Introducing Expected Returns into Risk Parity Portfolios: A New Framework for Asset Allocation*

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Abstract

Risk parity is an allocation method used to build diversified portfolios that does not rely on any assumptions of expected returns, thus placing risk management at the heart of the strategy. This explains why risk parity became a popular investment model after the global financial crisis in 2008. However, risk parity has also been criticized because it focuses on managing risk concentration rather than portfolio performance, and is therefore seen as being closer to passive management than active management. In this article, we show how to introduce assumptions of expected returns into risk parity portfolios. To do this, we consider a generalized risk measure that takes into account both the portfolio return and volatility. However, the trade-off between performance and volatility contributions creates some difficulty, while the risk budgeting problem must be clearly defined. After deriving the theoretical properties of such risk budgeting portfolios, we apply this new model to asset allocation. First, we consider long-term investment policy and the determination of strategic asset allocation. We then consider dynamic allocation and show how to build risk parity funds that depend on expected returns.

Keywords: Risk parity, risk budgeting, expected returns, ERC portfolio, value-at-risk, expected shortfall, active management, tactical asset allocation, strategic asset allocation.

JEL classification: G11.

1 Introduction

Although portfolio management did not change much in the 40 years following the seminal works of Markowitz and Sharpe, the development of risk budgeting techniques marked an important milestone in the deepening of the relationship between risk and asset management. Risk parity subsequently became a popular financial model for investment after the global financial crisis in 2008. Today, pension funds and institutional investors are using this

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approach in the development of smart beta and the redefinition of long-term investment policies (Roncalli, 2013).

In a risk budgeting (RB) portfolio, ex-ante risk contributions are equal to some given risk budgets. Generally, allocation is carried out by taking a volatility risk measure into account. It simplifies the computation, especially when a large number of assets are involved. However, the volatility risk measure has been criticized because it assumes that asset returns are normally distributed (Boudt *et al.*, 2013). There are now different approaches to extending the risk budgeting method by considering non-normal asset returns. However, in our view, these extensions do not generally produce better results and present a number of computational problems when implemented for large asset universes.

A more interesting extension is the introduction of expected returns into the risk budgeting approach. Risk parity is generally presented as an allocation method unrelated to the Markowitz approach. Most of the time, these are opposed, because risk parity does not depend on expected returns. This is the strength of such an approach. In particular, with an equal risk contribution (ERC) portfolio, the risk budgets are the same for all assets (Maillard *et al.*, 2010). This may be interpreted as a neutral portfolio applied when the portfolio manager has no convictions. However, the risk parity approach has also been strongly criticized, because some investment professionals consider this aspect a weakness, with some active managers having subsequently reintroduced expected returns in an ad hoc manner. For instance, they modify the weights of the risk parity portfolio in a second step by applying the Black-Litterman model or optimizing the tracking error. A second solution consists in linking risk budgets to expected returns. In this paper, we propose a third route considering a generalized standard deviation-based risk measure, which encompasses the Gaussian value-at-risk and expected shortfall risk measures. It is often forgotten that these risk measures depend on the vector of expected returns. In this case, the risk contribution of an asset has two components: a performance contribution and a volatility contribution. A positive view on one asset will reduce its risk contribution and increase its allocation. However, unlike in the mean-variance framework, the RB portfolio obtained remains relatively diversified.

The introduction of expected returns into risk parity portfolios is particularly relevant in strategic asset allocation (SAA). SAA is the central component of a long-term investment policy. It relates to the portfolio of equities, bonds and alternative assets that the investor wishes to hold over a long period (typically 10 to 30 years). Risk parity portfolios based on the volatility risk measure define well-diversified strategic portfolios. The use of a standard deviation-based risk measure allows the risk premiums of the different asset classes to be taken into account. Risk parity may also be relevant in tactical asset allocation (TAA), where it may be viewed as an alternative method to the Black-Litterman model. Active managers may then naturally incorporate their bets into the RB portfolio and continue to benefit from the diversification. This framework has already been used by Martellini *et al.* (2014) to understand the behavior of risk parity funds with respect to economic environments. In particular, they show how to improve a risk parity strategy against a backdrop of interest rate rises.

This article is organized as follows. In section two, we introduce the theoretical framework, showing in particular how the objective function of mean-variance optimization as a risk measure can be interpreted. We then move on to define the risk contribution and describe how it relates to performance and volatility contributions. In section three, we explain the specification of the risk budgeting portfolio, demonstrating that the problem is more complex than for the volatility risk measure and has a unique solution under some restrictions. Examples are provided in the fourth section, in which we apply the RB approach

to strategic asset allocation. We also compare RB portfolios with optimized portfolios in the case of tactical asset allocation. Section five offers some concluding remarks.

2 The framework

2.1 Combining performance allocation and risk allocation

We consider a universe of n risky assets. Let μ and Σ be the vector of expected returns and the covariance matrix of asset returns. We have $\Sigma_{i,j} = \rho_{i,j} \sigma_i \sigma_j$ where σ_i is the volatility of asset i and $\rho_{i,j}$ is the correlation between asset i and asset j. The mean-variance optimization (MVO) model is the traditional method for optimizing performance and risk (Markowitz, 1952). This is generally achived by considering the following quadratic programming problem:

$$x^\star (\gamma) = \arg\min \frac{1}{2} x^\top \Sigma x - \gamma x^\top (\mu - r\mathbf{1})$$

where x is the vector of portfolio weights, γ is a parameter to control the investor's risk aversion and r is the return of the risk-free asset. Sometimes restrictions are imposed to reflect the constraints of the investor, for instance, we could impose $\mathbf{1}^\top x = 1$ and $x \geq \mathbf{0}$ for a long-only portfolio. This framework is particularly appealing because the objective function has a concrete financial interpretation in terms of utility functions, with the investor facing a trade-off between risk and performance. To obtain a better expected return, the investor must then choose a riskier portfolio.

However, the mean-variance framework has been hotly debated for some time (Michaud, 1989) and the stability of the MVO allocation is an open issue, even if some methods can regularize the optimized portfolio (Ledoit and Wolf, 2004; DeMiguel *et al.*, 2009). The problem is that the Markowitz optimization is a very aggressive model of active management, detecting arbitrage opportunities that are sometimes false and may result from noise data. The model then transforms these arbitrage opportunities into investment bets in an optimistic way without considering adverse scenarios. This problem is particularly relevant when the input parameters are historical estimates. In this case, the Markowitz optimization is equivalent to optimizing the in-the-sample backtest.

