, u_N) = \psi\left(\sum_{n=1}^{N} \psi^{-1}(u_n)\right)$$

- Gamma case $X \sim \Gamma(\alpha, \beta)$
  - No approximation

$$\mathbf{C}'(u_1, u_2) = \left(1 + \frac{1}{\alpha}\right) u_1 u_2 - \frac{1}{\alpha}(u_1 + u_2 - 1)$$

  - Bernoulli-Poisson approximation

$$\mathbf{C}'(u_1, \ldots, u_N) = \left(u_1^{-\frac{1}{\alpha}} + \cdots + u_N^{-\frac{1}{\alpha}} - N + 1\right)^{-\alpha}$$

With approximation, the dependence function is a frailty model*.

Different risk factors (approximation case)

- General case

$$\mathbf{C}'(u_1, \ldots, u_N) = \prod_{k=1}^{K} \psi_k \left(\sum_{n=1}^{N} \psi_k^{-1} \left(u_n^{\theta_{n,k} \mu_k} \right) \right)$$

- one firm/one factor

$$\mathbf{C}'(u_1, \ldots, u_N) = \mathbf{C}^{\perp} \left(\mathbf{C}'_1(\mathbf{u}_1), \ldots, \mathbf{C}'_k(\mathbf{u}_k), \ldots, \mathbf{C}'_K(\mathbf{u}_K) \right)$$

	Credit migration approach	Actuarial approach
Downgrading risk	✓	
Default risk	✓	✓
Negative dependence	✓	
Stochastic representation		✓

*ψ is the Laplace transform associated of the distribution of X.

4 Contingent claims pricing

How to extend univariate pricing models
to multivariate pricing models?

$$\Downarrow$$

Distributions with Fixed Marginals

4.1 Two assets options

What is a conservative correlation?

$$\Downarrow$$

What is a conservative dependence function?

4.1.1 Multivariate RNDs and copulas

Let $\mathbb{Q}_n$ and $\mathbb{Q}$ be the risk-neutral probability distributions of $S_n(T)$ and $\mathbf{S}(T) = \begin{pmatrix} S_1(T) & \cdots & S_N(T) \end{pmatrix}^\top$. With arbitrage theory, we can show that

$$\mathbb{Q}(+\infty, \ldots, +\infty, S_n(T), +\infty, \ldots, +\infty) = \mathbb{Q}_n(S_n(T))$$

$\Rightarrow$ The margins of $\mathbb{Q}$ are the RNDs $\mathbb{Q}_n$ of Vanilla options.

Breeden et Litzenberger [1978] remark that European option prices permit to caracterize the probability distribution of $S_n(T)$

$$\begin{aligned} \phi(T, K) &:= 1 + e^{r(T-t_0)} \frac{\partial C(T, K)}{\partial K} \\ &= \Pr\{S_n(T) \leq K\} \\ &= \mathbb{Q}_n(K) \end{aligned}$$

Durrleman [2001] extends this result in the bivariate case:

1. for a call max option, $\phi(T,K)$ is the diagonal section of the copula

$$\phi(T,K) = \mathbf{C}(\mathbb{Q}_1(K), \mathbb{Q}_2(K))$$

2. for a spread option, we have

$$\phi(T,K) = \int_0^{+\infty} \partial_1 \mathbf{C}(\mathbb{Q}_1(x), \mathbb{Q}_2(x+K)) \, \mathrm{d}\mathbb{Q}_1(x)$$

⇒ Other results are derived in Durrleman [2001] (bounds, general pricing kernel, etc.) — see Coutant, Durrleman, Rapuch and Roncalli [2001].

4.1.2 Computation of the implied parameter $\widehat{\rho}$

- BS model: LN distribution calibrated with ATM options; Pricing kernel = LN distributions + Normal copula

