Modelling Dependence for Credit Derivatives with Copulae*

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*A former paper and these slides can be found on http://gro.creditlyonnais.fr.

Agenda

- The One-Credit Intensity Framework
- Copulae In A Nutshell
- How To Introduce Dependence in Intensity Models: A Copula Method.
- Pricing Credit Derivatives
- The Calibration Problem
- Hedging Credit Derivatives

1 The One-Credit Intensity Framework

This is an alternative to Merton's structural model. In Merton's model the default occurs when the stock price of the firm falls below a pre-specified deterministic threshold (debt of the firm). But the default time is then predictable.

Characteristics of the intensity model (Duffie, Lando):

- the Intensity model allows to add some randomness to the default threshold, in such a way that the default occurs as a complete surprise.
- this model loses the micro-economic interpretation of the default time (the model comes from reliability theory), but we do not care for the purpose of pricing.

1.1 Construction Of The Default Time

The **default time** of a firm is often defined by

$$\tau_1 := \inf\left\{t : \int_0^t \lambda_1(s)\,\mathrm{d}s \geq \theta_1\right\}, \qquad \theta_1 \perp\!\!\!\perp \mathcal{F}_\infty$$

- λ_1 a nonnegative, continuous, $\mathcal{F}$-adapted process called the **intensity process**. It contains the information on the credit quality of firm 1. Here, for simplicity, we will suppose it to be deterministic in the examples.
- θ_1 is a random threshold (an exponential r.v. of parameter 1), independent of the intensity.
- we assume the **recovery rate** R_1 is deterministic.

1.2 Some Properties Of The Model

- We have $\mathbb{P}(\tau_1 \leq s \mid \mathcal{F}_\infty) = \mathbb{P}(\tau_1 \leq s \mid \mathcal{F}_t)\,, s \leq t$, whence $\mathcal{F}_\infty \perp\!\!\!\perp \mathcal{G}_t | \mathcal{F}_t$, and the (H)-hypothesis (i.e. $\mathcal{F}$-martingales remain $\mathcal{G}$-martingales) and it is the more general model having this property – provided some continuity assumption (El Karoui, Jeanblanc).
- The Process $(\mathbf{1}_{\tau_1 \leq t})_{t \geq 0}$ is Markov.
- In some sense, we can identify the intensity process λ_1 and the instantaneous spread of firm 1 (until time of default).

1.3 Pricing Default Zero-Coupons

The zero-coupon of firm 1 is given by (when $\tau_1 > t$)

$$B_1(t,T) = \mathbb{E}\left[e^{-\int_t^T (r(s)+\lambda_1(s))\,\mathrm{d}s} \mid \mathcal{F}_t \right]$$

If we choose a deterministic intensity, we can calibrate it on Credit Default Swaps market prices. When the term structure is flat $s_1(T) = s_1$ (CDS prices are rather scarce!), a good approximation of the intensity is:

$$\lambda_1 = \frac{s_1}{1 - R_1}$$

When there is only one credit, we can identify the intensity process with the spread of the firm.

1.4 Multi-Credit Extensions

When dealing with more than one firm, there are many ways to incorporate dependence in the model. Example with two firms:

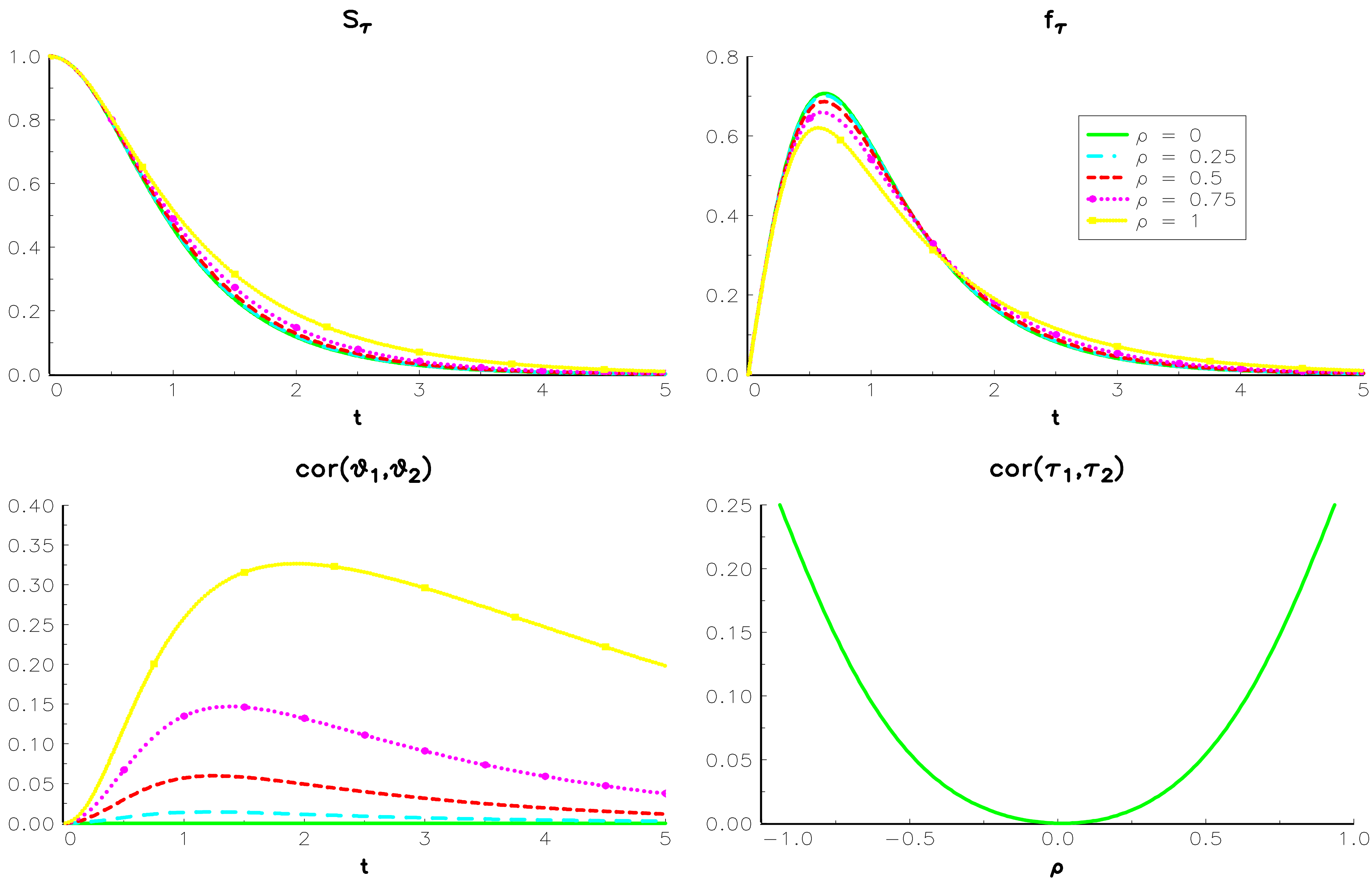
$$
\begin{array}{ccc}
\theta_1 & \overset{\perp\!\!\!\perp}{\longrightarrow} & \lambda_t^1, r_t \\
(2)\big\downarrow & & \big\downarrow(1) \\
\theta_2 & \underset{\perp\!\!\!\perp}{\longrightarrow} & \lambda_t^2, r_t
\end{array}
$$

- (1) correlating the intensity (stochastic) processes, but this method provides low correlations between the default times,
- (2) correlating the random thresholds with a survival copula $\breve{C}^{\theta}$ (Schönbucher and Schubert's approach, 2001),
- (3) a more intricate way: λ_2 may be correlated with θ_1 (Jarrow and Yu, 2001).

Example of (1): correlating the intensity processes.

- We choose for the intensities two Cox-Ingersoll-Ross processes driven by correlated Brownian motions (on the graphics, we choose two squared Brownian motions for simplicity's sake).
- We also draw two independent random thresholds.

When the correlation parameter ρ ranges from -1 to $+1$ the output correlation between default times is less than 25 %.



Influence of the correlation parameter on the first default time

2 Copulae In A Nutshell

Definition: a **copula** is the joint probability of any I-dimensional vector of uniform r.v. $(U_1, \ldots, U_I)$,

$$C^U(u_1, \ldots, u_I) := \mathbb{P}(U_1 \leq u_1, \ldots, U_I \leq u_I).$$

Example : the independent copula: $C^{\perp}(u_1, \ldots, u_I) = u_1 \ldots u_I$.

Now let $X = (X_1, \ldots, X_I)$ be any I-dimensional random variable.

Key idea: Copulae are used to split the margins of X and the dependence of the joint distribution.

