Some remarks on two-asset options pricing and stochastic dependence of asset prices*

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July 16, 2001

Abstract

In this short note, we consider some problems of two-asset options pricing. In particular, we investigate the relationship between options prices and the 'correlation' parameter in the Black-Scholes model. Then, we consider the general case in the framework of the copula construction of risk-neutral distributions. This extension involves results on the supermodular order applied to the Feynman-Kac representation. We show that it could be viewed as a generalization of a maximum principle for parabolic PDE.

1 Introduction

In this paper, we address the problem of the relationship between the dependence function and the price of two-asset options. For example, one might wonder if the price of a Spread option is a monotone (decreasing or increasing) function of the correlation parameter in the Black-Scholes model. Another question is related to the (lower and upper) bounds of the option price with respect to this correlation parameter. For the Black-Scholes model, we solve these two problems by using a maximum principle for parabolic PDE. In the general case where the risk-neutral distribution satisfies a copula decomposition, we directly use properties of the supermodular order to solve them. We remark that this method could be viewed as a generalization of the previous maximum principle.

As a matter of fact, the supermodular method may be used for more general problems than two-asset options pricing. These problems are special cases of Fokker-Planck equations and correspond to a Feynman-Kac representation (Friedman [1975]).

Theorem 1 (Feynman-Kac representation) *Let $x(t)$ be a diffusion process determined by the SDE of the form*

$$\mathrm{d}x(t) = \mu(x(t), t)\ \mathrm{d}t + \sigma(x(t), t)\ \mathrm{d}W(t) \tag{1}$$

where $W(t)$ is a n-dimensional Wiener process defined on the fundamental probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the covariance matrix $\mathbb{E}\left[W(t)W(t)^\top\right] = \boldsymbol{\rho} t$. Let $\mathcal{A}$ be the differential operator defined by

$$\mathcal{A}u(x,t) = \frac{1}{2}\operatorname{trace}\left[\sigma(x,t)^\top \partial^2_{x,x}u(x,t)\,\sigma(x,t)\,\boldsymbol{\rho}\right] + \mu(x,t)^\top \partial_x u(x,t) \tag{2}$$

$\mathcal{A}$ is called the 'infinitesimal generator' of the diffusion process $x(t)$. Under the following assumptions:

*We acknowledge helpful discussions with Jérôme Busca, Nicole El Karoui and Gaël Riboulet. All errors are our own.
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1. $\mu(x,t)$, $\sigma(x,t)$, $k(x,t)$ *and* $g(x,t)$ *are lipschitz and bounded functions on* $\mathbb{R}^m \times [0,T]$*;*

2. $f(x)$ *is continuous;*

3. $g(x,t)$ *and* $f(x)$ *satisfy the exponential growth condition;*

4. and the polynomial growth condition is verified for $u(x,t)$*;*

there exists then a unique solution to the Cauchy problem

$$\left\{ \begin{array}{l} -\partial_t u(x,t) + k(x,t)\, u(x,t) = \mathcal{A}u(x,t) + g(x,t) \\ u(x,T) = f(x) \end{array} \right. \tag{3}$$

This solution is given by the Feynman-Kac formula

$$u(x,t_0) = \mathbb{E}\left[f(x(T)) \exp\left(-\int_{t_0}^{T} k(x(t),t)\ \mathrm{d}t\right) + \int_{t_0}^{T} g(x(t),t) \exp\left(-\int_{t_0}^{t} k(x(s),s)\ \mathrm{d}s\right) \mathrm{d}t \right| x(t_0) = x \Bigg] \tag{4}$$

2 The case of the Black-Scholes model

In the BS model, the asset prices are correlated geometric Brownian motions

$$\left\{ \begin{array}{l} \mathrm{d}S_1(t) = \mu_1 S_1(t)\ \mathrm{d}t + \sigma_1 S_1(t)\ \mathrm{d}W_1(t) \\ \mathrm{d}S_2(t) = \mu_2 S_2(t)\ \mathrm{d}t + \sigma_2 S_2(t)\ \mathrm{d}W_2(t) \end{array} \right. \tag{5}$$

where $\mathbb{E}[W_1(t)\, W_2(t)] = \rho t$. Using Ito calculus and the arbitrage theory, the price $P(S_1,S_2,t)$ of the European two-asset option with the payoff function $G(S_1,S_2)$ is the (unique) solution of the following parabolic PDE

$$\left\{ \begin{array}{l} \frac{1}{2}\sigma_1^2 S_1^2 \partial_{1,1}^2 P + \rho\sigma_1\sigma_2 S_1 S_2 \partial_{1,2}^2 P + \frac{1}{2}\sigma_2^2 S_2^2 \partial_{2,2}^2 P + b_1 S_1 \partial_1 P + b_2 S_2 \partial_2 P - rP + \partial_t P = 0 \\ P(S_1,S_2,T) = G(S_1,S_2) \end{array} \right. \tag{6}$$

where b_1 and b_2 are the cost-of-carry parameters and r is the instantaneous constant interest rate. First, we consider the case of the Spread option and show that the price is a nonincreasing function of the correlation parameter ρ. Second, we extend this result to other two-asset options. Moreover, we give explicit lower and upper bounds for these option prices.

2.1 An example with the Spread option

Before to state the main proposition, we recall the weak maximum principle for parabolic PDE[1].

Theorem 2 (Phragmen-Lindeloff principle) *We consider the operator* $\mathcal{L}u(x,t) = \sum_{i,j} a_{i,j}(x,t)\, \partial_{i,j}^2 u(x,t) + \sum_i b_i(x,t)\, \partial_i u(x,t) + \partial_t u(x,t) + c(x,t)\, u(x,t)$ *where the functions* $a_{i,j}(x,t)$, $b_i(x,t)$ *and* $c(x,t)$ *are continuous and bounded. Moreover, we assume that* $(a_{i,j}(x,t))$ *is a symmetric, positive definite matrix for* $(x,t) \in \mathbb{R}^m \times [0,T)$*. Let* $w \in C^2(\mathbb{R}^m \times [0,T)) \cap C^0(\mathbb{R}^m \times [0,T])$ *with* $|w(x,t)| \leq \beta e^{\alpha\|\mathbf{x}\|}$ *for all* $(x,t) \in \mathbb{R}^m \times [0,T]$ *and some constants* α *and* β*. If* $\mathcal{L}w(x,t) \leq 0$ *for* $t < T$ *and* $w(x,T) \geq 0$*, then we have* $w(x,t) \geq 0$ *for all* $(x,t) \in \mathbb{R}^m \times [0,T]$*.*

In order to obtain qualitative properties of the Spread option, it is more convenient to deal with an elliptic operator. If we make the change of variables $\tilde{S}_1 = \ln S_1$ and $\tilde{S}_2 = \ln S_2$, it comes that

$$\left\{ \begin{array}{l} \frac{1}{2}\sigma_1^2 \partial_{1,1}^2 P + \rho\sigma_1\sigma_2 \partial_{1,2}^2 P + \frac{1}{2}\sigma_2^2 \partial_{2,2}^2 P + \left(b_1 + \frac{1}{2}\sigma_1^2\right)\partial_1 P + \left(b_2 + \frac{1}{2}\sigma_2^2\right)\partial_2 P - rP + \partial_t P = 0 \\ P\left(\tilde{S}_1, \tilde{S}_2, T\right) = \left(\exp \tilde{S}_2 - \exp \tilde{S}_1 - K\right)^+ \end{array} \right. \tag{7}$$

The operator $\mathcal{L}_\rho u = \frac{1}{2}\sigma_1^2 \partial_{1,1}^2 u + \rho\sigma_1\sigma_2 \partial_{1,2}^2 u + \frac{1}{2}\sigma_2^2 \partial_{2,2}^2 u + \left(b_1 + \frac{1}{2}\sigma_1^2\right)\partial_1 u + \left(b_2 + \frac{1}{2}\sigma_2^2\right)\partial_2 u - ru + \partial_t u$ is also elliptic for $\rho \in \left]-1,1\right[$. We can now establish the following proposition.

