Stock-Bond Correlation: Theory & Empirical Results*

Lorenzo Portelli
Amundi Investment Institute
lorenzo.portelli@amundi.com

Thierry Roncalli
Amundi Investment Institute
thierry.roncalli@amundi.com

April 2024

Abstract

Stock-bond correlation is an important component of portfolio allocation. It is widely used by institutional investors to determine strategic asset allocation, and is carefully monitored by multi-asset fund managers to implement tactical asset allocation. Over the past 20 years, the correlation between stock and bond returns in the US has been negative, while it was largely positive prior to the dot-com crisis. Investors currently believe that a negative stock-bond correlation is more beneficial than a positive stock-bond correlation because it reduces the risk of a balanced portfolio and limits drawdowns during periods of equity market distress.

In this study, we provide an overview of stock-bond correlation modeling. In the first part, we present several theoretical models related to the comovement of stock and bond returns. We distinguish between performance and hedging assets and show that negative correlation implies a negative bond risk premium due to the covariance risk premium component. In contrast, the payoff approach can explain that bonds can be both performance and hedging assets. In addition, a good understanding of the stock-bond correlation requires an assessment of the relationship between the aggregate stock-bond correlation at the portfolio level and the individual stock-bond correlation at the asset level. Macroeconomic models are also useful in interpreting the sign of the stock-bond correlation. They can be divided into three categories: inflation-centric, real-centric, and inflation-growth based.

The second part presents the empirical results. We find that the joint dynamics of stock and bond returns differ across countries. The negative stock-bond correlation is mainly associated with the North American market and the European market before the European debt crisis. When sovereign credit risk is a concern, we generally observe a positive stock-bond correlation. However, even in the US, we cannot speak of a unique stock-bond correlation, as the level depends strongly on the composition of the equity portfolio. We also confirm the influence of the inflation factor, but the results for the growth factor are not robust. Finally, we show that the stock-bond correlation is mainly explained by the extreme market regimes, since the stock-bond correlation can be assumed to be zero in normal market regimes.

Keywords: Stock-bond correlation, risk premium, payoff approach, growth, inflation.

JEL Classification: G11, C01, E3.

*The authors are very grateful to Mohamed Ben Slimane, Viviana Gisimundo, Karine Hervé, Frédéric Lepetit, Francesco Sandrini, Raphaël Sobotka, Lauren Stagnol and Eric Tazé-Bernard for their helpful comments. The opinions expressed in this research are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

1 Introduction

Asset correlation, along with expected return and volatility, is one of the key elements of portfolio optimization. It is therefore closely related to the concept of diversification. Among the various asset correlations, one is of primary interest: the stock-bond correlation. This is the linear (or Pearson) correlation between stock and bond returns. One of the big questions about the stock-bond correlation is its sign. Is it negative or positive? Until the beginning of this century, investors believed that the natural rule was a positive correlation. This is easily explained by valuation models because the present value of both stock and bond cash flows are similarly affected by changes in discount rates. Therefore, when interest rates fall (or rise), the prices of stocks and bonds rise (or fall), creating a positive covariance between stock and bond returns. A break is observed during the dot-com cycle, and the years 2000-2019 are associated in the minds of many investors with a negative stock-bond correlation. During these periods, investors believed that negative correlation was the natural rule. However, this new normal is being challenged as recent years have shown that both stocks and bonds can suffer at the same time. As a result, the estimated correlation has generally been positive over the past four years. The fear that the negative regime will disappear and lead to a regime of positive correlation has captured some investors, which explains the great debate on this parameter in the financial community.

To answer the question of whether the stock-bond correlation must be positive or negative, it is important to define what we mean by stock-bond correlation. In many studies, the concept of stock-bond correlation is generally not well defined. First, it is the correlation between the returns of long-duration government bonds and the returns of the country's stock market. Typically, the US stock-bond correlation refers to the correlation between the 10-year US Treasury bond and the S&P 500 Index. It is important to note that bond returns are calculated using a given maturity, while stock returns are calculated using a proxy for the market portfolio of large-cap stocks. The choice of 10 years is somewhat arbitrary, as we could use a shorter maturity, say 7 years, but the duration of the bond must be sufficiently long. When applying to other countries or regions, we generally use the same approach, taking into account local currency yields and calculating the stock-bond correlation from the perspective of local investors. Otherwise, it will be biased by currency risk and may reflect the dependence of equity and bond markets on the US dollar. Another important choice is the statistical method used to estimate the stock-bond correlation. In particular, we need to distinguish between short-term and long-term stock-bond correlations. In this paper, we refer to different time concepts. For the sake of clarity, we use the following definitions:

- The frequency or time frequency is the period used to calculate a given quantity, typically asset returns. A short-term stock-bond correlation is a correlation between stock and bond returns calculated at a short frequency, typically daily, weekly or monthly. A long-term stock-bond correlation is based on a longer time frequency, such as the correlation between annual, three-year or five-year returns.

- The time horizon refers to the period of analysis, and we distinguish between short-run and long-run analysis. Short-run analysis covers a few weeks or months, less than a year. Long-run analysis looks at a long period of time, such as ten, twenty, or thirty years. Medium-run analysis is not well defined (Blanchard, 1997), but can be thought of as the time period in the middle, and mainly concerns one- and two-year dynamics.

- The final time scale is the rolling window used to estimate a statistic. For example, we typically use a four-year rolling window to estimate the stock-bond correlation with monthly returns. This means that the estimator is based on 48 monthly observations.

The correlation between stocks and bonds has been extensively studied in the academic literature, with research tracing the influence of current valuation models, such as those developed by Shiller, on the movements of both stocks and bonds (Shiller, 1982; Barsky, 1986; Campbell and Shiller, 1988; Shiller and Beltratti, 1992). But the real turning point is the release of the seminal and unpublished paper by Li (2002). For this author, uncertainty about expected inflation (or inflation risk) is a more important factor than unexpected inflation and real interest rates in understanding the stock-bond correlation. Another important publication is Ilmanen (2003), who investigated why the stock-bond correlation can change sign and found that growth, inflation and volatility are the three most important factors. Other economic factors will be explored later such as monetary policy or the demand for cash. Uncertainties about growth and inflation continue to be seen as the primary channels linking the economy to stock and bond comovements, but it is too simplistic to reduce the correlation between stocks and bonds to a simple pass-through mechanism from the economy to the financial market. The concept of stock-bond correlation is more complex because it has many implications for investors' views. For example, it is related to diversification and thus to the implied risk premia priced in by the market. A negative stock-bond correlation is associated with a lower, and often negative, bond risk premium. In this context, bonds are seen as a hedge for equity exposure. At the same time, bonds are a source of carry. Does this mean that investors do not use bonds as a carry asset when the correlation between stocks and bonds is negative? Certainly not, otherwise the cost of hedging would be too high. This question is related to an important stylized fact. In the US, the stock-bond correlation tends to be positive when interest rates are high. This is also the case in many countries with high credit risk. This may imply that a negative stock-bond correlation reflects flight-to-quality episodes, *i.e.* the behavior of investors during periods of poor equity performance. The case of the US stock-bond correlation could not then be extended to countries where investors have little confidence in the management of economic risk and debt sustainability. These comments also raise the question of whether an exposure to local bonds is a better equity hedge than an exposure to US bonds.

The paper is organized as follows. In the second section, we review the various theoretical models related to the stock-bond correlation. We clarify the concepts of performance assets and hedging assets. To this end, we present the Black-Litterman approach to risk premia. We show how the sign of correlation affects equity and bond risk premia through the covariance risk premium. To reconcile a positive bond risk premium with a negative stock-bond correlation, we need to consider another framework based on nonlinear payoff functions. In this section, we also examine how the aggregate stock-bond correlation relates to individual stock-bond correlations and the composition of the equity portfolio. The last part of this section is devoted to macroeconomic models of the stock-bond covariance. Several economic factors can explain the magnitude and sign of the correlation, at least three of them: inflation risk, growth risk and real interest rates. In addition, the behavior of investors, especially during periods of flight-to-quality, has also been studied by academics. In a third section, we conduct an empirical analysis. First, we estimate the stock-bond correlation for the US, but also for other DM and EM countries. We find that the correlation cycle in the other countries does not necessarily coincide with the correlation cycle in the US. Thus, it is misleading to speak of a negative stock-bond correlation since the dot-com crisis. Second, we estimate the equity and bond risk premia priced in by the market and analyze the various components, in particular the importance of the covariance risk premium. In the case of the US market, these results are extended to the credit market and to sector and factor equity portfolios. This section ends with an econometric analysis of the relationship between the stock-bond correlation and economic factors and an investigation of the bond payoff. Finally, Section Four provides some concluding remarks.

2 Theoretical models

2.1 Performance assets versus hedging assets

2.1.1 Black-Litterman approach of risk premia

We follow the presentation of Roncalli (2013). We consider an investment universe with n assets. Let $x = (x_1, \ldots, x_n)$ be the vector of weights in the portfolio. Let μ and Σ be the vector of expected returns and the covariance matrix of asset returns, respectively. The risk-free rate is equal to r. It follows that the expected excess return and the volatility of the portfolio are equal to $\pi(x) = x^\top (\mu - r\mathbf{1}_n)$ and $\sigma(x) = \sqrt{x^\top \Sigma x}$. The Markowitz utility function is:

$$\mathcal{U}(x) := \pi(x) - \frac{\phi}{2}\sigma^2(x) = x^\top(\mu - r\mathbf{1}_n) - \frac{\phi}{2}x^\top \Sigma x$$

where ϕ is the risk-aversion parameter. Maximizing the quadratic utility function implies that the unconstrained tangent portfolio is given by[1]:

$$x^\star = \frac{1}{\phi}\Sigma^{-1}(\mu - r\mathbf{1}_n) = \frac{\Sigma^{-1}(\mu - r\mathbf{1}_n)}{\mathbf{1}_n^\top \Sigma^{-1}(\mu - r\mathbf{1}_n)}$$

Given an optimal portfolio $x^\star = x$, we can compute the implied risk premia $\tilde{\pi}$ using the previous formula. We have:

$$\tilde{\pi} := \tilde{\mu} - r\mathbf{1}_n = \phi \Sigma x$$

where $\tilde{\mu}$ is the vector of implied expected returns. Since we have $x^\top(\tilde{\mu} - r\mathbf{1}_n) = \phi x^\top \Sigma x$, we deduce that the associated risk-aversion parameter is equal to:

$$\phi = \frac{x^\top(\tilde{\mu} - r\mathbf{1}_n)}{x^\top \Sigma x} = \frac{\mathrm{SR}(x \mid r)}{\sqrt{x^\top \Sigma x}}$$

Finally, we get:

$$\tilde{\pi} = \mathrm{SR}(x \mid r)\frac{\Sigma x}{\sqrt{x^\top \Sigma x}} = \mathrm{SR}(x \mid r)\frac{\partial \sigma(x)}{\partial x} \tag{1}$$

$\tilde{\pi}$ is the vector of implied risk premia in the Black-Litterman model. It is equal to the Sharpe ratio of the portfolio times the vector of marginal volatilities. In other words, if one asset has a higher marginal volatility than another, then it has a higher expected risk premium:

$$\frac{\partial \sigma(x)}{\partial x_i} \geq \frac{\partial \sigma(x)}{\partial x_j} \Leftrightarrow \tilde{\pi}_i \geq \tilde{\pi}_j$$

2.1.2 Impact of correlation on risk premia

The case of negative correlations To understand Equation (1), we use the formula for the marginal volatility of asset i:

$$\frac{\partial \sigma(x)}{\partial x_i} = \frac{x_i\sigma_i^2 + \sigma_i \sum_{j \neq i} x_j \rho_{i,j}\sigma_j}{\sigma(x)}$$

where σ_j is the volatility of asset j and $\rho_{i,j}$ is the correlation between the returns of assets i and j. We deduce that:

$$\begin{aligned}\tilde{\pi}_i &= \mathrm{SR}(x \mid r)\frac{\partial \sigma(x)}{\partial x_i} \\ &= \frac{\mathrm{SR}(x \mid r)}{\sigma(x)}\left(x_i\sigma_i^2 + \sigma_i \sum_{j \neq i} x_j \rho_{i,j}\sigma_j\right)\end{aligned}$$

[1] See Roncalli (2013, pages 13-14).

We notice that all the terms are positive except for the correlations, which can be negative. In particular, if the correlations between asset i and the other assets are positive, the risk premium of asset i is necessarily positive:

$$\rho_{i,j} \geq 0 \Rightarrow \tilde{\pi}_i \geq 0$$

The risk premium is negative only if at least one of the correlations is negative:

$$\tilde{\pi}_i < 0 \Rightarrow \exists j \neq i : \rho_{i,j} < 0$$

To illustrate the previous two mathematical properties, we consider an example with an investment universe of three assets. The volatilities are 20%, 10% and 5% respectively. We consider three sets of parameters for the correlation values $(\rho_{1,2}, \rho_{1,3}, \rho_{2,3})$ and assume that the market portfolio is $(50\%, 40\%, 10\%)$ and its Sharpe ratio[2] is 0.25. The results are shown in Table 1. When correlations are positive, risk premia are positive and generally increase with volatility risk. When correlations are negative, we can obtain both positive and negative risk premia. For example, in the second case, the risk premium of the third asset is -55 bps. In the third case, the risk premium of the third asset is $+52$ bps, even though $\rho_{2,3}$ is negative.

Table 1: Implied risk premia (in %)

$(\rho_{1,2}, \rho_{1,3}, \rho_{2,3})$	$\tilde{\pi}_1$	$\tilde{\pi}_2$	$\tilde{\pi}_3$
$(50\%, 60\%, 70\%)$	4.79	1.82	0.90
$(60\%, -40\%, -50\%)$	4.85	1.94	−0.55
$(50\%, 60\%, -30\%)$	4.85	1.74	0.52

In finance, we learn that risk is rewarded, which means that the expected return must increase as we take on more risk. How is it possible for an asset to have a negative implied risk premium? Put another way, it means that assets can have an implied expected return that is lower than that of the risk-free asset: $\tilde{\pi}_i < 0 \Rightarrow \tilde{\mu}_i < r$ where r is the risk-free return. In our previous example, if $(\rho_{1,2}, \rho_{1,3}, \rho_{2,3}) = (60\%, -40\%, -50\%)$, we have $\tilde{\mu}_3 = r - 0.55$ bps. In fact, we need to distinguish between performance assets and hedging assets. According to Roncalli (2013), hedging assets correspond to assets whose implied risk premium is negative. Investors accept to include these assets in their portfolio because they would like to hedge the risk of the other assets. Since we know that hedging has a cost, it is therefore normal for the implied risk premium of hedging assets to be negative. In contrast, performance assets systematically have a positive implied risk premium.

Sensitivity of the risk premium to correlations In Appendix A.1 on page 92, we compute the derivative of $\tilde{\pi}_i$ with respect to the correlation $\rho_{i,j}$ and get:

$$\frac{\partial \tilde{\pi}_i}{\partial \rho_{i,j}} = \mathrm{SR}\left(x \mid r\right) \left(\frac{1}{\sigma\left(x\right)} - \frac{x_i \left(\Sigma x\right)_i}{\sigma^3\left(x\right)}\right) x_j \sigma_i \sigma_j$$

We have:

$$\begin{aligned}
\frac{\partial \tilde{\pi}_i}{\partial \rho_{i,j}} \geq 0 \quad & \Leftrightarrow \quad \frac{1}{\sigma\left(x\right)} - \frac{x_i \left(\Sigma x\right)_i}{\sigma^3\left(x\right)} \geq 0 \\
& \Leftrightarrow \quad \frac{x_i \left(\Sigma x\right)_i}{\sigma\left(x\right)} \leq \sigma\left(x\right) \\
& \Leftrightarrow \quad \mathcal{RC}_i \leq \sigma\left(x\right)
\end{aligned}$$

[2] In this section, we assume that the Sharpe ratio of the portfolio is always equal to 25%.

where $\mathcal{RC}_i$ is the risk contribution of asset i if the risk measure is the portfolio volatility. The previous inequality is almost always true because we have the Euler decomposition:

$$\sigma(x) = \sum_{i=1}^{n} \mathcal{RC}_i = \sum_{i=1}^{n} x_i \frac{\partial \sigma(x)}{\partial x_i}$$

$\mathcal{RC}_i > \sigma(x)$ can therefore only be obtained in a very specific configuration, when the universe of assets is not large and some assets have a negative risk contribution. Apart from this case, this means that the risk premium of the asset is an increasing function of its correlations with the other assets.

The relationship between $\tilde{\pi}_i$ and the correlation $\rho_{j,k}$ between two other assets is equal to:

$$\begin{aligned}\frac{\partial \tilde{\pi}_i}{\partial \rho_{j,k}} &= -\operatorname{SR}(x \mid r) \frac{(\Sigma x)_i}{\sigma^3(x)} x_j x_k \sigma_j \sigma_k \\ &= -\operatorname{SR}(x \mid r) \frac{x_j x_k \sigma_j \sigma_k}{\sigma^2(x)} \frac{\partial \sigma(x)}{\partial x_i}\end{aligned}$$

In general, the marginal volatility is positive, implying that the relationship is decreasing:

$$\frac{\partial \sigma(x)}{\partial x_i} \geq 0 \Rightarrow \frac{\partial \tilde{\pi}_i}{\partial \rho_{j,k}} \leq 0$$

The investment universe consists of four assets. The volatility parameters are $\sigma_1 = 20\%$, $\sigma_2 = 10\%$, $\sigma_3 = 5\%$, and $\sigma_4 = 40\%$, while the correlation matrix is:

$$\rho = \begin{pmatrix} 100\% & & & \\ 35\% & 100\% & & \\ 50\% & 60\% & 100\% & \\ 70\% & 80\% & 90\% & 100\% \end{pmatrix}$$

The composition of the market portfolio is $x = (20\%, 10\%, 30\%, 40\%)$. The derivative of the risk premium with respect to the correlations is given in Table 2. We check that the derivative is positive with intra-correlations and negative with inter-correlations.

Table 2: Calculation of the derivative $\dfrac{\partial \tilde{\pi}_i}{\partial \rho_{j,k}}$ (in %)

$\rho_{j,k}$	$\dfrac{\partial \tilde{\pi}_1}{\partial \rho_{j,k}}$	$\dfrac{\partial \tilde{\pi}_2}{\partial \rho_{j,k}}$	$\dfrac{\partial \tilde{\pi}_3}{\partial \rho_{j,k}}$	$\dfrac{\partial \tilde{\pi}_4}{\partial \rho_{j,k}}$
$\rho_{1,2}$	0.202	0.457	−0.010	−0.089
$\rho_{1,3}$	0.304	−0.026	0.222	−0.134
$\rho_{1,4}$	3.239	−0.275	−0.158	0.466
$\rho_{2,3}$	−0.013	0.171	0.056	−0.034
$\rho_{2,4}$	−0.139	1.829	−0.040	0.116
$\rho_{3,4}$	−0.209	−0.103	0.890	0.175

2.1.3 Calculating the implied correlation

For a given value of the risk premium $\pi_i^{\star}$, the implied correlation $\tilde{\rho}_{i,j}$ is the correlation value of $\rho_{i,j}$ such that $\tilde{\pi}_i = \pi_i^{\star}$, all else being equal. In Appendix A.2 on page 93, we show that the implied correlation, if it exists, is equal to:

$$\tilde{\rho}_{i,j} = \rho_{i,j} + \frac{-b \pm \sqrt{b^2 - ac}}{a}$$

where:

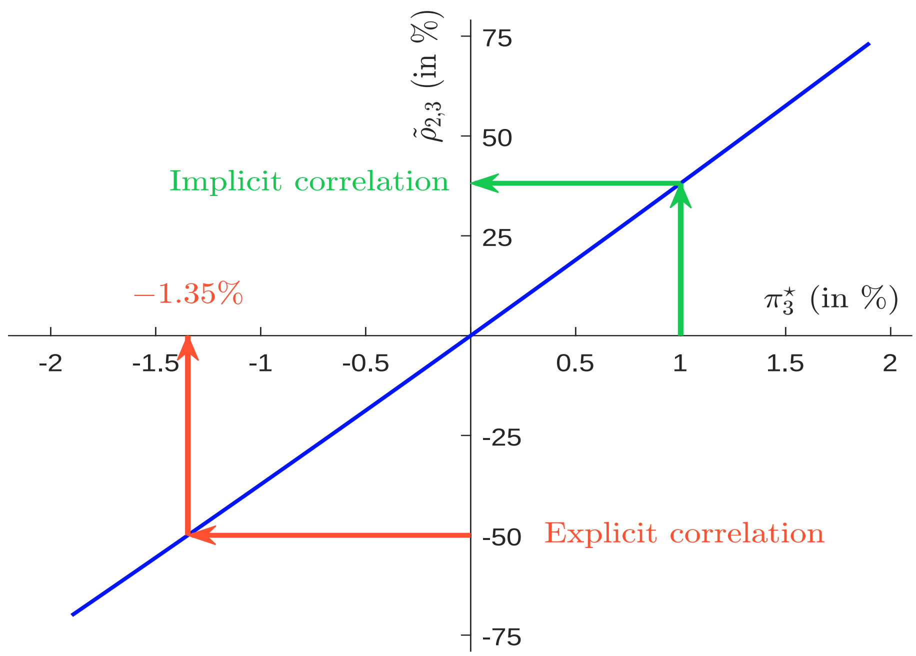
$$\begin{cases} a = x_j^2 \sigma_i^2 \sigma_j^2 \operatorname{SR}^2 \left(x \mid r\right) \\ b = x_j \sigma_i \sigma_j \left(\operatorname{SR}^2 \left(x \mid r\right) \left(\Sigma x\right)_i - x_i {\pi_i^\star}^2\right) \\ c = \left(\operatorname{SR} \left(x \mid r\right) \left(\Sigma x\right)_i\right)^2 - \left(\pi_i^\star \sigma \left(x \mid \rho\right)\right)^2 \end{cases}$$

This analysis is useful when the level of the implied risk premium is far from what market participants expect. Let us consider an example to illustrate this. We take an investment universe of 3 assets, whose volatilities are 20% and whose correlation matrix is:

$$\rho = \begin{pmatrix} 100\% & & \\ 60\% & 100\% & \\ -10\% & -50\% & 100\% \end{pmatrix}$$

The composition of the market portfolio is $\left(50\%, 45\%, 5\%\right)$. The calculation of the implied risk premia gives $\tilde{\pi}_1 = 4.58\%$, $\tilde{\pi}_2 = 4.34\%$ and $\tilde{\pi}_3 = -1.35\%$. We might think that the last risk premium is disturbing, because it is negative. Let us assume that the market prices a risk premium $\pi_3^\star = 1.00\%$. We can then estimate the implied correlation $\tilde{\rho}_{2,3}$ that corresponds to this level of risk premium. We obtain $\tilde{\rho}_{2,3} = 38.2\%$. The calculation is illustrated in Figure 1, which shows the relationship between $\pi_3^\star$ and $\tilde{\rho}_{2,3}$. The explicit approach to risk premia is to define the correlations and derive the risk premia, while the implicit approach is to derive the correlations from the risk premia.

Figure 1: Implied correlation $\tilde{\rho}_{2,3}$



2.1.4 Application to the stock-bond constant-mix strategy

The stock-bond constant-mix strategy is a special case of the previous framework when we consider two assets. Let $x = (x_S, x_B)$ be the portfolio, where x_S and x_B are the equity and bond allocations, respectively expressed in %. According to the analysis of Roncalli (2013, page 277), we have:

$$\tilde{\pi}_S = \frac{\mathrm{SR}\left(x \mid r\right)}{\sigma\left(x\right)} \left(\underbrace{x_S \sigma_S^2}_{\text{variance}} + \underbrace{x_B \rho \sigma_S \sigma_B}_{\text{covariance}} \right)$$

and:

$$\tilde{\pi}_B = \frac{\mathrm{SR}\left(x \mid r\right)}{\sigma\left(x\right)} \left(\underbrace{x_B \sigma_B^2}_{\text{variance}} + \underbrace{x_S \rho \sigma_S \sigma_B}_{\text{covariance}} \right)$$

The risk premium has then two components:

$$\text{risk premium} = \text{variance premium} + \text{covariance premium}$$

The variance premium is a reward to the investor for the specific risk of the asset, while the covariance premium is a reward only if the risk of the asset covaries with the other asset. Therefore, the variance premium is always positive, while the covariance risk can be positive or negative depending on the correlation between the two assets:

$$\left\{ \begin{array}{l} \rho \geq 0 \Leftrightarrow \text{covariance premium} \geq 0 \\ \rho < 0 \Leftrightarrow \text{covariance premium} < 0 \end{array} \right.$$

When the stock-bond correlation is positive, the bond risk premium benefits from the risk of stocks. The higher the equity volatility σ_S, the higher the bond risk premium $\tilde{\pi}_B$:

$$\frac{\partial \tilde{\pi}_B}{\partial \sigma_S} > 0$$

When the stock-bond correlation is negative, the bond risk premium is reduced because the bond exposure hedges part of the equity exposure. Indeed, we have[3]:

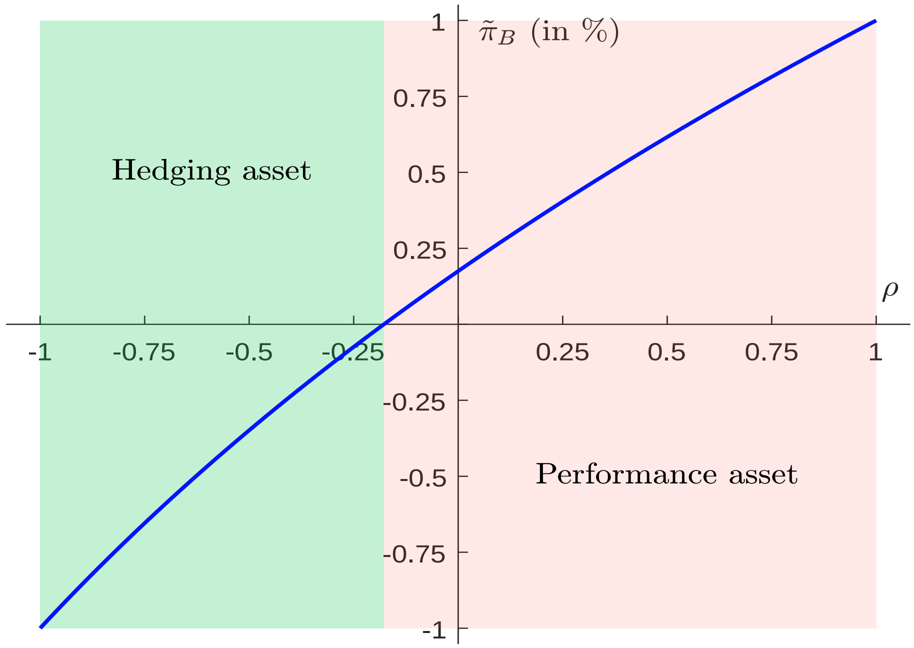
$$\tilde{\pi}_B = \frac{\mathrm{SR}\left(x \mid r\right)}{\sigma\left(x\right)} \left(\underbrace{x_B \sigma_B \left(\sigma_B - \rho \sigma_S\right)}_{\geq 0} + \underbrace{\rho \sigma_S \sigma_B}_{\leq 0} \right)$$

The bond investor then systematically pays the covariance risk $\rho \sigma_S \sigma_B$. The effect of a fly-to-quality regime then has a large impact on the risk premium. During an equity crisis episode, the demand for government bonds increases, leading to a decrease in the stock-bond correlation (Beber *et al.*, 2009; Brière *et al.*, 2012), which coincides with the increase in equity volatility:

$$\text{fly-to-quality} \Rightarrow \left\{ \begin{array}{c} \rho \searrow \\ \sigma_S \nearrow \end{array} \right. \Rightarrow \tilde{\pi}_B \searrow$$

Figure 2 shows the evolution of the bond risk premium $\tilde{\pi}_B$ relative to the correlation parameter ρ for a 60/40 constant-mix portfolio. The volatilities are $\sigma_S = 15\%$ and $\sigma_B = 4\%$. We check that the risk premium becomes negative when the correlation is significantly low. In this example, bonds are considered hedging assets if the correlation is less than -17.78%.

Figure 2: Bond risk premium relative to the stock-bond correlation (60/40 constant-mix)



In fact, we have the following property:

$$\tilde{\pi}_B \leq 0 \Leftrightarrow \rho \leq -\left(\frac{x_B}{1-x_B}\right)\frac{\sigma_B}{\sigma_S}$$

This means that a negative bond risk premium is associated with a negative stock-bond correlation, while a sufficiently negative stock-bond correlation induces a negative bond risk premium. This result seems inconsistent with multi-asset management, since fund managers want both a positive bond risk premium and a negative stock-bond correlation. They want a positive bond risk premium because they want to generate performance. They also want a negative stock-bond correlation to reduce portfolio risk and protect the fund from equity drawdowns. The preceding framework suggests that this vision of multi-asset management may be flawed. In this framework, when the stock-bond correlation is negative, the bond allocation must be viewed as a hedging strategy for the equity allocation. One implication of negative stock-bond correlation and negative bond premium is that bond exposure is bounded:

$$\tilde{\pi}_B \leq 0 \Leftrightarrow x_B \leq -\frac{\rho\sigma_S\sigma_B}{\sigma_B\left(\sigma_B-\rho\sigma_S\right)}$$

This means that the hedging ability of bonds increases with the equity allocation. At first glance, it seems easier to hedge the equity allocation of a 20/80 constant-mix portfolio than that of an 80/20 constant-mix portfolio because the former has fewer equities. However, this is not the case in this framework because investors in the 20/80 constant-mix portfolio view bonds as a performance asset, not a hedging asset. On the contrary, in an 80/20 constant-mix portfolio, the bond allocation can only be used as a hedging buffer because the

[3] Since we have $x_S + x_B = 1$, we obtain:

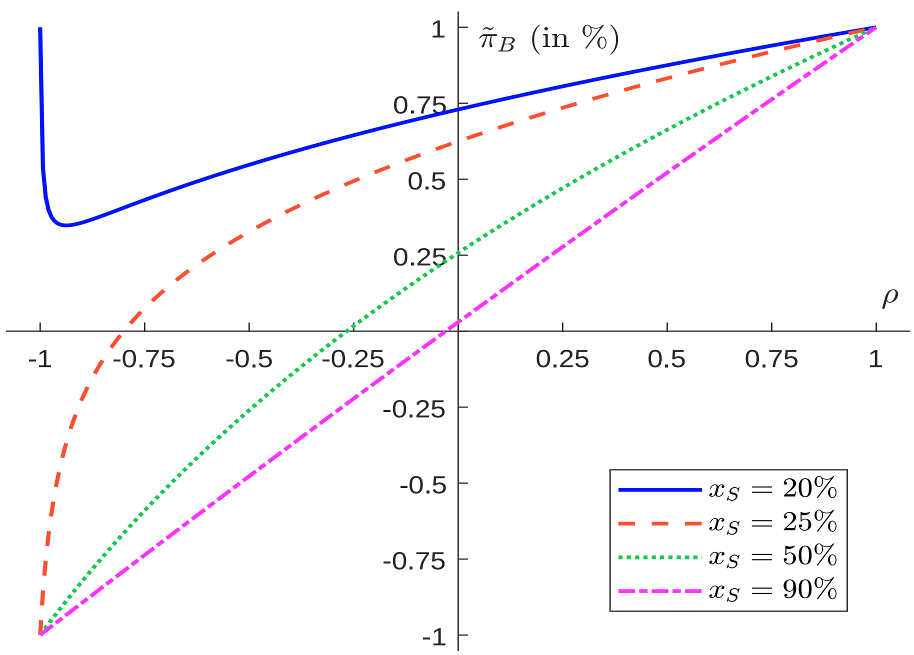
$$x_B\sigma_B^2 + x_S\rho\sigma_S\sigma_B = x_B\sigma_B^2 + (1-x_B)\rho\sigma_S\sigma_B = x_B\sigma_B\left(\sigma_B-\rho\sigma_S\right)+\rho\sigma_S\sigma_B$$

20% in bonds is not enough to significantly affect the performance of the overall portfolio. Figure 3 illustrates the dependence of the bond risk premium on the equity allocation. If the latter is low, the implied bond risk premium can only be positive even if the stock-bond correlation is negative. Since $\tilde{\pi}_B$ is a decreasing function with respect to x_S, we can get negative risk premia. For example, if $\sigma_S = 15\%$ and $\sigma_B = 4\%$, Figure 3 shows that the bond risk premium is negative when $x_S \geq 50\%$ and $\rho \leq -25\%$. These results can be summarized as follows:

1. When the stock-bond correlation is positive, a stock-bond constant-mix portfolio is a diversified strategy; the investor expects performance from both the bond and stock allocations;

2. When the stock-bond correlation is negative and the stock allocation is sufficiently high, a stock-bond constant-mix portfolio is a deleveraged equity strategy in which the bond allocation serves to hedge or protect some of the equity risk; the investor does not expect the bond allocation to participate in the performance of the portfolio and expects to pay a cost for this hedging exposure.

Figure 3: Bond risk premium relative to the stock allocation



So far, we have implicitly assumed that the bond allocation is in government bonds. Now let us see the effect of bond volatility. In the previous examples, we assumed that $\sigma_S = 15\%$ and $\sigma_B = 4\%$, which means that $\sigma_B \ll \sigma_S$. This indicates that bond volatility is low. If we look at a bond allocation that includes more credit risk, we see that the variance risk premium goes up, but the impact on the covariance risk premium depends on the sign of the stock-bond correlation:

$$\sigma_B \nearrow \Rightarrow \left\{ \begin{array}{ll} x_B \sigma_B^2 \nearrow & \\ \rho \sigma_S \sigma_B x_S \nearrow & \text{if } \rho \geq 0 \\ \rho \sigma_S \sigma_B x_S \searrow & \text{if } \rho < 0 \end{array} \right.$$

If $\rho \geq 0$, then it is obvious that $\tilde{\pi}_B$ is an increasing function of σ_B. If $\rho < 0$, the result is uncertain. In Appendix A.3 on page 94, we compute the derivative of $\tilde{\pi}_B$ relative to σ_B and find that:

$$\frac{\partial \tilde{\pi}_B}{\partial \sigma_B} = \frac{\mathrm{SR}\left(x \mid r\right)}{\sigma^3\left(x\right)} \left(\underbrace{x_S^2 x_B \left(2+\rho^2\right)\sigma_S^2\sigma_B + x_B^3\sigma_B^3}_{\geq 0} + \underbrace{\rho}_{<0} \cdot \underbrace{x_S^3\sigma_S^3 + 3x_S x_B^2 \sigma_S \sigma_B^2}_{\geq 0} \right)$$

Again, we have two opposite effects. Each of the terms is of order $\mathcal{O}\left(x^3\right)$ and $\mathcal{O}\left(\sigma^3\right)$. The residual order is $\mathcal{O}\left(3+\rho^2\right)$ for the first term, while it is $\mathcal{O}\left(4\rho\right)$ for the second term. Since $3+\rho^2 \geq 4\rho$, this means that the positive term dominates the negative term, except in some special cases.

Summary and main results

In the CAPM, a risk premium has two components: a variance risk premium, which is always positive, and a covariance risk premium, which is negative when the correlation risk is negative. Stocks always have a positive risk premium because the variance component largely dominates the covariance component. This explains why equities are performance assets. When bonds have high risks, such as default risk, or when interest rate risk is high, bonds have a positive risk premium. This explains why corporate bonds have a positive risk premium. For sovereign bonds, the sign of the risk premium depends strongly on two factors: the idiosyncratic risk of the bonds and the correlation of the bonds with stocks. When correlation is negative and idiosyncratic risk is low, the covariance risk premium dominates the variance risk premium and sovereign bonds have a negative risk premium. In this case, sovereign bonds are hedging assets. Sovereign bonds are performance assets and exhibit a positive risk premium only when the correlation with equities is positive or when the idiosyncratic risk is high enough to offset the negative correlation risk.

2.2 Payoff approach of the diversification

In the previous section, we analyzed the risk premium π_B of bonds, or equivalently the expected return μ_B of bonds, since we have the relationship $\mu_B = \pi_B + r$. We have seen that there is a relationship between the bond risk premium and the equity risk premium, because the market risk premium is a function of the two risk premia:

$$\pi_m = x_S \pi_S + x_B \pi_B = \mathrm{SR}\left(x_m \mid r\right)\sigma_m$$

where π_m and σ_m are the market risk premium and the market volatility associated with the market portfolio x_m. Therefore, we can write:

$$\pi_B = \frac{1}{x_B}\pi_m - \frac{x_S}{x_B}\pi_S$$

Below, we consider the payoff approach to the bond risk premium, *i.e.* we calculate the conditional bond risk premium with respect to the performance of the stock market:

$$\pi_{B|S} = \mathbb{E}\left[R_B\left(t\right) \mid R_S\left(t\right)\right]$$

2.2.1 The linear payoff case

Multi-asset investment universe Consider an investment universe with n assets. We assume that asset returns are Gaussian: $R(t) \sim \mathcal{N}(\mu, \Sigma)$. Let $R_i(t)$ be the return of asset i. In Appendix A.4 on page 95, we show that:

$$\begin{aligned} R_i(t) &= \alpha_i + \sum_{j \neq i} \beta_{(i,j)} R_j(t) + u_i(t) \\ &= \alpha_i + \beta_{(i)}^\top R_{(-i)}(t) + u_i(t) \end{aligned} \tag{2}$$

We use the notation $(-i)$ to denote a vector or a matrix with $(n-1)$ dimensions, where asset i is excluded. We have:

$$\begin{cases} \alpha_i = \mu_i - \beta_{(i)}^\top \mu_{(-i)} \\ \beta_{(i)} = \Sigma_{(-i,-i)}^{-1} \Sigma_{(-i,i)} \\ u_i(t) \sim \mathcal{N}\left(0, \Sigma_{(i,i)} - \Sigma_{(i,-i)} \Sigma_{(-i,-i)}^{-1} \Sigma_{(-i,i)}\right) \end{cases}$$

From Equation (2), we see that the unconditional expectation of $R_i(t)$ is the expected return μ_i:

$$\begin{aligned} \mathbb{E}\left[R_i(t)\right] &= \mathbb{E}\left[\alpha_i + \beta_{(i)}^\top R_{(-i)}(t) + u_i(t)\right] \\ &= \alpha_i + \beta_{(i)}^\top \mathbb{E}\left[R_{(-i)}(t)\right] \\ &= \mu_i - \beta_{(i)}^\top \mu_{(-i)} + \beta_{(i)}^\top \mu_{(-i)} \\ &= \mu_i \end{aligned}$$

From Equation (2), we can also estimate the conditional expectation of $R_i(t)$ with respect to the performance of the other assets:

$$\begin{aligned} \mathbb{E}\left[R_i(t) \mid R_{(-i)}(t)\right] &= \alpha_i + \beta_{(i)}^\top R_{(-i)}(t) \\ &= \alpha_i + \sum_{j \neq i} \beta_{(i,j)} R_j(t) \\ &\neq \mu_i \end{aligned} \tag{3}$$

Equation (3) describes the payoff relationship of the asset return $R_i(t)$. If we care about risk, the unconditional variance of $R_i(t)$ is the variance σ_i^2:

$$\begin{aligned} \operatorname{var}\left(R_i(t)\right) &= \mathbb{E}\left[\left(\left(\alpha_i + \beta_{(i)}^\top R_{(-i)}(t) + u_i(t)\right) - \mu_i\right)^2\right] \\ &= \mathbb{E}\left[\left(\beta_{(i)}^\top \left(R_{(-i)}(t) - \mu_{(-i)}\right) + u_i(t)\right)^2\right] \\ &= \beta_{(i)}^\top \Sigma_{(-i,-i)} \beta_{(i)} + \Sigma_{(i,i)} - \Sigma_{(i,-i)} \Sigma_{(-i,-i)}^{-1} \Sigma_{(-i,i)} \\ &= \Sigma_{(i,i)} + \Sigma_{(i,-i)} \Sigma_{(-i,-i)}^{-1} \Sigma_{(-i,-i)} \Sigma_{(-i,-i)}^{-1} \Sigma_{(-i,i)} - \Sigma_{(i,-i)} \Sigma_{(-i,-i)}^{-1} \Sigma_{(-i,i)} \\ &= \Sigma_{(i,i)} \end{aligned}$$

while the conditional variance takes the following expression:

$$\begin{aligned}
\operatorname{var}\left(R_i(t) \mid R_{(-i)}(t)\right) &= \mathbb{E}\left[\left(\left(\alpha_i + \beta_{(i)}^\top R_{(-i)}(t) + u_i(t)\right) - \left(\alpha_i + \beta_{(i)}^\top R_{(-i)}(t)\right)\right)^2\right] \\
&= \mathbb{E}\left[u_i(t)^2\right] \\
&= \Sigma_{(i,i)} - \Sigma_{(i,-i)}\Sigma_{(-i,-i)}^{-1}\Sigma_{(-i,i)}
\end{aligned}$$

We check that $\operatorname{var}\left(R_i(t) \mid R_{(-i)}(t)\right) \leq \sigma_i^2$.

Remark 1. *The previous analysis can be extended to risk premia. Indeed, we have* $R(t) - r \sim \mathcal{N}(\mu - r, \Sigma)$. *So we can replace* $R_i(t)$ *by* $R_i(t) - r$ *and* μ_i *by* $\pi_i = \mu_i - r$ *and the analysis remains valid.*

Two-asset investment universe Let us apply the previous analysis to the stock-bond constant-mix strategy. The payoff relationship is:

$$\mathbb{E}\left[R_B(t) \mid R_S(t)\right] = \alpha_B + \beta_B R_S(t) \tag{4}$$

where $\alpha_B = \mu_B - \beta_B \mu_S$ and $\beta_B = \rho \dfrac{\sigma_B}{\sigma_S}$, while the conditional variance is:

$$\operatorname{var}\left(R_B(t) \mid R_S(t)\right) = \sigma_B^2 - \frac{(\rho\sigma_B\sigma_S)^2}{\sigma_S^2} = \left(1 - \rho^2\right)\sigma_B^2$$

To analyze the payoff of bonds relative to stocks, we distinguish two cases:

- The stock-bond correlation is positive
  The bond payoff is positive when stock returns are greater than a threshold:

$$\begin{aligned}
\mathbb{E}\left[R_B(t) \mid R_S(t)\right] \geq 0 &\Leftrightarrow \alpha_B + \beta_B R_S(t) \geq 0 \\
&\Leftrightarrow R_S(t) \geq -\frac{\alpha_B}{\beta_B} \\
&\Leftrightarrow R_S(t) \geq \mu_S - \frac{\mu_B \sigma_S}{\rho \sigma_B}
\end{aligned}$$

  Since $\mu_S \geq 0$, if stock returns are positive, then the bond payoff is positive.

- The stock-bond correlation is negative
  Since the coefficient β_B is negative, the bond payoff is positive when stocks do not perform well:

$$\begin{aligned}
\mathbb{E}\left[R_B(t) \mid R_S(t)\right] \geq 0 &\Leftrightarrow \alpha_B + \beta_B R_S(t) \geq 0 \\
&\Leftrightarrow R_S(t) \leq -\frac{\alpha_B}{\beta_B} \\
&\Leftrightarrow R_S(t) \leq -\mu_S + \frac{\mu_B \sigma_S}{\rho \sigma_B}
\end{aligned}$$

  If stock returns are negative, then the bond payoff is positive.

We assume that $\mu_S = 8\%$, $\mu_B = 2\%$, $\sigma_S = 15\%$, $\sigma_B = 3\%$. Figure 4 shows the bond payoff for three values of the stock-bond correlation ρ. When $\rho = 0\%$, the payoff does not depend on stock returns. When $\rho = 50\%$, we obtain an upward payoff, which means that

Figure 4: Bond payoff relative to equity return

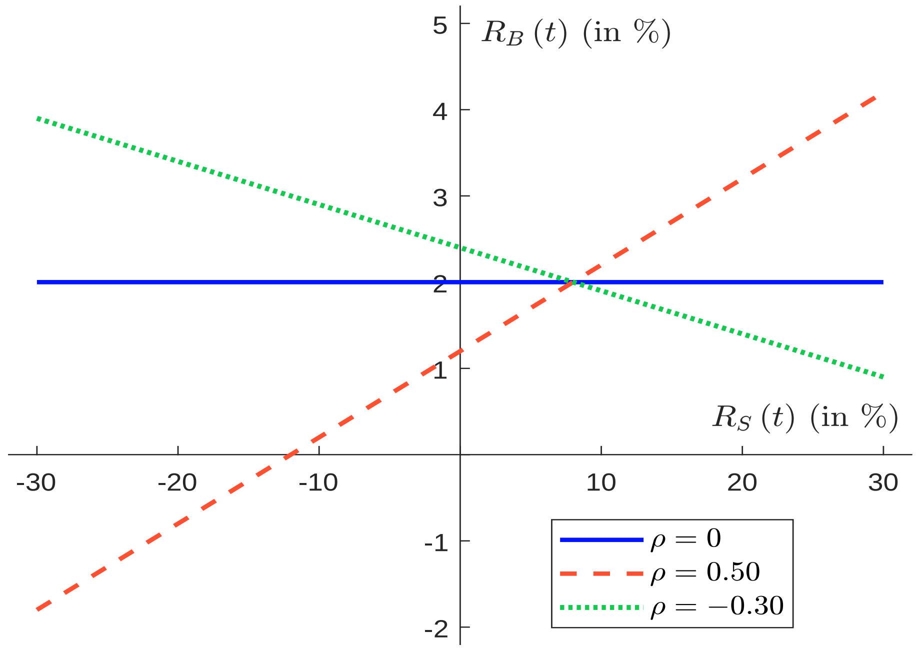
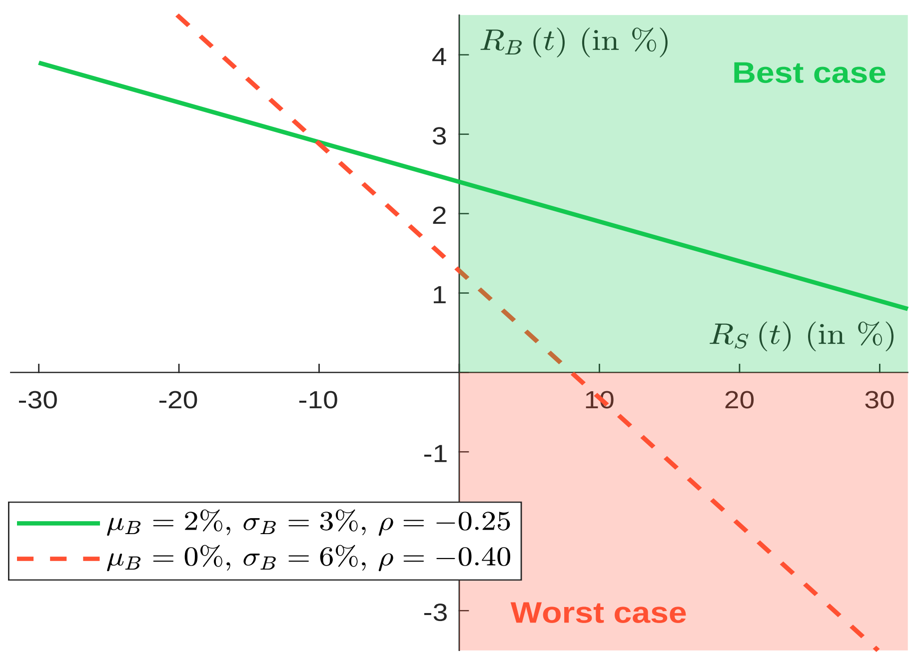


Figure 5: Best and worst cases of a bond payoff



the expected performance of bonds increases with the performance of the stock market. When $\rho = -30\%$, we obtain a downward payoff. In this case, when stocks perform very poorly, bonds perform very well. In a bear market for stocks, the performance of bonds is, on average, greater than the unconditional bond risk premium.

As noted by Roncalli (2018), the payoff of equity asset classes is an increasing affine function, and it is noteworthy that the payoffs of stocks and bonds cross in the upper right quadrant (green payoff[4] in Figure 5). The worst case of diversification is when the good return from stocks is offset by the bad return from bonds. This is the case, for example, for a bond payoff where the stock-bond correlation is highly negative or the bond risk premium is low (red payoff[5] in Figure 5). Therefore, a long-only diversified portfolio of stocks and bonds really makes sense when bonds diversify stocks in bad times and are a performance asset in good times.

The previous analysis then clarifies the concept of hedging assets in a diversified portfolio. In fact, investors do not need to be diversified all the time. In particular, they do not need diversification in good times because they do not want the positive returns of some assets to be offset by the negative returns of other assets. Therefore, diversification can destroy portfolio performance in good times. Investors only need diversification in bad economic times and stressed markets. Therefore, in the case of the stock-bond constant-mix strategy, investors prefer a negative stock-bond correlation in bad times, but they are happy with a positive stock-bond correlation in good times. From this perspective, bonds can be both a hedging asset and a performance asset, and the concept of a hedging asset only makes sense in bad times and not all the time. To characterize the hedging property of bonds in bad times, we compute the probability that the bond return is positive conditional on the stock return:

$$
\begin{aligned}
\Pr\left\{R_B(t) \geq 0 \mid R_S(t)\right\} &= \Pr\left\{u_B(t) \geq -\left(\alpha_B + \beta_B R_S(t)\right)\right\} \\
&= \Phi\left(\frac{\alpha_B + \beta_B R_S(t)}{\sqrt{1-\rho^2}\sigma_B}\right) \\
&= \Phi\left(\frac{\mu_B + \beta_B\left(R_S(t) - \mu_S\right)}{\sqrt{1-\rho^2}\sigma_B}\right) \\
&= \Phi\left(\frac{\mu_B\sigma_S + \rho\sigma_B\left(R_S(t) - \mu_S\right)}{\sqrt{1-\rho^2}\sigma_S\sigma_B}\right)
\end{aligned}
$$

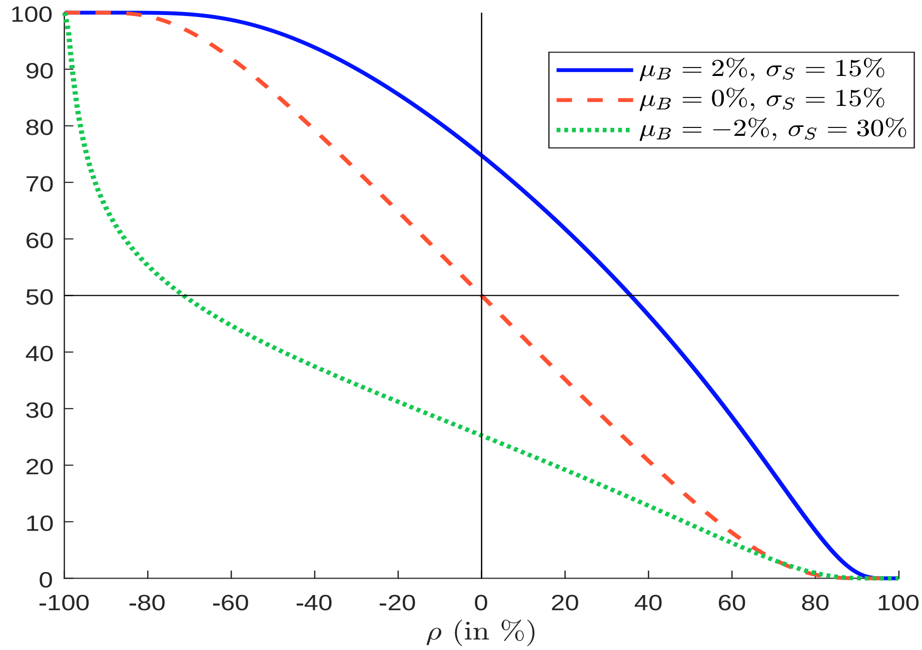
Figure 6 shows the evolution of $\Pr\left\{R_B(t) \geq 0 \mid R_S(t) = -20\%\right\}$ with respect to the stock-bond correlation ρ when the default parameters are $\mu_S = 8\%$, $\mu_B = 2\%$, $\sigma_S = 15\%$, and $\sigma_B = 3\%$. In this case, the probability is greater than 50% when the correlation is less than 40%. This is a good situation because bonds have a high probability of generating positive returns when stocks underperform, except when the correlation is high. When $\mu_B = 0\%$, this probability is shifted to the left. Finally, when $\mu_B = -2\%$ and $\sigma_S = 30\%$, the probability of overperformance is greater than the probability of underperformance only when the stock-bond correlation is highly negative, less than 70%.

Remark 2. *To observe a probability greater than* 50%*, the stock-bond correlation must be lower than the critical value* $\rho^\star$*, which is equal to:*

$$
\Pr\left\{R_B(t) \geq 0 \mid R_S(t)\right\} \geq 50\% \Leftrightarrow \rho \leq \rho^\star = \frac{\mu_B}{\sigma_B} \bigg/ \frac{\mu_S - R_S(t)}{\sigma_S}
$$

[4]The parameters are $\mu_S = 8\%$, $\mu_B = 2\%$, $\sigma_S = 15\%$, $\sigma_B = 3\%$ and $\rho = -25\%$.
[5]The parameters are $\mu_S = 8\%$, $\mu_B = 0\%$, $\sigma_S = 15\%$, $\sigma_B = 6\%$ and $\rho = -40\%$.

Figure 6: Probability $\Pr\left\{R_B(t) \geq 0 \mid R_S(t) = -20\%\right\}$ in % that the bond return is positive when the performance of the stock market is -20%



2.2.2 The nonlinear payoff case

We have seen that diversification is necessary in bad times, but not always interesting in good times. Therefore, a negative stock-bond correlation provides a good diversification for a stock-bond constant-mix allocation, because we can hedge some of the equity underperformance with the bond exposure. This property is related to the flight-to-quality regime, which can be interpreted as the opposite of the contagion regime (Baur and Lucey, 2009). Nevertheless, as seen in the first section, a positive stock-bond correlation helps to observe a positive bond risk premium. In order to reconcile these two contradictory statements, we need to consider that the stock-bond correlation is time-varying and not constant (Andersson *et al.*, 2008; Brixton *et al.*, 2023). For example, Figure 7 shows two nonlinear bond payoffs. The blue curve represents the fly-to-quality regime[6], while the red curve could be the bond payoff in a contagion regime[7].

Theory In statistical terms, a linear payoff is the conditional expectation of the random variable Y given the value of the random vector X:

$$y = \mathbb{E}\left[Y \mid X = x\right] = \alpha + \beta^\top x$$

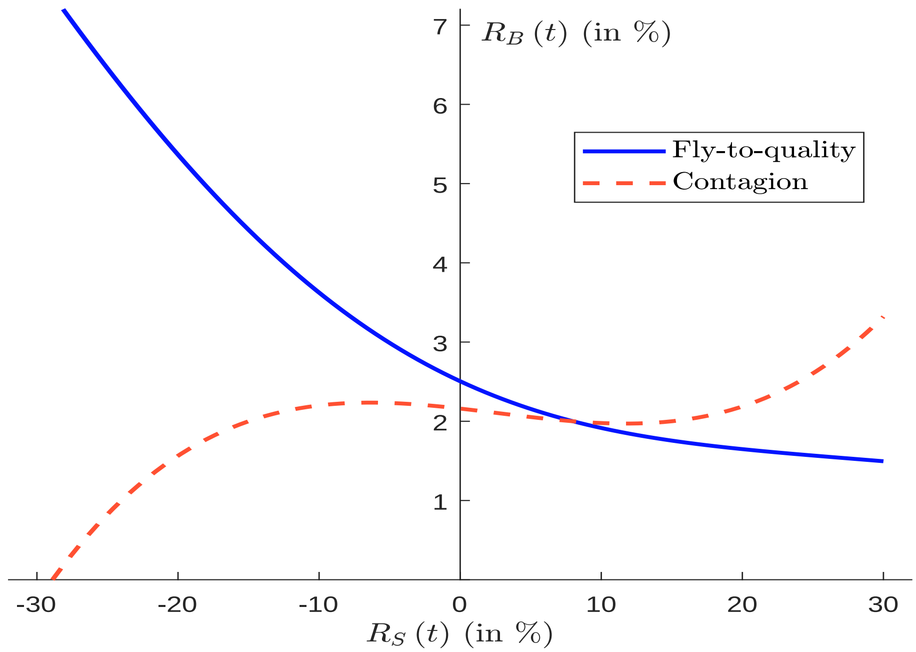
Here, the linear relationship between x and y justifies the name "*linear payoff*". A nonlinear payoff is a generalization of the previous framework when the relationship is nonlinear:

$$y = \mathbb{E}\left[Y \mid X = x\right] = m(x)$$

[6] This payoff was generated by assuming a parametric function of the stock-bond correlation: $\rho = -1.10 + \Phi\left(\sigma_S^{-1}(R_S - \mu_S)\right)^{0.20}$.

[7] The parametric function of the stock-bond correlation is $\rho = \left(\sigma_S^{-2} R_S^2 - 1\right)/10$.

Figure 7: Nonlinear bond payoff



Here, the function $m(x)$ is not necessarily linear. For example, a call option is a nonlinear payoff because we have $y = \max(0, x - c)$ where c is a constant. $y = (x - c)^2$ is a straddle option payoff. The concept of payoff is important in option theory because the PnL of the option buyer depends on the performance of an underlying asset (Figure 8). It is also important in alternative risk premia because it helps to understand how to position the performance of investment strategies relative to a reference asset, which in most cases is the performance of the stock market (Hamdan *et al.*, 2016; Roncalli, 2017). In Figure 9, we show different stylized payoff profiles. In general, the constant profile corresponds to the payoff of an alpha strategy that has no beta component or directional risk. The convex profile is the most common payoff implemented in investment portfolios and describes the conditional performance of carry strategies (*e.g.*, the short volatility strategy). This profile is similar to a short put payoff and has high downside risk. Another popular profile is the concave payoff, which we encounter when we consider a trend-following or time-series momentum strategy (Jusselin *et al.*, 2017).

Remark 3. *While we estimate linear payoffs using ordinary least squares and linear regression, estimating nonlinear payoffs requires more advanced statistical tools. According to Roncalli (2020), this can be done using Nadaraya-Watson regression, local polynomial regression, or local quantile regression.*

Bond payoff Let's apply the previous framework to the stock-bond allocation. We have:

$$\begin{aligned} y &= \mathbb{E}\left[R_B \mid R_S = x\right] \\ &= \int_{\mathbb{R}} y\, f_{B|S}(y; x)\,\mathrm{d}y \\ &= \int_{\mathbb{R}} y \frac{f_{S,B}(x,y)}{f_S(x)}\,\mathrm{d}y \end{aligned}$$

Figure 8: Option payoff profiles

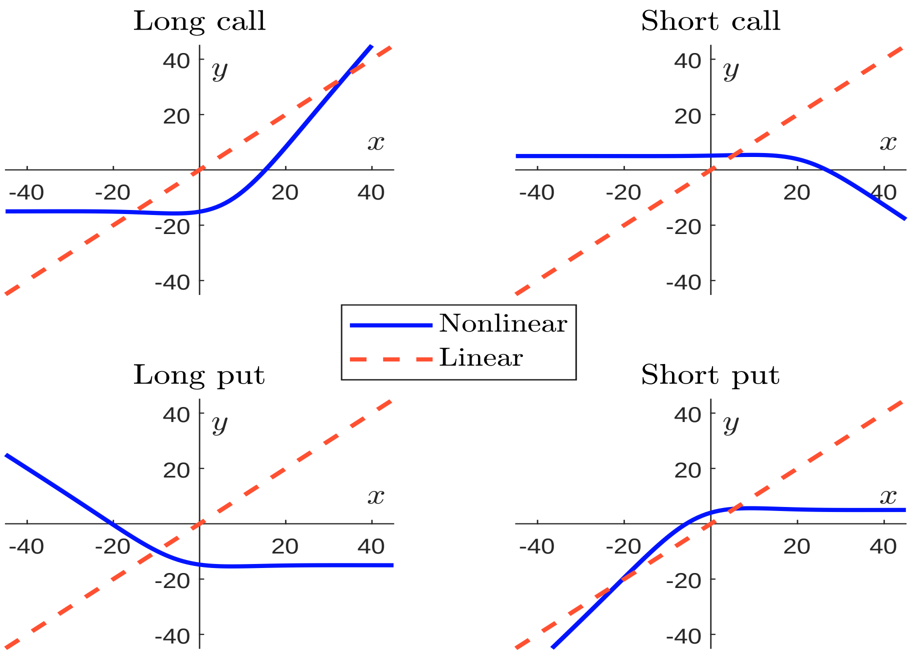
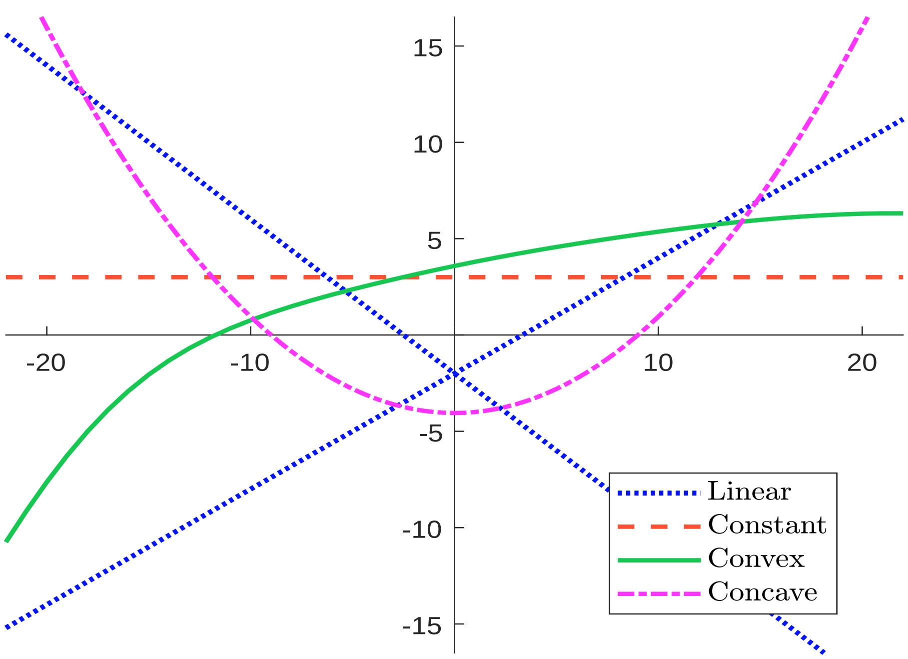


Figure 9: Stylized payoff profiles



where $f_{S,B}$ is the joint density of (R_S, R_B), f_S is the density function of R_S and $f_{B|S}$ is the conditional density of R_B given the value of R_S. If we consider a copula representation, we get:

$$y = \int_{\mathbb{R}} y\, c_{S,B}\left(\mathbf{F}_S(x), \mathbf{F}_B(y)\right) f_B(y)\, \mathrm{d}y$$

where $c_{S,B}$ is the copula density function of (R_S, R_B), $\mathbf{F}_S$ and $\mathbf{F}_B$ are the two cumulative distribution functions of R_S and R_B, and f_B is the density function of R_B. This means that the bond payoff depends on the dependence between bond returns and stock returns. In the case where $\mathbf{C}_{S,B}$ is the independent or product copula, we conclude that the bond payoff is constant and equal to the unconditional expectation of the bond return:

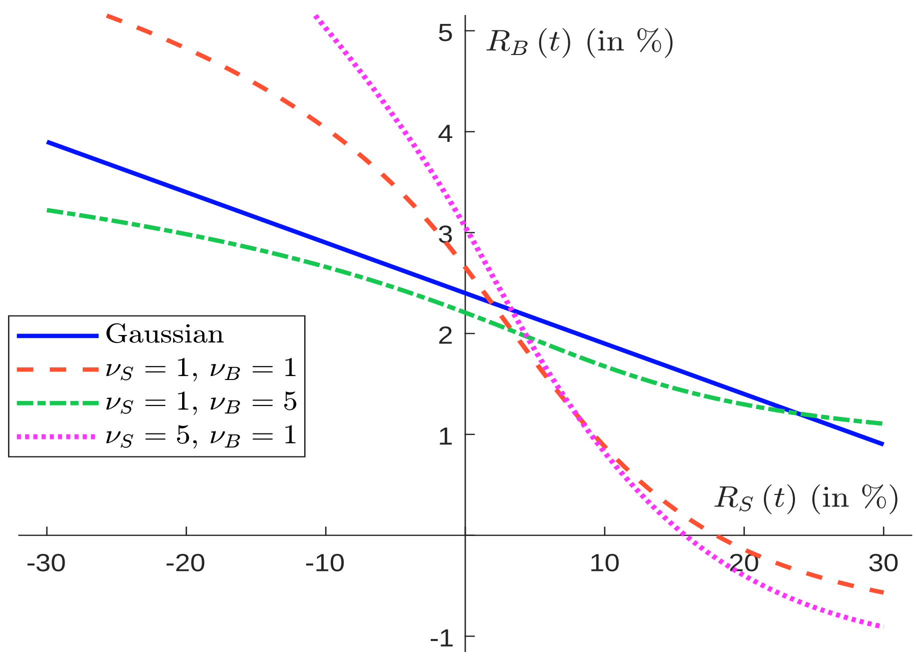
$$y = m(x) = \int_{\mathbb{R}} y\, f_B(y)\, \mathrm{d}y = \mathbb{E}[R_B]$$

In the case where $\mathbf{C}_{S,B}$ is the Gaussian copula and the margins are normal, we can show that the bond payoff is linear and equal to the previous formula (Roncalli, 2020, page 737):

$$y = m(x) = \alpha_B + \beta_B x$$

If the margins are not normal or if the copula is not Gaussian, the bond payoff is not linear.