2.1 Sklar's Representation Lemma

We denote for the marginal and joint distributions of X:

$$\begin{aligned} F(x_1, \ldots, x_I) &:= \mathbb{P}(X_1 \leq x_1, \ldots, X_I \leq x_I), \\ F_i(x_i) &:= \mathbb{P}(X_i \leq x_i), \quad i = 1, \ldots, I. \end{aligned}$$

As $F_i(X_i)$ are uniform r.v., they admit a copula, which we call the copula of X and write C_X. And we get Sklar's representation:

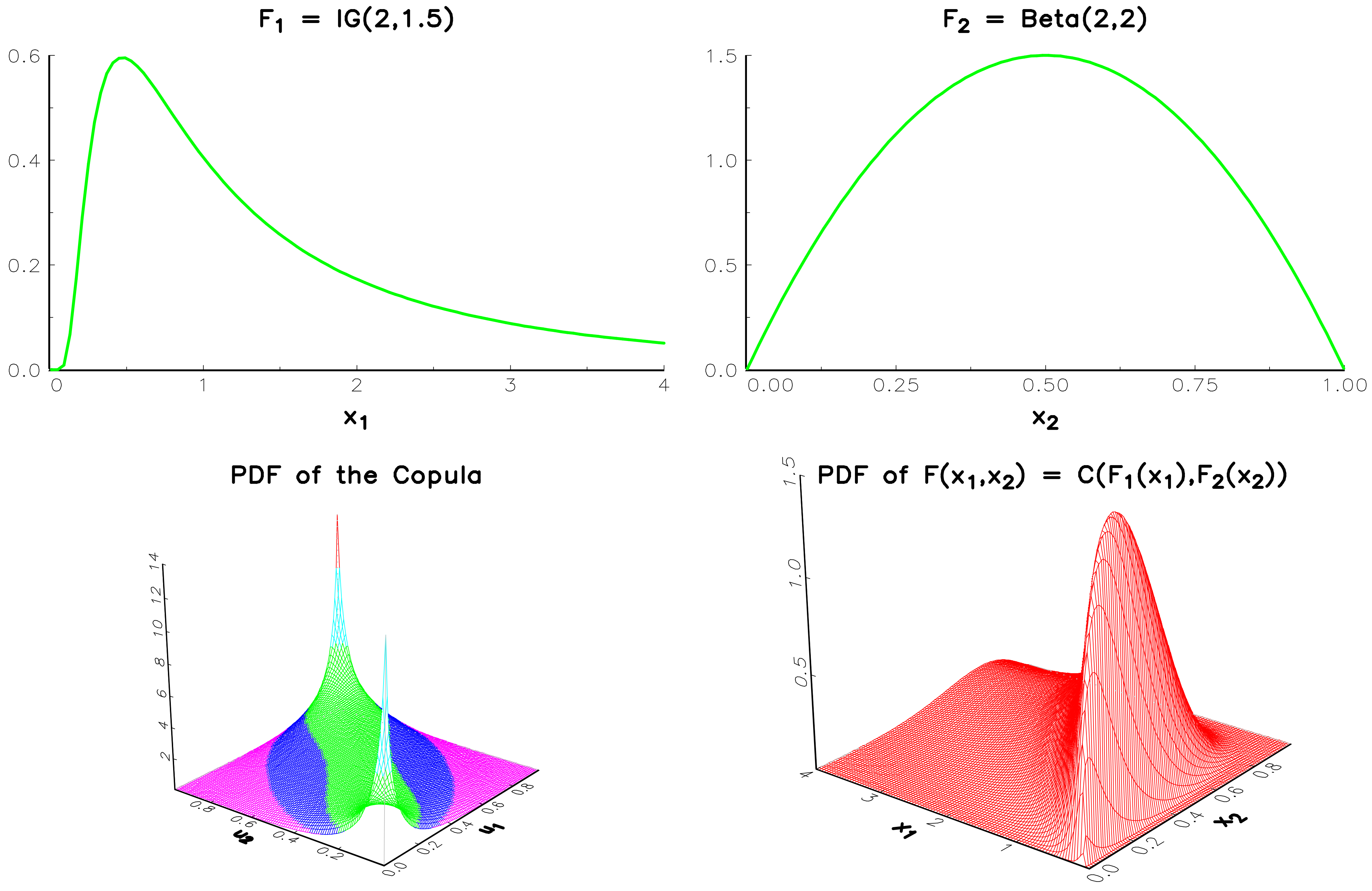
$$F(x_1, \ldots, x_I) = C_X \left(F_1(x_1), \ldots, F_I(x_I) \right).$$

Sometimes it is more convenient to use the joint (S) and marginal (S_i) survival distributions of X, so we can define $\bar{C}_X$, the survival copula of X with

$$S(x_1, \ldots, x_I) = \bar{C}_X \left(S_1(x_1), \ldots, S_I(x_I) \right).$$

How to use this result ?

- We can extract copulae from well known multi-variate distributions (e.g. the Gaussian, Student copula families).
- We can create new multi-variate distributions by joining arbitrary margins together with some given copulae.



