Solution of the Financial Risk Management Examination

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Remark 1 *The first five questions are corrected in TR-GDR*[1] *and in the document of exercise solutions, which is available on my web page*[2].

1 The BCBS regulation

2 Market risk

3 Credit risk

4 Counterparty credit risk

5 Operational risk

6 Value-at-risk of a long-short portfolio

The main reference for this exercise is TR-GDR on pages 61-63. We note $P_A(t)$ (resp. $P_B(t)$) the value of the stock A (resp. B) at time t. The portfolio value is:

$$P(t) = x_A \cdot P_A(t) + x_B \cdot P_B(t)$$

with x_A and x_B the number of stocks A and B. We deduce that the PnL between t and $t+1$ is:

$$\begin{aligned}
\mathrm{PnL}(t; t+1) &= P(t+1) - P(t) \\
&= x_A (P_A(t+1) - P_A(t)) + x_B (P_B(t+1) - P_B(t)) \\
&= x_A P_A(t) R_A(t; t+1) + x_B P_B(t) R_B(t; t+1) \\
&= W_A(t) R_A(t; t+1) + W_B(t) R_B(t; t+1)
\end{aligned}$$

where $R_A(t; t+1)$ and $R_B(t; t+1)$ are the asset returns of A and B between t and $t+1$, and $W_A(t)$ and $W_B(t)$ are the wealth invested in stocks A and B at time t.

1. We have $W_A = +2$ and $W_B = -1$. The PnL (expressed in millions of euros) has the following expression:

$$\mathrm{PnL} = 2R_A - R_B$$

We have $\mathrm{PnL} \sim \mathcal{N}(0, \sigma_{\mathrm{PnL}})$ with:

$$\begin{aligned}
\sigma_{\mathrm{PnL}} &= \sqrt{(2\sigma_A)^2 + (-\sigma_B)^2 + 2\rho_{A,B} \cdot (2\sigma_A) \cdot (-\sigma_B)} \\
&= \sqrt{4 \times 0.20^2 + (-0.20)^2 - 4 \times 0.5 \times 0.20^2} \\
&= \sqrt{3} \times 20\% \\
&\simeq 34.64\%
\end{aligned}$$

[1] Thierry Roncalli, *La Gestion des Risques Financiers*, Economica, deuxième édition, 2009.

[2] The direct link is `www.thierry-roncalli.com/download/gdr-correction.pdf`.

The annual volatility (expressed in millions of euros) of the long-short portfolio is then equal to 34.64%. To compute the value at risk for a time horizon of one day, we consider the square root rule (TR-GDR, page 74). We obtain[3]:

$$\begin{aligned}\mathrm{VaR}_{1D} &= \frac{1}{\sqrt{260}} \times \Phi^{-1}(0.99) \times \sqrt{3} \times 20\% \\ &= 5.01\%\end{aligned}$$

The probability to lose 50 100 euros per day is equal to 1%.

2. We use the historical data to calculate the scenarios of asset returns (R_A, R_B). We then deduce the empirical distribution of the PnL, which is equal to $2R_A - R_B$. Finally, we compute the corresponding empirical quantile. With 250 scenarios, the 1% decile is between the second and third worst cases:

$$\begin{aligned}\mathrm{VaR}_{1D} &= -\left[-56\,850 + \frac{1}{2}(-54\,270 - (-56\,850))\right] \\ &= 55\,560\end{aligned}$$

The probability to lose 55 560 euros per day is equal to 1%. We notice that the difference between the historical VaR and the gaussian VaR is equal to 11%.

3. If $\rho_{A,B} = -0.50$, the volatility of the PnL becomes:

$$\begin{aligned}\sigma_{\mathrm{PnL}} &= \sqrt{4 \times 0.20^2 + (-0.20)^2 - 4 \times (-0.5) \times 0.20^2} \\ &= \sqrt{7} \times 20\%\end{aligned}$$

We deduce that:

$$\frac{\mathrm{VaR}_{1D}(\rho_{A,B} = -50\%)}{\mathrm{VaR}_{1D}(\rho_{A,B} = +50\%)} = \frac{\sigma_{\mathrm{PnL}}(\rho_{A,B} = -50\%)}{\sigma_{\mathrm{PnL}}(\rho_{A,B} = +50\%)} = \sqrt{\frac{7}{3}} = 1.53$$

The value-at-risk increases because the hedging effect of the positive correlation vanishes. With a negative correlation, a long-short portfolio becomes more risky than a long-only portfolio.