Figure 10: Effect of marginal distributions and copula functions on bond payoff ($\rho = -25\%$, Clayton copula)



To illustrate the nonlinearity property, we consider several hypotheses concerning the marginal distributions of R_S and R_B, and the copula function of (R_S, R_B). First, we assume that the copula is Normal with $\rho = -25\%$, which corresponds to the copula of the bivariate Gaussian distribution with a correlation of -25%. The margins are Student's t distributions: $\sigma_S^{-1}(R_S - \mu_S) \sim t_{\nu_S}$ and $\sigma_B^{-1}(R_B - \mu_B) \sim t_{\nu_B}$. Using the following parameters $\mu_S = 8\%$, $\mu_B = 2\%$, $\sigma_S = 15\%$, and $\sigma_B = 3\%$, Figure 60 on page 106 shows the bond payoff for different values of ν_S and ν_B. The degree of nonlinearity depends on the fat tail of the asset returns. For example, if $\nu_S = 1$ and $\nu_B = 1$, the payoff is highly nonlinear. If $\nu_S = 5$ and $\nu_B = 1$, meaning that the fat tail of bond returns is larger than the fat tail of stock returns, the slope of the payoff is high. The pure effect of the copula functions[8] is shown in Figure 61 on page 107. Here, we assume that the margins are Gaussian: $\sigma_S^{-1}(R_S - \mu_S) \sim \mathcal{N}(0,1)$ and $\sigma_B^{-1}(R_B - \mu_B) \sim \mathcal{N}(0,1)$. The effect is not important, except in the bear market. We see a significant difference between the Clayton and Frank copulas. If the marginal distributions are the Student's t_1 distributions, the impact is more significant (Figure 62 on page 107). If we combine all these effects, we get the option profiles in Figure 10. This analysis emphasizes two important features: the dependence function of stock and bond returns and their marginal distributions. In particular, we know that the tail dependence is not symmetric, and that the stock-bond correlation is not a good metric for measuring the joint behavior of stock and bond returns locally at the extremes. This mean that the long-term stock-bond correlation is a mixture of the stock-bond correlation in bear, flat and bull markets.

One implication of the nonlinear payoff is that there is an asymmetry in excess correlation: large negative returns are not correlated in the same way as large positive returns. These stylized facts are well documented in the academic literature. Following the seminal work of Longin and Solnik (2001), we can find many references to these asymmetries between stock and bond returns (Garcia and Tsafack, 2011; Chui and Yang, 2012; Jammazi *et al.*, 2015; Shahzad *et al.*, 2017). This line of research, based on copula functions and extreme value theory, joins another area of research based on the time-varying covariance between stock and bond returns (Li, 2002; Scruggs and Glabadanidis, 2003). Whatever the factors explaining conditional heteroskedasticity, *e.g.*, stock market uncertainty (Connolly *et al.*, 2007; Andersson *et al.*, 2008; Baur and Lucey, 2009), the main conclusion is that a constant stock-bond correlation and hence a constant conditional bond risk premium does not make much sense. If this is true in the long run, it is also true in the short run. For example, we may wonder whether a one-year rolling correlation captures the correct empirical dependence between stock and bond returns.

Summary and main results

The payoff approach to diversification distinguishes between good and bad times. Contrary to the CAPM, a negative stock-bond correlation is consistent with bonds being both hedging and performance assets. Indeed, when the stock-bond correlation is negative, the payoff of the bond return is positive when stock returns are negative, but in some situations it can remain positive even when stock returns are positive. However, the best explanation for bonds being both a hedge and a performance asset is that the stock-bond correlation is time-varying, which implies that the conditional payoff of bonds is nonlinear. There is an apparent contradiction between measuring the stock-bond dependence with the stock-bond correlation and the fact that the bond payoff is nonlinear. To resolve this problem, we need to consider the local dependence function or the regime correlation between stock and bond returns. In fact, measuring the dependence with the stock-bond correlation cannot explain the asymmetric features between bad and good times. This implies going beyond the constant stock-bond correlation.

[8] The copula functions are calibrated to have the same correlation $\rho = -25\%$.

2.3 Portfolio composition and aggregate correlation

2.3.1 Individual vs. aggregate correlation

Let R_S and R_B be the stock and bond returns. The stock-bond correlation is defined as the ratio of the stock-bond covariance to the product of standard deviations:

$$\rho_{S,B} = \frac{\mathbb{E}\left[\left(R_S - \mathbb{E}\left[R_S\right]\right)\left(R_B - \mathbb{E}\left[R_B\right]\right)\right]}{\sigma\left(R_S\right)\sigma\left(R_B\right)}$$

The problem is that there is no security or asset that measures the performance of the stock market or the performance of the bond market. In practice, we proxy the performance of a market by the performance of an index or a basket of assets. For example, Rankin and Idil (2014) calculated the stock-bond correlation by looking at monthly changes over a rolling three-year window using the S&P 500 Index and the 10-year US Treasury bond[9]. This means that the calculated correlation is an aggregate correlation. Therefore, we are not sure that the behavior of this correlation can be compared to the correlation between two assets, that is to an individual correlation. For example, we know that an aggregate volatility, that is the volatility of a basket of assets, is different than an individual volatility, that is the volatility of an asset. The main reason is the impact of the diversification effect. We have the same results when considering aggregate and individual covariances. There is no reason that it is different for aggregate and individual correlations.

2.3.2 The mathematics of aggregate stock-bond correlation

We assume that the stock market return is the weighted average of a basket of n stocks:

$$R_S\left(t\right) = \sum_{i=1}^{n} w_i\left(t\right) R_i\left(t\right)$$

where $R_i\left(t\right)$ is the return of asset i and $w_i\left(t\right)$ is the weight of asset i at time t. We use the notations $\sigma_i\left(t\right) = \sigma\left(R_i\left(t\right)\right)$, $\sigma_S\left(t\right) = \sigma\left(R_S\left(t\right)\right)$ and $\rho_{i,B}\left(t\right) = \rho\left(R_i\left(t\right), R_B\left(t\right)\right)$. In Appendix A.5.1 on page 95, we show that the stock-bond correlation $\rho_{S,B}\left(t\right)$ is the weighted sum of the individual correlations:

$$\rho_{S,B}\left(t\right) = \sum_{i=1}^{n} \omega_i\left(t\right)\rho_{i,B}\left(t\right)$$

where:

$$\omega_i\left(t\right) = \frac{w_i\left(t\right)\sigma_i\left(t\right)}{\sigma_S\left(t\right)}$$

A first finding is that the aggregate stock-bond correlation $\rho_{S,B}\left(t\right)$ is not the weighted average of the individual correlations[10]:

$$\rho_{S,B}\left(t\right) \neq \sum_{i=1}^{n} w_i\left(t\right)\rho_{i,B}\left(t\right) := \bar{\rho}_{S,B}\left(t\right)$$

[9]Similarly, Li *et al.* (2022) used the daily returns of the stock market index from Ken French's data library and the 5-year US Treasury. Duffee (2023) used the 10-year nominal zero-coupon Treasury bond, but did not specify the definition of the US stock market. Brixton *et al.* (2023) did not explain how the stock-bond correlations were computed. The analysis of Molenaar *et al.* (2023) is close to that of Rankin and Idil (2014), but they used the stock market index from Ken French's data library.

[10]Except when all individual correlations $\rho_{i,B}\left(t\right)$ are equal to 1.

To illustrate the previous results, let us consider an example with four stocks and one bond. The volatilities are 20%, 10%, 15%, 25% and 5%. The correlation matrix is:

$$\rho = \left(\begin{array}{rr:rr:r} 1.00 & 0.80 & 0.25 & 0.25 & -0.50 \\ 0.80 & 1.00 & 0.25 & 0.25 & -0.30 \\ \hdashline 0.25 & 0.25 & 1.00 & 0.80 & 0.30 \\ 0.25 & 0.25 & 0.80 & 1.00 & 0.40 \\ \hdashline -0.50 & -0.30 & 0.30 & 0.40 & 1.00 \end{array}\right)$$

We observe two blocks of stocks. The first block, consisting of the first and second stocks, is negatively correlated with the bond, while the second block, consisting of the third and fourth stocks, is positively correlated with the bond. Using different baskets, we obtain the results shown in Table 3. The first basket is an equally weighted portfolio of the first block. The stock-bond correlation is -45.40%, while the average correlation is -40%. The second basket is an equally weighted portfolio of the second block. The stock-bond correlation is now positive and equal to $+38.08\%$. The last basket is an equally weighted portfolio of the entire stock investment universe. The stock-bond correlation is positive, while the average correlation is negative ($\rho_{S,B}(t) = +2.80\%$ vs. $\bar{\rho}_{S,B}(t) = -2.50\%$).

Table 3: Aggregate vs. average stock-bond correlation

Basket	#1		#2		#3	
i	w_i	ω_i	w_i	ω_i	w_i	ω_i
1	50.00%	69.84%			25.00%	37.33%
2	50.00%	34.92%			25.00%	18.67%
3			50.00%	39.39%	25.00%	28.00%
4			50.00%	65.65%	25.00%	46.67%
$\rho_{S,B}$	-45.40%		38.08%		2.80%	
$\bar{\rho}_{S,B}$	-40.00%		35.00%		-2.50%	

One lesson from the previous example is that two equity portfolios in the same investment universe can have two opposite stock-bond correlations. If we investigate further, we find that the stock-bond correlation can be written as:

$$\rho_{S,B}(t) = \sum_{i=1}^{n} w_i(t)\,\gamma_i(t)\,\rho_{i,B}(t)$$

where $\gamma_i(t) = \dfrac{\sigma_i(t)}{\sigma_S(t)}$ is the volatility ratio. The contribution of stock i to the stock-bond correlation is an increasing function of its weight and its volatility ratio. A second finding is that the stock-bond correlation is mainly driven by large-cap and highly volatile stocks.

To illustrate these results, we consider the composition of the S&P 500 Index at the end of each year from 1990 and 2023. We compute the Herfindahl index $\mathcal{H}(w) = \sum_{i=1}^{n} w_i(t)$ and the number of equivalent bets $\mathcal{H}^{-1}(w)$, which summarizes the concentration of portfolio weights. In fact, we have $1 \leq \mathcal{H}^{-1}(w) \leq n$, which means that the bounds are reached for the single stock concentrated portfolio and the equally weighted portfolio, respectively. Then, we estimate the cross-sectional dispersion of the stock volatilities:

$$\mathrm{CSV} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\sigma_i(t) - \bar{\sigma}(t)\right)^2}$$

Table 4: Statistics of the S&P 500 Index

Year	$\mathcal{H}^{-1}(w)$	CSV (in %)	ϕ	$\mathcal{L}(\omega)$	$\Omega_m(t)$ (in %) 5	10	25	$\eta_\alpha(t)$ 25%	50%	75%
1990	127.1	13.94	0.87	1.70	10.6	17.6	31.8	18	59	157
1991	127.9	12.79	0.89	1.84	10.6	18.0	31.1	17	62	162
1992	141.3	10.68	1.10	2.46	9.2	15.9	29.5	20	67	171
1993	154.1	9.76	1.13	2.85	9.1	15.9	29.2	20	65	167
1994	150.5	8.24	0.84	2.39	8.7	15.7	29.0	20	66	170
1995	150.7	8.96	1.15	2.88	9.2	16.3	29.7	19	62	160
1996	144.0	9.23	0.78	2.17	10.8	18.6	32.6	16	56	150
1997	137.9	9.77	0.54	1.70	10.8	18.8	34.6	15	49	134
1998	113.1	12.13	0.60	1.83	12.1	19.9	37.1	14	44	117
1999	86.1	13.36	0.74	2.30	15.1	25.6	44.1	10	33	96
2000	105.0	19.92	0.90	2.30	12.7	21.7	38.3	13	42	115
2001	99.5	22.50	1.05	1.90	15.4	24.5	39.3	11	43	126
2002	105.7	23.20	0.89	1.59	13.6	22.9	39.0	12	42	132
2003	114.3	12.44	0.73	1.63	12.6	21.2	35.4	14	50	148
2004	122.7	10.05	0.91	1.95	10.8	18.3	31.7	17	59	163
2005	130.3	8.26	0.80	2.04	10.3	16.5	30.1	19	64	168
2006	128.8	8.73	0.87	2.10	10.2	16.5	30.3	19	63	168
2007	123.1	9.04	0.57	1.57	11.8	18.4	32.9	16	60	160
2008	105.1	26.76	0.65	1.31	13.7	21.5	36.0	14	49	145
2009	126.7	24.01	0.88	1.61	14.8	22.4	36.3	13	51	151
2010	134.1	9.47	0.52	1.45	10.3	17.9	32.1	17	60	169
2011	124.1	10.56	0.45	1.28	11.1	18.0	32.4	17	61	171
2012	126.0	9.72	0.76	1.69	12.6	19.9	33.1	15	59	170
2013	145.0	7.59	0.69	1.85	11.6	17.5	31.1	18	62	169
2014	149.7	6.96	0.61	1.76	10.5	16.1	29.2	20	65	170
2015	142.6	8.76	0.57	1.56	11.8	18.7	32.2	17	58	161
2016	141.5	10.29	0.79	1.82	10.8	17.5	29.9	18	67	175
2017	129.5	7.58	1.14	2.66	11.9	18.1	31.0	18	63	174
2018	118.4	7.95	0.47	1.56	15.4	22.5	36.4	13	49	147
2019	103.0	7.95	0.64	1.83	15.9	22.3	35.4	13	52	150
2020	72.5	15.99	0.46	1.36	20.5	26.9	39.8	9	45	142
2021	67.5	9.20	0.70	1.99	22.6	30.5	42.0	7	39	135
2022	85.0	11.23	0.47	1.42	19.5	27.2	39.4	9	46	142
2023	61.3	8.50	0.65	2.03	25.3	35.3	46.7	5	32	123

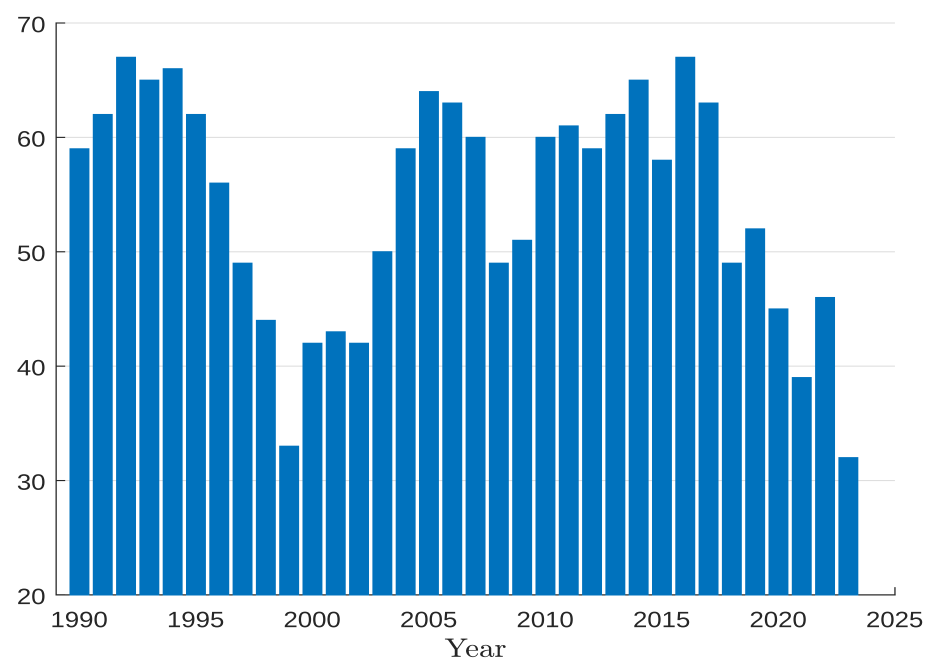
It corresponds to the standard deviations of the portfolio's stock volatilities for a given date t. To analyze this metric, we introduce the ratio $\phi = \dfrac{\text{CSV}}{\sigma_S(t)}$, which compares the cross-sectional dispersion of the stock volatilities with the volatility of the index portfolio. The leverage metric $\mathcal{L}(\omega)$ is the sum of the weights $\omega_i(t)$ and gives an idea of the amplification effect of the individual stock-bond correlations. We introduce the function $\Omega_m(t)$, which is the cumulative proportion of the m largest weights $\omega_i(t)$:

$$\Omega_m(t) = \frac{\sum_{i=n-m+1}^{n} \omega_{i:n}(t)}{\sum_{i=1}^{n} \omega_i(t)}$$

where $\omega_{i:n}(t)$ is the i^{th} order statistic of $\omega_i(t)$. $\Omega_m(t)$ measures the contribution of the

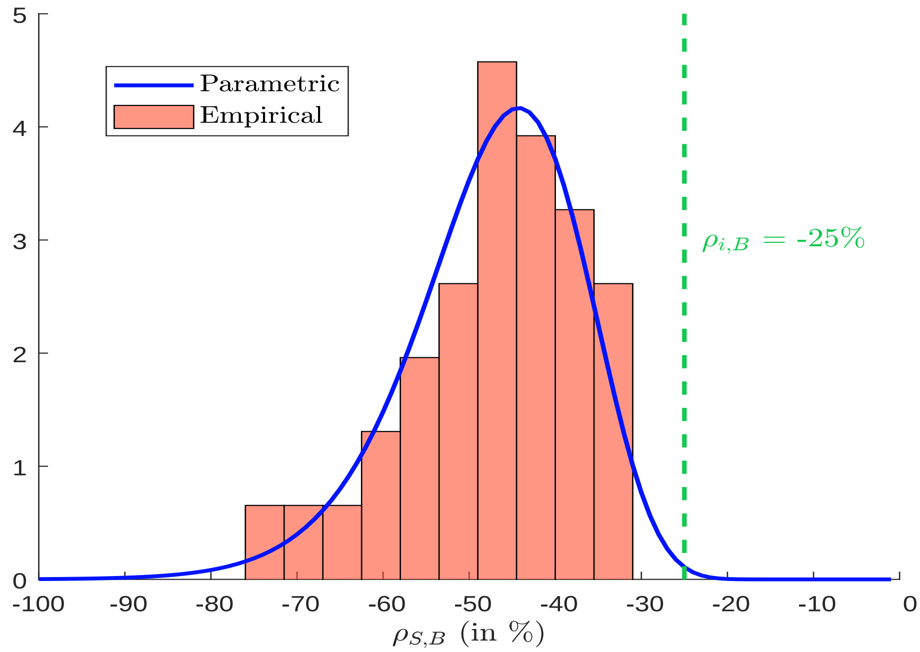
m most influential stocks to the formation of the stock-bond correlation. For the different dates, we compute the contribution indices $\Omega_5(t)$, $\Omega_{10}(t)$ and $\Omega_{25}(t)$. Finally, we estimate the number of stocks that explain a given fraction α of the stock-bond correlation: $\eta_\alpha(t) = \left\{\inf m : \Omega_m(t) \geq \alpha\right\}$. For example, if $\alpha = 50\%$, $\eta_\alpha(t)$ is the number of stocks that explain 50% of the stock-bond correlation. Results are shown in Table 4. We observe a trend in the number of equivalent bets. It increased in the early 1990s, but fell during the dot-com bubble and remained low until 2008. After the global financial crisis, it was between 130 and 140 stocks. Since 2017, it has fallen dramatically, reaching its lowest value in 2023. In fact, in that year, the number of equivalent bets was only 61 stocks because of the magnificent 7 stocks (Apple, Alphabet, Microsoft, Amazon, Meta, Tesla and Nvidia). Cross-sectional volatility changes a lot over time[11]. The lowest value was 6.96% in December 2014, while the highest value was 26.76% in December 2008. The ratio ϕ is generally less than 1, meaning that the cross-sectional volatility is lower than the market portfolio volatility. However, there are some exceptions, which concern the years 1992, 1993, 1995, 2001 and 2017. Finally, the leverage ratio takes its values between 1.28 and 2.88. The amplification effect is then completely different from one year to another. In Figure 12, we estimate the empirical and parametric distribution of the aggregate stock-bond correlation $\rho_{S,B}$ when the individual stock-bond correlation is set to -25%. On average, $\rho_{S,B}$ is close to -50%. Furthermore, we observe a high dispersion and a negative skewness. Therefore, the amplification effect, which is a consequence of the leverage ratio, can be significant and take its value between 1 and 3. All these results show that we must be careful in interpreting the stock-bond correlation, since a large part of its value is explained by the structure of the stock market and is not related to the dependence between the stock and bond markets.

Figure 11: Number of stocks that explain 50% of the US stock-bond correlation (S&P 500 Index)



[11] We use a one-year historical window to estimate stock volatility.

Figure 12: Amplification effect of the individual stock-bond correlation



Remark 4. *The concentration index $\Omega_m(t)$ and its inverse function $\eta_\alpha(t)$ confirm the previous statement. In 2023,* 25% *of the stock-bond correlation is explained by only* 5 *stocks and* 50% *by* 32 *stocks. In this context, the idiosyncratic behavior of some stocks can have a large impact on the stock-bond correlation, and this measure can give a false perception of the stock-bond dependence (see Figure 11).*

Assume that the individual correlations between stocks and the bond are equal: $\rho_{i,B} = \rho_{S,B}^{\text{stock}}$. We deduce that the aggregate stock-bond correlation is equal to the product of the individual stock-bond correlation and the diversification ratio:

$$\rho_{S,B}(t) = \rho_{S,B}^{\text{stock}} \cdot \mathcal{DR}\left(w(t)\right)$$

Since $\mathcal{DR}\left(w(t)\right) \geq 1$, we conclude that the aggregate stock-bond correlation is always greater than the individual stock-bond correlation in absolute value:

$$\left|\rho_{S,B}(t)\right| \geq \left|\rho_{S,B}^{\text{stock}}\right|$$

We face a paradox here, because if each individual stock is -10% correlated with the bond, we expect the stock-bond correlation to be -10%, but the previous result shows that the stock-bond correlation $\rho_{S,B}(t)$ could be -50%.

Consider again an example with four stocks and one bond. The volatilities are 20%, 10%, 15%, 25% and 5%. The correlation matrix is:

$$\rho = \left(\begin{array}{cccc:c}
1.00 & 0.80 & 0.45 & 0.35 & -0.30 \\
0.80 & 1.00 & 0.25 & 0.25 & -0.30 \\
0.45 & 0.25 & 1.00 & 0.50 & -0.30 \\
0.35 & 0.25 & 0.50 & 1.00 & -0.30 \\
\hdashline
-0.30 & -0.30 & -0.30 & -0.30 & 1.00
\end{array}\right)$$

We estimate four risk-based equity portfolios: the equally weighted (EW) portfolio (DeMiguel *et al.*, 2009), the long-only minimum variance (MV) portfolio (Clarke *et al.*, 2011; Richard and Roncalli, 2015), the equal risk contribution (ERC) portfolio (Maillard *et al.*, 2010), and the most diversified portfolio or MDP (Choueifaty and Coignard, 2008). The results are presented in Table 5. For each smart beta portfolio, we report the allocation w_i and the weight ω_i for each stock i. We notice that the minimum variance portfolio has the lowest stock-bond correlation ($\rho_{S,B} = -36.74\%$), while the most diversified portfolio has the highest stock-bond correlation ($\rho_{S,B} = -40.45\%$), as expected by theory. These results highlight the relationship between stock-bond correlation and stock portfolio diversification. In particular, the absolute level of the stock-bond correlation increases with the diversification of the benchmark equity portfolio. For example, the stock-bond correlation is different when we use the Dow Jones, S&P 500, and Russell 3000 indexes as benchmarks, even though they measure the stock market performance of the same country. The same is true when we compare the stock-bond correlation between two countries. For example, the magnitude of the stock-bond correlation will be different if the country's investment universe is small (*e.g.*, if we use the BEL 20 or DAX 30 indexes) or large (*e.g.*, if we use the Eurostoxx or Stoxx 600 indexes).

Table 5: Stock-bond correlation with smart beta portfolios

Basket	EW		MV		ERC		MDP	
i	w_i	ω_i	w_i	ω_i	w_i	ω_i		
1	25.00%	37.37%	0.00%	0.00%	17.44%	29.70%	0.00%	0.00%
2	25.00%	18.68%	75.00%	81.65%	39.36%	33.51%	55.55%	53.93%
3	25.00%	28.02%	25.00%	40.82%	26.66%	34.06%	27.78%	40.45%
4	25.00%	46.71%	0.00%	0.00%	16.54%	35.21%	16.67%	40.45%
$\rho_{S,B}$	−39.23%		−36.74%		−39.74%		−40.45%	

We further assume that the individual correlations between stocks and the bond are equal ($\rho_{i,B} = \rho_{S,B}^{\text{stock}}$), while the correlation matrix between stock returns is constant ($\rho_{i,j} = \rho^{\text{stock}}$). Earlier, we showed that the stock-bond correlation is bounded:

$$\left|\rho_{S,B}^{\text{stock}}\right| \leq \left|\rho_{S,B}\left(t\right)\right| \leq \left|\rho_{S,B}^{\text{stock}}\right| \cdot \mathcal{DR}\left(w^{\text{mdp}}\right)$$

The lower bound is reached when the diversification ratio of the stock portfolio is equal to 1. We can prove this:

$$\mathcal{DR}\left(w\right) = 1 \Leftrightarrow \left\{\begin{array}{c} \exists i : w_i = 1 \\ \text{or} \\ \rho_{i,j} = \rho^{\text{stock}} = 1 \end{array}\right.$$

This means that the lower bound is reached when the concentration is maximum (the stock portfolio is invested in one stock) or when there are no diversification opportunities (when the correlations between the stock returns are all equal to 1). The upper bound is reached when the diversification ratio is maximum, *i.e.* when the stock market portfolio is the most diversified portfolio as defined by Choueifaty and Coignard (2008). In Appendix A.6 on page 98, we compute the expression for the diversification ratio $\mathcal{DR}\left(w^{\text{mdp}}\right)$. We deduce that the ratio between the aggregate stock-bond correlation and the individual stock-bond correlation satisfies the following inequalities:

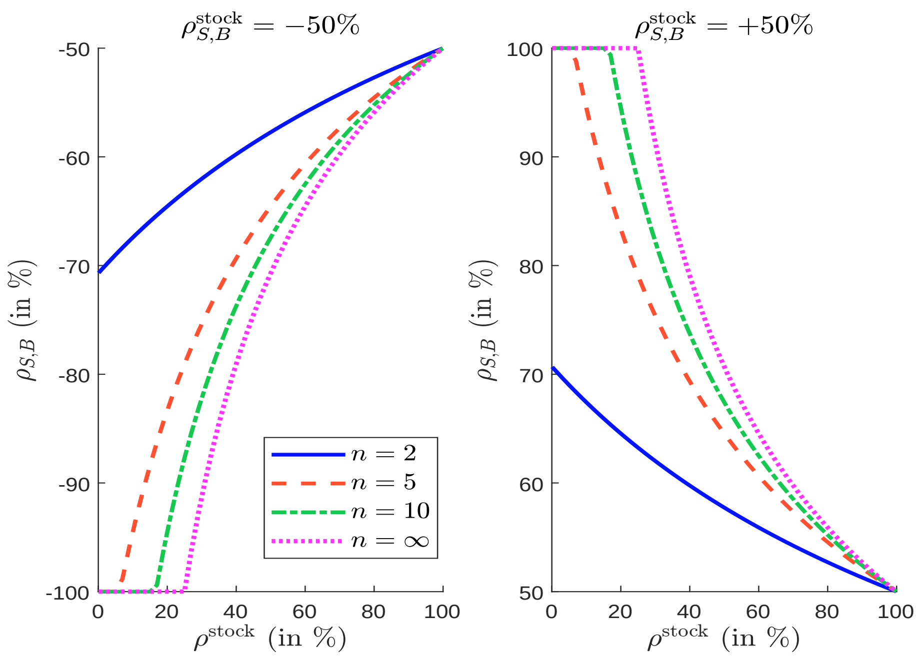
$$1 \leq \frac{\rho_{S,B}\left(t\right)}{\rho_{S,B}^{\text{stock}}} \leq \frac{1}{\sqrt{\rho^{\text{stock}} + \dfrac{\left(1 - \rho^{\text{stock}}\right)}{n}}}$$

When $n \to \infty$, we get:

$$1 \leq \frac{\rho_{S,B}(t)}{\rho_{S,B}^{\text{stock}}} \leq \frac{1}{\sqrt{\rho^{\text{stock}}}}$$

In Figure 13, we show the aggregate stock-bond correlation $\rho_{S,B}(t)$ of the most diversified portfolio with respect to the stock correlation ρ^{stock} when considering two values of $\rho_{S,B}^{\text{stock}}$ (-50% and $+50\%$). We can clearly see that diversification strongly influences and exacerbates the stock-bond correlation. Two factors play an important role: the level of average correlation in the equity market and the depth of the equity investment universe. However, the effect of diversification is less pronounced when the individual stock-bond correlation is low (Figure 65 on page 109).

Figure 13: Aggregate stock-bond correlation with the MDP ($\left|\rho_{S,B}^{\text{stock}}\right| = 50\%$)



2.3.3 Sector analysis

In Appendix A.7 on page 98, we show that:

$$\Delta\rho_{S,B}(t+1) = \sum_{i=1}^{n} \frac{w_i(t)\,\sigma_i(t)}{\sigma_S(t)} \left(\xi_i(t+1)\,\rho_{i,B}(t+1) - \rho_{i,B}(t)\right)$$

We deduce that:

$$\Delta\rho_{i,B}(t+1) \Rightarrow \Delta\rho_{S,B}(t+1) \neq 0$$

Even if the individual stock-bond correlations do not change, the stock-bond correlation changes because of the stock returns. In fact, we have:

$$\xi_i(t+1) = \frac{\dfrac{\left(1 + R_i(t+1)\right)\sigma_i(t+1)}{\sigma_i(t)}}{\dfrac{\left(1 + R_S(t+1)\right)\sigma_S(t+1)}{\sigma_S(t)}}$$

Assume that the stock and index volatilities remain constant between t and $t+1$. We obtain:

$$\xi_i(t+1) = \frac{1 + R_i(t+1)}{1 + R_S(t+1)}$$

Stocks that outperform the market contribute more than stocks that underperform the market. This result is very intuitive and consistent with the previous analysis. The implication is that the aggregate stock-bond correlation reflects the stock-bond dependence of the best performing stocks, sectors or factors. However, there is little research in the academic literature that examines the stock-bond correlation across sectors. We found only two references. First, Brixton *et al.* (2023) show that the stock-bond correlation may depend on the sector. In particular, they conclude that utilities, real estate, and consumer staples have a higher correlation than energy, industrials, and materials. Second, Duffee (2023) suggest that "*time variation in the stock-bond covariance may be driven by changing sector dynamics*", particularly in the real estate sector.

Summary and main results

Since there is no asset that represents the stock market, the stock-bond correlation is calculated using an index portfolio or a basket of stocks. Therefore, the calculated stock-bond correlation is an aggregate correlation, not an individual correlation. Like the volatility measure, the behavior of an aggregate correlation or covariance differs from the behavior of an individual correlation or covariance because of the diversification effect. This means that the stock-bond correlation is highly dependent on the composition of the stock basket and the weighting scheme. For example, the stock-bond correlation may be different if the basket is skewed toward growth, value, defensive, large-cap, or quality stocks. The assumption that the aggregate stock-bond correlation is a summary of the individual stock-bond correlations is incorrect because the cross-sectional variance of individual stock-bond correlations is large and the aggregate stock-bond correlation is a leveraged version of the individual stock-bond correlation. The amplification effect is significant and can range from 1 to 3, with an average of 2. This means that an individual correlation of -25% or $+25\%$ generally translates into an aggregate stock-bond correlation of -50% or $+50\%$, respectively. In addition, the amplification effect has a negative skewness, which raises some questions about the market's narrative of stock-bond comovement. In fact, investors may have a false sense of true diversification between the stock and bond markets. Using the same investment universe, we can show that the stock-bond correlation can be positive or negative simply because the basket weights are different. The big question in studying the dynamics of the aggregate stock-bond correlation is then the effect of the dynamics of the basket weights. Comparing the stock-bond correlation during the dot-com bubble, the subprime crisis, or the covid event is extremely misleading because the stock market portfolio was completely different during these three periods. In particular, we need to separate the correlation effects from the sector allocation effects. In this context, it is important to understand how the stock-bond correlation is formed and what the individual contributions are. For example, about thirty stocks explain 50% of the US stock-bond correlation at the end of 2023.

2.4 Macroeconomic models of the stock-bond covariance

So far, we have focused on the relationship between stock-bond correlation and portfolio construction. Let us now turn to the economic models. Li (2002) is certainly one of the first references to relate the stock-bond correlation to macroeconomic factors. It is not new to define pricing factors in terms of macroeconomic variables (Chen *et al.*, 1986), but the main result of Li (2002) is to consider a time-varying relationship. This publication opened the door to a significant research activity on macroeconomics and stock-bond comovement. The survey provided by Duffee (2023) distinguishes between inflation-centric and real-centric models. Inflation-centric models emphasize the role of expected inflation and inflation uncertainty in explaining the stock-bond correlation. Real-centric models assume that aggregate shocks affect real interest rates and the stock-bond correlation. These two types of models can be summarized by a stylized growth-inflation model.

2.4.1 Inflation-centric model

The model We consider the model developed by Li (2002). It is described in detail in Appendix A.8 on page 99. Li (2002) assumes that the real interest rate $r(t)$, the inflation rate $\pi(t)$ and the dividend yield $\delta(t)$ follow affine mean-reverting processes:

$$\begin{cases} r(t+1) = \bar{r} + \varrho_r \left(r(t) - \bar{r}\right) + \varepsilon_r(t+1) \\ \pi(t+1) = \bar{\pi} + \varrho_\pi \left(\pi(t) - \bar{\pi}\right) + \varepsilon_\pi(t+1) \\ \delta(t+1) = \bar{\delta} + \varrho_\delta \left(\delta(t) - \bar{\delta}\right) + \varepsilon_\delta(t+1) \end{cases}$$

where $\bar{r}$, $\bar{\pi}$ and $\bar{\delta}$ are the long-run equilibria, ϱ_r, ϱ_π and ϱ_δ are the adjustment velocities, and $\varepsilon_r(t+1)$, $\varepsilon_\pi(t+1)$ and $\varepsilon_\delta(t+1)$ are the innovation shocks distributed according to $\mathcal{N}\left(0,\sigma_r^2\right)$, $\mathcal{N}\left(0,\sigma_\pi^2\right)$, and $\mathcal{N}\left(0,\sigma_\delta^2\right)$. We note $\rho_{r,\pi}$, $\rho_{r,\delta}$ and $\rho_{\pi,\delta}$ the correlations between the innovation shocks.

Expression of the stock-bond correlation In Appendix A.8 on page 99, we show that the conditional correlation between the bond return R_B and the stock return R_S is equal to:

$$\rho_{S,B} = \frac{\sigma_{S,B}}{\sigma_B \sigma_S} = \frac{x_B^\top \Sigma x_S}{\sqrt{x_B^\top \Sigma x_B}\sqrt{x_S^\top \Sigma x_S}} \tag{5}$$

where $x_B = \left(x_r^B, x_\pi^B, x_\delta^B\right)$, $x_S = \left(x_r^S, x_\pi^S, x_\delta^S\right)$, $x_r^B = -\dfrac{1-\varrho_r^n}{1-\varrho_r}$, $x_\pi^B = -\dfrac{1-\varrho_\pi^n}{1-\varrho_\pi}\varrho_\pi$, $x_\delta^B = 0$, $x_r^S = -\dfrac{1}{1-\varrho_r}$, $x_\pi^S = 1$, $x_\delta^S = \dfrac{1}{1-\varrho_\delta}$ and n is the bond duration. The expression of the covariance is then equal to:

$$\begin{aligned} \sigma_{S,B} := \operatorname{cov}\left(R_B, R_S\right) &= \frac{\left(1-\varrho_r^n\right)}{\left(1-\varrho_r\right)^2}\sigma_r^2 - \frac{\left(1-\varrho_\pi^n\right)\varrho_\pi}{\left(1-\varrho_\pi\right)}\sigma_\pi^2 + \\ &\quad \frac{\left(1-\varrho_\pi^n\right)\varrho_\pi - \left(1-\varrho_r^n\right)\left(1-\varrho_\pi\right)}{\left(1-\varrho_r\right)\left(1-\varrho_\pi\right)}\rho_{r,\pi}\sigma_r\sigma_\pi - \\ &\quad \frac{\left(1-\varrho_r^n\right)}{\left(1-\varrho_r\right)\left(1-\varrho_\delta\right)}\rho_{r,\delta}\sigma_r\sigma_\delta - \frac{\left(1-\varrho_\pi^n\right)\varrho_\pi}{\left(1-\varrho_\pi\right)\left(1-\varrho_\delta\right)}\rho_{\pi,\delta}\sigma_\pi\sigma_\delta \end{aligned} \tag{6}$$

For the variance terms, we have:

$$\sigma_B^2 = \left(\frac{1-\varrho_r^n}{1-\varrho_r}\right)^2 \sigma_r^2 + \left(\frac{1-\varrho_\pi^n}{1-\varrho_\pi}\varrho_\pi\right)^2 \sigma_\pi^2 + \frac{2\left(1-\varrho_r^n\right)\left(1-\varrho_\pi^n\right)\varrho_\pi}{\left(1-\varrho_r\right)\left(1-\varrho_\pi\right)}\rho_{r,\pi}\sigma_r\sigma_\pi \tag{7}$$

and:

$$\begin{aligned}\sigma_S^2 &= \frac{1}{(1-\varrho_r)^2}\sigma_r^2+\sigma_\pi^2+\frac{1}{(1-\varrho_\delta)^2}\sigma_\delta^2-\frac{2}{(1-\varrho_r)}\rho_{r,\pi}\sigma_r\sigma_\pi- \\ &\quad \frac{2}{(1-\varrho_r)(1-\varrho_\delta)}\rho_{r,\delta}\sigma_r\sigma_\delta+\frac{2}{(1-\varrho_\delta)}\rho_{\pi,\delta}\sigma_\pi\sigma_\delta\end{aligned} \tag{8}$$

Interpretation of the model We generally assume that the adjustment rates are less than 1, meaning that there is a mean-reverting effect. For example, since $\mathbb{E}_t\left[r(t+1)\right]=r(t)+(\varrho_r-1)\left(r(t)-\bar{r}\right)$ and $\varrho_r\leq 1$, we have:

$$\begin{cases} r(t)\geq\bar{r}\Rightarrow\mathbb{E}_t\left[r(t+1)\right]\leq r(t) \\ r(t)\leq\bar{r}\Rightarrow\mathbb{E}_t\left[r(t+1)\right]\geq r(t)\end{cases}$$

We distinguish two cases:

1. Positive auto-correlation:

$$\varrho_r\in[0,1]\Rightarrow\begin{cases} r(t)\geq\bar{r}\Rightarrow\bar{r}\leq\mathbb{E}_t\left[r(t+1)\right]\leq r(t) \\ r(t)\leq\bar{r}\Rightarrow r(t)\leq\mathbb{E}_t\left[r(t+1)\right]\leq\bar{r}\end{cases}$$

   The expected real rate is between the equilibrium and the current real rate.

2. Negative auto-correlation:

$$\varrho_r\in\left[-\frac{1}{2},0\right]\Rightarrow\begin{cases} r(t)\geq\bar{r}\Rightarrow\mathbb{E}_t\left[r(t+1)\right]\leq\bar{r}\leq r(t) \\ r(t)\leq\bar{r}\Rightarrow r(t)\leq\bar{r}\leq\mathbb{E}_t\left[r(t+1)\right]\end{cases}$$

   There is an overreaction and the adjustment is too strong.

The volatility of real and inflation rates is generally low, between 0.5% and 2%, while the volatility of dividend yields is higher, typically between 2% and 20%. Economic theory also suggests that:

- The correlation between real interest rates and inflation is negative (Fisher equation);
- The correlation between real interest rates and dividend yields is negative;
- The correlation between inflation and dividend yields can be either negative or positive;

Assuming that the default values of the parameters are $\varrho_r=0.5$, $\varrho_\pi=0.6$, $\varrho_\delta=0.20$, $\sigma_r=1\%$, $\sigma_\pi=1\%$, $\sigma_\delta=5\%$, $\rho_{r,\pi}=-25\%$, $\rho_{r,\delta}=-25\%$ and $\rho_{\pi,\delta}=-25\%$, we find that the stock-bond correlation[12] is 51%. In Tables 6 and 7, we calculate the sensitivity of $\rho_{S,B}$ with respect to the speed of adjustment or volatility, holding all other parameters constant. It is noteworthy that the stock-bond correlation is almost systematically positive. The only case where $\rho_{S,B}$ is negative is when the volatility of inflation shocks is large[13]. If we consider the case of negative autocorrelation, we obtain the results in Tables 8 and 9. Again, all stock-bond correlations are positive. To obtain a negative correlation, we need a positive correlation between inflation and dividend yield shocks and a large value of inflation or dividend yield volatility (Table 10).

[12] We consider a 10-year bond, meaning that $n=10$.
[13] If $\sigma_\pi=5\%$, then $\rho_{S,B}=-34.3\%$.

Table 6: Effect of the adjustment speed ($\varrho > 0$)

ϱ_r	$\rho_{S,B}$	ϱ_π	$\rho_{S,B}$	ϱ_δ	$\rho_{S,B}$
0.10	34.9%	0.10	55.7%	0.10	51.9%
0.25	39.6%	0.25	56.7%	0.25	50.6%
0.50	51.1%	0.50	55.4%	0.50	47.9%
0.75	70.1%	0.75	37.1%	0.75	44.4%

The default parameters are $\varrho_r = 0.5$, $\varrho_\pi = 0.6$, $\varrho_\delta = 0.20$, $\sigma_r = 1\%$, $\sigma_\pi = 1\%$, $\sigma_\delta = 5\%$, $\rho_{r,\pi} = -25\%$, $\rho_{r,\delta} = -25\%$ and $\rho_{\pi,\delta} = -25\%$.

Table 7: Effect of the volatility ($\varrho > 0$)

σ_r	$\rho_{S,B}$	σ_π	$\rho_{S,B}$	σ_δ	$\rho_{S,B}$
1%	51.1%	1%	51.1%	1%	53.8%
2%	70.5%	2%	24.1%	5%	51.1%
3%	80.4%	3%	−0.2%	10%	46.6%
5%	89.9%	5%	−34.3%	30%	42.5%

The default parameters are $\varrho_r = 0.5$, $\varrho_\pi = 0.6$, $\varrho_\delta = 0.20$, $\sigma_r = 1\%$, $\sigma_\pi = 1\%$, $\sigma_\delta = 5\%$, $\rho_{r,\pi} = -25\%$, $\rho_{r,\delta} = -25\%$ and $\rho_{\pi,\delta} = -25\%$.

Table 8: Effect of the adjustment speed ($\varrho < 0$)

ϱ_r	$\rho_{S,B}$	ϱ_π	$\rho_{S,B}$	ϱ_δ	$\rho_{S,B}$
−0.10	47.8%	−0.10	46.2%	−0.10	42.9%
−0.20	46.2%	−0.20	44.9%	−0.20	44.9%
−0.30	44.9%	−0.30	43.6%	−0.30	46.8%
−0.50	42.5%	−0.50	41.3%	−0.50	50.4%

The default parameters are $\varrho_r = -0.3$, $\varrho_\pi = -0.2$, $\varrho_\delta = -0.2$, $\sigma_r = 1\%$, $\sigma_\pi = 1\%$, $\sigma_\delta = 5\%$, $\rho_{r,\pi} = -25\%$, $\rho_{r,\delta} = -25\%$ and $\rho_{\pi,\delta} = -25\%$.

Table 9: Effect of the volatility ($\varrho < 0$)

σ_r	$\rho_{S,B}$	σ_π	$\rho_{S,B}$	σ_δ	$\rho_{S,B}$
1%	44.9%	1%	44.9%	1%	83.5%
2%	58.4%	2%	52.8%	5%	44.9%
3%	67.9%	3%	61.8%	10%	32.2%
5%	79.9%	5%	76.4%	30%	23.0%

The default parameters are $\varrho_r = -0.3$, $\varrho_\pi = -0.2$, $\varrho_\delta = -0.2$, $\sigma_r = 1\%$, $\sigma_\pi = 1\%$, $\sigma_\delta = 5\%$, $\rho_{r,\pi} = -25\%$, $\rho_{r,\delta} = -25\%$ and $\rho_{\pi,\delta} = -25\%$.

Table 10: Effect of the volatility ($\rho_{\pi,\delta} > 0$)

σ_r	$\rho_{S,B}$	σ_π	$\rho_{S,B}$	σ_δ	$\rho_{S,B}$
1%	4.4	1%	4.4%	1%	28.3%
2%	45.6	2%	−33.4%	5%	4.4%
3%	64.9	3%	−53.1%	10%	−2.7%
5%	82.3	5%	−72.0%	30%	−8.3%

The default parameters are $\varrho_r = 0.5$, $\varrho_\pi = 0.6$, $\varrho_\delta = 0.20$, $\sigma_r = 1\%$, $\sigma_\pi = 1\%$, $\sigma_\delta = 5\%$, $\rho_{r,\pi} = -25\%$, $\rho_{r,\delta} = -25\%$ and $\rho_{\pi,\delta} = 50\%$.

Figure 14: Probability density function of $\rho_{S,B}$ (inflation-centric model)

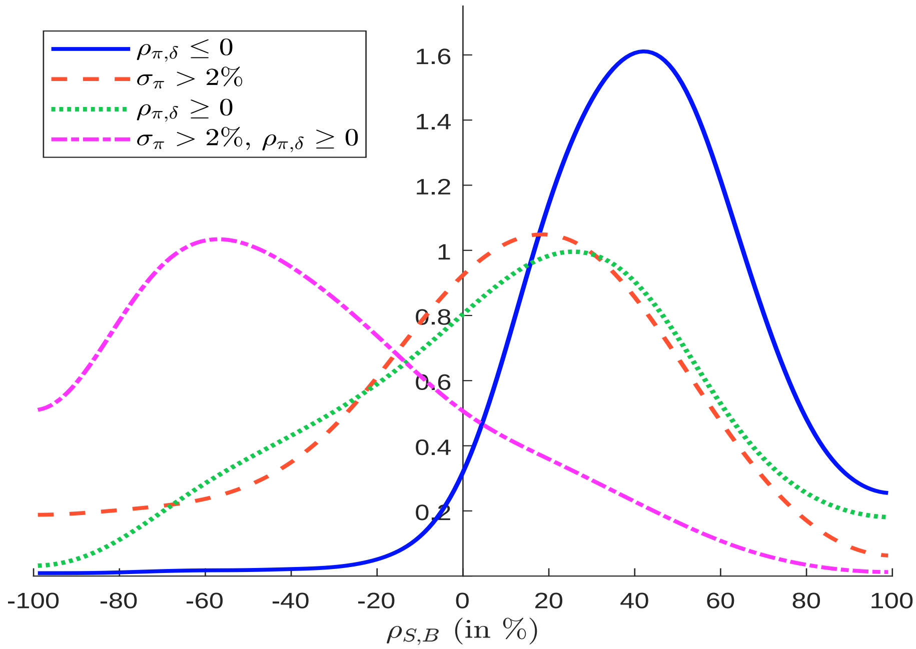
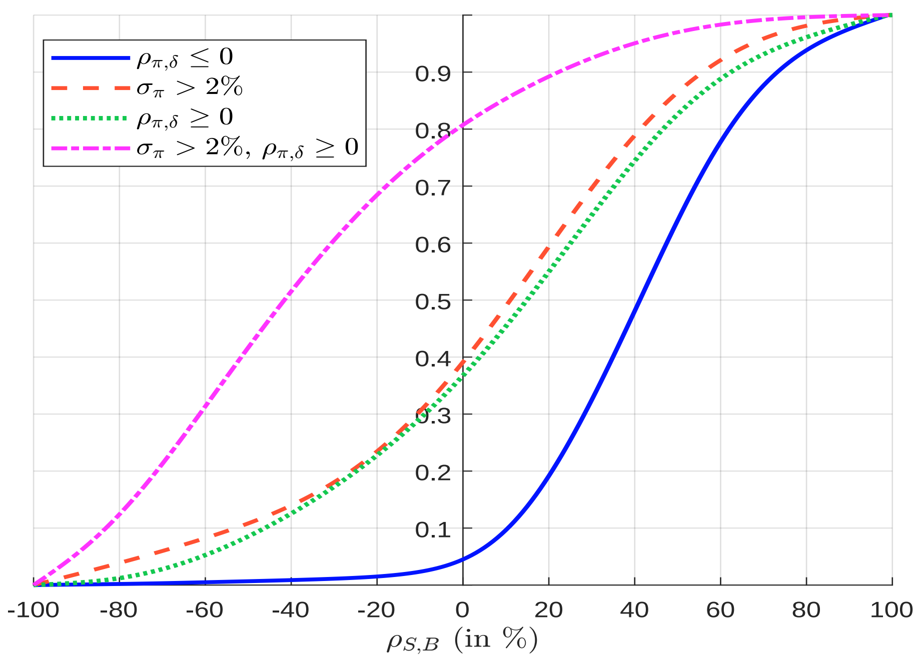


Figure 15: Cumulative distribution function of $\rho_{S,B}$ (inflation-centric model)



To verify that the stock-bond correlation is generally positive in the inflation-centric model, we run a Monte Carlo simulation. We assume that the model parameters are uniformly distributed: $\varrho_r \sim \mathcal{U}_{[0.1,0.9]}$, $\varrho_\pi \sim \mathcal{U}_{[0.1,0.9]}$, $\varrho_\delta \sim \mathcal{U}_{[0.1,0.9]}$, $\sigma_r \sim \mathcal{U}_{[0,2\%]}$, $\sigma_\pi \sim \mathcal{U}_{[0,2\%]}$, $\sigma_\delta \sim \mathcal{U}_{[0,2\%]}$, $\rho_{r,\pi} \sim \mathcal{U}_{[-50\%,0]}$, $\rho_{r,\delta} \sim \mathcal{U}_{[-50\%,0]}$ and $\rho_{\pi,\delta} \sim \mathcal{U}_{[-50\%,0]}$. The estimated probability density function is shown in Figure 14 and corresponds to the curve $\rho_{\pi,\delta} \leq 0$. We verify that the probability of observing a negative stock-bond correlation is low, less than 5% (Figure 15). If we assume that $\sigma_\pi \sim \mathcal{U}_{[2\%,10\%]}$, we obtain the curve $\sigma_\pi \geq 2\%$. Notice that the probability density function has shifted to the left. There is now a 40% probability of observing a negative stock-bond correlation. We have the same conclusion if we assume that the correlation $\rho_{\pi,\delta} \sim \mathcal{U}_{[0,50\%]}$ is positive. The main difference is the shape of the density function when the stock-bond correlation is close to -1. Let us now combine the two effects. We conclude that in an environment of high inflation volatility and a positive correlation between inflation and dividend yields, the stock-bond correlation is more likely to be negative than positive.

To get an economic interpretation of the stock-bond correlation, we can decompose $\sigma_{S,B}$ into three terms:

$$\sigma_{S,B} = \sigma_{S,B}(\sigma_r) + \sigma_{S,B}(\sigma_\pi) + \sigma_{S,B}\left(\rho_{\pi,\delta}, \sigma_\pi\right)$$

where:

$$\left\{\begin{array}{lcl}
\sigma_{S,B}(\sigma_r) & = & \left(\dfrac{(1-\varrho_r^n)(1-\varrho_\delta)\sigma_r - (1-\varrho_r)(1-\varrho_r^n)\rho_{r,\delta}\sigma_\delta}{(1-\varrho_r)^2(1-\varrho_\delta)}\right)\sigma_r + \\
& & \left(\dfrac{(1-\varrho_\pi^n)\varrho_\pi - (1-\varrho_r^n)(1-\varrho_\pi)}{(1-\varrho_r)(1-\varrho_\pi)}\rho_{r,\pi}\sigma_\pi\right)\sigma_r \geq 0 \\
\sigma_{S,B}(\sigma_\pi) & = & -\left(\dfrac{(1-\varrho_\pi^n)\varrho_\pi}{(1-\varrho_\pi)}\right)\sigma_\pi^2 \leq 0 \\
\sigma_{S,B}\left(\rho_{\pi,\delta},\sigma_\pi\right) & = & -\left(\dfrac{(1-\varrho_\pi^n)\varrho_\pi}{(1-\varrho_\pi)(1-\varrho_\delta)}\sigma_\delta\right)\rho_{\pi,\delta}\sigma_\pi \gtrless 0
\end{array}\right.$$

The first term depends on the uncertainty of real interest rates. The stock-bond covariance generally increases with the volatility risk σ_r of interest rates. As Li (2002) explains, this is intuitive "*because the real interest rate determines how an investor discounts stock and bond cash flows. Therefore, interest rate shocks are likely to move stock and bond prices in the same direction.*" The first component then implies a positive correlation due to asset valuation. The second term depends on the volatility of inflation shocks. It has a negative contribution. When the risk of inflation is high, bonds may be less attractive due to the erosion of purchasing power, while equities offer a better potential hedge. Finally, the last term depends on both inflation risk and the correlation between inflation and dividend yields. When $\rho_{\pi,\delta}$ is negative, the third component is positive. Indeed, as inflation rises, dividend yields tend to fall, creating a positive comovement between stock and bond returns. If $\rho_{\pi,\delta}$ is positive, the third component is negative, because as inflation rises, dividend yields tend to rise. In this case, bond returns tend to fall while stock returns tend to rise.

Let us illustrate the decomposition process with the previous Monte Carlo simulations. Results are provided in Table 11. We consider the case where the correlation between inflation and dividend yield is positive. Assuming that $\rho_{S,B} \geq 0$, the average stock-bond correlation is 43.8% and we have the following decomposition:

$$\underbrace{43.8\%}_{\bar{\rho}_{S,B}} = \underbrace{40.5\%}_{\bar{\rho}_{S,B}(\sigma_r)} + \underbrace{-18.5\%}_{\bar{\rho}_{S,B}(\sigma_\pi)} + \underbrace{21.7\%}_{\bar{\rho}_{S,B}\left(\rho_{\pi,\delta},\sigma_\pi\right)}$$

We verify that the interest rate component is positive and the inflation component is negative. The third component $\rho_{S,B}\left(\rho_{\pi,\delta},\sigma_\pi\right)$ is positive when $\rho_{\pi,\delta}\leq 0$ and negative when $\rho_{\pi,\delta}\geq 0$. Moreover, we notice that even thought contribution of $\rho_{S,B}\left(\sigma_r\right)$ is always positive, it tends to decrease as σ_π increases. Therefore, we can have a negative comovement between the interest rate component and the inflation component.

Table 11: Decomposition of the stock-bond correlation in the inflation-centric model

Sign	Condition	$\bar{\rho}_{S,B}$	$\bar{\sigma}_{S,B}\left(\sigma_r\right)$	$\bar{\sigma}_{S,B}\left(\sigma_\pi\right)$	$\bar{\sigma}_{S,B}\left(\rho_{\pi,\delta},\sigma_\pi\right)$
$\rho_{\pi,\delta}\geq 0$	$\rho_{S,B}\geq 0$	43.8%	40.5%	−18.5%	21.7%
	$\rho_{S,B}\geq 50\%$	66.0%	57.2%	−3.3%	12.1%
$\rho_{\pi,\delta}\leq 0$	$\rho_{S,B}\leq 0$	−50.0%	79.1%	−60.7%	−68.4%
	$\rho_{S,B}\leq -50\%$	−70.6%	5.2%	−40.1%	−35.7%

We can also consider another decomposition of $\sigma_{S,B}$:

$$\sigma_{S,B}=\sigma'_{S,B}\left(\sigma_r\right)+\sigma'_{S,B}\left(\sigma_\pi\right)$$

where:

$$\left\{\begin{array}{lcl}\sigma'_{S,B}\left(\sigma_r\right) & = & \left(\dfrac{\left(1-\varrho_r^n\right)\left(1-\varrho_\delta\right)\sigma_r-\left(1-\varrho_r\right)\left(1-\varrho_r^n\right)\rho_{r,\delta}\sigma_\delta}{\left(1-\varrho_r\right)^2\left(1-\varrho_\delta\right)}\right)\sigma_r\geq 0\\ \sigma'_{S,B}\left(\sigma_\pi\right) & = & -\left(\dfrac{\left(1-\varrho_\pi^n\right)\varrho_\pi}{\left(1-\varrho_\pi\right)}\right)\sigma_\pi^2+\left(\dfrac{\left(1-\varrho_\pi^n\right)\varrho_\pi-\left(1-\varrho_r^n\right)\left(1-\varrho_\pi\right)}{\left(1-\varrho_r\right)\left(1-\varrho_\pi\right)}\rho_{r,\pi}\sigma_r\right)\sigma_\pi\\ & & -\left(\dfrac{\left(1-\varrho_\pi^n\right)\varrho_\pi}{\left(1-\varrho_\pi\right)\left(1-\varrho_\delta\right)}\sigma_\delta\right)\rho_{\pi,\delta}\sigma_\pi\gtrless 0\end{array}\right.$$

The contribution of the unexpected inflation shocks can be positive if the correlation $\rho_{\pi,\delta}$ is negative and $\varrho_\pi\leq\dfrac{1}{2}$. This means that small adjustments to the inflation equilibrium increase the stock-bond correlation. Therefore, we must be careful with the model as it is very sensitive to the values of the parameters, especially the signs of the correlations, but also to the adjustment mechanisms of the macroeconomic factors to their equilibrium. Li (2002) complements his model with an empirical analysis. He finds empirically that uncertainty about expected inflation increases the comovement between stock and bond returns, while lower inflation risk decreases the stock-bond correlation. In summary, the changing nature of the stock-bond correlation is due to the level risk of macroeconomic factors and is mainly driven by inflation risk, more specifically expected inflation, as the effect of inflation shocks and real interest rates is less significant.

In the previous model, the macroeconomic variables are the real interest rate and inflation, while the financial variable is the dividend yield. However, the latter can be related to economic growth and can therefore also be considered as a macroeconomic variable. Hasseltoft (2009) and Burkhardt and Hasseltoft (2012) developed a model that explicitly refers to growth. Indeed, they assume that real consumption growth, inflation, stock and bond returns follow a vector autoregression process, while investor preferences are given by the recursive Epstein-Zin utility function. They then introduce inflation regimes, in particular two regimes:

1. A countercyclical inflation regime whose typical period is between 1965 and 2000;

2. A procyclical inflation regime, which occurs after 2000.

Again, they show that expected inflation and inflation volatility are important drivers of stock and bond returns. One of their key findings is that inflation reduces bond returns when inflation is countercyclical and increases bond returns when inflation is procyclical:

> "*[...] inflation risk is always negatively related to stock prices but can either decrease or increase bond prices depending on whether inflation is counter- or procyclical. In countercyclical inflation regimes, nominal bonds are risky assets and therefore perform badly as inflation risk increases. However, nominal bonds provide a hedge against bad times when inflation is procyclical. This produces a drop in nominal rates as inflation risk increases, generating positive bond returns. We find that this asymmetry in how inflation risk impacts asset prices helps explain why the stock-bond correlation switches sign over time.*" (Burkhardt and Hasseltoft, 2012).

These two seminal papers have led to a large body of research in which inflation dynamics and the correlation between inflation and growth are the main drivers of the sign of the stock-bond correlation. For example, Campbell *et al.* (2017) estimated a positive bond risk premium in the 1980s and a negative bond risk premium during the dot-com crisis and the 2008 global financial crisis. They attributed these changes to changes in the stock-bond correlation over time due to changes in the covariance between inflation and the real economy, specifically inflation and the output gap (Campbell *et al.*, 2020). They also mentioned the effect of monetary policy on the stock-bond correlation. This new explanatory factor was investigated by Song (2017) and Baele and Van Holle (2017). They both showed that accommodating monetary policy can be an important factor to explain the negative stock-bond correlation:

> "*Negative stock-bond correlations are associated with periods of accommodating monetary policy, but only in times of low inflation. Irrespective of the inflation and/or growth regime, stock-bond correlations are always positive when monetary policy is restrictive. Pure inflation and growth regimes instead have little explanatory power for stock-bond correlations.*" (Baele and Van Holle, 2017).

Even though empirical results have found a relationship between inflation dynamics and changes in the sign of the stock-bond correlation, the magnitude of the change remains puzzling, according to Duffee (2023).

2.4.2 Real-centric model

While inflation-centric models focus on inflation and are based on the nominal channel, *i.e.* changes in nominal yields, real-centric models focus on macroeconomic shocks and are based on the real channel, *i.e.* changes in real yields (Chernov *et al.*, 2021; Duffee, 2023). Chernov *et al.* (2021) considered two types of macroeconomic shocks, transitory and permanent shocks to consumption, but Duffee (2023) explained that any macroeconomic shocks can be considered. Some shocks can produce a positive stock-bond correlation, while other shocks can produce a negative stock-bond correlation, and the transition from one shock regime to another induces a time-varying evolution of the stock-bond correlation. From this perspective, real-centric models are more heterogeneous than inflation-centric models because the underlying factors are very different from one model to another. However, they generally shared a common framework, which is the long-run risk model of Bansal and Yaron (2004). This model relates consumption and dividend growth rates and analyzes two main situations: persistent expected growth and fluctuating economic uncertainty. Macroeconomic news and shocks affect investors' market sentiment and then change asset pricing. Here are some examples:

- Ermolov (2022) considers two different macroeconomic shocks: demand-like and supply-like.
- For Jones and Pyun (2023), it is the shocks to expected consumption growth that drive the stock-bond correlation.
- Laarits (2022) explores the impact of the precautionary savings channel and risk compensation on safe rates, which can explain the flight to quality behavior.

In Table 12, we report the different economic mechanisms of the stock-bond comovement identified by Duffee (2023).