[1] The next theorem could be linked to the submartingale property of $w(x(t),t)$.

Proposition 3 *The price of the* Spread *option in the Black-Scholes model is a nonincreasing function of ρ for $\rho \in [-1, 1]$.*

Proof. The complete proof is given in Appendix A. We just give here the main ideas to prove the proposition. We first verify the exponential growth condition. Then, we consider the case $\rho_1 < \rho_2$ and compute the difference function $\Delta\left(\tilde{S}_1, \tilde{S}_2, t\right) = P_{\rho_1}\left(\tilde{S}_1, \tilde{S}_2, t\right) - P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, t\right)$. It comes that Δ is the solution of the following PDE

$$\begin{cases} \mathcal{L}_{\rho_1}\Delta\left(\tilde{S}_1, \tilde{S}_2, t\right) = (\rho_2 - \rho_1)\,\sigma_1\sigma_2\partial^2_{1,2}P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, t\right) \\ \Delta\left(\tilde{S}_1, \tilde{S}_2, T\right) = 0 \end{cases}$$

In order to apply the maximum principle to $\Delta\left(\tilde{S}_1, \tilde{S}_2, t\right)$, we would like to show that $\partial^2_{1,2}P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, t\right) \leq 0$. We show this by using again the maximum principle to $\partial^2_{1,2}P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, t\right)$. It comes finally that $\Delta\left(\tilde{S}_1, \tilde{S}_2, t\right) \geq 0$. So, we conclude that

$$\rho_1 < \rho_2 \Rightarrow P_{\rho_1}\left(\tilde{S}_1, \tilde{S}_2, t\right) \geq P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, t\right) \tag{8}$$

■

Remark 4 *The* Spread *option price is a one to one mapping with respect to the parameter ρ. To one price corresponds one and only one parameter ρ. This is the implied BS correlation*[2]*.*

2.2 Other two-asset options

We may extend the previous proposition to other two-asset options. We remark that the key point in the proof is the sign of the cross derivative $\partial^2_{1,2}G(S_1, S_2)$ of the payoff function. In general, this differential is a measure and G does not depend on the parameter ρ.

Proposition 5 *Let G be the payoff function. If $\partial^2_{1,2}G$ is a nonpositive (resp. nonnegative) measure then the option price is nonincreasing (resp. nondecreasing) with respect to ρ.*

Let us investigate some examples. For the call option on the maximum of two assets, the payoff function is defined as $G(S_1, S_2) = (\max(S_1, S_2) - K)^+$. We have $\partial^2_{1,2}G(S_1, S_2) = -\delta_{\{S_1 = S_2, S_1 > K\}}$ which is a nonpositive measure. So, the option price nonincreases with respect to ρ. In the case of a BestOf call/call option, the payoff function is $G(S_1, S_2) = \max\left((S_1 - K_1)^+, (S_2 - K_2)^+\right)$ and we have $\partial^2_{1,2}G(S_1, S_2) = -\delta_{\{S_2 - K_2 - S_1 + K_1 = 0, S_1 > K_1, S_2 > K_2\}}$. We have the same behaviour than the Max option. For the Min option, we remark that $\min(S_1, S_2) = S_1 + S_2 - \max(S_1, S_2)$. So, the price is a nondecreasing function of ρ. Other results could be found in Table 1.

2.3 Bounds of two-asset options prices

The previous analysis leads us to define the lower and upper bounds of two-asset options price when the parameter ρ is unknown. Let $P^-(S_1, S_2, t)$ and $P^+(S_1, S_2, t)$ be respectively the lower and upper bounds

$$P^-(S_1, S_2, t) \leq P_\rho(S_1, S_2, t) \leq P^+(S_1, S_2, t) \tag{9}$$

We have the following result.

Proposition 6 *If $\partial^2_{1,2}G$ is a nonpositive (resp. nonnegative) measure then $P^-(S_1, S_2, t)$ and $P^+(S_1, S_2, t)$ correspond to the cases $\rho = 1$ (resp. $\rho = 1$) and $\rho = -1$ (resp. $\rho = 1$).*

[2] In fact we can show that increasingness is strict by using a **strong** maximum principle which is available as soon as the operator is strictly elliptic (Nirenberg [1953]).

Option type	Payoff	increasing	decreasing
Spread	$(S_2 - S_1 - K)^+$		$\checkmark$
Basket	$(\alpha_1 S_1 + \alpha_2 S_2 - K)^+$	$\alpha_1\alpha_2 > 0$	$\alpha_1\alpha_2 < 0$
Max	$(\max(S_1, S_2) - K)^+$		$\checkmark$
Min	$(\min(S_1, S_2) - K)^+$	$\checkmark$	
BestOf call/call	$\max\left((S_1 - K_1)^+, (S_2 - K_2)^+\right)$		$\checkmark$
BestOf put/put	$\max\left((K_1 - S_1)^+, (K_2 - S_2)^+\right)$		$\checkmark$
Worst call/call	$\min\left((S_1 - K_1)^+, (S_2 - K_2)^+\right)$	$\checkmark$	
Worst put/put	$\min\left((K_1 - S_1)^+, (K_2 - S_2)^+\right)$	$\checkmark$	

Table 1: Relationship between option prices and the parameter ρ

To be more precise, we have to study the special cases $\rho = -1$ and $\rho = 1$. Let ϵ be a constant which is equal to 1 if the distribution $\partial^2_{1,2}G$ is a nonnegative measure and -1 if the distribution $\partial^2_{1,2}G$ is a nonpositive measure. The bounds satisfy then the one-dimensional PDE

$$\begin{cases} \frac{1}{2}\sigma_1^2 S^2 \partial^2_{1,1} P^{\pm}(S,\theta) + b_1 S \partial_1 P^{\pm}(S,\theta) - rP^{\pm}(S,\theta) + \partial_\theta P^{\pm}(S,\theta) = 0 \\ P^{\pm}(S,T) = G(S, h_{\pm\epsilon}(S)) \end{cases} \tag{10}$$

where

$$h_\epsilon(S) = S_2(t)\left[\frac{S}{S_1(t)}\right]^{\epsilon\sigma_2/\sigma_1} \exp\left(\left(b_2 + \frac{1}{2}\epsilon\sigma_1\sigma_2 - \frac{1}{2}\sigma_2^2 - \epsilon\frac{\sigma_2}{\sigma_1}b_1\right)(T-t)\right) \tag{11}$$

3 The general case

In this section, we generalize the previous results. For that, we know that the European prices of two-asset options are given by

$$P(S_1, S_2, t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[G(S_1(T), S_2(T)) \mid \mathcal{F}_t\right] \tag{12}$$

with $\mathbb{Q}$ the *martingale* probability measure. Let $\mathbf{F}$ be the bivariate risk-neutral distribution at time t. Using the copula construction of Coutant, Durrleman, Rapuch and Roncalli [2001], we have

$$\mathbf{F}(S_1, S_2) = \mathbf{C}(\mathbf{F}_1(S_1), \mathbf{F}_2(S_2)) \tag{13}$$

where $\mathbf{F}_1$ and $\mathbf{F}_2$ are the two univariate risk-neutral distributions. $\mathbf{C}$ is called the *risk-neutral* copula.