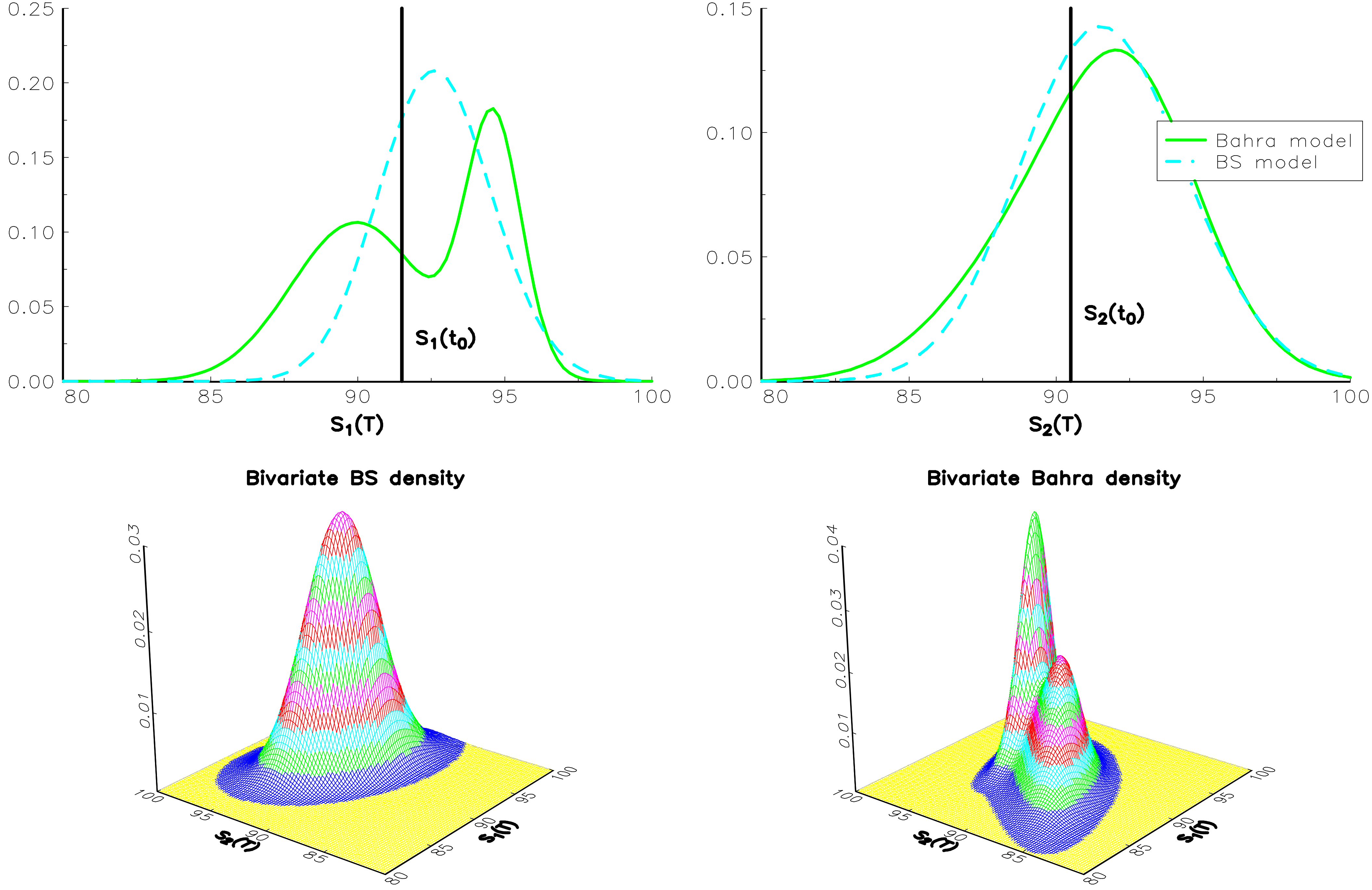
$$\widehat{\rho}_1 = -0.341$$

- Bahra model: mixture of LN distributions calibrated with eight European prices; Pricing kernel = MLN distributions + Normal copula

$$\widehat{\rho}_2 = 0.767$$

Remark 4 *$\widehat{\rho}_1$ and $\widehat{\rho}_2$ are parameters of the Normal Copula. $\widehat{\rho}_1$ is a Pearson correlation, not $\widehat{\rho}_2$.*

$\Rightarrow$ BS model: negative dependence / Bahra model: positive dependence.



A spread option example

4.1.3 Bounds of a spread option

For some two-assets options, bounds are related to Fréchet copulas (see Cherubini and Luciano [2000] for binary options and Coutant, Durrleman, Rapuch and Roncalli [2001] for BestOf/WorstOf options).

For spread options, bounds are more complicated, but can be related to Vanilla prices. For example, we obtain when $K > 0$

$$\int_0^K \sup_{u \geq x} \left(\partial_K C_1(T, u - x) - \partial_K C_2(T, u) \right)^+ \mathrm{d}x \leq K e^{-rT} - \mathrm{CS}\,(T, 0) + \mathrm{CS}\,(T, K)$$

$$K e^{-rT} - \mathrm{CS}\,(T, 0) + \mathrm{CS}\,(T, K) \leq K e^{rT} - \int_0^K \sup_{u \geq x} \left(\partial_K C_1(T, u - x) - \partial_K C_2(T, u) \right)$$

$\Rightarrow$ **What is a conservative dependence function ?**

4.2 Credit derivatives

A default is generally described by a *survival* function $\mathbf{S}(t) = \Pr\{T > t\}$. Let $\breve{\mathbf{C}}$ be a *survival* copula. A multivariate survival distributions $\mathbf{S}$ can be defined as follows

$$\mathbf{S}(t_1, \ldots, t_N) = \breve{\mathbf{C}}(\mathbf{S}_1(t_1), \ldots, \mathbf{S}_N(t_N))$$

where $(\mathbf{S}_1, \ldots, \mathbf{S}_N)$ are the marginal survival functions. Nelsen [1999] notices that "*$\bar{C}$ couples the joint survival function to its univariate margins in a manner completely analogous to the way in which a copula connects the joint distribution function to its margins*".

$\Rightarrow$ Introducing correlation between defaultable securities can then be done using the copula framework (see Li [2000] and Maccarinelli and Maggiolini [2000]).

4.2.1 First-to-Default valuation

Let us define the first-to-default τ as follows

$$\tau = \min\left(T_1, \ldots, T_N\right)$$

Nelsen [1999] shows that the survival function of τ is given by the *diagonal section* of the survival copula.

Let $\mathbf{C}$ be a copula. Its survival copula is given by the following formula

$$\breve{\mathbf{C}}\left(\mathbf{S}_1\left(t_1\right), \ldots, \mathbf{S}_N\left(t_N\right)\right) = \bar{\mathbf{C}}\left(1 - u_1, \ldots, 1 - u_n, \ldots, 1 - u_N\right)$$

with

$$\bar{\mathbf{C}}\left(u_1, \ldots, u_n, \ldots, u_N\right) = \sum_{n=0}^{N}\left[(-1)^n \sum_{\mathbf{u} \in \mathcal{Z}(N-n,N)} \mathbf{C}\left(\mathbf{u}\right)\right]$$

where $\mathcal{Z}\left(M, N\right)$ denotes the set $\left\{\mathbf{u} \in [0,1]^N \mid \sum_{n=1}^N \mathcal{X}_{\{1\}}\left(u_n\right) = M\right\}$.