4. The PnL formula becomes (TR-GDR, pages 91-95) :

$$\begin{aligned}\mathrm{PnL}(t; t+1) &= W_A(t) R_A(t; t+1) + W_B(t) R_B(t; t+1) - \\ &\quad (C_A(t+1) - C_A(t))\end{aligned}$$

with $C_A(t)$ the call option price. We have:

$$C_A(t+1) - C_A(t) \simeq \Delta_A \cdot (P_A(t+1) - P_A(t))$$

where Δ_A is the delta of the option. If the nominal of the option is 2 millions of euros, we obtain:

$$\begin{aligned}\mathrm{PnL} &= 2R_A - R_B - 2 \times 0.5 \times R_A \\ &= R_A - R_B \qquad (1)\end{aligned}$$

and:

$$\begin{aligned}\sigma_{\mathrm{PnL}} &= \sqrt{0.20^2 + (-0.20)^2 - 2 \times 0.5 \times 0.20^2} \\ &= 20\%\end{aligned}$$

If the nominal of the option is 4 millions of euros, we obtain:

$$\begin{aligned}\mathrm{PnL} &= 2R_A - R_B - 4 \times 0.5 \times R_A \\ &= -R_B \qquad (2)\end{aligned}$$

[3] because $\Phi^{-1}(0.99) = -\Phi^{-1}(0.01)$.

and:

$$\sigma_{\mathrm{PnL}} = 20\%$$

In both cases, we have:

$$\begin{aligned}
\mathrm{VaR}_{1D} &= \frac{1}{\sqrt{260}} \times \Phi^{-1}(0.99) \times 20\% \\
&= 28\,900\,€
\end{aligned}$$

The value-at-risk of the long-short portfolio (1) is then equal to the value-at-risk of the short portfolio (2) because of two effects: the absolute exposure of the long-short portfolio is higher than the absolute exposure of the short portfolio, but a part of the risk exposure of the long-short portfolio is hedged by the positive correlation between the two stocks.

5. We have:

$$\mathrm{PnL}(t; t+1) = W_A(t) R_A(t; t+1) - (C_B(t+1) - C_B(t))$$

with $C_B(t)$ the call option price. We have:

$$C_B(t+1) - C_B(t) \simeq \Delta_B \cdot (P_B(t+1) - P_B(t))$$

where Δ_B is the delta of the option. If the nominal of the option is x millions of euros, we obtain:

$$\begin{aligned}
\mathrm{PnL} &= 2R_A - x \cdot \Delta_B \cdot R_B \\
&= 2R_A - \frac{x}{2} R_B
\end{aligned}$$

We have[4]:

$$\begin{aligned}
\sigma^2_{\mathrm{PnL}} &= 4\sigma_A^2 + \frac{x^2}{4}\sigma_B^2 + 2\rho_{A,B} \cdot (2\sigma_A) \cdot \left(-\frac{x}{2}\sigma_B\right) \\
&= \frac{\sigma_A^2}{4}\left(x^2 - 8\rho_{A,B}x + 16\right)
\end{aligned}$$

Minimizing the Gaussian value-at-risk is equivalent to minimizing the variance of the PnL. We deduce that the first-order condition is:

$$\frac{\partial\, \sigma^2_{\mathrm{PnL}}}{\partial\, x} = \frac{\sigma_A^2}{4}(2x - 8\rho_{A,B}) = 0$$

We deduce that the minimum value-at-risk is reached when the nominal of the option is:

$$x = 4\rho_{A,B}$$

We finally obtain:

$$\begin{aligned}
\sigma_{\mathrm{PnL}} &= \frac{\sigma_A}{2}\sqrt{16\rho_{A,B}^2 - 32\rho_{A,B}^2 + 16} \\
&= 2\sigma_A\sqrt{1 - \rho_{A,B}^2}
\end{aligned}$$

and:

$$\begin{aligned}
\mathrm{VaR}_{1D} &= \frac{1}{\sqrt{260}} \times 2.33 \times 2 \times 20\% \times \sqrt{1 - \rho_{A,B}^2} \\
&\simeq 5.78\% \times \sqrt{1 - \rho_{A,B}^2}
\end{aligned}$$

If $\rho_{A,B}$ is negative (resp. positive), the exposure x is negative meaning that we have to buy (resp. to sell) a call option on stock B in order to hedge a part of the risk related to stock A. If $\rho_{A,B}$ is equal to zero, the exposure x is equal to zero because a position on stock B adds systematically a supplementary risk to the portfolio.