In the Black-Scholes model and for the Spread option, we have proved that if $\rho_1 < \rho_2$, then $P_{\rho_1}(S_1, S_2, t) \geq P_{\rho_2}(S_1, S_2, t)$ and that the lower and upper bounds are reached respectively for $\rho = 1$ and $\rho = -1$. Now, we are going to give similar results for the general case. For the Spread option, we will prove that if $\mathbf{C}_1 \prec \mathbf{C}_2$, then $P_{\mathbf{C}_1}(S_1, S_2, t) \geq P_{\mathbf{C}_2}(S_1, S_2, t)$ and that the lower and upper bounds are reached for the upper and lower Fréchet bounds. These results are all based on properties of the supermodular order.

3.1 Supermodular order

Following Müller and Scarsini [2000], we say that the function f is supermodular if and only if

$$\Delta^{(2)} f := f(x_1 + \varepsilon_1, x_2 + \varepsilon_2) - f(x_1 + \varepsilon_1, x_2) - f(x_1, x_2 + \varepsilon_2) + f(x_1, x_2) \geq 0 \tag{14}$$

for all $(x_1, x_2) \in \mathbb{R}^2$ and $(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_+$. If f is twice differentiable, then the condition (14) is equivalent to $\partial^2_{1,2} f(x_1, x_2) \geq 0$ for all $(x_1, x_2) \in \mathbb{R}^2$ (Marshall and Olkin [1979]). We can then show the following relationship between the concordance order and supermodular functions.

Theorem 7 *Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be the probability distribution functions of X_1 and X_2. Let $\mathbb{E}_{\mathbf{C}}[f(X_1,X_2)]$ denote the expectation of the function $f(X_1,X_2)$ when the copula of the random vector (X_1,X_2) is $\mathbf{C}$. If $\mathbf{C}_1 \prec \mathbf{C}_2$, then $\mathbb{E}_{\mathbf{C}_1}[f(X_1,X_2)] \leq \mathbb{E}_{\mathbf{C}_2}[f(X_1,X_2)]$ for all supermodular functions f such that the expectations exist.*

Proof. See Tchen [1980] and Müller and Scarsini [2000]. ∎

3.2 Concordance order and two-asset options prices

Using the previous theorem, we have directly the following proposition.

Proposition 8 *If the payoff function G is supermodular, then the option price nondecreases with respect to the concordance order. More explicitly, we have*

$$\mathbf{C}_1 \prec \mathbf{C}_2 \Rightarrow P_{\mathbf{C}_1}(S_1,S_2,t) \leq P_{\mathbf{C}_2}(S_1,S_2,t) \tag{15}$$

Let us consider the simple case of the Basket option where $G(S_1,S_2) = (S_1+S_2-K)^+$. We have

$$\begin{aligned}\Delta^{(2)}G &= (x_1+x_2+\varepsilon_1+\varepsilon_2-K)^+ - (x_1+x_2+\varepsilon_2-K)^+ - (x_1+x_2+\varepsilon_1-K)^+ + (x_1+x_2-K)^+ \\ &:= \max(\Delta_1,0) - \max(\Delta_2,0) - \max(\Delta_3,0) + \max(\Delta_4,0)\end{aligned} \tag{16}$$

To prove that G is supermodular, we consider the different cases. For example, if $\Delta_4 = x_1+x_2-K \geq 0$, we get $\Delta^{(2)}G = 0$. If only $\Delta_4 < 0$, then $\Delta^{(2)}G = \Delta_1 - \Delta_2 - \Delta_3 = -(x_1+x_2-K) = -\Delta_4 > 0$. The other cases may be verified in the same way. Using the previous proposition, if $\mathbf{C}_1\langle S_1(T), S_2(T)\rangle \prec \mathbf{C}_2\langle S_1(T), S_2(T)\rangle$, it comes that $\mathbb{E}_{\mathbf{C}_1}[G(S_1(T),S_2(T))] \leq \mathbb{E}_{\mathbf{C}_2}[G(S_1(T),S_2(T))]$. So, we deduce that the price of the Basket option nondecreases with respect to the concordance order and that the price of the Spread option **nonincreases** with respect to the concordance order. We can prove this last statement in two different ways. Indeed, for the Spread option, we have $\mathbf{C}_1\langle -S_1(T), S_2(T)\rangle \succ \mathbf{C}_2\langle -S_1(T), S_2(T)\rangle$ because $\mathbf{C}_{\langle -X_1,X_2\rangle}(u_1,u_2) = u_2 - \mathbf{C}_{\langle X_1,X_2\rangle}(1-u_1,u_2)$ (Nelsen [1999]). Hence, using the supermodularity of the Basket payoff function, $\mathbb{E}_{\mathbf{C}_1}[G(-S_1(T),S_2(T))] \geq \mathbb{E}_{\mathbf{C}_2}[G(-S_1(T),S_2(T))]$. Another way to derive the result is to remark that $\partial^2_{1,2}H$ is a nonpositive measure, where $H(S_1,S_2) = G(-S_1,S_2)$. Thus, $-H$ is supermodular and the result is a consequence of the next proposition.

We can then generalize results of Table 1 in this framework. For that, we state the main proposition of this section.

