Portfolio Optimization versus Risk-Budgeting Allocation[1]

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[1] The opinions expressed in this presentation are those of the author and are not meant to represent the opinions or official positions of Lyxor Asset Management.

Outline

1. Portfolio optimization
   - Some Models
   - Robustness of the Markowitz framework
   - Weights constraints and Portfolio Theory
2. Risk-budgeting techniques
   - Risk-budgeting principles
   - The ERC portfolio
3. Some illustrations
   - Risk-based indexation
   - Risk parity funds
   - Bond portfolios management

Some models

Most Popular Models in Asset Allocation

- Mean-variance portfolio selection (Markowitz, 1952)
(minimum-variance strategy, tangency portfolio, strategic asset allocation, market-cap indexation, etc.)

- Dynamic optimization (Merton, 1971)
(constant-mix strategy, liability-driven investment, lifecycle funds, target date funds, etc.)

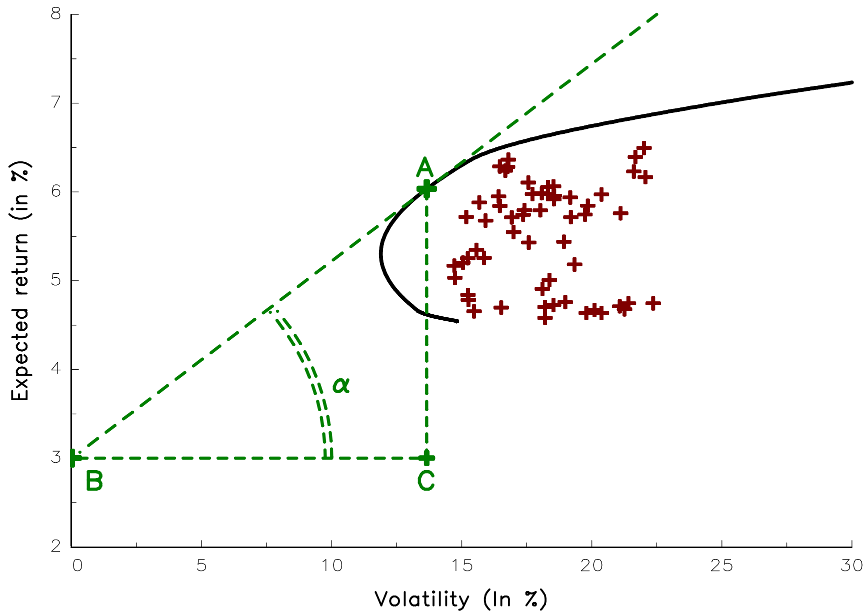
- Tactical asset allocation (Black-Litterman, 1992)
(equilibrium portfolios, flexible views, market timing, etc.)

$\Rightarrow$ These 3 models are based on optimization techniques.

The Markowitz framework

In the portfolio theory of Markowitz, we maximize the expected return for a given level of volatility:

$$\max \mu(w) = \mu^\top w \quad \text{u.c.} \quad \sigma(w) = \sqrt{w^\top \Sigma w} = \sigma^\star$$



- Optimized portfolios with respect to volatility and expected return
- The optimal portfolio is the tangency portfolio
- Confusion between volatility and diversification concepts
- The solution is not robust; it is highly sensitive to expected return inputs
- High turnover of the portfolio

Stability

We consider the minimum-variance portfolio (because it does not depend on expected returns).

2 assets with $\sigma_1(t) = \sigma_2(t) = 20\%$ and $\rho_{1,2}(t) = 100\%$:

$$w_1^\star(t) = w_2^\star(t) = 50\%$$

In $t+1$, if the volatility of the first $\sigma_1(t+1) = 19,9\%$, we obtain:

$$w_1^\star(t+1) = 100\% \text{ and } w_2^\star(t+1) = 0\%$$

An example

3 assets with $\sigma_1(t) = 20\%$, $\sigma_2(t) = 22\%$ and $\sigma_3(t) = 23\%$ and a uniform correlation ρ. We assume that the true correlation ρ is 90%.

Table: Optimal portfolios

		Estimated correlation					
		No short selling			With short selling		
Asset	EW	90%	85%	95%	90%	85%	95%
1	**33.33%**	95.65%	70.09%	100.00%	100.00%	70.09%	100.00%
2	**33.33%**	4.35%	23.78%	0.00%	17.61%	23.78%	36.82%
3	**33.33%**	0.00%	6.13%	0.00%	-17.61%	6.13%	-36.82%
Volatility	**20.93%**	20.00%	20.17%	20.00%	19.92%	20.17%	20.02%

Michaud (1989), *FAJ*:

The Markowitz Optimization Enigma: Is "Optimized" Optimal?

On the importance of the information matrix

Let μ and Σ be the vector of expected returns and the covariance matrix. Optimal solutions are of the following form:

$$w^\star \propto \Sigma^{-1}\mu$$

In the case of the minimum-variance portfolio, the form is:

$$w^\star \propto \Sigma^{-1}\mathbf{1}$$

The important quantity is $\mathscr{I} = \Sigma^{-1}$, which is called the information matrix.

Which factors are important?

Eigendecomposition of the information matrix

The eigendecomposition of $\mathscr{I} = \Sigma^{-1}$ is the same as the one of Σ, but with reverse order of eigenvectors and inverse eigenvalues:

$$\begin{aligned} V_i(\mathscr{I}) &= V_{n-i}(\Sigma) \\ \lambda_i(\mathscr{I}) &= \frac{1}{\lambda_{n-i}(\Sigma)} \end{aligned}$$

Table: Example with the previous covariance matrix (with correlation 90%)

	Covariance matrix			Information matrix		
Asset / Factor	1	2	3	1	2	3
1	52.88%	-13.02%	83.87%	83.87%	-13.02%	52.88%
2	58.56%	-65.93%	-47.16%	-47.16%	-65.93%	58.56%
3	61.44%	74.05%	-27.25%	-27.25%	74.05%	61.44%
Eigenvalues	0.1319	0.0051	0.0043	233.4190	197.1199	7.5790
% cumulated	93.4%	97.0%	100.0%	53.3%	98.3%	100.0%

Solutions

Because the optimal solution depends principally on the last factors of the covariance matrix, we have to introduce some regularization techniques:

- regularization of the objective function by using resampling techniques
- regularization of the covariance matrix:
  - Factor analysis
  - Shrinkage methods
  - Random matrix theory
  - etc.
- regularization of the program specification by introducing some constraints

Main result

We consider a universe of n assets. We denote by μ the vector of their expected returns and by Σ the corresponding covariance matrix. We specify the optimization problem as follows:

$$\min \frac{1}{2} w^\top \Sigma w$$
$$\text{u.c.} \begin{cases} \mathbf{1}^\top w = 1 \\ \mu^\top w \geq \mu^\star \\ w \in \mathbb{R}^n \bigcap \mathscr{C} \end{cases}$$

where w is the vector of weights in the portfolio and $\mathscr{C}$ is the set of weights constraints. We define:

- the unconstrained portfolio $w^\star$ or $w^\star(\mu, \Sigma)$:

$$\mathscr{C} = \mathbb{R}^n$$

- the constrained portfolio $\tilde{w}$:

$$\mathscr{C}\left(w^-, w^+\right) = \left\{w \in \mathbb{R}^n : w_i^- \leq w_i \leq w_i^+\right\}$$

Theorem

Jagannathan and Ma (2003) show that the constrained portfolio is the solution of the unconstrained problem:

$$\tilde{w} = w^{\star}\left(\tilde{\mu}, \tilde{\Sigma}\right)$$

with:

$$\left\{ \begin{array}{l} \tilde{\mu} = \mu \\ \tilde{\Sigma} = \Sigma + (\lambda^{+} - \lambda^{-})\mathbf{1}^{\top} + \mathbf{1}(\lambda^{+} - \lambda^{-})^{\top} \end{array} \right.$$

where λ^{-} and λ^{+} are the Lagrange coefficients vectors associated to the lower and upper bounds.

$\Rightarrow$ Introducing weights constraints is equivalent to introduce some relative views (similar to the **Black-Litterman** approach).

Proof for the global minimum-variance portfolio

We define the Lagrange function as $f(w;\lambda_0) = \frac{1}{2} w^\top \Sigma w - \lambda_0 \left(\mathbf{1}^\top w - 1\right)$ with $\lambda_0 \geq 0$. The first order conditions are $\Sigma w - \lambda_0 \mathbf{1} = 0$ and $\mathbf{1}^\top w - 1 = 0$. We deduce that the optimal solution is:

$$w^\star = \lambda_0^\star \Sigma^{-1}\mathbf{1} = \frac{1}{\mathbf{1}^\top \Sigma \mathbf{1}} \Sigma^{-1}\mathbf{1}$$

With weights constraints $\mathscr{C}(w^-, w^+)$, we have:

$$f\left(w;\lambda_0,\lambda^-,\lambda^+\right) = \frac{1}{2} w^\top \Sigma w - \lambda_0 \left(\mathbf{1}^\top w - 1\right) - {\lambda^-}^\top \left(w - w^-\right) - {\lambda^+}^\top \left(w^+ - w\right)$$

with $\lambda_0 \geq 0$, $\lambda_i^- \geq 0$ and $\lambda_i^+ \geq 0$. In this case, the first-order conditions becomes $\Sigma w - \lambda_0 \mathbf{1} - \lambda^- + \lambda^+ = 0$ and $\mathbf{1}^\top w - 1 = 0$. We have:

$$\tilde{\Sigma}\tilde{w} = \left(\Sigma + \left(\lambda^+ - \lambda^-\right)\mathbf{1}^\top + \mathbf{1}\left(\lambda^+ - \lambda^-\right)^\top\right)\tilde{w} = \left(2\tilde{\lambda}_0 - \tilde{w}^\top \Sigma \tilde{w}\right)\mathbf{1}$$

Because $\tilde{\Sigma}\tilde{w}$ is a constant vector, it proves that $\tilde{w}$ is the solution of the unconstrained optimisation problem with $\lambda_0^\star = \left(2\tilde{\lambda}_0 - \tilde{w}^\top \Sigma \tilde{w}\right)$.