Table 12: Covariance between stock returns and bond returns (real-centric models)

Economic mechanism	Sign
Positively serially correlated shocks to consumption growth rates	$-$
Flight to quality	$-$
Countercyclical flight to quality	$-$
Transitory shocks to consumption	$+$
Procyclical conditional volatility of the SDF	$+$
Slow-moving habit	$+$
Stochastic time rate of preference	$+$

Source: Duffee (2023, pages 11-12).

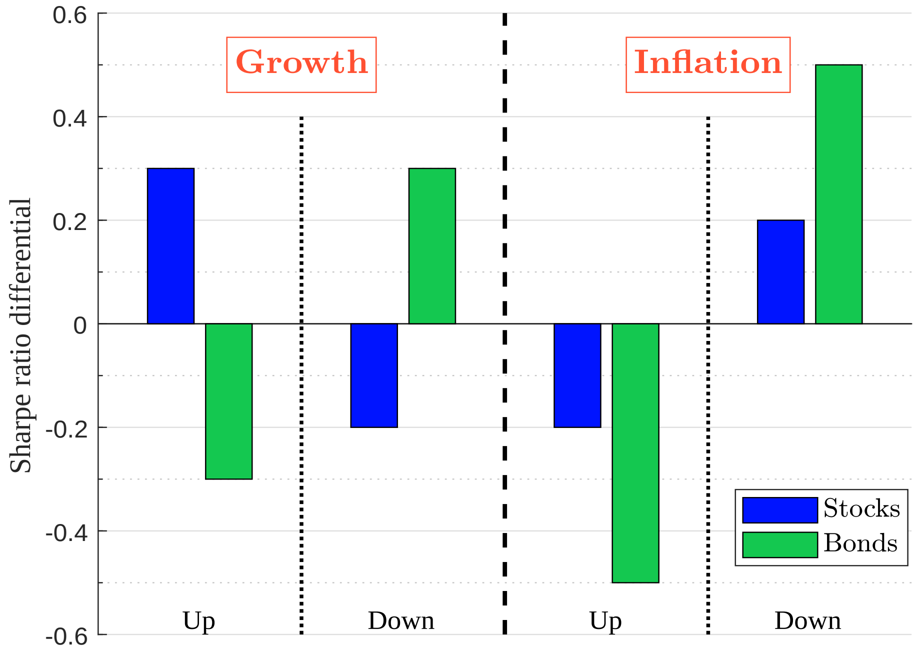
2.4.3 Growth-inflation model

Brixton *et al.* (2023) assume that the stock-bond correlation is driven by growth and inflation, the two main macroeconomic factors:

> "*[...] Positive growth news raises equity investors' expectations of future cash flows and, hence, equity prices. It also raises expectations for short-term interest rates, through both the systematic response of central banks [...], so bond prices fall. In other words, stocks and bonds have opposite-signed sensitivities to growth news. [...] Positive inflation news directly reduces the value of bonds' fixed nominal cash flows, as well as raising short-term interest rate expectations, so prices fall. Equities, in theory, give investors a claim on real cash flows, but in practice, rising inflation has usually been associated with falling stock prices. Stocks and bonds therefore have same-signed sensitivities to inflation news.*" (Brixton *et al.*, 2023, page 3).

To illustrate their assumption, Brixton *et al.* (2023) compute the Sharpe ratio of US stocks and bonds by dividing the period January 1972-June 2022 into up and down growth and inflation regimes. They then report the difference between the Sharpe ratio of a particular regime and the Sharpe ratio of the entire period. We reproduce their results in Figure 16. They find that stocks outperform in a growth up regime, while bond returns are higher in a growth down regime. In the case of inflation, both stocks and bonds prefer an inflation down regime. In this approach, the stock-bond correlation is negative when the growth component dominates, while the stock-bond correlation is positive when the inflation dimension drives the financial market.

Figure 16: Sharpe ratio differentials by macroeconomic environment (US, January 1972-June 2022)



Source: Brixton *et al.* (2023, Exhibit 3, page 5).

The model developed by Brixton *et al.* (2023) assumes a two-factor model for the asset return innovations:

$$\left\{ \begin{array}{l} R_S(t) - \mathbb{E}_{t-1}\left[R_S(t)\right] = \beta_{S,g}\varepsilon_g(t) + \beta_{S,\pi}\varepsilon_\pi(t) \\ R_B(t) - \mathbb{E}_{t-1}\left[R_B(t)\right] = \beta_{B,g}\varepsilon_g(t) + \beta_{B,\pi}\varepsilon_\pi(t) \end{array} \right.$$

where $\varepsilon_g(t) \sim \mathcal{N}\left(0, \sigma_g^2\right)$ and $\varepsilon_\pi(t) \sim \mathcal{N}\left(0, \sigma_\pi^2\right)$ are the growth and inflation shocks. According to the previous empirical results, the sensitivity parameters satisfy $\beta_{S,g} \geq 0$, $\beta_{S,\pi} \leq 0$, $\beta_{B,g} \leq 0$ and $\beta_{B,\pi} \leq 0$. In Appendix A.9 on page 104, we show that the stock-bond covariance has the following expression:

$$\sigma_{S,B} = \underbrace{\beta_{S,g}\beta_{B,g}}_{\text{negative}}\sigma_g^2 + \underbrace{\beta_{S,\pi}\beta_{B,\pi}}_{\text{positive}}\sigma_\pi^2 + \underbrace{\left(\beta_{S,g}\beta_{B,\pi} + \beta_{S,\pi}\beta_{B,g}\right)}_{\text{positive/negative}}\rho_{g,\pi}\sigma_g\sigma_\pi$$

The above expression can be written as:

$$\sigma_{S,B}(t) = \beta_g\sigma_g^2(t) + \beta_\pi\sigma_\pi^2(t) + \beta_{g,\pi}\rho_{g,\pi}(t)\,\sigma_g(t)\,\sigma_\pi(t)$$

where $\beta_g = \beta_{S,g}\beta_{B,g}$, $\beta_\pi = \beta_{S,\pi}\beta_{B,\pi}$ and $\beta_{g,\pi} = \beta_{S,g}\beta_{B,\pi} + \beta_{S,\pi}\beta_{B,g}$. The stock-bond covariance is then time-varying because growth and inflation risks change over time, as does the growth-inflation correlation. Let us first assume that $\rho_{g,\pi} = 0$. Note that $\beta_g < 0$ implies that an increase in growth risk reduces the stock-bond covariance, while an increase in inflation risk increases the stock-bond covariance because $\beta_\pi > 0$. When the growth-inflation correlation is not equal to zero, the effect of the third term depends on the signs of $\beta_{g,\pi}$ and $\rho_{g,\pi}(t)$. We have:

$$\beta_{g,\pi} \geq 0 \Leftrightarrow \beta_{S,g} \leq \beta^{\star}_{S,g} := \left| \frac{\beta_{S,\pi}\beta_{B,g}}{\beta_{B,\pi}} \right|$$

We conclude that the effect of the third term is ambiguous. Here is a summary of the effect of the three components on the stock-bond covariance:

Growth risk	Inflation risk	Growth-inflation correlation		
β_g	β_π	$\beta_{S,g} \leq \beta^{\star}_{S,g}$	$\beta_{S,g} \geq \beta^{\star}_{S,g}$	
$-$	$+$	$-$	$+$	$\rho_{g,\pi}(t) \leq 0$
$-$	$+$	$+$	$-$	$\rho_{g,\pi}(t) \geq 0$

Another interesting result is that the stock-bond correlation does not depend on the level of the growth and inflation volatilities, but only on the ratio of the growth risk to the inflation risk:

$$\rho_{S,B} = \frac{\beta_{S,g}\beta_{B,g}\varphi^2_{g,\pi} + \beta_{S,\pi}\beta_{B,\pi} + \left(\beta_{S,g}\beta_{B,\pi} + \beta_{S,\pi}\beta_{B,g}\right)\rho_{g,\pi}\varphi_{g,\pi}}{\sqrt{\beta^2_{B,g}\varphi^2_{g,\pi} + \beta^2_{B,\pi} + 2\beta_{B,g}\beta_{B,\pi}\rho_{g,\pi}\varphi_{g,\pi}}\sqrt{\beta^2_{S,g}\varphi^2_{g,\pi} + \beta^2_{S,\pi} + 2\beta_{S,g}\beta_{S,\pi}\rho_{g,\pi}\varphi_{g,\pi}}}$$

where:

$$\varphi_{g,\pi} = \frac{\sigma_g}{\sigma_\pi}$$

To illustrate the effect of each parameter, we plot the relationship between $\rho_{S,B}$ and one parameter while holding the other parameters constant. We assume that the standard parameters are $\beta_{S,g} = 1$, $\beta_{S,\pi} = \beta_{B,g} = \beta_{B,\pi} = -1$ and $\varphi_{g,\pi} = 2$. Results are shown in Figure 66 on page 109. Most of the time, the stock-bond correlation is expected to be negative except when $\beta_{S,g}$ or $\beta_{B,g}$ is very low. If we assume that $\beta_{S,g} = 0.20$, the correlation $\rho_{S,B}$ can be positive, especially if the correlation $\rho_{g,\pi}$ is positive (Figure 67 on page 110). The impact of $\varphi_{g,\pi}$ is given in Figure 68 on page 110. We observe a downward-slopping function. The higher the growth-inflation risk ratio, the lower the stock-bond correlation. The effect of $\rho_{g,\pi}$ is more complex to analyze, as shown in Figure 69 on page 111.

Table 13: Growth volatility, inflation volatility, growth-inflation correlation (in %) and growth-inflation ratio

Country	1960-2000				2000-2023			
	σ_g	σ_π	$\rho_{g,\pi}$	$\varphi_{g,\pi}$	σ_g	σ_π	$\rho_{g,\pi}$	$\varphi_{g,\pi}$
France	2.3	3.9	-13.9	0.6	3.2	1.3	20.4	2.4
Germany	2.3	1.9	-13.7	1.2	2.7	1.7	14.2	1.6
Japan	2.7	5.0	-4.5	0.5	2.5	1.2	3.8	2.1
UK	2.5	5.4	-49.1	0.5	4.6	1.8	8.7	2.5
US	2.4	3.1	-45.2	0.8	2.2	1.8	35.0	1.2

Source: OECD (2024), `data.oecd.org/gdp/quarterly-gdp.htm`, `data.oecd.org/price/inflation-cpi.htm` & Author's calculations.

In Figures 70-73 on pages 111-113, we have calculated the 4-year rolling window estimates of the four macroeconomic parameters (σ_g, σ_π, $\rho_{g,\pi}$ and $\varphi_{g,\pi}$) for five countries: France, Germany, Japan, the United Kingdom and the United States. As expected, these macroeconomic drivers change significantly over time. This is especially true for $\rho_{g,\pi}$ and $\varphi_{g,\pi}$. If we divide the period 1965-2023 into two subperiods before and after 2000, we obtain the results shown in Table 13. The first period is characterized by a negative growth-inflation correlation and a low growth-inflation risk ratio. In this case, the positive components of the stock-bond covariance dominate the negative components and the stock-bond correlation tends to be positive. In contrast, the period between 2000 and 2023 is characterized by a positive growth-inflation correlation and a high growth-inflation risk ratio. In this situation, the growth-inflation model predicts a negative correlation.

Summary and main results

Three types of models are useful for studying the stock-bond correlation: inflation-centric models, real-centric models, and growth-inflation models. In inflation-centric models, the key driver of the stock-bond correlation is inflation, specifically expected inflation and inflation risk (or inflation innovations). This can lead to both positive and negative stock-bond correlations, depending on the correlation between inflation, consumption growth and dividend yields. A negative (positive) correlation between inflation and dividend yields generally leads to a positive (negative) correlation. Modern inflation-centric theories emphasize the importance of the dynamics between inflation and the economy. In these approaches, countercyclical inflation is associated with a positive stock-bond correlation. Conversely, procyclical inflation generates a negative stock-bond correlation as bonds become hedging assets. The concept of the inflation regime is then central, but other factors may also play a role. In particular, accommodating monetary policy is a source of negative stock-bond correlation.

Factors	Sign
Accommodating monetary policy	−
Beta-sensitivity to growth	−
Consumption risk	−
Flight to quality	−
Growth risk	−
Growth-inflation risk ratio	−
Real interest rate	+
Inflation risk	+
Risk aversion	−

The dynamics of real interest rates and the real economy are the focus of real-centric models. In this approach, macroeconomic shocks are the drivers of the stock-bond correlation. In most cases, these shocks lead to a negative correlation, such as shocks to consumption growth or shocks to perceived risk. They explain the flight to quality phenomenon. In this type of models, we can observe a positive stock-bond correlation when real interest rates are high. The third type of model is a two-factor model, where the two main risk factors are growth uncertainty and inflation uncertainty. Growth risk pushes the stock-bond correlation to negative, while inflation risk pushes the stock-bond correlation to positive. In this approach, the correlation can be positive or negative, and its sign depends on the ratio of growth to inflation risk and the magnitude of the beta sensitivity of asset returns to growth and inflation shocks. A low beta sensitivity to growth tends to produce a positive correlation, while a high beta sensitivity to growth tends to produce a negative correlation. The table above summarizes the impact of various macroeconomic factors.

3 Empirical results

While the first part of the paper was dedicated to the theory of understanding the drivers of the stock-bond correlation, in this second part we conduct an empirical and econometric analysis in relation to the first part. As always in economics and finance, there is no single answer. It is an illusion to try to fine-tune a one-size-fits-all model. Investor behavior varies from period to period, market structure changes significantly over time, financial risk becomes more complex, and so on. Using the same model to explain the stock-bond correlation in the 1960s and the stock-bond correlation today has many limitations. Instead of testing the various theoretical models developed in the first part and choosing the best one, we take a different approach. We assume that each model is more or less correct and examine the empirical implications from an investor's perspective.

3.1 Estimation of the stock-bond correlation

3.1.1 Estimation method

Estimating the stock-bond correlation is not straightforward because it involves several methodological choices. The first choice concerns the selection of the bond and the index portfolio. In the case of the US, we can choose between several equity benchmarks, such as the S&P 500 Index, the MSCI USA Index or the Russell 1000 Index. For the bond, we have to choose the maturity (*e.g.*, 1, 3, 5, 7, 10 or 30 years) and the type of bond (rolling asset or generic asset). The second choice concerns the performance measure. For the stock portfolio, the choice is between price index performance and total return performance. For bonds, the choice is between total return and yield-to-maturity. The third choice is the frequency of the performance measure. Do we want to look at daily, weekly or monthly returns? Finally, the fourth choice is the estimation method itself. Three main approaches have been studied in the academic literature. The first considers the empirical estimator based on a rolling window, the second uses the exponentially weighted moving average estimator, while the third assumes a GARCH model family. Each model requires the calibration of some parameters. For example, for the empirical model we need to define the length of the window, while for the EWMA estimator we need to set the decay parameter. In the case of GARCH models, there are many models to choose from. For example, Andersson *et al.* (2008), Wu and Lin (2014), and Chiang *et al.* (2015) used DCC-GARCH, GJR-GARCH, and ADCC-GARCH, respectively.

3.1.2 US analysis

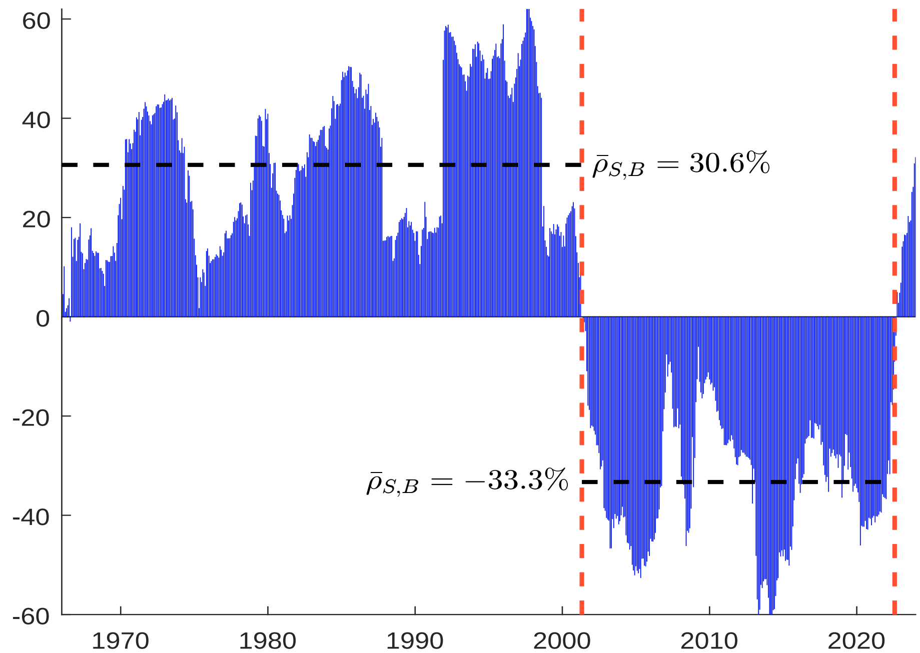
We look at monthly time series of the 10-year generic bond and the S&P 500 index provided by Bloomberg[14]. We compute the empirical correlation $\hat{\rho}_{S,B}(t)$ of the stock and bond returns with a 4-year rolling window[15]. Results are given in Figure 17. It is remarkable that we observe three regimes:

- The correlation is positive from 1965 to April 2001. During this period, the average $\bar{\rho}_{S,B}$ is 30.6%. It peaks in September 1997 and reaches a value of 64.7%.
- The correlation is then negative until July 2022. During this period, the average $\bar{\rho}_{S,B}$ is -33.3%. It peaks in April 2013 and reaches a value of -63.8%.
- Since August 2022, the correlation is positive again.

[14]The Bloomberg tickers are USGG10YR (US Generic Govt 10Yr) and SPX (S&P 500).

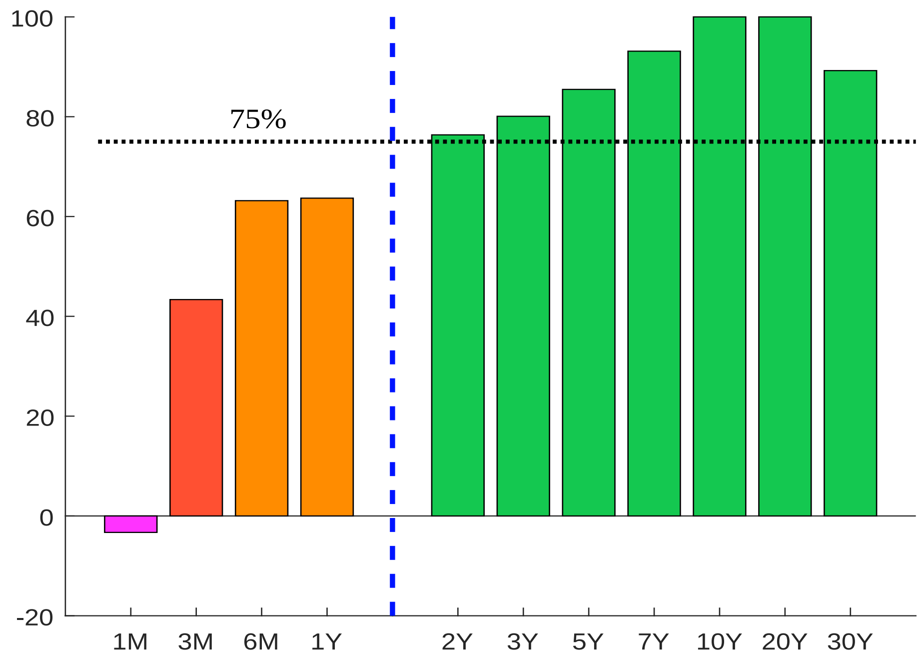
[15]Monthly bond return is calculated using the formula $R_B(t) = -\mathcal{S}(t) \cdot \big(y(t) - y(t-1)\big)$, where $\mathcal{S}(t)$ is the sensitivity of the bond at month t and $y(t)$ is the yield-to-maturity of the bond at month t.

Figure 17: Rolling 4-year stock-bond correlation (US, 10Y, 1965-2023, monthly frequency)



Source: Bloomberg (2024) & Authors' calculations.

Figure 18: Kendall correlation between $\hat{\rho}_{S,B}(t, m)$ and $\hat{\rho}_{S,B}(t, 10\text{Y})$ (US, 2065-2023, monthly frequency)



Source: Bloomberg (2024) & Authors' calculations.

The previous break dates are underestimated because the estimated correlation is a long-term estimate based on four years of historical data. Therefore, the turning point is certainly earlier[16]. The estimation of the correlation depends on several parameters. For example, it depends on the maturity of the generic bond. In Figure 75 on page 114, we show the evolution of $\hat{\rho}_{S,B}(t)$ when we consider the 3-month generic bond. We can observe a period before 2000 when the stock-bond correlation is negative and a period after 2000 when the stock-bond correlation is positive. Thus, the results obtained with the 3-month maturity are not always consistent with those obtained with the 10-year maturity. On the contrary, we observe a strong coherence between the 10-year and the 30-year maturities[17] (Figure 76 on page 114).

Let $\hat{\rho}_{S,B}(t,m)$ be the correlation for a given maturity m of the generic bond. Figure 18 shows the Kendall correlation between $\hat{\rho}_{S,B}(t,m)$ and $\hat{\rho}_{S,B}(t,10\text{Y})$ for several values of m. This correlation is greater than 75% when m is greater than or equal to 2 years. For shorter maturities, the correlation is lower. It is even negative for the one-month generic bond. Clearly, we need to distinguish between the short part of the yield curve and the rest of the yield curve. The reason is that the slope of the yield curve can be inverted and the correlation between stocks and bonds can change from one period to another. Therefore, in the following, we assume that the estimated stock-bond correlation with the 10-year generic bond is the benchmark.

The calculation of the correlation is also sensitive to the frequency and the estimation method. To illustrate this, we look at daily returns and show the rolling 4-year stock-bond correlation in Figure 19. The overall picture remains the same. Nevertheless, we generally find that the daily correlation is lower than the monthly correlation in more than 70% of the observations (Figure 77 on page 115). If we consider only the cases where the monthly correlation is positive, this frequency is more than 75%. Another popular estimation method is to use an exponentially-weighted moving average correlation. In this case, it depends on the decay factor λ, which controls the degree of weighting decrease[18]. When λ is equal to one, we get the traditional empirical estimator. When λ is less than one, the estimator gives more weight on the most recent period and we can obtain a very reactive short-term estimator when $\lambda \leq 0.95$. Figure 20 shows the EWMA stock-bond correlation when λ is set to 0.96, which implies that the half-life of the estimator is 25 days. We observe very rapid reversals when the correlation is measured at a daily frequency. The difference between the daily and monthly behavior is the result of the two main motivations for holding US bonds. On the one hand, investors want to capture a long-term bond premium. On the other hand, investors want to protect their equity exposure in the short run. These two states (risk premium and flight to quality) do not operate at the same frequency.

We may wonder if the calculation of the stock-bond correlation is different if we consider another equity investment universe. Table 14 shows the Kendall correlation matrix for 8 equity indices[19]. We note that the results are very consistent between the S&P 500 Index,

[16]Later, we obtain a more precise estimate of the turning point when we use time series with a higher frequency (daily and weekly).

[17]However, there are some significant differences between 2000 and 2016. Between 2001 and 2006, the stock-bond correlation for the 30-year bond is lower than for the 10-year bond, while it is higher between 2013 and 2015.

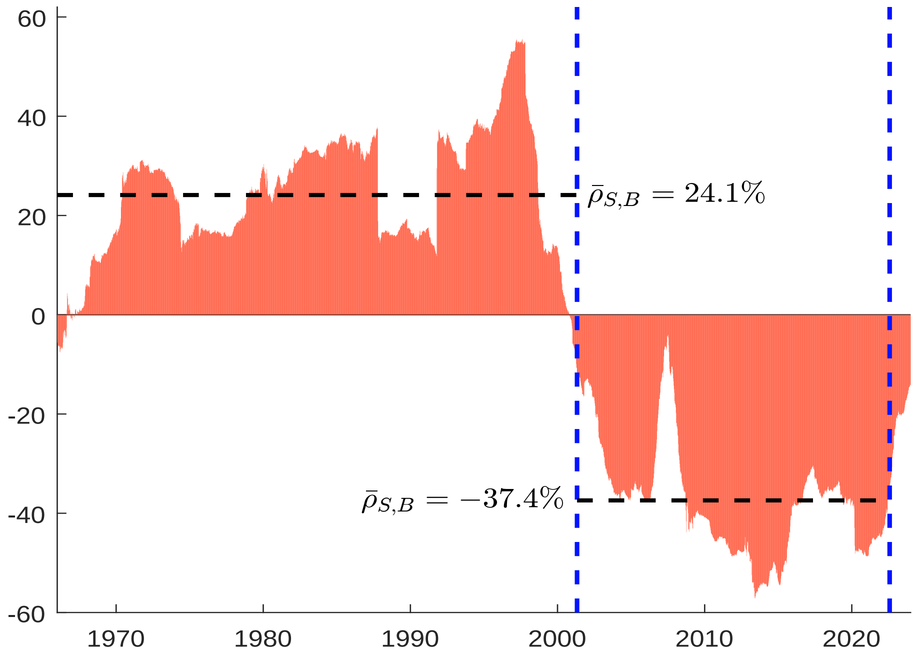
[18]The formula of the EWMA stock-bond correlation is:

$$\hat{\rho}_{S,B}(t) = \frac{\sum_{s=1}^{m} \omega_s \left(R_S(t-s) - \sum_{s=1}^{m} \omega_s R_S(t-s)\right)\left(R_B(t-s) - \sum_{s=1}^{m} \omega_s R_B(t-s)\right)}{\sqrt{\sum_{s=1}^{m} \omega_s \left(R_S(t-s) - \sum_{s=1}^{m} \omega_s R_S(t-s)\right)^2 \sum_{s=1}^{m} \omega_s \left(R_B(t-s) - \sum_{s=1}^{m} \omega_s R_B(t-s)\right)^2}}$$

where $\omega_s = (1-\lambda)\lambda^{s-1}/(1-\lambda^m)$.

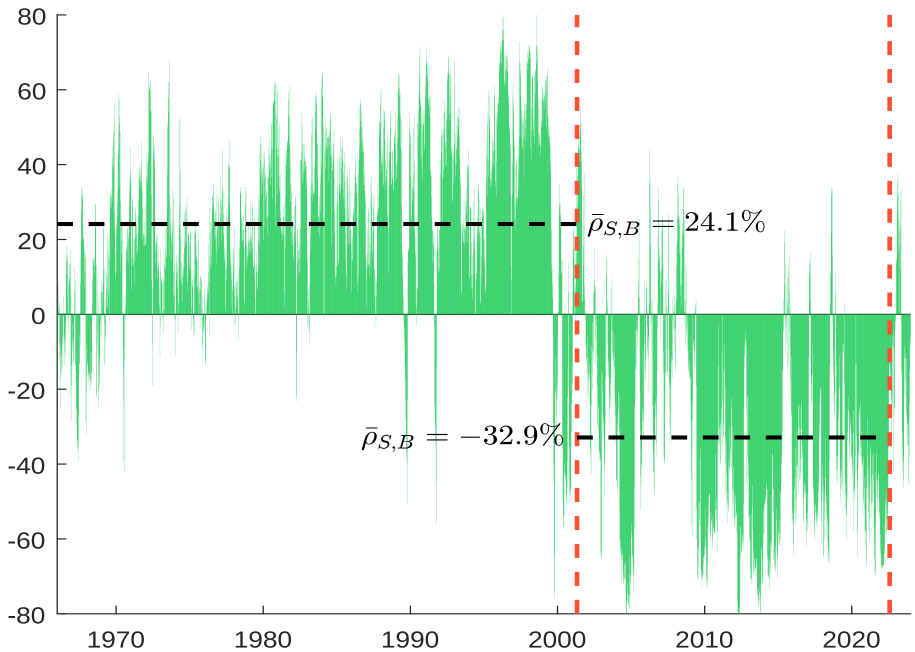
[19]In Figures 78 and 79 on page 115, we also show the scatterplot of the stock-bond correlation using two different stock indices. We can see that the results can be very different from one stock index to another.

Figure 19: Rolling 4-year stock-bond correlation (US, 10Y, 1965-2023, daily frequency)



Source: Bloomberg (2024) & Authors' calculations.

Figure 20: EWMA stock-bond correlation (US, 10Y, 1965-2023, daily frequency, $\lambda = 0.96$)



Source: Bloomberg (2024) & Authors' calculations.

Table 14: Kendall correlation matrix of the stock-bond correlation (in %)

#	Index	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
(1)	S&P 500	100.0	82.4	78.1	88.6	98.2	98.0	83.6	97.2
(2)	Nasdaq	82.4	100.0	89.3	77.1	83.1	83.4	83.2	83.7
(3)	Nasdaq 100	78.1	89.3	100.0	69.7	78.6	78.0	72.5	77.7
(4)	Dow Jones	88.6	77.1	69.7	100.0	88.8	88.8	83.9	89.0
(5)	MSCI USA	98.2	83.1	78.6	88.8	100.0	98.1	83.8	97.4
(6)	Russell 1000	98.0	83.4	78.0	88.8	98.1	100.0	85.0	98.8
(7)	Russell 2000	83.6	83.2	72.5	83.9	83.8	85.0	100.0	86.0
(8)	Russell 3000	97.2	83.7	77.7	89.0	97.4	98.8	86.0	100.0

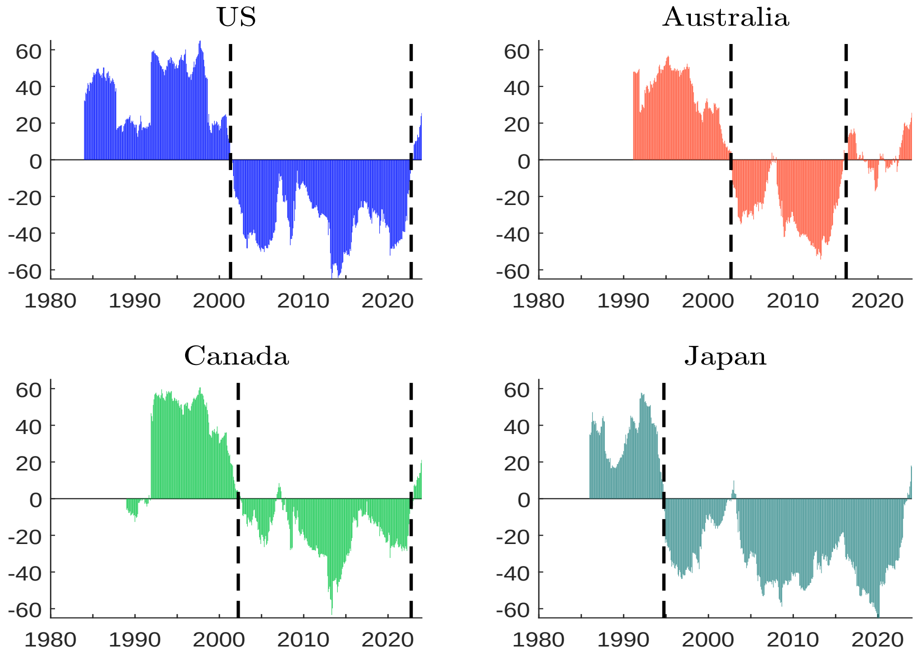
Source: Datastream (2024) & Authors' calculations.

the MSCI USA Index, and the Russell 1000 Index. However, the stock-bond correlation profile changes when we use the Nasdaq 100 Index or the Russell 2000 Index instead of the S&P 500 Index. The former is heavily weighted toward technology companies, while the Russell 2000 Index measures the performance of small-cap US companies. These empirical results support the theoretical results obtained in Section 2.3 on page 21, which show that the stock-bond correlation is sensitive to the equity portfolio construction, in particular to the sector and factor weights.

3.1.3 Other countries

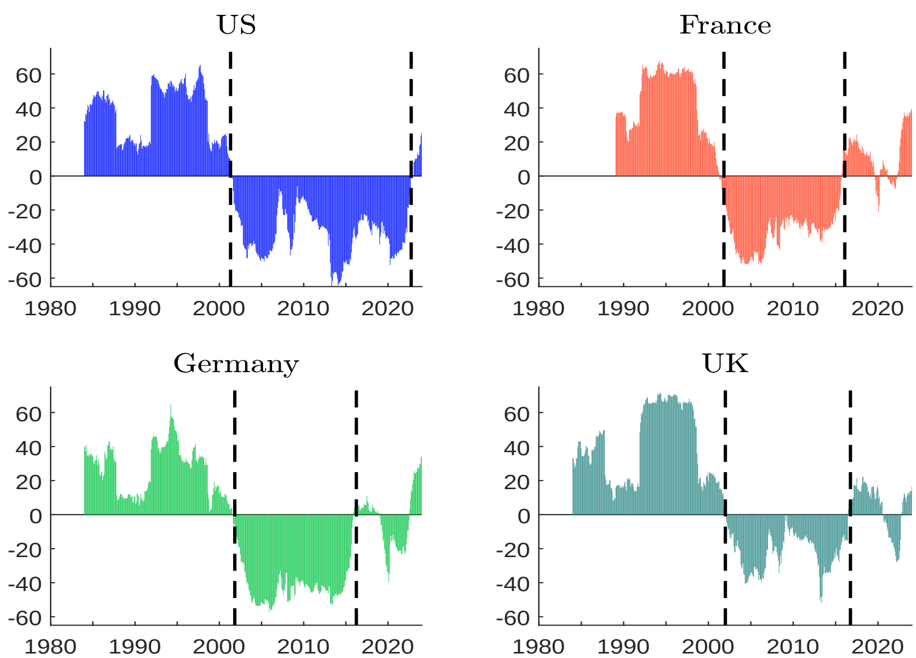
On pages 117 to 122, we estimated the stock-bond correlation for a variety of countries, including developed, developing, emerging, European, American, and Asian nations. We note that the patterns observed for the United States are far from being common to the other countries. In fact, only Canada has a stock-bond correlation behavior that is in phase with the US cycle (Figure 21). Looking at the other developed countries, we generally observe a reversal of the positive correlation during the dot-com crisis, as in the US. In Japan, however, the reversal occurs earlier, in 1994. Another big difference is the reversal of the negative stock-bond correlation, which happens between 2016 and 2017 for many developed countries, including Australia, France, Germany and the UK (Figure 22). We also note that the European debt crisis has a strong impact on the southern European countries: Greece, Italy, Portugal and Spain (Figure 23). Indeed, the stock-bond correlation for these countries turns positive at the onset of the European debt crisis and has remained positive to date. The case of Ireland is more complex, with a correlation close to zero in the 10 years following the European debt crisis. Developing and emerging countries present a unique case. We observe two distinct clusters: one where the stock-bond correlation is consistently positive, and another where it can be either positive or negative. For example, the stock-bond correlation is systematically positive for Brazil, South Africa and Turkey (Figure 24), while the stock-bond correlation has been negative in some periods for China, India and Singapore (Figure 25). One reason for this dichotomy is the impact of sovereign credit risk and currency risk on stock prices. As we have seen in Section 2, the stock-bond correlation rises with the credit risk of the country. Therefore, the normal situation for emerging countries is to observe a positive stock-bond correlation. In the case of some countries with strong economies, the credit risk component has become less important. It is also interesting to note that these empirical results are coherent with the status of the currency in foreign exchange markets (Baku *et al.*, 2019). All these results show that the US-centric view of the stock-bond correlation that we find among academics and professionals cannot be generalized to other countries.

Figure 21: Rolling 4-year stock-bond correlation (Australia, Canada, Japan)



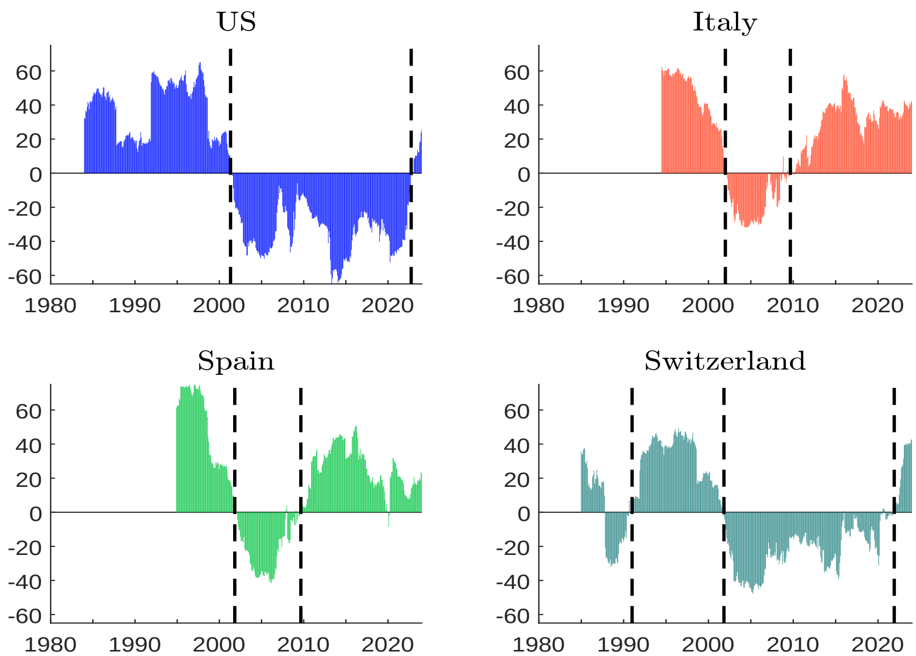
Source: Datastream (2024) & Authors' calculations.

Figure 22: Rolling 4-year stock-bond correlation (France, Germany, UK)



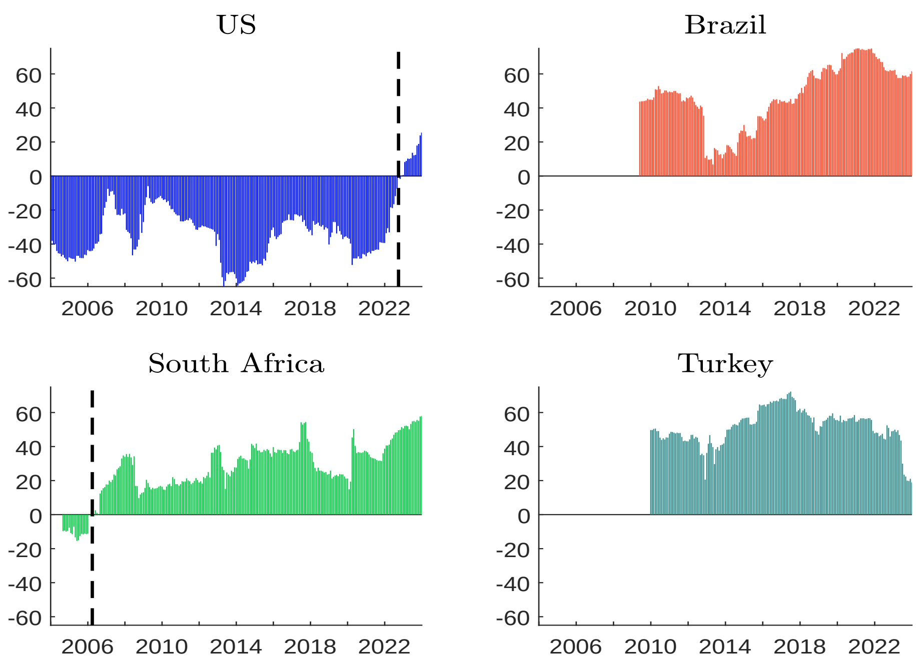
Source: Datastream (2024) & Authors' calculations.

Figure 23: Rolling 4-year stock-bond correlation (Italy, Spain, Switzerland)



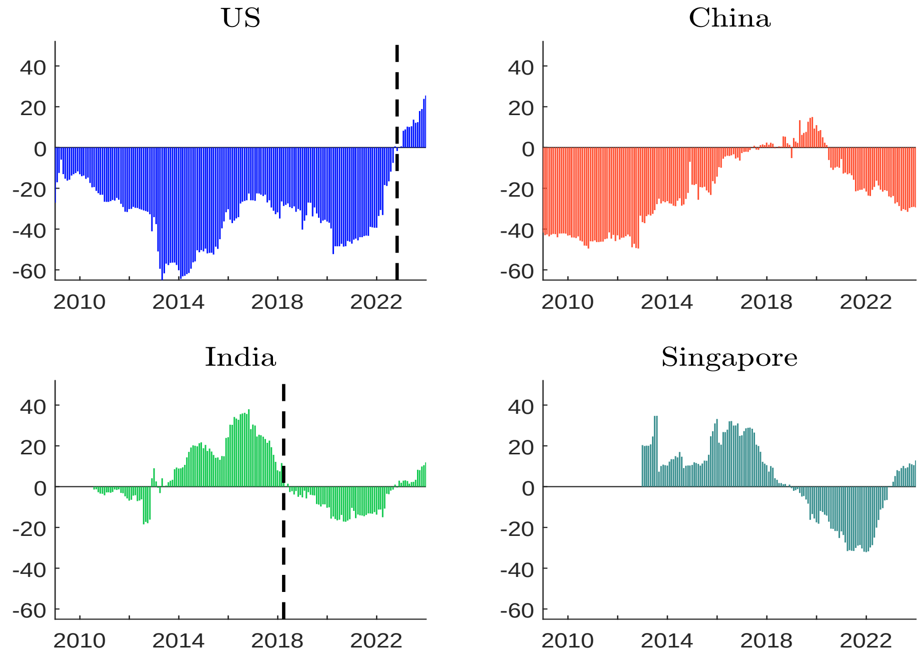
Source: Datastream (2024) & Authors' calculations.

Figure 24: Rolling 4-year stock-bond correlation (Brazil, South Africa, Turkey)



Source: Datastream (2024) & Authors' calculations.

Figure 25: Rolling 4-year stock-bond correlation (China, India, Singapore)



Source: Datastream (2024) & Authors' calculations.

Summary and main results

It is generally accepted that the stock-bond correlation was positive between 1965 and 2000 and has been negative since then, except for the last years. This view is shared by academics and professionals. We confirm these empirical patterns when we focus on the North American market (US and Canada), but it is not true for other regions and countries. In Europe, the most recent reversal occurred at the beginning of 2015 for countries with low credit risk (France, Germany, UK), while the stock-bond correlation has been positive since the European debt crisis for countries with high credit risk (Greece, Italy, Portugal, Spain). In the case of developing countries, we find two groups: a group of countries for which the stock-bond correlation is systematically positive and a group of countries for which the stock-bond correlation can change sign. In this case, the factor that determines the sign of the correlation depends on the country risk perceived by the market through credit risk and currency risk. Below is the average stock-bond correlation for the period 2014-2023:

AUS	16.9	AUT	8.0	BEL	22.7	BRA	58.3
BGR	11.9	CAN	8.5	CHL	16.2	CHN	−9.5
COL	25.8	CZE	−1.7	DNK	31.4	FIN	28.3
FRA	25.5	DEU	17.1	GRC	32.0	HKG	29.2
HUN	26.5	IND	3.8	IDN	49.8	IRL	27.2
ISR	32.5	ITA	36.5	JPN	−17.7	KOR	22.7
MYS	29.5	MEX	29.6	NLD	40.3	NZL	42.1
NOR	−28.2	PER	41.8	PHL	41.2	POL	21.0
PRT	27.7	ROU	37.0	RUS	−1.0	SGP	5.4
ZAF	47.8	ESP	18.7	SWE	18.1	CHE	16.5
TWN	−4.4	TUR	31.2	GBR	12.8	USA	4.6

3.2 Risk premium analysis

3.2.1 US analysis

Global risk premium Using the results obtained in Section 2.1.4 on page 8, we calculate the equity and bond risk premia assuming that the market portfolio has a Sharpe ratio of 30%. At each date t, we estimate $\sigma_S(t)$, $\sigma_B(t)$, and $\rho_{S,B}(t)$ using a three-year rolling window, and calculate the equity allocation in the market portfolio as follows:

$$x_S(t) = \frac{\mathrm{MV}_S(t)}{\mathrm{MV}_S(t) + \mathrm{MV}_B(t)}$$

where $\mathrm{MV}_S(t)$ and $\mathrm{MV}_B(t)$ are the market values of all US stocks and US Treasury bonds with maturities greater than one year. The bond allocation in the market portfolio is then $x_B(t) = 1 - x_S(t)$. To calculate all of these metrics, we use Datastream indices: US DS Market Index (TOTMKUS) and US Total All Lives DS Govt Index (AUSGVAL). The equity and bond risk premia priced by the market are:

$$\begin{aligned}\tilde{\pi}_S(t) &= \tilde{\pi}_S^{(\mathrm{var})}(t) + \tilde{\pi}_S^{(\mathrm{cov})}(t) \\ &= \frac{\mathrm{SR}\left(x(t) \mid r\right) x_S(t)\,\sigma_S^2(t)}{\sigma\left(x(t)\right)} + \frac{\mathrm{SR}\left(x(t) \mid r\right) x_B(t)\,\rho_{S,B}(t)\,\sigma_S(t)\,\sigma_B(t)}{\sigma\left(x(t)\right)}\end{aligned} \quad (9)$$

and:

$$\begin{aligned}\tilde{\pi}_B(t) &= \tilde{\pi}_B^{(\mathrm{var})}(t) + \tilde{\pi}_B^{(\mathrm{cov})}(t) \\ &= \frac{\mathrm{SR}\left(x(t) \mid r\right) x_B(t)\,\sigma_B^2(t)}{\sigma\left(x(t)\right)} + \frac{\mathrm{SR}\left(x(t) \mid r\right) x_S(t)\,\rho_{S,B}(t)\,\sigma_S(t)\,\sigma_B(t)}{\sigma\left(x(t)\right)}\end{aligned} \quad (10)$$

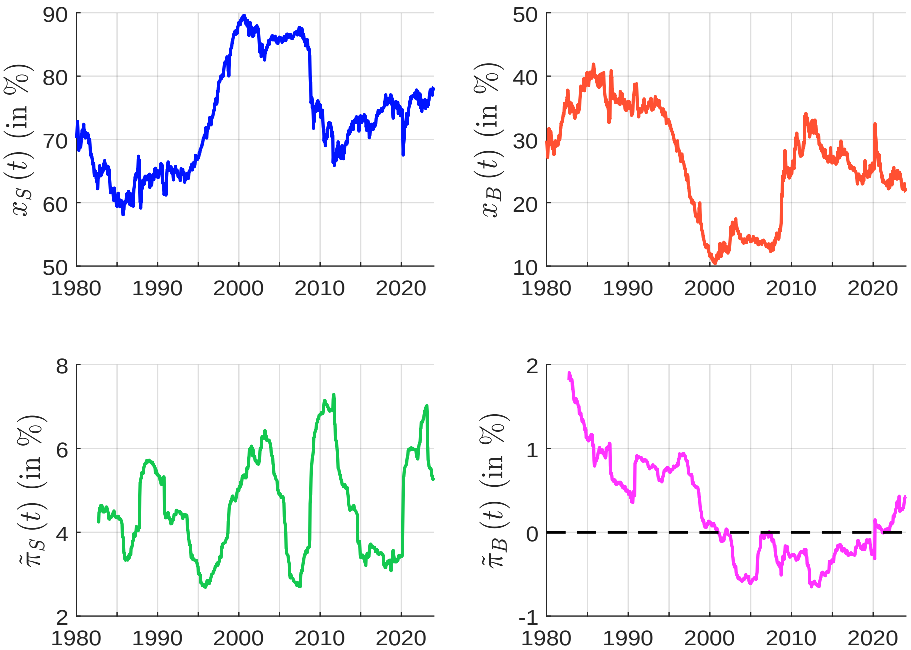
where:

$$\sigma\left(x(t)\right) = \sqrt{x_S^2(t)\,\sigma_S^2(t) + x_B^2(t)\,\sigma_B^2(t) + 2\rho_{S,B}(t)\,x_S(t)\,x_B(t)\,\sigma_S(t)\,\sigma_B(t)}$$

The results are shown in Figure 26. We see that the stock-bond allocation has evolved significantly over the past 50 years. The equity allocation was 70% in the early 1980s and then fell to 60% in the mid-1980s. It then rose to 90% during the dot-com bubble and remained high at 85% until the global financial crisis. After the crisis, it stabilized between 70% and 80%. The equity risk premium ranged from 2.7% to 7.3%, with an average of 4.6%, while the bond risk premium ranged from −65 bps to 190 bps, with an average of 22 bps. As expected by the theory of the consumption-based model (Lucas, 1978; Cochrane, 2001), the equity risk premium has increased in each bad period, *i.e.* it is a skewness risk premium (Roncalli, 2017). The behavior of the bond risk premium is different. It decreases from 1980 to 2005 to become negative between 2000 and 2020. Therefore, the nature of US bonds changes over this long period. While they were performance assets before 2000, they become hedging assets after the dot-com bubble. Only recently have they returned to their performance asset status. Figure 27 shows the decomposition of the risk premium into its variance and covariance components. While the variance risk premium has the same impact on equity and bond risk premia, the covariance risk premium does not. The latter has a small impact on the equity risk premium, as its contribution is less than 5%. The story is different for the bond risk premium. On average, the covariance risk premium explains 68% of the bond risk premium and can reach a level of 90% in some periods.

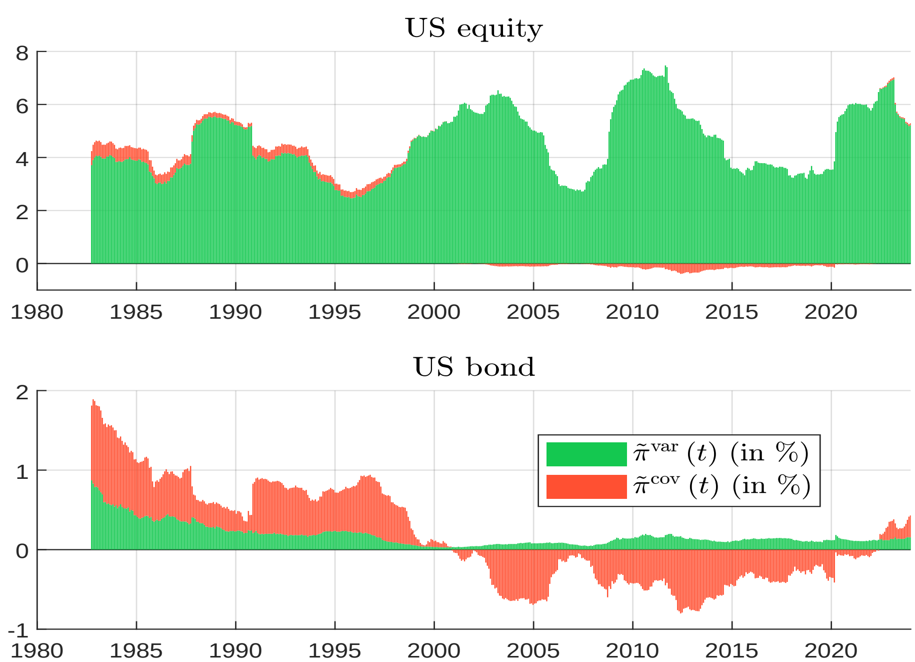
The main drivers of the bond risk premium are then the bond volatility and the stock-bond correlation. Let us compute the risk premia priced in by the market when we assume

Figure 26: US stock and bond risk premia



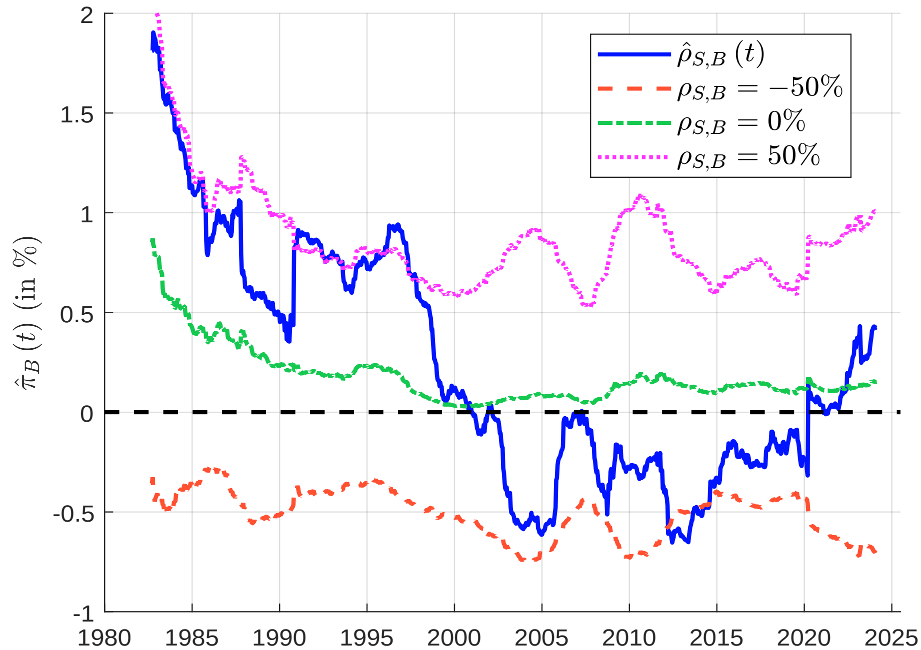
Source: Datastream (2024) & Authors' calculations.

Figure 27: Variance and covariance components of US equity and bond risk premia



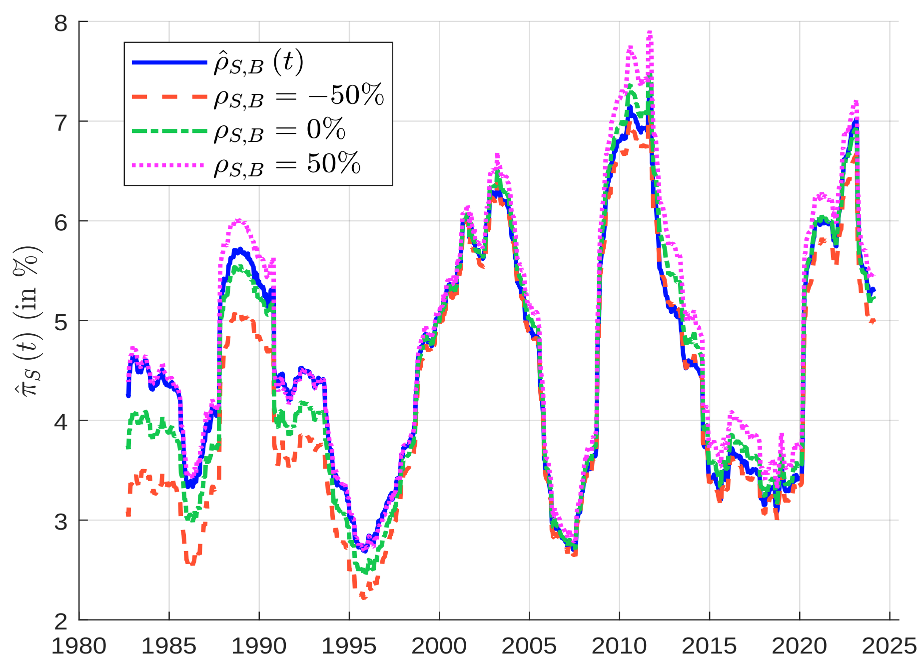
Source: Datastream (2024) & Authors' calculations.

Figure 28: US bond risk premium under different hypothesis on the stock-bond correlation



Source: Datastream (2024) & Authors' calculations.

Figure 29: US equity risk premium under different hypothesis on the stock-bond correlation



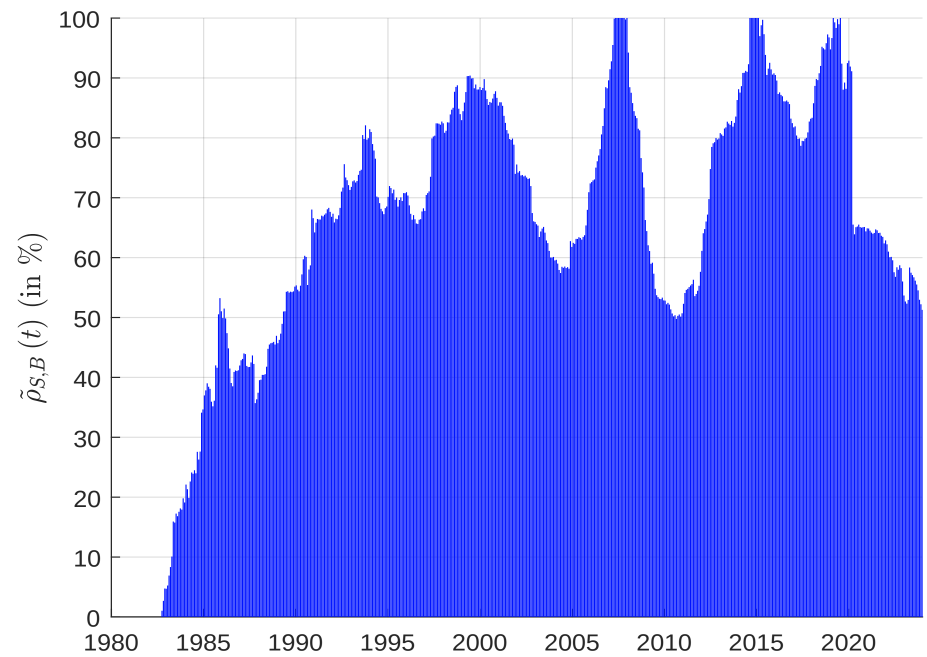
Source: Datastream (2024) & Authors' calculations.

different values of $\rho_{S,B}(t)$. Results are given in Figures 28 and 29. We find that the stock-bond correlation has a small effect on the equity risk premium, especially since 1995. In contrast, the effect on the bond risk premium is massive. While the average bond risk premium was 22 bps with the empirical stock-bond correlation, it is -51 bps when $\rho_{S,B}(t) = -50\%$ and $+86$ bps when $\rho_{S,B}(t) = +50\%$. Again, we see the importance of the stock-bond correlation on the bond risk premium.

Implied stock-bond correlation Given a level $\pi^{\star}_{B}(t)$ of the bond risk premium, we can calculate the corresponding implied stock-bond correlation $\tilde{\rho}_{S,B}(t)$. Figure 30 shows the evolution of $\tilde{\rho}_{S,B}(t)$ when $\pi^{\star}_{B}(t)$ is equal to 100 bps. As expected, the implied stock-bond correlation is positive. Since 1990, it is above 50%. In Figure 31, we plot the evolution of the break-even stock-bond correlation when $\pi^{\star}_{B}(t)$ is equal to 0 bps. Contrary to some ideas we find in the market, a positive bond risk premium requires the stock-bond correlation to be above a threshold. Looking at the period 1997-2023, we get the following result:

$$\pi_B(t) \geq 0 \Leftrightarrow \rho_{S,B}(t) \geq -10\%$$

Figure 30: US implied stock-bond correlation when $\pi^{\star}_{B}(t) = 100$ bps

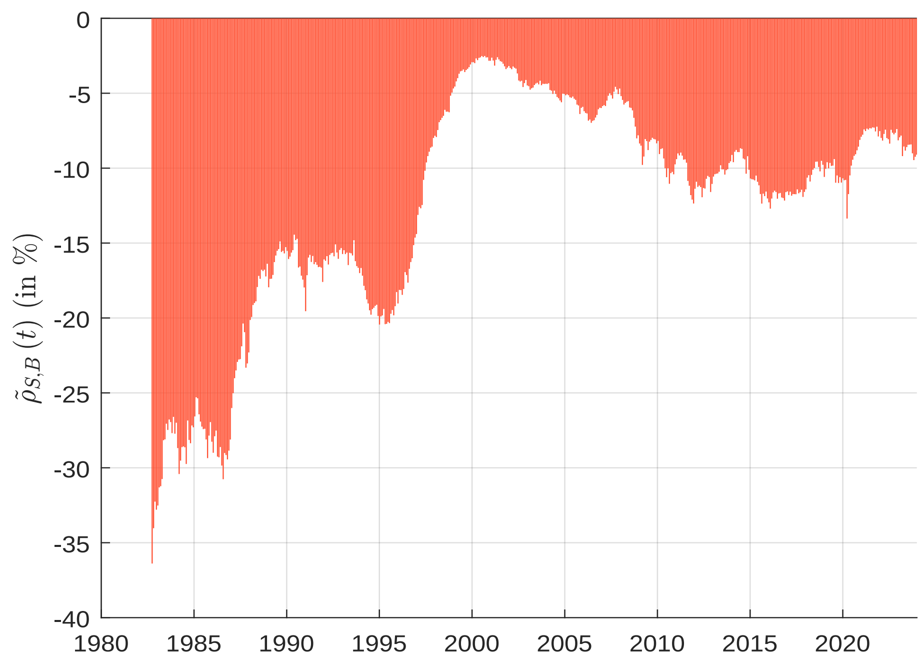


Source: Datastream (2024) & Authors' calculations.

Term risk premium The previous estimate of the bond risk premium does not take into account the term structure of interest rates. Therefore, $\hat{\pi}_B(t)$ can be considered as an average of the different maturities issued by the US government. However, we can split the total market value of US bonds by considering several maturity bands. Figure 32 shows the evolution of the implied bond risk premium $\hat{\pi}_B(t;m)$ for the maturity band m. As expected, we check that:

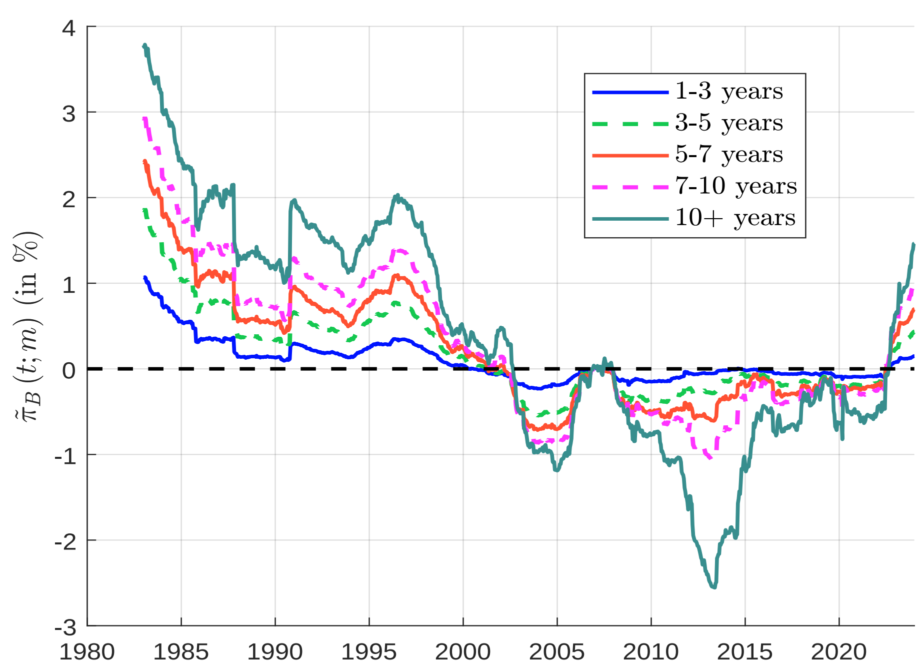
$$m_2 > m_1 \Rightarrow \left|\hat{\pi}_B(t;m_2)\right| \geq \left|\hat{\pi}_B(t;m_1)\right|$$

Figure 31: US break-even stock-bond correlation



Source: Datastream (2024) & Authors' calculations.

Figure 32: US term structure of bond risk premia



Source: Datastream (2024) & Authors' calculations.