Proposition 9 *Let G be a continuous payoff function. If the distribution $\partial^2_{1,2}G$ is a nonnegative (resp. nonpositive) measure then the option price is nondecreasing (resp. nonincreasing) with respect to the concordance order.*

Proof. See Appendix B. ∎

Remark 10 *This last proposition is interesting because families of copulas are generally totally (positively or negatively) ordered. For example, we know that the parametric family $\mathbf{C}_\rho$ of bivariate Normal copulas is positively ordered*

$$\rho_1 < \rho_2 \Rightarrow \mathbf{C}_{\rho_1} \prec \mathbf{C}_{\rho_2} \tag{17}$$

As for the Black-Scholes model, the option price nonincreases or nondecreases with respect to the 'correlation' parameter ρ depending on the submodular or supermodular property of the payoff function. So, the results of the Black-Scholes model can be viewed as a special case of this framework. Moreover, they remain true if the bivariate risk-neutral distribution is not gaussian, but only has a Normal copula. Other cases are considered in Coutant, Durrleman, Rapuch *and* Roncalli *[2001]. For example, we obtain similar relationships with Ornstein-Uhlenbeck diffusions.*

3.3 Bounds of two-asset options prices

Let us introduce the lower and upper Fréchet copulas $\mathbf{C}^{-}(u_1,u_2)=\max(u_1+u_2-1,0)$ and $\mathbf{C}^{+}(u_1,u_2)=\min(u_1,u_2)$. We can prove that for any copula $\mathbf{C}$, we have $\mathbf{C}^{-}\prec\mathbf{C}\prec\mathbf{C}^{+}$. For any distribution $\mathbf{F}$ with given marginals $\mathbf{F}_1$ and $\mathbf{F}_2$, it comes that $\mathbf{C}^{-}(\mathbf{F}_1(x_1),\mathbf{F}_2(x_2))\leq\mathbf{F}(x_1,x_2)\leq\mathbf{C}^{+}(\mathbf{F}_1(x_1),\mathbf{F}_2(x_2))$ for all $(x_1,x_2)\in\mathbb{R}_+^2$. The probabilistic interpretation of these Fréchet bounds are the following:

- two random variables X_1 and X_2 are **countermonotonic** — or $\mathbf{C}\langle X_1,X_2\rangle=\mathbf{C}^{-}$ — if there exists a random variable X such that $X_1=f_1(X)$ and $X_2=f_2(X)$ with f_1 nonincreasing and f_2 nondecreasing;
- two random variables X_1 and X_2 are **comonotonic** — or $\mathbf{C}\langle X_1,X_2\rangle=\mathbf{C}^{+}$ — if there exists a random variable X such that $X_1=f_1(X)$ and $X_2=f_2(X)$ where the functions f_1 and f_2 are nondecreasing.

We can now state the following proposition.

Proposition 11 *If $\partial^2_{1,2}G$ is a nonpositive (resp. nonnegative) measure then $P^{-}(S_1,S_2,t)$ and $P^{+}(S_1,S_2,t)$ correspond to the cases $\mathbf{C}=\mathbf{C}^{+}$ (resp. $\mathbf{C}=\mathbf{C}^{-}$) and $\mathbf{C}=\mathbf{C}^{-}$ (resp. $\mathbf{C}=\mathbf{C}^{+}$).*

Let ϵ be a constant which is equal to 1 if the distribution $\partial^2_{1,2}G$ is a nonnegative measure and -1 if the distribution $\partial^2_{1,2}G$ is a nonpositive measure. We have

$$P^{\pm}(S_1,S_2,t)=e^{-r(T-t)}\left(\frac{1\mp\epsilon}{2}\mathbb{E}_{\mathbf{C}^{-}}\left[G(S_1(T),S_2(T))\right]+\frac{1\pm\epsilon}{2}\mathbb{E}_{\mathbf{C}^{+}}\left[G(S_1(T),S_2(T))\right]\right)\tag{18}$$

If the two univariate risk-neutral distributions $\mathbf{F}_1$ and $\mathbf{F}_2$ are continuous, these bounds become more tractable because

$$\mathbf{C}=\mathbf{C}^{-}\Leftrightarrow S_2(T)=\mathbf{F}_2^{-1}(1-\mathbf{F}_1(S_1(T)))\tag{19}$$

and

$$\mathbf{C}=\mathbf{C}^{+}\Leftrightarrow S_2(T)=\mathbf{F}_2^{-1}(\mathbf{F}_1(S_1(T)))\tag{20}$$

For example, if $\epsilon=1$, we have $P^{-}(S_1,S_2,t)=e^{-r(T-t)}\mathbb{E}\left[G\left(S_1(T),\mathbf{F}_2^{-1}(1-\mathbf{F}_1(S_1(T)))\right)\right]$ and $P^{+}(S_1,S_2,t)=e^{-r(T-t)}\mathbb{E}\left[G\left(S_1(T),\mathbf{F}_2^{-1}(\mathbf{F}_1(S_1(T)))\right)\right]$. For the Black-Scholes model, we can either solve the one-dimensional PDE (10) or compute the one-dimensional expectation. For example, in the case of the Spread option, we obtain

$$P^{\pm}(S_1,S_2,t)=e^{-r(T-t)}\int_{-\infty}^{\infty}\left(h_{\mp}(\xi(x))-\xi(x)-K\right)^{+}\frac{1}{\sigma_1\sqrt{2\pi(T-t)}}\exp\left(-\frac{1}{2}\frac{x^2}{\sigma_1^2(T-t)}\right)\mathrm{d}x\tag{21}$$

where $\xi(x)=S_1(t)\,e^{x+\left(b_1-\frac{1}{2}\sigma_1^2\right)(T-t)}$ and h is defined by the equation (11).

4 Numerical illustrations

Let us first consider the case of the Spread option in the Black and Scholes model. We use the following parameters: $b_1=6\%$, $b_2=5\%$, $\sigma_1=25\%$, $\sigma_2=20\%$ and $r=5\%$. The maturity T of the Spread option is one month. We note $\Delta_{\rho_1,\rho_2}(S_1,S_2)=P_{\rho_1}(S_1,S_2,0)-P_{\rho_1}(S_1,S_2,0)$. In Figures 1 and 2, we have reported the values of $\Delta_{\rho_1,\rho_2}(S_1,S_2)$ when the strike K is respectively equal to 0 and 15. We verify that $\Delta_{\rho_1,\rho_2}(S_1,S_2)\geq 0$ when $\rho_1<\rho_2$. Figure 3 shows how the price $P_{\rho}(S_1,S_2,0)$ moves with respect to ρ. We remark that the relationship between the option price and the 'correlation' parameter is almost linear. For the Min option, we obtain Figure 4.

When the cross derivatives of the payoff function is neither a nonpositive measure neither a nonnegative measure, the relationship between the price and the parameter ρ may be less simple. In this case, it depends on