Examples

Table: Specification of the covariance matrix Σ (in %)

σ_i	$\rho_{i,j}$			
15.00	100.00			
20.00	10.00	100.00		
25.00	40.00	70.00	100.00	
30.00	50.00	40.00	80.00	100.00

Given these parameters, the global minimum variance portfolio is equal to:

$$w^{\star} = \begin{pmatrix} 72.742\% \\ 49.464\% \\ -20.454\% \\ -1.753\% \end{pmatrix}$$

Table: Global minimum variance portfolio when $w_i \geq 10\%$

$\tilde{w}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
56.195	0.000	0.000	15.000	100.000			
23.805	0.000	0.000	20.000	10.000	100.000		
10.000	1.190	0.000	19.671	10.496	58.709	100.000	
10.000	1.625	0.000	23.980	17.378	16.161	67.518	100.000

Table: Global minimum variance portfolio when $0\% \leq w_i \leq 50\%$

$\tilde{w}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
50.000	0.000	1.050	20.857	100.000			
50.000	0.000	0.175	20.857	35.057	100.000		
0.000	0.175	0.000	24.290	46.881	69.087	100.000	
0.000	0.000	0.000	30.000	52.741	41.154	79.937	100.000

Table: MSR tangency portfolio when $0\% \leq w_i \leq 40\%$ and $\mathrm{sh}^\star = 0.5$

$\tilde{w}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
40.000	0.000	0.810	19.672	100.000			
40.000	0.000	0.540	22.539	37.213	100.000		
0.000	0.000	0.000	25.000	46.970	71.698	100.000	
20.000	0.000	0.000	30.000	51.850	43.481	80.000	100.000

We obtain:

$$\widetilde{\mathrm{sh}} = \begin{pmatrix} 0.381 \\ 0.444 \\ 0.5 \\ 0.5 \end{pmatrix}$$

Euler decomposition of risk measures

Let $\mathscr{R}(w_1,\ldots,w_n)$ be a coherent convex risk measure. We have:

$$\mathscr{R}(w_1,\ldots,w_n) = \sum_{i=1}^{n} \underbrace{w_i \cdot \frac{\partial\,\mathscr{R}(w_1,\ldots,w_n)}{\partial\, w_i}}_{RC_i}$$

RC_i is the risk contribution of the i^{th} asset: it is the product of its weight by its marginal risk.

Let us consider a set of **given** risk budgets $\{\mathrm{RB}_1,\ldots,\mathrm{RB}_n\}$, the risk-budgeted portfolio is defined by:

$$\left\{\begin{array}{c} RC_1(w_1,\ldots,w_n) = \mathrm{RB}_1 \\ \vdots \\ RC_i(w_1,\ldots,w_n) = \mathrm{RB}_i \\ \vdots \\ RC_n(w_1,\ldots,w_n) = \mathrm{RB}_n \end{array}\right.$$

When the risk measure is the volatility of the portfolio

Let Σ and w be the covariance matrix and the portfolio weights. We have:

$$\mathscr{R}(w) = \sigma(w) = \sqrt{w^\top \Sigma w}$$

We deduce that:

$$\frac{\partial\, \mathscr{R}(w)}{\partial\, w} = \frac{\Sigma w}{\sqrt{w^\top \Sigma w}}$$

The risk contribution of the i^{th} asset is then:

$$RC_i = w_i \cdot \frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}}$$

We verify that this risk measure is convex:

$$\sum_{i=1}^n RC_i = \sum_{i=1}^n w_i \cdot \frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}} = w^\top \frac{\Sigma w}{\sqrt{w^\top \Sigma w}} = \sqrt{w^\top \Sigma w} = \mathscr{R}(w)$$

Illustration

- 3 assets
- Volatilities are respectively 20%, 30% and 15%
- Correlations are set to 60% between the 1st asset and the 2nd asset and 10% between the first two assets and the 3rd asset

The marginal risk for the first asset is:

$$\lim_{\varepsilon \to 0} \frac{\sigma(w_1+\varepsilon, w_2, w_3) - \sigma(w_1, w_2, w_3)}{(w_1+\varepsilon) - w_1}$$

If $\varepsilon = 1\%$, we have:

$$\frac{\partial \mathscr{R}(x)}{\partial w_1} \simeq \frac{16.72\% - 16.54\%}{1\%} = 18.01\%$$

Traditional approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	**50.00%**	17.99%	9.00%	54.40%
2	**25.00%**	25.17%	6.29%	38.06%
2	**25.00%**	4.99%	1.25%	7.54%
Volatility			16.54%	

Risk budgeting approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	41.62%	16.84%	7.01%	**50.00%**
2	15.79%	22.19%	3.51%	**25.00%**
2	42.58%	8.23%	3.51%	**25.00%**
Volatility			14.02%	

ERC approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	30.41%	15.15%	4.61%	**33.33%**
2	20.28%	22.73%	4.61%	**33.33%**
2	49.31%	9.35%	4.61%	**33.33%**
Volatility			13.82%	

Definition

The Euler decomposition gives us:

$$\sigma(w) = \sum_{i=1}^{n} w_i \times \frac{\partial\, \sigma(w)}{\partial\, w_i} = \sum_{i=1}^{n} \mathrm{RC}_i$$

The ERC portfolio is the risk-balanced portfolio such that the risk contribution is the same for all assets of the portfolio:

$$\mathrm{RC}_i = \mathrm{RC}_j \quad \text{for all } i,j$$

The ERC portfolio is then the solution of the following non-linear system:

$$\left\{\begin{array}{c} w_2 \times (\Sigma w)_2 = w_1 \times (\Sigma w)_1 \\ w_3 \times (\Sigma w)_3 = w_1 \times (\Sigma w)_1 \\ \vdots \\ w_n \times (\Sigma w)_n = w_1 \times (\Sigma w)_1 \\ w_1 + w_2 + \cdots + w_n = 1 \\ w_1 > 0, w_2 > 0, \ldots, w_n > 0 \end{array}\right.$$

Properties

Consider the following optimization problem:

$$\begin{aligned} w^\star(c) &= \arg\min \sqrt{w^\top \Sigma w} \\ & \text{u.c.} \quad \left\{ \begin{array}{l} \sum_{i=1}^n \ln w_i \geq c \\ \mathbf{1}^\top w = 1 \\ \mathbf{0} \leq w \leq \mathbf{1} \end{array} \right. \end{aligned}$$

We have $w^\star(-\infty) = w_{\mathrm{mv}}$ and $w^\star(-n \ln n) = w_{\mathrm{1/n}}$. The ERC portfolio corresponds to a particular value of c such that $-\infty \leq c \leq -n \ln n$.

1. The solution of the ERC problem exists and is **unique**.
2. We also obtain the following inequality:

$$\sigma_{\mathrm{mv}} \leq \sigma_{\mathrm{erc}} \leq \sigma_{\mathrm{1/n}}$$

   because if $c_1 \leq c_2$, we have $\sigma(w^\star(c_1)) \leq \sigma(w^\star(c_2))$. The ERC portfolio may be viewed as a portfolio "between" the 1/n portfolio and the minimum-variance portfolio.
3. The ERC portfolio is a form of variance-minimizing portfolio subject to a constraint of sufficient diversification in terms of weights.

- **If the correlations are the same**, the solution is:

$$w_i = \frac{\sigma_i^{-1}}{\sum_{j=1}^n \sigma_j^{-1}}$$

The weight allocated to each component i is given by the ratio of the inverse of its volatility with the harmonic average of the volatilities.

- **If the volatilities are the same**, we have:

$$w_i \propto \left(\sum_{k=1}^n w_k \rho_{ik} \right)^{-1}$$

The weight of the asset i is proportional to the inverse of the weighted average of correlations of component i with other components.

- **In the general case**, we obtain:

$$w_i \propto \beta_i^{-1}$$

The weight of the asset i is proportional to the inverse of its beta.

- The ERC portfolio is the tangency portfolio if all the assets have the same Sharpe ratio and if the correlation is uniform (one-factor model).
- Let us consider the minimum variance portfolio with a constant correlation matrix $C_n(\rho)$. The solution is:

$$w_i = \frac{-((n-1)\rho+1)\sigma_i^{-2} + \rho \sum_{j=1}^n (\sigma_i \sigma_j)^{-1}}{\sum_{k=1}^n \left(-((n-1)\rho+1)\sigma_k^{-2} + \rho \sum_{j=1}^n (\sigma_k \sigma_j)^{-1}\right)}$$

  The lower bound of $C_n(\rho)$ is achieved for $\rho = -(n-1)^{-1}$ and we have:

$$w_i = \frac{\sum_{j=1}^n (\sigma_i \sigma_j)^{-1}}{\sum_{k=1}^n \sum_{j=1}^n (\sigma_k \sigma_j)^{-1}} = \frac{\sigma_i^{-1}}{\sum_{k=1}^n \sigma_k^{-1}} \rightarrow \mathrm{erc}$$

- The ERC portfolio minimizes the Gini and Herfindal indexes applied to the risk measure.

Pros and cons of market-cap indexation

Pros of market-cap indexation

- A convenient and **recognized approach** to participate to broad equity markets.
- **Management simplicity**: low turnover & transaction costs.

Cons of market-cap indexation

- Trend-following strategy: momentum bias leads to bubble risk exposure as weight of best performers ever increases.
  $\Rightarrow$ Mid 2007, financial stocks represent 40% of the Eurostoxx 50 index.
- Growth biais as high valuation multiples stocks weight more than low-multiple stocks with equivalent realised earnings.
  $\Rightarrow$ Mid 2000, the 8 stocks of the technology/telecom sectors represent 35% of the Eurostoxx 50 index.
  $\Rightarrow$ $2^1/_2$ years later after the dot.com bubble, these two sectors represent 12%.
- Concentrated portfolios.
  $\Rightarrow$ The top 100 market caps of the S&P 500 account for around 70%.
- Lack of risk diversification and high drawdown risk: no portfolio construction rules leads to concentration issues (e.g. sectors, stocks).

Alternative-weighted indexation

Alternative-weighted indexation aims at building passive indexes where the weights are not based on market capitalization.