When the copula is radially symmetric, we have

$$\breve{\mathbf{C}} = \mathbf{C}$$

The survival distribution $\mathbf{S}$ of τ is

$$\mathbf{S}(t) = \mathbf{C}(\mathbf{S}_1(t), \ldots, \mathbf{S}_N(t))$$

It comes that the density of τ is given by

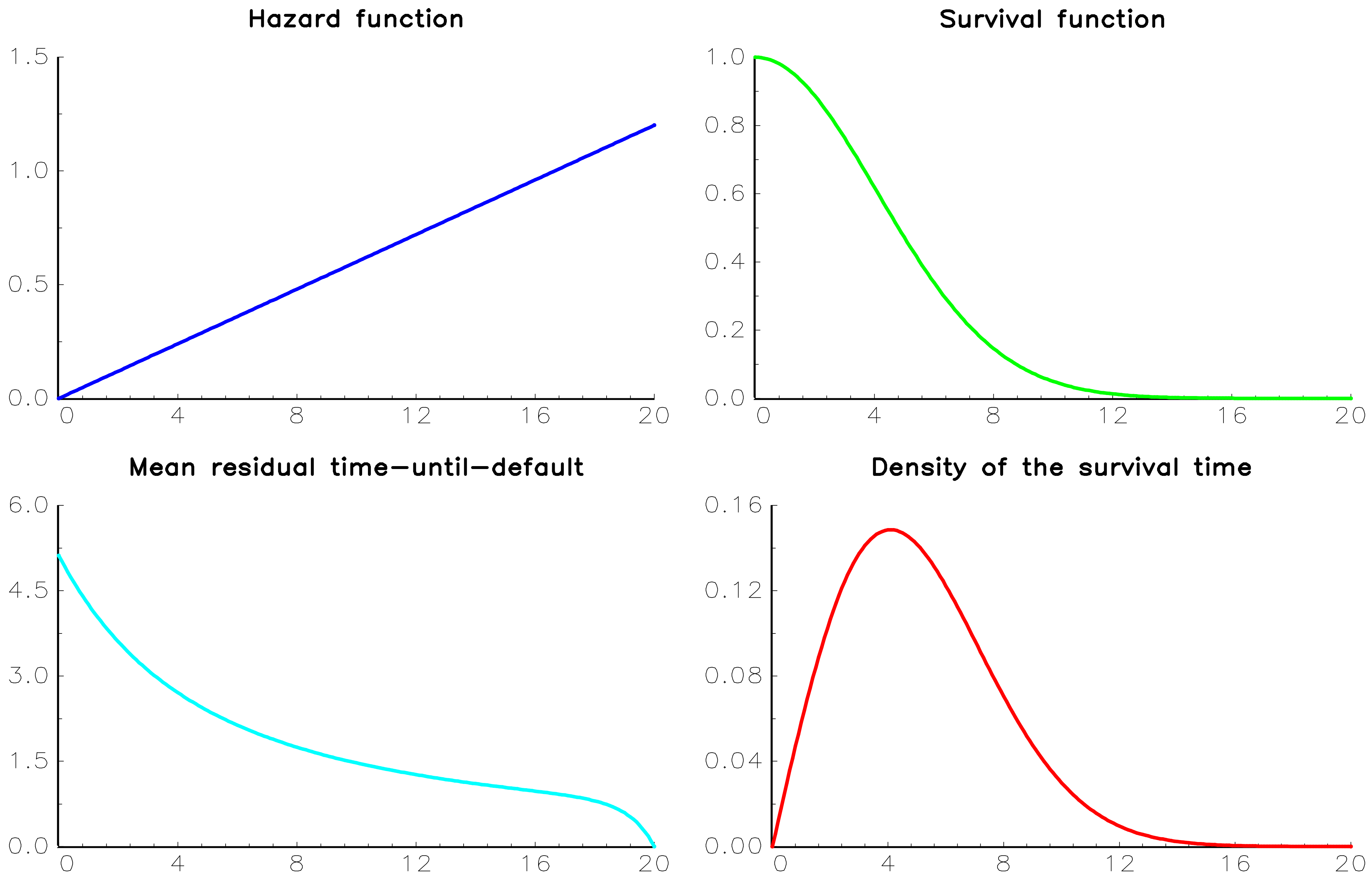
$$\begin{aligned} f(t) &= -\partial_t \mathbf{S}(t) \\ &= \sum_{n=1}^{N} \partial_n \mathbf{C}(\mathbf{S}_1(t), \ldots, \mathbf{S}_N(t)) \times f_n(t) \end{aligned}$$

4.2.2 Example

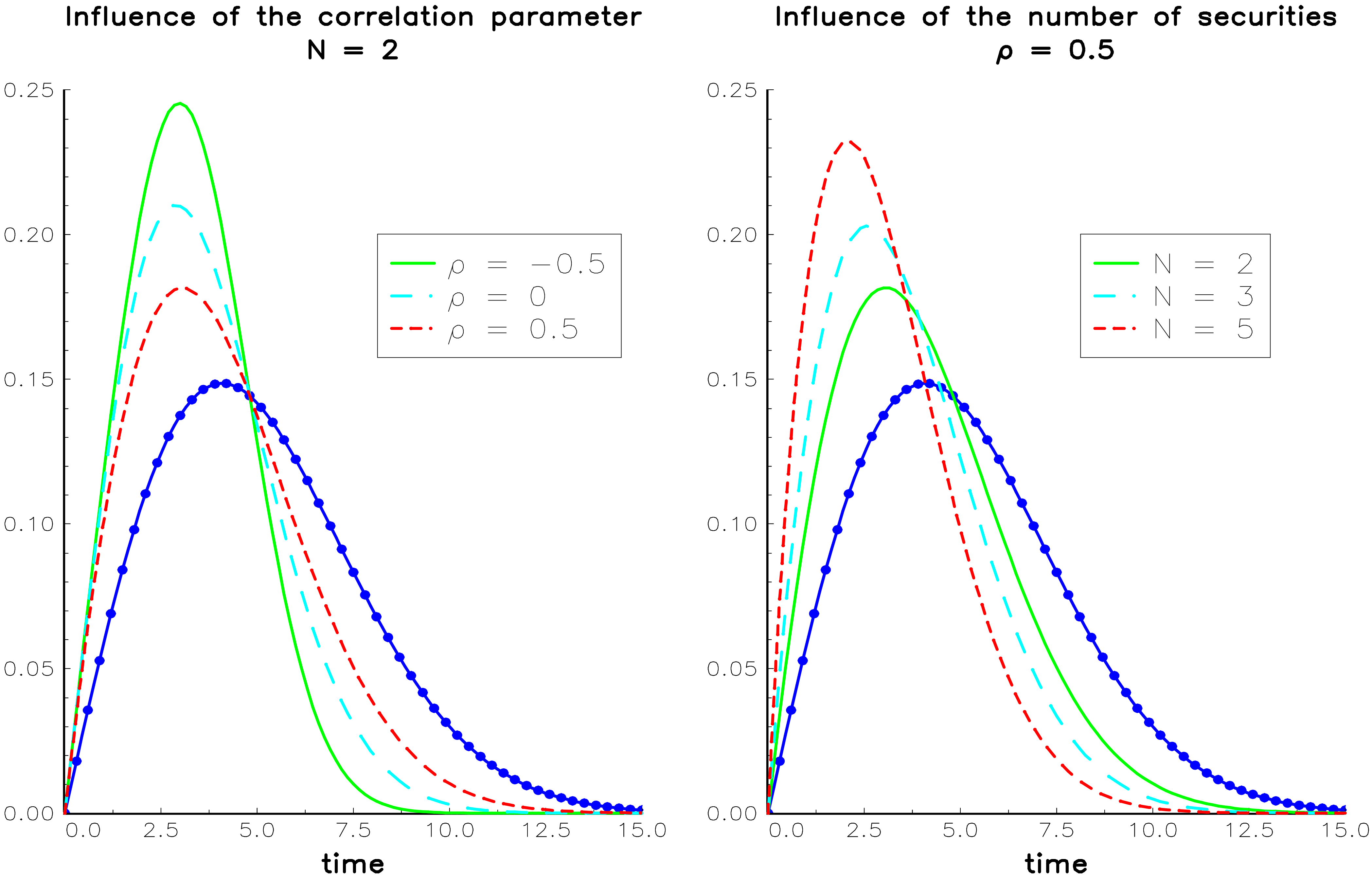
N credit events, default of each credit event given by a Weibull survival function (the baseline hazard is constant and equal to 3% per year and the Weibull parameter is 2).