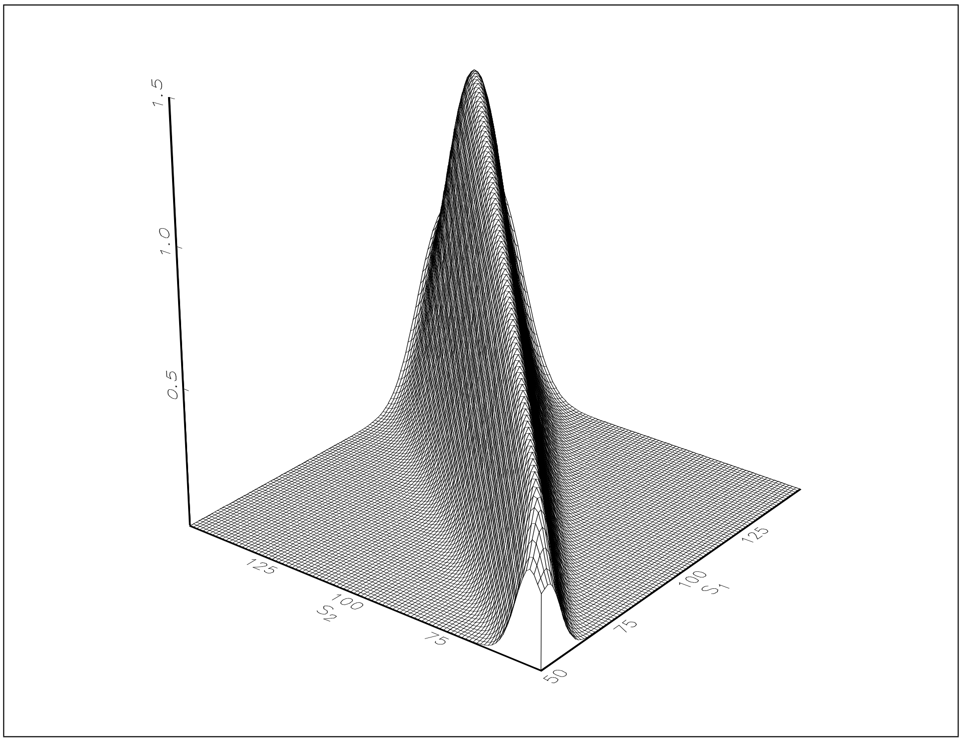

Figure 1: Difference $\Delta_{\rho_1,\rho_2}(S_1, S_2)$ with $\rho_1 = 0$ and $\rho_2 = 0.5$ ($K = 0$)

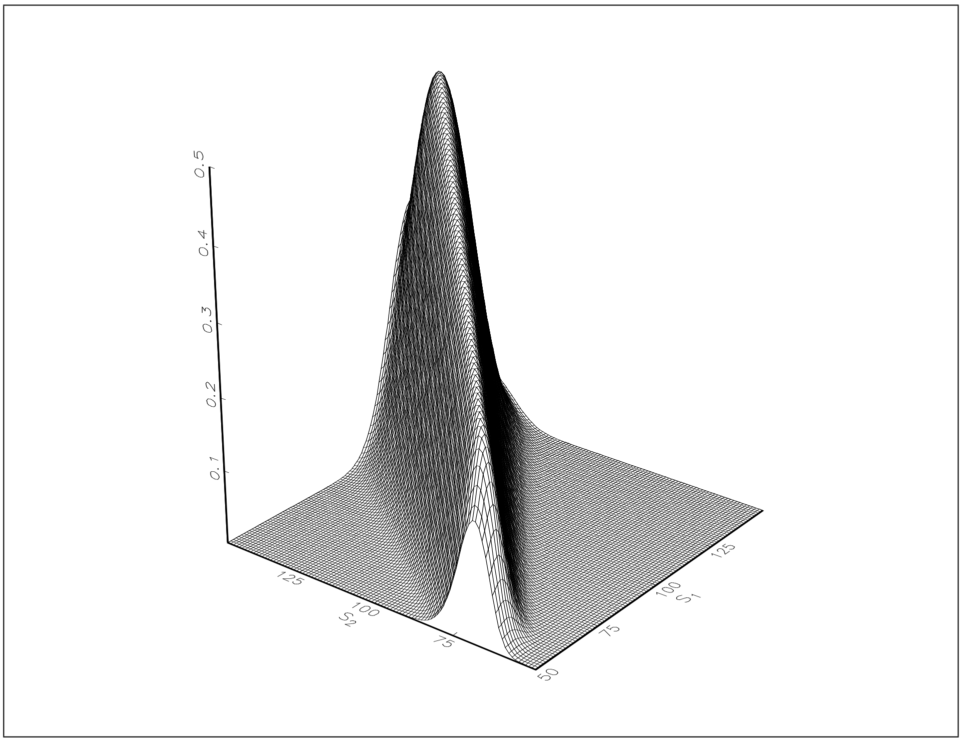

Figure 2: Difference $\Delta_{\rho_1,\rho_2}(S_1, S_2)$ with $\rho_1 = -0.25$ and $\rho_2 = 0$ ($K = 15$)

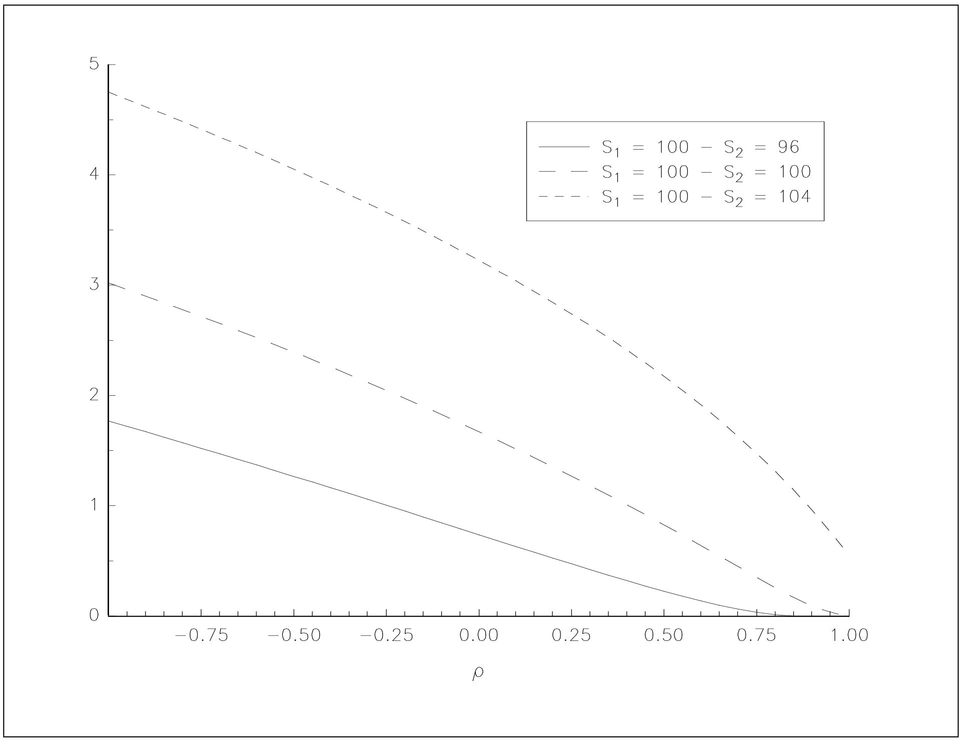

Figure 3: Relationship between the price of the Spread option and the parameter ρ ($K = 5$)