Two sets of responses:

1. Fundamental indexation $\Rightarrow$ promising alpha
    1. Dividend yield indexation
    2. RAFI indexation
2. Risk-based indexation $\Rightarrow$ promising diversification
    1. Equally weighted (1/n)
    2. Minimum-variance portfolio
    3. ERC portfolio
    4. MDP/MSR portfolio

Risk-based methods

Equally-weighted (1/n)

- Weights are equal
- Easy to understand
- Contrarian strategy with a take-profit scheme
- The least concentrated in terms of weights
- Do not depend on risks

Most Diversified Portfolio (MDP)

- Also known as the Max Sharpe Ratio (MSR) portfolio of EDHEC
- Based on the assumption that sharpe ratio is equal for all stocks
- It is the tangency portfolio if the previous assumption is verified
- Sensitive to the covariance matrix

Minimum-variance (MV)

- Low volatility portfolio
- The only optimal portfolio not depending on expected returns assumptions
- Good out of sample performance
- Concentrated portfolios
- Sensitive to the covariance matrix

Equal-Risk Contribution (ERC)

- Risk contributions are equal
- Highly diversified portfolios
- Less sensitive to the covariance matrix (than the MV and MDP portfolios)
- Not efficient for universe with a large number of stocks (equivalent to the 1/n portfolio)

Comparison of the 4 Methods

In terms of bets

$$\begin{array}{ll} \exists i : w_i = 0 & \text{(MV - MDP)} \\ \forall i : w_i \neq 0 & \text{(1/n - ERC)} \end{array}$$

In terms of risk factors

$$\begin{array}{rcll} w_i & = & w_j & (1/n) \\ \dfrac{\partial\,\sigma(w)}{\partial\,w_i} & = & \dfrac{\partial\,\sigma(w)}{\partial\,w_j} & (\text{MV}) \\ w_i \times \dfrac{\partial\,\sigma(w)}{\partial\,w_i} & = & w_j \times \dfrac{\partial\,\sigma(w)}{\partial\,w_j} & (\text{ERC}) \\ \dfrac{1}{\sigma_i} \times \dfrac{\partial\,\sigma(w)}{\partial\,w_i} & = & \dfrac{1}{\sigma_j} \times \dfrac{\partial\,\sigma(w)}{\partial\,w_j} & (\text{MDP}) \end{array}$$

Application to the Eurostoxx 50 index

Table: Composition in % (January 2010)

	CW	MV	ERC	MDP	1/n	MV 10%	MDP 10%	MV 5%	MDP 5%
TOTAL	6.1		2.1		2			5.0	
BANCO SANTANDER	5.8		1.3		2				
TELEFONICA SA	5.0	31.2	3.5		2	10.0		5.0	5.0
SANOFI-AVENTIS	3.6	12.1	4.5	15.5	2	10.0	10.0	5.0	5.0
E.ON AG	3.6		2.1		2				1.4
BNP PARIBAS	3.4		1.1		2				
SIEMENS AG	3.2		1.5		2				
BBVA(BILB-VIZ-ARG)	2.9		1.4		2				
BAYER AG	2.9		2.6	3.7	2	2.2	5.0	5.0	5.0
ENI	2.7		2.1		2				
GDF SUEZ	2.5		2.6	4.5	2		5.4	5.0	5.0
BASF SE	2.5		1.5		2				
ALLIANZ SE	2.4		1.4		2				
UNICREDIT SPA	2.3		1.1		2				
SOC GENERALE	2.2		1.2	3.9	2		3.7		5.0
UNILEVER NV	2.2	11.4	3.7	10.8	2	10.0	10.0	5.0	5.0
FRANCE TELECOM	2.1	14.9	4.1	10.2	2	10.0	10.0	5.0	5.0
NOKIA OYJ	2.1		1.8	4.5	2		4.8		5.0
DAIMLER AG	2.1		1.3		2				
DEUTSCHE BANK AG	1.9		1.0		2				
DEUTSCHE TELEKOM	1.9		3.2	2.6	2	5.7	3.7	5.0	5.0
INTESA SANPAOLO	1.9		1.3		2				
AXA	1.8		1.0		2				
ARCELORMITTAL	1.8		1.0		2				
SAP AG	1.8	21.0	3.4	11.2	2	10.0	10.0	5.0	5.0

	CW	MV	ERC	MDP	1/n	MV 10%	MDP 10%	MV 5%	MDP 5%
RWE AG (NEU)	1.7	2.7	2.7		2	7.0		5.0	
ING GROEP NV	1.6		0.8	0.4	2				
DANONE	1.6	1.9	3.4	1.8	2	8.7	3.3	5.0	5.0
IBERDROLA SA	1.6		2.5		2	5.1		5.0	1.2
ENEL	1.6		2.1		2			5.0	2.9
VIVENDI SA	1.6	2.8	3.1	4.5	2	10.0	5.9	5.0	5.0
ANHEUSER-BUSCH INB	1.6	0.2	2.7	10.9	2	2.1	10.0	5.0	5.0
ASSIC GENERALI SPA	1.6		1.8		2				
AIR LIQUIDE(L')	1.4		2.1		2			5.0	
MUENCHENER RUECKVE	1.3		2.1	2.1	2		3.1	5.0	5.0
SCHNEIDER ELECTRIC	1.3		1.5		2				
CARREFOUR	1.3	1.0	2.7	1.3	2	3.7	2.5	5.0	5.0
VINCI	1.3		1.6		2				
LVMH MOET HENNESSY	1.2		1.8		2				
PHILIPS ELEC(KON)	1.2		1.4		2				
L'OREAL	1.1	0.8	2.8		2	5.5		5.0	5.0
CIE DE ST-GOBAIN	1.0		1.1		2				
REPSOL YPF SA	0.9		2.0		2			5.0	
CRH	0.8		1.7	5.1	2		5.2		5.0
CREDIT AGRICOLE SA	0.8		1.1		2				
DEUTSCHE BOERSE AG	0.7		1.5		2				1.9
TELECOM ITALIA SPA	0.7		2.0		2				2.5
ALSTOM	0.6		1.5		2				
AEGON NV	0.4		0.7		2				
VOLKSWAGEN AG	0.2		1.8	7.1	2		7.4		5.0
Total of components	50	11	50	17	50	14	16	20	23

Application to the Eurostoxx 50 index

Figure: Performance of the ERC Eurozone Index (Ticker: SGIXERCE Index)

	ERC Index	Euro Stoxx 50 NR
Cumulated Return	583.8%	264.9%
Annualized Return	10.7%	7.1%
Volatility	21.2%	22.9%
Sharpe Ratio	0.34	0.16
Tracking Error	6.0%	NA
Information Ratio	0.60	NA
Beta	0.89	1.00
Max DrawDown	-55.1%	-64.6%

Justification of diversified funds

Investor Profiles

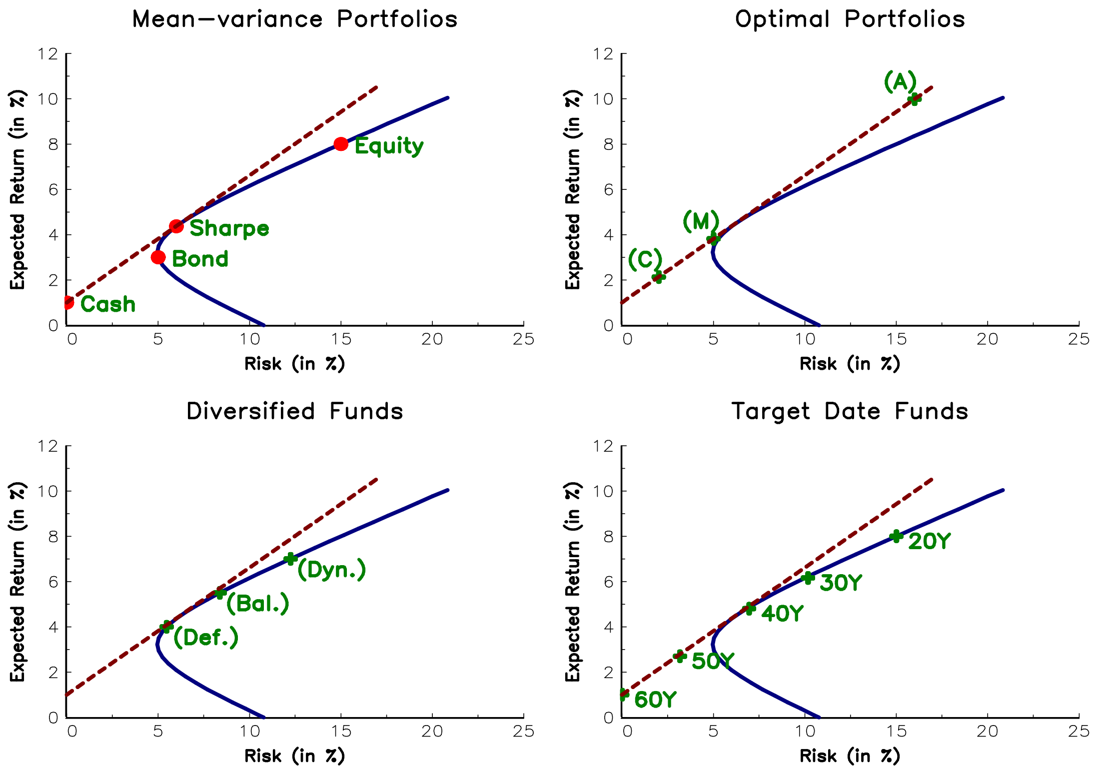
1. Moderate (medium risk tolerance)
2. Conservative (low risk tolerance)
3. Aggressive (high risk tolerance)

Fund Profiles

1. Defensive (80% bonds and 20% equities)
2. Balanced (50% bonds and 50% equities)
3. Dynamic (20% bonds and 80% equities)