[4] Because $\sigma_A = \sigma_B = 20\%$.

7 Estimating frequency and severity distributions for operational risk

1. (a) The probability density function of $\mathcal{LN}(\mu,\sigma)$ is (TR-GDR, page 239):

$$f(x)=\frac{1}{\sigma x\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2\right)$$

For $m\geq 1$, the non-centered moment is:

$$\mathbb{E}\left[L_i^m\right]=\int_0^\infty x^m\frac{1}{\sigma x\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2\right)\,\mathrm{d}x$$

By considering the change of variables $y=\sigma^{-1}(\ln x-\mu)$ and $z=y-\sigma$, we obtain:

$$\begin{aligned}
\mathbb{E}\left[L_i^m\right] &= \int_{-\infty}^{\infty}\exp(m\mu+m\sigma y)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}y^2\right)\,\mathrm{d}y \\
&= \exp(m\mu)\times\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}y^2+m\sigma y\right)\,\mathrm{d}y \\
&= \exp(m\mu)\times\exp\left(\frac{1}{2}m^2\sigma^2\right)\times\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(y-m\sigma)^2\right)\,\mathrm{d}y \\
&= \exp(m\mu)\times\exp\left(\frac{1}{2}m^2\sigma^2\right)\times\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z^2\right)\,\mathrm{d}z \\
&= \exp(m\mu)\times\exp\left(\frac{1}{2}m^2\sigma^2\right)\times\left[\Phi(z)\right]_{-\infty}^{\infty} \\
&= \exp\left(m\mu+\frac{1}{2}m^2\sigma^2\right)
\end{aligned}$$

We can deduce that:

$$\mathbb{E}\left[L_i\right]=e^{\left(\mu+\frac{1}{2}\sigma^2\right)}$$

and:

$$\begin{aligned}
\operatorname{var}(L_i) &= \mathbb{E}\left[L_i^2\right]-\mathbb{E}^2\left[L_i\right] \\
&= e^{\left(2\mu+2\sigma^2\right)}-e^{\left(2\mu+\sigma^2\right)} \\
&= e^{\left(2\mu+\sigma^2\right)}\left(e^{\sigma^2}-1\right)
\end{aligned}$$

We can estimate the parameters μ and σ with the generalized method of moments by using the following empirical moments (TR-GDR, pages 239):

$$\begin{aligned}
h_{i,1}(\mu,\sigma) &= L_i-e^{\left(\mu+\frac{1}{2}\sigma^2\right)} \\
h_{i,2}(\mu,\sigma) &= \left(L_i-e^{\left(\mu+\frac{1}{2}\sigma^2\right)}\right)^2-e^{\left(2\mu+\sigma^2\right)}\left(e^{\sigma^2}-1\right)
\end{aligned}$$

(b) The log-likelihood function is:

$$\begin{aligned}
\ell(\mu,\sigma) &= \sum_{i=1}^n\ln f(L_i) \\
&= -\frac{n}{2}\ln\sigma^2-\frac{n}{2}\ln 2\pi-\sum_{i=1}^n\ln L_i-\frac{1}{2}\sum_{i=1}^n\left(\frac{\ln L_i-\mu}{\sigma}\right)^2
\end{aligned}$$

To find the ML estimators $\hat{\mu}$ and $\hat{\sigma}$, we can proceed in two different ways:

#1 $Y \sim \mathcal{LN}(\mu, \sigma)$ implies that $X = \ln Y \sim \mathcal{N}(\mu, \sigma)$. We know that the ML estimators $\hat{\mu}$ and $\hat{\sigma}$ associated to X are:

$$\begin{aligned}
\hat{\mu} &= \frac{1}{n}\sum_{i=1}^{n} x_i \\
\hat{\sigma} &= \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2}
\end{aligned}$$

We deduce that the ML estimators $\hat{\mu}$ and $\hat{\sigma}$ associated to the sample $\{L_1, \ldots, L_n\}$ are:

$$\begin{aligned}
\hat{\mu} &= \frac{1}{n}\sum_{i=1}^{n} \ln L_i \\
\hat{\sigma} &= \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\ln L_i - \hat{\mu})^2}
\end{aligned}$$