However, this inequality does not hold if we consider nominal values rather than absolute values. Thus, we are faced with the paradox that the implied bond risk premium is higher for short-term maturities than for long-term maturities when stock-bond correlations are negative. And this result does not require an inverted yield curve. This paradox is easy to understand. Indeed, we recall that the covariance risk premium for bonds is:

$$\tilde{\pi}_B^{(\mathrm{cov})}(t) = \frac{\mathrm{SR}\left(x(t) \mid r\right) x_S(t)\, \sigma_S(t)}{\sigma\left(x(t)\right)} \cdot \rho_{S,B}(t) \cdot \sigma_B(t)$$

Since bond volatility is an increasing function of bond maturity, we deduce that the absolute value of the bond premium is an increasing function of bond maturity. The term structure of bond premia is respected when the stock-bond correlation is positive, but it is inverted when the stock-bond correlation is negative. This paradox shows that a negative stock-bond correlation is not a natural situation. It can only occur during short-term periods corresponding to a flight-to-quality episode. When a negative stock-bond correlation persists for a long time, it clearly indicates that investors are very risk-averse and use bonds as a hedge of the stock market.

Considering the credit asset class Let us now introduce other asset classes in the market portfolio. We denote them by the letter C. Let $\left(x_S(t), x_B(t), x_C(t)\right)$ be the composition of the market portfolio at time t. In this case, the bond premium is $\tilde{\pi}_B(t) = \tilde{\pi}_B^{(\mathrm{var})}(t) + \tilde{\pi}_B^{(\mathrm{cov})}(t)$ where $\tilde{\pi}_B^{(\mathrm{var})}(t) = \dfrac{\mathrm{SR}\left(x(t) \mid r\right)}{\sigma\left(x(t)\right)} \cdot x_B(t) \cdot \sigma_B^2(t)$ and:

$$\tilde{\pi}_B^{(\mathrm{cov})}(t) = \frac{\mathrm{SR}\left(x(t) \mid r\right)}{\sigma\left(x(t)\right)} \cdot \left(\underbrace{x_S(t)\, \rho_{S,B}(t)\, \sigma_S(t)}_{\text{Equity}} + \underbrace{x_C(t)\, \rho_{C,B}(t)\, \sigma_C(t)}_{\text{Other asset classes}} \right) \cdot \sigma_B(t)$$

Note that the formula for the variance premium is the same, but the covariance premium has changed and depends on the correlation between the returns on government bonds and the returns on asset class C. In Figure 33, in addition to the US equity and US government bond markets, we consider the US corporate bond market and divide it into investment grade (IG) and high yield (HY) bonds[20]. The empirical results are consistent with the theory, since the following inequalities are systematically verified:

$$\tilde{\pi}_{\mathrm{Gov}}(t) \leq \tilde{\pi}_{\mathrm{IG}}(t) \leq \tilde{\pi}_{\mathrm{HY}}(t)$$

The ex-ante risk premium is higher for high yield bonds, then for investment grade bonds and finally for government bonds. The main reason is that we observe an ordering between the stock-bond correlations, as shown in Figure 34. Indeed, we have:

$$\rho_{S,\mathrm{Gov}}(t) \leq \rho_{S,\mathrm{IG}}(t) \leq \rho_{S,\mathrm{HY}}(t)$$

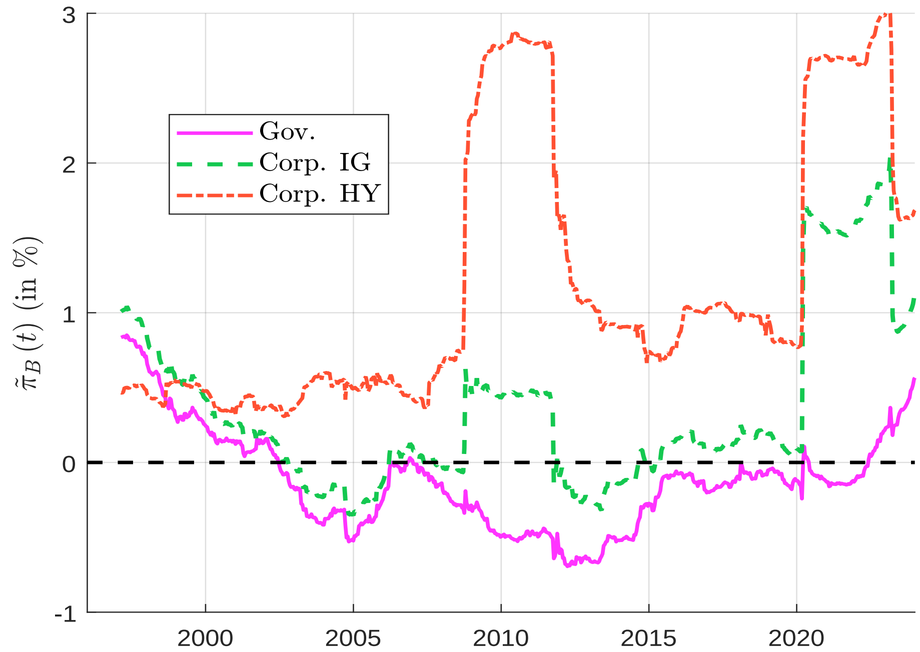
Figure 35 shows the boxplot of the stock-bond correlation when considering credit rating classes of corporate bonds. We verify the following inequality ordering:

$$\bar{\rho}_{S,\mathrm{Gov}}(t) \leq \underbrace{\bar{\rho}_{S,\mathrm{AAA}}(t) \leq \bar{\rho}_{S,\mathrm{AA}}(t) \leq \bar{\rho}_{S,\mathrm{A}}(t) \leq \bar{\rho}_{S,\mathrm{BBB}}(t)}_{\text{Corp.IG}} \leq \underbrace{\bar{\rho}_{S,\mathrm{BB}}(t) \leq \bar{\rho}_{S,\mathrm{B}}(t) \leq \bar{\rho}_{S,\mathrm{CCC}}(t)}_{\text{Corp. HY}}$$

This confirms that the stock-bond correlation highly depends on the credit risk of the country. It can only be negative if the market perceives government bonds as a safe haven asset. Again, this shows that the negative stock-bond correlation is mainly explained by fly-to-quality behavior.

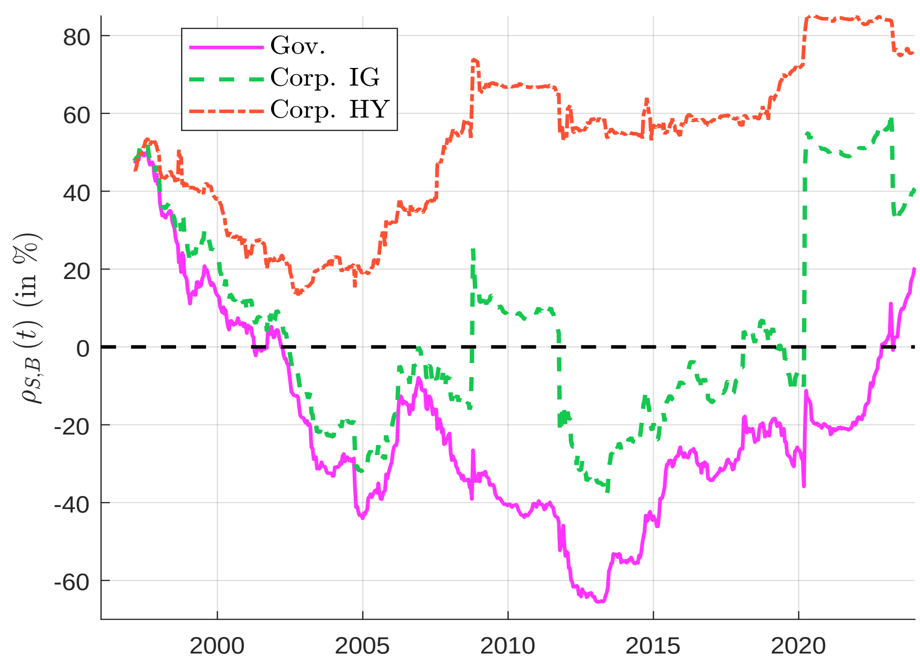
[20] We use the ICE BofA US Corporate Index (C0AO) and ICE BofA US High Yield Index (H0A0).

Figure 33: US Gov., Corp. IG and Corp. HY bond risk premia



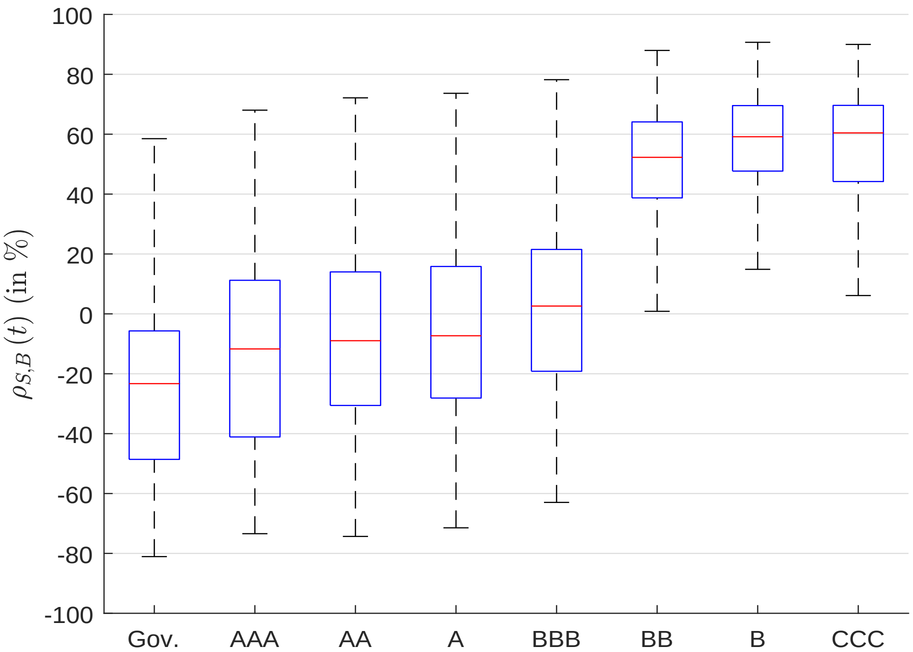
Source: Datastream (2024) & Authors' calculations.

Figure 34: US stock-bond correlation (Gov., Corp. IG and Corp. HY)



Source: Datastream (2024) & Authors' calculations.

Figure 35: Boxplot of the US stock-bond correlation by credit rating (2000-2023)



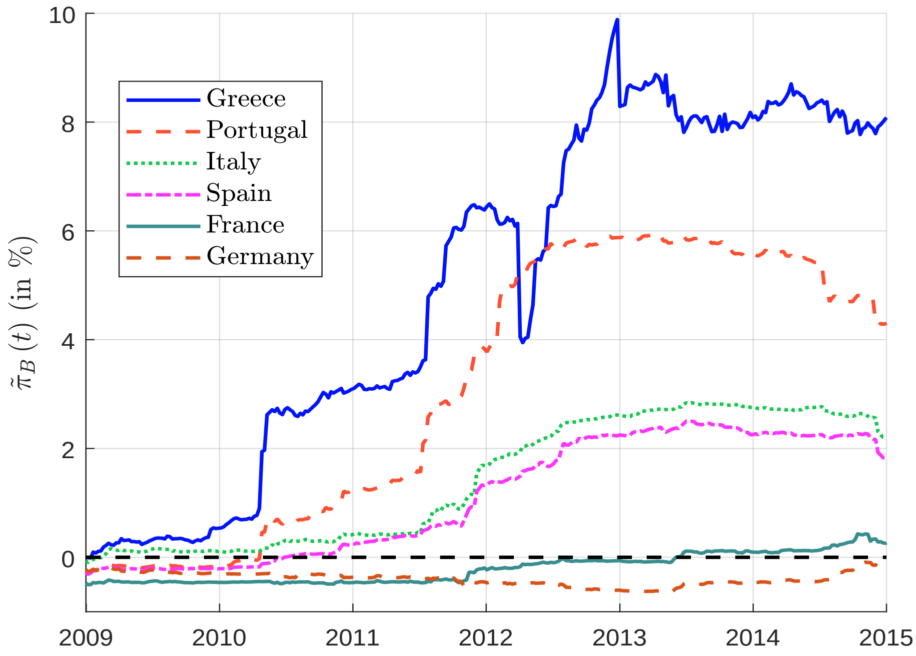
Source: Datastream (2024) & Authors' calculations.

3.2.2 Other countries

Figures 91-100 on pages 122 to 127 show the equity and bond risk premia priced in by the market for various countries. These results confirm those obtained for the United States. The bond risk premium can be negative (and is generally negative) when the stock-bond correlation is negative. In Figure 36, we focus on the period of the European debt crisis. It is interesting to note the jump in the Greek bond premium in 2010. The Greek bond premium reached a peak in November 2012 and coincide with the agreement with the IMF. The second country to see an increase in its bond risk premium is Portugal, followed by Italy and Spain. Conversely, the impact of the European debt crisis has little effect on the implied bond risk premium of France and Germany. These results are counterintuitive given the risk sentiment toward these two countries at the time. However, they can be easily understood as the bonds of these two countries became the safe haven assets in the European Union.

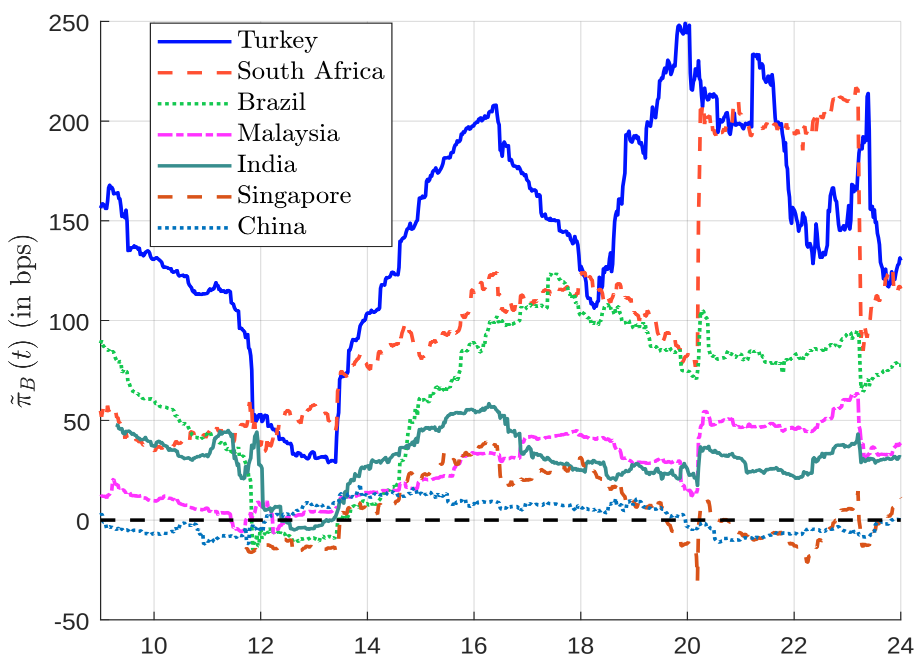
Let us now look at emerging markets. The evolution of the bond risk premium is shown in Figure 37. The reader may be surprised by the size of these risk premia. One reason is that we use the same Sharpe ratio of 30% for all countries. We know that investors in EM countries require a higher Sharpe ratio because they do not have the same risk appetite and utility function as investors in DM countries. The second reason is that the estimates are for an investment in local currency bonds, not hard currency. In this case, the perception of the country's credit risk is different. This reason is the main factor explaining the relatively low level of EM bond risk premia. More interesting is the ranking of bond risk premia among countries. At the end of December 2023, we have the following order: Turkey, South Africa, Brazil, Malaysia, India, Singapore and China. We recall that the implied risk premium corresponds to the expected return required by the investor to invest in the asset. Therefore, we conclude that local investors have less confidence in Turkish, South African and Brazilian government bonds than in Malaysian, Indian, Singaporean and Chinese government bonds.

Figure 36: Bond risk premium during the European debt crisis



Source: Datastream (2024) & Authors' calculations.

Figure 37: Bond risk premium of EM countries (local currency)



Source: Datastream (2024) & Authors' calculations.

Table 15: Average risk premia in % as priced in by the market (local currency)

Country	Equity				Bond			
	1990s	2000s	2010s	2020s	1990s	2000s	2010s	2020s
Australia	4.76	4.17	4.56	4.72	0.66	−0.18	−0.18	0.28
Brazil			6.30	6.16			0.56	0.82
Canada	3.49	4.79	4.28	4.53	0.94	−0.06	−0.23	0.10
China			7.19	5.73			0.05	−0.06
France	5.00	5.96	5.57	5.79	0.82	−0.24	0.04	0.41
Germany	4.79	6.11	5.46	5.41	0.47	−0.19	−0.20	0.13
Greece		6.55	9.68	7.06		0.11	6.93	2.35
India			5.89	5.52			0.29	0.30
Italy	6.70	5.66	6.24	5.57	0.91	0.03	1.58	2.13
Japan	5.74	6.32	6.36	4.80	0.09	−0.14	−0.04	0.06
Malaysia			3.23	3.59			0.20	0.45
Poland			5.65	6.05			0.33	0.31
Portugal		4.72	5.24	4.47		0.03	2.76	1.23
Singapore			3.54	4.32			0.11	−0.04
South Africa			5.13	6.20			0.81	1.74
Spain	5.44	5.60	6.42	5.21	0.75	−0.13	1.11	0.95
Switzerland	4.39	5.41	4.69	4.18	0.16	−0.33	−0.28	0.10
Turkey			6.90	8.40			1.33	1.76
UK	4.12	4.96	4.89	4.45	0.84	−0.23	−0.10	1.11
US	3.96	5.11	4.58	5.29	0.69	−0.22	−0.35	0.01

Source: Datastream (2024) & Authors' calculations.

In Table 15, we report the average equity and risk premia as priced in by the market for the different countries and different decades. For developed countries, we also report the implied risk premia in 2023 in Table 16. We distinguish between different maturities of government bonds. As expected from theory, we find that the risk premium on bonds increases with maturity. However, these results are valid because most stock-bond correlations are positive in 2023. All these findings show the importance of the stock-bond risk premium and its relationship with the bond risk premium. Setting a stock-bond correlation to be negative has then a high impact when designing a strategic asset allocation (see Section 3.5.3 on page 78).

Table 16: 2023 risk premia in % as priced in by the market

Country	Equity	1-3 years	3-5 years	5-7 years	7-10 years	10+ years
Australia	4.27	0.19	0.33	0.56	0.85	1.19
Canada	4.43	0.05	0.13	0.24	0.48	0.80
France	5.77	0.13	0.31	0.52	0.73	1.16
Germany	5.40	0.11	0.34	0.52	0.75	1.10
Italy	5.12	0.58	1.24	1.81	2.41	3.57
Japan	4.57	0.01	0.03	0.08	0.12	0.26
Portugal	4.57	0.28	0.82	1.32	1.83	2.60
Spain	4.78	0.28	0.68	1.03	1.49	2.12
Switzerland	4.23	0.10	0.27	0.40	0.68	0.93
UK	3.94	0.40	0.82	1.17	1.62	3.69
US	5.53	0.12	0.34	0.54	0.75	0.99

Source: Datastream (2024) & Authors' calculations.

Summary and main results

In the United States, the equity risk premium priced in by the market varies widely between 1983 and 2023, ranging from 2.7% to 7.3%. We observe four similar cycles over this period. The bond risk premium has also varied widely, from −65 bps to 190 bps, but unlike the equity risk premium, it does not exhibit cyclical behavior. The level of the implied bond risk premium highly depends on the stock-bond correlation, while this latter has little influence on the equity risk premium. This is because, on average, about 70% of the US bond risk premium is explained by the covariance risk premium, while the variance risk premium is the main component of the US equity risk premium, accounting for about 95%. As expected, we find the effects of maturity and credit rating on the bond risk premium. These different results also hold for the other countries. In local currency terms, there is no significant difference between developed and emerging market government bonds. This means that local investors do not demand a higher risk premium when investing in EM bonds than in DM bonds. Nevertheless, we observe a significant difference between DM and EM bonds. Since the stock-bond correlation is generally positive in EM markets, the bond risk premium is generally positive. In DM markets, the implied bond risk premium can be negative due to the impact of the stock-bond correlation through the covariance risk premium component. This again shows that a negative long-term stock-bond correlation is not a normal situation, as investors are willing to accept a negative bond risk premium. In the long run, this is not sustainable, especially from the perspective of a strategic asset allocation framework.

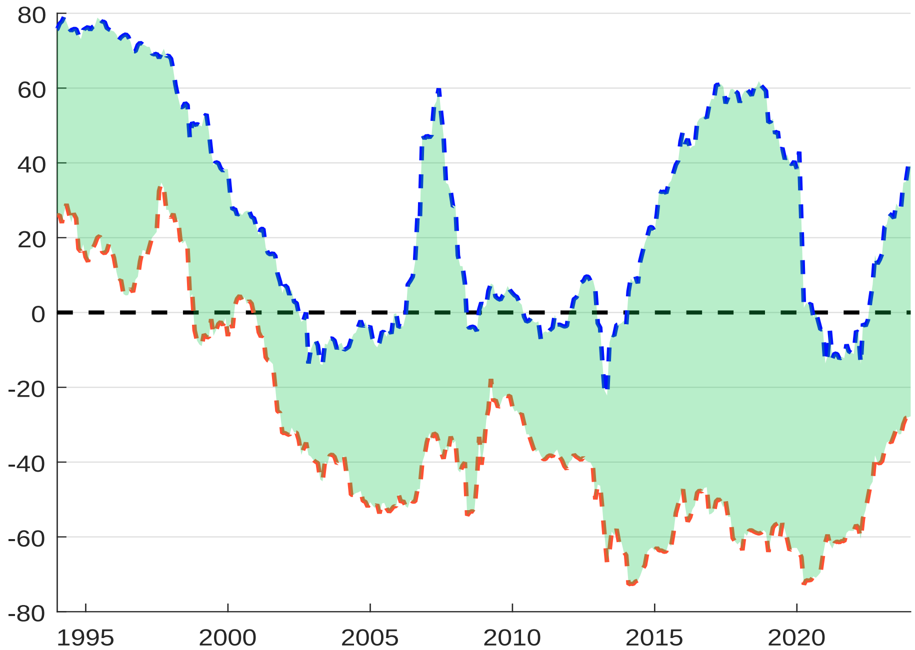
3.3 How the structure of the stock market affects the stock-bond correlation

3.3.1 Sectors

We have already seen that the structure of the stock index has a strong impact on the stock-bond correlation. We can therefore see that the stock-bond correlation may be different if we look at a sub-sample of the stock market universe. A natural way to sample the market is to look at sectors. Using the S&P 500 sector indices based on the GICS classification, we have calculated the stock-bond correlation for each sector. The first thing we notice is that the range of the stock-bond correlation calculated using the sector indices is wide. In Figure 38, we show the minimum and maximum values of the 11 correlations for each date. The difference is never less than 25%. The sector range peaks at the beginning of 2018 with a difference of 125% between the maximum and minimum correlation (see Figure 101 on page 127).

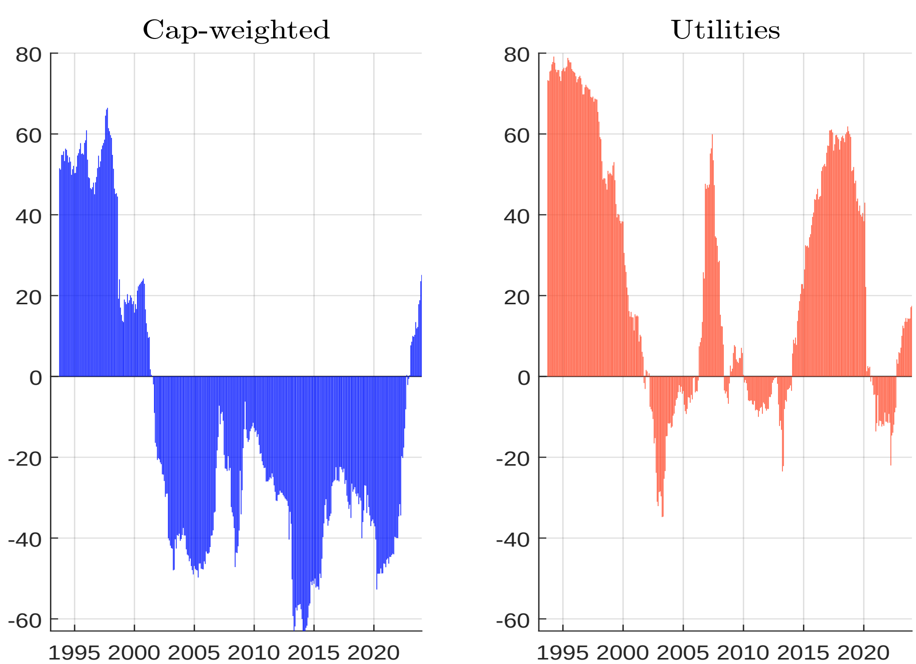
In Table 17, we give the average difference of $\rho_{S,B}^{\text{Sector}} - \rho_{S,B}^{\text{Index}}$ for different periods, where $\rho_{S,B}^{\text{Sector}}$ is the stock-bond correlation calculated with the S&P 500 sector index and $\rho_{S,B}^{\text{Index}}$ is the traditional stock-bond correlation calculated with the S&P 500 cap-weighted index. We find that the stock-bond correlation is systematically higher for real estate and utilities. This is also the case for communication services, consumer staples and health care, but only since 2000, when the stock-bond correlation turns negative. The case of utilities is particularly interesting because there are very few periods when we observe a negative correlation. Most of the time, the correlation between utilities and bonds is positive. These results are consistent because sectors with stable cash flows and bond-like behavior tend to have higher correlations, and inflation-resistant sectors tend to have higher correlations (Brixton *et al.*, 2023).

Figure 38: Sector range of US stock-bond correlation



Source: Datastream (2024) & Authors' calculations.

Figure 39: US stock-bond correlation (cap-weighted vs. utilities)



Source: Datastream (2024) & Authors' calculations.

Table 17: Difference $\rho_{S,B}^{\text{Sector}} - \rho_{S,B}^{\text{Index}}$ in % (S&P 500)

Sector	1995 1999	2000 2004	2005 2009	2010 2014	2015 2019	2020 2023	1995 2003
Communication Services	−0.9	7.8	12.3	34.6	36.2	13.7	17.4
Consumer Discretionary	−21.4	−6.9	2.9	5.8	2.0	9.2	−1.7
Consumer Staples	−12.8	14.2	19.6	26.2	43.7	10.6	17.1
Energy	−11.7	13.8	9.5	−1.1	−3.0	−26.2	−2.5
Financials	3.1	2.7	8.1	0.7	−23.9	−21.6	−4.7
Health Care	−14.5	20.7	19.5	16.5	18.8	9.8	11.9
Industrials	−8.6	−3.1	−4.4	3.8	−2.3	−10.5	−4.0
Information Technology	−27.3	2.7	1.0	−5.0	5.1	14.9	−1.9
Materials	−25.3	1.9	−9.2	1.0	−1.7	−5.0	−6.4
Real Estate			23.0	16.1	71.1	24.4	34.5
Utilities	18.7	16.5	42.4	35.9	82.4	21.4	36.7

Remark 5. *These sector results also apply to the MSCI EMU Index and the MSCI Europe Index (see Tables 30 and 31 on page 137).*

3.3.2 Factors

We do the same exercise with equity factors[21]. Results are given in Table 18. As expected, the low volatility factor has the highest correlation with bonds (Cazalet *et al.*, 2014; Stagnol *et al.*, 2021). On average, we observe a difference of +20% with respect to the traditional stock-bond correlation over the period 1995-2003. Between 2015 and 2019, the difference is even more greater than +40%. We also find a higher correlation for high dividend and quality factors, although it is more modest for the latter. However, the lower correlation observed between 2020 and 2023 for the quality factor is puzzling. In theory, growth stocks are more correlated than value stocks. We confirm these patterns on average, but the relationship is not stable and is mainly due to the period 2015-2023. More surprising are the results for the momentum factor, which has a higher correlation than the overall market.

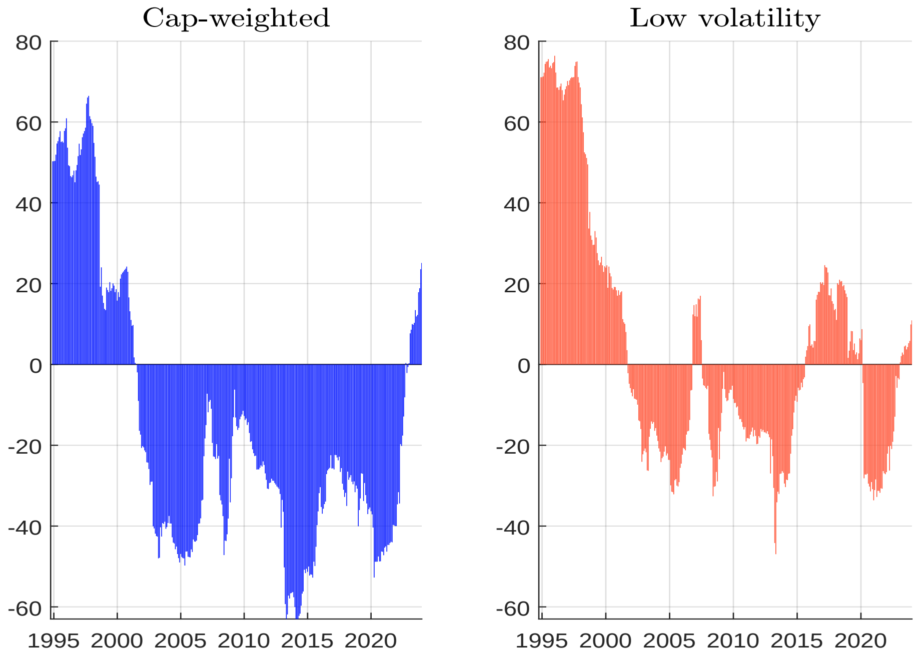
Table 18: Difference $\rho_{S,B}^{\text{Factor}} - \rho_{S,B}^{\text{Index}}$ in % (S&P 500)

Period	Pure Value	Pure Growth	High beta	Low Vol.	Momen-tum	High Div.	Qua-lity
1995-1999	−4.1	−5.5	−8.3	6.2	5.0	0.6	8.4
2000-2004	7.3	−2.8	−4.6	14.2	11.8	7.9	8.5
2005-2009	7.7	3.8	3.3	16.9	0.6	16.0	1.9
2010-2014	−1.0	−2.0	−5.3	18.1	−3.2	16.1	3.4
2015-2019	−13.6	7.9	−15.2	43.6	9.7	33.3	3.7
2020-2023	−27.4	9.2	−12.5	6.0	10.2	−22.1	2.8
2000-2023	−4.6	2.8	−6.7	19.9	5.6	11.1	4.2

Remark 6. *These factor results between two regions are less consistent than the sector results. For instance, Table 32 on page 138 shows the results for the MSCI Europe. One reason could be that the sector allocations in each factor differ from region to region. Another reason could be the country bias due to currency risk.*

[21] We use the following S&P indices: S&P 500/CITIGROUP Pure Value Index, S&P 500/CITIGROUP Pure Growth Index, S&P 500 High Beta Index, S&P 500 Low Volatility Index, S&P 500 Momentum Index, S&P 500 High Dividend Index and S&P 500 Quality Index.

Figure 40: US stock-bond correlation (cap-weighted vs. low volatility)



Source: Datastream (2024) & Authors' calculations.

3.3.3 Individual stocks

As shown in Section 2.3 on page 21, the aggregate stock-bond correlation computed for a stock index differs from the individual stock-bond correlations of the components of the stock index. For example, in Figure 41 we compare the stock-bond correlation computed for the S&P 500 Index with the range of individual stock-bond correlations. If we compute the confidence interval at the α level, we obtain Figure 42. We see that the aggregate stock-bond correlation can be outside the confidence interval of the individual correlations. The reason is the leverage effect due to the diversification of the portfolio, as shown in Figure 43. We obtain an average leverage factor of 2.06 with a standard deviation of 0.35. These empirical results confirm the previous theoretical results. However, leverage cannot explain the sign of the aggregate stock-bond correlation. In each period, we can find stocks that have positive and negative correlations, and the sign of the aggregate correlation depends strongly on the frequency of positive and negative correlations. Again, market structure does not seem to explain why more stocks are positively correlated with the US 10-year bond.

3.3.4 Equity duration

The price of a financial asset is the expected value of the stochastic discounted value of the cash flow leg. Under some assumptions (Roncalli, 2020, chapter 3), we can show that:

$$P(t) = \sum_{t_m > t} B(t, t_m) \mathbb{E}\left[\mathrm{CF}(t_m) \mid \mathcal{F}_t\right]$$

where $\mathcal{F}_t$ is the filtration under the risk-neutral probability measure $\mathbb{Q}$, $B(t, t_m)$ is the discount factor for the maturity date t_m and $\left\{\mathrm{CF}(t_m), t_m \geq t\right\}$ is the stream of stochastic

Figure 41: Confidence interval of the individual stock-bond correlation (US, monthly return, $\alpha = 0\%$)

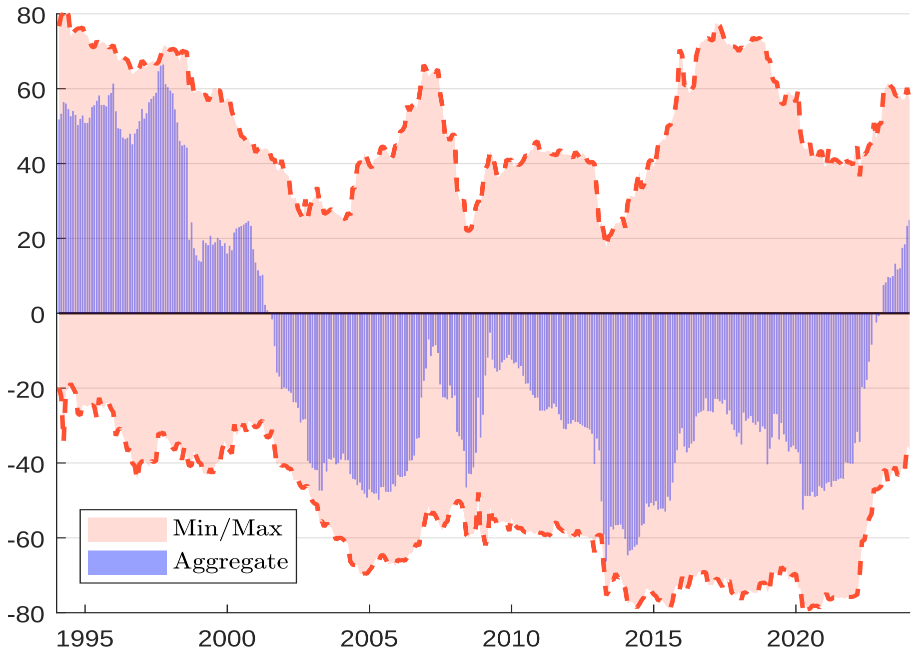


Figure 42: Confidence interval of the individual stock-bond correlation (US, monthly return, $\alpha = 25\%$)

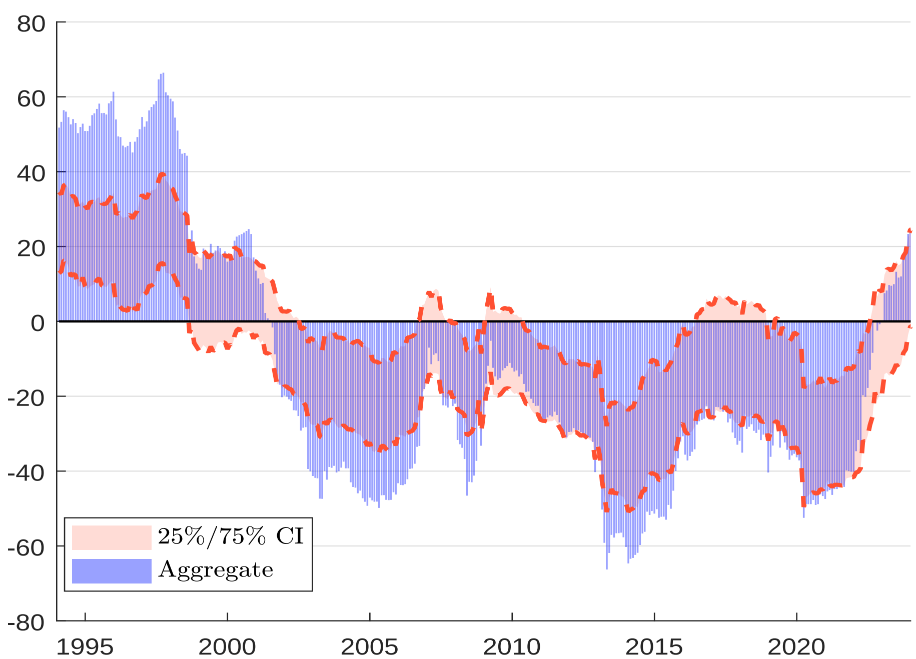
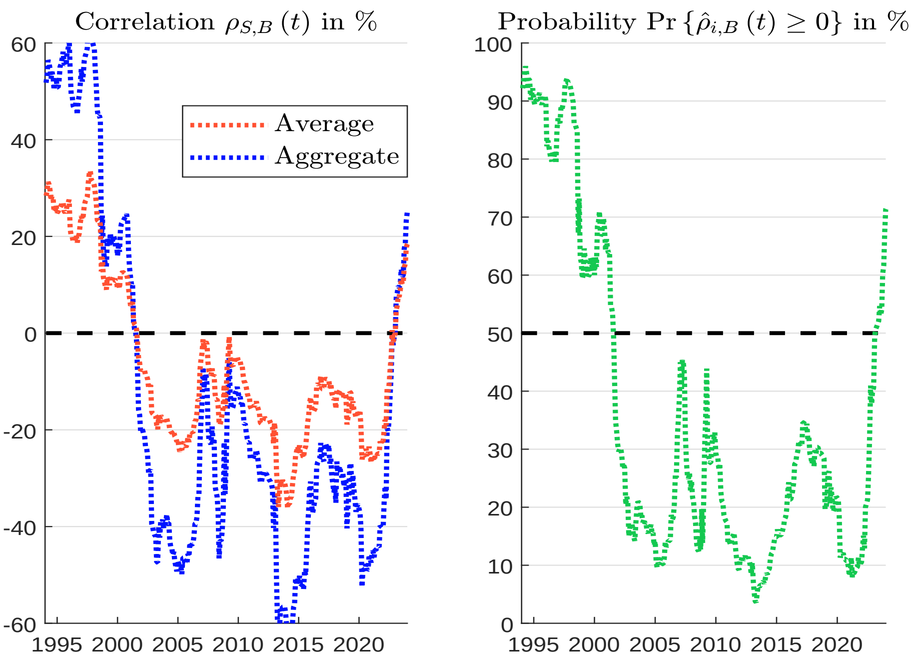


Figure 43: Average stock-bond correlation vs. aggregate stock-bond correlation (US, monthly return)



cash flows. The duration is defined as the time weighted average of the expected cash flows:

$$D(t) = \sum_{t_m > t} (t_m - t) B(t, t_m) \mathbb{E}\left[\mathrm{CF}(t_m) \mid \mathcal{F}_t\right]$$

By assuming a flat yield curve, we can show that:

$$\frac{\partial P(t)}{\partial r(t)} = -P(t) \frac{D(t)}{1 + r(t)}$$

where $r(t)$ is the short-term interest rate. The impact of the yield curve on the price changes is then:

$$\frac{\Delta P(t)}{P(t)} = -\tilde{D}(t) \Delta r(t)$$

where $\tilde{D}(t) = \dfrac{D(t)}{1 + r(t)}$ is the modified duration. We can use the previous framework for both bonds and equities. In the case of a bond, we have[22] $\mathbb{E}\left[\mathrm{CF}(t_m) \mid \mathcal{F}_t\right] = C(t_m) \mathbf{S}(t, t_m)$ where t_m is the coupon date, $C(t_m)$ is the coupon value (including the notional amount to be repaid at maturity), and $\mathbf{S}(t, t_m)$ is the issuer's survival function (because of the credit risk). In the case of a stock, the expected cash flows are the projected earnings of the company that will be distributed to shareholders, and the pricing formula corresponds to the dividend discount model (DDM). If we assume that the main risk factor is the yield curve, we obtain the following first-order approximation:

$$\operatorname{cov}\left(\frac{\Delta P_S(t)}{P_S(t)}, \frac{\Delta P_B(t)}{P_B(t)}\right) \approx \tilde{D}_S(t) \tilde{D}_B(t) \sigma_r^2(t)$$

[22] We assume that the recovery rate in the event of default is zero.

The DDM model then implies that the stock-bond correlation is positive and depends mainly on interest rate volatility and equity duration, since bond duration does not change much. However, equity risk factors cannot be reduced to the yield curve factor, and the previous approximation is not generally true.

The previous analysis highlights the role of equity duration in the stock-bond correlation. This confirms the empirical results we have obtained with sector and factor analysis. Portfolios with long-duration stocks tend to have higher stock-bond correlations than portfolios with short-duration stocks. One of the difficulties is the calculation of equity duration, since it involves the series of projected earnings. However, we can obtain some approximate formulas that do not require the forecast of companies' cash flows. The Gordon growth model assumes that:

$$P(t) = \sum_{t_m = t+1}^{\infty} \frac{\mathrm{CF}(t_m)}{\left(1 + r(t)\right)^{(t_m - t)}}$$

where $\mathrm{CF}(t_m) = \left(1 + g(t)\right)^{(t_m - t)} \mathrm{CF}(t)$ and $g(t)$ is the growth rate. We deduce that:

$$\begin{aligned} P(t) &= \mathrm{CF}(t) \sum_{t_m = t+1}^{\infty} \frac{\left(1 + g(t)\right)^{(t_m - t)}}{\left(1 + r(t)\right)^{(t_m - t)}} \\ &= \mathrm{CF}(t) \frac{1 + g(t)}{r(t) - g(t)} \\ &= \frac{\mathrm{CF}(t+1)}{r(t) - g(t)} \end{aligned}$$

If the discount rate $r(t)$ and the growth rate $g(t)$ are independent, we obtain[23]:

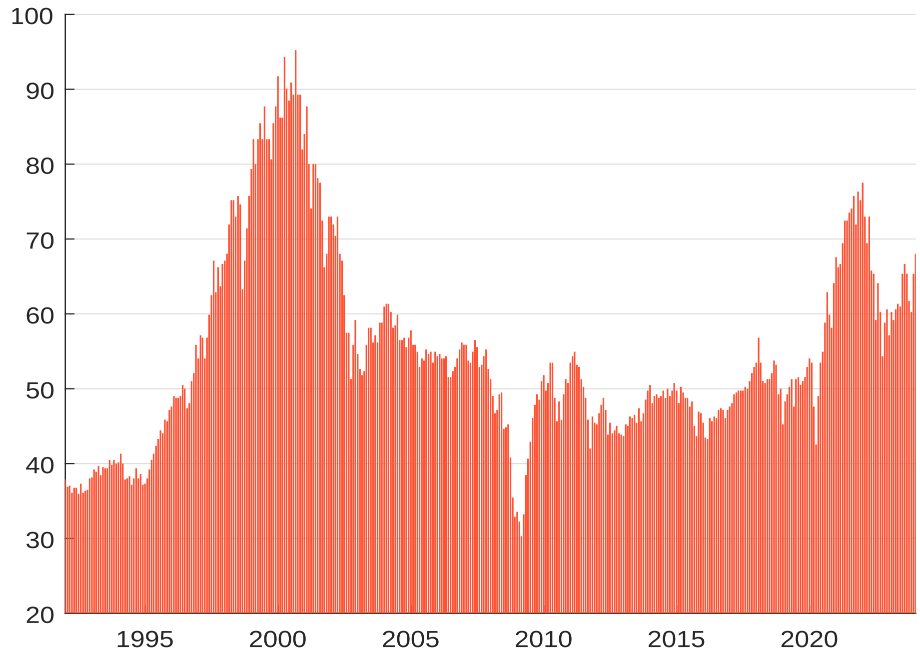
$$\begin{aligned} \tilde{D}(t) &= -\frac{1}{P(t)} \cdot \frac{\partial P(t)}{\partial r(t)} \\ &= \frac{1}{r(t) - g(t)} \\ &= \frac{1}{\mathrm{DY}(t)} \end{aligned} \tag{12}$$

where $\mathrm{DY}(t) = \dfrac{\mathrm{CF}(t+1)}{P(t)}$ is the dividend to price ratio or dividend yield. Figure 44 shows the evolution of the equity duration of the S&P 500 Index. The Kendall correlation between the stock-bond correlation and equity duration is zero. If we use the change in equity duration instead of its contemporaneous level, the Kendall correlation increases. For example, it is 16% and 33% for the one-year and three-year changes, respectively. If we calculate the equity duration with Equation (11), the results are different (see Figure 104 on page 129). The Kendall correlation is equal to zero. So, the relationship between the equity duration and the stock-bond correlation is highly sensitive to the computation of $\tilde{D}(t)$.

[23]Otherwise, we have the following general formula:

$$\tilde{D}(t) = \frac{1}{r(t) - g(t)} \left(1 - \frac{\partial g(t)}{\partial r(t)}\right) \tag{11}$$

Figure 44: Equity duration in year (US, Equation 12)



Summary and main results

As shown in the theoretical part, the stock-bond correlation depends on the composition of the equity portfolio. Therefore, we naturally observe differences when we look at a subset of the market rather than the global market. This is true when we analyze the stock-bond correlation at the sector level or when we use equity factors. For example, real estate and utilities have a higher correlation than the overall market. The same is true for communication services, consumer staples and health care, but to a lesser extent. On the other hand, industrials and materials have a lower correlation than the overall market. The case of financials is more complex, as the correlation was higher before the 2008 global financial crisis and lower after. Among equity factors, value stocks have a more negative correlation with bonds, while growth, low volatility, high dividend and quality stocks have a more positive correlation with bonds. We confirm the theoretical results on the relationship between individual and aggregate stock-bond correlations. On average, the stock-bond correlation calculated for the S&P 500 Index is twice as high as the average stock-bond correlation calculated for the individual stocks that make up the S&P 500 Index. The multiplication factor depends on the concentration of the stock market because diversification reduces variance risk faster than covariance risk. We also examine the empirical relationship between the stock-bond correlation and the equity duration of the stock market. Defining equity duration as the inverse of the dividend yield, we observe a positive relationship between the level of the stock-bond correlation and the change in equity duration. That is, an increase in equity duration tends to increase the stock-bond correlation. This result is consistent with the findings for growth stocks, but puzzling for bond-like equity portfolios such as the low-volatility risk factor or the utilities sector. However, the relationship between stock-bond correlation and equity duration is not robust and disappears when we consider other ways of calculating the latter.

3.4 Stock-bond correlation and the macroeconomy

We now examine the relationship between the stock-bond correlation and macroeconomic variables. As we have seen, growth and inflation are the most important factors from a theoretical point of view. We also include the interest rate component because it affects equity duration, is one of the two main risk factors in the dividend discount model, and is the central variable in real-centric models.

3.4.1 Yield factors

We have several choices for the interest rate factor. If we believe that the bond carry has an impact on the stock-bond correlation, then the ten-year bond yield to maturity $y(t)$ is the right choice. If we believe that the interest rate channel is mainly driven by the discount factor, we can use a short-term interest rate such as the fed funds rate $r(t)$. However, if we are more interested in the effect of monetary policy, it is better to use the target rate $r^{\star}(t)$ set by the Federal Open Market Committee (FOMC) of the Federal Reserve.

Figure 45: Dynamics of the stock-bond correlation $\rho_{S,B}(t)$ and the ten-year bond yield to maturity $y(t)$ (US, 1965-2023)

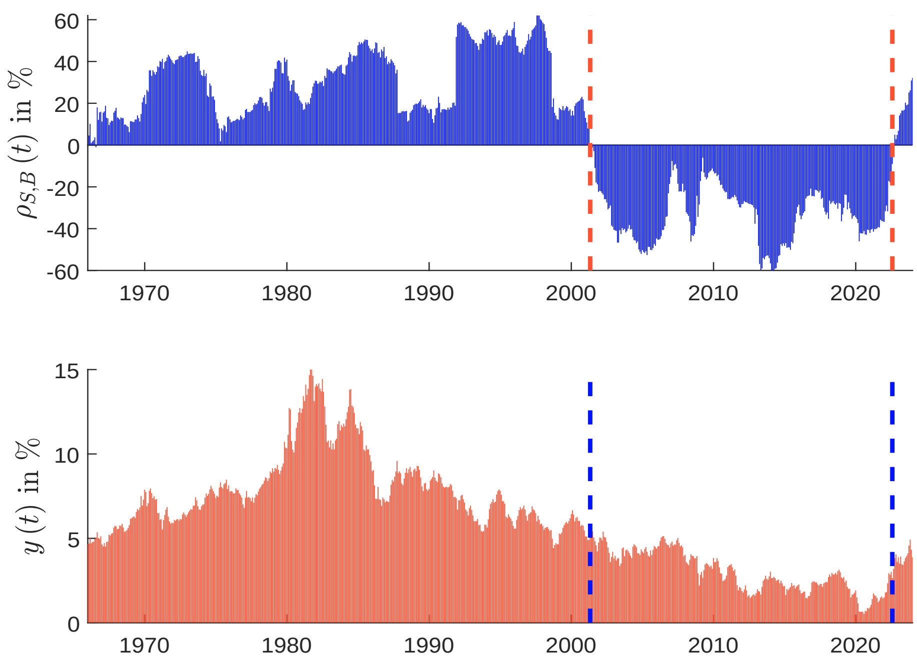


Figure 45 compares the dynamics of the stock-bond correlation $\rho_{S,B}(t)$ and the dynamics of the ten-year bond yield to maturity $y(t)$. It is not obvious to relate the two dynamics. However, we can derive a very simple rule of thumb. The stock-bond correlation is positive when the bond carry is high (*e.g.*, $y(t) \gg 5\%$) and negative when the bond carry is low (*e.g.*, $y(t) \ll 5\%$). An exception is the most recent period (Figure 105 on page 129). If we look at the fed funds rate $r(t)$ or the FOMC target rate $r^{\star}(t)$, the rule of thumb is less robust (Figures 106 and 107 on page 130). To confirm that the level of stock-bond correlation depends on the bond yield, we run the following linear regression:

$$\rho_{S,B}(t) = \beta_0 + \beta_1 y(t) + \varepsilon(t)$$

where $\varepsilon(t)$ is a white noise process. Results are given in Table 19. The coefficient of determination $\mathfrak{R}_c^2$ is equal to 48.72%. We verify that the estimated coefficients $\hat{\beta}_0$ and $\hat{\beta}_1$ are significant at the 1% confidence level. In addition, $\hat{\beta}_1$ has the correct sign. A 1% increase in the yield to maturity induces a 7.75% increase in the stock-bond correlation. If we replace the bond yield $y(t)$ with the fed funds rate $r(t)$ and the FMOC target rate $r^\star(t)$, we get similar results, but with a lower coefficient of determination $\mathfrak{R}_c^2$ of about 41%.

Table 19: Linear regression of the stock-bond correlation on interest rate factors (US, 1965-2023)

	$y(t)$	$r(t)$	$r^\star(t)$
$\hat{\beta}_0$	-0.40^{***}	-0.22^{***}	-0.23^{***}
$\hat{\beta}_1$	7.75^{***}	5.73^{***}	5.99^{***}
$\mathfrak{R}_c^2$	48.72%	41.36%	41.84%

3.4.2 Inflation factors

To test the relationship between the stock-bond correlation and the inflation factors, we calculate the inflation rate as:

$$\pi(t) = \frac{\mathrm{CPI}(t)}{\mathrm{CPI}(t-h)} - 1$$

where $\mathrm{CPI}(t)$ is the consumer price index and h is the time frequency. To be consistent with the estimation of the stock-bond correlation, we calculate the inflation volatility $\sigma_\pi(t)$ with a 4-year rolling window. Following Brixton *et al.* (2023), we run the linear regression:

$$\rho_{S,B}(t) = \beta_0 + \beta_\pi \pi(t) + \beta_{\sigma_\pi}\sigma_\pi(t) + \varepsilon(t)$$

Theoretically, we expect that $\beta_\pi \geq 0$ and $\beta_{\sigma_\pi} \geq 0$. Results are given in Table 20. First, note that β_{σ_π} has the correct sign only when h is greater than or equal to one year. Moreover, β_{σ_π} is not significant when the frequency h is equal to one year. Second, we note that the coefficient of determination $\mathfrak{R}_c^2$ is relatively low, but it increases with the frequency h. In this case, the level of explanatory power is mainly explained by the inflation volatility $\sigma_\pi(t)$. These results are disappointing compared to those obtained by Brixton *et al.* (2023). The difference between the two approaches is that Brixton *et al.* (2023) used a 10-year rolling window to estimate the stock-bond correlation and inflation volatility, while we use a 4-year rolling window. Again, the distinction between short-term and long-term estimates is important.

Table 20: Linear regression of the stock-bond correlation on inflation factors (US, 1965-2023)

h	1M	1Y	2Y	3Y	5Y
$\hat{\beta}_0$	0.19^{***}	-0.10^{***}	-0.14^{***}	-0.17^{***}	-0.25^{***}
$\hat{\beta}_\pi$	25.57^{***}	3.53^{***}	1.70^{***}	1.02^{***}	0.87^{***}
$\hat{\beta}_{\sigma_\pi}$	-24.78^{***}	1.57	4.21^{***}	7.28^{***}	9.96^{***}
$\mathfrak{R}_c^2$	10.44%	10.26%	15.16%	21.65%	34.56%

3.4.3 Growth factors

We conduct the same analysis with the growth factors. In this case, we replace the inflation level and inflation volatility with the quarterly level of GDP and its standard deviation.

Results are given in Table 21. We find that the growth level is an important factor, but not the growth volatility, which is not significant and has an explanatory power of less than 6%. Since the data are quarterly and not monthly, it is common to use another economic indicator to measure growth. The main approach is to look at monthly industrial production. In this case, the growth volatility is significant but the coefficient of determination is less than 15% and the estimated sign of β_{σ_g} is opposite to the expected sign of β_{σ_g}. In addition, we note that the volatility calculated with industrial production is very different from the volatility calculated with GDP (see Figure 110 on page 132). Therefore, we consider a third measure of growth. We perform a linear regression of GDP on several economic variables: industrial production, personal income, personal consumption, and the unemployment rate. We then use the forecast of GDP at the monthly level to construct the monthly GDP proxy. In this case, we obtain the correct sign for growth volatility. However, the explanatory power of $\sigma_g(t)$ remains low, less than 5%.

Table 21: Linear regression of the stock-bond correlation on growth factors (US, 1965-2023)

	h	1M/1Q	1Y	2Y	3Y	5Y
GDP	$\hat{\beta}_0$	-0.00	-0.18^{***}	-0.28^{***}	-0.34^{***}	-0.44^{***}
	$\hat{\beta}_g$	6.39^{***}	4.02^{***}	2.85^{***}	2.15^{***}	1.30^{***}
	$\hat{\beta}_{\sigma_g}$	-1.02	-0.24	-0.98	-1.44	0.83
	$\mathfrak{R}_c^2$	5.35%	14.75%	24.55%	30.55%	35.27%
Industrial production	$\hat{\beta}_0$	0.10	0.00	-0.06^{**}	-0.13^{***}	-0.06^{*}
	$\hat{\beta}_g$	3.50^{***}	1.78^{***}	1.58^{***}	1.57^{***}	1.08^{***}
	$\hat{\beta}_{\sigma_g}$	-1.47^{*}	0.75	1.54^{**}	2.72^{***}	-0.44
	$\mathfrak{R}_c^2$	1.47%	6.29%	11.35%	15.73%	14.46%
Monthly GDP proxy	$\hat{\beta}_0$	-0.05	-0.33^{***}	-0.54^{***}	-0.72^{***}	-0.91^{***}
	$\hat{\beta}_g$	9.39^{***}	3.98^{***}	2.87^{***}	2.33^{***}	1.58^{***}
	$\hat{\beta}_{\sigma_g}$	2.73	-3.90^{**}	-4.82^{***}	-3.64^{**}	1.71
	$\mathfrak{R}_c^2$	3.65%	18.51%	29.28%	36.79%	41.63%

3.4.4 Combined factors

In Table 22, we give the evolution of the coefficient of determination $\mathfrak{R}_c^2$ as we add supplementary factors[24]. We start with the yield factor $y(t)$, then we add the inflation factors $\pi(t)$ and $\sigma_\pi(t)$ and finally we include the growth factors $g(t)$ and $\sigma_g(t)$. These results show that yield is the most important factor, followed by inflation volatility, growth level, inflation level and finally growth volatility. Thus, economic factors such as inflation and growth explain less than 10% of the dynamics of the stock-bond correlation. Moreover, we do not get the correct sign of β_{σ_π} and β_{σ_g}. Our results are therefore not consistent with those of Brixton *et al.* (2023).

Table 22: Coefficient of determination $\mathfrak{R}_c^2$ in % (US, 1965-2023)

Factors	1M	1Y	2Y	3Y	5Y
Yield	48.72%	48.72%	48.72%	48.72%	48.72%
+ Inflation	55.01%	52.41%	52.32%	51.49%	52.92%
+ Growth	57.90%	55.51%	55.66%	56.92%	56.44%

[24]The estimates are given in Tables 33 and 34 on page 138.

Summary and main results

Econometric analysis indicates that growth uncertainties have low explanatory power in modeling the stock-bond correlation. Indeed, carry emerges as the most significant economic factor, followed by inflation. Nevertheless, the econometric analysis reveals that the relationship between economic factors and the stock-bond correlation is unstable and highly dependent on the time horizon.

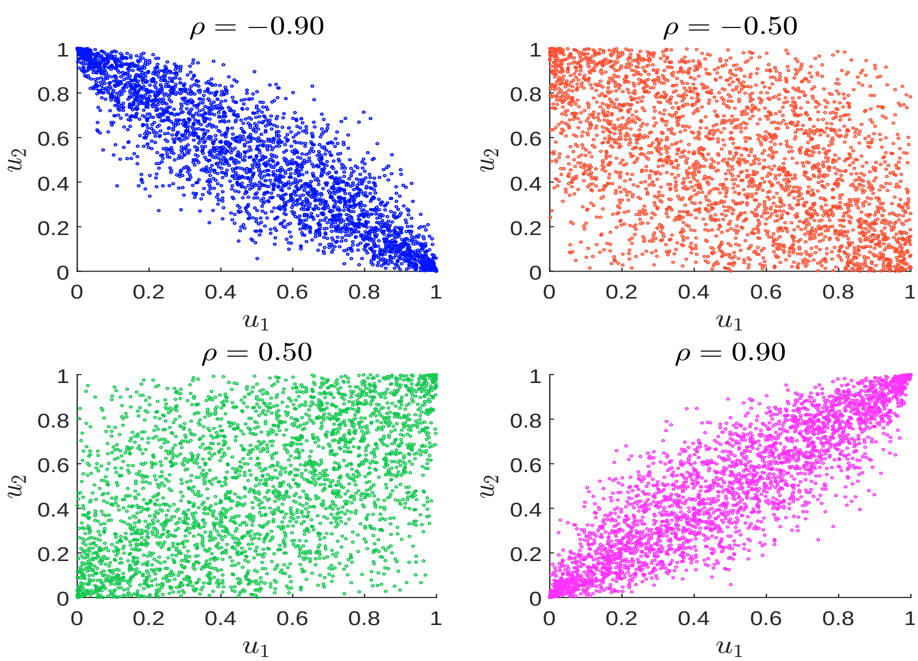
3.5 Diversification, payoff and stock-bond correlation

The previous empirical analyses provide a global picture of the stock-bond correlation. However, they do not distinguish between market regimes, such as bear and bull markets, and nonlinearity in the joint stock-bond dynamics. In this section, we extend the study by considering the local dynamics between stock and bond returns.

3.5.1 Average, local and extreme dependence

The stock-bond correlation computed in the previous analyses corresponds to the Pearson correlation and can be seen as an average measure of the dependence between stock and bond returns, assuming they are normally distributed. To go further, we must first remove the assumption that returns are Gaussian, since we know that asset returns are fat-tailed and have a kurtosis greater than 3. We can then calibrate the local dependence function between stock and bond returns by considering copula theory. However, in order to interpret the local dependence as a Pearson correlation and thus as a traditional stock-bond correlation, we will implicitly assume that the copula function remains Gaussian.

Figure 46: Dependogram of the Gaussian copula



To analyze the local dependence, we extensively use the concept of the dependogram (Roncalli, 2020, page 743). The dependogram between two random variables is the empirical copula of the realizations of these random variables. This is also the scatterplot between the two random variables normalized by their marginals. Like the copula function, the dependogram does not depend on the univariate probability distributions. In Figure 46, we show the dependogram of 3 000 random variates following the bivariate Gaussian copula with parameter ρ. If the correlation ρ is negative, the contour density plot corresponds to an ellipsoid with the major axis oriented from top/left to bottom/right. This means that the extremes are generally located at these two corners. Conversely, if the correlation ρ is positive, the extremes will be at the bottom/left and top/right corners due to the axis orientation of the ellipsoid. We consider the daily stock and bond returns $R_S(t)$ and $R_B(t)$ for the US market and the 40-year period 1980-2019. Since we have seen that the correlation changed sign around the beginning of the dot-com crisis, we split the sample into two equal subperiods 1980-1999 and 2000-2019. Then we compute the empirical copula of $R_S(t)$ and $R_B(t)$ by considering the probability integral transform and the empirical distribution $\hat{\mathbf{F}}_S$ and $\hat{\mathbf{F}}_B$: $u_S(t) = \hat{\mathbf{F}}_S\left(R_S(t)\right)$ and $u_B(t) = \hat{\mathbf{F}}_B\left(R_B(t)\right)$. The dependogram of $\left(R_S(t), R_B(t)\right)$, *i.e.* the empirical copula between $u_S(t)$ and $u_B(t)$, is given in Figures 47 and 48. First, we notice that the subperiods have two different dependencies. The 1980-1999 subperiod is characterized by a positive dependence, while the 2000-2019 subperiod has a negative dependence. This result confirms the previous finding that the short-term stock-bond correlation in the US turns from positive to negative between 1980 and 2020. In the first period, the copula correlation was +34.2% and changes to −33.7% in the second period. Since the second value is almost the opposite of the first value, we would expect a complete symmetry between the two dependograms. This is not the case. In particular, we see that the local dependence is higher in bad and good times between 2000 and 2019 than between 1980 and 1999. To confirm this observation, we compute the local probability. Let U_S and U_B be the normalized stock and bond returns, respectively. We note:

$$\begin{aligned}
p\left(\left[u'_S, u''_S\right] \times \left[u'_B, u''_B\right]\right) &= \Pr\left(u'_S \le U_S \le u''_S, u'_B \le U_B \le u''_B\right) \\
&= \mathbf{C}\left(u''_S, u''_B\right) - \mathbf{C}\left(u''_S, u'_B\right) - \mathbf{C}\left(u'_S, u''_B\right) + \mathbf{C}\left(u'_S, u'_B\right)
\end{aligned}$$

where $\mathbf{C}$ is the copula function of U_S and U_B. Using the scale invariance property, we also have:

$$p\left(\left[u'_S, u''_S\right] \times \left[u'_B, u''_B\right]\right) = \Pr\left(\mathbf{F}_S^{-1}\left(u'_S\right) \le R_S(t) \le \mathbf{F}_S^{-1}\left(u''_S\right), \mathbf{F}_B^{-1}\left(u'_B\right) \le R_B(t) \le \mathbf{F}_B^{-1}\left(u''_B\right)\right)$$

$p\left(\left[u'_S, u''_S\right] \times \left[u'_B, u''_B\right]\right)$ calculates then the joint local probability of $\left(R_S(t), R_B(t)\right)$. We define bad times as the observations corresponding to the worst returns and good times as the observations corresponding to the best returns. Normal times correspond to all other observations. Let $\alpha \in [0, 0.5]$ be the frequency of bad or good times. We have $t \in \mathcal{B}ad \Leftrightarrow R(t) \le \mathbf{F}^{-1}(\alpha)$, $t \in \mathcal{G}ood \Leftrightarrow R(t) \ge \mathbf{F}^{-1}(1-\alpha)$ and $t \in \mathcal{N}ormal \Leftrightarrow \mathbf{F}^{-1}(\alpha) \le R(t) \le \mathbf{F}^{-1}(1-\alpha)$. For example, if α is set to 10%, the bad and good times correspond to the 10% worst and best returns, respectively. In Figures 47 and 48, the boundary of the bad and good times is shown by the dashed black lines.