Bivariate distribution with given marginals

2.2 The Fréchet Bounds

We have an inequality which generalizes $-1 \leq \rho \leq +1$ (where ρ is the linear Gaussian correlation) to copulae:

$$
\begin{aligned}
C^{+}(u_1, \ldots, u_I) &= \min(u_1, \ldots, u_I) \\
C^{-}(u_1, \ldots, u_I) &= \left(\sum_{i=1}^{I} u_i - I + 1 \right)^{+}.
\end{aligned}
$$

Then for any copula C we have:

$$
C^{-}(u_1, \ldots, u_I) \leq C(u_1, \ldots, u_I) \leq C^{+}(u_1, \ldots, u_I)
$$

3 A Copula Multi-Default Model

Common philosophy of all copula models for credit risk:

- Provide a simple extension of the single-credit framework.
- Split the calibration of the margins and the dependence.

Common shortcomings:

- One must choose an arbitrary copula family (dependency structure of the default times).
- The calibration of the dependence is not easy.

3.1 The Threshold Approach

Here, following Gesiecke, Schönbucher and Schubert we put a copula $\bar{C}_\theta$ directly on the random thresholds θ_i (and keep the same construction of default times).

One has to be cautious with this modelling:

- Keep in mind that the thresholds θ_i are not directly observable market variables, whence the threshold survival copula $\bar{C}_\theta$ has a priori no economic interpretation.
- The model is not Markov.

3.2 Construction Of The Default Times

The **default time** of a firm is often defined by, for any $i \in [[1, I]]$

$$\tau_i := \inf\left\{t : \int_0^t \lambda_i(s)\,\mathrm{d}s \geq \theta_i\right\}.$$

- λ^i are non-negative, continuous, $\mathcal{F}$-adapted processes called the **'intensity processes'**.
- $\theta_i \sim \mathcal{E}(1) \qquad \forall i = 1, \dots, I;$
- $(\theta_1, \dots, \theta_I)$ has a survival copula $\breve{C}_\theta$;
- $(\theta_1, \dots, \theta_I) \perp\!\!\!\perp \mathcal{F}_\infty$
- we assume the **recovery rates** R_i are deterministic.

3.3 Mathematical Properties

As usual, we consider the three following filtrations:

- the default-free filtration $(\mathcal{F}_t)_{t\geq 0}$;
- the filtration generated by the defaults $(\mathcal{H}_t)_{t\geq 0} = (\sigma(\tau_i \wedge s, s \leq t, i = 1, \dots, I))_{t\geq 0}$;
- the market filtration (made right-continuous if necessary) $(\mathcal{G}_t)_{t\geq 0} = (\mathcal{F}_t \vee \mathcal{H}_t)_{t\geq 0}$

We have for all $t \geq 0$ and $s_i \in [0, t]\,, i = 1, \dots, I$.

$$\mathbb{P}\left(\tau_1 \leq s_1, \dots, \tau_I \leq s_I \mid \mathcal{F}_\infty\right) = \mathbb{P}\left(\tau_1 \leq s_1, \dots, \tau_I \leq s_I \mid \mathcal{F}_t\right)$$

Therefore we get as in the one-firm case : $\mathcal{F}_\infty \perp\!\!\!\perp \mathcal{G}_t | \mathcal{F}_t$, whence we get the (H)-hypothesis. But here we do not know if the converse statement holds.

3.4 The Survival Approach

Li's approach is a special case of the threshold model useful when 'intensities' are deterministic. It is the case that is mostly used in practical applications.

We define the random default times as if they were independent:

$$\tau_i := \inf\left\{t \geq 0 : \int_0^t \lambda_s^i \, \mathrm{d}s \geq \theta_i\right\}, \qquad i = 1, \ldots, I.$$

Now, using Sklar's lemma, S_τ has a copula representation

$$S_\tau(t_1, \ldots, t_I) = \bar{C}_\tau \left(S_1(t_1), \ldots, S_I(t_I)\right).$$

Comparing with the survival function obtained in the threshold model, we get $\bar{C}_\tau = \bar{C}_\theta$, so what we choose is directly the copula of the defaults.

4 Pricing Basket Credit Derivatives

We are only concerned here with default-linked credit derivatives written on a basket of companies.

Two main methods of computing the price of a credit derivative :

- Closed Formulae (CDS, F2D, N2D),
- Monte-Carlo simulations (N2D, CDO)

The choice of the methodology has an impact on the choice of the copula: Gaussian or Student copulae are easy to simulate but not so tractable in closed formulae.