Despite the above drawback, the Markowitz model remains an excellent tool for combining performance allocation and risk allocation. Moreover, as noted by Roncalli (2013), "*there are no other serious and powerful models to take into account return forecasts*". The only other model that is extensively used in active management is the Black-Litterman model, but this may in fact be viewed as an extension of the Markowitz model. In both cases, the trade-off between return and risk is highlighted. Let $\mu(x) = x^\top \mu$ and $\sigma(x) = \sqrt{x^\top \Sigma x}$ be the expected return and the volatility of portfolio x. We also note $\pi(x) = \mu(x) - r$ its risk premium. It is obvious that the optimization problem can also be formulated as follows[1]:

$$x^\star (c) = \arg\min -\pi(x) + c \cdot \sigma(x)$$

Remark 1 *The Markowitz optimization problem can be interpreted as a risk minimization problem:*

$$x^\star (c) = \arg\min \mathcal{R}(x)$$

where $\mathcal{R}(x)$ is the risk measure defined as follows:

$$\mathcal{R}(x) = -\pi(x) + c \cdot \sigma(x)$$

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[1]The mapping between the solutions $x^\star (\gamma)$ and $x^\star (c)$ is given by the relationship: $c = \gamma^{-1} \sigma(x^\star (\gamma))$.

Notably, the Markowitz model is therefore equivalent to minimizing a risk measure that encompasses both the performance dimension and the risk dimension.

2.2 Interpretation of the Markowitz risk measure

The above analysis suggests that we can use the Markowitz risk measure in a risk parity model taking expected returns into account. Let $L(x)$ be the portfolio excess loss. We have $L(x) = -x^\top (R - r)$ where R is the random vector of returns. We consider the generalized standard deviation-based risk measure:

$$\begin{aligned} \mathcal{R}(x) &= \mathbb{E}[L(x)] + c \cdot \sigma(L(x)) \\ &= -(\mu(x) - r) + c \cdot \sigma(x) \end{aligned}$$

Assuming that asset returns are normally distributed: $R \sim \mathcal{N}(\mu, \Sigma)$, we have $\mu(x) = x^\top \mu$ and $\sigma(x) = \sqrt{x^\top \Sigma x}$. Let π be the vector of risk premiums[2]. It follows that:

$$\mathcal{R}(x) = -x^\top \pi + c \cdot \sqrt{x^\top \Sigma x}$$

The result is the Markowitz risk measure. This formulation encompasses two well-known risk measures:

- Gaussian value-at-risk:

$$\mathrm{VaR}_\alpha(x) = -x^\top \pi + \Phi^{-1}(\alpha)\sqrt{x^\top \Sigma x}$$

  In this case, the scaling factor c is equal to $\Phi^{-1}(\alpha)$.

- Gaussian expected shortfall:

$$\mathrm{ES}_\alpha(x) = -x^\top \pi + \frac{\sqrt{x^\top \Sigma x}}{(1 - \alpha)} \phi\left(\Phi^{-1}(\alpha)\right)$$

  Like the value-at-risk measure, this is a standard deviation-based risk measure where the scaling factor c is equal to $\phi\left(\Phi^{-1}(\alpha)\right)/(1 - \alpha)$.

We can deduce that the expression of the marginal risk is:

$$\mathcal{MR}_i = -\pi_i + c\frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}}$$

This means that:

$$\begin{aligned} \mathcal{RC}_i &= x_i \cdot \left(-\pi_i + c\frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}}\right) \\ &= -x_i \pi_i + c\frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \end{aligned}$$

This confirms that the standard deviation-based risk measure satisfies the Euler decomposition:

$$\mathcal{R}(x) = \sum_{i=1}^n \mathcal{RC}_i$$

This implies that it is a good option for a risk budgeting approach.

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[2]We have $\pi_i = \mu_i - r$.

2.3 Relationship between the risk contribution, performance contribution and volatility contribution

Here the risk contribution has two components. The first component is the opposite of the performance contribution $\pi_i(x)$, while the second component corresponds to the standard risk contribution $\sigma_i(x)$ based on the volatility risk measure. We can therefore reformulate $\mathcal{RC}_i$ as follows:

$$\mathcal{RC}_i = -\pi_i(x) + c\sigma_i(x)$$

with $\pi_i(x) = x_i\pi_i$ and $\sigma_i(x) = x_i \cdot (\Sigma x)_i / \sigma(x)$.

The normalized risk contribution of asset i can be defined as follows:

$$\mathcal{RC}_i^\star = \frac{-\pi_i(x) + c\sigma_i(x)}{\mathcal{R}(x)}$$

Similarly, the normalized performance (or excess return) contribution is:

$$\mathcal{PC}_i^\star = \frac{\pi_i(x)}{\pi(x)} = \frac{x_i\pi_i}{\sum_{j=1}^n x_j\pi_j}$$

while the volatility contribution is:

$$\mathcal{VC}_i^\star = \frac{\sigma_i(x)}{\sigma(x)} = \frac{x_i \cdot (\Sigma x)_i}{x^\top\Sigma x}$$

We thus obtain the following proposition.

Proposition 1 *The risk contribution of asset i is the weighted average of the excess return contribution and the volatility contribution:*

$$\mathcal{RC}_i^\star = (1 - \omega)\,\mathcal{PC}_i^\star + \omega\mathcal{VC}_i^\star \tag{1}$$

where the weight ω is:

$$\omega = \frac{c\sigma(x)}{-\pi(x) + c\sigma(x)} \tag{2}$$

The range of ω is $]-\infty, \infty[$. If $c = 0$, ω is equal to 0. ω is then a decreasing function with respect to c until the value $c^\star = (\mu(x) - r)/\sigma(x)$, which is the ex-ante Sharpe ratio $\mathrm{SR}(x \mid r)$ of the portfolio. If $c > c^\star$, ω is positive and tends towards one, when c tends towards ∞. We conclude that the risk contribution is a return-based (or volatility-based) contribution if c is lower (or higher) than the Sharpe ratio of the portfolio. The singularity around the Sharpe ratio implies that the value of c must be carefully calibrated.

3 Risk budgeting portfolios

Roncalli (2013) defines the RB portfolio using the following non-linear system:

$$\begin{cases} \mathcal{RC}_i(x) = b_i\mathcal{R}(x) \\ b_i > 0 \\ x_i \geq 0 \\ \sum_{i=1}^n b_i = 1 \\ \sum_{i=1}^n x_i = 1 \end{cases} \tag{3}$$

where b_i is the risk budget of asset i expressed in relative terms. The constraint $b_i > 0$ implies that we cannot set some risk budgets to zero. This restriction is necessary in order to ensure that the RB portfolio is unique (Roncalli, 2013).