In Table 23, we calculate the local probability of each stock-bond market regime[25] when α is set to 10%. How do we read these numbers? Let us first focus on the first period from 1980 to 1999. For example, the theoretical probability[26] of observing bad times in both stock and bond markets is 2.37%. The probability of observing bad times in the stock market and good times in the bond market is 0.24%. We get a lower number, which is normal since the stock-bond correlation is positive. If we look at the empirical copula, we get different

[25] The rows correspond to the stock regimes, while the columns correspond to the bond regimes. For example, the bad stock market regime is in the first row, while the good bond market regime is in the third column.

[26] The theoretical probability is calculated by assuming a Gaussian copula with a correlation of +34.2%.

Figure 47: Dependogram of $R_S(t)$ and $R_B(t)$ (US, daily returns, 1980-1999)

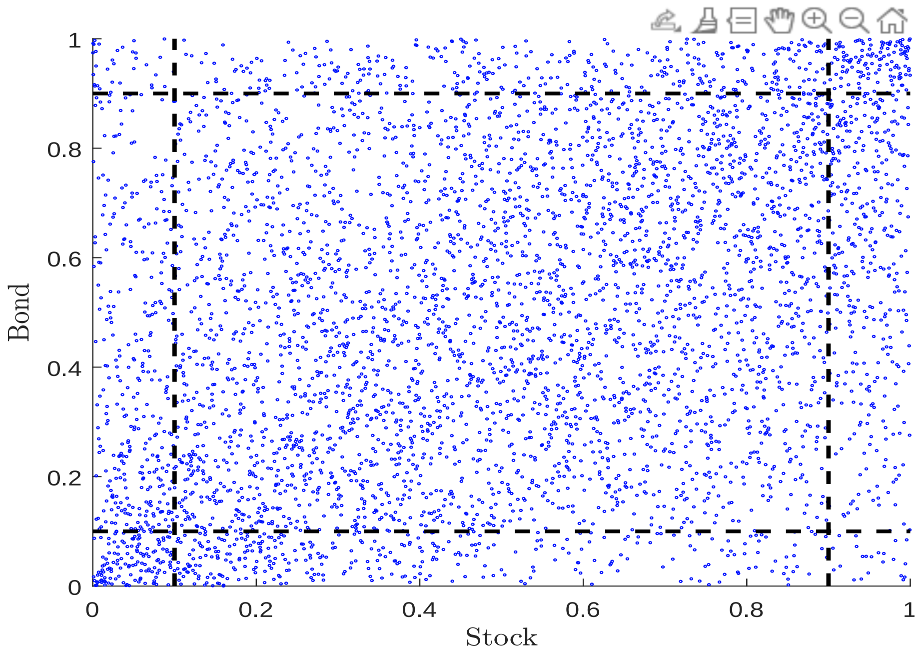


Figure 48: Dependogram of $R_S(t)$ and $R_B(t)$ (US, daily returns, 2000-2019)

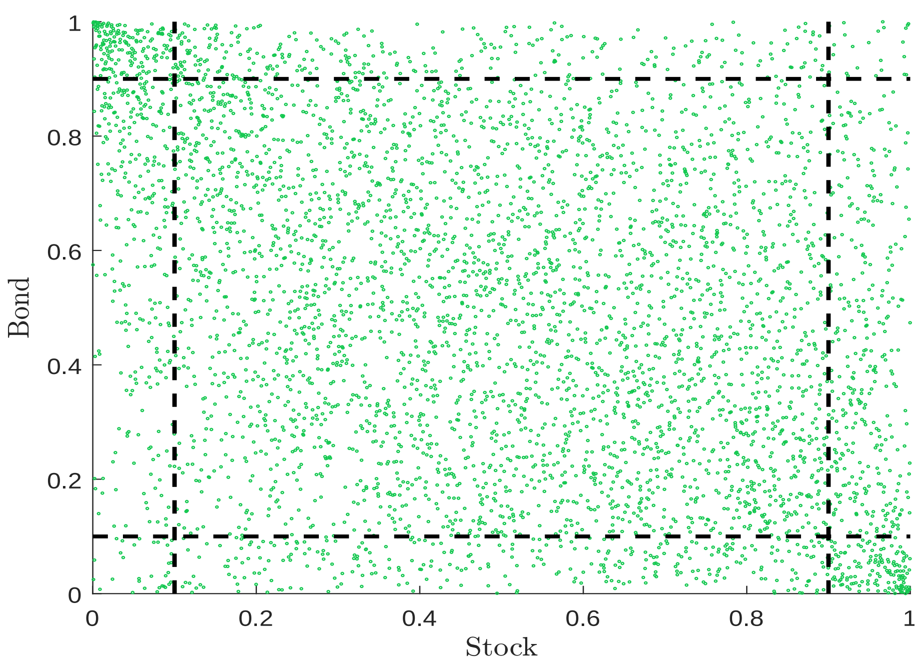


Table 23: Probability in % of stock-bond market regimes (US, daily returns, $\alpha = 10\%$)

Copula	Stock-bond market regime	1980-1999 $\mathcal{B}ad$	1980-1999 $\mathcal{N}ormal$	1980-1999 $\mathcal{G}ood$	2000-2019 $\mathcal{B}ad$	2000-2019 $\mathcal{N}ormal$	2000-2019 $\mathcal{G}ood$
Theoretical	$\mathcal{B}ad$	2.37	7.40	0.24	0.24	7.42	2.34
	$\mathcal{N}ormal$	7.40	65.20	7.40	7.42	65.17	7.42
	$\mathcal{G}ood$	0.24	7.40	2.37	2.34	7.42	0.24
Empirical	$\mathcal{B}ad$	2.90	6.45	0.65	0.66	5.77	3.56
	$\mathcal{N}ormal$	6.66	66.52	6.82	6.39	67.70	5.93
	$\mathcal{G}ood$	0.44	7.04	2.52	2.95	6.55	0.50

results. In fact, the empirical probability of observing bad times in both the stock and bond markets over the period 1980-1999 is now 2.90%, which is higher than the theoretical probability. Similarly, the probability of observing bad times in the stock market and good times in the bond market is 0.65%, again higher than the theoretical probability. If we focus on the second period from 2000 to 2019, we obtain the following results. The theoretical[27] and empirical probabilities of observing bad times in both the stock and bond markets are now 0.24% and 0.66%, respectively. If we consider bad times in the stock market and good times in the bond market, the theoretical and empirical probabilities become 2.34% and 3.56%, respectively.

Table 24: Pearson correlation of the market regime ($\mathcal{N}ormal$,$\mathcal{N}ormal$) (US, daily returns)

α	1980-1999	2000-2019
0%	34.2%	-33.7%
1%	33.0%	-32.0%
5%	27.9%	-24.1%
10%	21.9%	-20.4%
25%	11.4%	-6.3%

For each market regime, we can compute the local correlation such that the theoretical probability of observing a regime is exactly equal to its empirical frequency. In this case, we can show that the calibration cannot be done for all regimes. For example, if we consider the market regime characterized by normal times in the stock market and normal times in the bond market, we can show that the calibrated correlation is mostly undefined because it can range from -50% to $+50\%$. Thus, zero correlation is certainly the best estimate. This result is confirmed if we calculate the Pearson correlation of the market regime ($\mathcal{N}ormal$,$\mathcal{N}ormal$). As shown in Table 24, the correlation decreases as α increases.

Table 25: Local correlation in % of stock-bond market regimes (US, daily returns, $\alpha = 10\%$)

Stock-bond market regime	1980-1999 $\mathcal{B}ad$	1980-1999 $\mathcal{G}ood$	2000-2019 $\mathcal{B}ad$	2000-2019 $\mathcal{G}ood$	2021-2023 $\mathcal{B}ad$	2021-2023 $\mathcal{G}ood$
$\mathcal{B}ad$	44.17	12.49	-12.36	-55.06	26.38	-20.63
$\mathcal{G}ood$	22.35	37.22	-45.05	-19.35	4.28	11.05
Full period	34.2		-33.7		5.83	

In our cases, only four regimes can then be calibrated, those that include both bad and good times and do not rely on normal times. Results are given in Table 25. Since the

[27] The theoretical probability is calculated by assuming a Gaussian copula with a correlation of -33.7%.

Figure 49: Pearson correlation of the market regime ($\mathcal{N}ormal$,$\mathcal{N}ormal$) (daily returns, 2000-2019)

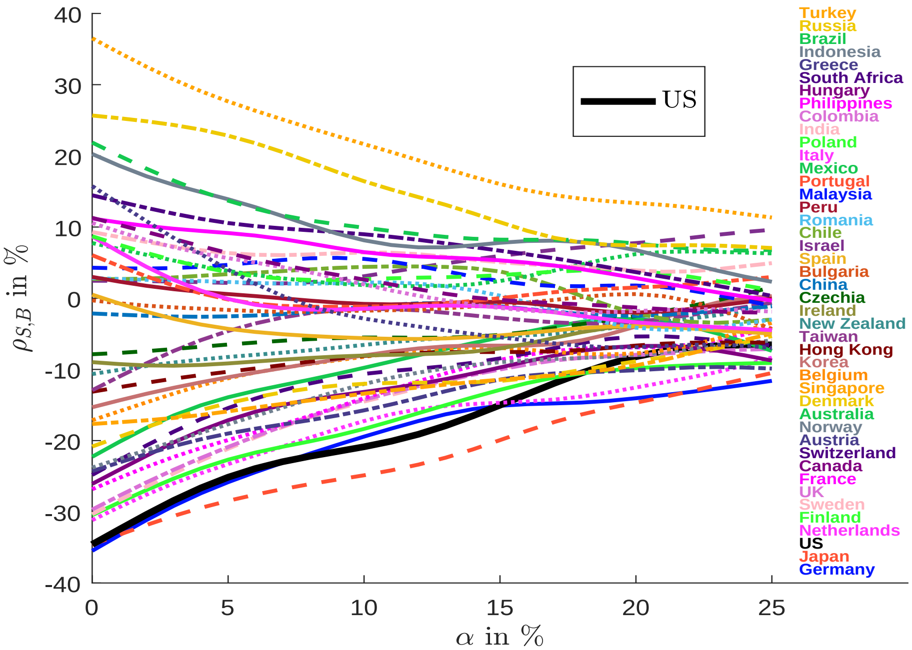
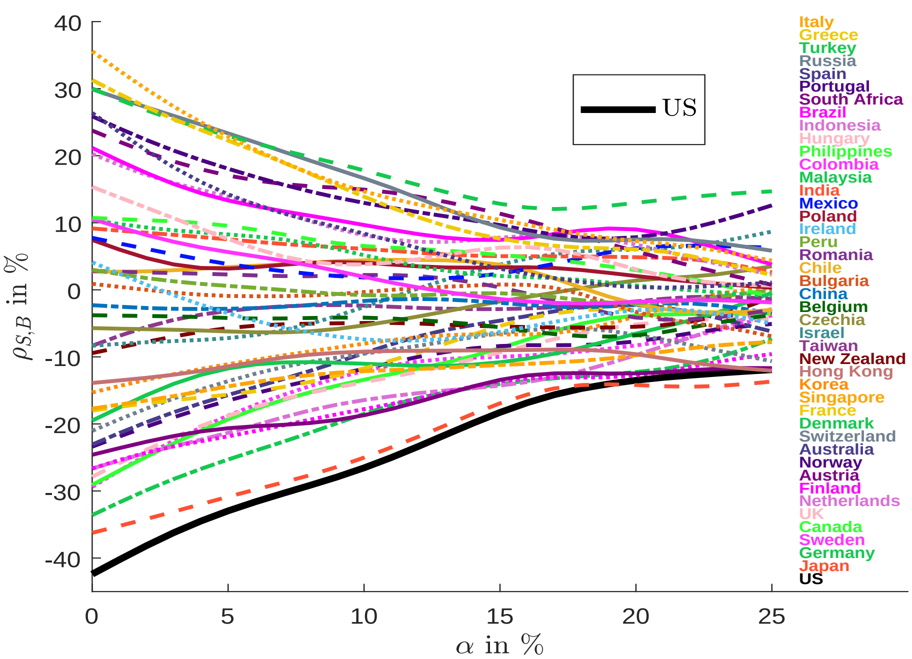
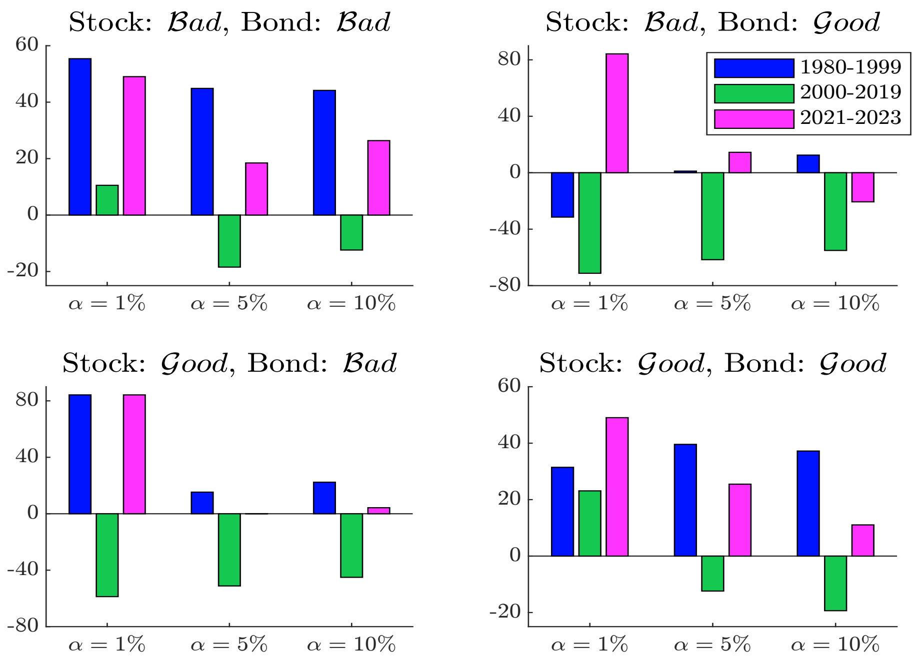


Figure 50: Pearson correlation of the market regime ($\mathcal{N}ormal$,$\mathcal{N}ormal$) (daily returns, 2010-2019)



stock-bond correlation was 34.2% for the whole period 1980-1999, it is 44.17% if we consider the stock-bond regime ($\mathcal{B}ad$,$\mathcal{B}ad$). We note that the stock-bond correlation is actually an average of the correlation of the four regimes, weighted by their frequency of occurrence. Between 1980 and 1999, the stock-bond correlation is then mainly driven by two market regimes: (1) bad times in both the stock and bond markets and, (2) good times in both the stock and bond markets. For the period 2000 and 2019, the contribution of the market regimes is different. The stock-bond correlation is mainly driven by periods when the stock and bond markets are asynchronous. In particular, the contribution is highest when the stock market is in bad times and the bond market is in good times. More recently, the positive stock-bond correlation is explained by the contribution when both stock and bond markets are in bad times. The short-term stock-bond correlation remains negative when stocks are in bad times and bonds are in good times.

Figure 51: Local correlation in % with respect to α (US, daily returns)



Remark 7. *The previous results question the origin of the bad and good times of each market regime. In fact, we can assume that one market can lead the other, i.e. the second market reacts to the bad or good performance of the first market. For example, a bad performance of the stock market can result in a good performance of the bond market due to the increase in risk aversion and the flight-to-quality behavior of investors. However, a poor stock market performance can also lead to a poor bond market performance due to the anticipation of negative growth and a rise in interest rates due to an increase in public debt. We can also assume that the stock market reacts to the bond market. In the previous analysis, we computed the stock-bond correlation conditional on the state of the market regardless of the cause of the state, but we could also compute the stock-bond correlation conditional on the trigger of the bad and good times.*

Figure 52: Estimated linear stock-bond payoff and top 4% influential observations (US, daily return, 1980-1999)

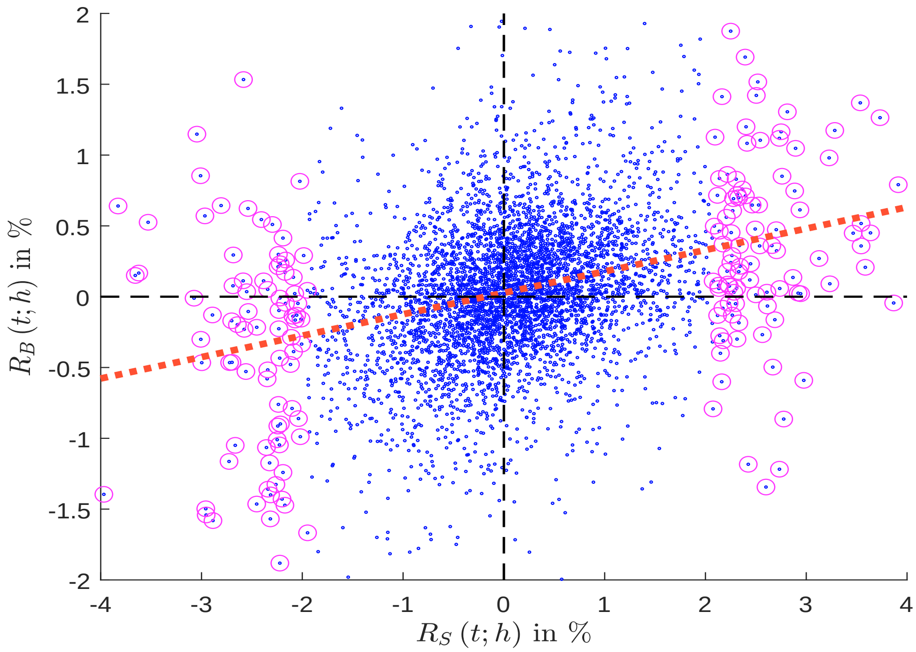
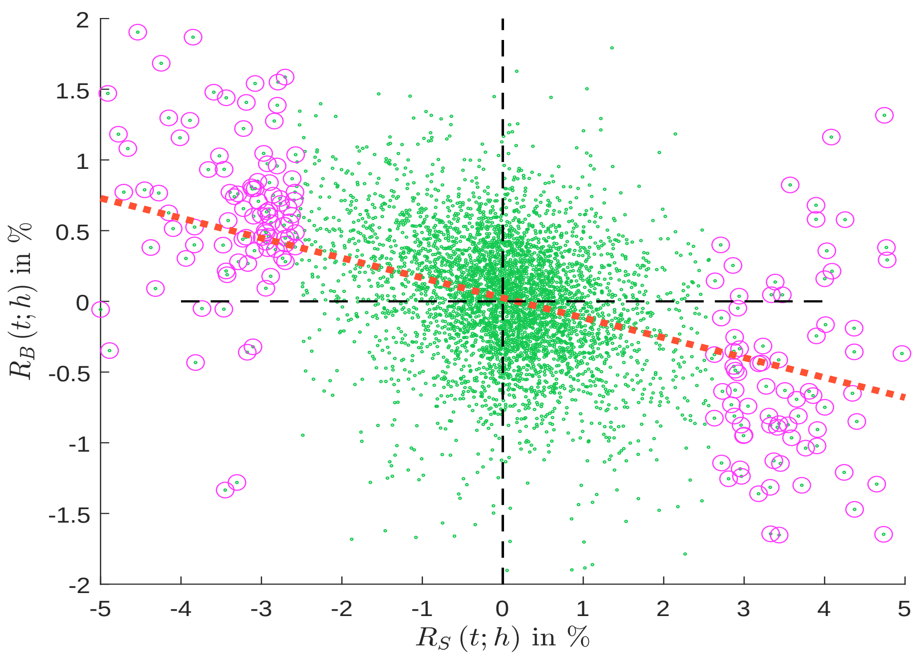


Figure 53: Estimated linear stock-bond payoff and top 4% influential observations (US, daily return, 2000-2019)



3.5.2 Payoff of the stock-bond correlation

In this section, we continue to examine the conditional comovement of stock and bond returns. We apply the payoff theory defined in Section 2.2 on page 11. By assuming that the stock market leads the bond market, the bond payoff is given by the parametric function $r_B = m(r_S)$:

$$m(r_S) = \mathbb{E}\left[R_B(t;h) \mid R_S(t;h) = r_S\right]$$

where $R_S(t;h)$ and $R_B(t;h)$ are the stock and bond returns calculated with frequency h.

Linear payoff To estimate the linear function $m(r_S)$, we use the linear regression:

$$r_B(t;h) = \beta_0 + \beta_1 r_S(t;h) + u(t)$$

and we have $\hat{m}(r_S) = \hat{\beta}_0 + \hat{\beta}_1 r_S$. The estimated payoff functions are shown in Figures 52 and 53. As expected, the slope of the payoff was upward for the period 1980-1999 and downward for the period 2000-2019. More surprising is the fact that the bond payoff is systematically negative when stock returns are positive in the second period. This finding again highlights the hedging status of US bonds between 2000 and 2019. For each linear payoff, we have also reported the top 4% influential observations[28], which account for 25% of the explanatory power of the linear regression.

Nonlinear payoff To estimate the nonlinear function $m(r_S)$, we use the local polynomial regression:

$$r_B(t;h) = \beta_0 + \sum_{j=1}^{p} \beta_j \left(r_S(t;h) - r_S\right)^j + u(t)$$

where p is the degree of the polynomial. The least squares problem becomes:

$$\left(\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_p\right) = \arg\min \sum_{t=1}^{T} \mathcal{K}\left(\frac{r_S - r_S(t;h)}{\kappa}\right)\left(r_B(t;h) - \beta_0 - \sum_{j=1}^{p} \beta_j \left(r_S(t;h) - r_S\right)^j\right)^2$$

where $\mathcal{K}(x)$ is the kernel. We can then show that $\hat{m}(r_S) = \hat{\beta}_0$ (Roncalli, 2020, page 643). In the following, we use the Gaussian kernel with a bandwidth κ equal to $2 \cdot \hat{\sigma} \cdot T^{-0.20}$, where $\hat{\sigma}$ is the standard deviation of $r_S(t;h)$.

Figure 54 shows the scatter plot of the joint daily returns $\left(R_S(t;h), R_B(t;h)\right)$ and the estimated payoff corresponding to the dotted red line for the period 1980-1999. The slope of the payoff is negative for very bad stock market periods, otherwise it is positive. This confirms that the unconditional stock-bond correlation was positive during this period. For the second period, the negativity of the unconditional stock-bond correlation is also confirmed, but we note that the slope of the payoff is flat or positive for very good stock market periods, when the daily return of the S&P 500 index is greater than 3% (Figure 55). These results are then consistent with the empirical analysis of the short-term stock-bond correlation. When we look at monthly returns[29], the story is different. For the 1980-1999 period, the slope of the payoff is first strongly negative when monthly stock returns are negative, and then strongly positive when monthly stock returns are positive. And the slope increases during large bull markets. For the second period 2000-2019, the slope remains

[28] We compute the hat matrix $H = X\left(X^\top X\right)^{-1} X^\top$ where X is the exogenous matrix of the regression model $Y = X\beta + U$. The most influential observations are those with the highest leverage $H_{i,i}$.

[29] See Figures 112 and 113 on page 133.

Figure 54: Estimated nonlinear stock-bond payoff (US, daily return, 1980-1999)

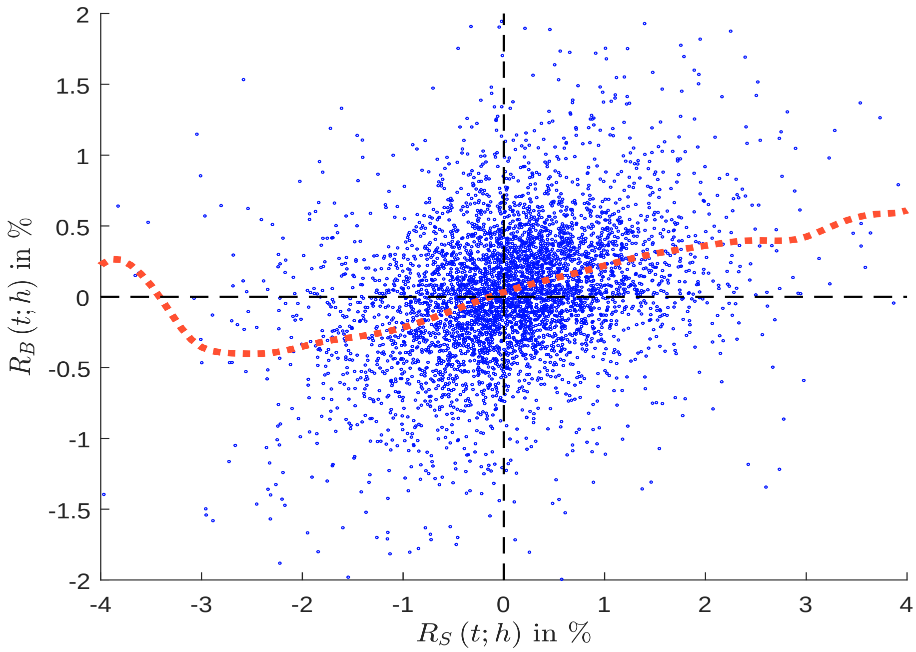
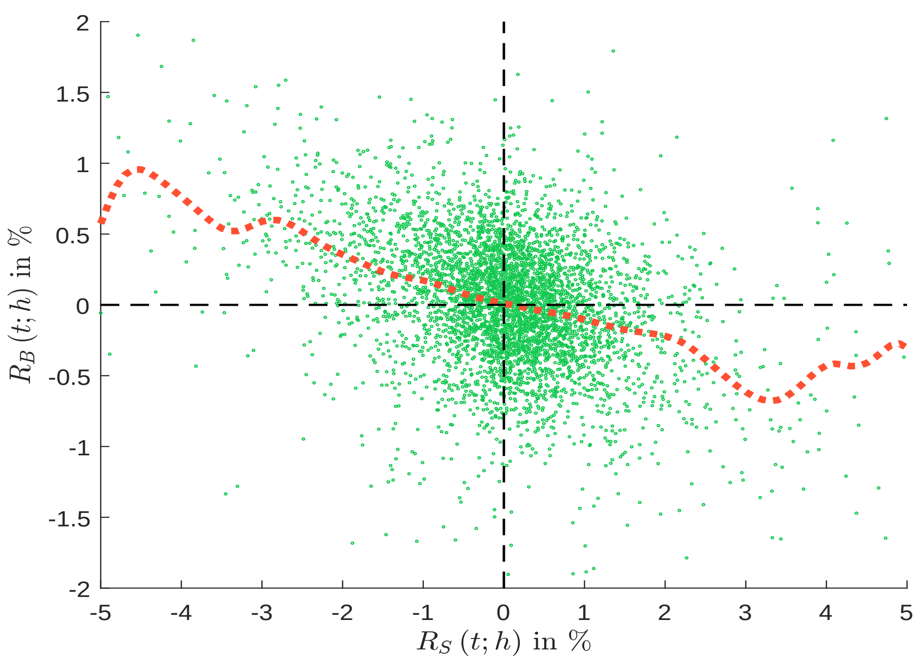
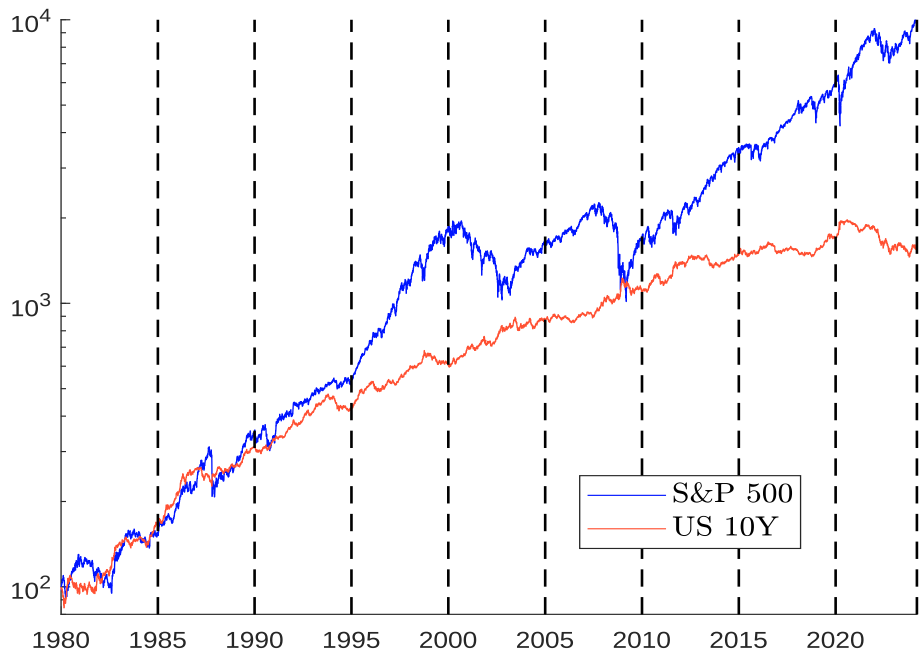


Figure 55: Estimated nonlinear stock-bond payoff (US, daily return, 2000-2019)



negative. If we take a longer time frequency, the payoff for 3-year returns is flat for the first period (Figure 114 on page 134). For the second period, the slope of the payoff is negative, then flat and finally positive (Figure 115 on page 134). Because we are constrained by the size of the period, we cannot explore 10-year or 30-year time frequencies. However, we can make the hypothesis that the slope of the payoff is close to zero: $\lim_{h\to\infty} \dfrac{\partial\, m\,(r_S)}{\partial\, h} \approx 0$. In the long run, it is difficult to assume that the unconditional stock-bond correlation is negative, because this would imply a negative long-term bond risk premium[30]. In fact, the long-term correlation between stocks and bonds is the comovement of the trends[31], which have been clearly up for forty years. For example, the non-overlapping five-year stock-bond correlation has been equal to +19% between 1980 and 2023 (Figure 56).

Figure 56: Cumulative performance of the S&P 500 Index and the generic US 10Y bond



3.5.3 Implications for strategic asset allocation

The stock-bond correlation is an important component that drives strategic asset allocation. In particular, we can find some expectations in the reports of asset owners and managers, called Capital Market Assumptions (CMA). Below, we list some of the values suggested by asset managers[32]: BlackRock Investment Institute: −16%, BNY Mellon: +3%, Capital

[30] If we consider reasonable figures of stock and bond volatilities.
[31] The long-term stock-bond correlation is highly related to the concept of cointegration.
[32] Sources are the following CMA reports found on asset manager websites:

- BlackRock Investment Institute (2024), *Capital Market Assumptions*, February 2024.
- BNY Mellon (2023), *2024 Capital Market Assumptions: The Path to Normalization*, November 2023.
- Capital Group (2024), *Capital Market Assumptions*, January 2024.
- Invesco (2023), *2024 Long-term Capital Market Assumptions*, September 2023.
- J.P. Morgan Asset Management (2023), *2024 Long-term Capital Market Assumptions*, December 2023.

Group: -9%, Invesco: -4% and J.P. Morgan AM: -11%. Several observations can be made. First, correlations are not always defined in CMA reports. Some CMA reports focus exclusively on expected returns, others include volatility figures, but only a third of CMA reports provide assumptions on correlations. One of the reasons is that correlations are certainly more difficult to predict because there are no economically or statistically proven models. In the case of expected returns and volatilities, we can choose from several existing models that have been used for many years. Second, we note that even if the CMA report contains correlation figures, this does not mean that we can find the expected stock-bond correlation as defined in this study, *i.e.* the correlation between the 10-year government bond and the large-cap stock market. The final comment concerns the time horizon. Do the figures reported refer to a short-term or a long-term stock-bond correlation?

The last question echoes the question raised in the introduction about the concept of time. Since it is relatively clear that the time horizon of the CMA reports corresponds to a long-run analysis, typically 10 years, the time frequency of the figures reported is generally not defined. The choice between short-term and long-term stock-bond correlation depends on the investor's approach to portfolio management. If the investor is a large institution that does not have the ability to rebalance its portfolio due to the size of the portfolio and the cost of market impacts, it is better to use a long-term stock-bond correlation. On the other hand, if the investor is very active and implements a tactical asset allocation, it is better to use a short-term stock-bond correlation. Therefore, we cannot use the same correlation for a buy-and-hold strategy and a constant-mix strategy. For example, in the current context, we expect sovereign wealth funds to use a higher correlation value than active multi-asset fund managers.

Summary and main results

The stock-bond correlation is an unconditional Pearson correlation, which is an average of different correlations conditional on different market regimes. The stock-bond correlation is indeterminate in normal times, which is not the case in bad times and good times. Therefore, the stock-bond correlation is largely the result of the comovement of stock and bond returns in bad and good times, since the contribution of normal times is small and we can assume that the stock-bond correlation is close to zero in normal times. Because of the averaging formula, the magnitude of the stock-bond correlation is generally underestimated in the two extreme market conditions. When we compute the conditional stock-bond correlation, the payoff analysis shows that negative and positive correlations can coexist in the same period. We note that the slope of the bond payoff is highly dependent on time frequency. Therefore, the stock-bond correlation can be very different when we look at daily returns, monthly returns, annual returns, five-year returns, etc. The choice of time frequency for calculating the stock-bond correlation is then an important element in the design of the portfolio allocation. Indeed, it is better to consider a long-term correlation for buy-and-hold strategies, while a short-term correlation is more appropriate when the portfolio is managed on an active basis, as in constant-mix strategies.

3.6 Economic narratives of the stock-bond correlation

Investors believe that there are at least 3 dimensions that need to be squared to define an asset allocation based on the economic cycle: growth, inflation and monetary policy. Growth is important for optimal risk exposure across asset classes, while inflation tends to drive asset allocation within asset classes. Central banks and monetary policy are affected by growth and inflation and can influence asset allocation at different levels. Using a proprietary tool, we disentangle this multi-dimensional framework between an investment cycle and an inflation cycle, identifying the relevant phases and their determinants. As we have seen over the past 50 years, inflation obviously affects growth and monetary policy actions through conventional and unconventional instruments, adapting their tone to the different upward and downward pressures in the economic and social system. In the 1970s and 1980s, Volker's FED was able to lower long-term inflation expectations by aggressively raising interest rates, while after the GFC, central banks developed unconventional monetary policy tools to boost liquidity and escape deflationary gravity for more than a decade. The extraordinary inflation rebalancing in 2022 introduced a new regime and monetary policy had to change, affecting growth and market dynamics.

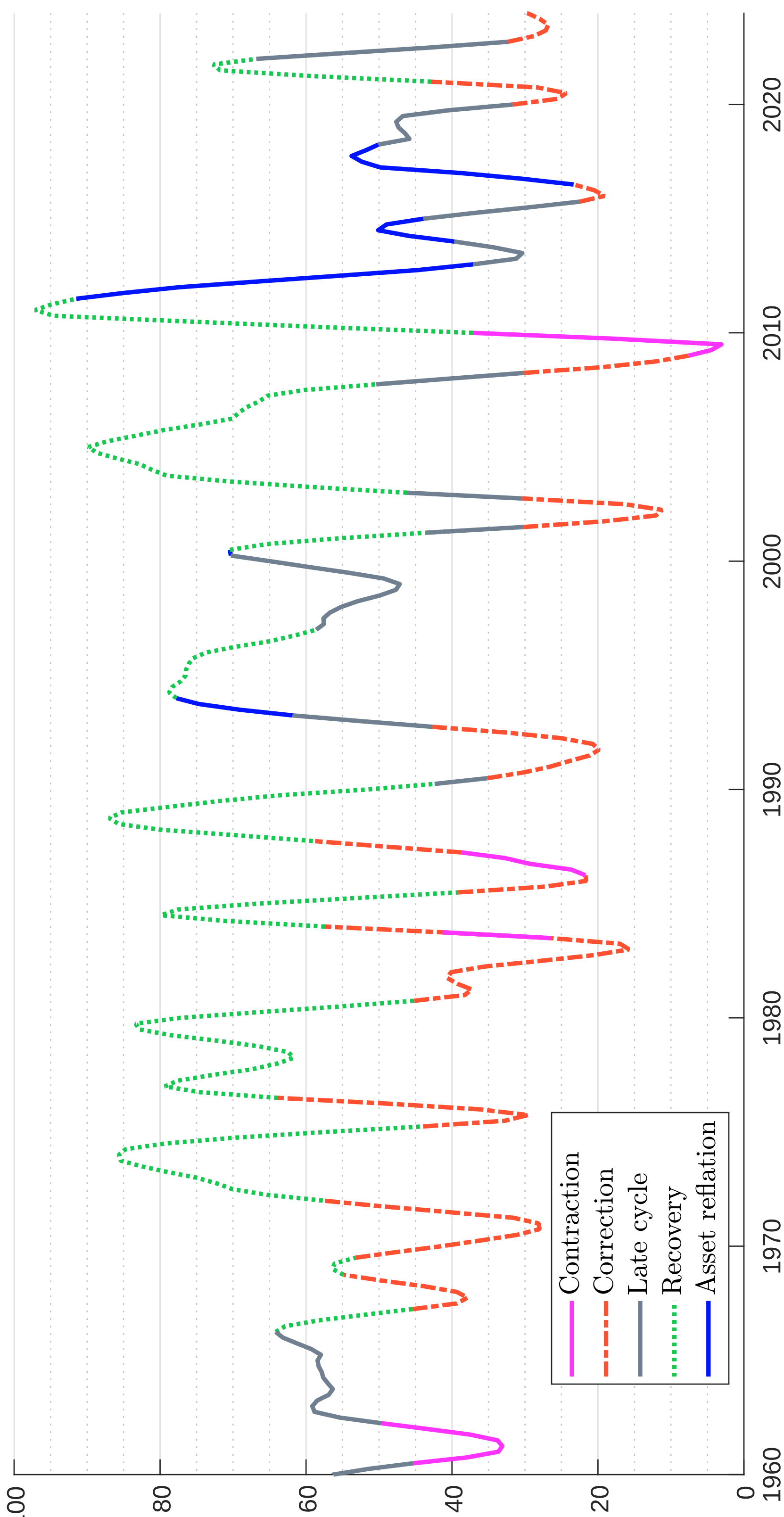
An economic cycle typically has four phases: expansion, peak, contraction and correction. However, we can also identify an additional economic regime, known as asset reflation, in which monetary policy interventions support financial asset returns and dampen market volatility despite the absence of an additional boost from growth. This phase can explain what has happened several times over the past decade, *i.e.* the economy growing on average at trend rates while inflation remains stubbornly below target despite central bank balance sheet expansion. Figure 57 shows the evolution of the business cycle index[33] and the identification of the five regimes. It can be compared with the inflation patterns[34] reported in Figure 58.

Under normal conditions, the sequence of growth fluctuations is the driver of inflation and central banks react with interest rates in a countercyclical manner, as stated in their official statement. The early recovery phase is driven by inventory restocking and inflation starts to rise due to the demand for goods in the production space. Typically, producer prices spike, but tend to normalise relatively quickly and regularly. Therefore, central banks are in a wait-and-see mode and are quite accommodative. The yield curve tends to steepen, discounting higher growth and inflation expectations. In such a phase, equities outperform govies and the stock-bond correlation is negative. If central banks are successful in their vigilance, they will begin to react when the economy enters an overheating phase in which growth is above trend, inflation crosses the trough from producer prices to consumer prices, paving the way for a potential imbalance, and central banks become uncomfortable with inflation above their target. The normal response is to start raising interest rates in line with the forces of the business cycle. Typically, at this stage, valuation adjustments occur, volatilities rise, equity and government bond yields move closer together and correlations increase. The economy then begins to cool (growth and inflation), central banks begin to adjust interest rates to focus more on full employment and, if successful, the economy emerges from recession. Government bonds become a very effective hedge against risky assets and correlations turn significantly negative. The bottom line investment implications are as follows. The normal sequence of expansionary and contracting phases in the economic

[33]The business cycle index is calculated using a clustering algorithm applied to a set of economic and financial variables distributed along four dimensions: growth (*e.g.*, GDP, unemployment, sales or EPS), inflation (*e.g.*, CPI, PPI or unit labour costs), monetary policy (*e.g.*, monetary aggregates, policy rates or central bank balance sheets) and leverage (*e.g.*, public and private debt).

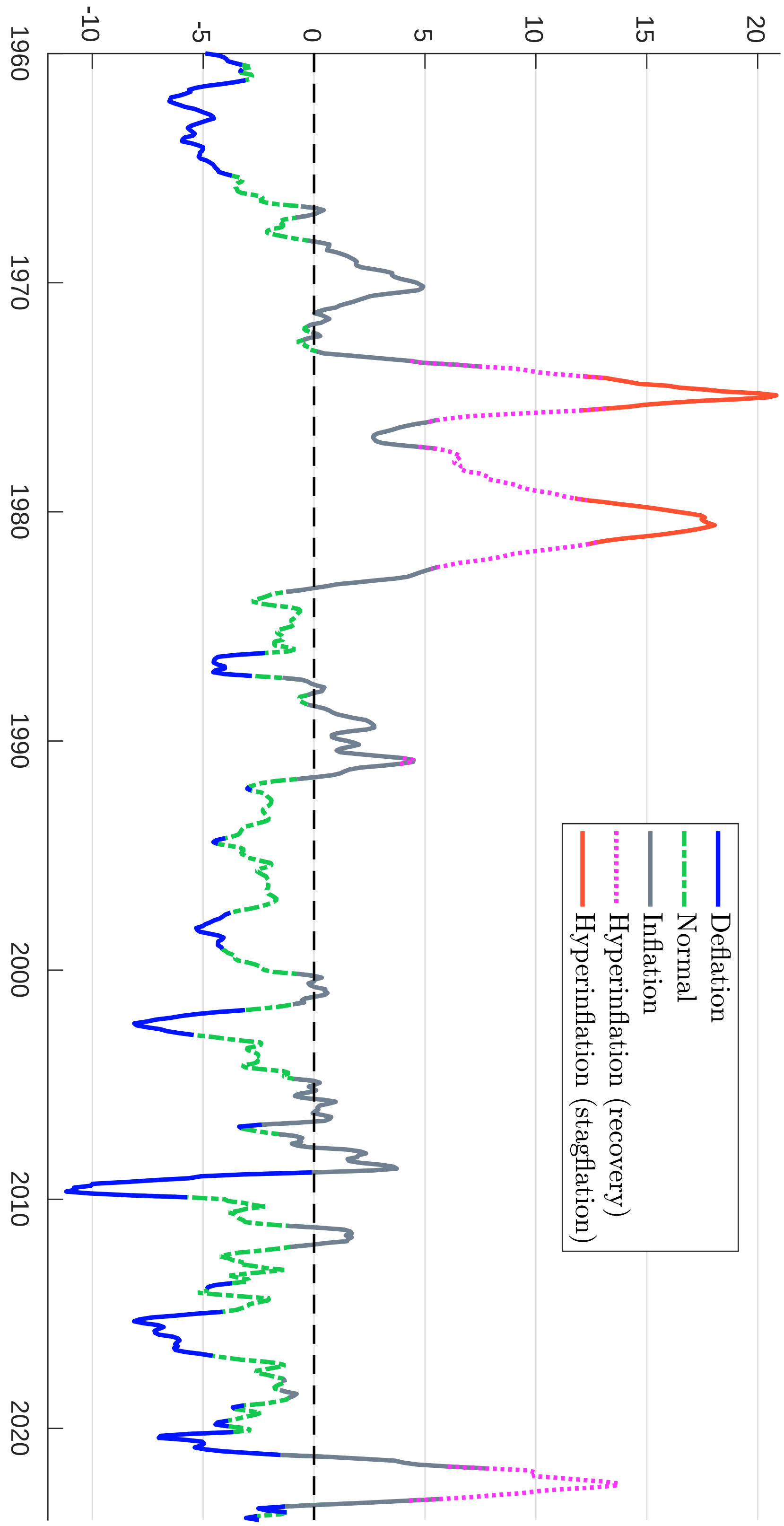
[34]The inflation index is the first principal component of four inflation indicators: US CPI, US PPI, ULC and Core PCE.

Figure 57: Smoothed cycle index (scale from 0 to 100)



Source: Amundi Investment Institute, Data as of March 2024.

Figure 58: Inflation index (in %)



Source: Amundi Investment Institute, Data as of March 2024.

cycle, with a normal response from central banks, will obviously produce different returns for equities and government bonds and for a balanced portfolio, but the stock-bond correlation will work in the right way, mitigating overall volatility and large drawdowns.

However, we can identify two exceptional conditions: asset reflation (*Goldilocks*) and inflationary shocks (*pain*). In the case of asset reflation, the unconventional monetary policies implemented since the first quarter of 2009 in the aftermath of the GFC have provided unprecedented injections of liquidity into financial markets, leading to unusual inflation: asset prices have risen across the board. The expansion of central banks' balance sheets was ultimately aimed at ensuring the stability of financial intermediaries and creating pricing power by increasing the supply of wealth effects and, to some extent, circumventing the significant fragility of the financial sector in the aftermath of the GFC. It has been sustained for more than a decade by real deflationary pressures from the sovereign debt crisis, low growth and global credit deleveraging. Overall, it has been an extraordinary decade for markets, with very positive returns for equities and govies, with negative stock-bond correlation during periods of risk asset sell-offs. In short, a strong hedge with positive carry (Goldilocks).

However, since the 1990s, and even more so since the asset reflation, investors have tended to forget some important aspects:

1. The dynamics of inflation are less regular than those of growth, so fluctuations and shifts tend to be more persistent and less predictable.

2. Inflation isn't stationary mainly because it's multifaceted; there are different forces that generate inflation and they naturally have different effects on the persistence of fluctuations.

3. More importantly, some forces are endogenous and others are exogenous, *i.e.* some are internal to the economic system and others are external and act as a shock (positive or negative) to the economic system.

4. Central banks are more effective in dealing with endogenous forces and less effective in absorbing external shocks; if at all, they very often tend to exacerbate and amplify shocks in the economic system rather than absorb them.

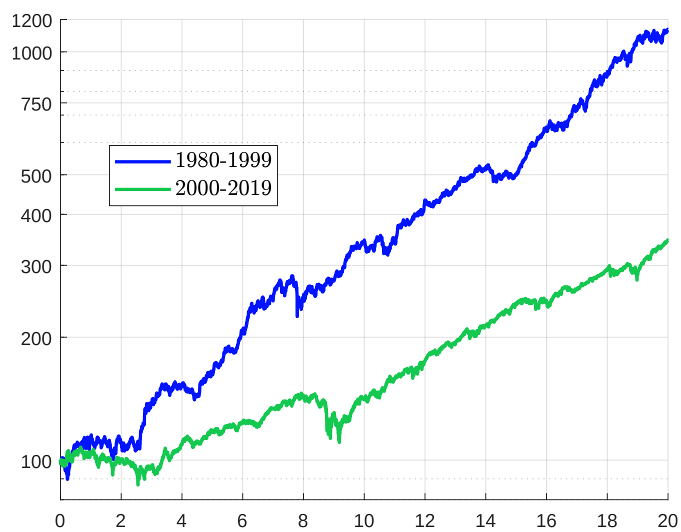
From an investment perspective, there are four important components of inflation: cyclical (demand driven), services, energy and food. Energy and food are often associated with external commodity shocks that cause severe damage to the economy, financial markets and asset class behaviour. In fact, shocks tend to be persistent and move across all inflation components, such as services and core, creating hyperinflationary regimes. In such an environment, central banks have to suppress some growth to counter the risk of de-anchoring long-term inflation expectations, drain liquidity and tighten financial conditions more than the growth dimension would suggest, often pushing the economy into stagflation. The overall disorderly financial deleveraging and huge discounts in cross-asset valuations are creating twin bear markets with negative returns for equities and bonds, positive stock-bond correlation and limited diversification in cash.

From an ex-post point of view, we can always understand the behaviour of the stock-bond correlation and the previous narratives show that the concept of inflation and growth risks goes beyond the concept of inflation and growth volatilities. They are more complex because we cannot simplify these patterns into two regimes: high-risk and low-risk regimes. This may explain the poor econometric results in section 3.4 on page 66. Surely a more sophisticated or non-linear model is needed. However, we must be cautious about predicting the stock-bond correlation in the long run, as new regimes may emerge.

4 Conclusion

Stock-bond correlation is a fundamental part of portfolio allocation. Investors consider that the last two decades, 2000-2019, have been a fortunate period for diversification. They have benefited from falling interest rates, which imply good mark-to-market bond returns, and negative stock-bond correlation, which helps during equity market drawdowns. The most recent period has been more challenging, as we have seen a rise in stock-bond correlation, and many investors are concerned about a lack of diversification due to less attractive comovements between bonds and stocks. However, it is misleading to say that the bond market is less attractive because it has generated higher carry. If we look at the period before 2000, the correlation between stocks and bonds is positive. So we can conclude that it was not a good period in terms of diversification. However, if we compare the period 1980-1999 (positive stock-bond correlation) with the period 2000-2019 (negative stock-bond correlation), the nominal performance of a diversified stock-bond portfolio is better in the first period than in the second[35] (Figure 59). Thus, investors' concerns about correlation are not performance concerns, but risk concerns. In fact, we know that the expected return of portfolios does not depend on correlations, which only affect the volatility and higher statistical moments of the P&L. Therefore, investors demand a negative stock-bond correlation because they want to hedge their equity exposure, reduce the drawdown of diversified funds, and smooth short-term performance. In a sense, negative stock-bond correlation is a management tool for active investors, while buy-and-hold investors care little about the level and sign of stock-bond correlation.

Figure 59: Cumulative performance of US 50/50 equity-bond constant-mix portfolio



Our empirical results show that the US stock-bond correlation was positive between 1965 and 2000, negative since 2000 after the dot-com crisis, and positive again in recent years. This confirms the empirical evidence of many empirical studies (Brixton *et al.*, 2023; Molenaar *et al.*, 2023). These results were obtained using the 10-year US Treasury bond and the S&P 500 Index, and hold for other bond maturities greater than one year. However, the behavior of the stock-bond correlation is different at shorter maturities. In this case, we observe more frequent periods of positive and negative correlations. For example, for the

[35] This is also true when we look at real performance adjusted for inflation, as there is an annualized difference of 4.5% between the two periods.

3-month US Treasury Bill bond, we have 13 different sub-periods instead of the previous 3 sub-periods. Looking at other countries, the behavior of the stock-bond correlation is different. In fact, we found that only Canada seems to have the same behavior as the US. Japan experienced a change in the sign of the stock-bond correlation long before, and this structural break coincides with the introduction of the zero interest rate policy in 1995. Southern European countries, which were more affected during the European debt crisis, have recorded a positive correlation since 2010 (Greece, Italy, Portugal and Spain). For some other European Union countries, the return to a positive correlation occurred later in 2015-2016 (Denmark, Finland, France, Germany, Ireland and the Netherlands). Most emerging economies didn't experience a negative correlation between stocks and bonds, with the exception of China, India and Singapore. In this context, it is clear that credit risk and local investors' confidence in the sustainability of the country's debt are two important factors explaining the sign of the stock-bond correlation in both developed and developing countries. In summary, the fact that the stock-bond correlation was positive between 1980 and 1999 and then negative between 2000 and 2019 is a purely US-centric view. However, if you talk to European portfolio managers, most of them explain that they use US bonds to hedge non-US equities, for instance European equities, because it is a better instrument than, for example, European bonds. So the US stock-bond correlation and US bonds have a special status compared to other stock-bond correlations and other sovereign bonds. The challenge for the next few years will be for US government bonds to continue to have the confidence of investors that they are a good hedge for broad equities.

Since the stock-bond correlation does not affect the expected return of a diversified stock-bond portfolio, it is a key component in calculating the ex-ante equity and bond risk premia. In particular, we can show that the required risk premium in the Capital Asset Pricing Model has two components: a variance risk premium and a covariance risk premium. Theoretically, the stock-bond correlation has little effect on the equity risk premium, while it has a large effect on the bond risk premium. These findings are confirmed by our empirical results. On average, about 70% of the US bond risk premium is explained by the covariance risk premium, while the variance risk premium is the main component of the US equity risk premium, accounting for about 95%. Another theoretical finding is that the bond risk premium is an increasing function of both the stock-bond correlation and the indiosyncratic risk of bonds. Therefore, a necessary condition for the bond risk premium to be negative is that the stock-bond correlation is negative and below a certain value, called the break-even stock-bond correlation. This condition is not sufficient because the idiosyncratic risk of bonds can offset the effect of the negative stock-bond correlation. Our empirical results show that the market priced in a negative bond risk premium between 2000 and 2020 due to the negative US stock-bond correlation. More recently, the US bond risk premium has turned positive, mainly due to the reversal of the stock-bond correlation. Indeed, the break-even stock-bond correlation has been roughly the same since 2010, suggesting that we are not observing a shift in the market's perception of the idiosyncratic risk of US bonds. Our results demonstrate that investors are willing to pay a cost in the form of a negative bond risk premium in order to benefit from the hedging characteristics of US bonds. Turning to the credit asset class, we observe an increasing relationship between the implied bond risk premium and corporate bond ratings, as expected by theory. This implies that credit risk is an important determinant of both the stock-bond correlation and the required bond risk premium. For example, in the 2020s, the highest bond risk premia are observed for Greece, Italy, Portugal, South Africa and Turkey. Thus, the highest risk premia are not necessarily observed for EM economies from the perspective of local investors. For example, China and Singapore have the lowest ex-ante bond risk premia between 2020 and 2023, comparable to the levels observed in Japan, Switzerland and the US.

Just as the choice of bond affects the level of stock-bond correlation, the choice of stock portfolio affects the correlation measure. First, we need to distinguish between the aggregate stock-bond correlation, which is calculated for a portfolio of stocks, and the individual stock-bond correlation, which is calculated for individual stocks. Like the volatility risk measure, the correlation of the portfolio return with the bond return is not an average of individual correlations. This is because diversification increases correlation risk, which is the opposite of what we observe for portfolio volatility, where diversification reduces volatility risk. In other words, portfolio diversification has a greater effect on volatility than on covariance, resulting in a leverage effect on correlation. If we assume that the individual stock-bond correlations are homogeneous and equal, then the aggregate stock-bond correlation depends on the diversification ratio of the equity portfolio. In this case, the absolute value of the stock-bond correlation is obtained for the most diversified portfolio (MDP). Portfolio construction and composition matters when computing the stock-bond correlation. Theoretically, the leverage between individual and aggregate stock-bond correlations is between a factor of 1 and 3. On average, we get a leverage of two for the S&P 500 Index. Since the composition of the portfolio is important, we get different values if we look at small-cap indices, sectors or factor portfolios. In the US, for example, real estate and utilities have a higher correlation than the overall market. The same is true for communication services, consumer staples and health care, but to a lesser extent. On the other hand, industrials and materials have a lower correlation than the overall market. Among equity factors, value stocks have a more negative correlation with bonds, while growth, low volatility, high dividend and quality stocks have a more positive correlation with bonds. It is tempting to interpret these results in terms of the equity duration of the portfolio, but the relationship between stock-bond correlation and equity duration is not always robust.

Academics have identified several factors that drive the stock-bond correlation and have proposed many macroeconomic models to describe its dynamics. We can group them into three families: inflation-centric models, real-centric models, and growth-inflation models. Except for a few real-centric models, inflation is the main driver of the stock-bond correlation. It takes several forms and can be the level of inflation, expected inflation, inflation volatility, inflation risk, and inflation innovation. Real-centric models may use other trigger variables, but they focus more or less on the flight-to-quality behavior of investors. The third type of model is more of a battle between growth uncertainty and inflation uncertainty. In theory, the stock-bond correlation rises with inflation risk and falls with growth risk. Therefore, the negative stock-bond correlation is explained when investors are concerned about growth risk. Our empirical results are not convincing. While the econometric results show that inflation has a higher explanatory power than growth, the coefficient of determination is relatively low and, more importantly, the regression coefficients do not always have the correct sign predicted by the theory. In fact, carry (or yield) turns out to be the most important economic factor. This is not surprising if US bonds are used as an equity hedge and there is a cost to dynamically hedging the equity exposure, as this cost is a function of the carry.

The final issue examined in this study is the payoff function of government bonds. In the payoff approach, a negative stock-bond correlation can be consistent with a positive bond risk premium for several reasons. First, we need to distinguish time horizons. A negative short-term stock-bond correlation can coexist with a long-term stock-bond correlation. And our empirical results show that the bond payoff is different when we consider short-term and long-term returns. Second, the payoff may be nonlinear. In this case, we need to distinguish the analysis in bad, normal and good times. Our empirical results confirm that local and conditional correlations are different from a global correlation and show that we can assume that the stock-bond correlation is close to zero in normal times. The third reason, which is related to the second, is that not all observations are equal when calculating the stock-bond correlation, and a few extreme observations can make a large contribution. This is the case for the US stock-bond correlation between 2000 and 2019, which is mainly driven by flight-to-quality episodes.

The sign and magnitude of the stock-bond correlation is clearly related to the status of bonds. In general, we believe that a negative stock-bond correlation indicates a situation where investors can benefit from the hedging status of bonds. From this perspective, we can say that the hedging property comes from the sign of the stock-bond correlation. In this way, the stock-bond correlation is an exogenous parameter determined by the economy, inflation, monetary policy, etc., and investors observe this parameter in order to hedge or not hedge their equity portfolios. But we can reverse the relationship (or causality). Perhaps the level of correlation between stocks and bonds is a consequence of investors' decisions. If investors believe that bonds can hedge their equity exposure, this behaviour may imply a negative stock-bond correlation. From this perspective, the stock-bond correlation is a parameter determined by both macro-financial fundamentals and investor perceptions. This assumption would explain why the stock-bond correlation varies so much across countries.

References

ANDERSSON, M., KRYLOVA, E., and VÄHÄMAA, S. (2008). Why Does the Correlation between Stock and Bond Returns Vary over Time?. *Applied Financial Economics*, 18(2), pp. 139-151.

BAELE, L., and VAN HOLLE, F. (2017). Stock-Bond Correlations, Macroeconomic Regimes and Monetary Policy. *SSRN*, 3075816.

BAKU, E. FORTES, R., HERVÉ, K., LEZMI, E., MALONGO, H., RONCALLI, T. and XU, J. (2019). Factor Investing in Currency Markets: Does it Make Sense?. *Journal of Portfolio Management*, 46(2), pp. 141-155.

BANSAL, R., and YARON, A. (2004). Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles. *Journal of Finance*, 59(4), pp. 1481-1509.

BARSKY, R. B. (1986). Why Don't the Prices of Stocks and Bonds Move Together?. *American Economic Review*, 79(5), pp. 1132-1145.

BAUR, D. G., and LUCEY, B. M. (2009). Flights and Contagion: An Empirical Analysis of Stock-Bond Correlations. *Journal of Financial Stability*, 5(4), pp. 339-352.

BEBER, A., BRANDT, M. W., and KAVAJECZ, K. A. (2009). Flight-to-quality or Flight-to-liquidity? Evidence from the Euro-area Bond Market. *Review of Financial Studies*, 22(3), pp. 925-957.

BLANCHARD, O. J. (1997). The Medium Run. *Brookings Papers on Economic Activity*, 28(2), pp. 89-158.

BRIÈRE, M., CHAPELLE, A., and SZAFARZ, A. (2012). No Contagion, Only Globalization and Flight to Quality. *Journal of international Money and Finance*, 31(6), pp. 1729-1744.

BRIXTON, A., BROOKS, J., HECHT, P., ILMANEN, A., MALONEY, T., and MCQUINN, N. (2023). A Changing Stock-Bond Correlation: Drivers and Implications. *Journal of Portfolio Management*, forthcoming.

BURKHARDT, D., and HASSELTOFT, H. (2012). Understanding Asset Correlations. *SSRN*, 1879855.

CAMPBELL, J. Y., PFLUEGER, C., and VICEIRA, L. M. (2020). Macroeconomic Drivers of Bond and Equity Risks. *Journal of Political Economy*, 128(8), pp. 3148-3185.

CAMPBELL, J. Y., and SHILLER, R. J. (1988). The Dividend-price Ratio and Expectations of Future Dividends and Discount Factors. *Review of Financial Studies*, 1(3), pp. 195-228.

CAMPBELL, J. Y., SUNDERAM, A., and VICEIRA, L. M. (2017). Inflation Bets or Deflation Hedges? The Changing Risks of Nominal Bonds. *Critical Finance Review*, 6(2), pp. 263-301.

CAZALET, Z., GRISON, P., and RONCALLI, T. (2014). The Smart Indexing Puzzle. *Journal of Index Investing*, 5(1), pp. 97-119.

CAZALET, Z., and RONCALLI, T. (2014). Facts and Fantasies About Factor Investing. *SSRN*, 2524547.

CHEN, N. F., ROLL, R., and ROSS, S. A. (1986). Economic Forces and the Stock Market. *Journal of Business*, 59(3), pp. 383-403.

Chernov, M., Lochstoer, L. A., and Song, D. (2021). The Real Channel for Nominal Bond-Stock Puzzles. *NBER*, w29085.

Chiang, T. C., Li, J., and Yang, S. Y. (2015). Dynamic Stock-Bond Return Correlations and Financial Market Uncertainty. *Review of Quantitative Finance and Accounting*, 45, pp. 59-88.

Choueifaty, Y., and Coignard, Y. (2008). Toward Maximum Diversification. *Journal of Portfolio Management*, 35(1), pp. 40-51.

Cieslak, A., and Pflueger, C. (2023). Inflation and Asset Returns. *Annual Review of Financial Economics*, 15, pp. 433-448.

Clarke, R., De Silva, H., and Thorley, S. (2011). Minimum-variance Portfolio Composition. *Journal of Portfolio Management*, 37(2), pp. 31-45.

Chui, C. M., and Yang, J. (2012). Extreme Correlation of Stock and Bond Futures Markets: International Evidence. *Financial Review*, 47(3), pp. 565-587.

Cochrane, J. H. (2001). *Asset Pricing*. Princeton University Press.

Connolly, R., Stivers, C., and Sun, L. (2005). Stock Market Uncertainty and the Stock-Bond Return Relation. *Journal of Financial and Quantitative Analysis*, 40(1), pp. 161-194.

Connolly, R., Stivers, C., and Sun, L. (2007). Commonality in the Time-variation of Stock-Stock and Stock-Bond Return Comovements. *Journal of Financial Markets*, 10(2), pp. 192-218.

David, A., and Veronesi, P. (2013). What Ties Return Volatilities to Price Valuations and Fundamentals?. *Journal of Political Economy*, 121(4), pp. 682-746.

David, A., and Veronesi, P. (2016). The Economics of the Comovement of Stocks and Bonds. In Veronesi, P. (Ed.), *Handbook of Fixed-Income Securities*, Chapter 15, pp. 313-326.

Demey, P., Maillard, S., and Roncalli, T. (2010). Risk-based Indexation. *SSRN*, 1582998.

DeMiguel, V., Garlappi, L., and Uppal, R. (2009). Optimal versus Naive Diversification: How Inefficient is the $1/N$ Portfolio Strategy?. *Review of Financial Studies*, 22(5), pp. 1915-1953.

Duffee, G. R. (2023). Macroeconomic News and Stock-Bond Comovement. *Review of Finance*, 27(5), pp. 1859-1882.