4.1 Pricing Default Zero-Coupons (1)

We can derive a pricing formula for firm 1's zero-coupon of maturity T at time t, as long as no firm has defaulted $-\ \tau_i > t, \quad i = 1, \ldots, I\ -$,

$$B_1(t,T) = \mathbb{E}\left[e^{-\int_t^T r_s \,\mathrm{d}s} \frac{\bar{C}_\theta\left(e^{-\int_0^T \lambda_1(s)\,\mathrm{d}s}, \ldots, e^{-\int_0^t \lambda_I(s)\,\mathrm{d}s}\right)}{\bar{C}_\theta\left(e^{-\int_0^t \lambda_1(s)\,\mathrm{d}s}, \ldots, e^{-\int_0^t \lambda_I(s)\,\mathrm{d}s}\right)} \mid \mathcal{F}_t \right]$$

- We notice that firm j's ($j \neq 1$) `intensities' intervene in the pricing of firm 1's zero-coupon (in particular, default of any firm changes firm 1's pricing formula).
- When $\bar{C}_\theta = \bar{C}^\perp$, we retrieve the usual formula.

4.2 Pricing Default Zero-Coupons (2)

When firms $k, \ldots, I$ have defaulted the price of firm 1's zero-coupon becomes (for $\tau_1 > t, \ldots, \tau_{k-1} > t, \tau_k \leq t, \ldots, \tau_I \leq t$):

$$B_1(t,T) = \mathbb{E}\left[e^{-\int_t^T r_s \,\mathrm{d}s} \frac{\partial_{k,\ldots,I} \bar{C}_\theta \left(e^{-\int_0^T \lambda_1(s)\,\mathrm{d}s}, \ldots, e^{-\int_0^{\tau_I} \lambda_I(s)\,\mathrm{d}s} \right)}{\partial_{k,\ldots,I} \bar{C}_\theta \left(e^{-\int_0^t \lambda_1(s)\,\mathrm{d}s}, \ldots, e^{-\int_0^{\tau_I} \lambda_I(s)\,\mathrm{d}s} \right)} \mid \mathcal{F}_t \right]$$

So we observe a jump of the price of zero-coupon of firm 1 when some of the other firms defaults, which corresponds to a jump of the spread of firm 1.

Important: in this model one cannot identify the 'intensity' with the spread at time $t > 0$.

4.3 Implied Dynamics of Spreads

At each time t, we can find the spread (forward CDS rate) of firm 1 for a given maturity δ:

$$\begin{aligned} s_1(t, t+\delta) &= (1-R_1)\frac{\int_t^{t+\delta} B_0(0,u)\mathbb{P}(\tau_1 \in \mathrm{d}u \mid \mathcal{G}_t)}{\int_t^{t+\delta} \mathrm{d}u B_0(0,u)\mathbb{P}(\tau_1 \geq u \mid \mathcal{G}_t)}, \\ s_1(t,t) &= \lim_{\delta \to 0} s_1(t, t+\delta) = (1-R_1)\mathbb{P}(\tau_1 \in \mathrm{d}t \mid \mathcal{G}_t). \end{aligned}$$

The instantaneous spread is related to the 'density of the compensator' of the process $\mathbf{1}_{\tau_1 \leq t}$.

We observe that even in the case of deterministic 'intensities', spreads are decreasing between default times and suffer from jumps at each time of default.

4.4 An Example with Nth-To-Default Contracts

We choose a Normal copula and we price first- and Nth-to-default contracts for different values of the (unique) correlation parameter.

We choose two baskets of $I = 4$ credits with the following characteristics ($R = 50\%$). Basket 1 is homogeneous but it is not the case for basket 2.

credit	basket1	basket2
1	100 bp	50 bp
2	100 bp	100 bp
3	100 bp	100 bp
4	100 bp	150 bp

We give here some approximation formulae for the margin of the Nth to default (N = 1...4). We note $s_1, \dots, s_4$ the spreads of the firms and $m_1, \dots, m_4$ the fair margins of the first-, ..., fourth-to-default contract.

In case of the independent copula, $\breve{\mathbf{C}}^{\tau} = \breve{\mathbf{C}}^{\perp}$, we have:

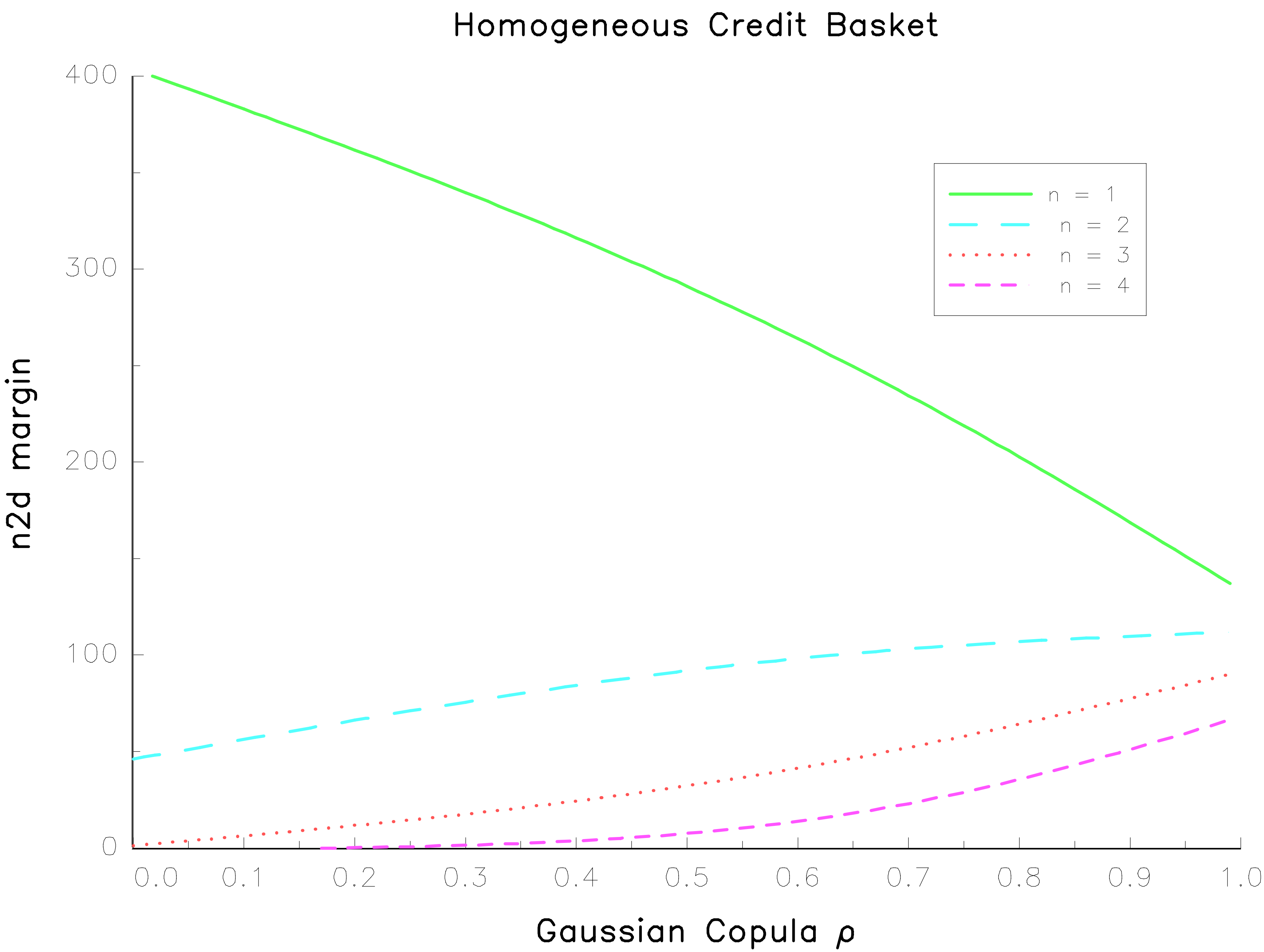
$$m_1 \approx \sum_{i=1}^{4} s_i \qquad m_2 \approx m_3 \approx m_4 \approx 0.$$

In case of the upper Fréchet copula, $\breve{\mathbf{C}}_{\tau} = \breve{\mathbf{C}}^{+}$, we have:

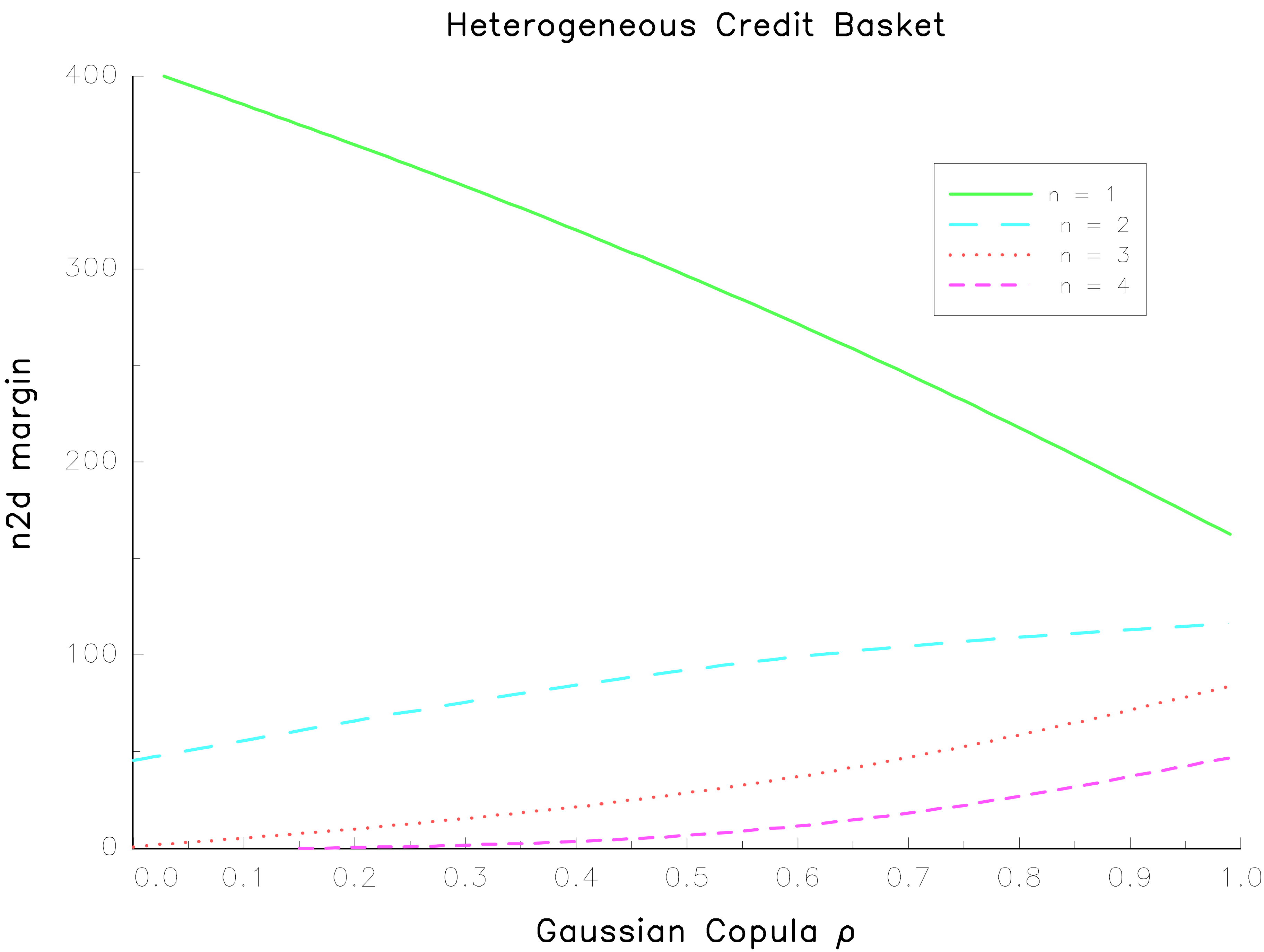
$$m_1 \approx s_{\sigma(1)}, \dots, m_4 \approx s_{\sigma(4)}.$$

where we have sorted the corresponding intensities

$$\frac{s_{\sigma(1)}}{1 - R_{\sigma(1)}} \geq \dots \geq \frac{s_{\sigma(4)}}{1 - R_{\sigma(4)}}$$



Homogeneous Credit Basket
n2d margin
Gaussian Copula ρ
n = 1
n = 2
n = 3
n = 4
400
300
200
100
0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0



Heterogeneous Credit Basket
400
300
200
100
0
n2d margin
n = 1
n = 2
n = 3
n = 4
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Gaussian Copula ρ

4.5 The Problem of Repricing

At time $t > 0$ when repricing the product in a Monte-Carlo methodology, it may be tempting to price a new product of maturity $T - t$ and draw new default times with the same dependence function.

But in this way, we would overlook the fact that the process $(\mathbf{1}_{\tau_1 \le t}, \ldots, \mathbf{1}_{\tau_I \le t})$ is **not Markov**.

The model imposes a shape of dependence at time 0, but a priori we can say nothing about the (conditional) dependence at time $t > 0$ given the information $\mathcal{G}_t$.