#2 We maximize the log-likelihood function:

$$\{\hat{\mu}, \hat{\sigma}\} = \arg\max \ell(\mu, \sigma)$$

The first-order condition is:

$$\left\{\begin{array}{l} \partial_{\mu}\, \ell(\mu, \sigma) = 0 \\ \partial_{\sigma}\, \ell(\mu, \sigma) = 0 \end{array}\right.$$

We have:

$$\partial_{\mu}\, \ell(\mu, \sigma) = \frac{1}{\sigma^2}\sum_{i=1}^{n}(\ln L_i - \mu) = 0$$

and:

$$\partial_{\sigma}\, \ell(\mu, \sigma) = -\frac{n}{\sigma} + \sum_{i=1}^{n}\frac{(\ln L_i - \mu)^2}{\sigma^3} = 0$$

We finally obtain:

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} \ln L_i$$

and:

$$\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\ln L_i - \hat{\mu})^2}$$

(c) The probability density function is:

$$\begin{aligned}
f(x) &= \frac{\partial \Pr\{L \le x\}}{\partial x} \\
&= \theta \frac{x^{-(\theta+1)}}{x_-^{-\theta}}
\end{aligned}$$

For $m \ge 1$, we have:

$$\begin{aligned}
\mathbb{E}[L_i^m] &= \int_{x_-}^{\infty} x^m \theta \frac{x^{-(\theta+1)}}{x_-^{-\theta}}\, \mathrm{d}x \\
&= \frac{\theta}{x_-^{-\theta}} \int_{x_-}^{\infty} x^{m-\theta-1}\, \mathrm{d}x \\
&= \frac{\theta}{x_-^{-\theta}} \left[\frac{x^{m-\theta}}{m-\theta}\right]_{x_-}^{\infty} \\
&= \frac{\theta}{\theta - m} x_-^m
\end{aligned}$$

We deduce that:

$$\mathbb{E}[L] = \frac{\theta}{\theta - 1} x_-$$

and:

$$\begin{aligned} \operatorname{var}(L) &= \mathbb{E}\left[L^2\right] - \mathbb{E}^2[L] \\ &= \frac{\theta}{\theta - 2} x_-^2 - \left(\frac{\theta}{\theta - 1} x_-\right)^2 \\ &= \frac{\theta}{(\theta - 1)^2 (\theta - 2)} x_-^2 \end{aligned}$$

We can then estimate the parameter θ by considering the following empirical moments:

$$\begin{aligned} h_{i,1}(\theta) &= L_i - \frac{\theta}{\theta - 1} x_- \\ h_{i,2}(\theta) &= \left(L_i - \frac{\theta}{\theta - 1} x\right)^2 - \frac{\theta}{(\theta - 1)^2 (\theta - 2)} x_-^2 \end{aligned}$$

The generalized method of moments can consider either the first moment $h_{i,1}(\theta)$, the second moment $h_{i,2}(\theta)$ or the joint moments $(h_{i,1}(\theta), h_{i,2}(\theta))$. In the first case, the estimator is:

$$\hat{\theta} = \frac{\sum_{i=1}^n L_i}{\sum_{i=1}^n L_i - n x_-}$$

(d) The log-likelihood function is:

$$\begin{aligned} \ell(\theta) &= \sum_{i=1}^n \ln f(L_i) \\ &= n \ln \theta - (\theta + 1) \sum_{i=1}^n \ln L_i + n\theta \ln x_- \end{aligned}$$

The first-order condition is:

$$\partial_\theta \, \ell(\theta) = \frac{n}{\theta} - \sum_{i=1}^n \ln L_i + \sum_{i=1}^n \ln x_- = 0$$

We deduce that:

$$n = \theta \sum_{i=1}^n \ln \frac{L_i}{x_-}$$

The ML estimator is then:

$$\hat{\theta} = \frac{n}{\sum_{i=1}^n (\ln L_i - \ln x_-)}$$

(e) The probability density function of *(iii)* is:

$$\begin{aligned} f(x) &= \frac{\partial \Pr\{L \leq x\}}{\partial x} \\ &= \frac{\beta^\alpha x^{\alpha - 1} e^{-\beta x}}{\Gamma(\alpha)} \end{aligned}$$

It follows that the log-likelihood function is:

$$\begin{aligned} \ell(\alpha, \beta) &= \sum_{i=1}^n \ln f(L_i) \\ &= -n \ln \Gamma(\alpha) + n\alpha \ln \beta + (\alpha - 1) \sum_{i=1}^n \ln L_i - \beta \sum_{i=1}^n L_i \end{aligned}$$