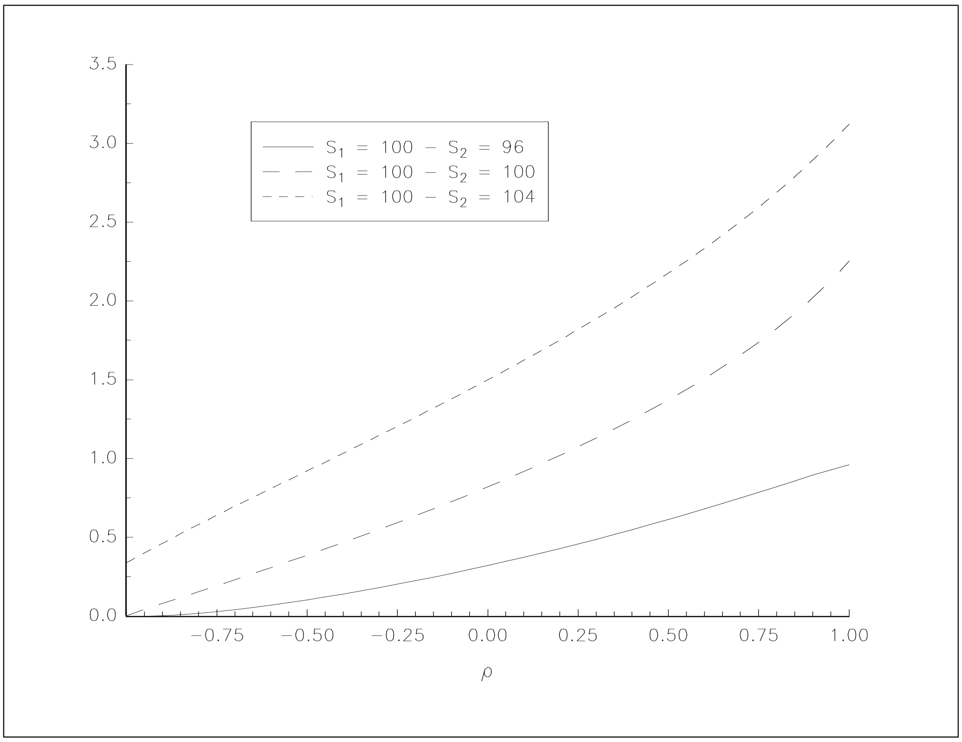

Figure 4: Relationship between the price of the Min option and the parameter ρ ($K = 100$)

the parameters of the asset prices **and** on the characteristics of the option. Let us consider the following case WorstOf call/put option

$$G(S_1,S_2)=\min\left((S_1-K_1)^+,(K_2-S_2)^+\right) \tag{22}$$

In Figure 5, we have reported the values of the option prices for $K_1=105$ and $K_2=95$. We remark that the price is not necessarily a monotonous function of ρ. Moreover, the bounds do not always correspond to the cases $\rho=1$ and $\rho=-1$.

We consider now the Heston model. The asset prices $S_n(t)$ are given by the following SDE

$$\left\{\begin{array}{lcl} \mathrm{d}S_n(t) & = & \mu_n S_n(t)\,\mathrm{d}t+\sqrt{V_n(t)}S_n(t)\,\mathrm{d}W_n^1(t) \\ \mathrm{d}V_n(t) & = & \kappa_n(V_n(\infty)-V_n(t))\,\mathrm{d}t+\sigma_n\sqrt{V_n(t)}\,\mathrm{d}W_n^2(t)\end{array}\right. \tag{23}$$

with $\mathbb{E}\left[W_n^1(t)\,W_n^2(t)\mid\mathcal{F}_{t_0}\right]=\rho_n(t-t_0)$, $\kappa_n>0$, $V_n(\infty)>0$ and $\sigma_n>0$. The market prices of risk processes are $\lambda_n^1(t)=(\mu_n-r)/\sqrt{V_n(t)}$ and $\lambda_n^2(t)=\lambda_n\sigma_n^{-1}\sqrt{V_n(t)}$. To compute prices of two-asset options, we consider that the risk-neutral copula is the Normal copula with parameter ρ. In Figure 6, we have reported the values of option prices[3].

5 Discussion

We conclude this paper with some remarks.

- We recall that main results depend on the sign of $\partial_{1,2}^2 G$. Using two different points of view, we obtain the same condition. It appears that results obtained with a maximum principle for the Black-Scholes model are a special case of the supermodular order. It could be explained by the Feynman-Kac representation of risk-neutral valuation.

- Similar problems have been already studied in actuarial sciences. For example, Dhaene and Goovaerts [1996] shows that the bounds of the stop-loss problem are reached for the Fréchet bounds (see Genest, Marceau and Mesfioui [2000] for a survey).

- We have discussed here about two-asset options. The natural following step is to consider more than two assets. In the case of the Black-Scholes model, the PDE becomes

$$\left\{\begin{array}{l} \frac{1}{2}\sum\limits_i \sigma_i^2 S_i^2\partial_{i,i}^2 P+\sum\limits_{i<j}\rho_{i,j}\sigma_i\sigma_j S_i S_j\partial_{i,j}^2 P+\sum\limits_i b_i S_i\partial_i P-rP+\partial_t P=0 \\ P(S_1,..,S_N,T)=G(S_1,..,S_N)\end{array}\right. \tag{24}$$

  where $\rho_{i,j}$ is the correlation between the Brownian motions of S_i and S_j. If we fix all the correlations $\rho_{i,j}$ except one, we retrieve the same condition as in the two-assets options case. Sometimes, the trader uses the same values for all $\rho_{i,j}$. Let us denote ρ this parameter, which could be interpreted as the mean correlation. We can give the following result.

  Proposition 12 *Assume that G is continuous. If $\sum_{i<j}\sigma_i\sigma_j\partial_{i,j}^2 G$ is a nonnegative (resp. nonpositive) measure, then the price is nondecreasing (resp. nonincreasing) with respect to ρ.*

  In the case of the three-asset option with $G(S_1,S_2,S_3)=(S_1+S_2-S_3-K)^+$, we have $\sum_{i<j}\sigma_i\sigma_j\partial_{i,j}^2 G=(\sigma_1\sigma_2-\sigma_1\sigma_3-\sigma_2\sigma_3)\,\delta_{(S_1+S_2-S_3-K=0)}$. Hence, if $\sigma_1\sigma_2-\sigma_1\sigma_3-\sigma_2\sigma_3>0$, the price nondecreases with ρ, and if $\sigma_1\sigma_2-\sigma_1\sigma_3-\sigma_2\sigma_3<0$, the price nonincreases. In addition, if $\sigma_1\sigma_2-\sigma_1\sigma_3-\sigma_2\sigma_3=0$, the price does not depend on ρ. We could of course give similar results for the Max and for more general Basket options.

[3] The numerical values are $S_n(t_0)=100$, $\tau=1/12$, $b_n=r=5\%$, $V_n(t_0)=V_n(\infty)=\sqrt{20\%}$, $\kappa_n=0.5$, $\sigma_n=90\%$ and $\lambda_n=0$.