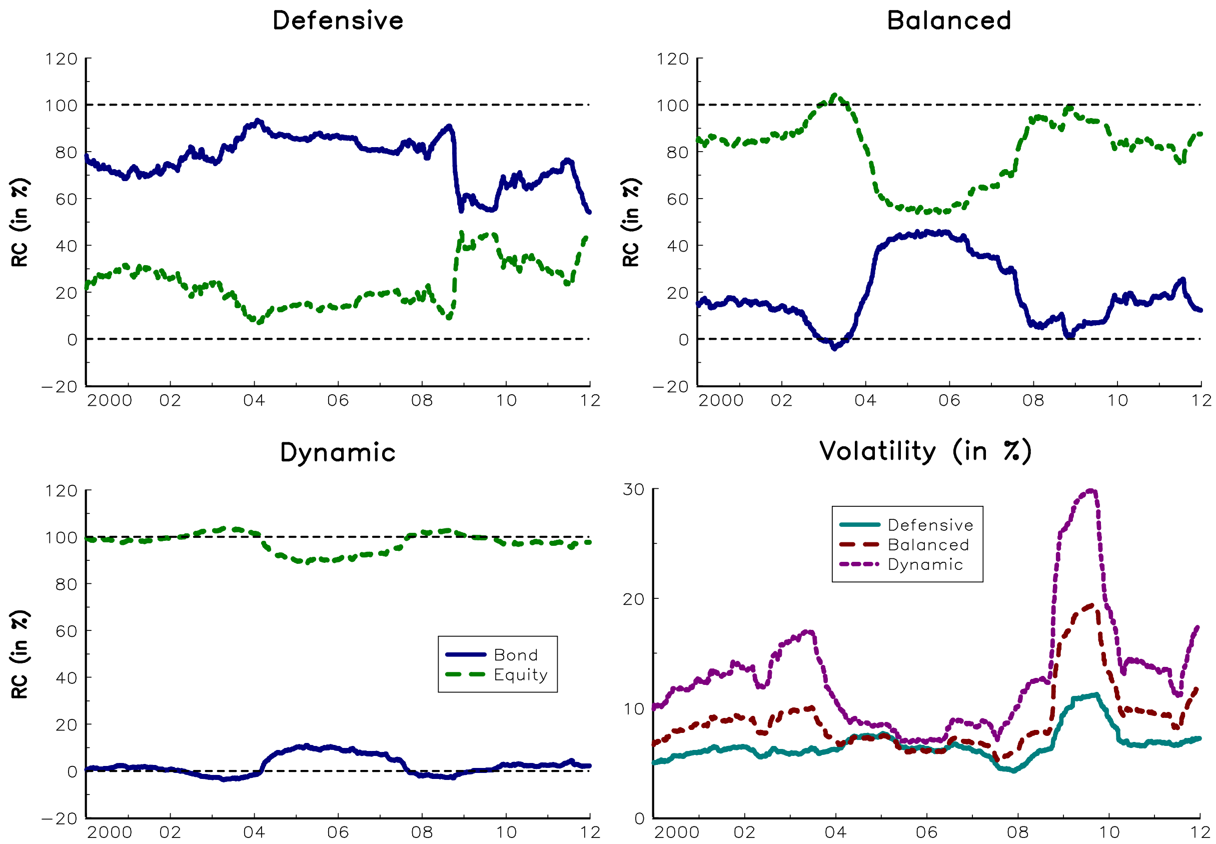
Relationship with portfolio theory?

Figure: The asset allocation puzzle



What type of diversification offer diversified funds?

Figure: Risk contribution of diversified funds[a]



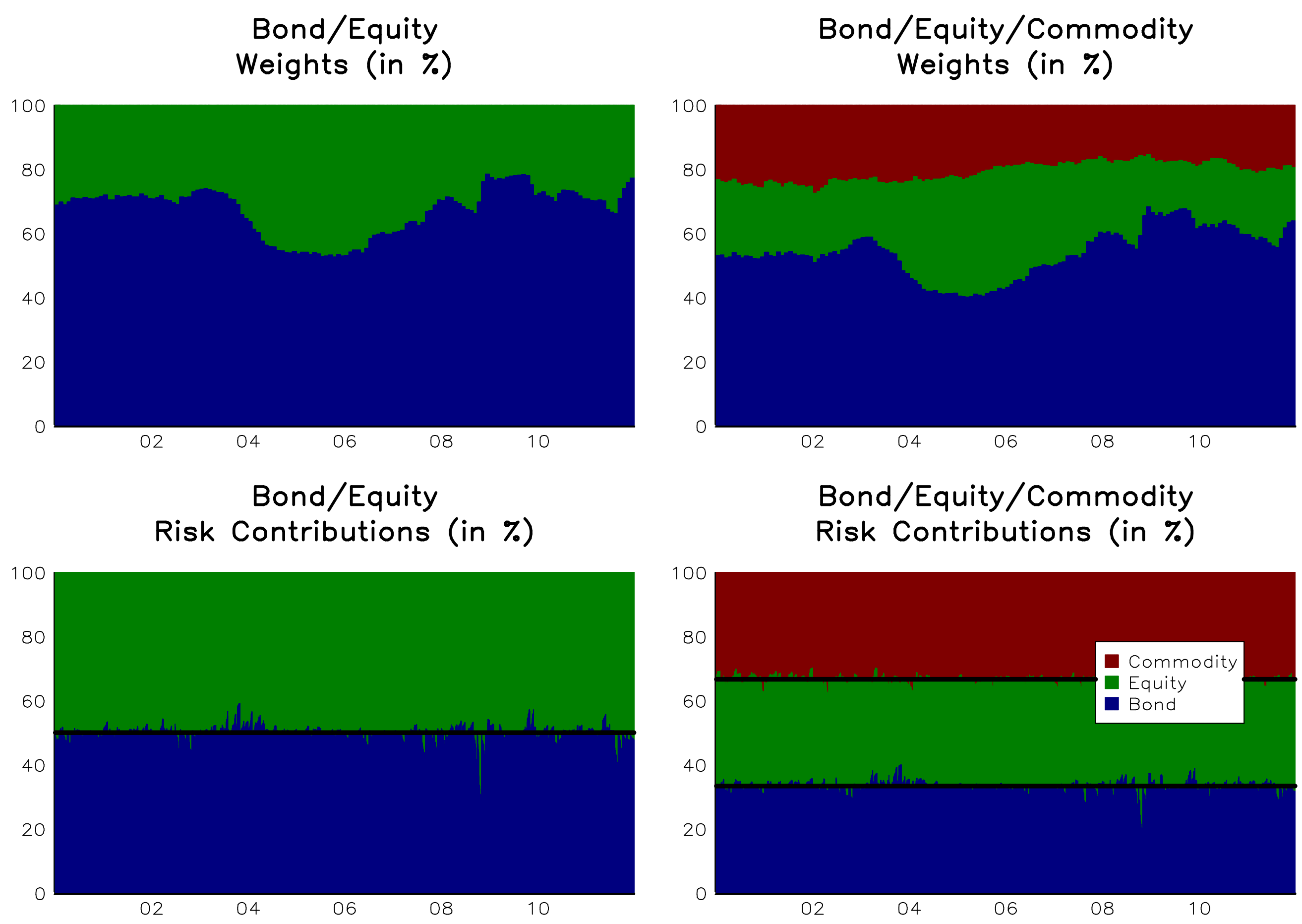
[a] Backtest with CG WGBI Index and MSCI World

Diversified funds
=
Marketing idea?

- Deleverage of an equity exposure
- Diversification in weights $\neq$ Risk diversification
- No mapping between fund profiles and volatility profiles
- No mapping between fund profiles and investor profiles

ERC diversified funds

Figure: Weights and risk contributions of ERC funds[2]



[2] Backtest with CG WGBI Index, MSCI World and DJ UBS Commodity Index

Risk parity funds

Definition

A risk parity fund is an ERC startegy on multi-assets classes.

Some examples

- AQR Capital Management
- Bridgewater
- Invesco
- Lyxor Asset Management
- PanAgora Asset Management
- Wegelin Asset Management

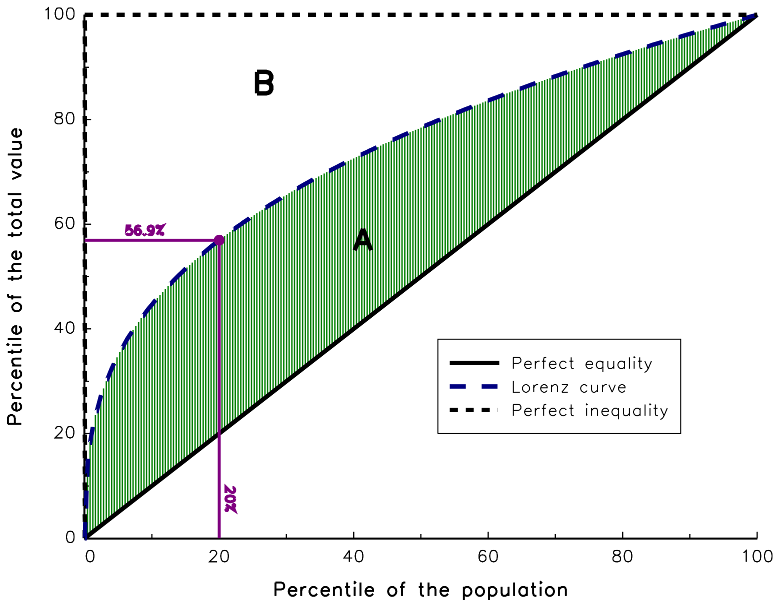
Statistical measures of concentration

- The Lorenz curve $\mathscr{L}(x)$
  It is a graphical representation of the concentration. It represents the cumulative weight of the first $x\%$ most representative stocks.

- The Gini coefficient
  It is a dispersion measure based on the Lorenz curve:

$$G = \frac{A}{A+B} = 2\int_0^1 \mathscr{L}(x)\,\mathrm{d}x - 1$$

  G takes the value 1 for a perfectly concentrated portfolio and 0 for the equally-weighted portfolio.



- The risk concentration of a portfolio is analyzed using Lorenz curve and Gini coefficient applied to risk contributions.

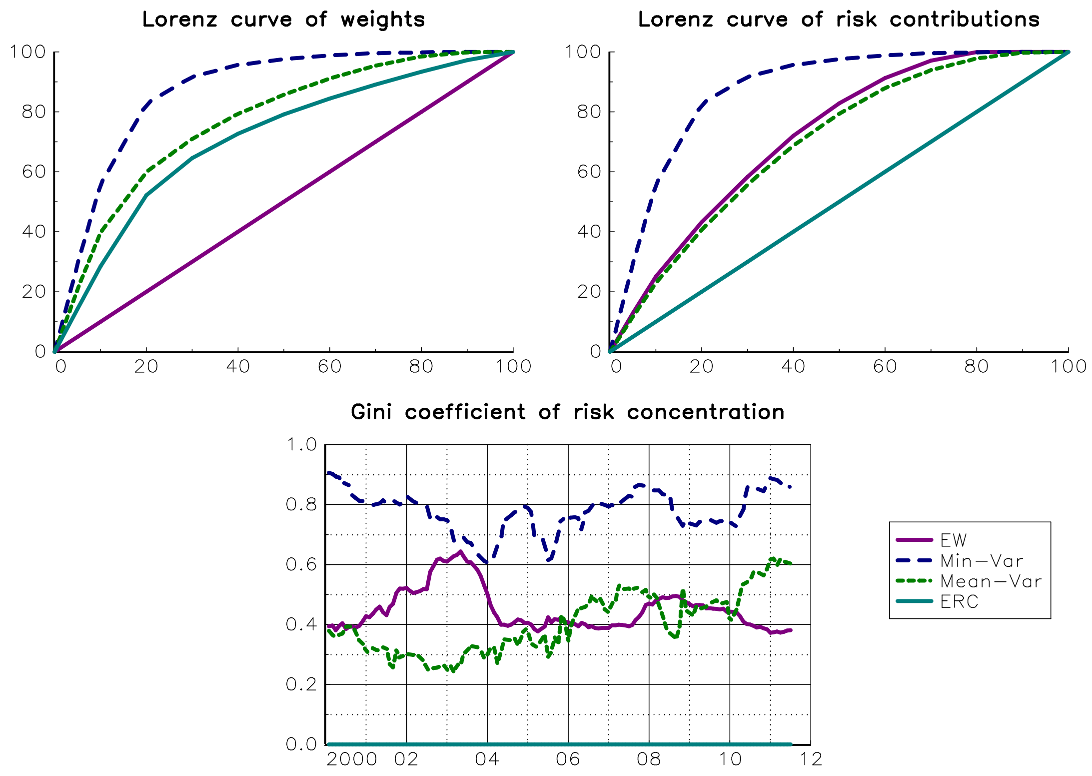
Diversification of multi-assets classes funds