Ermolov, A. (2022). Time-varying Risk of Nominal Bonds: How Important are Macroeconomic Shocks?. *Journal of Financial Economics*, 145(1), pp. 1-28.

Garcia, R., and Tsafack, G. (2011). Dependence Structure and Extreme Comovements in International Equity and Bond Markets. *Journal of Banking & Finance*, 35(8), pp. 1954-1970.

Hamdan, R., Pavlowsky, F., Roncalli, T., and Zheng, B. (2016). A Primer on Alternative Risk Premia. *SSRN*, 2766850.

Hasseltoft, H. (2009). The Fed Model and the Changing Correlation of Stock and Bond Returns: An Equilibrium Approach. *SSRN*, 1361489.

Ilmanen, A. (2003). Stock-bond Correlations. *Journal of Fixed Income*, 13(2), pp. 55-66.

Jammazi, R., Tiwari, A. K., Ferrer, R., and Moya, P. (2015). Time-varying Dependence between Stock and Government Bond Returns: International Evidence with Dynamic Copulas. *North American Journal of Economics and Finance*, 33, pp. 74-93.

Jones, C. S., and Pyun, S. (2023). Consumption Growth Persistence and the Stock-bond Correlation. *Working Paper*.

Jusselin, P., Lezmi, E., Malongo, H., Masselin, C., Roncalli, T. and Dao, T-L. (2017). Understanding the Momentum Risk premium: An In-Depth Journey Through Trend-Following Strategies. *SSRN*, 3042173.

Laarits, T. (2022). Precautionary Savings and the Stock-bond Covariance. *SSRN*, 3741486.

Li, L. (2002). Macroeconomic Factors and the Correlation of Stock and Bond Returns. *SSRN*, 363641.

Li, E. X., Zha, T., Zhang, J., and Zhou, H. (2022). Does Fiscal Policy Matter for Stock-bond Return Correlation?. *Journal of Monetary Economics*, 128, pp. 20-34.

Longin, F., and Solnik, B. (2001). Extreme Correlation of International Equity Markets. *Journal of Finance*, 56(2), pp. 649-676.

Lucas R. E. (1978). Asset Prices in an Exchange Economy. *Econometrica*, 46(6), pp. 1429-1445.

Maillard, S., Roncalli, T., and Teïletche, J. (2010). The Properties of Equally Weighted Risk Contribution Portfolios. *Journal of Portfolio Management*, 36(4), pp. 60-70.

Maloney, T., and Moskowitz, T. J. (2021). Value and Interest Rates: Are Rates to Blame for Value's Torments?. *Journal of Portfolio Management*, 47(6) pp. 65-87.

Molenaar, R., Senechal, E., Swinkels, L., and Wang, Z. (2023). Empirical Evidence on the Stock-Bond Correlation. *SSRN*, 4514947.

Rankin, E., and Idil, M. S. (2014). A Century of Stock-Bond Correlations. *Reserve Bank of Austalia Bulletin*, September, pp. 67-74.

Richard, J. C., and Roncalli, T. (2015). Smart Beta: Managing Diversification of Minimum Variance Portfolios. in Jurczenko, E. (Ed.), *Risk-based and Factor Investing*, Elsevier.

Roncalli, T. (2013). *Introduction to Risk Parity and Budgeting*. Chapman & Hall/CRC Financial Mathematics Series.

Roncalli, T. (2017). Alternative Risk Premia: What Do We Know?. In Jurczenko, E. (Ed.), *Factor Investing and Alternative Risk Premia*, Elsevier.

Roncalli, T. (2018). Keep Up The Momentum. *Journal of Asset Management*, 19(5), pp. 351-361.

Roncalli, T. (2020). *Handbook of Financial Risk Management*. Chapman and Hall/CRC Financial Mathematics Series.

SCRUGGS, J. T., and GLABADANIDIS, P. (2003). Risk Premia and the Dynamic Covariance between Stock and Bond Returns. *Journal of Financial and Quantitative Analysis*, 38(2), pp. 295-316.

SHAHZAD, S. J. H., RAZA, N., SHAHBAZ, M., and ALI, A. (2017). Dependence of Stock Markets with Gold and Bonds under Bullish and Bearish Market States. *Resources Policy*, 52, pp. 308-319.

SHILLER, R. J. (1982). Consumption, Asset Markets and Macroeconomic Fluctuations. *Carnegie-Rochester Conference Series on Public Policy*, 17, pp. 203-238.

SHILLER, R. J., and BELTRATTI, A. E. (1992). Stock Prices and Bond Yields: Can their Comovements be Explained in terms of Present Value Models?. *Journal of Monetary Economics*, 30(1), pp. 25-46.

SONG, D. (2017). Bond Market Exposures to Macroeconomic and Monetary Policy Risks. *Review of Financial Studies*, 30(8), pp. 2761-2817.

STAGNOL, L., LOPEZ, C., RONCALLI, T., and TAILLARDAT, B. (2021). Understanding the Performance of the Equity Value Factor. *SSRN*, 3813572.

WU, C. C., and LIN, Z. Y. (2014). An Economic Evaluation of Stock-Bond Return Comovements with Copula-based GARCH models. *Quantitative Finance*, 14(7), pp. 1283-1296.

YANG, J., ZHOU, Y., and WANG, Z. (2009). The Stock-Bond Correlation and Macroeconomic Conditions: One and a half Centuries of Evidence. *Journal of Banking & Finance*, 33(4), pp. 670-680.

A Mathematical results

A.1 First derivative of implied risk premium with respect to correlation

A.1.1 The case $\rho_{i,j}$ with $j \neq i$

We have:

$$\begin{aligned}\frac{\partial \tilde{\pi}_i}{\partial \rho_{i,j}} &= \mathrm{SR}\left(x \mid r\right) \frac{\partial^2 \sigma\left(x\right)}{\partial x_i \partial \rho_{i,j}} \\ &= \mathrm{SR}\left(x \mid r\right) \frac{\partial}{\partial \rho_{i,j}} \left(\frac{x_i \sigma_i^2 + \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j}{\sigma\left(x\right)}\right)\end{aligned}$$

and:

$$\frac{\partial}{\partial \rho_{i,j}} \left(\frac{x_i \sigma_i^2 + \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j}{\sigma\left(x\right)}\right) = \frac{x_j \sigma_i \sigma_j \sigma\left(x\right) - \left(x_i \sigma_i^2 + \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j\right) \dfrac{\partial \sigma\left(x\right)}{\partial \rho_{i,j}}}{\sigma^2\left(x\right)}$$

It follows that:

$$\frac{\partial \sigma\left(x\right)}{\partial \rho_{i,j}} = \frac{\partial \sqrt{x^\top \Sigma x}}{\partial \rho_{i,j}} = \frac{1}{2\sigma\left(x\right)} \frac{\partial \left(x^\top \Sigma x\right)}{\partial \rho_{i,j}} = \frac{x_i x_j \sigma_i \sigma_j}{\sigma\left(x\right)}$$

because:

$$\frac{\partial \left(x^\top \Sigma x\right)}{\partial \rho_{i,j}} = \frac{\partial \left(\sum_{i=1}^n \sum_{j=1}^n x_i x_j \rho_{i,j} \sigma_i \sigma_j\right)}{\partial \rho_{i,j}} = 2 x_i x_j \sigma_i \sigma_j$$

Therefore, we get:

$$\begin{aligned}\frac{\partial^2 \sigma\left(x\right)}{\partial x_i \partial \rho_{i,j}} &= \frac{x_j \sigma_i \sigma_j \sigma\left(x\right) - \left(x_i \sigma_i^2 + \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j\right) \dfrac{x_i x_j \sigma_i \sigma_j}{\sigma\left(x\right)}}{\sigma^2\left(x\right)} \\ &= \left(\frac{1}{\sigma\left(x\right)} - \frac{x_i \left(\Sigma x\right)_i}{\sigma^3\left(x\right)}\right) x_j \sigma_i \sigma_j\end{aligned}$$

Finally, we conclude that:

$$\frac{\partial \tilde{\pi}_i}{\partial \rho_{i,j}} = \mathrm{SR}\left(x \mid r\right) \left(\frac{1}{\sigma\left(x\right)} - \frac{x_i \left(\Sigma x\right)_i}{\sigma^3\left(x\right)}\right) x_j \sigma_i \sigma_j$$

A.1.2 The case $\rho_{j,k}$ with $j \neq i \wedge k \neq i$

We have:

$$\begin{aligned}\frac{\partial \tilde{\pi}_i}{\partial \rho_{j,k}} &= \mathrm{SR}\left(x \mid r\right) \frac{\partial^2 \sigma\left(x\right)}{\partial x_i \partial \rho_{j,k}} \\ &= \mathrm{SR}\left(x \mid r\right) \frac{\partial}{\partial \rho_{j,k}} \left(\frac{x_i \sigma_i^2 + \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j}{\sigma\left(x\right)}\right)\end{aligned}$$

and:

$$\frac{\partial}{\partial \rho_{j,k}} \left(\frac{x_i \sigma_i^2 + \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j}{\sigma\left(x\right)}\right) = \frac{0 - \left(x_i \sigma_i^2 + \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j\right) \dfrac{\partial \sigma\left(x\right)}{\partial \rho_{j,k}}}{\sigma^2\left(x\right)}$$

It follows that:

$$\frac{\partial \sigma(x)}{\partial \rho_{j,k}} = \frac{\partial \sqrt{x^\top \Sigma x}}{\partial \rho_{j,k}} = \frac{1}{2\sigma(x)} \frac{\partial \left(x^\top \Sigma x\right)}{\partial \rho_{j,k}} = \frac{x_j x_k \sigma_j \sigma_k}{\sigma(x)}$$

Therefore, we get:

$$\begin{aligned} \frac{\partial^2 \sigma(x)}{\partial x_i \partial \rho_{j,k}} &= -\frac{\left(x_i \sigma_i^2 + \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j\right)}{\sigma^2(x)} \frac{x_j x_k \sigma_j \sigma_k}{\sigma(x)} \\ &= -\frac{(\Sigma x)_i}{\sigma^3(x)} x_j x_k \sigma_j \sigma_k \end{aligned}$$

Finally, we conclude that:

$$\frac{\partial \tilde{\pi}_i}{\partial \rho_{j,k}} = -\operatorname{SR}\left(x \mid r\right) \frac{(\Sigma x)_i}{\sigma^3(x)} x_j x_k \sigma_j \sigma_k$$

A.2 Implied correlation

Let ρ be the current correlation matrix and $\tilde{\rho}$ the implied correlation matrix. By definition, we have $\tilde{\rho}_{i,k} = \rho_{i,k}$ for $k \neq j$. Let $\theta = \tilde{\rho}_{i,j}$ be the implied correlation to be computed. We have:

$$\sum_{k=1}^{n} x_k \tilde{\rho}_{i,k} \sigma_i \sigma_k = \sum_{k=1}^{n} x_k \rho_{i,k} \sigma_i \sigma_k + x_j \sigma_i \sigma_j \left(\theta - \rho_{i,j}\right)$$

and:

$$\begin{aligned} \sigma^2\left(x \mid \tilde{\rho}\right) &= \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} x_{k_1} x_{k_2} \tilde{\rho}_{k_1,k_2} \sigma_{k_1} \sigma_{k_2} \\ &= \sigma^2\left(x \mid \rho\right) + 2 x_i x_j \left(\theta - \rho_{i,j}\right) \sigma_i \sigma_j \end{aligned}$$

We deduce that:

$$\begin{aligned} & \operatorname{SR}\left(x \mid r\right) \frac{\partial \sigma\left(x \mid \tilde{\rho}\right)}{\partial x_i} = \pi_i^\star \\ \Leftrightarrow \quad & \operatorname{SR}\left(x \mid r\right) \left(\sum_{k=1}^{n} x_k \rho_{i,k} \sigma_i \sigma_k + x_j \sigma_i \sigma_j \left(\theta - \rho_{i,j}\right)\right) = \pi_i^\star \sqrt{\sigma^2\left(x \mid \rho\right) + 2 x_i x_j \left(\theta - \rho_{i,j}\right) \sigma_i \sigma_j} \\ \Leftrightarrow \quad & \operatorname{SR}\left(x \mid r\right) (A + B\delta) = \pi_i^\star \sqrt{\sigma^2\left(x \mid \rho\right) + 2 x_i B \delta} \end{aligned}$$

where $\delta = \theta - \rho_{i,j}$, $A = \sum_{k=1}^{n} x_k \rho_{i,k} \sigma_i \sigma_k$ and $B = x_j \sigma_i \sigma_j$. We obtain a quadratic equation:

$$\begin{aligned} (*) \quad \Leftrightarrow \quad & \operatorname{SR}^2\left(x \mid r\right) (A + B\delta)^2 - 2 x_i \pi_i^{\star^2} B \delta - \pi_i^{\star^2} \sigma^2\left(x \mid \rho\right) = 0 \\ \Leftrightarrow \quad & a\delta^2 + 2b\delta + c = 0 \end{aligned}$$

where:

$$\left\{ \begin{array}{l} a = \operatorname{SR}^2\left(x \mid r\right) B^2 \geq 0 \\ b = \left(\operatorname{SR}^2\left(x \mid r\right) A - x_i \pi_i^{\star^2}\right) B \\ c = \operatorname{SR}^2\left(x \mid r\right) A^2 - \pi_i^{\star^2} \sigma^2\left(x \mid \rho\right) \end{array} \right.$$

If there is a solution, it is given by:

$$\delta^\star = \frac{-b \pm \sqrt{b^2 - ac}}{a}$$

We deduce that:

$$\tilde{\rho}_{i,j} = \rho_{i,j} + \delta^\star = \rho_{i,j} + \frac{-b \pm \sqrt{b^2 - ac}}{a}$$

The existence of a solution implies that:

1. $b^2 - ac \geq 0 \Leftrightarrow x_i^2 \pi_i^{\star^4} + \mathrm{SR}^2 \left(x \mid r\right) \pi_i^{\star^2} \left(\sigma^2 \left(x \mid \rho\right) - 2x_i A\right) \geq 0$;
2. $\tilde{\rho}_{i,j} \in [-1, 1] \Leftrightarrow -1 \leq \rho_{i,j} + \dfrac{-b \pm \sqrt{b^2 - ac}}{a} \leq 1$;
3. $\tilde{\rho}$ is a correlation matrix.

A.3 First derivative of implied risk premium with respect to bond volatility

We have:

$$\tilde{\pi}_B = \frac{\mathrm{SR}\left(x \mid r\right)}{\sigma\left(x\right)} \left(x_B \sigma_B^2 + x_S \rho \sigma_S \sigma_B\right)$$

and:

$$\sigma^2\left(x\right) = x_S^2 \sigma_S^2 + x_B^2 \sigma_B^2 + 2 x_S x_B \rho \sigma_S \sigma_B$$

Let us first compute the derivative of $\sigma\left(x\right)$:

$$\frac{\partial \sigma\left(x\right)}{\partial \sigma_B} = \frac{x_B^2 \sigma_B + x_S x_B \rho \sigma_S}{\sigma\left(x\right)}$$

We thus get:

$$\frac{\partial \tilde{\pi}_B}{\partial \sigma_B} = \frac{\mathrm{SR}\left(x \mid r\right)}{\sigma^3\left(x\right)} \left(A - B\right)$$

where:

$$\begin{aligned}
A &= \left(2 x_B \sigma_B + x_S \rho \sigma_S\right) \sigma^2\left(x\right) \\
&= \left(2 x_B \sigma_B + x_S \rho \sigma_S\right) \left(x_S^2 \sigma_S^2 + x_B^2 \sigma_B^2 + 2 x_S x_B \rho \sigma_S \sigma_B\right) \\
&= 2 x_S^2 x_B \sigma_S^2 \sigma_B + 2 x_B^3 \sigma_B^3 + 4 x_S x_B^2 \rho \sigma_S \sigma_B^2 + \\
&\quad\; x_S^3 \rho \sigma_S^3 + x_S x_B^2 \rho \sigma_S \sigma_B^2 + 2 x_S^2 x_B \rho^2 \sigma_S^2 \sigma_B
\end{aligned}$$

and:

$$\begin{aligned}
B &= \left(x_B \sigma_B^2 + x_S \rho \sigma_S \sigma_B\right) \left(x_B^2 \sigma_B + x_S x_B \rho \sigma_S\right) \\
&= x_B^3 \sigma_B^3 + 2 x_S x_B^2 \rho \sigma_S \sigma_B^2 + x_S^2 x_B \rho^2 \sigma_S^2 \sigma_B
\end{aligned}$$

We deduce that:

$$\begin{aligned}
A - B &= 2 x_S^2 x_B \sigma_S^2 \sigma_B + 2 x_B^3 \sigma_B^3 + 4 x_S x_B^2 \rho \sigma_S \sigma_B^2 + x_S^3 \rho \sigma_S^3 + x_S x_B^2 \rho \sigma_S \sigma_B^2 + \\
&\quad\; 2 x_S^2 x_B \rho^2 \sigma_S^2 \sigma_B - x_B^3 \sigma_B^3 - 2 x_S x_B^2 \rho \sigma_S \sigma_B^2 - x_S^2 x_B \rho^2 \sigma_S^2 \sigma_B \\
&= x_S^2 x_B \left(2 + \rho^2\right) \sigma_S^2 \sigma_B + x_B^3 \sigma_B^3 + x_S^3 \rho \sigma_S^3 + 3 x_S x_B^2 \rho \sigma_S \sigma_B^2
\end{aligned}$$

Finally, we conclude that:

$$\frac{\partial \tilde{\pi}_B}{\partial \sigma_B} = \frac{\text{SR}\left(x \mid r\right)}{\sigma^3\left(x\right)} \left(x_S^2 x_B \left(2+\rho^2\right)\sigma_S^2\sigma_B + x_B^3\sigma_B^3 + x_S^3\rho\sigma_S^3 + 3x_S x_B^2 \rho\sigma_S\sigma_B^2\right)$$

A.4 Relationship between the conditional normal distribution and the linear regression

Consider a Gaussian random vector defined as follows:

$$\left(\begin{array}{c} X \\ Y \end{array}\right) \sim \mathcal{N}\left(\left(\begin{array}{c} \mu_x \\ \mu_y \end{array}\right), \left(\begin{array}{cc} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{array}\right)\right)$$

The conditional distribution of Y given $X = x$ is a multivariate normal distribution:

$$Y \mid X = x \sim \mathcal{N}\left(\mu_{y|x}, \Sigma_{yy|x}\right)$$

where:

$$\mu_{y|x} = \mathbb{E}\left[Y \mid X = x\right] = \mu_y + \Sigma_{yx}\Sigma_{xx}^{-1}\left(x - \mu_x\right)$$

and:

$$\Sigma_{yy|x} = \sigma^2\left[Y \mid X = x\right] = \Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy}$$

It follows that $Y = \mu_{y|x} + U$ where U is a centered Gaussian random variable with variance $s^2 = \Sigma_{yy|x}$. We recognize the linear regression of Y on X:

$$\begin{array}{rcl} Y & = & \mu_y + \Sigma_{yx}\Sigma_{xx}^{-1}\left(x - \mu_x\right) + U \\ & = & \left(\mu_y - \Sigma_{yx}\Sigma_{xx}^{-1}\mu_x\right) + \Sigma_{yx}\Sigma_{xx}^{-1}x + U \\ & = & \alpha + \beta^\top x + U \end{array}$$

where $\alpha = \mu_y - \Sigma_{yx}\Sigma_{xx}^{-1}\mu_x$ and $\beta = \Sigma_{yx}\Sigma_{xx}^{-1}$. Moreover, we have:

$$\begin{array}{rcl} \mathfrak{R}^2 & = & 1 - \dfrac{\text{var}\left(U\right)}{\text{var}\left(Y\right)} \\ & = & 1 - \dfrac{s^2}{\Sigma_{yy}} \\ & = & \dfrac{\Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy}}{\Sigma_{yy}} \end{array}$$

A.5 Stock-bond correlation and aggregation effects

A.5.1 Covariance formula

We assume that the stock market return is the weighted average of a basket of stocks:

$$R_S\left(t\right) = \sum_{i=1}^{n} w_i\left(t\right) R_i\left(t\right)$$

where $R_i\left(t\right)$ is the return of stock i and $w_i\left(t\right)$ is the weight of asset i at time t. Here, asset i denotes an individual stock or a sector. We deduce that the stock-bond covariance at time

t is equal to the weighted average of the covariance values:

$$\begin{aligned}
\operatorname{cov}\left(R_S\left(t\right), R_B\left(t\right)\right) &= \mathbb{E}\left[\left(R_S\left(t\right)-\mathbb{E}\left[R_S\left(t\right)\right]\right)\left(R_B\left(t\right)-\mathbb{E}\left[R_B\left(t\right)\right]\right)\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^{n} w_i\left(t\right) R_i\left(t\right)-\mathbb{E}\left[\sum_{i=1}^{n} w_i\left(t\right) R_i\left(t\right)\right]\right)\left(R_B\left(t\right)-\mathbb{E}\left[R_B\left(t\right)\right]\right)\right] \\
&= \sum_{i=1}^{n} w_i\left(t\right) \mathbb{E}\left[\left(R_i\left(t\right)-\mathbb{E}\left[R_i\left(t\right)\right]\right)\left(R_B\left(t\right)-\mathbb{E}\left[R_B\left(t\right)\right]\right)\right] \\
&= \sum_{i=1}^{n} w_i\left(t\right) \operatorname{cov}\left(R_i\left(t\right), R_B\left(t\right)\right) \\
&= \sum_{i=1}^{n} w_i\left(t\right) \rho_{i,B}\left(t\right) \sigma_i\left(t\right) \sigma_B\left(t\right)
\end{aligned}$$

where $\operatorname{cov}\left(R_i\left(t\right), R_B\left(t\right)\right)$ is the covariance between asset i and bond returns. We use the notations $\sigma_i\left(t\right)=\sigma\left(R_i\left(t\right)\right)$, $\sigma_B\left(t\right)=\sigma\left(R_B\left(t\right)\right)$ and $\rho_{i,B}\left(t\right)=\rho\left(R_i\left(t\right), R_B\left(t\right)\right)$. We deduce that:

$$\begin{aligned}
\rho_{S,B}\left(t\right) &= \frac{\sum_{i=1}^{n} w_i\left(t\right) \rho_{i,B}\left(t\right) \sigma_i\left(t\right) \sigma_B\left(t\right)}{\sigma_S\left(t\right) \sigma_B\left(t\right)} \\
&= \sum_{i=1}^{n}\left(\frac{w_i\left(t\right) \sigma_i\left(t\right)}{\sigma_S\left(t\right)}\right) \rho_{i,B}\left(t\right) \\
&= \sum_{i=1}^{n} \omega_i\left(t\right) \rho_{i,B}\left(t\right)
\end{aligned}$$

where $\omega_i\left(t\right)$ depends on the volatility ratio $\gamma_i\left(t\right)$:

$$\omega_i\left(t\right)=\frac{w_i\left(t\right) \sigma_i\left(t\right)}{\sigma_S\left(t\right)}=w_i\left(t\right) \gamma_i\left(t\right)$$

A.5.2 Variance formula

We have:

$$\begin{aligned}
\sigma_S^2\left(t\right) &= \sigma^2\left(\sum_{i=1}^{n} w_i\left(t\right) R_i\left(t\right)\right) \\
&= \sum_{i=1}^{n} w_i^2\left(t\right) \sigma_i^2\left(t\right)+2 \sum_{i>j} w_i\left(t\right) w_j\left(t\right) \rho_{i,j}\left(t\right) \sigma_i\left(t\right) \sigma_j\left(t\right)
\end{aligned}$$

where $\rho_{i,j}\left(t\right)=\rho\left(R_i\left(t\right), R_j\left(t\right)\right)$ is the cross-correlation between assets i and j. If $\rho_{i,j}\left(t\right)=0$, we get:

$$\sigma_S\left(t\right)=\sqrt{\sum_{i=1}^{n} w_i^2\left(t\right) \sigma_i^2\left(t\right)}$$

When $\rho_{i,j}(t) = 1$, the standard deviation becomes:

$$\begin{aligned}
\sigma_S(t) &= \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i(t)\, w_j(t)\, \sigma_i(t)\, \sigma_j(t)} \\
&= \sqrt{\sum_{i=1}^{n} w_i(t)\, \sigma_i(t) \sum_{j=1}^{n} w_j(t)\, \sigma_j(t)} \\
&= \sum_{i=1}^{n} w_i(t)\, \sigma_i(t)
\end{aligned}$$

In the other cases, we can show that the previous expression is an upper bound:

$$\sigma_S(t) \leq \sum_{i=1}^{n} w_i(t)\, \sigma_i(t)$$

A.5.3 Weight formula

By construction, we have $\sum_{i=1}^{n} w_i(t) = 1$. If $\rho_{i,j}(t) = 0$, we get:

$$\omega_i(t) = \frac{w_i(t)\, \sigma_i(t)}{\sigma_S(t)} = \frac{\sigma_i(t)}{\sqrt{\sum_{j=1}^{n} w_j^2(t)\, \sigma_j^2(t)}} w_i(t)$$

In particular, in the homogeneous case ($\sigma_i(t) = \sigma_j(t)$), we have:

$$\omega_i(t) = \frac{w_i(t)}{\sqrt{\mathcal{H}_S(t)}}$$

where $\mathcal{H}_S(t) = \sum_{i=1}^{n} w_i^2(t)$ is the Herfindahl index of the stock market at time t. We deduce that:

$$\sum_{i=1}^{n} \omega_i(t) = \sum_{i=1}^{n} \frac{w_i(t)}{\sqrt{\mathcal{H}_S(t)}} = \frac{1}{\sqrt{\mathcal{H}_S(t)}} \geq 1$$

When $\rho_{i,j}(t) = 1$, the correlation weight $\omega_i(t)$ becomes:

$$\omega_i(t) = \frac{w_i(t)\, \sigma_i(t)}{\sum_{i=1}^{n} w_j(t)\, \sigma_j(t)} \neq w_i(t)$$

and we check that the sum of the correlation weights is equal to one:

$$\sum_{i=1}^{n} \omega_i(t) = \frac{\sum_{i=1}^{n} w_i(t)\, \sigma_i(t)}{\sum_{i=1}^{n} w_j(t)\, \sigma_j(t)} = 1$$

In the other cases we have:

$$\begin{aligned}
\sum_{i=1}^{n} \omega_i(t) &= \frac{\sum_{i=1}^{n} w_i(t)\, \sigma_i(t)}{\sigma_S(t)} \\
&= \mathcal{DR}\left(w(t)\right) \\
&\geq 1
\end{aligned}$$

where $\mathcal{DR}(w)$ is the diversification ratio introduced by Choueifaty and Coignard (2008).

A.6 Maximum diversification ratio with constant correlation matrix

In the case of a constant correlation matrix $\rho_{i,j} = \rho$, we have $w^\top \sigma = \sum_{i=1}^n w_i \sigma_i$ and:

$$\begin{aligned} w^\top \Sigma w &= \sum_{i=1}^n \sum_{j=1}^n w_i w_j \rho_{i,j} \sigma_i \sigma_j \\ &= \rho \left(\sum_{i=1}^n w_i \sigma_i \right)^2 + (1-\rho) \left(\sum_{i=1}^n w_i^2 \sigma_i^2 \right) \end{aligned}$$

We deduce that the expression of the diversification ratio is equal to:

$$\begin{aligned} \mathcal{DR}(w) &= \frac{w^\top \sigma}{\sqrt{w^\top \Sigma w}} \\ &= \frac{\sum_{i=1}^n w_i \sigma_i}{\sqrt{\rho \left(\sum_{i=1}^n w_i \sigma_i \right)^2 + (1-\rho) \left(\sum_{i=1}^n w_i^2 \sigma_i^2 \right)}} \\ &= \frac{1}{\sqrt{\rho + (1-\rho) \varphi^2(w)}} \end{aligned}$$

where $\varphi(w)$ is the coefficient of variation of volatilities using the weighting scheme w:

$$\varphi(w) = \frac{\sqrt{\sum_{i=1}^n w_i^2 \sigma_i^2}}{\sum_{i=1}^n w_i \sigma_i}$$

Since the most diversified portfolio is equal to the ERC portfolio when the correlation matrix is constant (Demey *et al.*, 2010; Roncalli, 2013), we have:

$$w_i^{\mathrm{mdp}} = \frac{\sigma_i^{-1}}{\sum_{i=1}^n \sigma_i^{-1}}$$

We deduce that:

$$\varphi\left(w^{\mathrm{mdp}}\right) = \frac{1}{\sqrt{n}}$$

We conclude that:

$$\mathcal{DR}\left(w^{\mathrm{mdp}}\right) = \frac{1}{\sqrt{\rho + \dfrac{(1-\rho)}{n}}}$$

A.7 Aggregation with time-varying equity baskets

We have seen that the stock-bond correlation formula is:

$$\rho_{S,B}(t) = \sum_{i=1}^n \frac{w_i(t)\, \sigma_i(t)}{\sigma_S(t)} \rho_{i,B}(t)$$

In a capitalization-weighted index, the weights are given by:

$$w_i(t) = \frac{N_i P_i(t)}{\sum_{j=1}^n N_j P_j(t)}$$

where N_i is the number of shares outstanding for the i^{th} stock and $P_i(t)$ is the price at time t. The stock and index returns $R_i(t+1)$ and $R_S(t+1)$ between t and $t+1$ satisfy the following equations:

$$\left\{\begin{array}{l} P_i(t+1) = \big(1 + R_i(t+1)\big) P_i(t) \\ \sum_{j=1}^n N_j P_j(t+1) = \big(1 + R_S(t+1)\big) \sum_{j=1}^n N_j P_j(t) \end{array}\right.$$

It follows that the weights at time $t+1$ becomes:

$$\begin{aligned} w_i(t+1) &= \frac{N_i P_i(t+1)}{\sum_{j=1}^n N_j P_j(t+1)} \\ &= \frac{N_i P_i(t+1)}{N_i P_i(t)} \cdot \frac{N_i P_i(t)}{\sum_{j=1}^n N_j P_j(t)} \cdot \frac{\sum_{j=1}^n N_j P_j(t)}{\sum_{j=1}^n N_j P_j(t+1)} \\ &= \frac{1 + R_i(t+1)}{1 + R_S(t+1)} w_i(t) \end{aligned}$$

We deduce that:

$$\begin{aligned} \rho_{S,B}(t+1) &= \sum_{i=1}^n \frac{w_i(t+1)\,\sigma_i(t+1)}{\sigma_S(t+1)} \rho_{i,B}(t+1) \\ &= \sum_{i=1}^n \left(\frac{1 + R_i(t+1)}{1 + R_S(t+1)}\right) \left(\frac{\sigma_i(t+1)}{\sigma_S(t+1)}\right) w_i(t)\,\rho_{i,B}(t+1) \end{aligned}$$

and:

$$\rho_{S,B}(t+1) - \rho_{S,B}(t) = \sum_{i=1}^n \frac{w_i(t)\,\sigma_i(t)}{\sigma_S(t)} \big(\xi_i(t+1)\,\rho_{i,B}(t+1) - \rho_{i,B}(t)\big)$$

where:

$$\xi_i(t+1) = \frac{\dfrac{\big(1 + R_i(t+1)\big)\,\sigma_i(t+1)}{\sigma_i(t)}}{\dfrac{\big(1 + R_S(t+1)\big)\,\sigma_S(t+1)}{\sigma_S(t)}}$$

A.8 The inflation-centric model of Li (2002)

A.8.1 The economy

The real interest rate $r(t)$, the inflation rate $\pi(t)$ and the dividend yield $\delta(t)$ follow affine mean-reverting processes:

$$\left\{\begin{array}{l} r(t+1) = \bar{r} + \varrho_r\big(r(t) - \bar{r}\big) + \varepsilon_r(t+1) \\ \pi(t+1) = \bar{\pi} + \varrho_\pi\big(\pi(t) - \bar{\pi}\big) + \varepsilon_\pi(t+1) \\ \delta(t+1) = \bar{\delta} + \varrho_\delta\big(\delta(t) - \bar{\delta}\big) + \varepsilon_\delta(t+1) \end{array}\right.$$

where $\bar{r}$, $\bar{\pi}$ and $\bar{\delta}$ are the long-run equilibria, ϱ_r, ϱ_π and ϱ_δ are the adjustment velocities, and $\varepsilon_r(t+1)$, $\varepsilon_\pi(t+1)$ and $\varepsilon_\delta(t+1)$ are the innovation shocks distributed according to $\mathcal{N}\left(0, \sigma_r^2\right)$, $\mathcal{N}\left(0, \sigma_\pi^2\right)$, and $\mathcal{N}\left(0, \sigma_\delta^2\right)$. We note $\rho_{r,\pi}$, $\rho_{r,\delta}$ and $\rho_{\pi,\delta}$ the correlations between the innovation shocks. The logarithm of the real pricing kernel of the economy $M(t+1)$ follows the standard process:

$$\begin{aligned} m(t+1) &= \ln M(t+1) \\ &= -r(t) - \mu_m + \varepsilon_m(t+1) \end{aligned}$$

where $\varepsilon_m(t+1) \sim \mathcal{N}\left(0, \sigma_m^2\right)$ and $\varepsilon_m(t)$ is a linear function of the innovation shocks:

$$\begin{aligned}
\varepsilon_m(t+1) &= \sum_{x=r,\pi,\delta} \phi_x \varepsilon_x(t+1) \\
&= \phi_r \varepsilon_r(t+1) + \phi_\pi \varepsilon_\pi(t+1) + \phi_\delta \varepsilon_\delta(t+1) \\
&= \phi^\top \varepsilon(t+1)
\end{aligned}$$

ϕ_r, ϕ_π and ϕ_δ are the weights of the innovation processes to form the kernel innovation. We have $\phi = (\phi_r, \phi_\pi, \phi_\delta)$ and $\varepsilon(t+1) = \left(\varepsilon_r(t+1), \varepsilon_\pi(t+1), \varepsilon_\delta(t+1)\right)$. The non-arbitrage condition implies that the real interest rate is the return of the one-period real bond:

$$\begin{aligned}
r(t) &= -\ln \mathbb{E}_t\left[M(t+1)\right] \\
&= r(t) + \mu_m - \frac{1}{2}\operatorname{var}\left(\varepsilon_m(t+1)\right)
\end{aligned}$$

We deduce that:

$$\begin{aligned}
\mu_m &:= \frac{1}{2}\sigma_m^2 \\
&= \frac{1}{2}\sum_{x=r,\pi,\delta}\sum_{y=r,\pi,\delta} \phi_x \phi_y \rho_{x,y} \sigma_x \sigma_y \\
&= \frac{1}{2}\phi^\top \Sigma \phi
\end{aligned}$$

where Σ is the covariance matrix of $\varepsilon(t+1)$. Finally, we get:

$$m(t+1) = -r(t) - \mu_m + \varepsilon_m(t+1)$$

The pricing kernel is then distributed as a log-normal distribution:

$$M(t+1) = e^{m(t+1)} \sim \mathcal{LN}\left(-\left(r(t) + \mu_m\right), \sigma_m^2\right)$$

A.8.2 Asset returns

Preliminary results We have:

$$\begin{aligned}
\vartheta(t+1) &= m(t+1) + c + a_r r(t+1) + a_\pi \pi(t+1) + a_\delta \delta(t+1) \\
&= -r(t) - \mu_m + \varepsilon_m(t+1) + c + \\
&\quad a_r\left(\left(\bar{r} + \varrho_r\left(r(t) - \bar{r}\right) + \varepsilon_r(t+1)\right)\right) + \\
&\quad a_\pi\left(\bar{\pi} + \varrho_\pi\left(\pi(t) - \bar{\pi}\right) + \varepsilon_\pi(t+1)\right) + \\
&\quad a_\delta\left(\bar{\delta} + \varrho_\delta\left(\delta(t) - \bar{\delta}\right) + \varepsilon_\delta(t+1)\right) \\
&= -\mu_m + c + \\
&\quad (a_r\varrho_r - 1) r(t) + a_\pi \varrho_\pi \pi(t) + a_\delta \varrho_\delta \delta(t) + \\
&\quad a_r(1-\varrho_r)\bar{r} + a_\pi(1-\varrho_\pi)\bar{\pi} + a_\delta(1-\varrho_\delta)\bar{\delta} + \\
&\quad (\phi_r + a_r)\varepsilon_r(t+1) + (\phi_\pi + a_\pi)\varepsilon_\pi(t+1) + (\phi_\delta + a_\delta)\varepsilon_\delta(t+1)
\end{aligned}$$

We deduce that $\vartheta(t+1)$ is Gaussian with:

$$\begin{aligned}
\mathbb{E}_t\left[\vartheta(t+1)\right] &= -\mu_m + c + \\
&\quad (a_r\varrho_r - 1) r(t) + a_\pi \varrho_\pi \pi(t) + a_\delta \varrho_\delta \delta(t) + \\
&\quad a_r(1-\varrho_r)\bar{r} + a_\pi(1-\varrho_\pi)\bar{\pi} + a_\delta(1-\varrho_\delta)\bar{\delta}
\end{aligned}$$

and:

$$\begin{aligned}\operatorname{var}_t\left(\vartheta\left(t+1\right)\right) &= \sum_{x=r,\pi,\delta}\sum_{y=r,\pi,\delta}\left(\phi_x+a_x\right)\left(\phi_y+a_y\right)\rho_{x,y}\sigma_x\sigma_y \\ &= \left(\phi+a\right)^\top\Sigma\left(\phi+a\right)\end{aligned}$$

where $a=\left(a_r,a_\pi,a_\delta\right)$. We note:

$$\mu_\vartheta=\frac{1}{2}\left(\phi+a\right)^\top\Sigma\left(\phi+a\right) \tag{13}$$

It follows that:

$$\begin{aligned}\mathbb{E}_t\left[e^{\vartheta(t+1)}\right] &= \exp\left(\mathbb{E}_t\left[\vartheta\left(t+1\right)\right]+\frac{1}{2}\operatorname{var}_t\left(\vartheta\left(t+1\right)\right)\right) \\ &= \exp\left(\mu_\vartheta-\mu_m+c+a_r\left(1-\varrho_r\right)\bar{r}+a_\pi\left(1-\varrho_\pi\right)\bar{\pi}+a_\delta\left(1-\varrho_\delta\right)\bar{\delta}\right)\cdot \\ &\quad \exp\left(\left(a_r\varrho_r-1\right)r\left(t\right)\right)\cdot\exp\left(a_\pi\varrho_\pi\pi\left(t\right)\right)\cdot\exp\left(a_\delta\varrho_\delta\delta\left(t\right)\right)\end{aligned} \tag{14}$$

In the sequel, we will also use the following identities[36]:

$$\left\{\begin{array}{l}\alpha'_r r\left(t+1\right)-\alpha_r r\left(t\right)=\alpha'_r\left(1-\varrho_r\right)\bar{r}+\left(\alpha'_r\varrho_r-\alpha_r\right)r\left(t\right)+\alpha'_r\varepsilon_r\left(t+1\right)\\ \alpha'_\pi\pi\left(t+1\right)-\alpha_\pi\pi\left(t\right)=\alpha'_\pi\left(1-\varrho_\pi\right)\bar{\pi}+\left(\alpha'_\pi\varrho_\pi-\alpha_\pi\right)\pi\left(t\right)+\alpha'_\pi\varepsilon_\pi\left(t+1\right)\\ \alpha'_\delta\delta\left(t+1\right)-\alpha_\delta\delta\left(t\right)=\alpha'_\delta\left(1-\varrho_\delta\right)\bar{r}+\left(\alpha'_\delta\varrho_\delta-\alpha_\delta\right)\delta\left(t\right)+\alpha'_\delta\varepsilon_\delta\left(t+1\right)\end{array}\right. \tag{15}$$

Bond return We consider a nominal bond, whose remaining maturity is n period. We assume that its price $B\left(t\right)$ has an exponential form:

$$B^n\left(t\right)=\exp\left(\alpha_0^n+\alpha_r^n r\left(t\right)+\alpha_\pi^n\pi\left(t\right)\right)$$

Li (2002) showed that the log pricing kernel of this nominal bond is $m\left(t+1\right)-\pi\left(t+1\right)$, implying that:

$$B^n\left(t\right)=\mathbb{E}_t\left[e^{m(t+1)-\pi(t+1)}B^{n-1}\left(t+1\right)\right]$$

We have:

$$B^n\left(t\right)=\mathbb{E}_t\left[\exp\left(m\left(t+1\right)-\pi\left(t+1\right)+\alpha_0^{n-1}+\alpha_r^{n-1}r\left(t+1\right)+\alpha_\pi^{n-1}\pi\left(t+1\right)\right)\right]$$

Using the results (14) on page 101 with $c=\alpha_0^{n-1}$, $a_r=\alpha_r^{n-1}$, $a_\pi=\alpha_\pi^{n-1}-1$ and $a_\delta=0$, we deduce that:

$$\begin{aligned}B^n\left(t\right) &= \exp\left(\mu_\vartheta-\mu_m+\alpha_0^{n-1}+\alpha_r^{n-1}\left(1-\varrho_r\right)\bar{r}+\left(\alpha_\pi^{n-1}-1\right)\left(1-\varrho_\pi\right)\bar{\pi}\right)\cdot \\ &\quad \exp\left(\left(\alpha_r^{n-1}\varrho_r-1\right)r\left(t\right)\right)\cdot\exp\left(\left(\alpha_\pi^{n-1}-1\right)\varrho_\pi\pi\left(t\right)\right)\end{aligned}$$

Li (2002) obtained the following recursive relationships:

$$\left\{\begin{array}{l}\alpha_0^n=\mu_\vartheta-\mu_m+\alpha_0^{n-1}+\alpha_r^{n-1}\left(1-\varrho_r\right)\bar{r}+\left(\alpha_\pi^{n-1}-1\right)\left(1-\varrho_\pi\right)\bar{\pi}\\ \alpha_r^n=\alpha_r^{n-1}\varrho_r-1\\ \alpha_\pi^n=\left(\alpha_\pi^{n-1}-1\right)\varrho_\pi\end{array}\right.$$

[36] Because we have:

$$\alpha'_r r\left(t+1\right)-\alpha_r r\left(t\right)=\alpha'_r\left(\bar{r}+\varrho_r\left(r\left(t\right)-\bar{r}\right)+\varepsilon_r\left(t+1\right)\right)-\alpha_r r\left(t\right)$$

We deduce that:

$$\alpha_r^n = -\frac{1-\varrho_r^n}{1-\varrho_r}$$

and:

$$\alpha_\pi^n = -\frac{1-\varrho_\pi^n}{1-\varrho_\pi}\varrho_\pi$$

The bond return is defined as:

$$\begin{aligned} R_B\left(t+1\right) &= \ln B^{n-1}\left(t+1\right) - \ln B^n\left(t\right) \\ &= \left(\alpha_0^{n-1} - \alpha_0^n\right) + \left(\alpha_r^{n-1} r\left(t+1\right) - \alpha_r^n r\left(t\right)\right) + \left(\alpha_\pi^{n-1}\pi\left(t+1\right) - \alpha_\pi^n \pi\left(t\right)\right) \end{aligned}$$

Since we have:

$$\alpha_0^{n-1} - \alpha_0^n = \mu_m - \mu_\vartheta - \alpha_r^{n-1}\left(1-\varrho_r\right)\bar{r} - \left(\alpha_\pi^{n-1} - 1\right)\left(1-\varrho_\pi\right)\bar{\pi}$$

we deduce that:

$$\begin{aligned} R_B\left(t+1\right) &= \mu_m - \mu_\vartheta - \alpha_r^{n-1}\left(1-\varrho_r\right)\bar{r} - \left(\alpha_\pi^{n-1} - 1\right)\left(1-\varrho_\pi\right)\bar{\pi} + \\ &\quad \alpha_r^{n-1}\left(1-\varrho_r\right)\bar{r} + \left(\alpha_r^{n-1}\varrho_r - \alpha_r^n\right) r\left(t\right) + \alpha_r^{n-1}\varepsilon_r\left(t+1\right) + \\ &\quad \alpha_\pi^{n-1}\left(1-\varrho_\pi\right)\bar{\pi} + \left(\alpha_\pi^{n-1}\varrho_\pi - \alpha_\pi^n\right)\pi\left(t\right) + \alpha_\pi^{n-1}\varepsilon_\pi\left(t+1\right) \\ &= \mu_m - \mu_\vartheta + r\left(t\right) + \left(1-\varrho_\pi\right)\bar{\pi} + \varrho_\pi \pi\left(t\right) + \\ &\quad \alpha_r^{n-1}\varepsilon_r\left(t+1\right) + \alpha_\pi^{n-1}\varepsilon_\pi\left(t+1\right) \\ &= r\left(t\right) + \hat{\pi}\left(t\right) + \mu_m - \mu_\vartheta + \varepsilon_B\left(t+1\right) \end{aligned}$$

where:

$$\hat{\pi}\left(t\right) = \mathbb{E}_t\left[\pi\left(t+1\right)\right] = \bar{\pi} + \varrho_\pi\left(\pi\left(t\right) - \bar{\pi}\right)$$

and:

$$\varepsilon_B\left(t+1\right) = \alpha_r^{n-1}\varepsilon_r\left(t+1\right) + \alpha_\pi^{n-1}\varepsilon_\pi\left(t+1\right)$$

The conditional distribution of $R_B\left(t+1\right)$ is then Gaussian:

$$R_B\left(t+1\right) \sim \mathcal{N}\left(\mu_B\left(t\right), \sigma_B^2\left(t\right)\right)$$

where $\mu_B\left(t\right) = r\left(t\right) + \hat{\pi}\left(t\right) + \mu_m - \mu_\vartheta$, $\sigma_B^2\left(t\right) = \alpha^\top \Sigma \alpha$ and $\alpha = \left(\alpha_r^{n-1}, \alpha_\pi^{n-1}, 0\right)$.

Stock return We assume that the stock pays a dividend until the period n. Li (2002) showed that the stock price $S\left(t\right)$ is equal to:

$$\begin{aligned} S\left(t\right) &= \mathbb{E}_t\left[M\left(t+1\right)\left(S\left(t+1\right) + D\left(t+1\right)\right)\right] \\ &= \mathbb{E}_t\left[M\left(t+1\right)\left(1+\delta\left(t+1\right)\right) S\left(t+1\right)\right] \end{aligned}$$

where $D\left(t\right)$ is the nominal dividend and $\delta\left(t\right)$ is the dividend yield. Let us assume that $S\left(t\right)$ has an exponential form:

$$S\left(t\right) = \exp\left(\alpha_0^n + \alpha_r^n r\left(t\right) + \alpha_\delta^n \delta\left(t\right)\right)$$

Since $\ln\left(1+\delta\left(t+1\right)\right) \approx \delta\left(t+1\right)$, we deduce that:

$$\begin{aligned} S\left(t\right) &= \mathbb{E}_t\left[\exp\left(\ln M\left(t+1\right)+\ln\left(1+\delta\left(t+1\right)\right)+\ln S\left(t+1\right)\right)\right] \\ &= \mathbb{E}_t\left[\exp\left(m\left(t+1\right)+\alpha_0^{n-1}+\alpha_r^{n-1}r\left(t+1\right)+\left(\alpha_\delta^{n-1}+1\right)\delta\left(t+1\right)\right)\right] \end{aligned}$$

Using the results (14) on page 101 with $c=\alpha_0^{n-1}$, $a_r=\alpha_r^{n-1}$, $a_\pi=0$ and $a_\delta=\alpha_\delta^{n-1}+1$, we deduce that:

$$\begin{aligned} S\left(t\right) &= \exp\left(\mu_\vartheta-\mu_m+\alpha_0^{n-1}+\alpha_r^{n-1}\left(1-\varrho_r\right)\bar{r}+\left(\alpha_\delta^{n-1}+1\right)\left(1-\varrho_\delta\right)\bar{\delta}\right)\cdot \\ & \quad \exp\left(\left(\alpha_r^{n-1}\varrho_r-1\right)r\left(t\right)\right)\cdot\exp\left(\left(\alpha_\delta^{n-1}+1\right)\varrho_\delta\delta\left(t\right)\right) \end{aligned}$$

Li (2002) obtained the following recursive relationships:

$$\left\{\begin{array}{l} \alpha_0^n=\mu_\vartheta-\mu_m+\alpha_0^{n-1}+\alpha_r^{n-1}\left(1-\varrho_r\right)\bar{r}+\left(\alpha_\delta^{n-1}+1\right)\left(1-\varrho_\delta\right)\bar{\delta} \\ \alpha_r^n=\alpha_r^{n-1}\varrho_r-1 \\ \alpha_\delta^n=\left(\alpha_\delta^{n-1}+1\right)\varrho_\delta \end{array}\right.$$

We deduce that:

$$\alpha_r=\lim_{n\to\infty}\alpha_r^n=-\frac{1}{1-\varrho_r}$$

and:

$$\alpha_\delta=\lim_{n\to\infty}\alpha_\delta^n=\frac{\varrho_\delta}{1-\varrho_\delta}$$

The stock return is defined as:

$$\begin{aligned} R_S\left(t+1\right) &= \ln S\left(t+1\right)-\ln S\left(t\right)+\delta\left(t+1\right)+\pi\left(t+1\right) \\ &= \left(\alpha_0^{n-1}-\alpha_0^n\right)+\alpha_r r\left(t+1\right)-\alpha_r r\left(t\right)+ \\ & \quad \pi\left(t+1\right)+\left(\alpha_\delta+1\right)\delta\left(t+1\right)-\alpha_\delta\delta\left(t\right) \end{aligned}$$

We have:

$$\begin{aligned} \alpha_r\left(r\left(t+1\right)-r\left(t\right)\right) &= \alpha_r\left(1-\varrho_r\right)\bar{r}+\left(\alpha_r\varrho_r-\alpha_r\right)r\left(t\right)+\alpha_r\varepsilon_r\left(t+1\right) \\ &= -\bar{r}+r\left(t\right)+\alpha_r\varepsilon_r\left(t+1\right) \end{aligned}$$

and:

$$\begin{aligned} \left(\alpha_\delta+1\right)\delta\left(t+1\right)-\alpha_\delta\delta\left(t\right) &= \left(\alpha_\delta+1\right)\left(\bar{\delta}+\varrho_\delta\left(\delta\left(t\right)-\bar{\delta}\right)+\varepsilon_\delta\left(t+1\right)\right)-\alpha_\delta\delta\left(t\right) \\ &= \bar{\delta}+\left(\alpha_\delta+1\right)\varepsilon_\delta\left(t+1\right) \end{aligned}$$

Since we have $\alpha_0^{n-1}-\alpha_0^n=\mu_m-\mu_\vartheta+\bar{r}-\bar{\delta}$, we deduce that:

$$R_S\left(t+1\right)=r\left(t\right)+\hat{\pi}\left(t\right)+\mu_m-\mu_\vartheta+\varepsilon_S\left(t+1\right)$$

where:

$$\varepsilon_S\left(t+1\right)=\alpha_r\varepsilon_r\left(t+1\right)+\varepsilon_\pi\left(t+1\right)+\left(\alpha_\delta+1\right)\varepsilon_\delta\left(t+1\right)$$

The conditional distribution of $R_S\left(t+1\right)$ is then Gaussian:

$$R_S\left(t+1\right)\sim\mathcal{N}\left(\mu_S\left(t\right),\sigma_S^2\left(t\right)\right)$$

where $\mu_S\left(t\right)=r\left(t\right)+\hat{\pi}\left(t\right)+\mu_m-\mu_\vartheta$, $\sigma_S^2\left(t\right)=\alpha^\top\Sigma\alpha$ and $\alpha=\left(\alpha_r,1,\alpha_\delta+1\right)$.

A.8.3 The stock-bond covariance

We note $x_r^B = -\dfrac{1-\varrho_r^n}{1-\varrho_r}$, $x_\pi^B = -\dfrac{1-\varrho_\pi^n}{1-\varrho_\pi}\varrho_\pi$, $x_\delta^B = 0$, $x_r^S = -\dfrac{1}{1-\varrho_r}$, $x_\pi^S = 1$ and $x_\delta^S = \dfrac{1}{1-\varrho_\delta}$. We deduce that the conditional stock-bond covariance is given by:

$$\begin{aligned}\operatorname{cov}\left(R_B(t+1), R_S(t+1)\right) &= \mathbb{E}\left[\varepsilon_B(t+1)\,\varepsilon_S(t+1)\right] \\ &= \mathbb{E}\left[\left(x_B^\top \varepsilon_B(t+1)\right)\left(x_S^\top \varepsilon_S(t+1)\right)\right] \\ &= x_B^\top \Sigma x_S \end{aligned} \tag{16}$$

where $x_B = \left(x_r^B, x_\pi^B, x_\delta^B\right)$ and $x_S = \left(x_r^S, x_\pi^S, x_\delta^S\right)$, while the stock-bond correlation is:

$$\rho_{S,B} = \frac{x_B^\top \Sigma x_S}{\sqrt{x_B^\top \Sigma x_B}\sqrt{x_S^\top \Sigma x_S}} \tag{17}$$

The calculation gives:

$$\begin{aligned}\operatorname{cov}\left(R_B(t), R_S(t)\right) &= x_r^B x_r^S \sigma_r^2 + x_\pi^B \sigma_\pi^2 + \\ &\left(x_r^B + x_\pi^B x_r^S\right)\rho_{r,\pi}\sigma_r\sigma_\pi + x_r^B x_\delta^S \rho_{r,\delta}\sigma_r\sigma_\delta + x_\pi^B x_\delta^S \rho_{\pi,\delta}\sigma_\pi\sigma_\delta\end{aligned}$$

The expression of the covariance is then equal to:

$$\begin{aligned}\operatorname{cov}\left(R_B(t), R_S(t)\right) &= \frac{\left(1-\varrho_r^n\right)}{\left(1-\varrho_r\right)^2}\sigma_r^2 - \frac{\left(1-\varrho_\pi^n\right)\varrho_\pi}{\left(1-\varrho_\pi\right)}\sigma_\pi^2 + \\ &\frac{\left(1-\varrho_\pi^n\right)\varrho_\pi - \left(1-\varrho_r^n\right)\left(1-\varrho_\pi\right)}{\left(1-\varrho_r\right)\left(1-\varrho_\pi\right)}\rho_{r,\pi}\sigma_r\sigma_\pi - \\ &\frac{\left(1-\varrho_r^n\right)}{\left(1-\varrho_r\right)\left(1-\varrho_\delta\right)}\rho_{r,\delta}\sigma_r\sigma_\delta - \frac{\left(1-\varrho_\pi^n\right)\varrho_\pi}{\left(1-\varrho_\pi\right)\left(1-\varrho_\delta\right)}\rho_{\pi,\delta}\sigma_\pi\sigma_\delta\end{aligned} \tag{18}$$

For the variance terms, we have:

$$\begin{aligned}\sigma_B^2(t) &= x_B^\top \Sigma x_B \\ &= \left(\frac{1-\varrho_r^n}{1-\varrho_r}\right)^2\sigma_r^2 + \left(\frac{1-\varrho_\pi^n}{1-\varrho_\pi}\varrho_\pi\right)^2\sigma_\pi^2 + 2\frac{\left(1-\varrho_r^n\right)\left(1-\varrho_\pi^n\right)\varrho_\pi}{\left(1-\varrho_r\right)\left(1-\varrho_\pi\right)}\rho_{r,\pi}\sigma_r\sigma_\pi\end{aligned} \tag{19}$$

and:

$$\begin{aligned}\sigma_S^2(t) &= x_S^\top \Sigma x_S \\ &= \left(\frac{1}{1-\varrho_r}\right)^2\sigma_r^2 + \sigma_\pi^2 + \left(\frac{\varrho_\delta}{1-\varrho_\delta}\right)^2\sigma_\delta^2 - \\ &2\left(\frac{1}{1-\varrho_r}\right)\rho_{r,\pi}\sigma_r\sigma_\pi - 2\frac{1}{\left(1-\varrho_r\right)\left(1-\varrho_\delta\right)}\rho_{r,\delta}\sigma_r\sigma_\delta + \\ &2\left(\frac{1}{1-\varrho_\delta}\right)\rho_{\pi,\delta}\sigma_\pi\sigma_\delta\end{aligned} \tag{20}$$

A.9 Calculating the stock-bond correlation in the growth-inflation model

In the growth-inflation model, the dynamics of asset returns are given by a two-factor model:

$$\left\{\begin{array}{l} R_S(t) - \mathbb{E}_{t-1}\left[R_S(t)\right] = \beta_{S,g}\varepsilon_g(t) + \beta_{S,\pi}\varepsilon_\pi(t) \\ R_B(t) - \mathbb{E}_{t-1}\left[R_B(t)\right] = \beta_{B,g}\varepsilon_g(t) + \beta_{B,\pi}\varepsilon_\pi(t) \end{array}\right.$$

where $\varepsilon_g(t) \sim \mathcal{N}\left(0, \sigma_g^2\right)$ and $\varepsilon_\pi(t) \sim \mathcal{N}\left(0, \sigma_\pi^2\right)$ are the growth and inflation shocks, respectively. We also assume that $\varepsilon_g(t)$ and $\varepsilon_\pi(t)$ are two correlated processes and denote $\rho\left(\varepsilon_g(t), \varepsilon_\pi(t)\right) = \rho_{g,\pi}$. We deduce that:

$$\left\{\begin{array}{l} \sigma_B^2 = \operatorname{var}\left(R_B(t)\right) = \beta_{B,g}^2\sigma_g^2 + \beta_{B,\pi}^2\sigma_\pi^2 + 2\beta_{B,g}\beta_{B,\pi}\rho_{g,\pi}\sigma_g\sigma_\pi \\ \sigma_S^2 = \operatorname{var}\left(R_S(t)\right) = \beta_{S,g}^2\sigma_g^2 + \beta_{S,\pi}^2\sigma_\pi^2 + 2\beta_{S,g}\beta_{S,\pi}\rho_{g,\pi}\sigma_g\sigma_\pi \\ \sigma_{S,B}^2 = \operatorname{cov}\left(R_S(t), R_B(t)\right) = \beta_{S,g}\beta_{B,g}\sigma_g^2 + \beta_{S,\pi}\beta_{B,\pi}\sigma_\pi^2 + \left(\beta_{S,g}\beta_{B,\pi} + \beta_{S,\pi}\beta_{B,g}\right)\rho_{g,\pi}\sigma_g\sigma_\pi \end{array}\right.$$

Let $\varphi_{g,\pi} = \sigma_g/\sigma_\pi$ be the ratio of the growth volatility to the inflation volatility. We have $\sigma_g = \varphi_{g,\pi}\sigma_\pi$. We deduce that:

$$\left\{\begin{array}{l} \sigma_B^2 = \left(\beta_{B,g}^2\varphi_{g,\pi}^2 + \beta_{B,\pi}^2 + 2\beta_{B,g}\beta_{B,\pi}\rho_{g,\pi}\varphi_{g,\pi}\right)\sigma_\pi^2 \\ \sigma_S^2 = \left(\beta_{S,g}^2\varphi_{g,\pi}^2 + \beta_{S,\pi}^2 + 2\beta_{S,g}\beta_{S,\pi}\rho_{g,\pi}\varphi_{g,\pi}\right)\sigma_\pi^2 \\ \sigma_{S,B}^2 = \left(\beta_{S,g}\beta_{B,g}\varphi_{g,\pi}^2 + \beta_{S,\pi}\beta_{B,\pi} + \left(\beta_{S,g}\beta_{B,\pi} + \beta_{S,\pi}\beta_{B,g}\right)\rho_{g,\pi}\varphi_{g,\pi}\right)\sigma_\pi^2 \end{array}\right.$$

We conclude that the stock-bond correlation does not depend on the level of the growth and inflation volatilities, but only on the ratio $\varphi_{g,\pi}$:

$$\rho_{S,B} = \frac{\beta_{S,g}\beta_{B,g}\varphi_{g,\pi}^2 + \beta_{S,\pi}\beta_{B,\pi} + \left(\beta_{S,g}\beta_{B,\pi} + \beta_{S,\pi}\beta_{B,g}\right)\rho_{g,\pi}\varphi_{g,\pi}}{\sqrt{\beta_{B,g}^2\varphi_{g,\pi}^2 + \beta_{B,\pi}^2 + 2\beta_{B,g}\beta_{B,\pi}\rho_{g,\pi}\varphi_{g,\pi}}\sqrt{\beta_{S,g}^2\varphi_{g,\pi}^2 + \beta_{S,\pi}^2 + 2\beta_{S,g}\beta_{S,\pi}\rho_{g,\pi}\varphi_{g,\pi}}}$$

B Additional results

B.1 Figures

Figure 60: Effect of marginal distributions on bond payoff ($\rho = -25\%$, Normal copula)

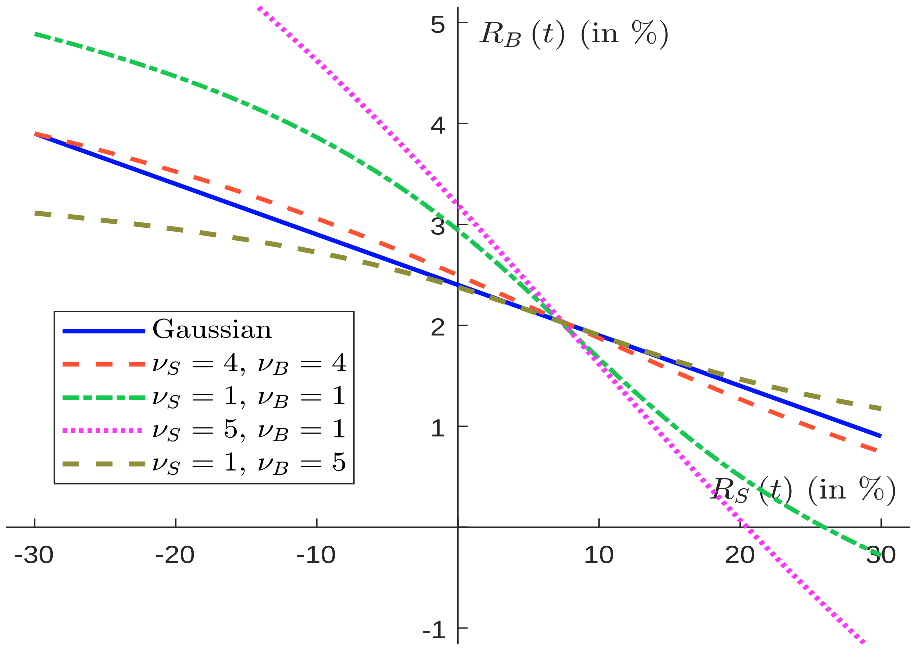


Figure 61: Effect of copula functions on bond payoff ($\rho = -25\%$, Gaussian margins)

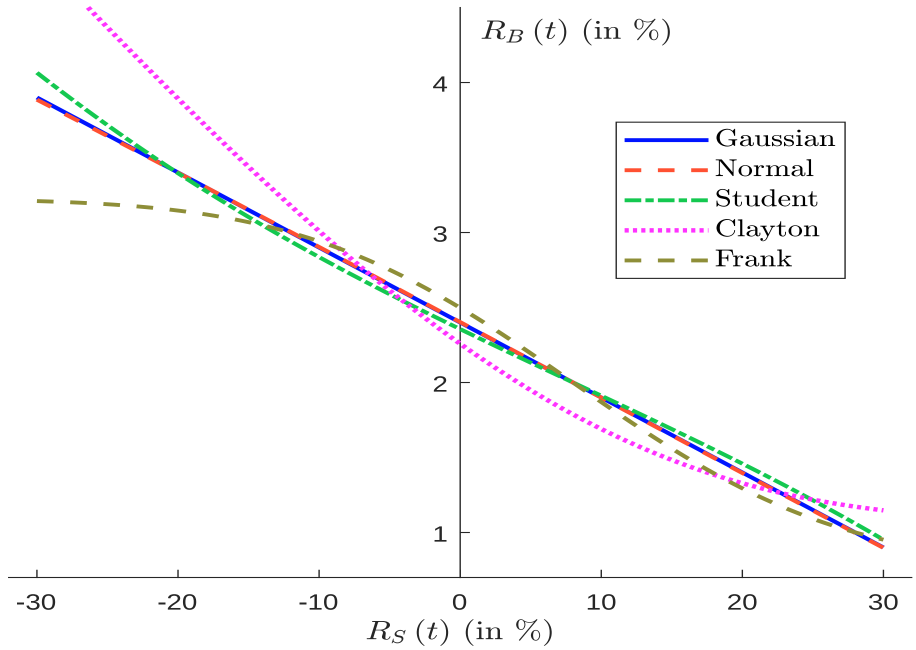


Figure 62: Effect of copula functions on bond payoff ($\rho = -25\%$, Student t_1 margins)