3.1 Existence and uniqueness of the RB portfolio

When using a standard deviation-based risk measure, we have to impose a second restriction in order to ensure the existence of the RB portfolio:

$$\mathcal{R}(x) \geq 0 \tag{4}$$

If $\{0\} \in \operatorname{Im}\mathcal{R}(x)$, it implies that the risk measure can take both positive and negative values. We then face a singularity problem, meaning that there may be no solution to the system (3). The restriction (4) is equivalent to requiring the scaling factor c to be larger than the bound $c^\star$ defined as follows:

$$
\begin{aligned}
c^\star &= \mathrm{SR}^+ \\
&= \max\left(\sup_{x \in [0,1]^n} \mathrm{SR}(x \mid r), 0\right)
\end{aligned}
$$

3.1.1 The case $c > \mathrm{SR}^+$

Theorem 1 *If $c > \mathrm{SR}^+$, the RB portfolio exists and is unique. It is the solution of the following optimization program:*

$$
\begin{aligned}
x^\star(\kappa) &= \arg\min \mathcal{R}(x) \\
u.c. \quad &\left\{
\begin{array}{l}
\sum_{i=1}^n b_i \ln x_i \geq \kappa \\
\mathbf{1}^\top x = 1 \\
x \geq \mathbf{0}
\end{array}
\right.
\end{aligned}
\tag{5}
$$

where κ is a constant to be determined.

We will now consider a slight modification to the optimization program (5):

$$
\begin{aligned}
y^\star &= \arg\min \mathcal{R}(y) \\
u.c. \quad &\left\{
\begin{array}{l}
\sum_{i=1}^n b_i \ln y_i \geq \kappa \\
y \geq \mathbf{0}
\end{array}
\right.
\end{aligned}
\tag{6}
$$

with κ as an arbitrary constant. Following Roncalli (2013), the optimal solution satisfies the first-order conditions:

$$y_i \frac{\partial \mathcal{R}(y)}{\partial y_i} = \lambda b_i$$

where λ is the Lagrange coefficient associated with the constraint $\sum_{i=1}^n b_i \ln y_i \geq \kappa$. As we have a standard optimization problem (minimization of a convex function subject to convex bounds), it can be deduced that:

  (i) The solution to the problem (6) exists and is unique if the objective function $\mathcal{R}(y)$ is bounded from below.

  (ii) The solution to the problem (6) may not exist if the objective function $\mathcal{R}(y)$ is not bounded from below.

For the volatility risk measure, we have $\mathcal{R}(y) \geq 0$, meaning that the solution always exists (case *i*). The existence of a solution is more complex when we consider the standard deviation-based risk measure. Indeed, we may have:

$$\lim_{y \to \infty} \mathcal{R}(y) = -\infty$$

6

because the expected return component may be negative and may not be offset by the volatility component. In this instance, the solution may not exist (case *ii*). However, if we require that $c > \mathrm{SR}^+$, we obtain case *i* because $\mathcal{R}(y) \geq 0$. More generally, the solution exists if there is a constant $\mathcal{R}^-$ such that $\mathcal{R}(y) > \mathcal{R}^-$. In this case, the RB portfolio corresponds to the normalized optimal portfolio $y^\star$. We can therefore deduce that there is a constant κ such that the RB portfolio is the only solution to the problem (5).

We now understand why restriction $\mathcal{R}(x) \geq 0$ is important for defining the RB portfolio. Indeed, a coherent convex risk measure satisfies the homogeneity property:

$$\mathcal{R}(\lambda x) = \lambda \mathcal{R}(x)$$

where λ is a positive scalar. Suppose that there is a portfolio $x \in [0,1]^n$ such that $\mathcal{R}(x) < 0$. We can then leverage the portfolio by a scaling factor $\lambda > 1$, and we obtain $\mathcal{R}(\lambda x) < \mathcal{R}(x) < 0$. It follows that:

$$\lim_{\lambda \to \infty} \mathcal{R}(\lambda x) = -\infty$$

This is why it is necessary for the risk measure to always be positive.

There is an issue with Theorem 1, however, as we have to calculate the upper bound SR^+. To do this, we can explore all the portfolios in set $[0,1]^n$ and calculate the supremum of the Sharpe ratios. This approach may be time-consuming, in particular when the asset universe is large. It is better to use the following result:

Theorem 2 *Let x_{mr} be the minimum risk long-only portfolio defined as follows:*

$$
\begin{aligned}
x_{\mathrm{mr}} \quad &= \quad \arg\min \mathcal{R}(x) \\
u.c. \quad &\left\{ \begin{array}{l} \mathbf{1}^\top x = 1 \\ x \geq \mathbf{0} \end{array} \right.
\end{aligned}
$$

The RB portfolio exists and is unique if $\mathcal{R}(x_{\mathrm{mr}}) > 0$.

3.1.2   The case $c \leq \mathrm{SR}^+$

This case is not relevant from a financial point of view, because the risk measure of some portfolios may be negative. By leveraging these portfolios, the risk measure may be infinitely negative. From a mathematical point of view, we can however show these results:

- Make SR^- equal to $\max\left(\inf_{x \in [0,1]^n} \mathrm{SR}(x \mid r), 0\right)$. If $c < \mathrm{SR}^-$, the RB portfolio exists and is unique.

- If $c \in \left[\mathrm{SR}^-, \mathrm{SR}^+\right]$, the RB portfolio may not exist.

Example 1 *We consider four assets. Their volatilities are equal to 15%, 20%, 25% and 30% while the correlation matrix of asset returns is provided by the following matrix:*

$$
C = \begin{pmatrix}
1.00 & & & \\
0.10 & 1.00 & & \\
0.40 & 0.70 & 1.00 & \\
0.50 & 0.40 & 0.80 & 1.00
\end{pmatrix}
$$

We will calculate the ERC portfolio by considering different values for the scaling factor. The results are reported in Table 1. If the risk premium of each asset is set to 7%, the maximum Sharpe ratio is equal to 0.56. In this case, the ERC portfolio does not exist if c is equal to 0.40. However, there is a solution if c is equal to 1.00, 1.65 or 2.33. If we consider higher risk premiums ($\pi_i = 25\%$), the bounds become $\mathrm{SR}^- = 0.83$ and $\mathrm{SR}^+ = 1.99$. We can confirm that there is a solution if c is equal to 0.40 or 2.33. Curiously, there is also a solution if c is equal to 1.00, but not if c is equal to 1.65.

Table 1: ERC portfolios for different values of c

c	$\pi_i = 7\%$				$\pi_i = 25\%$			
	0.40	1.00	$\Phi^{-1}(0.95)$	$\Phi^{-1}(0.99)$	0.40	1.00	$\Phi^{-1}(0.95)$	$\Phi^{-1}(0.99)$
	$\mathrm{SR}^- = 0.23$ / $\mathrm{SR}^+ = 0.56$				$\mathrm{SR}^- = 0.83$ / $\mathrm{SR}^+ = 1.99$			
1		47.71	43.54	42.06	19.78	0.09		56.82
2		28.40	28.18	28.11	21.89	0.16		29.75
3		12.83	15.05	15.82	27.63	94.41		7.34
4		11.06	13.23	14.01	30.70	5.34		6.08