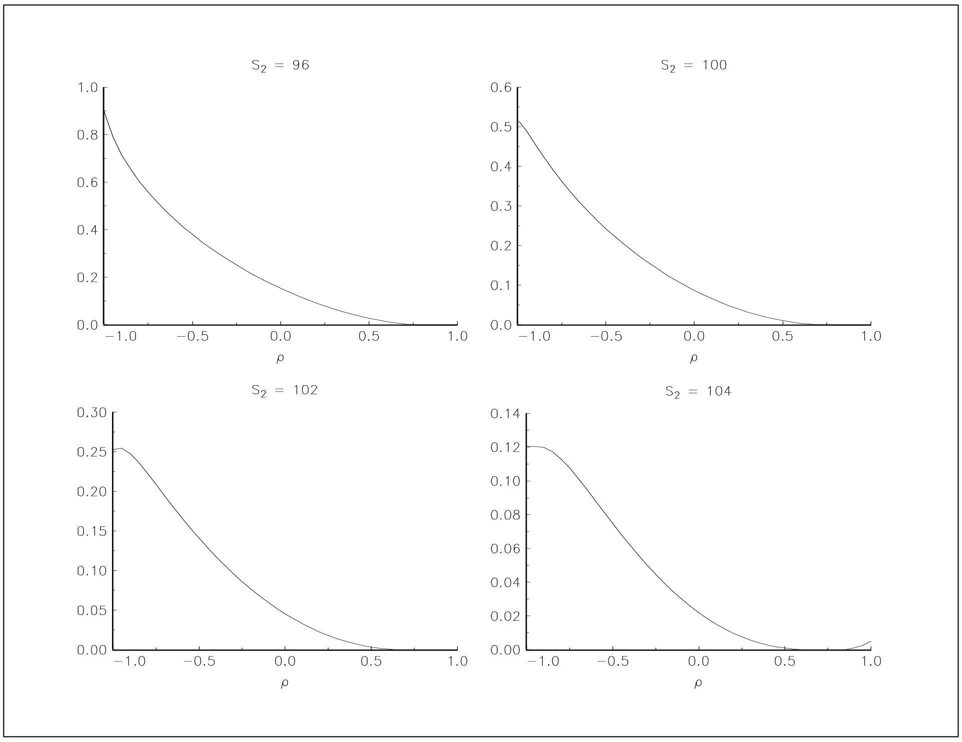

Figure 5: Relationship between the price of the WorstOf call/put option and the parameter ρ ($S_1 = 100$)

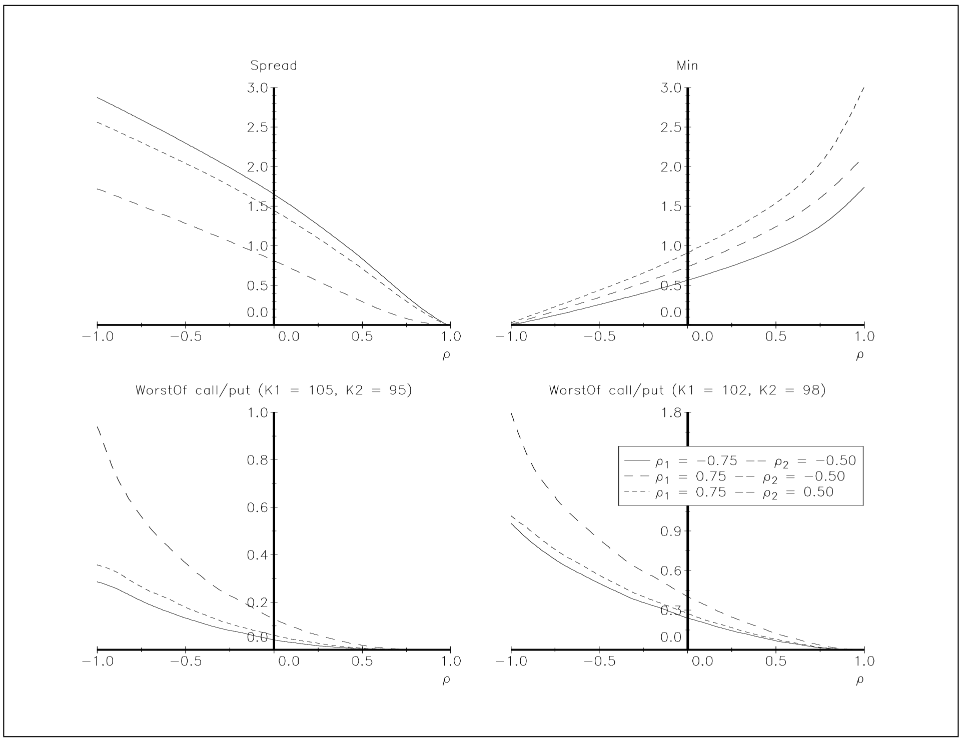

Figure 6: Relationship between the Heston prices and the parameter ρ

- TCHEN [1980] shows that if $\mathbb{E}_{\mathbf{C}_1}[f(X_1, X_2)] \geq \mathbb{E}_{\mathbf{C}_2}[f(X_1, X_2)]$ for all $(\mathbf{C}_1, \mathbf{C}_2) \in \mathcal{C}^2$ with $\mathbf{C}_1 \prec \mathbf{C}_2$ **then** f is supermodular. However, TCHEN [1980] establishes this property only for discrete margins. One may then wonder if $\partial^2_{1,2} G$ is neither positive nor negative implies that the price of the option is not monotone. This is an open problem. Moreover, one might wonder if the method using the concordance order can be generalized with more than two assets. MÜLLER and SCARSINI [2000] show that the supermodular order is strictly stronger than the concordance order for dimension bigger than three. So the method used for two-asset options cannot be generalized here. This is not surprising if we consider the example above: the condition about the sign of $\sum_{i<j} \sigma_i \sigma_j \partial^2_{i,j} G$ involves the values of the volatilities which are independent of the payoff function.

As a result, it is more difficult to define conservative price for multi-asset options. Understanding the relationship between the stochastic dependence and the price of equity structured products is then a challenge for both the front office and the risk management.

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A Proof of Proposition 3

We first prove that there exist two constants α and β such that $\left|P\left(\tilde{S}_1, \tilde{S}_2, t\right)\right| \leq \beta \exp\left(\alpha \left\|\tilde{S}_1, \tilde{S}_2\right\|\right)$. Following VILLENEUVE [1999], for all $M > 0$, there exists a constant C such that $\mathbb{E}\left[\sup_{t \geq s} \exp\left(M \left\|\tilde{S}_t^{s,(\tilde{s}_1, \tilde{s}_2)}\right\|\right)\right] \leq C \exp\left(M \left\|\tilde{s}_1, \tilde{s}_2\right\|\right)$ where $\tilde{S}_t^{s,(\tilde{s}_1, \tilde{s}_2)} = \left(\tilde{S}_1(t) \mid \tilde{S}_1(s) = \tilde{s}_1, \tilde{S}_2(t) \mid \tilde{S}_2(s) = \tilde{s}_2\right)$. By noting that $\left|P\left(\tilde{S}_1, \tilde{S}_2, T\right)\right| \leq \beta \exp\left(\alpha \left\|\tilde{S}_1, \tilde{S}_2\right\|\right)$, the majoration follows from the Feynman-Kac representation.