Backtest (monthly rebalancing) with 10 asset classes[a]:

- EuroStoxx, S&P 500, TOPIX, MSCI EM
- EuroMTS 10-15Y, EuroMTS Inflation, JPMorgan EMBI
- GSCI, GOLD, EPRA/NAREIT Dev. Europe

[a] Vol(Mean-Var) = Vol(ERC)

Figure: Concentration statistics



Diversification of multi-assets classes funds

Table: Range of weights and risk contributions

		EW		Min-Var		Mean-Var		ERC	
		min	max	min	max	min	max	min	max
Weights	Eurostoxx	10.0%	10.0%	0.0%	5.6%	0.0%	10.3%	2.3%	7.4%
	S&P 500	10.0%	10.0%	0.0%	5.9%	0.0%	18.2%	2.7%	8.8%
	TOPIX	10.0%	10.0%	0.0%	8.5%	3.5%	17.4%	3.1%	8.8%
	MSCI EM	10.0%	10.0%	0.0%	2.3%	0.0%	4.0%	1.7%	5.0%
	EMTS 10/15	10.0%	10.0%	0.0%	39.0%	14.7%	65.7%	13.7%	35.7%
	EMTS INF	10.0%	10.0%	34.3%	84.5%	0.0%	37.5%	18.7%	37.0%
	EMBI	10.0%	10.0%	0.0%	48.3%	0.0%	11.2%	6.5%	18.0%
	GSCI	10.0%	10.0%	0.0%	3.7%	0.7%	12.2%	2.8%	8.2%
	GOLD	10.0%	10.0%	0.0%	6.5%	4.5%	22.6%	4.2%	11.7%
	RE	10.0%	10.0%	0.0%	18.8%	0.8%	14.4%	2.9%	12.1%
RC	Eurostoxx	10.9%	37.1%	0.0%	5.6%	0.0%	21.2%	10.0%	10.0%
	S&P 500	7.5%	21.7%	0.0%	5.9%	0.0%	20.2%	10.0%	10.0%
	TOPIX	5.8%	18.4%	0.0%	8.5%	9.2%	28.6%	10.0%	10.0%
	MSCI EM	13.7%	27.9%	0.0%	2.3%	0.0%	18.3%	10.0%	10.0%
	EMTS 10/15	-2.5%	2.8%	0.0%	39.0%	9.2%	34.5%	10.0%	10.0%
	EMTS INF	-2.0%	1.5%	34.3%	84.5%	0.0%	11.9%	10.0%	10.0%
	EMBI	1.5%	8.8%	0.0%	48.3%	0.0%	9.3%	10.0%	10.0%
	GSCI	5.4%	22.0%	0.0%	3.7%	1.4%	22.6%	10.0%	10.0%
	GOLD	-3.7%	13.7%	0.0%	6.5%	7.4%	30.4%	10.0%	10.0%
	RE	3.7%	18.7%	0.0%	18.8%	1.3%	19.7%	10.0%	10.0%
Turnover		0%		101%		125%		42%	

$\Rightarrow$ High turnover of min- and mean-variance portfolios (stability issue)

$\Rightarrow$ ERC = risk-balanced dynamic allocation with moderate turnover

Bond portfolios management

1. Cap-weighted indexation for equities $\Rightarrow$ debt-weighted indexation for bonds
2. Fund management driven by the search of yield

$\Rightarrow$ Time to rethink bond indexes? (Toloui, 2010)

Bond portfolios management

Debt weighting

It is defined by[a]:

$$w_i = \frac{\mathrm{DEBT}_i}{\sum_{i=1}^{n} \mathrm{DEBT}_i}$$

[a] Two forms of debt-weighting are considered : DEBT (with the 11 countries) and DEBT* (without Greece after July 2010). This last one corresponds to the weighting scheme of the EGBI index.

Alternative weighting

1. Fundamental indexation
The GDP-weighting is defined by:

$$w_i = \frac{\mathrm{GDP}_i}{\sum_{i=1}^{n} \mathrm{GDP}_i}$$

2. Risk-based indexation
The DEBT-RB and GDP-RB weightings are defined by:

$$\begin{aligned} \mathrm{RB}_i &= \frac{\mathrm{DEBT}_i}{\sum_{i=1}^{n} \mathrm{DEBT}_i} \\ \mathrm{RB}_i &= \frac{\mathrm{GDP}_i}{\sum_{i=1}^{n} \mathrm{GDP}_i} \end{aligned}$$

Choosing the right measure of credit risk

- Volatility of price returns $\neq$ a good measure of credit risk
- Correlation of price returns $\neq$ a good measure of contagion
- A better measure is the yield spread, but it is difficult to compute because it is difficult to define the reference (risk-free) rate.

$\Rightarrow$ One of the best measure is the CDS spread (it does not depend on the currency, the yield curve or the duration).

The CDS model

Let $S_i(t)$ be the spread of the i^{th} issuer. We have:

$$\mathrm{d}S_i(t) = \sigma_i^S \cdot S_i(t) \cdot \mathrm{d}W_i(t)$$

Moreover, we assume that the correlation between the brownian motions $W_i(t)$ and $W_j(t)$ is $\Gamma_{i,j}$.

Computing the credit risk measure of a bond portfolio

Let $w = (w_1, \ldots, w_n)$ be the weights of bonds in the portfolio. The risk measure is:

$$\mathscr{R}(x) = \sqrt{w^\top \Sigma w} = \sqrt{\sum_{i=1}^n \sum_{j=1}^n w_i w_j \Sigma_{i,j}}$$

with $\Sigma_{i,j} = \Gamma_{i,j} \cdot \sigma_i^B \cdot \sigma_j^B$ and $\sigma_i^B = D_i \cdot \sigma_i^S \cdot S_i(t)$, where D_i is the duration of the bond i, σ_i^S is the CDS volatility of the corresponding issuer, $S_i(t)$ is the CDS level and $\Gamma_{i,j}$ is the correlation between the CDS relative variations of issuers corresponding to the bonds i and j.

$\mathscr{R}(w)$ is the volatility of the CDS basket which would perfectly hedge the credit risk of the bond portfolio.

$\mathscr{R}(w)$ depends on 3 "CDS" parameters $S_i(t)$, σ_i^S and $\Gamma_{i,j}$ and two "portfolio" parameters w_i and D_i.

Statistics as of December 1$^{\text{st}}$, 2011

Table: CDS levels $S_i(t)$, volatilities σ_i^S and correlations $\Gamma_{i,j}$

Country	Spread	Vol.	AT	BE	FI	FR	DE	GR	IE	IT	NL	PT	ES
Austria	184	68.4%	100%										
Belgium	299	72.7%	78%	100%									
Finland	69	64.6%	72%	69%	100%								
France	192	69.8%	81%	79%	70%	100%							
Germany	98	64.1%	81%	74%	70%	79%	100%						
Greece	6,000	63.3%	26%	36%	27%	26%	28%	100%					
Ireland	702	52.2%	63%	71%	56%	65%	66%	43%	100%				
Italy	467	73.3%	72%	85%	67%	73%	69%	38%	72%	100%			
Netherlands	105	65.1%	78%	76%	76%	78%	83%	32%	64%	71%	100%		
Portugal	1,048	55.3%	60%	73%	52%	62%	59%	44%	82%	75%	58%	100%	
Spain	390	70.6%	72%	79%	60%	69%	65%	32%	71%	80%	64%	70%	100%

Defining the risk contribution

Our credit risk measure $\mathscr{R}(w) = \sqrt{w^\top \Sigma w}$ is a convex risk measure. It means that:

$$\begin{aligned}
\mathscr{R}(w_1, \ldots, w_n) &= \sum_{i=1}^{n} w_i \cdot \frac{\partial \, \mathscr{R}(w_1, \ldots, w_m)}{\partial \, w_i} \\
&= \sum_{i=1}^{n} RC_i
\end{aligned}$$

We can then decompose the risk measure exactly by n individual sources of risk.

The risk contribution RC_i is an increasing function of the parameters D_i, $S_i(t)$ and σ_i^S.