3.2 Comparing WB and RB portfolios

Let $\{b_1, \ldots, b_b\}$ be a vector of budgets. In a weight budgeting portfolio, the weight of asset i is equal to its budget b_i. In a risk budgeting portfolio, it is the risk contribution of asset i that is equal to its budget b_i. Roncalli (2013) shows that the following inequalities hold:

$$\mathcal{R}(x_{\mathrm{mr}}) \leq \mathcal{R}(x_{\mathrm{rb}}) \leq \mathcal{R}(x_{\mathrm{wb}})$$

where x_{mr} is the (long-only) minimum risk portfolio, x_{rb} is the risk budgeting portfolio and x_{wb} is the weight budgeting portfolio.
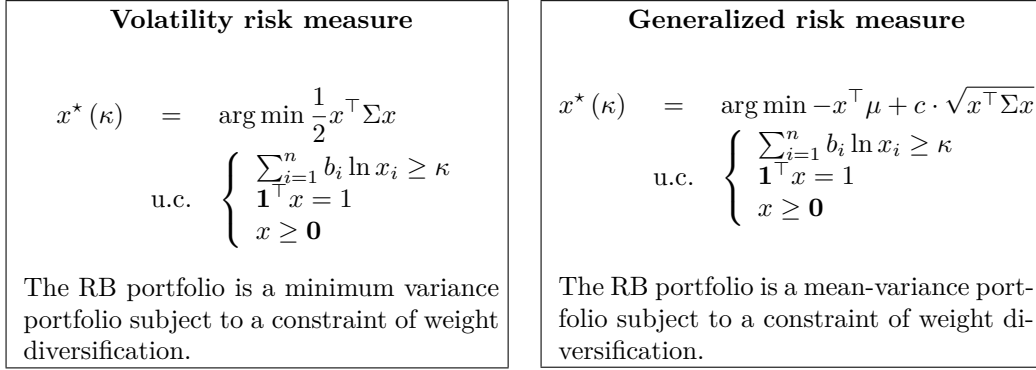
This result is important because it implies that the RB portfolio is located between these two portfolios. It has a lower risk than the WB portfolio and remains more diversified than the MR portfolio.

3.3 Comparing MVO and RB portfolios

By using the generalized risk measure instead of volatility, RB portfolios can be interpreted as MVO portfolios subject to a diversification constraint (see Figure 1). It is therefore tempting to view a risk parity portfolio as a diversified mean-variance portfolio, which implicitly corresponds to a shrinkage approach of the covariance matrix (Jagannathan and Ma, 2003). The issue is then whether these RB portfolios are simply a reformulation of MVO portfolio with regularization? This is in fact not the case, as the expected returns do not have a return dimension, but rather a risk dimension in the risk budgeting framework. They therefore constitute directional risks, which have to be managed. This is very different to the mean-variance approach, which considers expected returns as one of the two dimensions of the risk/return trade-off. This difference can be demonstrated by considering the stability issue of these two approaches.

Example 2 *We consider a universe of three assets. The risk premiums are $\pi_1 = \pi_2 = 8\%$ and $\pi_3 = 5\%$ respectively. The volatilities are $\sigma_1 = 20\%$, $\sigma_2 = 21\%$ and $\sigma_3 = 10\%$. Moreover, we assume that the cross-correlations are the same: $\rho_{i,j} = \rho = 80\%$.*

Figure 1: Comparing MVO and RB portfolios

Volatility risk measure	**Generalized risk measure**
$$x^\star(\kappa) = \arg\min \frac{1}{2}x^\top \Sigma x$$ $$\text{u.c.} \begin{cases} \sum_{i=1}^n b_i \ln x_i \geq \kappa \\ \mathbf{1}^\top x = 1 \\ x \geq \mathbf{0} \end{cases}$$	$$x^\star(\kappa) = \arg\min -x^\top \mu + c \cdot \sqrt{x^\top \Sigma x}$$ $$\text{u.c.} \begin{cases} \sum_{i=1}^n b_i \ln x_i \geq \kappa \\ \mathbf{1}^\top x = 1 \\ x \geq \mathbf{0} \end{cases}$$
The RB portfolio is a minimum variance portfolio subject to a constraint of weight diversification.	The RB portfolio is a mean-variance portfolio subject to a constraint of weight diversification.

Portfolio managers using the risk budgeting approach are motivated to obtain a diversified portfolio that changes in line with market conditions, but remains relatively stable over time. In Example 2, we calculate the long-only MVO portfolio by targeting a volatility of 15%. The solution is $(38.3\%, 20.2\%, 41.5\%)$. We will now change the input parameters slightly, and provide the results in Table 2. For instance, if the volatility of the second asset is 18% instead of 21%, the solution becomes $(13.7\%, 56.1\%, 30.2\%)$ and we can then see a substantial decrease in the weight of the first asset. This confirms that the MVO portfolio is highly sensitive to input parameters. We shall then consider the RB portfolios by targeting the risk budgets $(50.0\%, 26.4\%, 23.6\%)$, which correspond to the risk contributions of the initial MVO portfolio. Of course, the RB and MVO portfolios coincide if we use the original values of the parameters. If the volatility of the second asset is 18% instead of 21%, the RB solution becomes $(36.7\%, 23.5\%, 39.7\%)$. In this case, the RB portfolio is very different to the MVO portfolio. In fact, we observe that RB portfolios are more stable, even if we change the expected returns. This stability is important and explains why risk budgeting produces lower turnover than mean-variance optimization.

Table 2: Sensitivity of MVO and RB portfolios to input parameters

	ρ		70%	90%		90%		
	σ_2				18%	18%		
	π_1						20%	-10%
MVO	x_1	38.3%	38.3%	44.6%	13.7%	0.0%	56.4%	0.0%
	x_2	20.2%	25.9%	8.9%	56.1%	65.8%	0.0%	51.7%
	x_3	41.5%	35.8%	46.5%	30.2%	34.2%	43.6%	48.3%
RB	x_1	38.3%	37.5%	39.2%	36.7%	37.5%	49.1%	28.8%
	x_2	20.2%	20.4%	20.0%	23.5%	23.3%	16.6%	23.3%
	x_3	41.5%	42.1%	40.8%	39.7%	39.1%	34.2%	47.9%

3.4 Numerical algorithm to find the RB portfolio

There have been a number of studies into the most efficient way to numerically solve the RB portfolio (Chaves *et al.*, 2012; Spinu, 2013; Griveau-Billion *et al.*, 2014). In this article, we propose using the cyclical coordinate descent (or CCD) algorithm developed by Tseng

(2001). The main idea behind the CCD algorithm is to minimize a function $f(x_1, \ldots, x_n)$ by minimizing only one direction per step, whereas traditional descent algorithms consider all the directions at the same time. In this case, we find the value of x_i which minimizes the objective function by considering the values taken by x_j for $j \neq i$ to be fixed. The procedure is repeated for each direction until the global minimum is reached.