4.6 Is There Any Stationary Copula ?

To overcome the lack of Markov property, we look for a copula family $\bar{C}_\rho$ such that the survival copula of the conditional distribution at any time $t > 0$ belongs to the same family $\bar{C}_\kappa$ with a different parameter, so we need for all $T_i > t$, when no default has occurred

$$\mathbb{P}\left(\tau_1 > T_1, \ldots, \tau_I > T_I \mid \mathcal{G}_t\right) = \bar{C}_\kappa\left(\mathbb{P}\left(\tau_1 > T_1 \mid \mathcal{G}_t\right), \ldots, \mathbb{P}\left(\tau_I > T_I \mid \mathcal{G}_t\right)\right).$$

This leads to a functional equation in $\bar{C}_\rho$: $\forall \alpha, u \in [0,1]^I$,

$$\frac{\bar{C}_\rho(\alpha_1 u_1, \ldots, \alpha_I u_I)}{\bar{C}_\rho(\alpha_1, \ldots, \alpha_I)} = \bar{C}_{\kappa(\alpha,\rho)}\left(\frac{\bar{C}_\rho(\alpha_1 u_1, \ldots, \alpha_I)}{\bar{C}_\rho(\alpha_1, \ldots, \alpha_I)}, \ldots, \frac{\bar{C}_\rho(\alpha_1, \ldots, \alpha_I u_I)}{\bar{C}_\rho(\alpha_1, \ldots, \alpha_I)}\right),$$

with $\kappa(\alpha, \rho)$ some unknown function.

In case of $I = 2$, one solution is Gumbel-Barnett's family

$$\bar{C}_\rho(u_1, u_2) = u_1 u_2 e^{-\rho \log(u_1) \log(u_2)}, \quad \kappa(\alpha, \rho) = \rho/(1-\rho \log \alpha_1)(1-\rho \log \alpha_2).$$

5 The Calibration Problem

Description of a theoretical calibration procedure and why it cannot be carried out:

- Calibrating each firm's individual spread curve with Today's Credit Default Swaps Prices.
- Choosing a copula family (this is constrained by the use of Monte-Carlo simulations).
- Calibrating the parameter of the copula (e.g. the correlation in case of Gaussian dependence) with the prices of First-to-default.

First-to-default market is too much illiquid to perform such a true calibration.

5.1 Calibrating with Spread Jumps

As it is impossible to perform a calibration or a statistical estimation of the correlation between default times, we turn to another procedure.

Schönbucher and Schubert suggest to observe spread jumps to find the correlation parameter.

It is not so easy to use this method in daily practical applications.

6 Hedging Credit Derivatives

Identification of the risk factors:

- Spread movements
- Events of default

In the case of basket products contingent to occurrence of defaults, we consider that the second risk factor is the most important to be hedged against (the only one in a model with deterministic ̀intensities').

Moreover, we are constrained by the fact that only CDS are available to hedge both risks.

6.1 Mathematical Framework

For the hedging problem, we look at the model as a particular case of a multi-variate point process:

- We observe the successive times of defaults: $0 = T_0 < T_1 < \ldots < T_I$ with T_i the time of the i^{th} default;
- At each time of default T_i, we mark the name of the defaulting company X_i.

Then we map the model into a random measure model.

$$\mu(\omega, dt, dx) = \sum_{i=1}^{I} \delta_{T_i(\omega), X_i(\omega)}(dt, dx) \mathbf{1}_{T_i(\omega) < \infty}$$

Now we get the existence of a predictable representation of martingales – in the filtration $(\mathcal{F}_\infty \vee \mathcal{H}_t)_{t\geq 0}$ – from Jacod (1974).

We compute the conditional distribution $G_i(dt, dx)$ of T_{i+1} given $\mathcal{F}_\infty \vee \mathcal{H}_{T_i}$ and the following compensation measure in terms of intensities and copula

$$\nu(dt, dx) = \sum_{i=0}^{I} \frac{G_i(dt, dx)}{G_i(]t, \infty] \times [[1, I]])} \mathbf{1}_{T_i < t < T_{i+1}}$$

Now we have for every martingale $(M_t)_{t\geq 0}$

$$M_t = M_0 + \int_0^t \int_{[[1,I]]} \phi(s, x) \left(\mu(ds, dx) - \nu(ds, dx)\right) = M_0 + \sum_{i=1}^{I} \int_0^t \phi(s, i) dM_i(s)$$

In the case where intensities are deterministic, it gives the representation result.

6.2 What CDS Portfolio Should We Choose ?

In case of deterministic intensities we hedge with a portfolio of CDS. What quantity of each name should we buy/sell ?

From Jacod's representation theorem, we get, since each martingale M_i has one single jump of size 1 at time τ_i:

$$\Delta M_{T_i} = \phi(T_i, X_i).$$

The representation process with respect to firm i is thus the jump of the martingale in case of instantaneous default of firm i.

From this formula, as an approximation of the hedging strategy in the multi-firm model, we compute the representation processes both for the derivative we want to hedge and for the CDS, and get the hedging strategy through a linear system resolution.

Again, we have to be cautious with the computation of conditional expectations.