The first-order conditions $\partial_\alpha \ell(\alpha,\beta)=0$ and $\partial_\beta \ell(\alpha,\beta)=0$ are:

$$n\left(\ln\beta-\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\right)+\sum_{i=1}^{n}\ln L_i=0$$

and:

$$n\frac{\alpha}{\beta}-\sum_{i=1}^{n}L_i=0$$

(f) Let $Y\sim\Gamma(\alpha,\beta)$ and $X=\exp Y$. We have:

$$f_X(x)\,|\mathrm{d}x|=f_Y(y)\,|\mathrm{d}y|$$

where f_X and f_Y are the probability density functions of X and Y. We deduce that:

$$\begin{aligned}
f_X(x) &= \frac{\beta^\alpha y^{\alpha-1}e^{-\beta y}}{\Gamma(\alpha)}\cdot\frac{1}{e^y}\\
&= \frac{\beta^\alpha(\ln x)^{\alpha-1}e^{-\beta\ln x}}{x\Gamma(\alpha)}\\
&= \frac{\beta^\alpha(\ln x)^{\alpha-1}}{\Gamma(\alpha)\,x^{\beta+1}}
\end{aligned}$$

The support of this probability density function is $[0,+\infty)$. The log-likelihood function associated to the sample of individual losses $\{L_1,\ldots,L_n\}$ is:

$$\begin{aligned}
\ell(\alpha,\beta) &= \sum_{i=1}^{n}\ln f(L_i)\\
&= -n\ln\Gamma(\alpha)+n\alpha\ln\beta+(\alpha-1)\sum_{i=1}^{n}\ln\ln L_i-(\beta+1)\sum_{i=1}^{n}\ln L_i
\end{aligned}$$

2. (a) The conditional probability density function is (TR-GDR, page 242):

$$\begin{aligned}
f(L_i=x\mid L_i\geq H) &= \frac{f(x)}{1-\mathbf{F}(H)}\\
&= \left(\theta\frac{x^{-(\theta+1)}}{x_-^{-\theta}}\right)\Big/\left(\frac{H^{-\theta}}{x_-^{-\theta}}\right)\\
&= \theta\frac{x^{-(\theta+1)}}{H^{-\theta}}
\end{aligned}$$

The conditional probability function is then a Pareto distribution with the same parameter θ but with a new threshold $x_-=H$. We can then deduce that the ML estimator $\hat{\theta}$ is:

$$\hat{\theta}=\frac{n}{\left(\sum_{i=1}^{n}\ln L_i\right)-n\ln H}$$

(b) The conditional probability density function is (TR-GDR, page 242):

$$\begin{aligned}
f(L_i=x\mid L_i\geq H) &= \frac{f(x)}{1-\mathbf{F}(H)}\\
&= \left(\frac{\beta^\alpha x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}\right)\Big/\int_H^\infty\frac{\beta^\alpha t^{\alpha-1}e^{-\beta t}}{\Gamma(\alpha)}\,\mathrm{d}t\\
&= \frac{\beta^\alpha x^{\alpha-1}e^{-\beta x}}{\int_H^\infty\beta^\alpha t^{\alpha-1}e^{-\beta t}\,\mathrm{d}t}
\end{aligned}$$

The log-likelihood function is:

$$\begin{aligned}
\ell(\alpha,\beta) &= \sum_{i=1}^{n}\ln f(L_i\mid L_i\geq H)\\
&= n\alpha\ln\beta-n\ln\left(\int_H^\infty\beta^\alpha t^{\alpha-1}e^{-\beta t}\,\mathrm{d}t\right)+(\alpha-1)\sum_{i=1}^{n}\ln L_i-\beta\sum_{i=1}^{n}L_i
\end{aligned}$$

3. (a) We have:

$$\Pr\{N = m\} = e^{-\lambda_Y} \frac{\lambda_Y^m}{m!}$$

We deduce that the expression of the log-likelihood is:

$$\begin{aligned} \ell(\lambda_Y) &= \sum_{t=1}^{T} \ln \Pr\{N = N_{Y_t}\} \\ &= -\lambda_Y T + \left(\sum_{t=1}^{T} N_{Y_t}\right) \ln \lambda_Y - \sum_{t=1}^{T} \ln N_{Y_t}! \end{aligned}$$

The first-order condition is:

$$\frac{\partial\, \ell(\lambda_Y)}{\partial\, \lambda_Y} = -T + \frac{1}{\lambda_Y}\left(\sum_{t=1}^{T} N_{Y_t}\right) = 0$$