The Lagrangian function of the problem (6) is given by:

$$\mathcal{L}(x; \lambda) = \arg\min -x^\top \pi + c \cdot \sqrt{x^\top \Sigma x} - \lambda \sum_{i=1}^{n} b_i \ln x_i$$

Without loss of generality, we can fix $\lambda = 1$. The first-order conditions are

$$\frac{\partial \mathcal{L}(x; \lambda)}{\partial x_i} = -\pi_i + c \frac{(\Sigma x)_i}{\sigma(x)} - \frac{b_i}{x_i}$$

At the optimum, we have $\partial_{x_i} \mathcal{L}(x; \lambda) = 0$ or:

$$c\sigma_i^2 x_i^2 + \left(c\sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j - \pi_i \sigma(x) \right) x_i - b_i \sigma(x) = 0$$

By definition of the RB portfolio we have $x_i > 0$. We notice that the polynomial function is convex because we have $\sigma_i^2 > 0$. Since the product of the roots is negative, we always have two solutions with opposite signs. It can be deduced that the solution is the positive root of the second degree equation:

$$x_i^\star = \frac{-c\left(\sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j\right) + \pi_i \sigma(x) + \sqrt{\left(c\left(\sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j\right) - \pi_i \sigma(x)\right)^2 + 4cb_i\sigma_i^2 \sigma(x)}}{2c\sigma_i^2}$$

If the values of $(x_1, \cdots, x_n)$ are strictly positive and if $c > \mathrm{SR}^+$, $x_i^\star$ should be strictly positive. The positivity of the solution is then achieved after each iteration if the starting values are positive. The coordinate-wise descent algorithm consists in iterating the previous equation until convergence.

4 Applications to asset allocation

4.1 A defensive model of active management

If expected returns are added to the risk measure, the risk budgeting approach becomes an active management model. Indeed, the portfolio manager may then express their views and check how their bets modify the RB portfolio. However, contrary to the mean-variance approach which is a decidedly aggressive model of active management, the risk budgeting approach defines more conservative portfolios. In some ways, this framework is a defensive active management model. This property can be seen in the following example.

Example 3 *We consider an investment universe of three assets. The volatility is equal to 15%, 20% and 25% respectively, whereas the correlation matrix C is equal to:*

$$C = \begin{pmatrix} 1.00 & & \\ 0.30 & 1.00 & \\ 0.50 & 0.70 & 1.00 \end{pmatrix}$$

We also define six parameter sets of risk premia:

Set	#1	#2	#3	#4	#5	#6
π_1	0%	0%	20%	0%	0%	25%
π_2	0%	10%	10%	−20%	30%	25%
π_3	0%	20%	0%	−20%	−30%	−30%

Table 3 shows the optimized portfolios if we use the standard deviation-based risk measure and a scaling factor c of 2. For the risk budgeting portfolio, it is assumed that the risk budgets are the same for all three assets, meaning that we actually obtain an ERC portfolio. We then move on to consider two optimized mean-variance portfolios: the long/short portfolio (or L-S MVO) and the long-only portfolio (or L-O MVO). Detailed results are presented in Tables 6, 7 and 8 in Appendix B. For the MVO portfolios, risk aversion γ is calibrated in order to obtain the volatility of the corresponding ERC portfolio. For instance, the volatility of the three optimized portfolios (ERC, L-S MVO and L-O MVO) is the same for the parameter set #2 and is equal to 16.22%.

Table 3: Optimized portfolios

Set		#1	#2	#3	#4	#5	#6
	x_1	45.25	37.03	64.58	53.30	29.66	66.50
ERC	x_2	31.65	33.11	24.43	26.01	63.11	31.91
	x_3	23.10	29.86	10.98	20.69	7.24	1.59
	x_1	76.00	36.98	96.69	111.53	74.00	82.95
L-S MVO	x_2	43.11	33.20	48.36	23.70	94.41	57.13
	x_3	−19.11	29.81	−45.05	−35.23	−68.42	−40.07
	x_1	69.66	36.98	88.03	98.76	29.43	69.66
L-O MVO	x_2	30.34	33.20	11.97	1.24	70.57	30.34
	x_3	0.00	29.81	0.00	0.00	0.00	0.00

Applying parameter set #1, we obtain the traditional ERC portfolio based on the volatility risk measure. Indeed, all the risk premiums π_i are equal to zero, implying that the risk contributions $\mathcal{RC}_i^\star$ are exactly equal to the volatility contribution $\mathcal{VC}_i^\star$. For the mean-variance approach, the optimal portfolio is the minimum variance portfolio because the expected returns are equal to zero. We can therefore see several major differences between the RB portfolio and the MVO portfolios. If we move on to apply parameter set #2, the differences are minimal because the risk premiums are in line with volatilities[3]. In the three other cases (#3, #4 and #5), the ERC portfolio differs from the MVO portfolios significantly. This result can be explained by the fact that the expected returns are not ranked in the same order as the volatilities. In the last case #6, the ERC portfolio is very similar to the L-O MVO portfolio because the third asset has a strong negative risk premium[4] and the risk premiums of the first two assets are in line with the volatilities.

The above numerical results demonstrate that the RB portfolio is less aggressive than the MVO portfolio, and that the MVO portfolio is more concentrated than the RB portfolio. The framework presented in this article is therefore an interesting alternative to the mean-variance framework when portfolio managers want active management without concentrating their portfolios on a small number of bets.

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[3]In Appendix A, we show that the ERC portfolio is optimal under some conditions. In fact, the parameter set #2 is not so far to verify these conditions.

[4]In this case, it represents a low weight for the ERC portfolio whereas it is not selected by the L-O MVO portfolio.

4.2 Strategic asset allocation

We shall now consider the example given on page 287 in Roncalli (2013). The investment universe is composed of seven asset classes covering bonds, equities and commodities. Appendix C shows the values of the input parameters (expected return, volatility and cross-correlation) necessary to define a strategic asset allocation. Below we compare the RB and MVO approaches. The RB portfolio is defined using the following risk budgets: 20%, 10%, 15%, 20%, 10%, 15% and 10%. Concerning MVO portfolios, the objective function is to target an ex-ante volatility of 4.75% or 5%.

The results when the risk-free rate is equal to 3% are provided in Table 4. If $c = \infty$, we get the RB portfolio based on the volatility risk measure. In this case, volatility contributions are exactly equal to the risk budgets. In order to take the expected returns into account, we consider the RB portfolio with $c \neq \infty$. For instance, if $c = 1.5$, we overweight bonds overall and underweight equities. We can compare the allocation of the RB portfolio with those of MVO portfolios when we target a volatility $\sigma^\star$. MVO portfolios visibly improve the Sharpe ratio, but are more concentrated than the RB portfolios, both in weighting (US Bonds 10Y) and in risk (EM Equities). Therefore, even if the breakdown between bonds, equities and commodities is similar for RB and MVO portfolios, the RB allocation is clearly more appropriate for a strategic asset allocation as it is more diversified within each asset class.