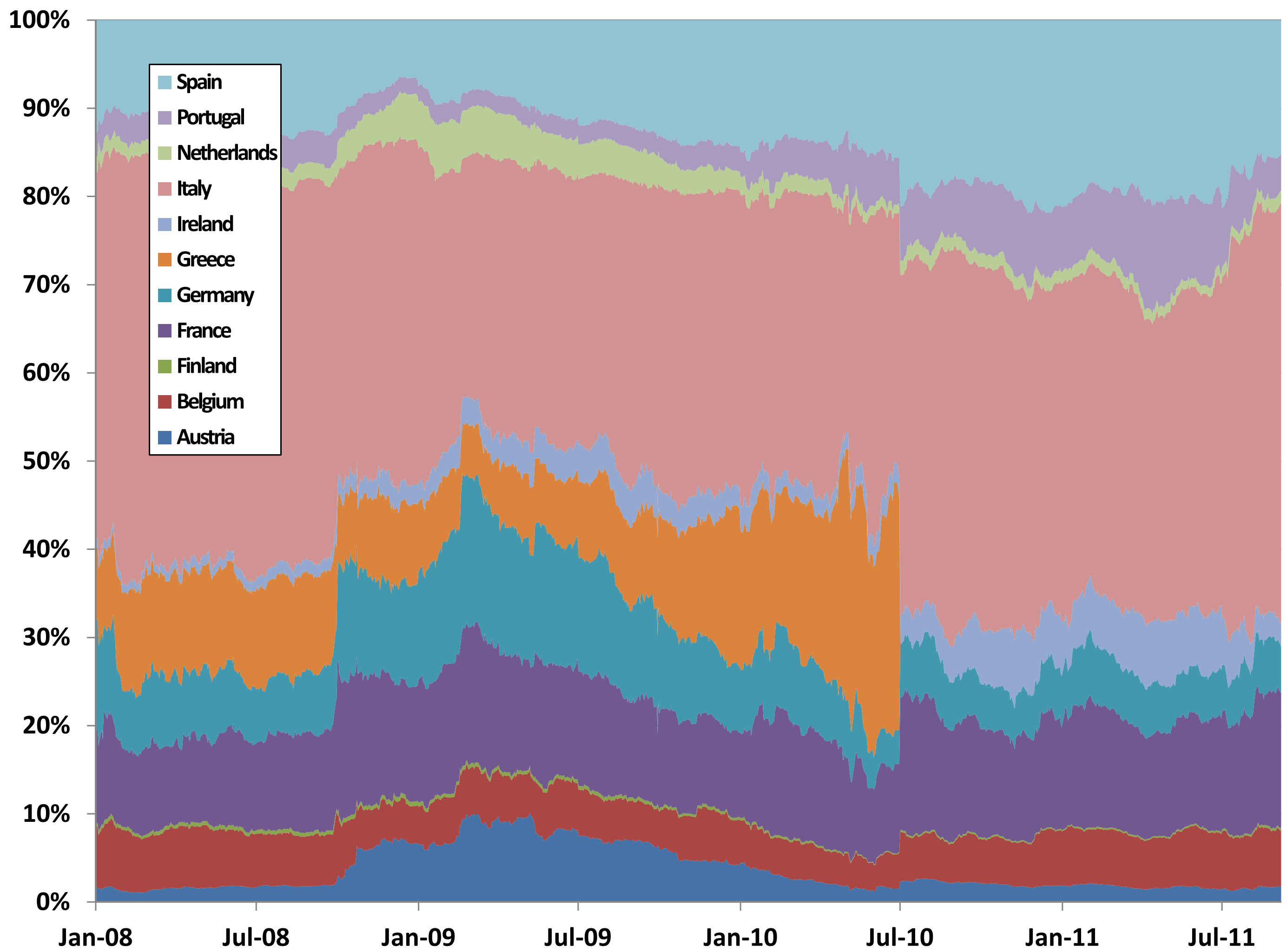
Some results for the EGBI index

Figure: EGBI weights and risk contributions

Country	July-08		July-09		July-10		July-11		September-11	
	Weights	RC	Weights	RC	Weights	RC	Weights	RC	Weights	RC
Austria	4.1%	1.7%	3.6%	7.7%	4.1%	2.3%	4.3%	1.5%	4.3%	1.8%
Belgium	6.2%	6.1%	6.5%	5.1%	6.3%	5.7%	6.4%	6.5%	6.4%	6.7%
Finland	1.2%	0.4%	1.3%	0.5%	1.3%	0.2%	1.6%	0.2%	1.6%	0.3%
France	20.5%	9.8%	20.4%	13.2%	22.2%	15.1%	23.1%	13.3%	23.1%	15.5%
Germany	24.4%	6.1%	22.3%	13.0%	22.9%	6.0%	22.1%	5.3%	22.1%	5.5%
Greece	4.9%	11.4%	5.4%	8.5%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Ireland	1.0%	1.3%	1.5%	4.3%	2.1%	3.3%	1.4%	5.4%	1.4%	2.7%
Italy	22.1%	45.2%	22.4%	29.5%	23.4%	38.7%	23.1%	38.5%	23.1%	46.3%
Netherlands	5.3%	1.7%	5.3%	4.1%	6.1%	1.6%	6.2%	1.2%	6.2%	1.5%
Portugal	2.4%	3.9%	2.3%	2.3%	2.1%	6.3%	1.6%	6.6%	1.6%	4.4%
Spain	7.8%	12.4%	9.1%	11.8%	9.6%	20.9%	10.3%	21.3%	10.3%	15.5%
Sovereign Risk	0.70%		2.59%		6.12%		4.02%		8.12%	

Some results for the EGBI index

Evolution of risk contributions



GDP indexation

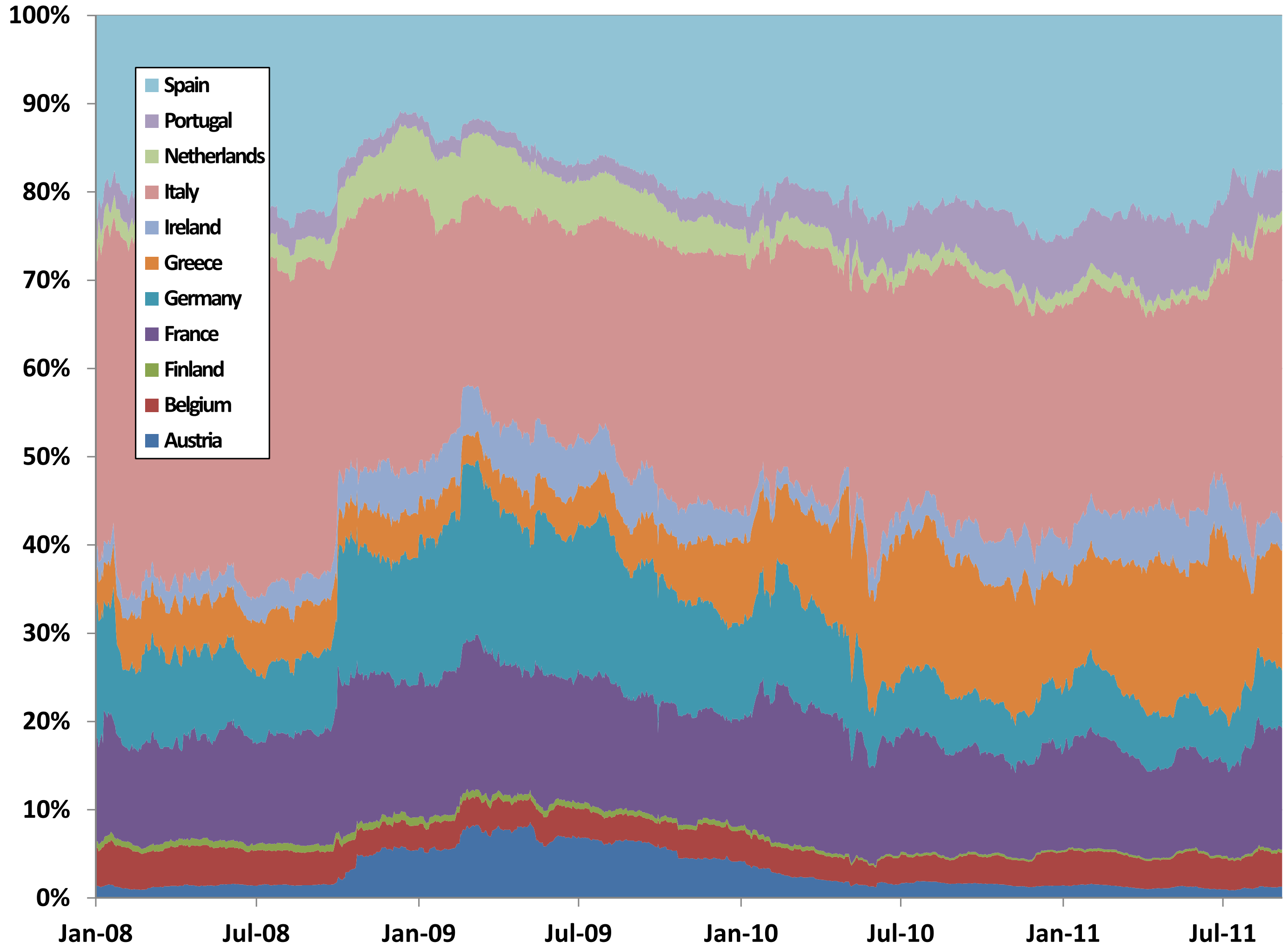
Figure: Weights and risk contributions of the GDP indexation

Country	July-08		July-09		July-10		July-11		September-11	
	GDP	RC	GDP	RC	GDP	RC	GDP	RC	GDP	RC
Austria	3.1%	1.4%	3.1%	7.0%	3.1%	1.7%	3.2%	1.0%	3.2%	1.3%
Belgium	3.8%	4.0%	3.8%	3.2%	3.9%	3.3%	4.0%	3.5%	4.0%	4.0%
Finland	2.0%	0.8%	1.9%	0.7%	2.0%	0.3%	2.1%	0.3%	2.1%	0.3%
France	21.2%	11.2%	21.5%	14.9%	21.4%	13.4%	21.5%	10.6%	21.5%	13.8%
Germany	27.4%	7.6%	27.2%	17.0%	27.7%	6.7%	27.9%	5.8%	27.9%	6.7%
Greece	2.6%	6.2%	2.7%	4.4%	2.6%	15.7%	2.4%	19.8%	2.4%	13.8%
Ireland	2.0%	3.0%	1.9%	5.6%	1.8%	2.6%	1.7%	5.9%	1.7%	3.3%
Italy	17.4%	37.5%	17.3%	23.5%	17.2%	25.8%	17.0%	23.9%	17.0%	32.6%
Netherlands	6.5%	2.5%	6.5%	5.3%	6.5%	1.6%	6.6%	1.2%	6.6%	1.6%
Portugal	1.9%	3.3%	1.9%	2.0%	1.9%	5.3%	1.9%	6.7%	1.9%	5.1%
Spain	12.0%	22.6%	12.0%	16.5%	11.8%	23.7%	11.8%	21.4%	11.8%	17.5%
Sovereign Risk	0.64%		2.47%		6.59%		4.56%		8.26%	

$\Rightarrow$ Debt and GDP indexations produces similar sovereign credit risk measures.