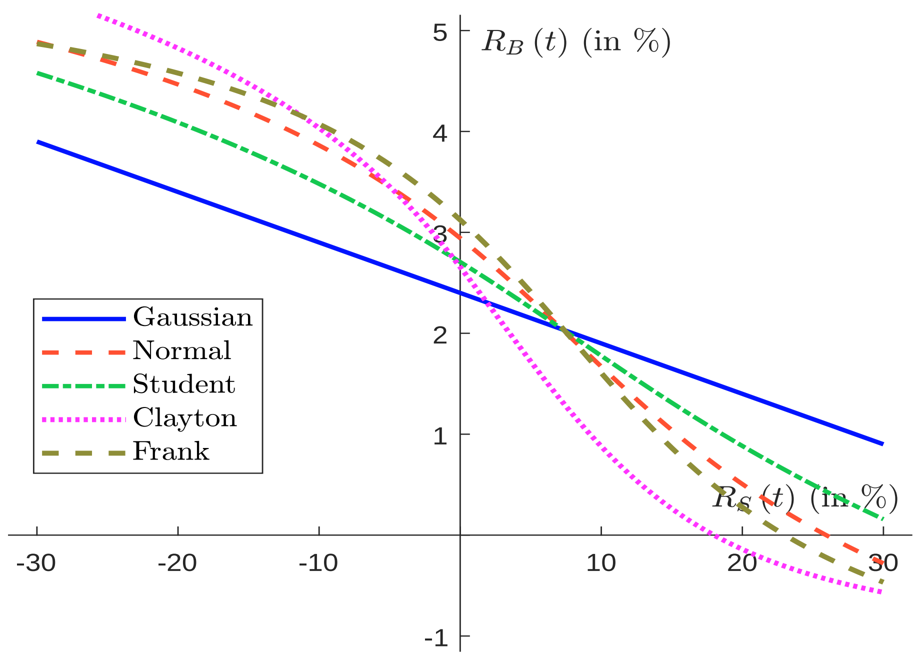


Figure 63: Number of stocks that explain 25% of the US stock-bond correlation (S&P 500 Index)

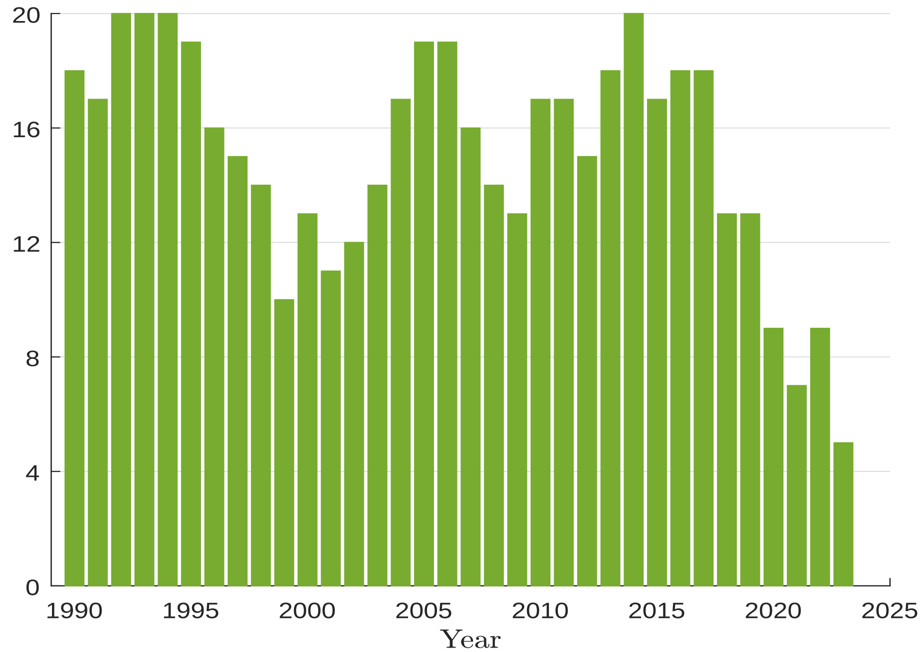


Figure 64: Number of stocks that explain 75% of the US stock-bond correlation (S&P 500 Index)

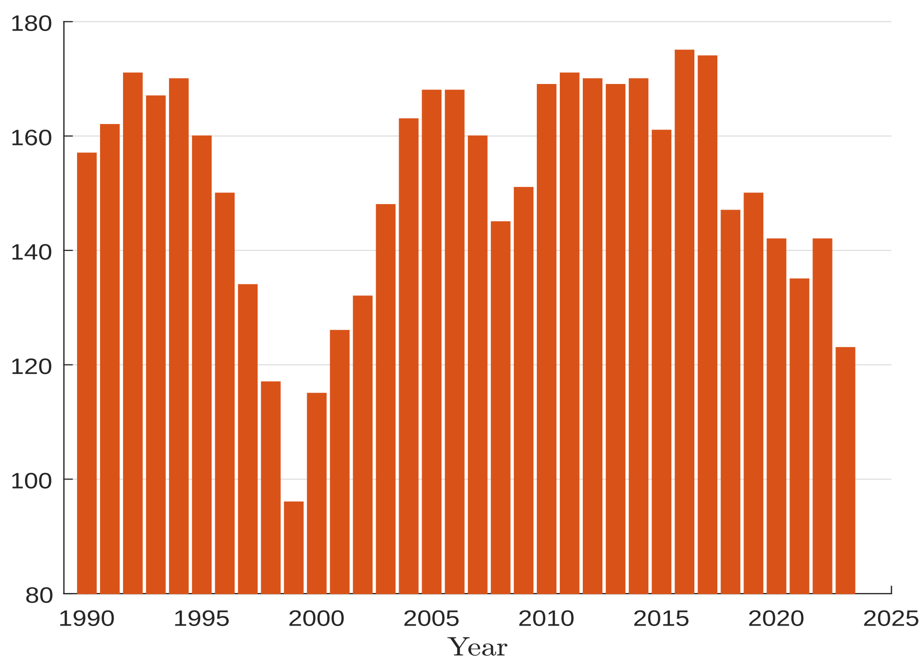


Figure 65: Aggregate stock-bond correlation of the MDP ($\left|\rho_{S,B}^{\text{stock}}\right| = 10\%$)

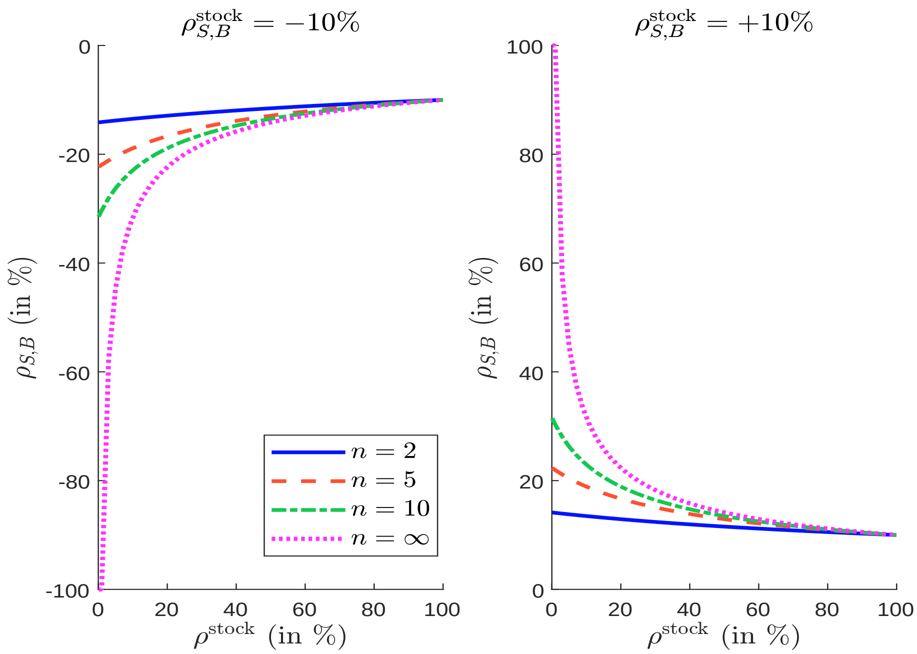


Figure 66: Stock-bond correlation $\rho_{S,B}$ in the growth-inflation model ($\beta_{S,g} = 1$, $\beta_{S,\pi} = \beta_{B,g} = \beta_{B,\pi} = -1$ and $\varphi_{g,\pi} = 2$)

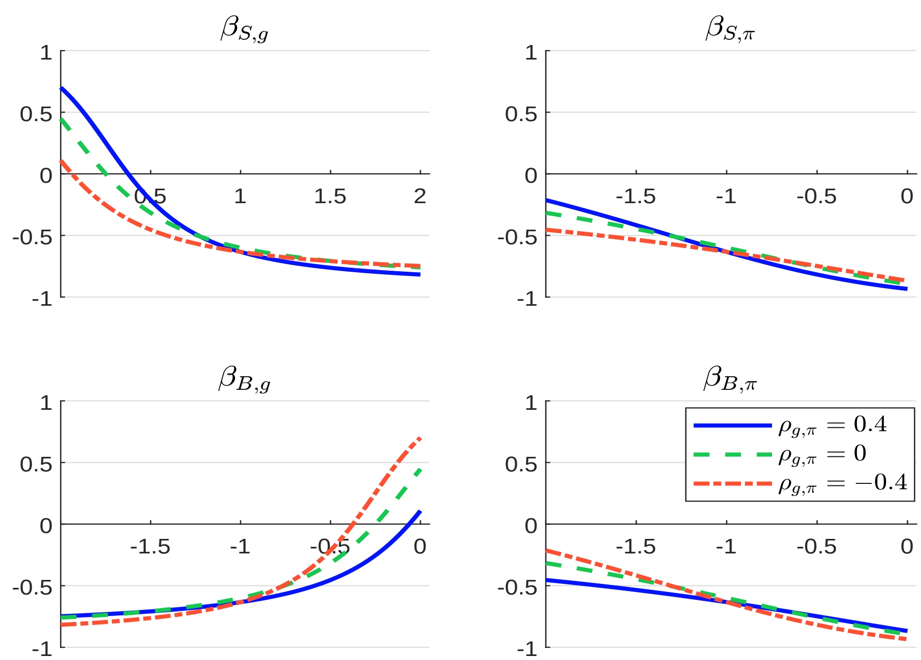


Figure 67: Stock-bond correlation $\rho_{S,B}$ in the growth-inflation model ($\beta_{S,g} = 0.2$, $\beta_{S,\pi} = \beta_{B,g} = \beta_{B,\pi} = -1$ and $\varphi_{g,\pi} = 2$)

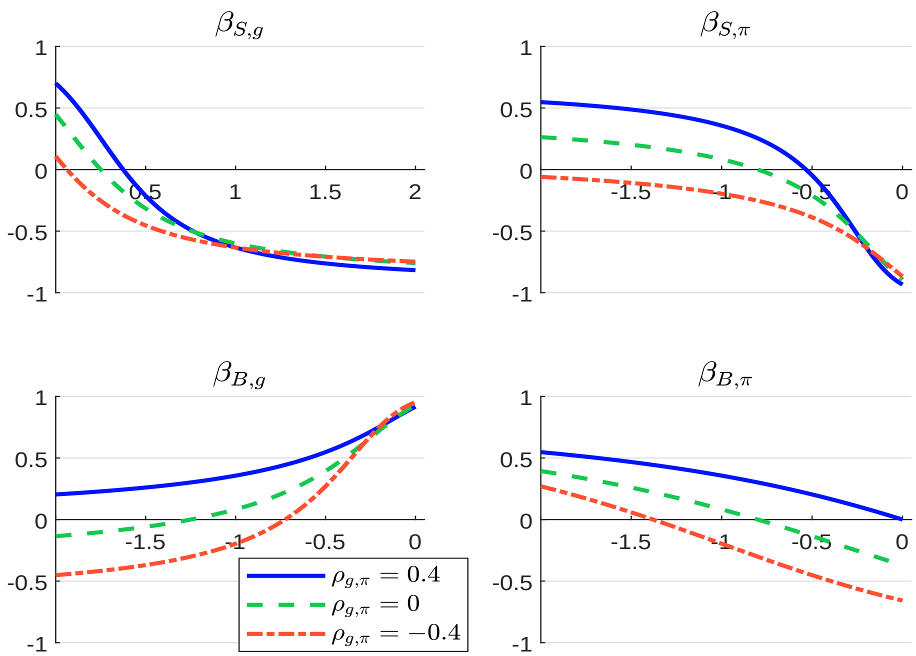


Figure 68: Stock-bond correlation $\rho_{S,B}$ in the growth-inflation model ($\beta_{S,\pi} = \beta_{B,g} = \beta_{B,\pi} = -1$ and $\rho_{g,\pi} = -40\%$)

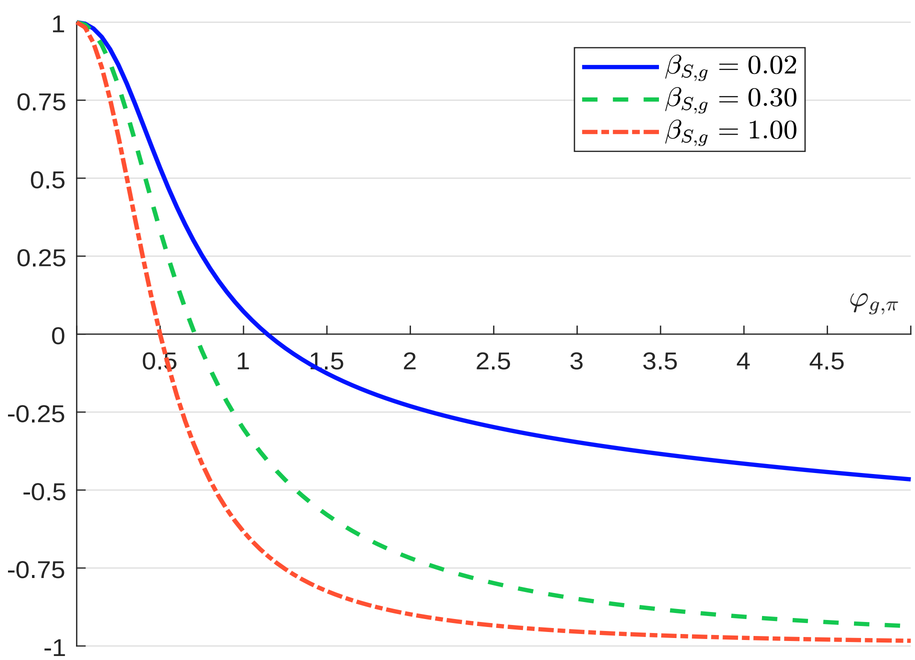


Figure 69: Stock-bond correlation $\rho_{S,B}$ in the growth-inflation model ($\beta_{S,\pi} = \beta_{B,g} = \beta_{B,\pi} = -1$ and $\varphi_{g,\pi} = 2$)

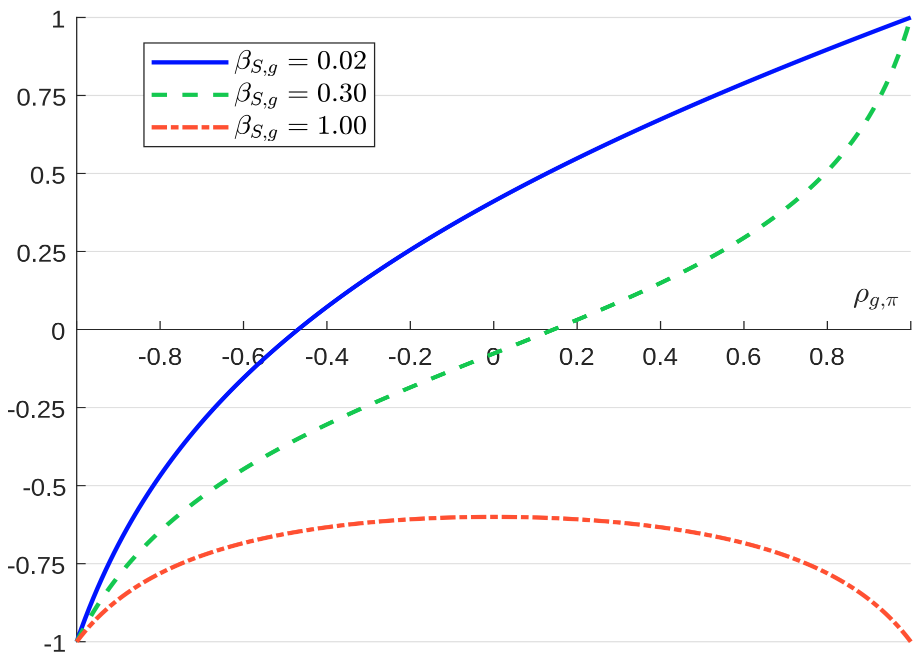
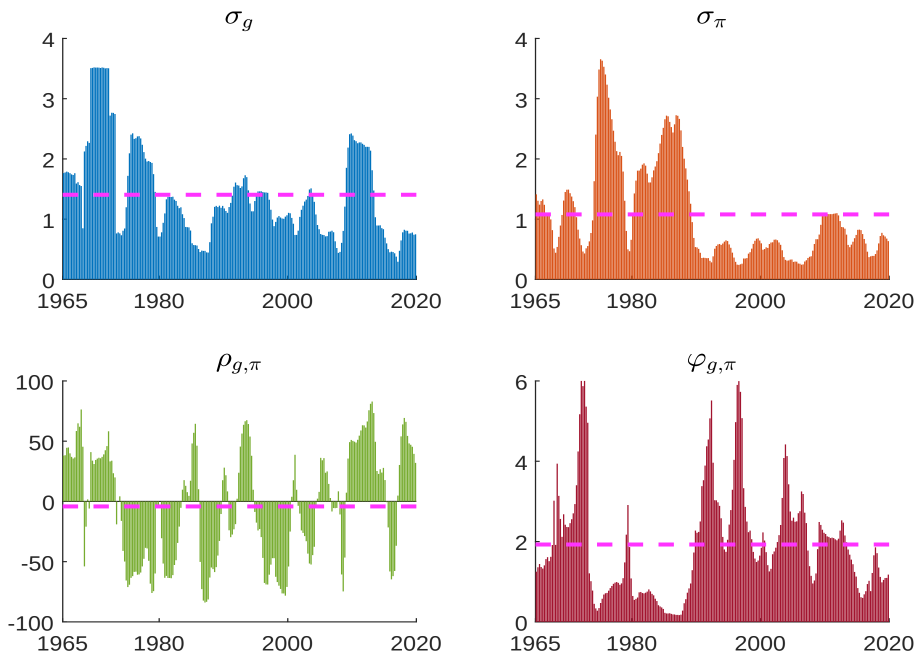
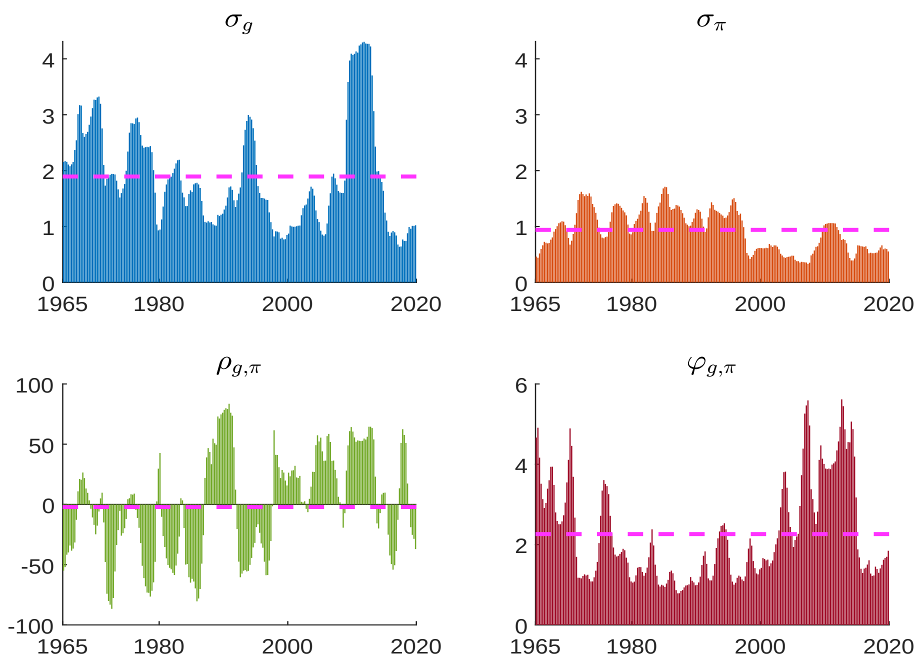


Figure 70: 4-year rolling window estimates of σ_g, σ_π, $\rho_{g,\pi}$ and $\varphi_{g,\pi}$ (France)



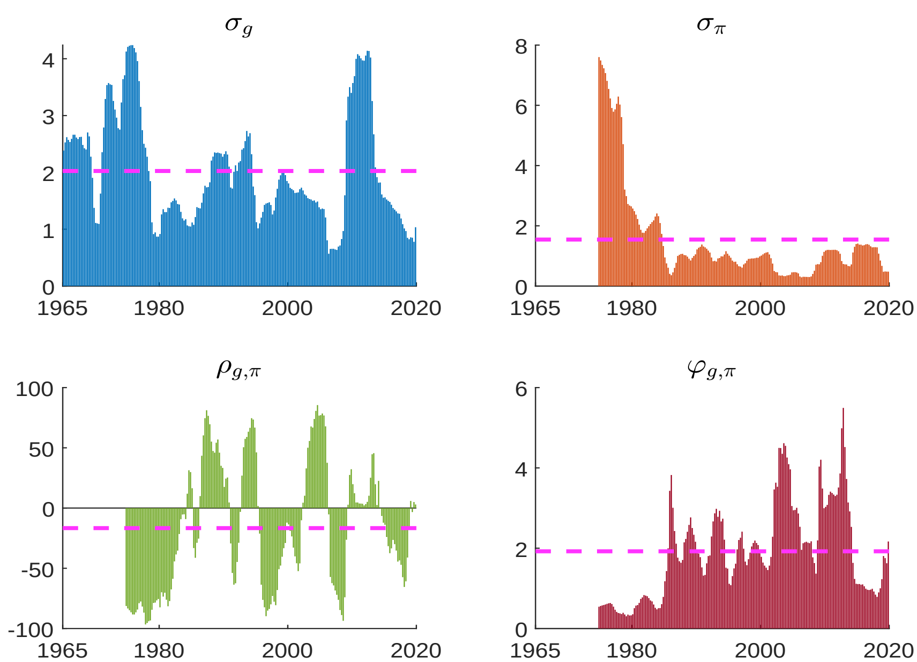
Source: OECD (2024), data.oecd.org/gdp/quarterly-gdp.htm, data.oecd.org/price/inflation-cpi.htm & Author's calculations.

Figure 71: 4-year rolling window estimates of σ_g, σ_π, $\rho_{g,\pi}$ and $\varphi_{g,\pi}$ (Germany)



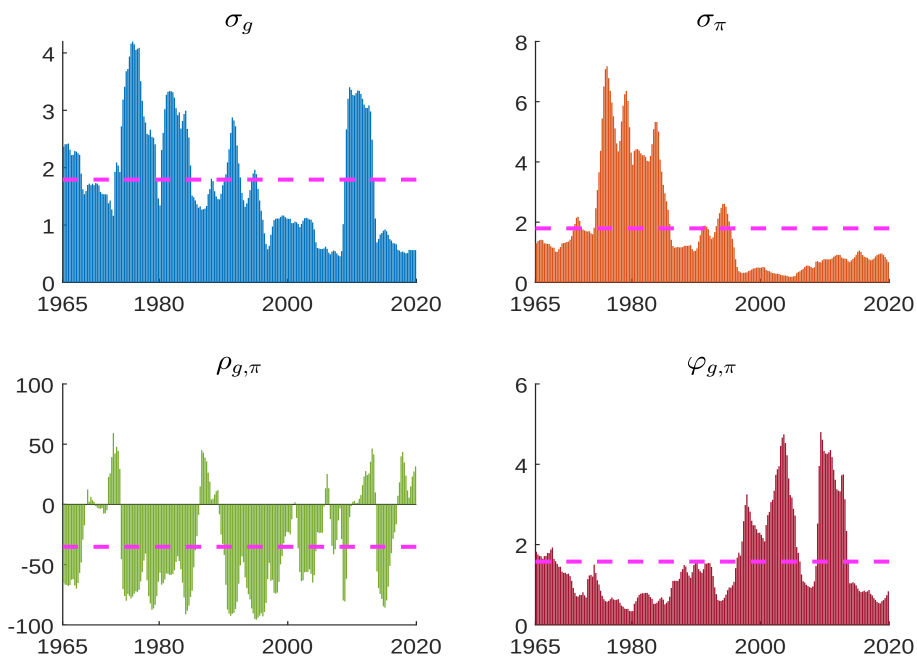
Source: OECD (2024), data.oecd.org/gdp/quarterly-gdp.htm, data.oecd.org/price/inflation-cpi.htm & Author's calculations.

Figure 72: 4-year rolling window estimates of σ_g, σ_π, $\rho_{g,\pi}$ and $\varphi_{g,\pi}$ (Japan)



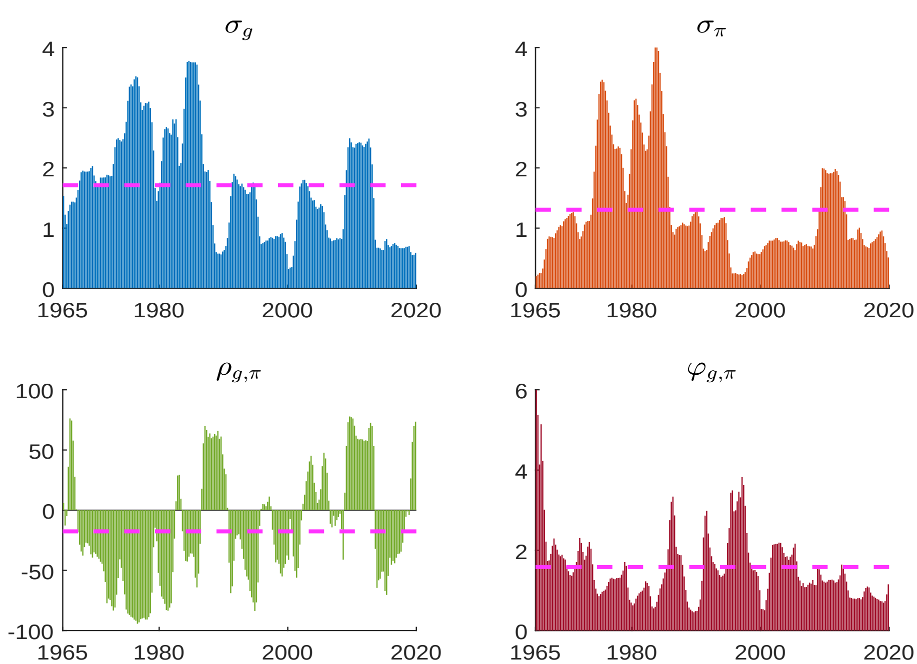
Source: OECD (2024), data.oecd.org/gdp/quarterly-gdp.htm, data.oecd.org/price/inflation-cpi.htm & Author's calculations.

Figure 73: 4-year rolling window estimates of σ_g, σ_π, $\rho_{g,\pi}$ and $\varphi_{g,\pi}$ (United Kingdom)



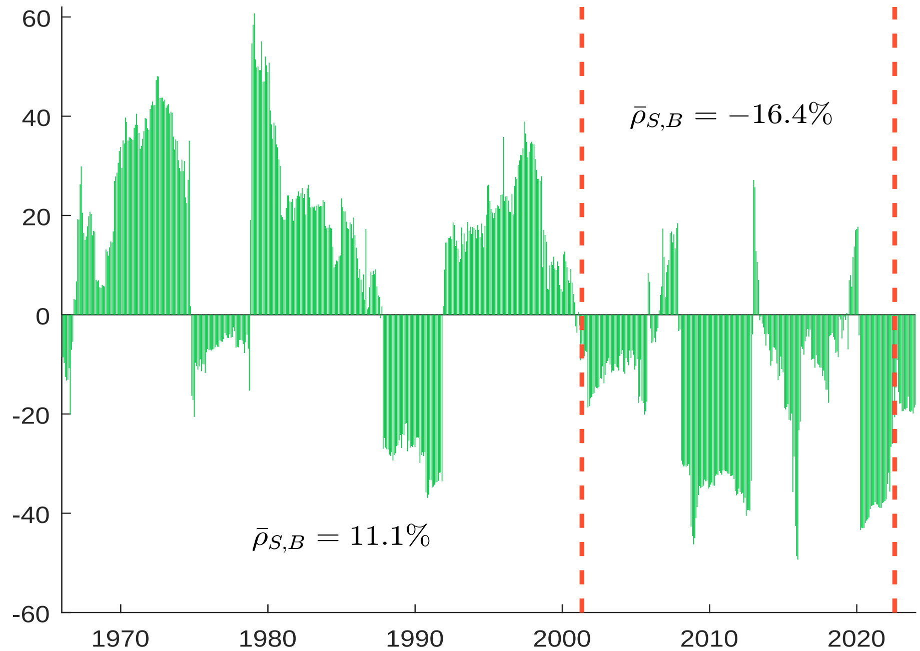
Source: OECD (2024), `data.oecd.org/gdp/quarterly-gdp.htm`, `data.oecd.org/price/inflation-cpi.htm` & Author's calculations.

Figure 74: 4-year rolling window estimates of σ_g, σ_π, $\rho_{g,\pi}$ and $\varphi_{g,\pi}$ (United States)



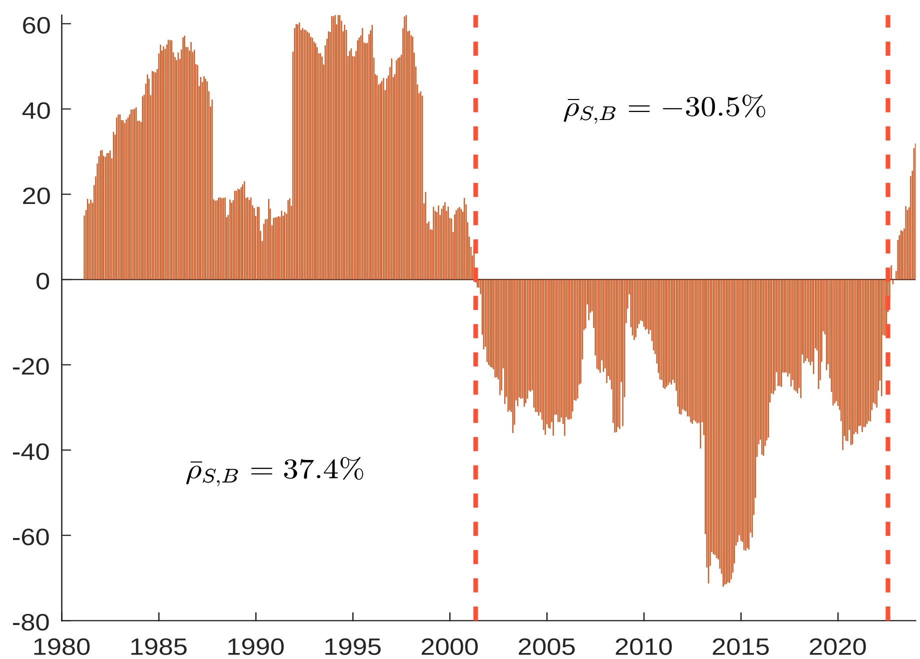
Source: OECD (2024), `data.oecd.org/gdp/quarterly-gdp.htm`, `data.oecd.org/price/inflation-cpi.htm` & Author's calculations.

Figure 75: Rolling 4-year stock-bond correlation (US, 3M, 1965-2023, monthly frequency)



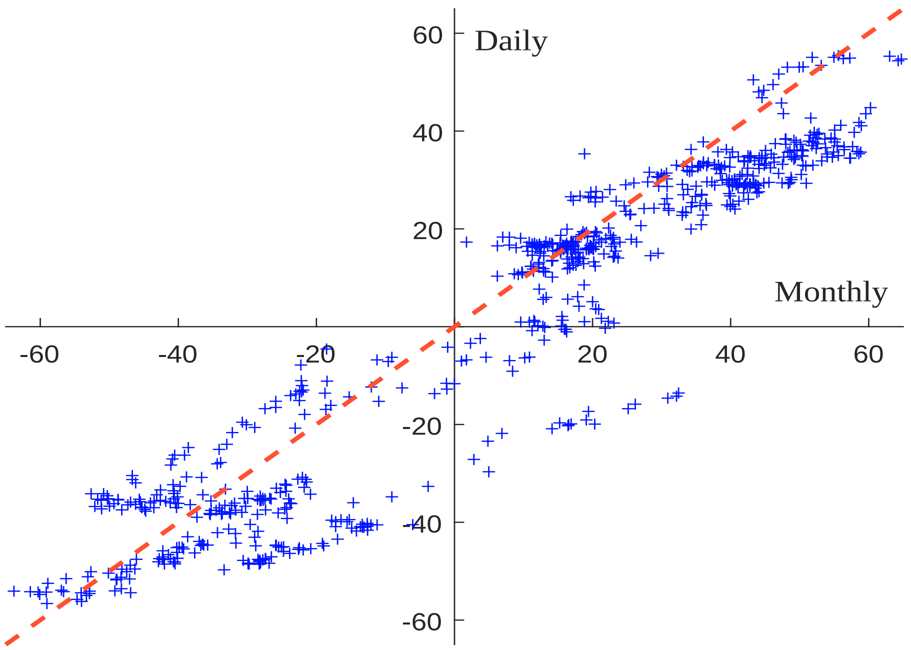
Source: Bloomberg (2024) & Authors' calculations.

Figure 76: Rolling 4-year stock-bond correlation (US, 30Y, 1980-2023, monthly frequency)



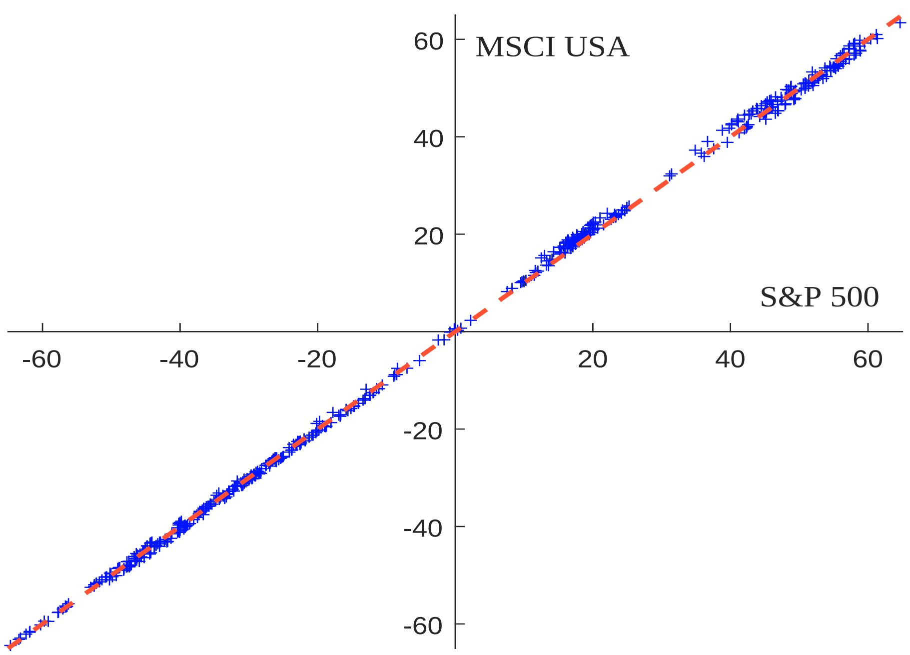
Source: Bloomberg (2024) & Authors' calculations.

Figure 77: Scatterplot of rolling 4-year stock-bond monthly and daily correlations (US, 10Y, 1965-2023)



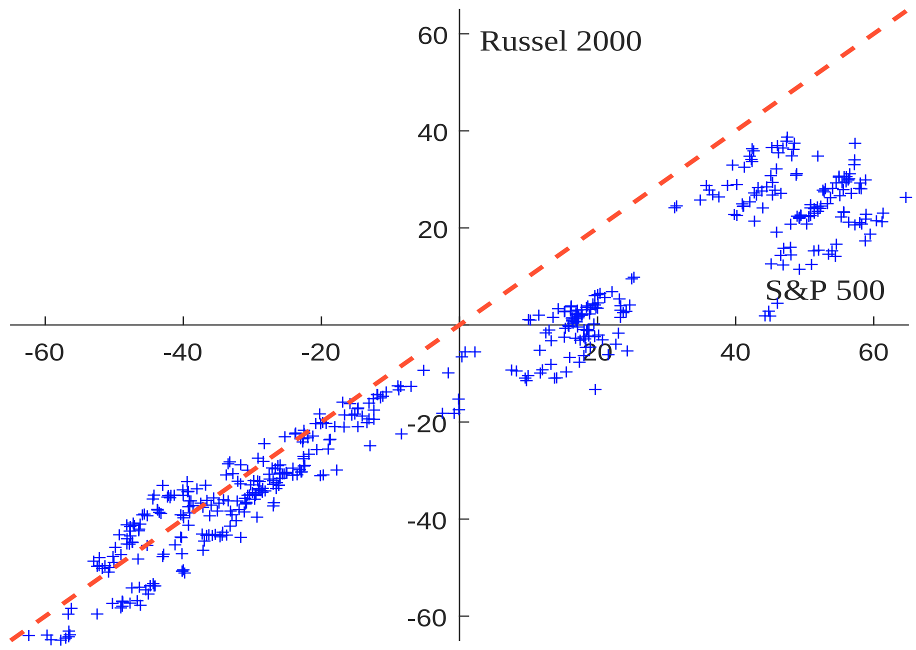
Source: Bloomberg (2024) & Authors' calculations.

Figure 78: Scatterplot of rolling 4-year stock-bond correlations (US, 10Y, 1980-2023)



Source: Datastream (2024) & Authors' calculations.

Figure 79: Scatterplot of rolling 4-year stock-bond correlations (US, 10Y, 1980-2023)



Source: Datastream (2024) & Authors' calculations.

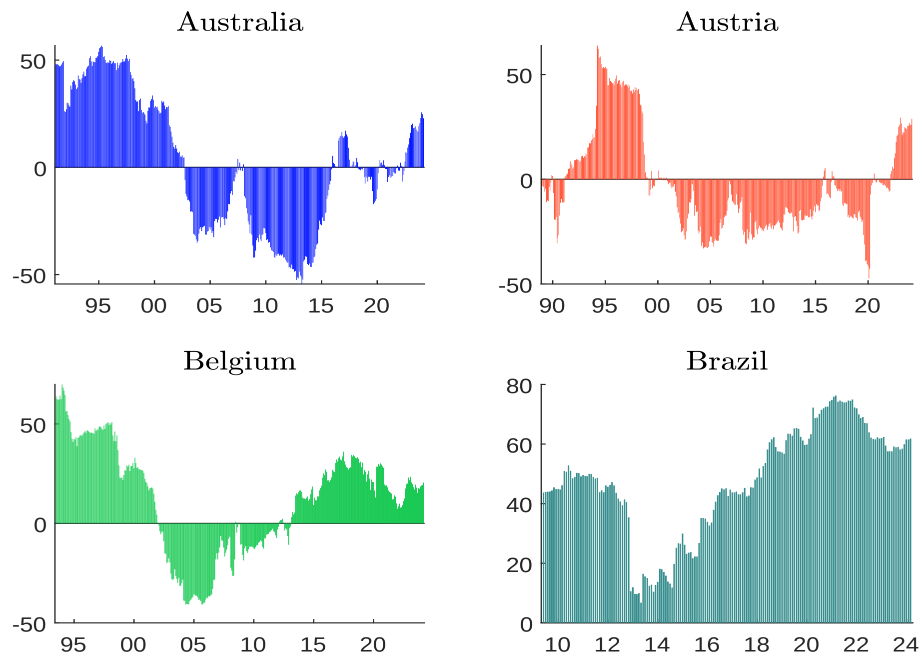
Below we report the estimated stock-bond correlation for the different countries. We use the Benchmark 10 Year DS Govt. Index (Total Return) for all countries except the following:

- Bulgaria, China, Colombia, Hong Kong, India, Israel, Peru, Philippines, Romania, Russia and Turkey: ICE BofA Government Index (Total Return);
- Brazil: Bloomberg EM Gov IL 7-10Y (Total Return);
- Chile, Malaysian and Taiwan: FTSE Government Bond Index 7+ Year (Total Return).

For the equity index, we use the MSCI Country Local Currency (Total Return Index) for all countries except the following:

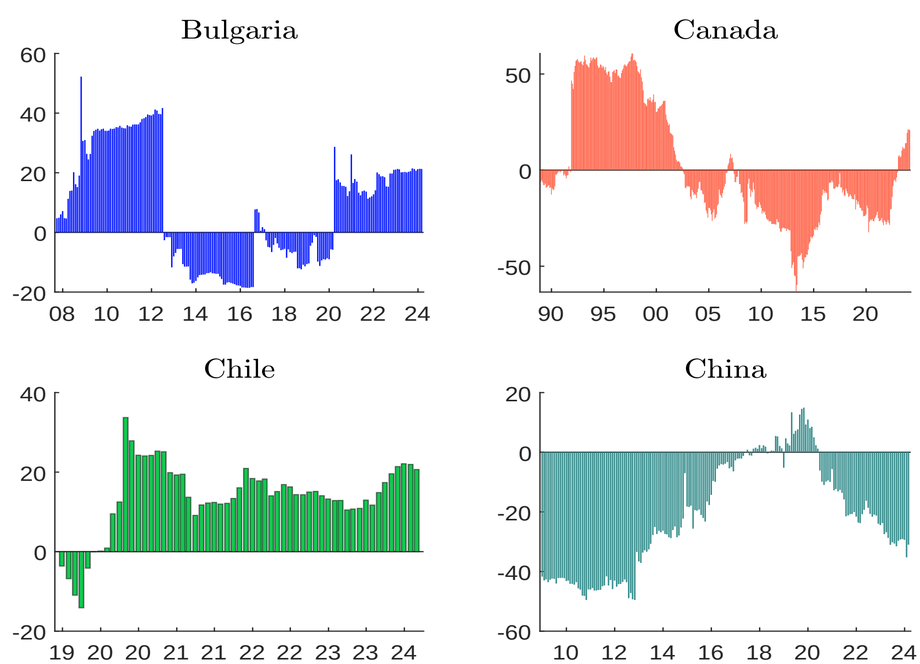
- Bulgaria: Bulgaria SE SOFIX (Price Index);
- Peru: S&P/BVL General(IGBVL) (Price Index);
- Romania: Romania BET (L) (Price Index);
- Russia: MOEX Russia Index (Total Return).

Figure 80: Rolling 4-year stock-bond correlation (Australia, Austria, Belgium, Brazil)



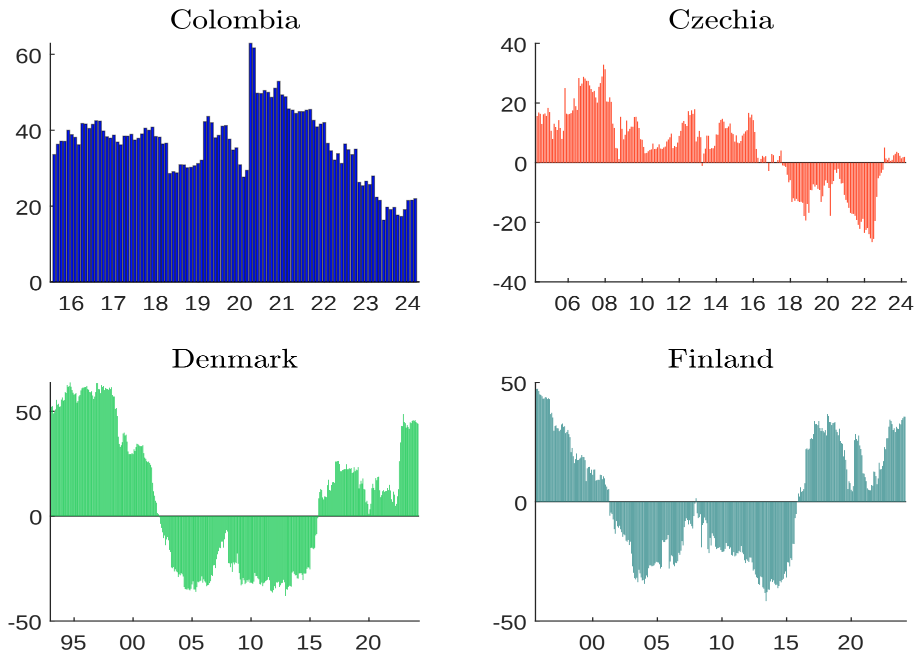
Source: Datastream (2024) & Authors' calculations.

Figure 81: Rolling 4-year stock-bond correlation (Bulgaria, Canada, Chile, China)



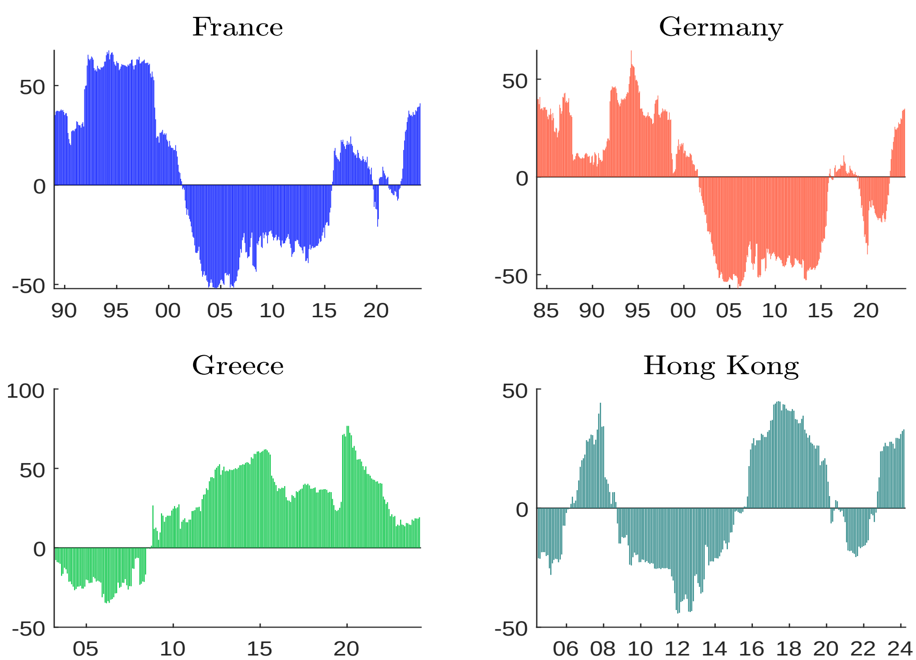
Source: Datastream (2024) & Authors' calculations.

Figure 82: Rolling 4-year stock-bond correlation (Colombia, Czechia, Denmark, Finland)



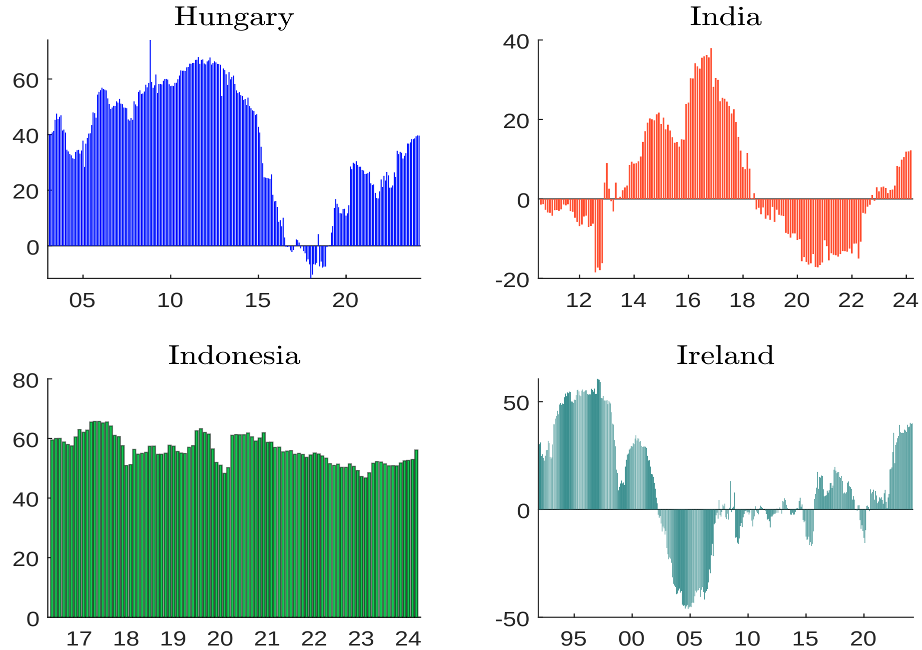
Source: Datastream (2024) & Authors' calculations.

Figure 83: Rolling 4-year stock-bond correlation (France, Germany, Greece, Hong Kong)



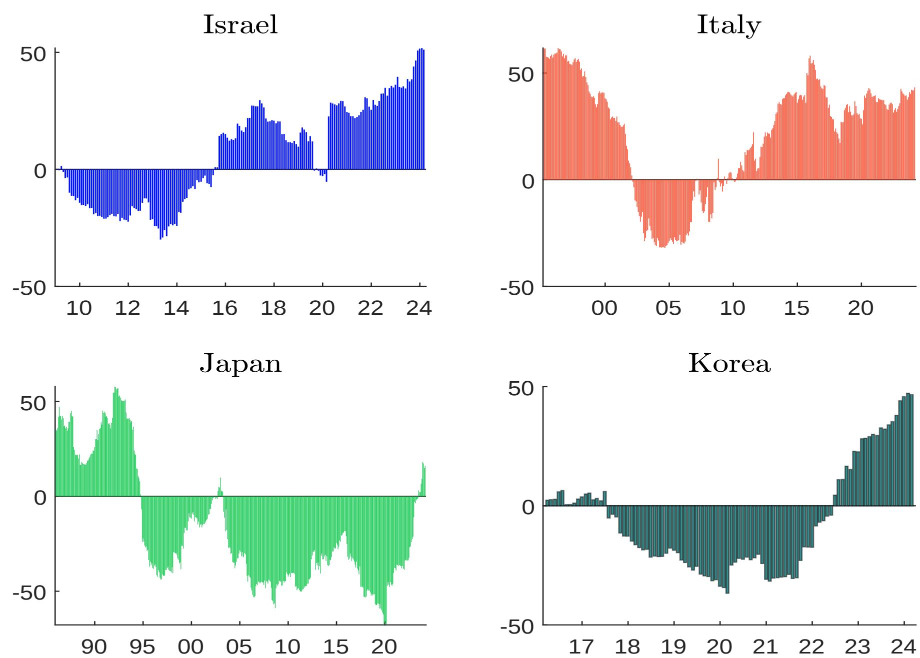
Source: Datastream (2024) & Authors' calculations.

Figure 84: Rolling 4-year stock-bond correlation (Hungary, India, Indonesia, Ireland)



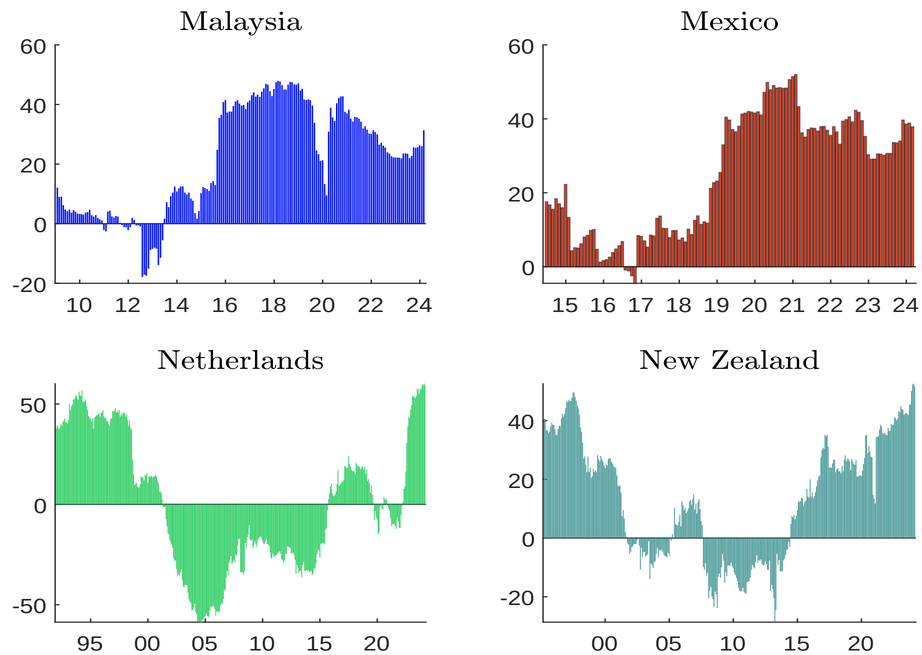
Source: Datastream (2024) & Authors' calculations.

Figure 85: Rolling 4-year stock-bond correlation (Israel, Italy, Japan, Korea)



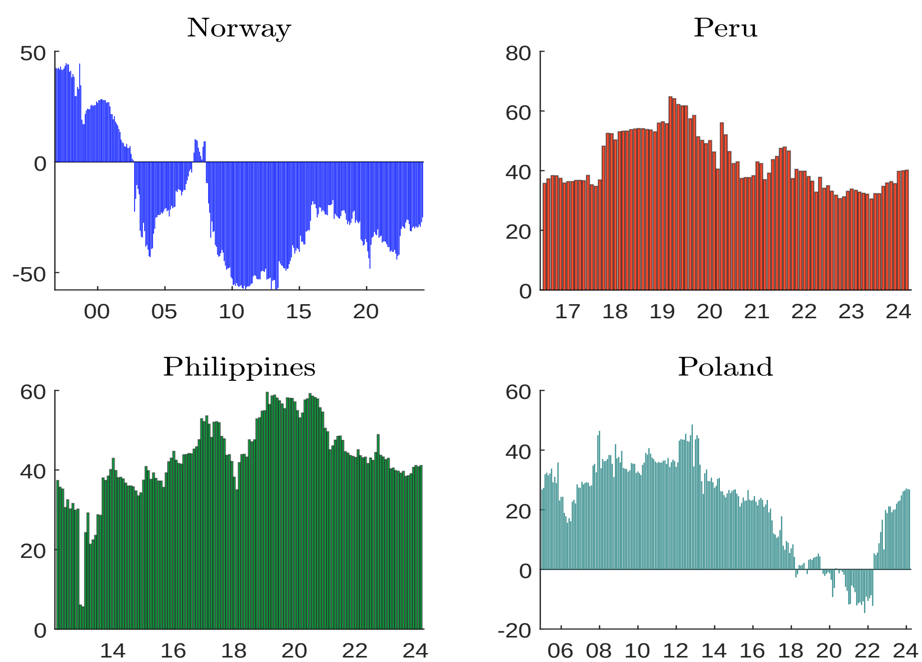
Source: Datastream (2024) & Authors' calculations.

Figure 86: Rolling 4-year stock-bond correlation (Malaysia, Mexico, Netherlands, New Zealand)



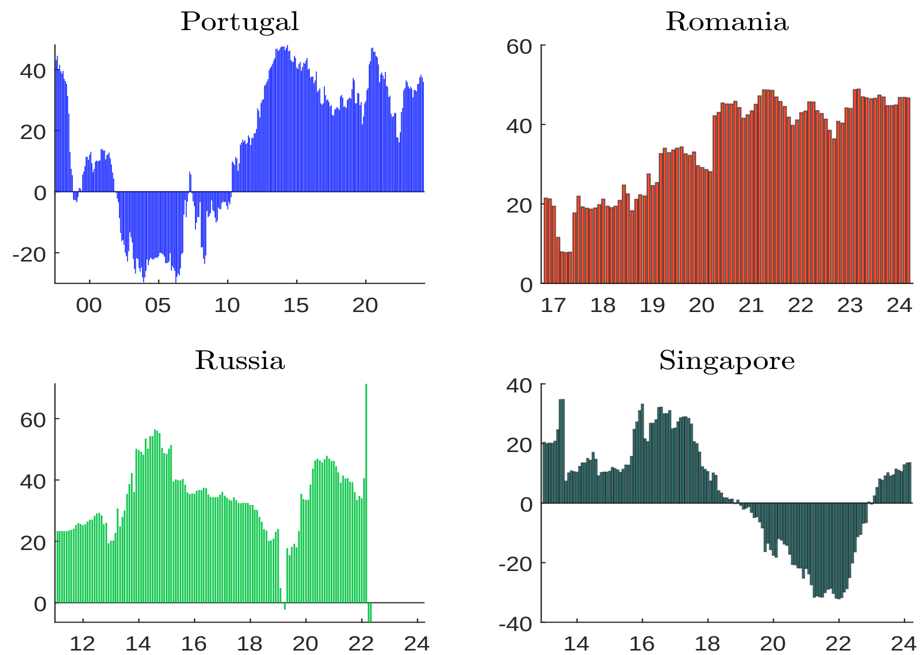
Source: Datastream (2024) & Authors' calculations.

Figure 87: Rolling 4-year stock-bond correlation (Norway, Peru, Philippines, Poland)



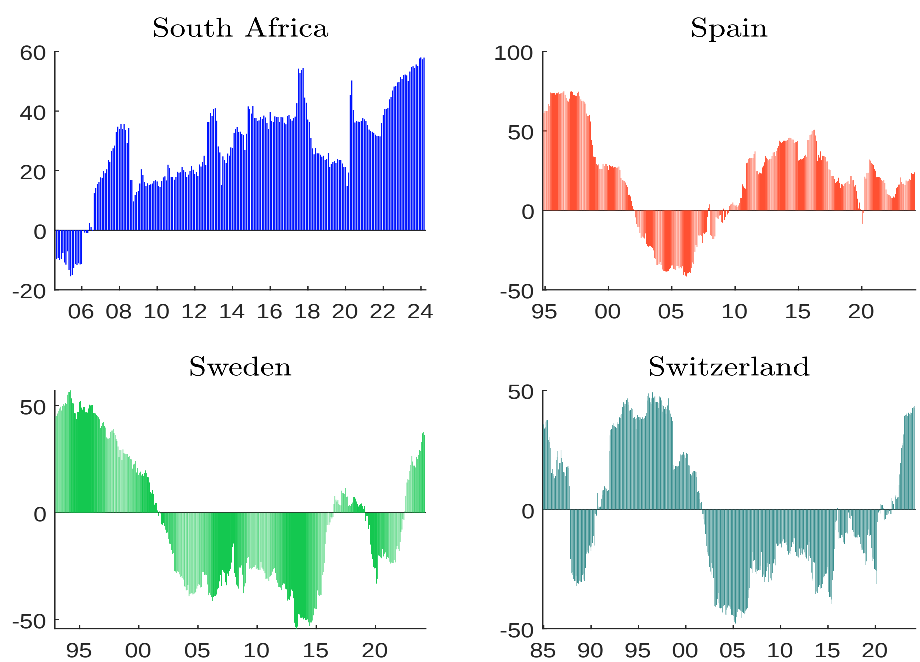
Source: Datastream (2024) & Authors' calculations.

Figure 88: Rolling 4-year stock-bond correlation (Portugal, Romania, Russia, Singapore)



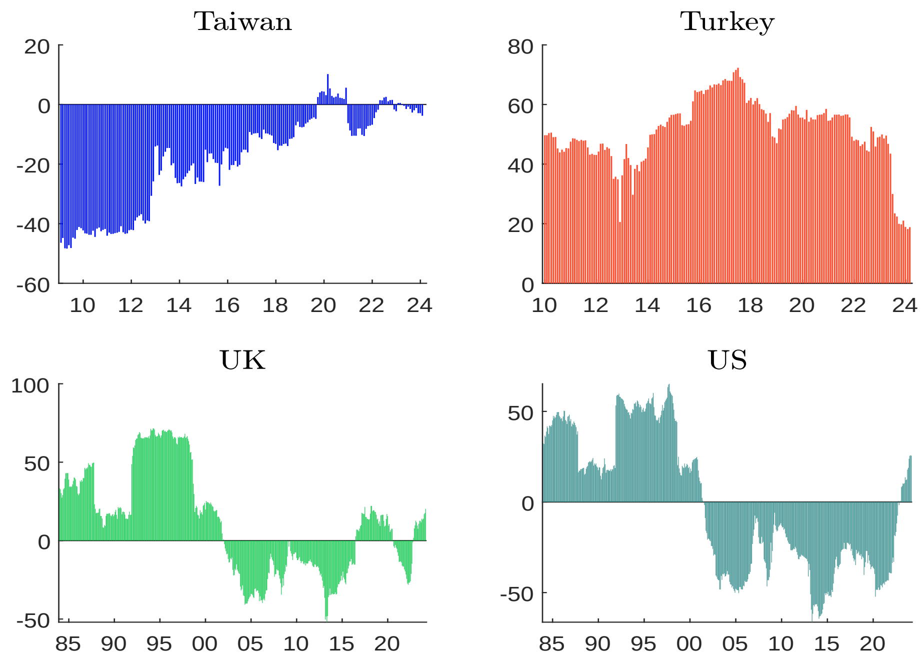
Source: Datastream (2024) & Authors' calculations.

Figure 89: Rolling 4-year stock-bond correlation (South Africa, Spain, Sweden, Switzerland)



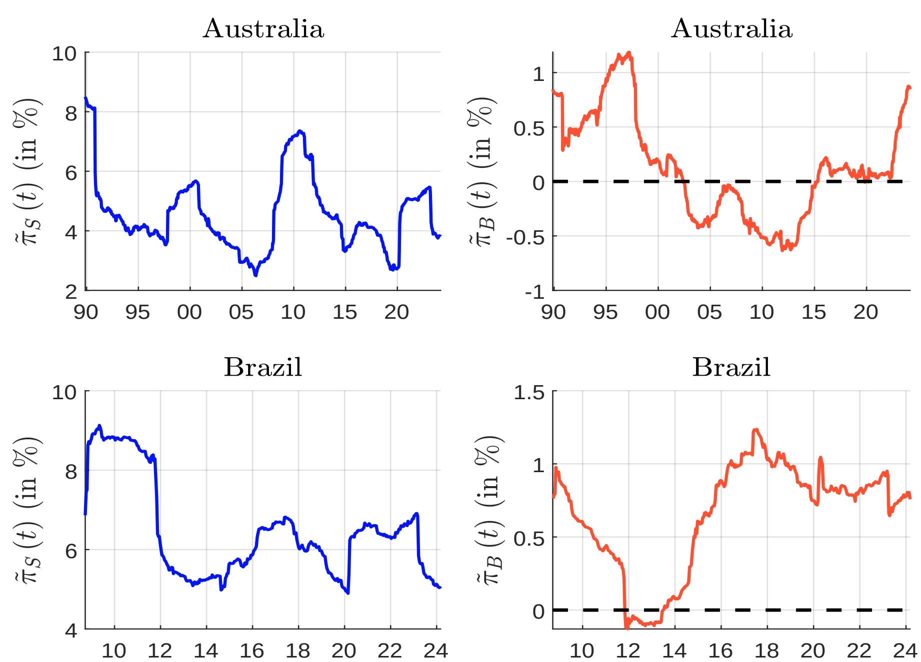
Source: Datastream (2024) & Authors' calculations.