Table 4: Long-term strategic portfolios with $r = 3\%$

| | RB | | | | | | MVO | | | |
| | $c = \infty$ | | $c = 3$ | | $c = 1.5$ | | $\sigma^\star = 4.75\%$ | | $\sigma^\star = 5\%$ | |
	x_i	$\mathrm{VC}_i^\star$	x_i	$\mathrm{VC}_i^\star$	x_i	$\mathrm{VC}_i^\star$	x_i	$\mathrm{VC}_i^\star$	x_i	$\mathrm{VC}_i^\star$
(1)	36.8	20.0	36.9	19.3	37.2	18.6	60.5	38.1	64.3	34.6
(2)	21.8	10.0	21.2	9.3	20.5	8.5	14.0	7.4	7.6	3.2
(3)	14.7	15.0	14.5	14.5	14.0	13.8	0.0	0.0	0.0	0.0
(4)	10.2	20.0	10.4	20.7	10.7	21.4	5.2	10.0	5.5	10.8
(5)	5.5	10.0	5.6	10.3	5.7	10.5	5.2	9.2	5.5	9.8
(6)	7.0	15.0	7.5	16.3	8.2	18.1	14.2	33.7	16.0	39.5
(7)	3.9	10.0	3.9	9.7	3.8	9.2	1.0	1.7	1.1	2.1
$\mu(x)$	5.69		5.73		5.79		5.64		5.83	
$\sigma(x)$	5.03		5.08		5.14		4.75		5.00	
$\mathrm{SR}(x \mid r)$	0.53		0.54		0.54		0.56		0.57	

In Figure 2, all these portfolios are located in the mean-variance (MV) and risk budgeting (RB) efficient frontiers[5]. We also report efficient frontiers when the risk-free rate is equal to 0% and 4%. Depending on the level of this parameter, the Sharpe ratio of the RB portfolio is either a decreasing or an increasing function of the scaling factor c. The RB efficient frontier is more limited than the MV efficient frontier. For instance, it is not possible to obtain an ex-ante volatility lower than 5% when the risk-free rate is equal to 3%. Moreover, the RB efficient frontier depends on the specification of the risk budgets.

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[5]The RB efficient frontier is obtained by considering $c \in \left]\mathrm{SR}^+, \infty\right[$.

Figure 2: Strategic asset allocation
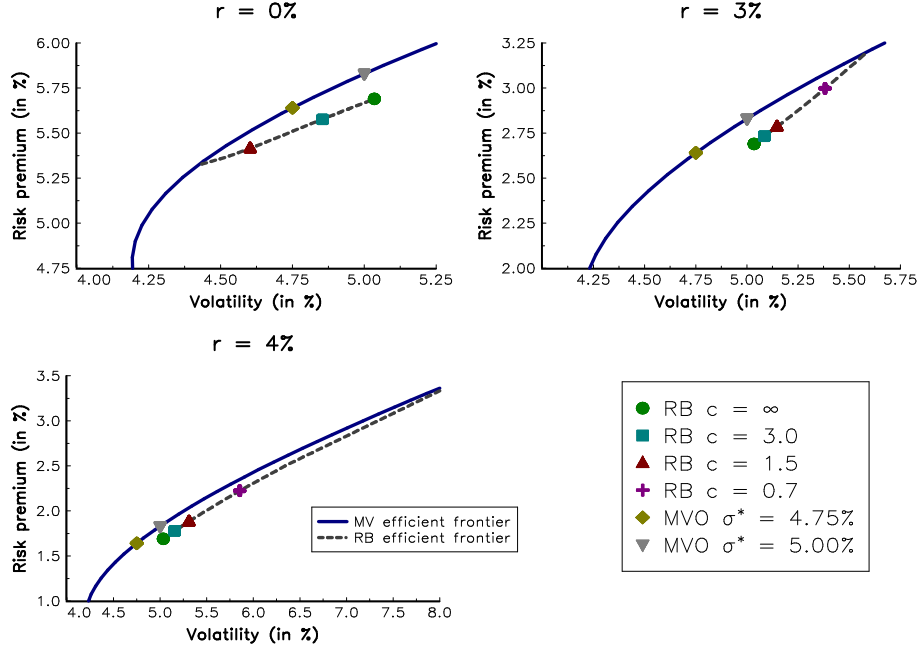
4.3 Tactical asset allocation

While strategic asset allocation refers to long-term investment horizons, tactical asset allocation deals with short to medium-term investment horizons. The aim is to define a dynamic allocation in order to modify the neutral portfolio and to enhance its performance. Trading signals based on fundamental analysis, medium-term market sentiment, business cycle forecasts or momentum patterns may be developed to achieve this.

In a TAA model, the risk measure is no longer static. At time t, it becomes:

$$\mathcal{R}_t\left(x_t\right) = -x_t^\top\left(\mu_t - r_t\right) + c_t \cdot \sqrt{x_t^\top \Sigma_t x_t} \tag{7}$$

In this framework, μ_t and Σ_t are time-varying statistics. The vector x_t corresponds to the portfolio weights at time t and generally depends on the previous allocation x_{t-1}. Let b be the vector of risk budgets. The risk parity strategy then consists of computing the RB portfolio for each time t:

$$\mathcal{RC}_{i,t}\left(x_t\right) = b_i \mathcal{R}_t\left(x_t\right)$$

By imposing $\pi_t = \mu_t - r_t = \mathbf{0}$, this framework is already used to design 'simple' risk parity equity/bond funds. Some portfolio managers also use this framework to define a neutral portfolio which is then modified according to their views. This construction of 'active' risk parity funds is similar to the TAA approach of Black and Litterman (1992). A more natural approach is to introduce expected returns into risk parity portfolios directly using Equation (7). However, an issue exists concerning the calibration of the scaling factor c_t. Indeed, we have previously seen that it is not possible to rely on the scaling factor c_t at a given confidence level α of the value-at-risk (or the expected shortfall) because the RB portfolio may not exist. It is therefore better to define c_t endogenously. For each time t, we compute

the maximum Sharpe ratio SR_t^+. We must then define a rule such that c_t is greater than the lower bound SR_t^+.

Let us consider an application with equities and bonds[6] for the period January 2000 – December 2011. The empirical covariance matrix is estimated using a lag window of 260 trading days. To calculate the vector of expected returns, we will use a simple moving average based on the daily returns for the last 260 trading days. We will also assume that the portfolio is rebalanced every week and that the risk budgets are equal. We will use the simple risk parity strategy[7] RP #0 and two versions of the active risk parity strategy[8] RP #1 and RP #2. Whereas the first RP fund is a pure risk parity strategy, the two other funds mix risk parity and trend-following strategies. Table 5 states different statistics[9] for the period January 2000 – December 2012. Our goal is not to say that active risk parity strategies are better than simple risk parity strategies, but to show that they are different. In this example, the trend-following signals have been positive, meaning that the performance, the Sharpe ratio and the drawdown of the simple risk parity strategy have all been improved. By taking expected returns into account, the turnover has been multiplied by a factor of 10. This justifies the tactical (or active) nature of this allocation method.