GDP indexation

Evolution of risk contributions



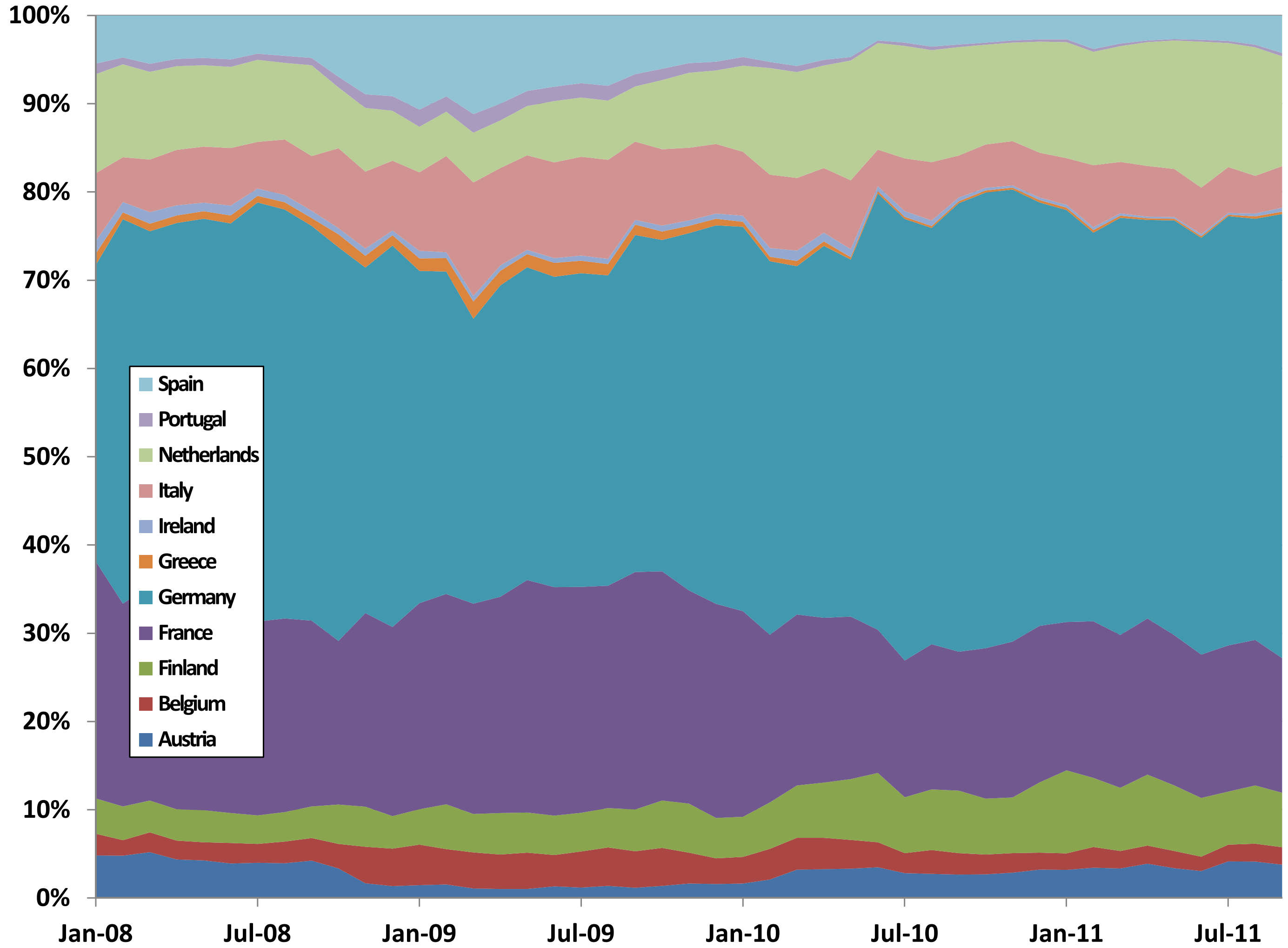
GDP-RB indexation

Figure: Weights and risk contributions of the GDP-RB indexation

Country	July-08		July-09		July-10		July-11		September-11	
	RC	Weights	RC	Weights	RC	Weights	RC	Weights	RC	Weights
Austria	3.1%	3.9%	3.1%	1.2%	3.1%	2.9%	3.2%	4.2%	3.2%	3.8%
Belgium	3.8%	2.1%	3.8%	4.1%	3.9%	2.2%	4.0%	1.9%	4.0%	2.0%
Finland	2.0%	3.2%	1.9%	4.4%	2.0%	6.3%	2.1%	6.0%	2.1%	6.1%
France	21.2%	22.0%	21.5%	25.6%	21.4%	15.5%	21.5%	16.5%	21.5%	15.3%
Germany	27.4%	47.8%	27.2%	35.5%	27.7%	50.0%	27.9%	48.7%	27.9%	50.2%
Greece	2.6%	0.7%	2.7%	1.4%	2.6%	0.2%	2.4%	0.2%	2.4%	0.3%
Ireland	2.0%	0.8%	1.9%	0.6%	1.8%	0.6%	1.7%	0.2%	1.7%	0.5%
Italy	17.4%	5.3%	17.3%	11.2%	17.2%	6.0%	17.0%	5.2%	17.0%	4.7%
Netherlands	6.5%	9.2%	6.5%	6.7%	6.5%	12.8%	6.6%	14.0%	6.6%	12.5%
Portugal	1.9%	0.7%	1.9%	1.6%	1.9%	0.4%	1.9%	0.2%	1.9%	0.4%
Spain	12.0%	4.2%	12.0%	7.7%	11.8%	3.1%	11.8%	2.9%	11.8%	4.2%
Sovereign Risk	0.39%		2.10%		3.25%		1.91%		4.13%	