We can then use the maximum principle. This part of the proof is adapted from BERESTICKY [1999]. Let $-1 \leq \rho_1 < \rho_2 \leq 1$ and P_ρ be the solution of the PDE. We consider $\Delta\left(\tilde{S}_1, \tilde{S}_2, t\right) = P_{\rho_1}\left(\tilde{S}_1, \tilde{S}_2, t\right) - P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, t\right)$. It comes that Δ is the solution of the following PDE

$$\left\{\begin{array}{l} \mathcal{L}_{\rho_1} \Delta\left(\tilde{S}_1, \tilde{S}_2, t\right) = \left(\rho_2 - \rho_1\right) \sigma_1 \sigma_2 \partial_{1,2}^2 P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, t\right) \\ \Delta\left(\tilde{S}_1, \tilde{S}_2, T\right) = 0 \end{array}\right.$$

The weak maximum principle asserts that if $\mathcal{L}_{\rho_1} \Delta \leq 0$ for $\left(\tilde{S}_1, \tilde{S}_2, t\right) \in \mathbb{R}^2 \times [0, T)$ and if $\Delta\left(\tilde{S}_1, \tilde{S}_2, T\right) \geq 0$ for $\left(\tilde{S}_1, \tilde{S}_2\right) \in \mathbb{R}^2$, then $\Delta\left(\tilde{S}_1, \tilde{S}_2, t\right) \geq 0$ for $\left(\tilde{S}_1, \tilde{S}_2, t\right) \in \mathbb{R}^2 \times [0, T]$. We assume that the solution is smooth (say C^∞ in the domain where $t < T$). We can differentiate with respect to $\tilde{S}_1$ and $\tilde{S}_2$ the equation $\mathcal{L}_{\rho_2} P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, t\right) = 0$ and we get $\mathcal{L}_{\rho_2} \partial_{1,2}^2 P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, t\right) = 0$. For the terminal condition, we use a convolution product with an identity approximation because the payoff is not smooth. Let $\theta\left(x_1, x_2\right)$ be a positive function $C^\infty(\mathbb{R}^2)$ with its support in $B(0, 1)$ satisfying $\iint_{B(0,1)} \theta\left(x_1, x_2\right) \mathrm{d}x_1 \mathrm{d}x_2 = 1$. We define $\theta_m\left(x_1, x_2\right) = m^{-2} \theta\left(m^{-1} x_1, m^{-1} x_2\right)$. We consider now $\psi_m\left(\tilde{S}_1, \tilde{S}_2, t\right) = \left(\partial_{1,2}^2 P_{\rho_2} * \theta_m\right)\left(\tilde{S}_1, \tilde{S}_2, t\right) = \left(\partial_{1,2}^2 \theta_m * P_{\rho_2}\right)\left(\tilde{S}_1, \tilde{S}_2, t\right)$. We know that ψ_m is C^∞. By using the properties of the convolution product, we get $\mathcal{L}_{\rho_2} \psi_m\left(\tilde{S}_1, \tilde{S}_2, t\right) = 0$. We just have to prove that $\psi_m\left(\tilde{S}_1, \tilde{S}_2, T\right) \leq 0$ and $\left|\psi_m\left(\tilde{S}_1, \tilde{S}_2, t\right)\right| \leq \beta \exp\left(\alpha \left\|\tilde{S}_1, \tilde{S}_2\right\|\right)$.

The first step can be done by calculating $\partial_{1,2}^2 P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, T\right)$ using the jump formula and the relationship $\partial_{1,2}^2 P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, T\right) = S_1 S_2 \partial_{1,2}^2 P_{\rho_2}\left(S_1, S_2, T\right)$. We get also $\partial_{1,2}^2 P_{\rho_2}\left(S_1, S_2, T\right) = -\delta_{\{S_2 - S_1 - K = 0\}}$ where δ is the dirac measure. Because $\partial_{1,2}^2 P_{\rho_2}$ is a nonpositive measure, it comes that $\psi_m\left(\tilde{S}_1, \tilde{S}_2, T\right) \leq 0$.

To show the majoration, we remark that the support of the function $\partial_{1,2}^2 \theta_m$ is included in $B(0, R)$ for some constant R and that there exists a constant M such that $\left|\partial_{1,2}^2 \theta_m\left(\tilde{S}_1, \tilde{S}_2\right)\right| \leq M$. It comes that

$$\begin{aligned} \left|\psi_m\left(\tilde{S}_1, \tilde{S}_2, t\right)\right| &= \left|\iint_{B(0,R)} \partial_{1,2}^2 \theta_m\left(x_1, x_2\right) P_{\rho_2}\left(\tilde{S}_1 - x_1, \tilde{S}_2 - x_2, t\right) \mathrm{d}x_1 \mathrm{d}x_2\right| \\ &\leq \iint_{B(0,R)} M \left|P_{\rho_2}\left(\tilde{S}_1 - x_1, \tilde{S}_2 - x_2, t\right)\right| \mathrm{d}x_1 \mathrm{d}x_2 \\ &\leq M \iint_{B(0,R)} \beta \exp\left(\alpha \left\|\tilde{S}_1 - x_1, \tilde{S}_2 - x_2\right\|\right) \mathrm{d}x_1 \mathrm{d}x_2 \\ &\leq M \iint_{B(0,R)} \beta \exp\left(\alpha \left\|\tilde{S}_1, \tilde{S}_2\right\| + \alpha \left\|x_1, x_2\right\|\right) \mathrm{d}x_1 \mathrm{d}x_2 \\ &\leq \beta' \exp\left(\alpha' \left\|\tilde{S}_1, \tilde{S}_2\right\|\right) \end{aligned} \tag{25}$$

and α' and β' do not depend on time. So, $\psi_m\left(\tilde{S}_1, \tilde{S}_2, t\right) \leq 0$ because of the maximum principle. Moreover, $\psi_m\left(\tilde{S}_1, \tilde{S}_2, t\right)$ converges pointwise to $\partial^2_{1,2} P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, t\right)$ because $\partial^2_{1,2} P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, t\right)$ is continuous for $t < T$. We finally obtain that $\mathcal{L}_{\rho_1} \Delta\left(\tilde{S}_1, \tilde{S}_2, t\right) \leq 0$ because $\partial^2_{1,2} P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, t\right) \leq 0$. This completes the proof.

B Proof of Proposition 9

We just have to remark that when $\partial^2_{1,2} G$ exists and is continous, we have $\Delta^{(2)} G = \int_{S_2}^{S_2+\varepsilon_2} \int_{S_1}^{S_1+\varepsilon_1} \partial^2_{1,2} G \geq 0$. When $\partial^2_{1,2} G$ is a distribution, we use the same method than in Appendix A. We consider the same kernel θ_m so that $G * \theta_m$ is smooth enough. We get $\Delta^{(2)}\left(G * \theta_m\right) = \int_{S_2}^{S_2+\varepsilon_2} \int_{S_1}^{S_1+\varepsilon_1} \partial^2_{1,2}\left(G * \theta_m\right) = \int_{S_2}^{S_2+\varepsilon_2} \int_{S_1}^{S_1+\varepsilon_1} \left(\partial^2_{1,2} G\right) * \theta_m \geq 0$. When m tends to ∞, we obtain $\Delta^{(2)}\left(G * \theta_m\right) \to \Delta^{(2)} G \geq 0$ because G is continuous. So the function G is supermodular. Using the proposition 8, we get the first result.

If the distribution $\partial^2_{1,2} G$ is a nonpositive measure, $-G$ is supermodular and we have directly the result.