Figure 90: Rolling 4-year stock-bond correlation (Taiwan, Turkey, United Kingdom, United States)



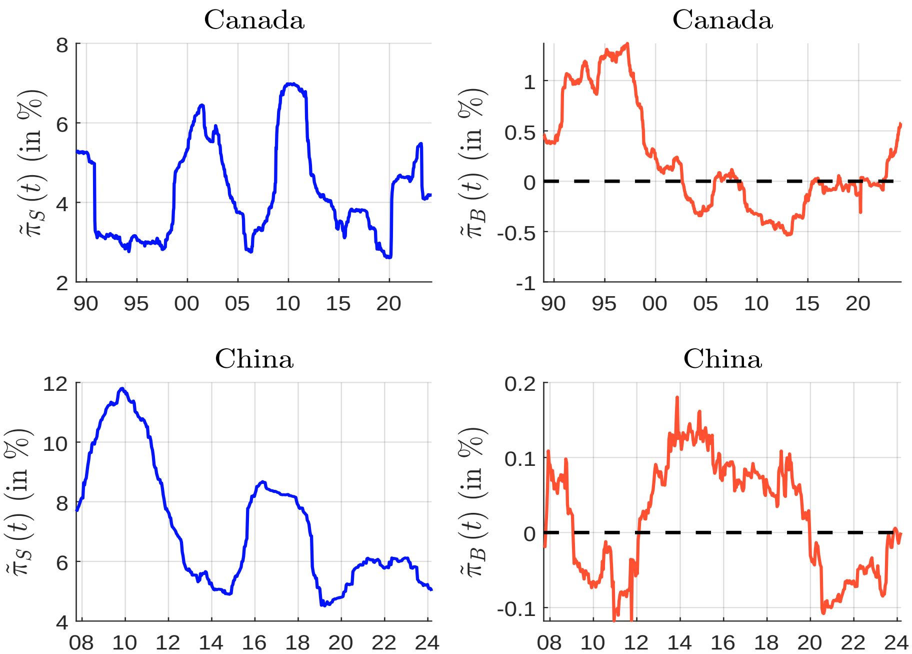
Source: Datastream (2024) & Authors' calculations.

Figure 91: Stock and bond risk premia (Australia, Brazil)



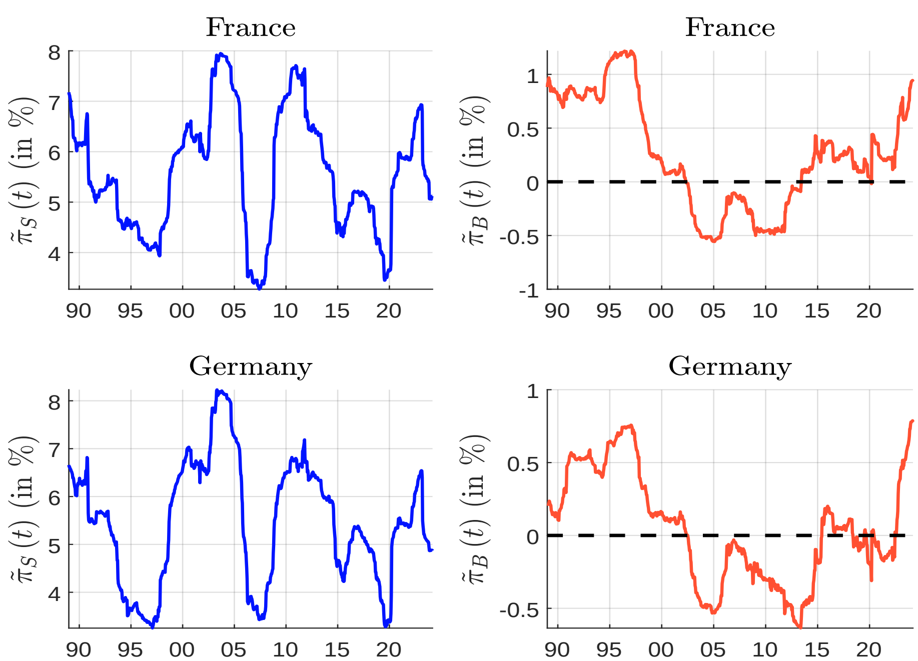
Source: Datastream (2024) & Authors' calculations.

Figure 92: Stock and bond risk premia (Canada, China)



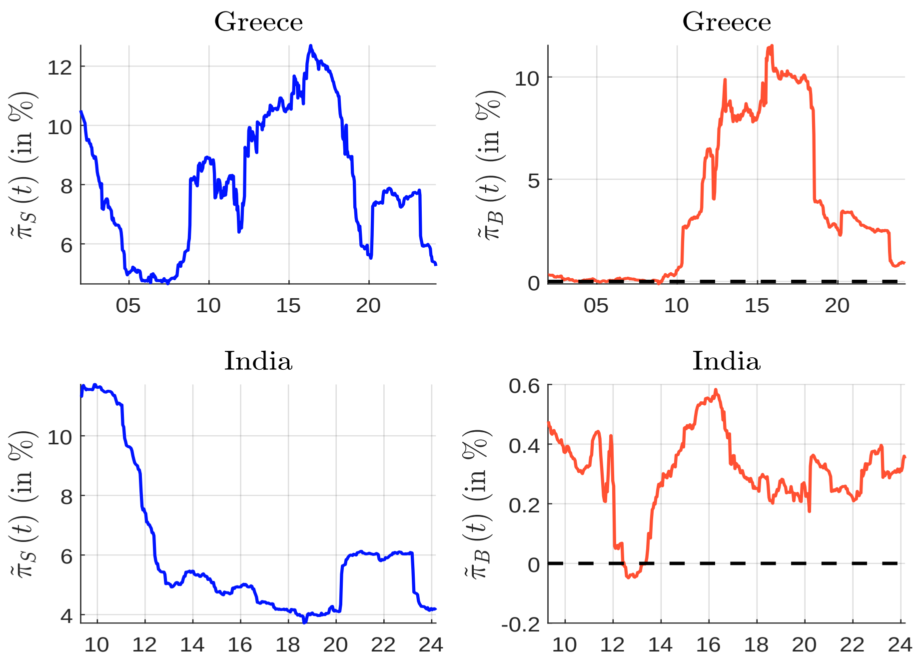
Source: Datastream (2024) & Authors' calculations.

Figure 93: Stock and bond risk premia (France, Germany)



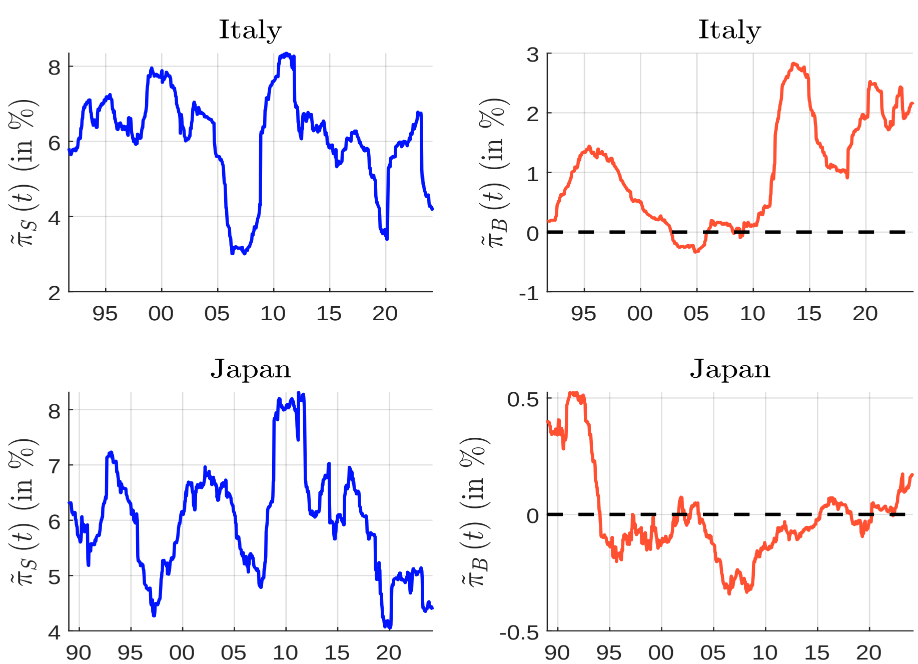
Source: Datastream (2024) & Authors' calculations.

Figure 94: Stock and bond risk premia (Greece, India)



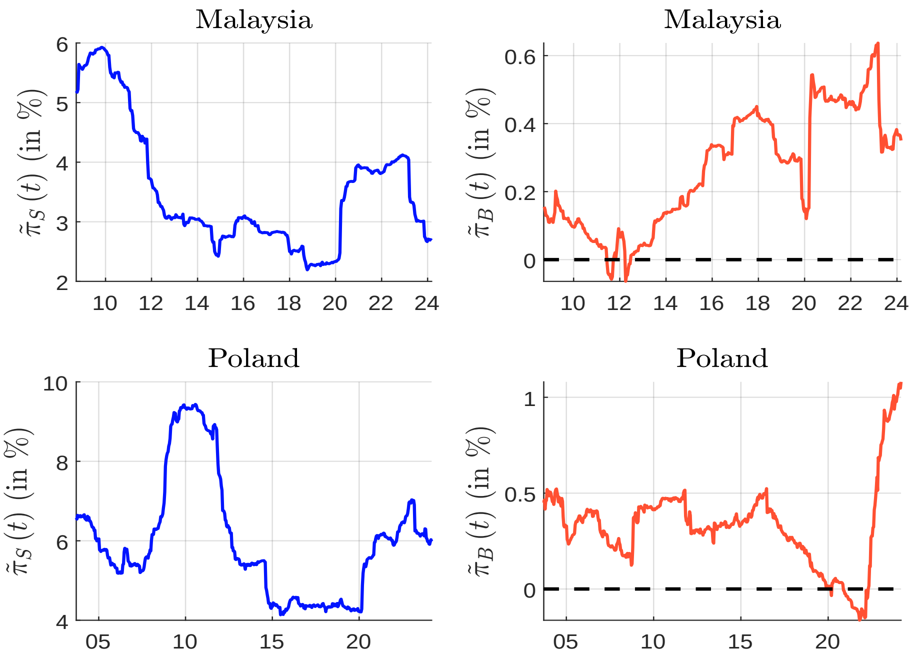
Source: Datastream (2024) & Authors' calculations.

Figure 95: Stock and bond risk premia (Italy, Japan)



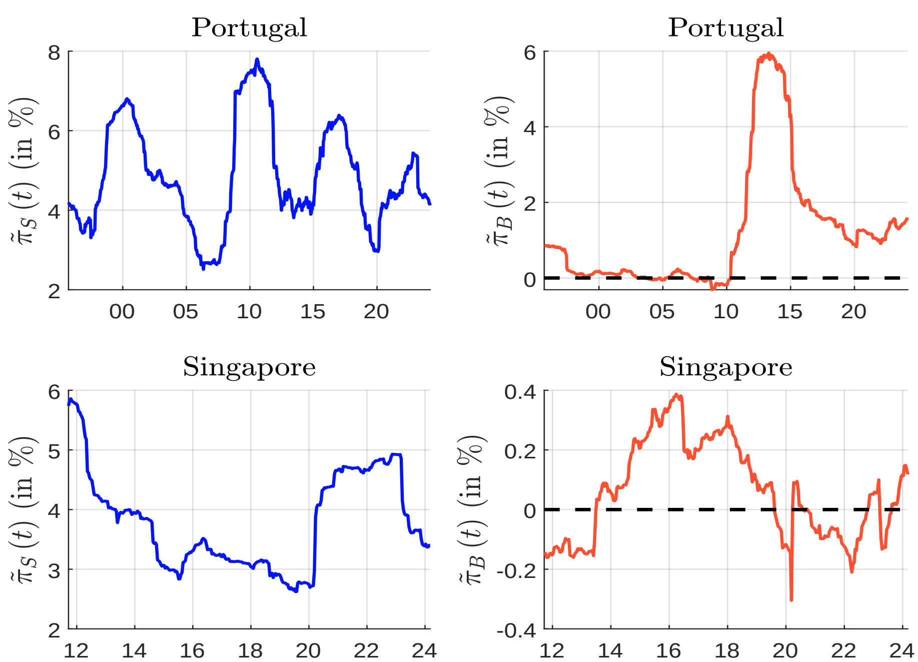
Source: Datastream (2024) & Authors' calculations.

Figure 96: Stock and bond risk premia (Malaysia, Poland)



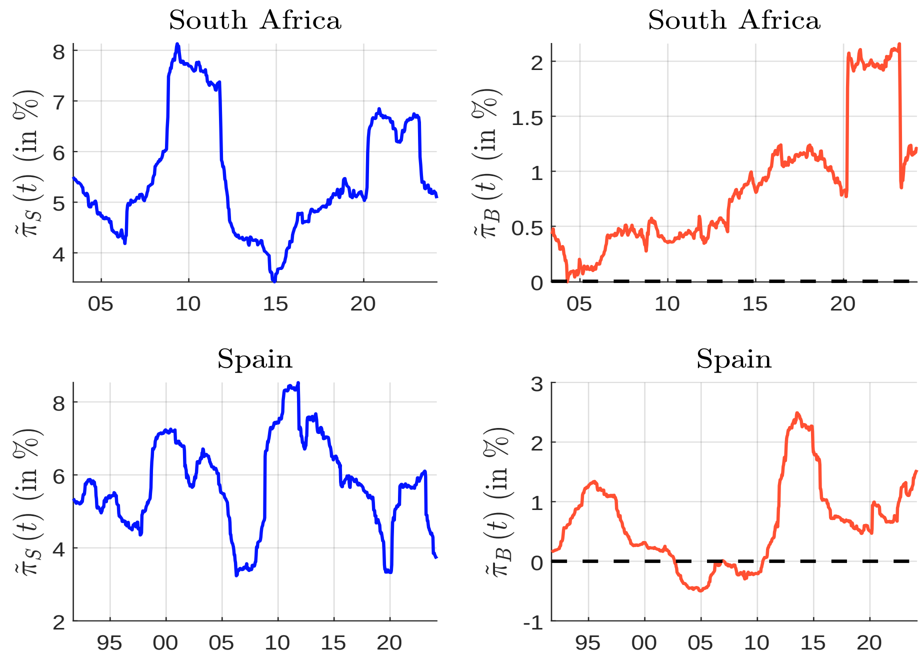
Source: Datastream (2024) & Authors' calculations.

Figure 97: Stock and bond risk premia (Portugal, Singapore)



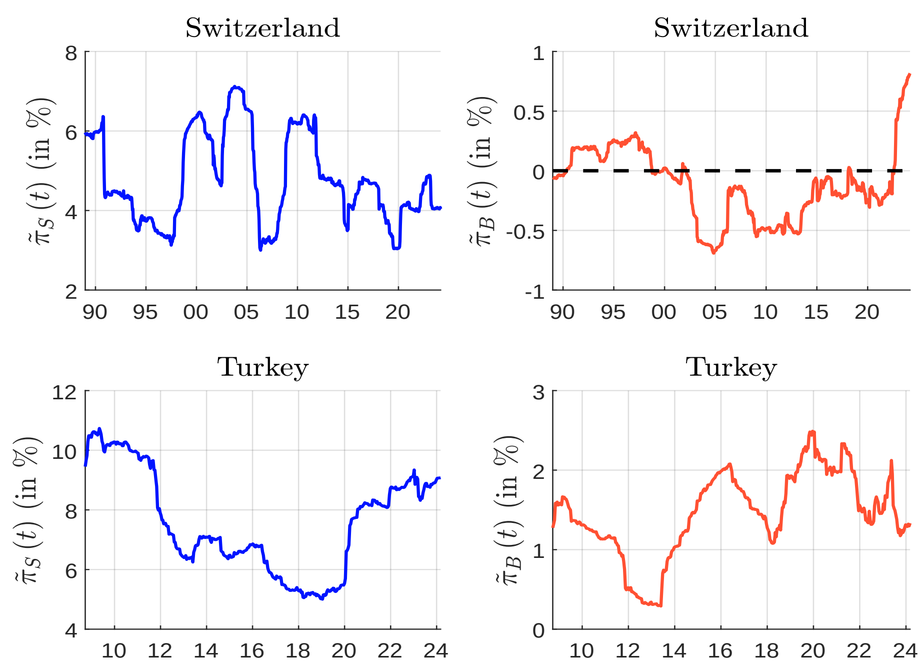
Source: Datastream (2024) & Authors' calculations.

Figure 98: Stock and bond risk premia (South Africa, Spain)



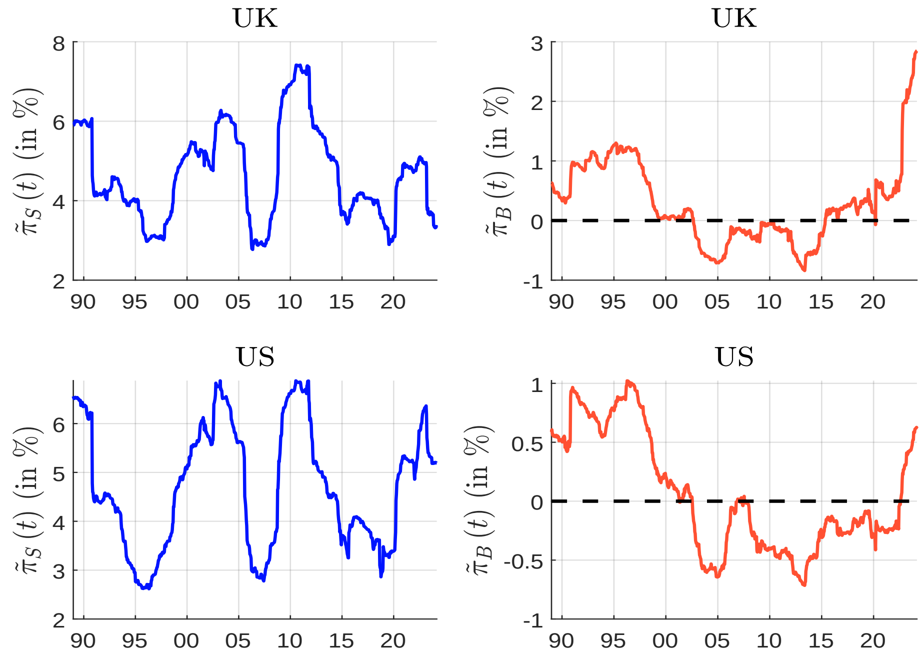
Source: Datastream (2024) & Authors' calculations.

Figure 99: Stock and bond risk premia (Switzerland, Turkey)



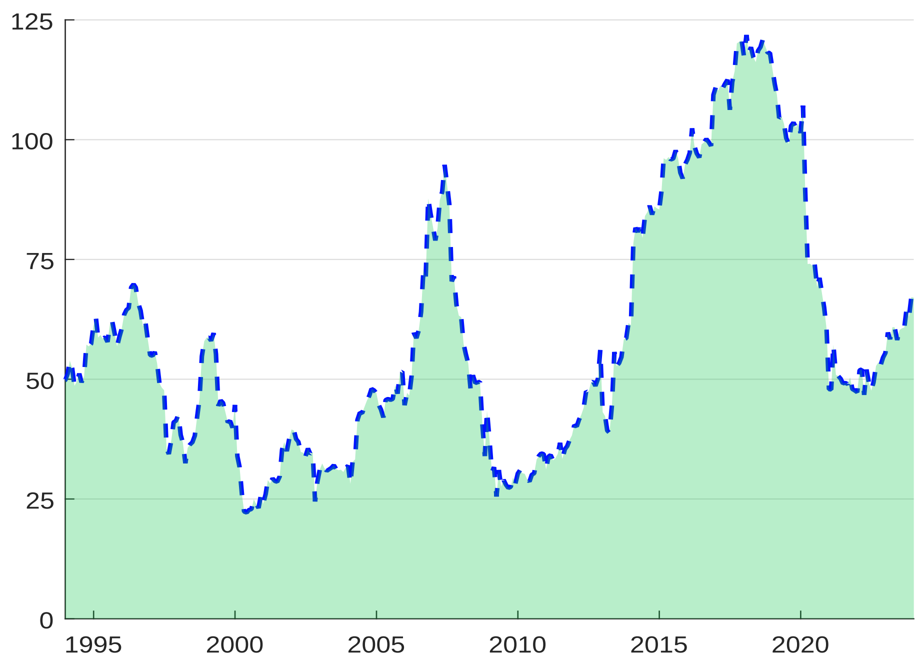
Source: Datastream (2024) & Authors' calculations.

Figure 100: Stock and bond risk premia (United Kingdom, United States)



Source: Datastream (2024) & Authors' calculations.

Figure 101: Difference in % between the maximum and minimum US sector-level stock-bond correlation



Source: Datastream (2024) & Authors' calculations.

Figure 102: Confidence interval of the individual stock-bond correlation (US, monthly return, $\alpha = 5\%$)

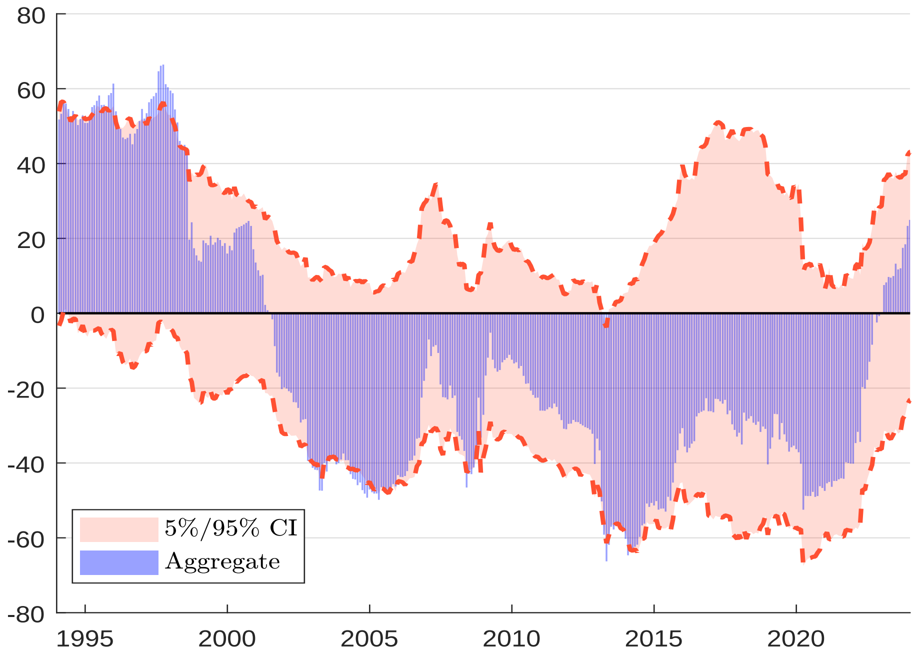


Figure 103: Confidence interval of the individual stock-bond correlation (US, monthly return, $\alpha = 10\%$)

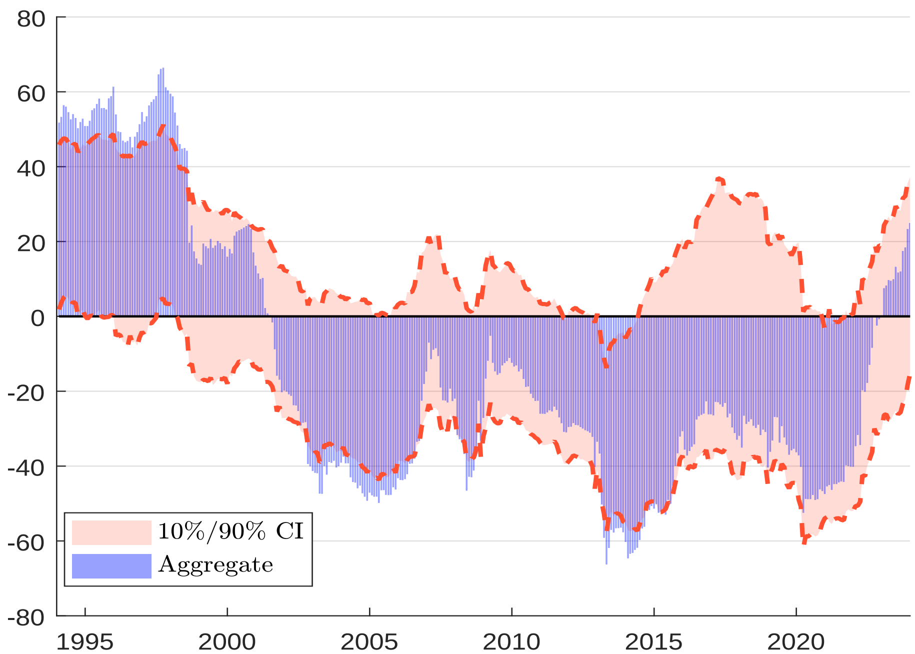


Figure 104: Equity duration (US, Equation 11)

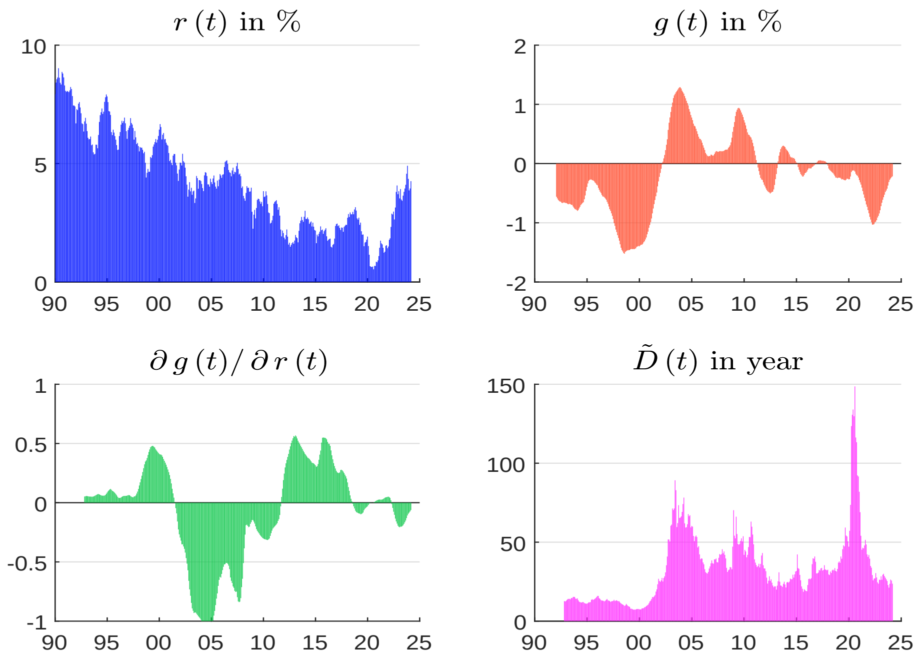


Figure 105: Dynamics of $\rho_{S,B}(t)$ and $y(t) - 5\%$ (US, 1965-2023)

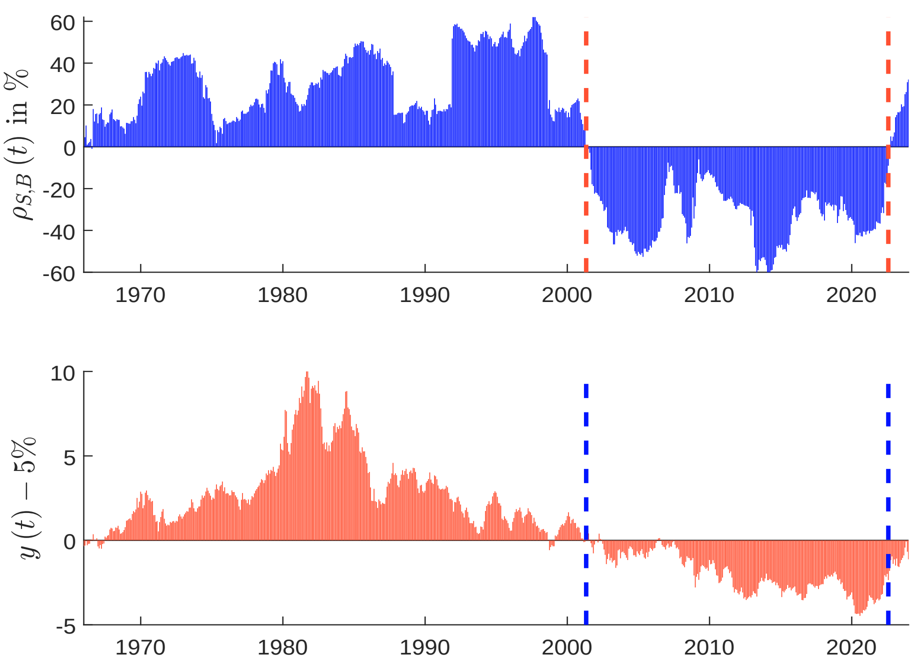


Figure 106: Dynamics of the stock-bond correlation $\rho_{S,B}(t)$ and the fed funds rate $r^\star(t)$ (US, 1965-2023)

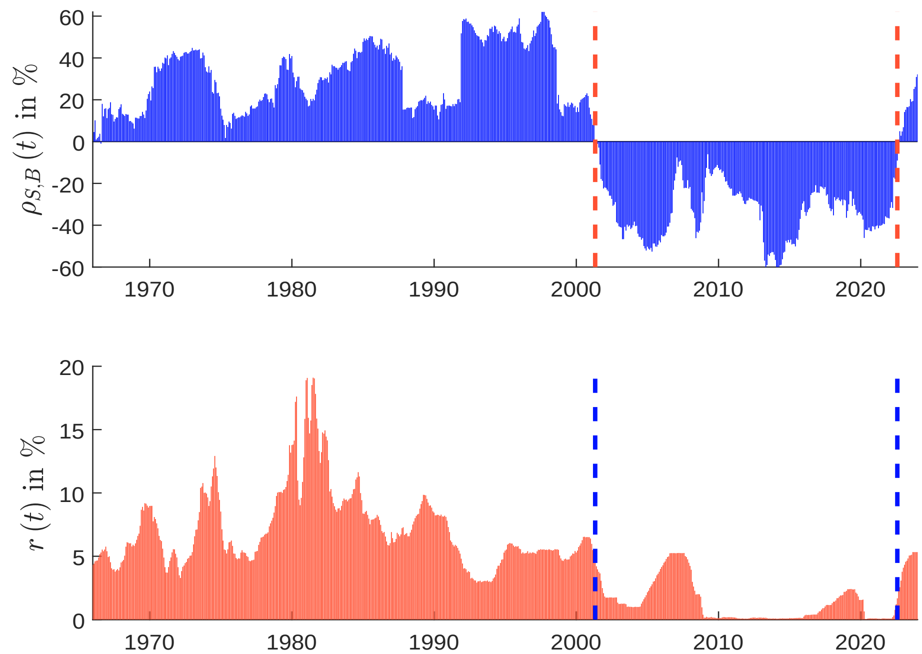


Figure 107: Dynamics of the stock-bond correlation $\rho_{S,B}(t)$ and the FOMC target rate $r^\star(t)$ (US, 1965-2023)

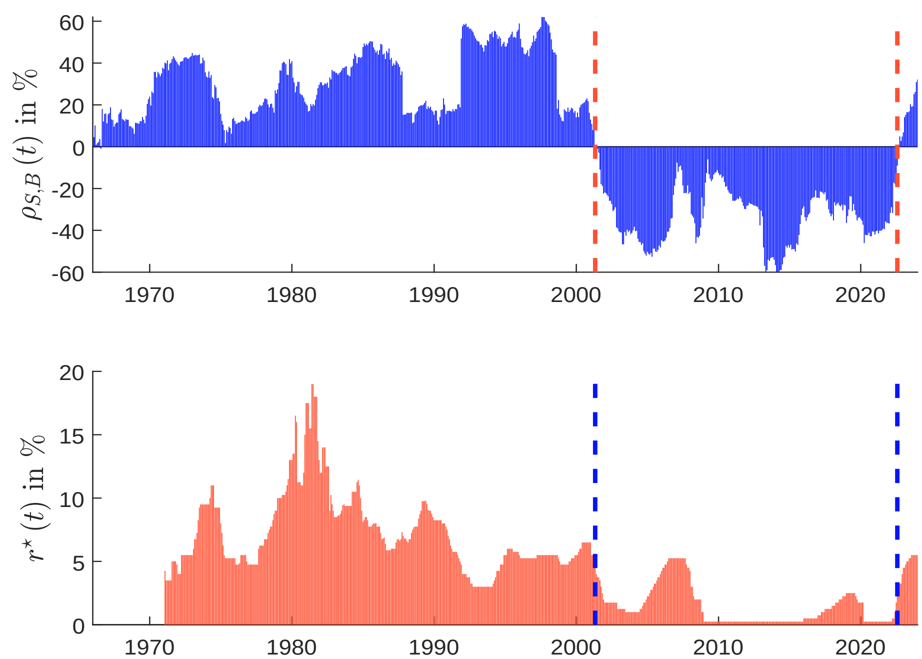


Figure 108: Dynamics of the stock-bond correlation $\rho_{S,B}(t)$ and the inflation level $\pi(t)$ (US, 1965-2023)

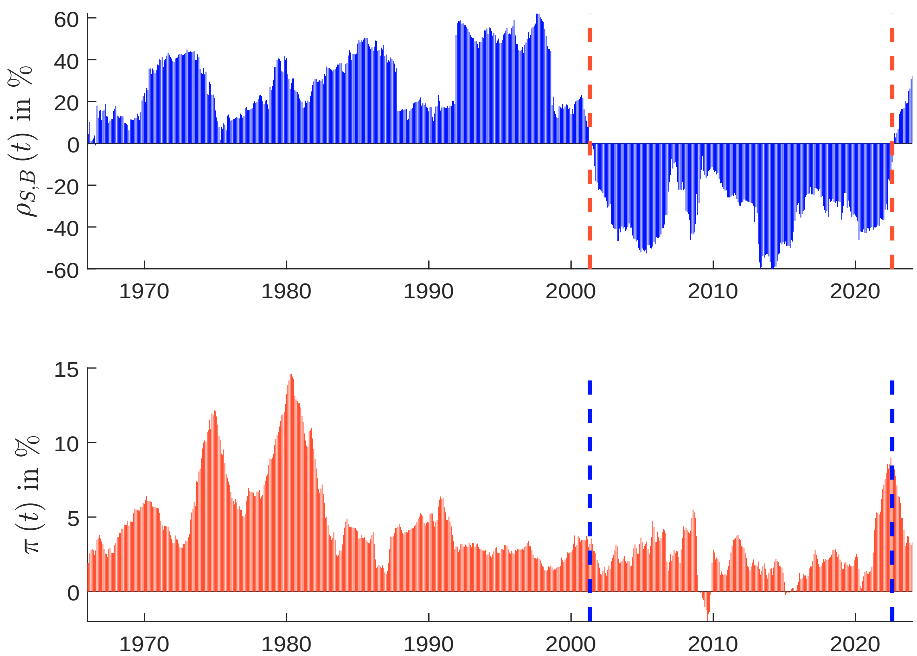


Figure 109: Dynamics of the stock-bond correlation $\rho_{S,B}(t)$ and the inflation volatility $\sigma_\pi(t)$ (US, 1965-2023)

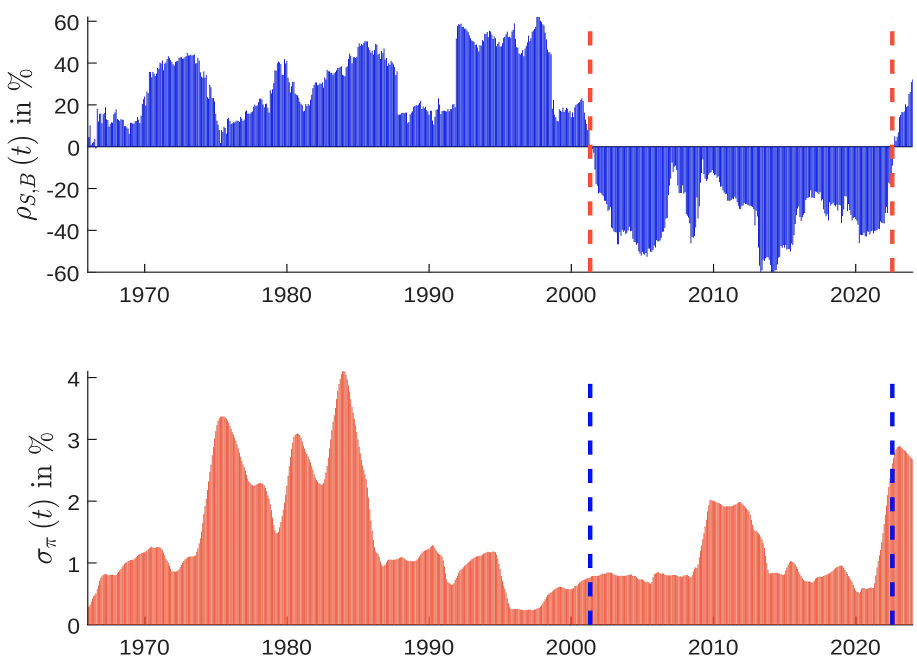


Figure 110: Three measures of growth volatility $\sigma_g(t)$ (US, 1965-2023)

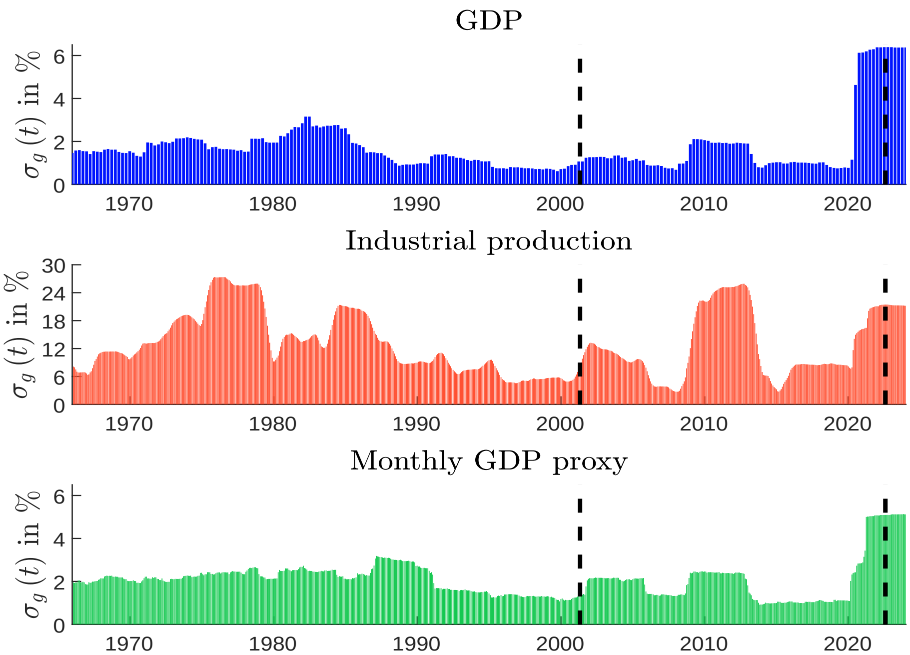


Figure 111: Pearson correlation of the market regime ($\mathcal{N}ormal$, $\mathcal{N}ormal$) (US, daily returns)

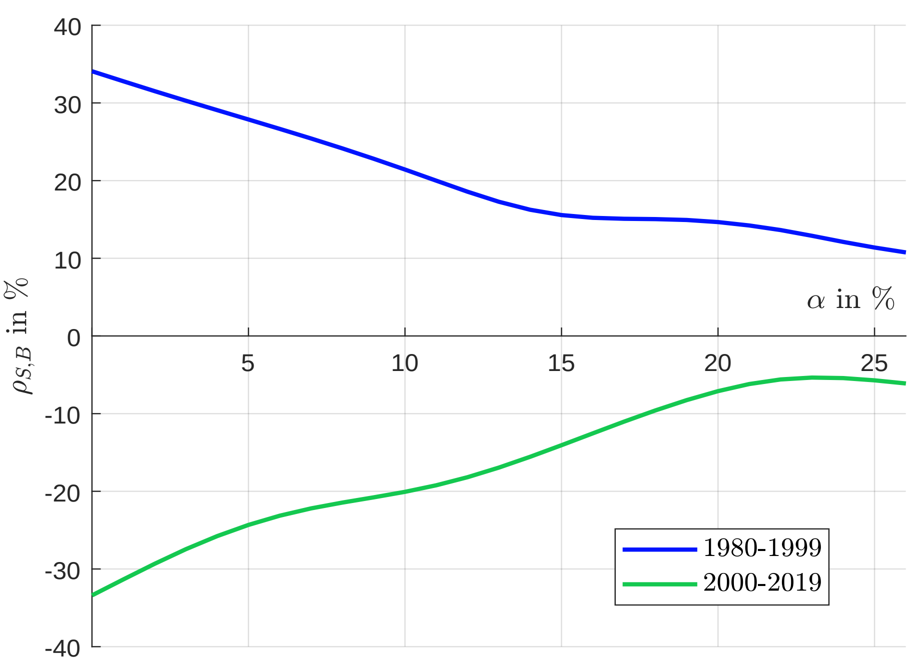


Figure 112: Estimated stock-bond payoff (US, monthly return, 1980-1999)

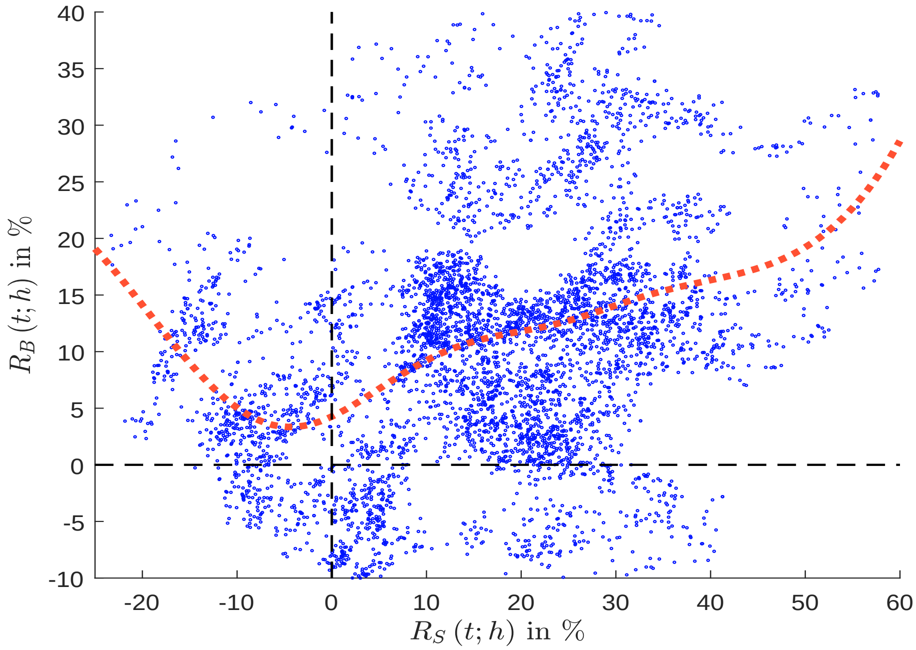


Figure 113: Estimated stock-bond payoff (US, monthly return, 2000-2019)

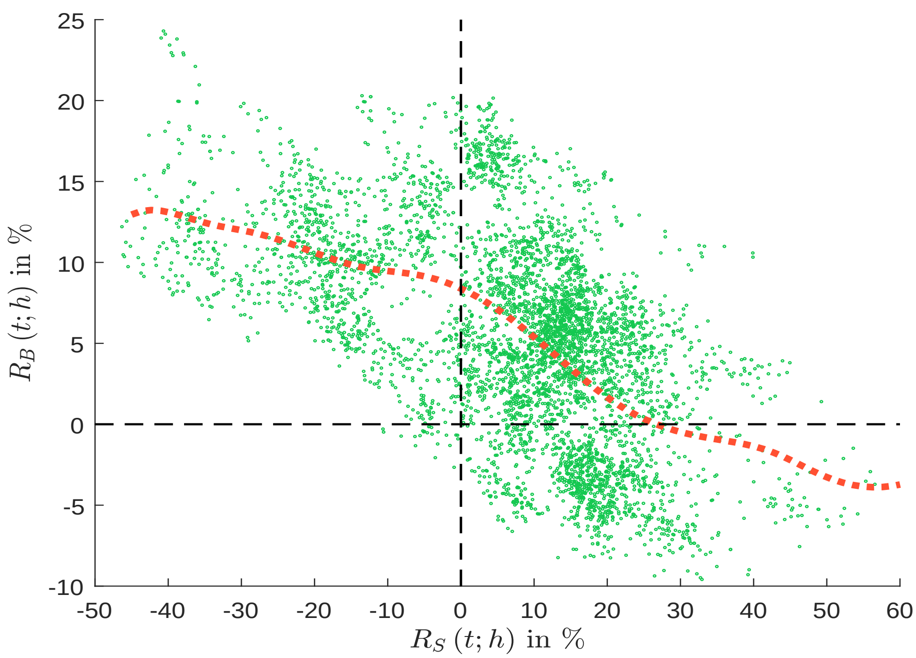


Figure 114: Estimated stock-bond payoff (US, 3-year return, 1980-1999)

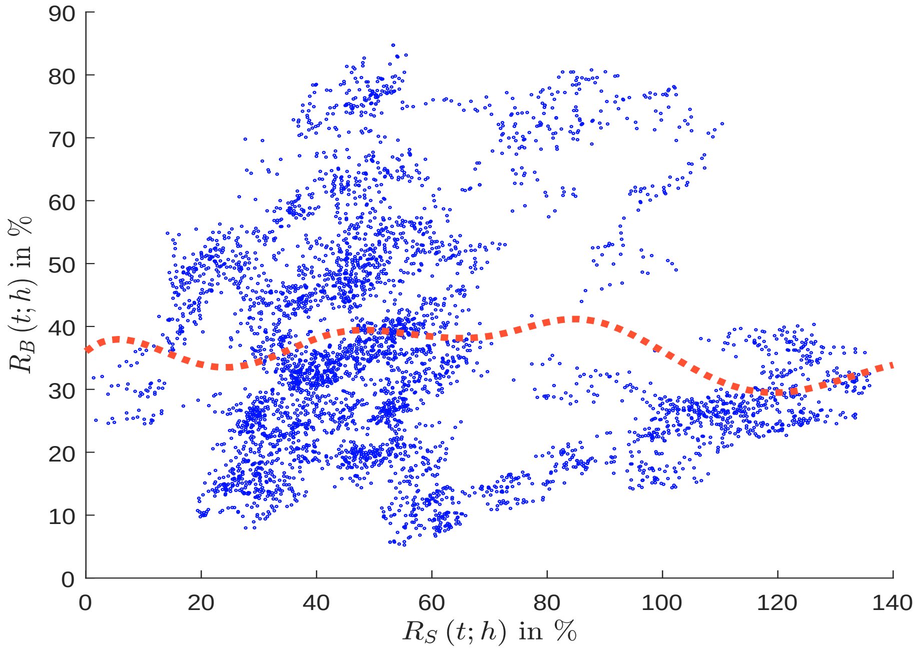


Figure 115: Estimated stock-bond payoff (US, 3-year return, 2000-2019)

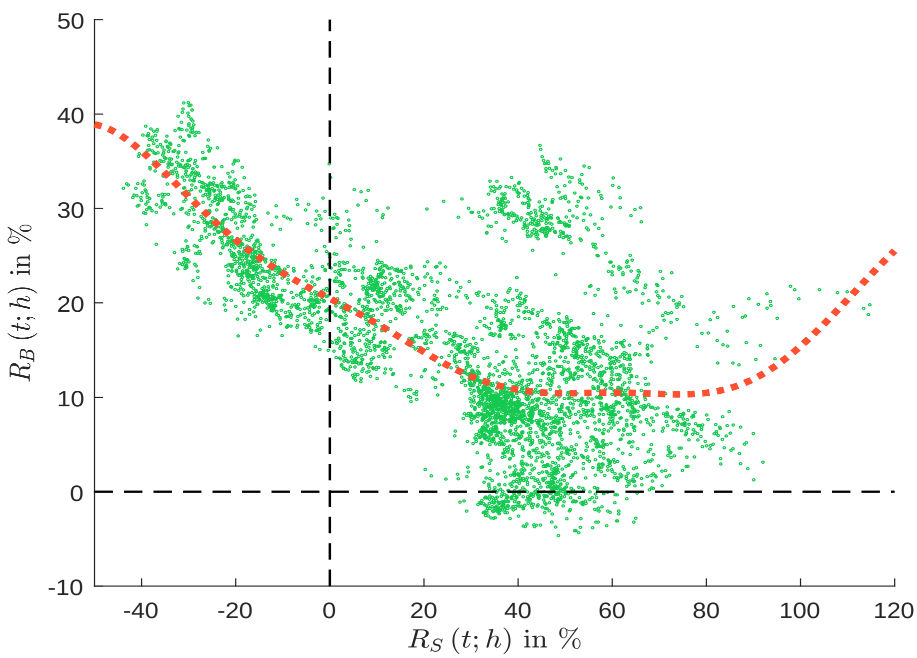


Figure 116: Estimated stock-bond payoff (Italy, daily return, 2010-2023)

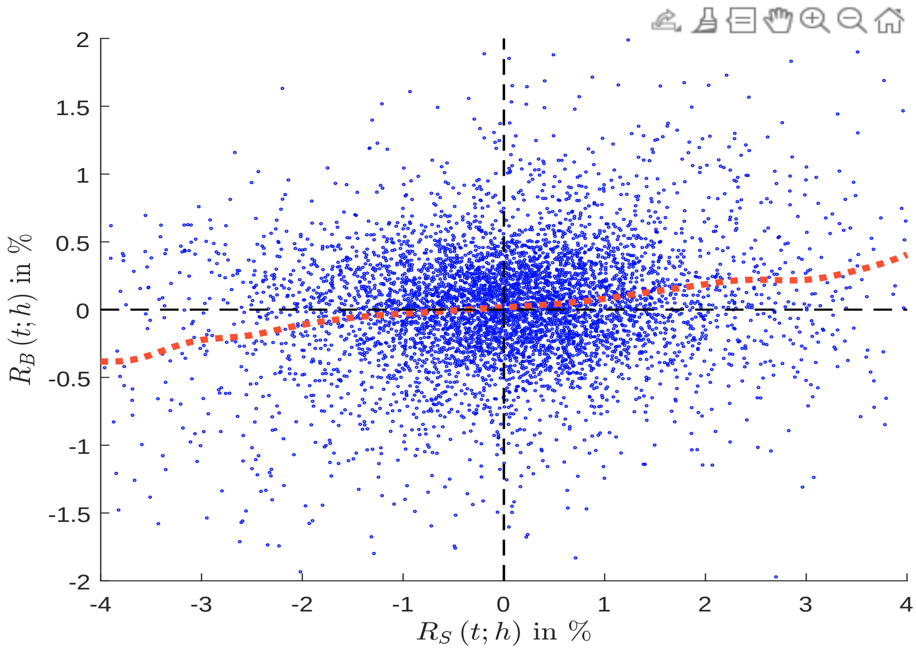
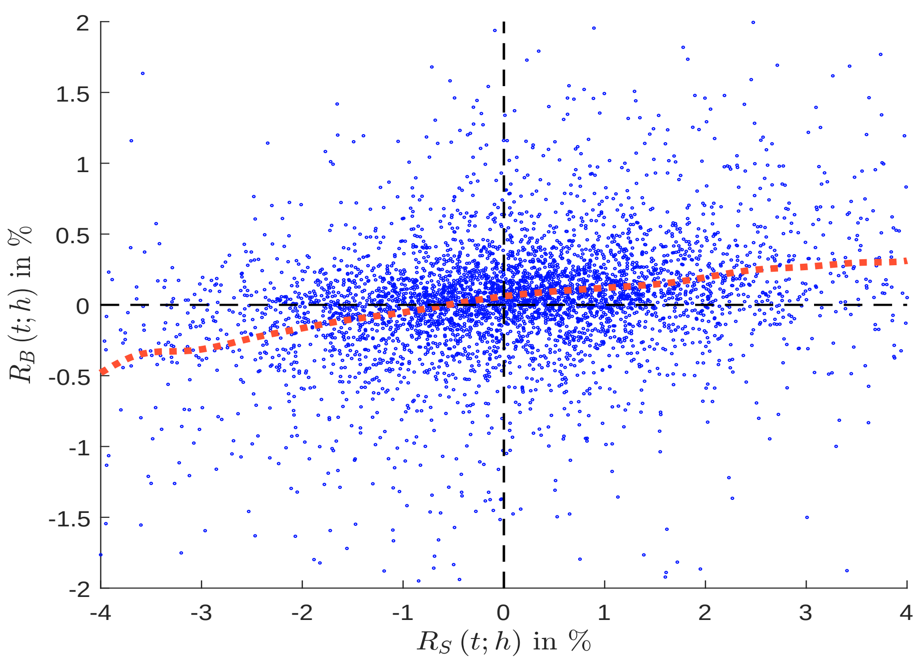


Figure 117: Estimated stock-bond payoff (Turkey, daily return, 2010-2023)



B.2 Tables

Table 26: Stock-bond correlation (1990-1999)

AUS	41.7	AUT	15.6	BEL	44.6	BRA	
BGR		CAN	40.5	CHL		CHN	
COL		CZE		DNK	46.0	FIN	27.9
FRA	44.0	DEU	27.8	GRC		HKG	
HUN		IND		IDN		IRL	40.2
ISR		ITA	47.0	JPN	4.8	KOR	
MYS		MEX		NLD	29.7	NZL	32.6
NOR	33.8	PER		PHL		POL	
PRT	30.5	ROU		RUS		SGP	
ZAF		ESP	45.9	SWE	36.7	CHE	33.9
TWN		TUR		GBR	55.1	USA	35.5

Source: Datastream (2024) & Authors' calculations.

Table 27: Stock-bond correlation (2000-2009)

AUS	−26.0	AUT	−17.3	BEL	−16.9	BRA	
BGR	31.0	CAN	−10.5	CHL		CHN	−43.4
COL		CZE	16.6	DNK	−25.8	FIN	−20.8
FRA	−33.7	DEU	−42.1	GRC	5.8	HKG	−16.5
HUN	52.9	IND		IDN		IRL	−9.4
ISR		ITA	−11.1	JPN	−38.1	KOR	
MYS		MEX		NLD	−31.6	NZL	−6.5
NOR	−31.7	PER		PHL		POL	31.0
PRT	−11.8	ROU		RUS		SGP	
ZAF	4.5	ESP	−12.4	SWE	−27.5	CHE	−26.4
TWN		TUR		GBR	−16.7	USA	−24.6

Source: Datastream (2024) & Authors' calculations.

Table 28: Stock-bond correlation (2010-2019)

AUS	−21.2	AUT	−20.5	BEL	20.4	BRA	42.2
BGR	−12.9	CAN	−29.7	CHL		CHN	−9.4
COL	37.2	CZE	0.6	DNK	−5.8	FIN	−11.9
FRA	−12.6	DEU	−27.9	GRC	44.6	HKG	3.1
HUN	31.3	IND	4.8	IDN	57.7	IRL	1.5
ISR	0.5	ITA	36.1	JPN	−41.8	KOR	−16.7
MYS	23.8	MEX	20.2	NLD	−12.9	NZL	9.3
NOR	−38.6	PER	33.6	PHL	43.8	POL	15.5
PRT	35.5	ROU	24.5	RUS	35.7	SGP	3.8
ZAF	30.9	ESP	34.6	SWE	−28.1	CHE	−23.7
TWN	−14.4	TUR	49.4	GBR	−11.3	USA	−48.2

Source: Datastream (2024) & Authors' calculations.

Table 29: Stock-bond correlation (2020-2023)

AUS	24.8	AUT	26.0	BEL	19.3	BRA	61.2
BGR	21.2	CAN	21.1	CHL	21.9	CHN	−32.0
COL	21.2	CZE	1.6	DNK	45.4	FIN	35.8
FRA	39.5	DEU	34.4	GRC	18.5	HKG	31.9
HUN	40.1	IND	11.9	IDN	53.0	IRL	40.3
ISR	52.1	ITA	41.8	JPN	17.8	KOR	46.8
MYS	26.1	MEX	38.7	NLD	60.3	NZL	52.8
NOR	−29.1	PER	39.7	PHL	41.1	POL	27.1
PRT	38.4	ROU	46.8	RUS		SGP	12.8
ZAF	57.8	ESP	22.9	SWE	37.3	CHE	43.2
TWN	−3.8	TUR	18.9	GBR	17.2	USA	25.8

Source: Datastream (2024) & Authors' calculations.

Table 30: Difference $\rho_{S,B}^{\text{Sector}} - \rho_{S,B}^{\text{Index}}$ in % (MSCI EMU)

Sector	2000 2004	2005 2009	2010 2014	2015 2019	2020 2023	1995 2003
Communication Services	7.8	25.5	26.3	11.4	5.3	14.7
Consumer Discretionary	4.2	−2.9	8.9	2.3	−0.0	2.8
Consumer Staples	11.9	12.4	21.0	40.6	29.6	22.3
Energy	11.0	10.3	−11.6	1.7	−28.0	−2.4
Financials	4.9	2.5	4.2	−28.9	−25.5	−7.7
Health Care	21.2	13.3	21.5	20.5	11.3	17.4
Industrials	−4.2	−0.3	5.9	4.3	3.0	1.4
Information Technology	−5.3	9.9	8.7	13.0	14.9	7.1
Materials	0.5	−14.5	−0.3	−3.5	−4.6	−4.9
Real Estate					24.2	24.2
Utilities	23.0	23.5	0.6	15.8	31.8	18.1

Source: Datastream (2024) & Authors' calculations.

Table 31: Difference $\rho_{S,B}^{\text{Sector}} - \rho_{S,B}^{\text{Index}}$ in % (MSCI Europe in \$)

Sector	2000 2004	2005 2009	2010 2014	2015 2019	2020 2023	1995 2003
Communication Services	3.4	16.7	15.4	7.1	1.9	8.8
Consumer Discretionary	0.1	−1.2	4.1	1.0	1.9	1.3
Consumer Staples	19.3	16.7	12.8	34.7	23.7	20.6
Energy	13.2	7.4	−9.7	5.2	−22.4	−0.4
Financials	4.0	1.9	4.2	−21.8	−18.9	−5.4
Health Care	22.3	9.6	20.8	22.1	13.4	17.6
Industrials	−11.4	−4.2	2.8	3.5	5.6	−1.6
Information Technology	−13.5	4.7	2.7	14.4	13.3	3.4
Materials	−3.4	−11.9	−2.6	−2.5	−5.7	−5.7
Real Estate			14.1	28.6	15.6	19.8
Utilities	31.4	27.5	−0.9	18.1	30.4	21.0

Source: Datastream (2024) & Authors' calculations.

Table 32: Difference $\rho_{S,B}^{\text{Factor}} - \rho_{S,B}^{\text{Index}}$ in % (MSCI Europe in \$)

Period	Value	Growth	Low Vol.	Momen-tum	High Div.	Qua-lity
2005-2009	−4.3	−8.9	15.1	2.3	5.7	2.0
2010-2014	3.1	1.3	4.7	−2.6	4.0	1.8
2015-2019	9.3	32.5	15.0	12.8	4.4	16.5
2020-2023	−8.3	23.6	14.8	7.7	−3.8	12.8
2000-2023	0.5	12.6	12.2	5.1	2.7	8.4

Source: Datastream (2024) & Authors' calculations.

Table 33: Linear regression of the stock-bond correlation on economic factors (US, 1965-2023, yield-inflation)

h	1M/1Q	1Y	2Y	3Y	5Y
$\hat{\beta}_0$	-0.18^{***}	-0.35^{***}	-0.39^{***}	-0.39^{***}	-0.41^{***}
$\hat{\beta}_y$	8.03^{***}	9.33^{***}	9.98^{***}	10.08^{***}	9.25^{***}
$\hat{\beta}_\pi$	-3.68	-0.89^{**}	-0.73^{***}	-0.95^{***}	-0.64^{***}
$\hat{\beta}_{\sigma_\pi}$	-25.61^{***}	-7.40^{***}	-5.35^{***}	-1.42	4.74^{***}
$\mathfrak{R}_c^2$	55.01%	52.41%	52.32%	51.49%	52.92%

Table 34: Linear regression of the stock-bond correlation on economic factors (US, 1965-2023, yield-inflation-growth)

	h	1M/1Q	1Y	2Y	3Y	5Y
GDP	$\hat{\beta}_0$	-0.19^{***}	-0.44^{***}	-0.48^{***}	-0.42^{***}	-0.43^{***}
	$\hat{\beta}_y$	8.30^{***}	9.40^{***}	9.24^{***}	8.73^{***}	8.11^{***}
	$\hat{\beta}_\pi$	-9.96^{*}	-1.58^{*}	-1.52^{***}	-1.50^{***}	-0.76^{**}
	$\hat{\beta}_{\sigma_\pi}$	-31.30^{***}	-11.54^{***}	-6.20^{***}	-0.20	5.33^{**}
	$\hat{\beta}_g$	-0.20	1.26^{**}	1.43^{***}	1.05^{***}	0.36
	$\hat{\beta}_{\sigma_g}$	2.14^{***}	2.68^{**}	0.75	-2.23^{**}	-1.05
	$\mathfrak{R}_c^2$	56.03%	53.77%	53.83%	54.36%	53.13%
Industrial production	$\hat{\beta}_0$	-0.23^{***}	-0.43^{***}	-0.43^{***}	-0.37^{***}	-0.40^{***}
	$\hat{\beta}_y$	8.50^{***}	9.59^{***}	9.30^{***}	7.51^{***}	6.80^{***}
	$\hat{\beta}_\pi$	-8.47^{***}	-0.95^{**}	-0.61^{**}	-0.31	-0.23^{*}
	$\hat{\beta}_{\sigma_\pi}$	-34.33^{***}	-11.56^{***}	-4.82^{***}	2.52^{**}	6.08^{***}
	$\hat{\beta}_g$	1.80^{**}	0.88^{***}	0.88^{***}	0.95^{***}	0.61^{***}
	$\hat{\beta}_{\sigma_g}$	3.84^{***}	2.73^{***}	0.83	-2.52^{***}	-1.90^{**}
	$\mathfrak{R}_c^2$	57.90%	55.51%	55.66%	56.92%	56.44%
Monthly GDP proxy	$\hat{\beta}_0$	-0.24^{***}	-0.59^{***}	-0.64^{***}	-0.63^{***}	-0.73^{***}
	$\hat{\beta}_y$	7.95^{***}	9.97^{***}	9.26^{***}	8.19^{***}	7.65^{***}
	$\hat{\beta}_\pi$	-7.02^{**}	-1.44^{***}	-1.33^{***}	-1.33^{***}	-0.87^{***}
	$\hat{\beta}_{\sigma_\pi}$	-29.86^{***}	-13.93^{***}	-6.22^{***}	-0.32	3.98^{***}
	$\hat{\beta}_g$	2.15	1.52^{***}	1.39^{***}	1.17^{***}	0.72^{***}
	$\hat{\beta}_{\sigma_g}$	7.14^{***}	9.49^{***}	2.72^{*}	-3.29^{**}	3.64^{*}
	$\mathfrak{R}_c^2$	56.13%	56.59%	55.97%	56.47%	55.35%