Table 5: Statistics of simple and active risk parity strategies

RP	$\hat{\mu}_{1Y}$	$\hat{\sigma}_{1Y}$	SR	$\mathcal{MDD}$	γ_1	γ_2	τ
#0	5.10	7.30	0.35	-21.39	0.07	2.68	0.30
#1	6.36	7.74	0.50	-14.97	0.04	2.76	3.00
#2	7.40	8.00	0.61	-12.84	0.04	2.74	3.64

Remark 2 *By considering a simple risk parity strategy, we implicitly assume that the risk is fully represented by the volatility. For instance, in the above example, the allocation is the same in the first quarter of 2003 and in summer 2010. This implies that these two periods present the same patterns in terms of volatility risk. By taking expected returns into account, we assume that directional risk is another component for measuring overall risk. From this point of view, the two periods do not present the same patterns, as they differ in terms of directional risks (trend, valuation, risk premium, etc.). Changes in market conditions are important to mitigate risk parity allocations, which are highly sensitive to bond risk even though the risk of interest rate rises is significant (Martellini et al., 2014).*

5 Conclusion

In this article, we consider the risk budgeting approach when the risk measure depends on expected returns, showing that the problem is more complex than when the risk measure is the portfolio volatility because there is a trade-off between performance contributions and

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[6]These correspond to the MSCI World TR Net index and the Citigroup WGBI All Maturities index.

[7]RP #0 corresponds to the classical ERC portfolio by considering the portfolio volatility as the risk measure.

[8]In the case of RP #1, we have $c_t = \max\left(1.10 \cdot \text{SR}_t^+, \Phi^{-1}\left(95\%\right)\right)$ whereas the scaling factor is $c_t = 1.10 \cdot \text{SR}_t^+ \cdot \mathbf{1}\left\{\text{SR}_t^+ > 0\right\} + \infty \cdot \mathbf{1}\left\{\text{SR}_t^+ \leq 0\right\}$ in the case of RP #2

[9]$\hat{\mu}_{1Y}$ is the annualized performance, $\hat{\sigma}_{1Y}$ is the yearly volatility and $\mathcal{MDD}$ is the maximum drawdown observed for the entire period. These statistics are expressed in %. SR is the Sharpe ratio, τ is the portfolio turnover, whereas the skewness and excess kurtosis correspond to γ_1 and γ_2.

volatility contributions. It appears that risk budgeting makes sense only when the scaling factor is higher than a specific value.

This framework is of particular interest when building a strategic asset allocation. The traditional way to apply the risk budgeting approach to an SAA is to link the risk budgets to risk premiums. With the new framework, risk premiums may be used to directly define the risk contributions of the SAA portfolio. Another important application concerns tactical asset allocations. To date, risk parity has been used to define a neutral portfolio that was improved using the Black-Litterman model. We can now incorporate expected returns into the risk budgeting step, in some ways making it an active management strategy.

By introducing expected returns, we nonetheless face the risk of incorporating bad forecasts or views, meaning that the robustness and simplicity of the original ERC portfolio has been lost. In our view, the framework presented here would therefore be more suitable for building risk parity portfolios with moderate bets than for creating very active trading strategies.

References

[1] BLACK F. and LITTERMAN R.B. (1992), Global Portfolio Optimization, *Financial Analysts Journal*, 48(5), pp. 28-43.

[2] BOUDT K., CARL P. and PETERSON B. (2013), Asset allocation with Conditional Value-at-Risk Budgets, *Journal of Risk*, 15(3), pp. 39-68.

[3] CHAVES D.B., HSU J.C., LI F. and SHAKERNIA O. (2012), Efficient Algorithms for Computing Risk Parity Portfolio Weights, *Journal of Investing*, 21(3), pp. 150-163.

[4] DEMIGUEL V., GARLAPPI L., NOGALES F.J. and UPPAL R. (2009), A Generalized Approach to Portfolio Optimization: Improving Performance by Constraining Portfolio Norms, *Management Science*, 55(5), pp. 798-812.

[5] EYCHENNE K., MARTINETTI S. and RONCALLI T. (2011), Strategic Asset Allocation, *SSRN*, www.ssrn.com/abstract=2154021.

[6] GRIVEAU-BILLION T., RICHARD J-C. and RONCALLI T. (2011), A Fast Algorithm for Computing High-Dimensional Risk Parity Portfolios, *SSRN*, www.ssrn.com/abstract=2325255.

[7] JAGANNATHAN R. and MA T. (2003), Risk Reduction in Large Portfolios: Why Imposing the Wrong Constraints Helps, *Journal of Finance*, 58(4), pp. 1651-1684.

[8] LEDOIT O. and WOLF M. (2004), Honey, I Shrunk the Sample Covariance Matrix, *Journal of Portfolio Management*, 30(4), pp. 110-119.

[9] MAILLARD S., RONCALLI T. and TEÏLETCHE J. (2010), The Properties of Equally Weighted Risk Contribution Portfolios, *Journal of Portfolio Management*, 36(4), pp. 60-70.

[10] MARKOWITZ H. (1952), Portfolio Selection, *Journal of Finance*, 7(1), pp. 77-91.

[11] MARTELLINI L., MILHAU V. and TARELLI A. (2014), Towards Conditional Risk Parity – Improving Risk Budgeting Techniques in Changing Economic Environments, *EDHEC Working Paper*, April.

[12] MICHAUD R.O. (1989), The Markowitz Optimization Enigma: Is 'Optimized' Optimal?, *Financial Analysts Journal*, 45(1), pp. 31-42.

[13] RONCALLI T. (2013), *Introduction to Risk Parity and Budgeting*, Chapman & Hall/CRC Financial Mathematics Series.

[14] SPINU F. (2013), An Algorithm for the Computation of Risk Parity Weights, *SSRN*, www.ssrn.com/abstract=2297383.

[15] TSENG P. (2001), Convergence of a Block Coordinate Descent Method for Nondifferentiable Minimization, *Journal of Optimization Theory and Applications*, 109(3), pp. 475-494.