$\mathbf{C}$ is a Normal copula of dimension N = very tractable (N can be very large) and $\partial_n \mathbf{C}$ is **almost** a Normal copula of dimension $N-1$.

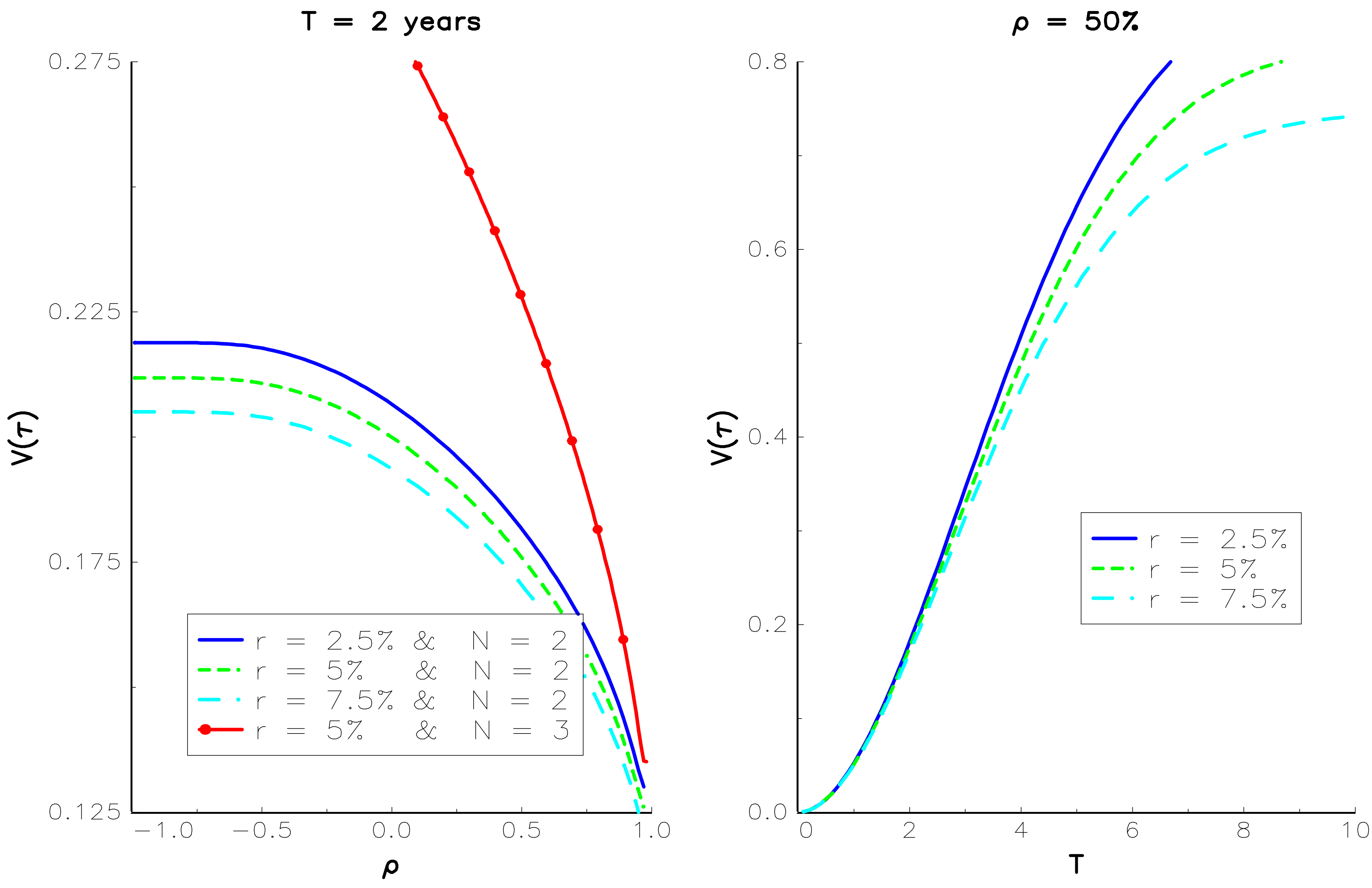
Two cases: constant interest rates and 'Vasicek' interest rates.



Weibull survival time



Density of the first-to-default τ



Premium of the first-to-default claim

5 Copula: a mathematical tool

Howard Sherwood in the AMS-IMS-SIAM Conference of 1993:

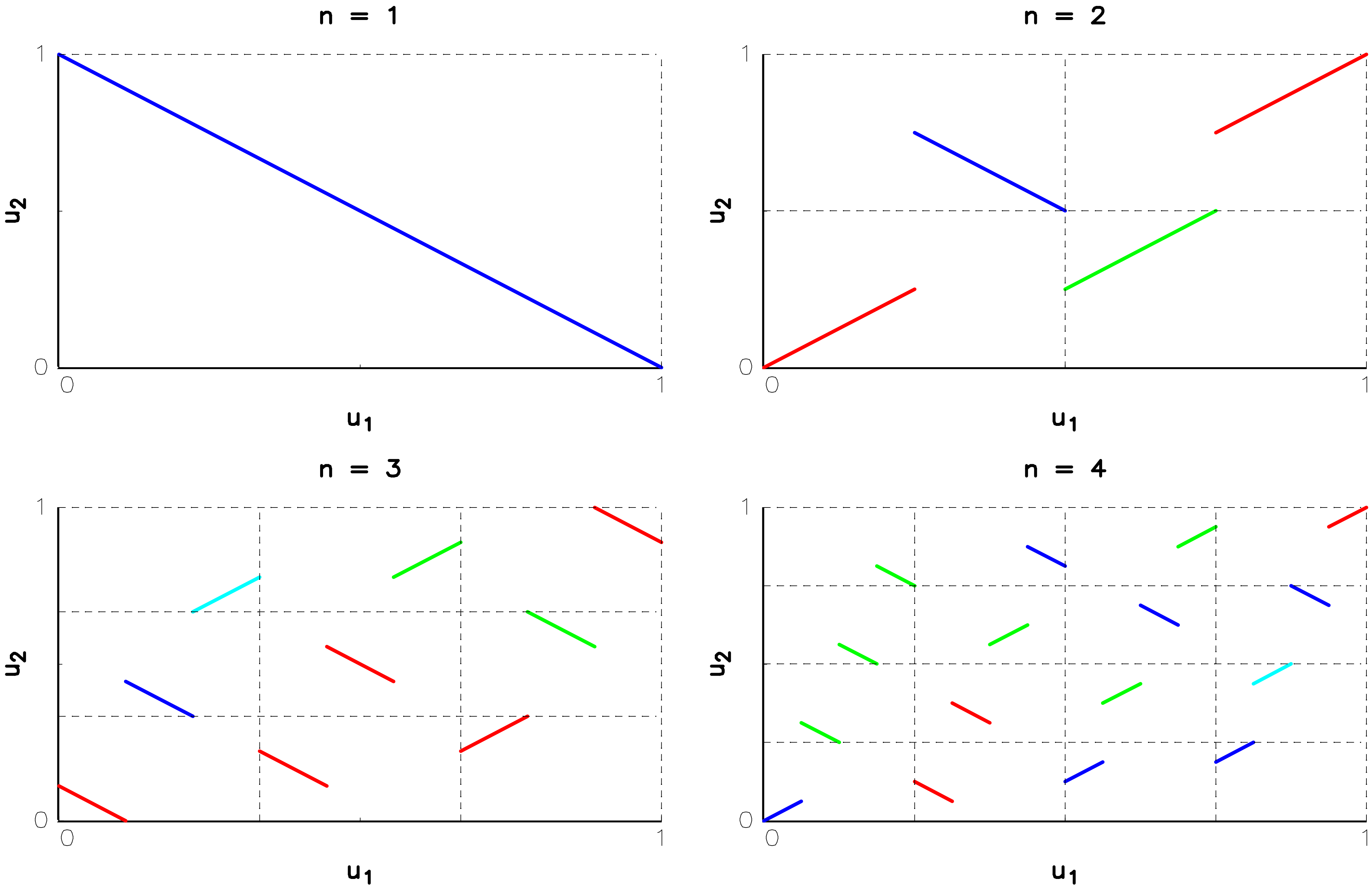
> *The subject matter of these conference proceedings comes in many guises. Some view it as the study of* **probability distributions with fixed marginals***; those coming to the subject from probabilistic geometry see it as the study of* **copulas***; experts in real analysis think of it as the study of* **doubly stochastic measures***; functional analysts think of it as the study of* **Markov operators***; and statisticians say it is the study of possible* **dependence** *relations between pairs of random variables. All are right since all these topics are isomorphic.*

5.1 Uniform versus strong convergence

Kimeldorf and Sampson [1978] show that one can pass from stochastic dependence to complete dependence in the natural sense of weak convergence:

> *Partition the unit square into n^2 congruent squares and denote by (i,j) the square whose upper right corner is the point with coordinates $x = i/n$, $y = j/n$. Similarly, partition each of these n^2 squares into n^2 subsquares and let (i,j,p,q) denote subsquare (p,q) of square (i,j). Now let the bivariate rv (U_n, V_n) distribute mass n^{-2} uniformly on either one of the diagonals of each of the n^2 subsquares of the form (i,j,j,i) for $1 \leq i \leq n$, $1 \leq j \leq n$.*

$$\lim_{n \to \infty} \sup_{u,v \in [0,1]} \left| \mathbf{C}_n \langle U_n, V_n \rangle (u,v) - \mathbf{C}^{\perp}(u,v) \right| = 0$$