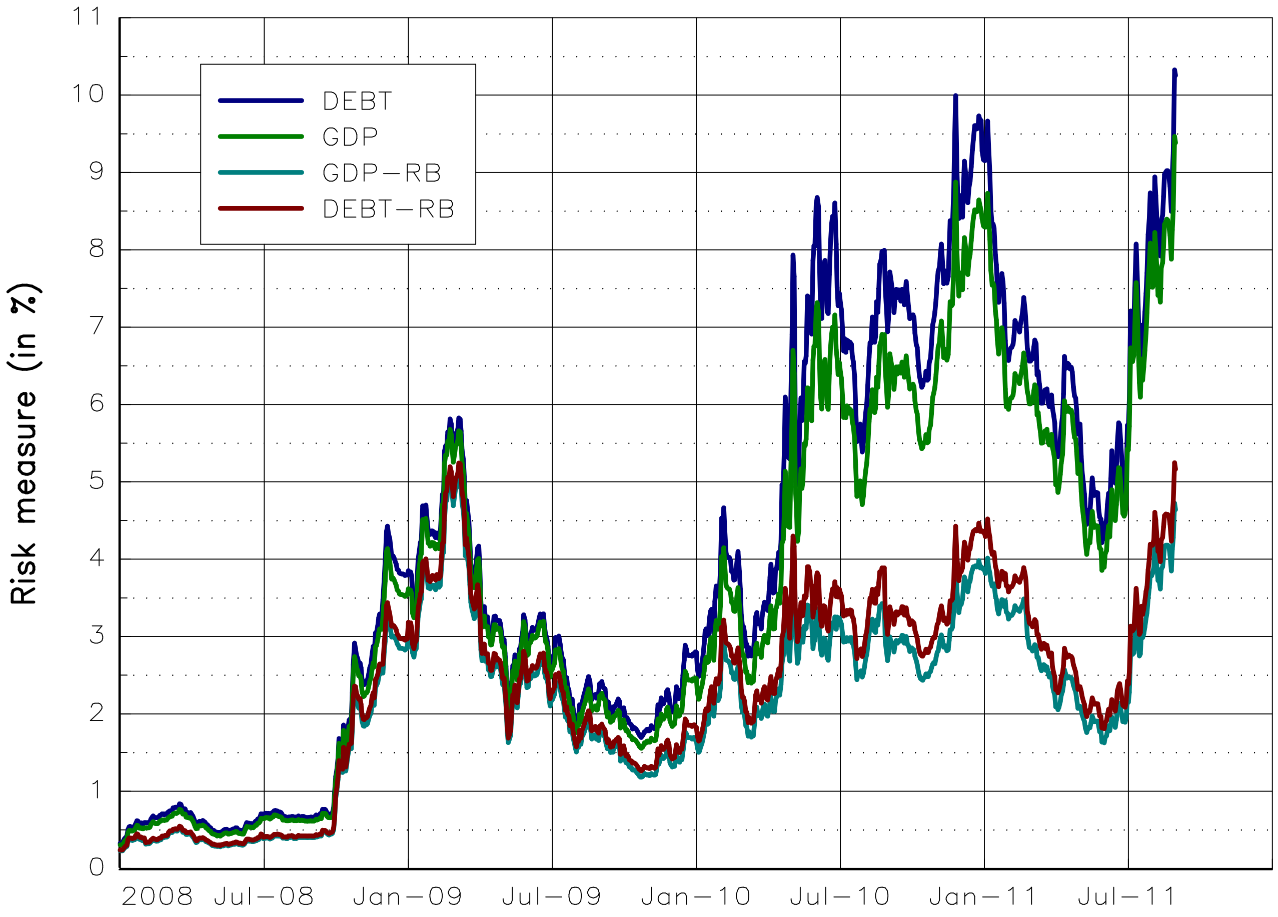
GDP-RB indexation

Evolution of weights



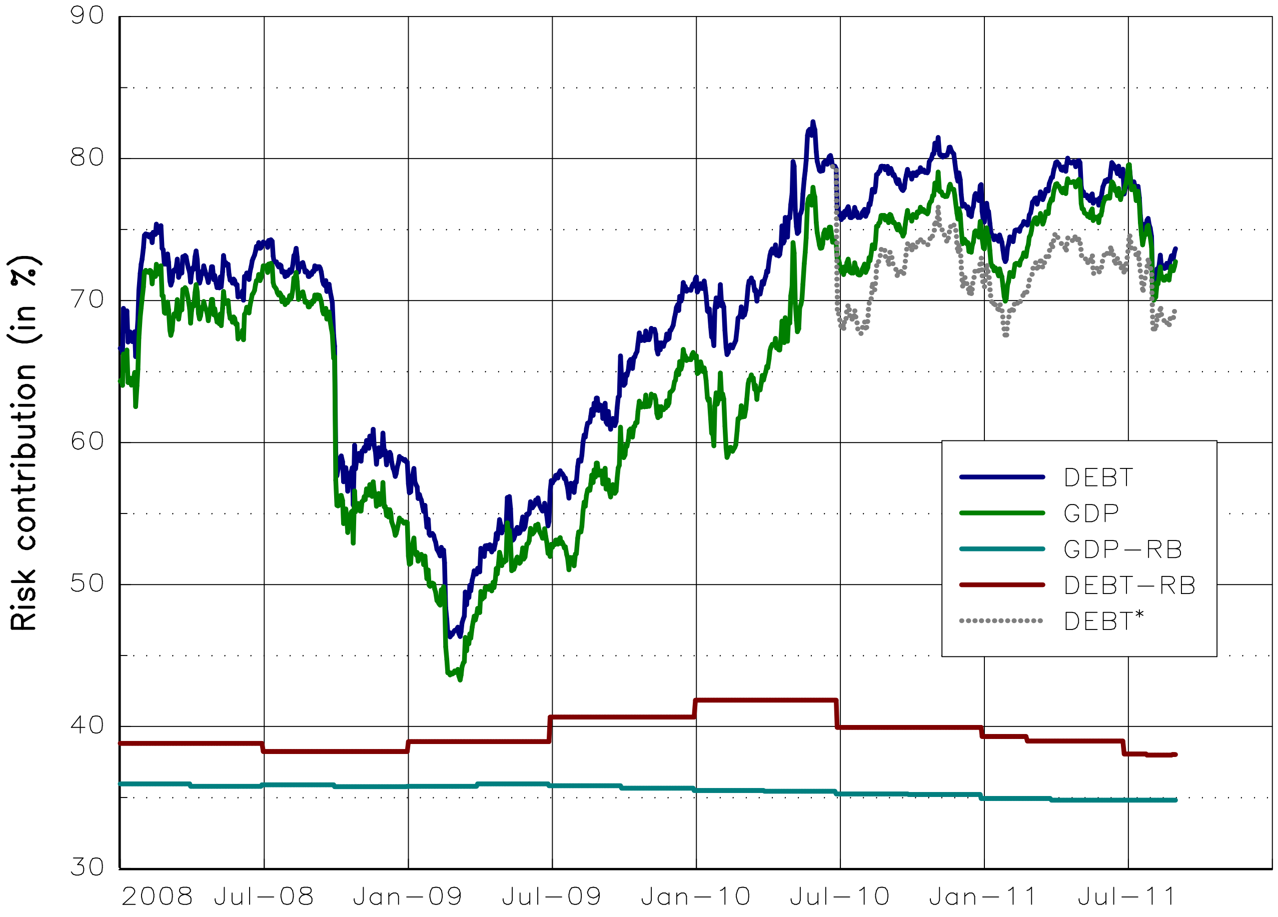
Comparison of the indexing schemes

Evolution of the risk measure



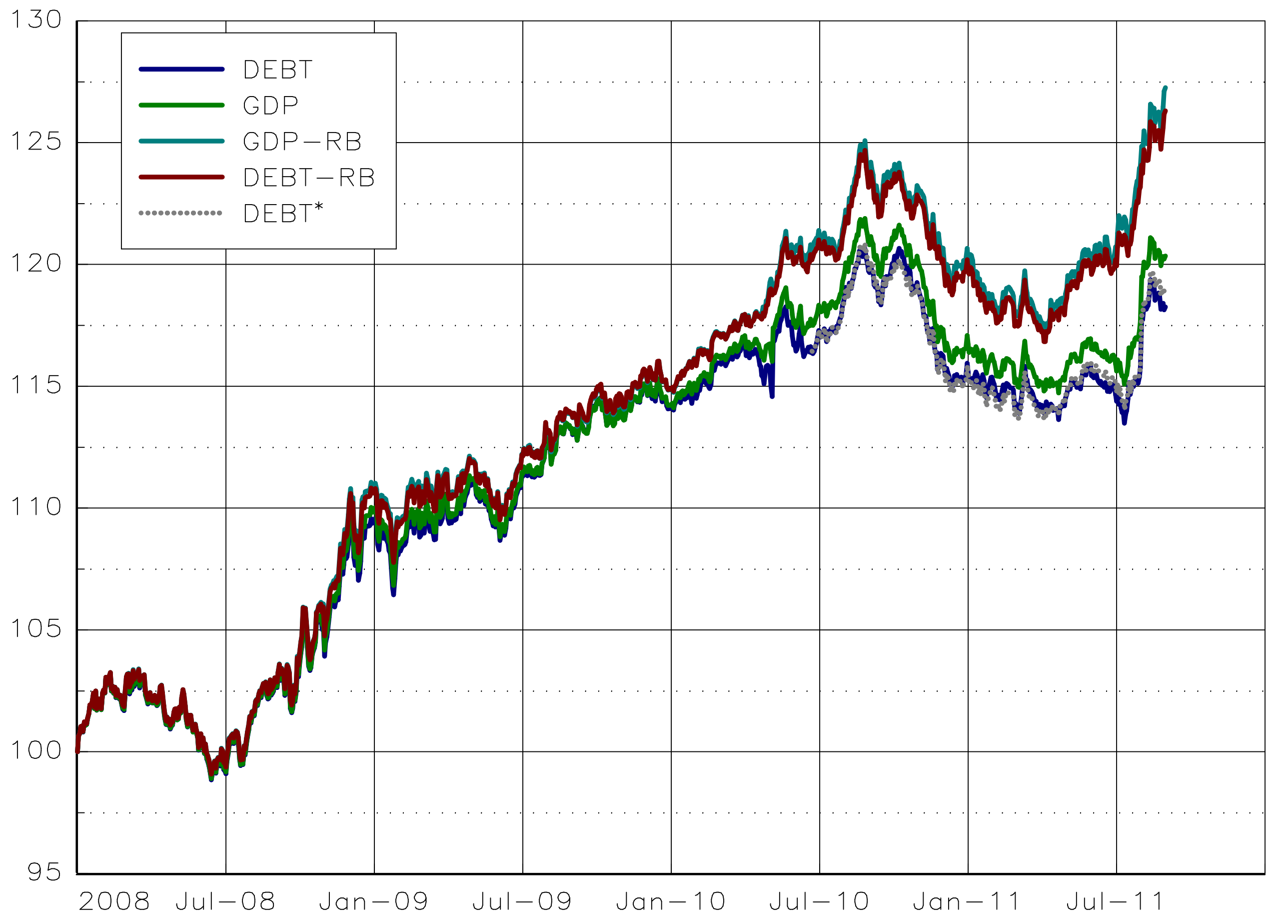
Comparison of the indexing schemes

Evolution of the GIIPS risk contribution



Comparison of the indexing schemes

Simulation of the performance



Comparison with active management

- Database: Morningstar
- Category: Bond EURO Government
- 218 funds

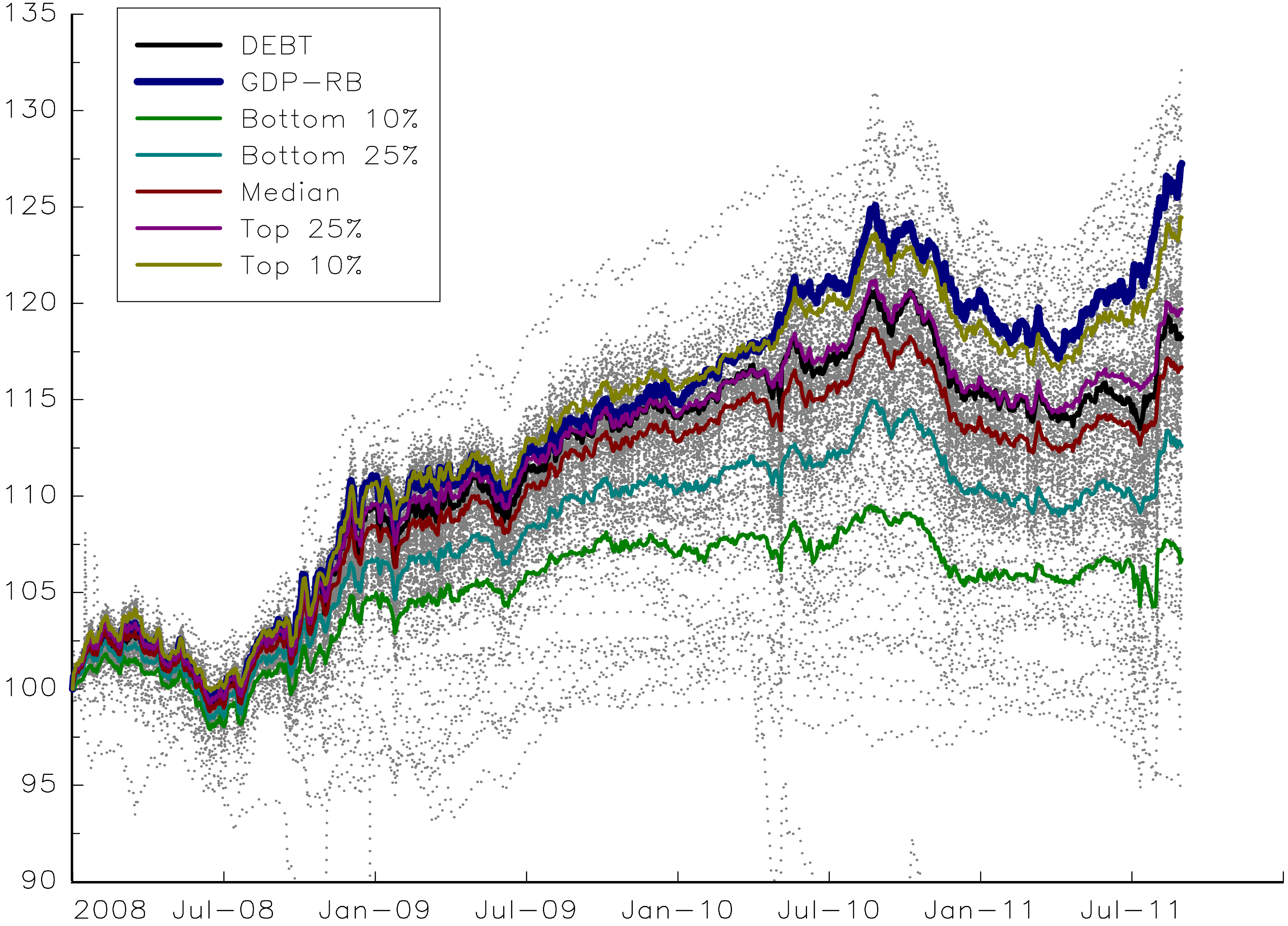
The Academic Rule[3]:

$$\text{Average Performance of Active Management} \\ = \\ \text{Performance of the Index} - \text{Management Fees}$$

⇒ Implied fees for Bond EURO Government: 36 bps / year

[3] There is a large literature on this subject, see e.g. Blake *et al.* (1993).

Comparison with active management



Conclusion

- Portfolio optimization leads to concentrated portfolios in terms of weights and risk
- The use of weights constraints to diversify is equivalent to a discretionary shrinkage method
- Risk-budgeting techniques is a better approach to diversify portfolios
- Strong focus on ERC portfolios last years from the asset management industry (equity indexes, risk parity funds, commodity funds, multi-strategy funds)

For Further Reading I

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