A Optimality of the ERC portfolio

Let us consider the ERC portfolio in the case of the volatility risk measure. If we assume a constant correlation matrix, Maillard *et al.* (2010) show that the optimal weights are given by $x_i \propto \sigma_i^{-1}$. They also show that it is the tangency portfolio if all the assets have the same Sharpe ratio: $\mu_i = r + s \cdot \sigma_i$. This implies that the volatility contributions are equal: $\mathcal{VC}_i^\star = \mathcal{VC}_j^\star$. Because $x_i \propto \sigma_i^{-1}$ and $\pi_i = s \cdot \sigma_i$, we can deduce that the performance contributions are also equal: $\mathcal{PC}_i^\star = \mathcal{PC}_j^\star$. It follows that the composition of the ERC portfolio is the same for both the volatility and standard deviation-based risk measures. Finally, we can confirm that the result obtained by Maillard *et al.* (2010) remains valid in the case of the standard deviation-based risk measure:

Theorem 3 *The ERC portfolio is optimal if we assume a constant correlation matrix and that all the assets have the same Sharpe ratio.*

B Additional results of the active management example

The composition x of the optimized portfolios is provided in Tables 6, 7 and 8 for the different sets of parameters. For each portfolio, we have also computed the normalized risk contribution $\mathcal{RC}_i^\star$, the normalized volatility contribution $\mathcal{VC}_i^\star$ and the normalized performance contribution $\mathcal{PC}_i^\star$. The parameter ω is defined by Equation (2) and measures the weight of the volatility contribution relative to the risk contribution. Finally, we show the risk premium $\pi(x)$, the volatility $\sigma(x)$ and the Sharpe ratio $\mathrm{SR}(x \mid r)$ of the optimized portfolio x. All the statistics are expressed in %, except the parameter ω and the Sharpe ratio $\mathrm{SR}(x \mid r)$ which are measured in decimals.

Table 6: ERC portfolios

Set	#1	#2	#3	#4	#5	#6
x_1	45.25	37.03	64.58	53.30	29.66	66.50
x_2	31.65	33.11	24.43	26.01	63.11	31.91
x_3	23.10	29.86	10.98	20.69	7.24	1.59
$\mathcal{RC}_1^\star$	33.33	33.33	33.33	33.33	33.33	33.33
$\mathcal{RC}_2^\star$	33.33	33.33	33.33	33.33	33.33	33.33
$\mathcal{RC}_3^\star$	33.33	33.33	33.33	33.33	33.33	33.34
$\mathcal{VC}_1^\star$	33.33	23.80	60.96	43.79	15.88	64.79
$\mathcal{VC}_2^\star$	33.33	34.00	23.85	26.32	75.03	33.10
$\mathcal{VC}_3^\star$	33.33	42.20	15.19	29.89	9.09	2.11
$\mathcal{PC}_1^\star$		0.00	84.09	0.00	0.00	68.92
$\mathcal{PC}_2^\star$		35.66	15.91	55.69	112.95	33.06
$\mathcal{PC}_3^\star$		64.34	0.00	44.31	-12.95	-1.98
ω	1.00	1.40	2.19	0.76	2.10	8.63
$\pi(x)$	0.00	9.28	15.36	-9.34	16.76	24.12
$\sigma(x)$	15.35	16.22	14.11	14.89	16.00	13.64
$\mathrm{SR}(x \mid r)$	0.00	0.57	1.09	-0.63	1.05	1.77

Table 7: Long/short MVO portfolios

Set	#1	#2	#3	#4	#5	#6
x_1	76.00	36.98	96.69	111.53	74.00	82.95
x_2	43.11	33.20	48.36	23.70	94.41	57.13
x_3	-19.11	29.81	-45.05	-35.23	-68.42	-40.07
$\mathcal{RC}_1^\star$	76.00	33.27	120.37	112.45	-67.64	4.69
$\mathcal{RC}_2^\star$	43.11	33.46	88.59	25.58	24.68	3.23
$\mathcal{RC}_3^\star$	-19.11	33.27	-108.96	-38.03	142.96	92.08
$\mathcal{VC}_1^\star$	76.00	23.75	85.78	103.74	35.60	72.61
$\mathcal{VC}_2^\star$	43.11	34.12	29.84	7.68	75.51	50.01
$\mathcal{VC}_3^\star$	-19.11	42.13	-15.62	-11.42	-11.11	-22.61
$\mathcal{PC}_1^\star$		0.00	79.99	0.00	0.00	44.08
$\mathcal{PC}_2^\star$		35.77	20.01	-205.47	57.98	30.36
$\mathcal{PC}_3^\star$		64.23	0.00	305.47	42.02	25.56
ω	1.00	1.40	6.98	1.08	-1.90	-1.38
$\pi(x)$	0.00	9.28	24.17	2.31	48.85	47.04
$\sigma(x)$	13.19	16.22	14.11	14.89	16.00	13.64
$\mathrm{SR}(x \mid r)$	0.00	0.57	1.71	0.15	3.05	3.45

Table 8: Long-only MVO portfolios

Set	#1	#2	#3	#4	#5	#6
x_1	69.66	36.98	88.03	98.76	29.43	69.66
x_2	30.34	33.20	11.97	1.24	70.57	30.34
x_3	0.00	29.81	0.00	0.00	0.00	0.00
$\mathcal{RC}_1^\star$	69.66	33.27	89.81	98.65	44.05	69.66
$\mathcal{RC}_2^\star$	30.34	33.46	10.19	1.35	55.95	30.34
$\mathcal{RC}_3^\star$	0.00	33.27	0.00	0.00	0.00	0.00
$\mathcal{VC}_1^\star$	69.66	23.75	92.36	99.47	14.91	69.66
$\mathcal{VC}_2^\star$	30.34	34.12	7.64	0.53	85.09	30.34
$\mathcal{VC}_3^\star$	0.00	42.13	0.00	0.00	0.00	0.00
$\mathcal{PC}_1^\star$		0.00	93.63	0.00	0.00	69.66
$\mathcal{PC}_2^\star$		35.77	6.37	100.00	100.00	30.34
$\mathcal{PC}_3^\star$		64.23	0.00	0.00	0.00	0.00
ω	1.00	1.40	3.00	0.99	2.95	12.72
$\pi(x)$	0.00	9.28	18.80	-0.25	21.17	25.00
$\sigma(x)$	13.57	16.22	14.11	14.89	16.00	13.57
$\mathrm{SR}(x \mid r)$	0.00	0.57	1.33	-0.02	1.32	1.84

C   Input parameters of the SAA example

We shall now consider the example given on page 287 in Roncalli (2013). The investment universe is composed of seven asset classes: US Bonds 10Y (1), EURO Bonds 10Y (2), Investment Grade Bonds (3), US Equities (4), Euro Equities (5), EM Equities (6) and Commodities (7). In Tables 9 and 10, we provide the long-run statistics used to compute the strategic asset allocation[10]. We have assumed that the long-term investor has decided to define the strategic portfolio using the risk budgets b_i provided in Table 9.

Table 9: Expected returns, volatility and risk budgets for the SAA approach (in %)

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
μ_i	4.2	3.8	5.3	9.2	8.6	11.0	8.8
σ_i	5.0	5.0	7.0	15.0	15.0	18.0	30.0
b_i	20.0	10.0	15.0	20.0	10.0	15.0	10.0

Table 10: Correlation matrix of asset returns for the SAA approach (in %)

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
(1)	100						
(2)	80	100					
(3)	60	40	100				
(4)	−10	−20	30	100			
(5)	−20	−10	20	90	100		
(6)	−20	−20	30	70	70	100	
(7)	0	0	10	20	20	30	100

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[10]These figures are taken from Eychenne *et al.* (2011).