Shuffles of Min

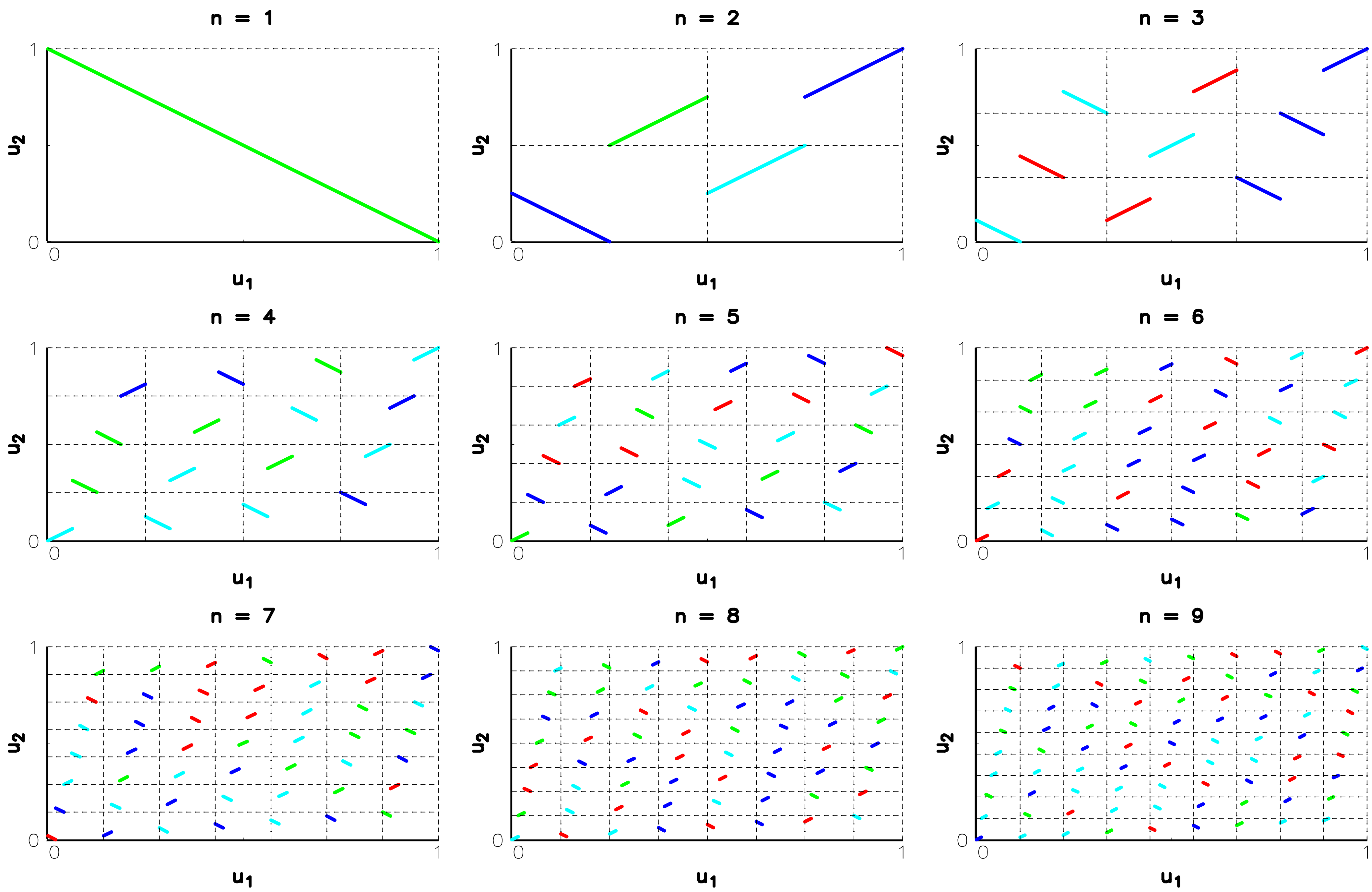


Illustration of the convergence to the product copula

Vitale [1991] extends this result:

Theorem 5 (Vitale [1991, theorem 1, p. 461]) *Let U and V be two uniform variables. There is a sequence of cyclic permutations $T_1, T_2, \ldots, T_n$ such that $(U, T_n U)$ converges in distribution to (U, V) as $n \to \infty$.*

These cyclic permutations are the "Shuffles of Min" defined by Mikusiński, Sherwood and Taylor [1992]:

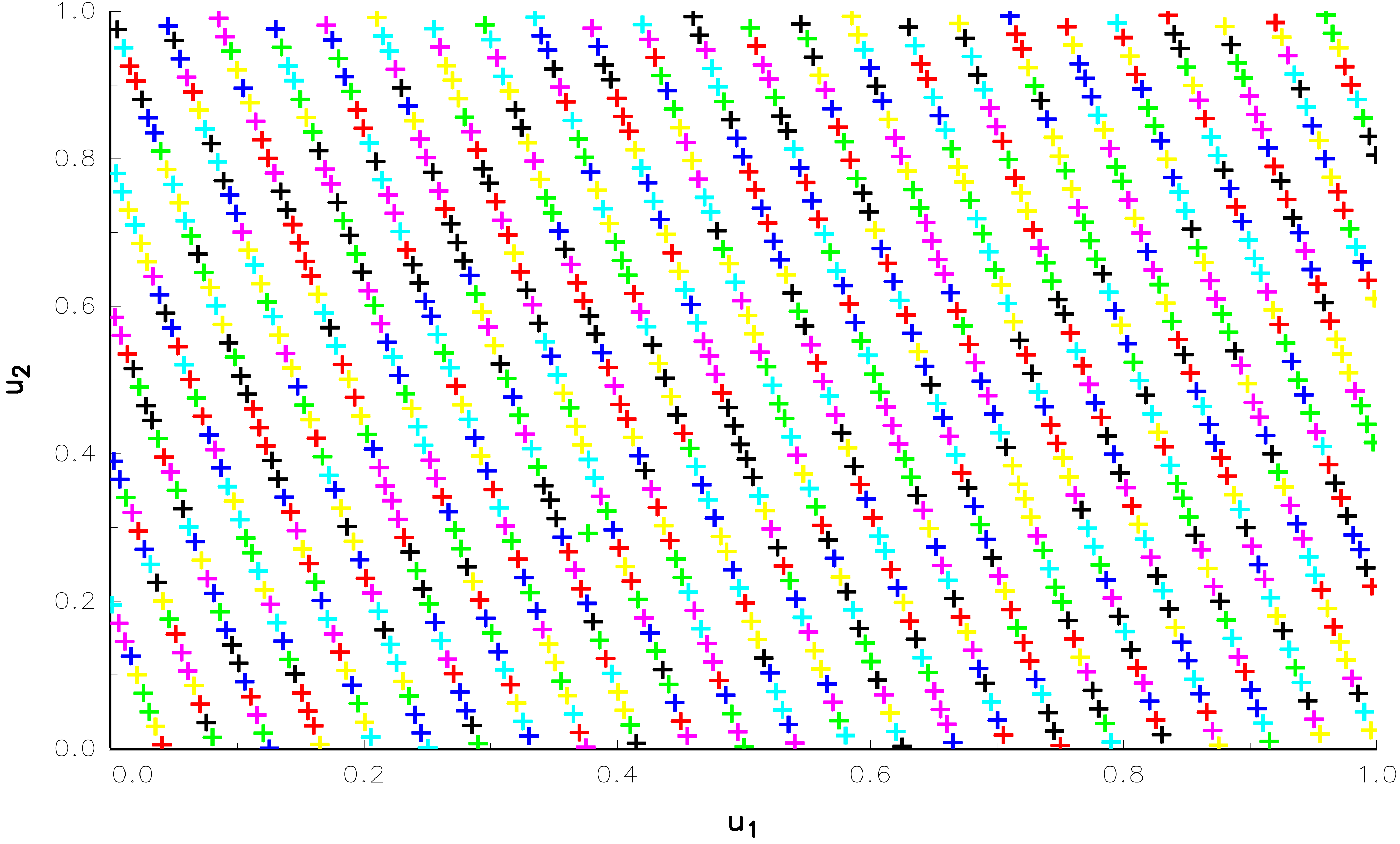
> *The mass distribution for a shuffle of Min can be obtained by (1) placing the mass for $\mathbf{C}^{+}$ on $[0, 1]^2$, (2) cutting $[0, 1]^2$ vertically into a finite number of strips, (3) shuffling the strips with perhaps some of them flipped around their vertical axes of symmetry, and then (4) reassembling them to form the square again. The resulting mass distribution will correspond to a copula called a shuffle of Min.*

Two remarks:

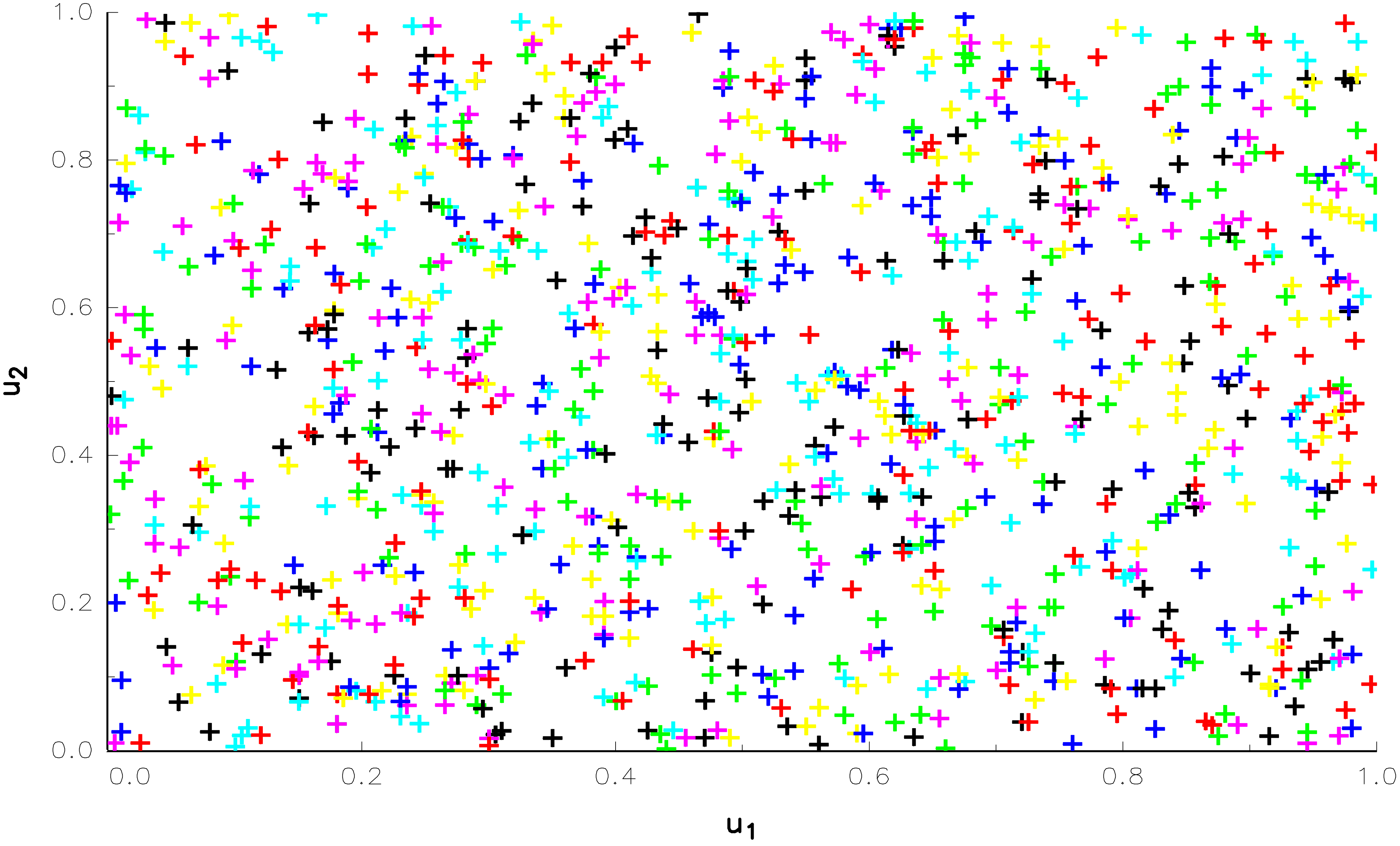
1. The first one concerns obviously the problem of multivariate uniform random generation. The theorem says us that it can be performed using an appropriate complete dependence framework.
2. The second one concerns the mode of convergence, and so the question of approximations:

   [...] with respect to uniform convergence, it is essentially impossible to distinguish between situations in which one random variable completely determines another and a situation in which a pair of random variables is independent (Li, Mikusiński and Taylor [2000]).

   ⇒ Li, Mikusiński, Sherwood and Taylor [1998] introduced strong convergence of copulas, which is defined to be strong convergence of the corresponding Markov operators.



Quasi-random bivariate uniform numbers $C^{\perp}$
generated by T_{200} and Sobol



Pseudo-random bivariate uniform numbers $C^{\perp}$
generated by T_{200} and LC

5.2 Working with distributions or with rv?

Modern probability theory is based on the measure theory of Kolmogorov [1993].

> *[...] clearly shows that the distinction between working directly with distribution functions (as we generally do in the theory of probabilistic metric spaces) rather than with random variables, is intrinsic and not just a matter of taste. It further shows that there are topics in probability which are not encompassed by the standard measure-theoretic model of the theory* (Schweizer and Sklar [1974]).
>
> *[...] it again points out that the classical model for probability theory [...] has its limitations* (Alsina, Nelsen and Schweizer [1974]).

Characterisation of the class of binary operations ψ on distribution functions which are induced pointwise

$$\psi\left\langle \mathbf{F}_1, \mathbf{F}_2 \right\rangle (x) = \Psi\left(\mathbf{F}_1(x), \mathbf{F}_2(x)\right)$$

and derivable from functions on random variables

$$X = \Upsilon\left(X_1, X_2\right)$$

Example 1 *Convolutions are derivable*

$$X\left\langle \mathbf{F}_1 \star \mathbf{F}_2 \right\rangle = X_1 + X_2$$

but not induced pointwise (see Frank *[1975] for more details).*

Example 2 *Mixtures are induced pointwise*

$$\mathbf{F} = p\mathbf{F}_1 + (1-p)\,\mathbf{F}_2$$

but not derivable.

Genest, Quesada Molina, Rodríguez Lallena and Sempi [1999] characterize quasi-copulas in the following way:

Theorem 6 *A function* $\mathbf{Q} : \mathbf{I}^2 \to \mathbf{I}$ *is a quasi-copula if and only if*

1. $\mathbf{Q}(0, u) = \mathbf{Q}(u, 0) = 0$ *and* $\mathbf{Q}(1, u) = \mathbf{Q}(u, 1) = 1$*;*
2. $\mathbf{Q}$ *is non-decreasing in each of its arguments;*
3. $\mathbf{Q}$ *satisfies Lipschitz's condition*

$$|\mathbf{Q}(u_2, v_2) - \mathbf{Q}(u_1, v_1)| \leq |u_2 - u_1| + |v_2 - v_1|$$

Main result

Nelsen, Quesada-Molina, Schweizer and Sempi [1996] show that the class of functions induced pointwise and derivable are order statistics.

$$\psi = \mathbf{Q}$$

6 References

[1] Alsina, C., R.B. Nelsen and B. Schweizer [1993], On the characterization of a class of binary operations on distribution functions, *Statistics & Probability Letters*, **17**, 85-89

[2] Beneš, V. and J. Štēpán [1997], Distributions with Given Marginals and Moment Problems, Kluwer Academic Publishers, Dordrecht

[3] Block, H.W., A.R. Sampson and T.H. Savits [1991], Topics in Statistical Dependence, Institute of Mathematical Statistics, Hayward, CA

[4] Bouyé, E., V. Durrleman, A. Nikeghbali, G. Riboulet and T. Roncalli [2000], Copulas for finance — a reading guide and some applications, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*

[5] Breeden, D. and R. Litzenberger [1978], State contingent prices implicit in option prices, *Journal of Business*, **51**, 621-651

[6] Chan, K.C., G.A. Karolyi, F.A. Longstaff et A.B. Sanders [1992], An empirical comparaison of alternative models of the short-term interest rate, *The Journal of Finance*, **47**, 1209-1227

[7] Cherubini, U. and E. Luciano [2000], Multivariate option pricing with copulas, University of Turin, *Working Paper*

[8] Coles, S., J. Currie and J. Tawn [1999], Dependence measures for extreme value analyses, Department of Mathematics and Statistics, Lancaster University, *Working Paper*

[9] Costinot, A., T. Roncalli and J. Teiletche [2000], Revisiting the dependence between financial markets with copulas, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*

[10] Coutant, S., V. Durrleman, G. Rapuch and T. Roncalli [2001], Copulas, multivariate risk-neutral distributions and implied dependence functions, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*

[11] Coutant, S., P. Martineu, J. Messines, G. Riboulet and T. Roncalli [2001], Revisiting the dependence in credit risk models, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*

[12] Darsow, W.F., B. Nguyen and E.T. Olsen [1992], Copulas and markov processes, *Illinois Journal of Mathematics*, **36-4**, 600-642

[13] Deheuvels, P. [1978], Caractérisation complète des lois extrêmes multivariées et de la convergence des types extrêmes, *Publications de l'Institut de Statistique de l'Université de Paris*, **23**, 1-36

[14] Durrleman, V. [2001], Implied correlation, Princeton University, *report*

[15] Durrleman, V., A. Nikeghbali and T. Roncalli [2000], Which copula is the right one?, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*

[16] Durrleman, V., A. Nikeghbali, and T. Roncalli [2000], Copulas approximation and new families, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*

[17] Genest, C., J.J. Quesada Molina, J.A. Rodríguez Lallena and C. Sempi [1999], A characterization of quasi-copulas, *Journal of Multivariate Analysis*, **69**, 193-205

[18] Georges, P., A-G. Lamy, E. Nicolas, G. Quibel and T. Roncalli [2001], Multivariate survival modelling: a unified approach with copulas, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*

[19] Joe, H. [1997], Multivariate Models and Dependence Concepts, *Monographs on Statistics and Applied Probability*, **73**, Chapmann & Hall, London

[20] Kimeldorf, G. and A. Sampson [1978], Monotone dependence, *Annals of Statistics*, **6(4)**, 895-903

[21] Li, D.X. [2000], On default correlation: a copula function approach, *Journal of Fixed Income*, March, 43-54

[22] Li, X., P. Mikusiński, H. Sherwood and M.D. Taylor [1997], On approximation of copulas, in V. Beneš and J. Štēpán (Eds.), *Distributions with Given Marginals and Moment Problems*, Kluwer Academic Publishers, Dordrecht

[23] Li, X., P. Mikusiński and M.D. Taylor [1998], Strong approximation of copulas, *Journal of Mathematical Analysis and Applications*, **225**, 608-623

[24] Li, X., P. Mikusiński and M.D. Taylor [2000], Remarks on convergence of Markov operators, *to appear*

[25] Maccarinelli, M. and V. Maggiolini [2000], The envolving practice of credit risk management in global financial institutions, *Risk Conference*, 26/27 September, Paris

[26] Mikusiński, P., H. Sherwood and M.D. Taylor [1992], Shuffles of min, *Stochastica*, **13**, 61-74

[27] Nelsen, R.B. [1999], An Introduction to Copulas, *Lectures Notes in Statistics*, **139**, Springer Verlag, New York

[28] Nelsen, R.B., J.J. Quesada Molina, B. Schweizer and C. Sempi [1996], Derivability of some operations on distribution functions, in L. Rushendorf, B. Schweizer and M.D. Taylor (Eds.), *Distributions with Fixed Marginals and Related Topics*, Institute of Mathematical Statistics, Hayward, CA

[29] Portnoy, S. and R. Koenker [1997], The Gaussian hare and the Laplacean tortoise: computability of ℓ_1 vs ℓ_2 regression estimators, *Statistical Science*, **12**, 279-300

[30] Rüschendorf, L., B. Schweizer and M.D. Taylor [1996], Distributions with Fixed Marginals and Related Topics, Institute of Mathematical Statistics, Hayward, CA

[31] Schweizer, B. and A. Sklar [1974], Operations on distribution functions not derivable from operations on random variables, *Studia Mathematica*, **LII**, 43-52

[32] Schweizer, B. and E. Wolff [1981], On nonparametric measures of dependence for random variables, *Annals of Statistics*, **9**, 879-885

[33] Song, P. X-K. [2000], Multivariate dispersion models generated from Gaussian copula, *Scandinavian Journal of Statistics,* **27-2**, 305-320

[34] Vitale, R.A. [1991], On stochastic dependence and a class of degenerate distributions, in H.W. Block, A.R. Sampson and T.H. Savits (Eds.), *Topics in Statistical Dependence,* Institute of Mathematical Statistics, Hayward, CA

[35] Vitale, R.A. [1991], Approximation by mutually completely dependent processes, *Journal of Approximation Theory*, **66**, 225-228