We deduce that the ML estimator is:

$$\hat{\lambda}_Y = \frac{1}{T}\sum_{t=1}^{T} N_{Y_t} = \frac{n}{T}$$

(b) Using the same arguments, we obtain:

$$\hat{\lambda}_Q = \frac{1}{4T}\sum_{t=1}^{4T} N_{Q_t} = \frac{n}{4T} = \frac{\hat{\lambda}_Y}{4}$$

(c) Considering a quarterly or annual basis has no impact on the capital charge. Indeed, the capital charge is computed for a one-year time horizon. If we use a quarterly basis, we have to find the distribution of the annual loss number. In this case, the annual loss number is the sum of the four quarterly loss numbers:

$$N_Y = N_{Q_1} + N_{Q_2} + N_{Q_3} + N_{Q_4}$$

We know that each quarterly loss number follows a Poisson distribution $\mathcal{P}\left(\hat{\lambda}_Q\right)$ and that they are independent. Because the Poisson distribution is infinitely divisible, we obtain:

$$N_{Q_1} + N_{Q_2} + N_{Q_3} + N_{Q_4} \sim \mathcal{P}\left(4\hat{\lambda}_Q\right)$$

We deduce that the annual loss number follows a Poisson distribution $\mathcal{P}\left(\hat{\lambda}_Y\right)$ in both cases.

(d) This result remains valid if we consider the first moment because the MM estimator is exactly the ML estimator.

(e) Because $\operatorname{var}(\mathcal{P}(\lambda)) = \lambda$, the MM estimator in the case of annual loss numbers is:

$$\hat{\lambda}_Y = \frac{1}{T}\sum_{t=1}^{T} N_{Y_t}^2 - \frac{n^2}{T^2}$$

If we use a quarterly basis, we obtain:

$$\begin{aligned} \hat{\lambda}_Q &= \frac{1}{4}\left(\frac{1}{T}\sum_{t=1}^{4T} N_{Q_t}^2 - \frac{n^2}{4T^2}\right) \\ &\neq \frac{\hat{\lambda}_Y}{4} \end{aligned}$$

There is no reason that $\hat{\lambda}_Y = 4\hat{\lambda}_Q$ meaning that the capital charge will not be the same.

8 Counterparty credit risk

1. See TR-GDR on pages 214-215.

2. The exposure-at-default is random because it is not known at the starting date t_0. If the counterparty defaults at time t, we distinguish two cases. If the mark-to-market is negative, the exposure-at-default is equal to 0. If the mark-to-market is positive, the bank has to buy another OTC product with the same characteristics as previously in order to replace the existing OTC product. The exposure-at-default is then equal to the replacement value of the OTC product or the value of the mark-to-market. We finally obtain:

$$e(t) = \max(\mathrm{MTM}(t), 0)$$

where $\mathrm{MTM}(t)$ is the mark-to-market of the OTC contract at the future date t. The exposure-at-default $e(t)$ is then a random variable.

3. (a) Let $\mathrm{MtM}_A(\mathcal{C})$ and $\mathrm{MTM}_B(\mathcal{C})$ be the mark-to-market values of Bank A and Bank B for the contract $\mathcal{C}$. We must theoretically verify that:

$$\begin{aligned} \mathrm{MtM}_{A+B}(\mathcal{C}) &= \mathrm{MTM}_A(\mathcal{C}) + \mathrm{MTM}_B(\mathcal{C}) \\ &= 0 \end{aligned} \tag{3}$$

In the case of listed products, the previous relationship is verified. In the case of OTC products, there is no market prices and the bank uses models to valuate them. The mark-to-market value is then a mark-to-model value. Because the two banks do not use the same model with the same parameters, we notice a mismatch between the two mark-to-market values:

$$\mathrm{MTM}_A(\mathcal{C}) + \mathrm{MTM}_B(\mathcal{C}) \neq 0$$

For instance, we obtain:

$$\begin{aligned} \mathrm{MTM}_{A+B}(\mathcal{C}_1) &= 10 - 11 = -1 \\ \mathrm{MTM}_{A+B}(\mathcal{C}_2) &= -5 + 6 = 1 \\ \mathrm{MTM}_{A+B}(\mathcal{C}_3) &= 6 - 3 = 3 \\ \mathrm{MTM}_{A+B}(\mathcal{C}_4) &= 17 - 12 = 5 \\ \mathrm{MTM}_{A+B}(\mathcal{C}_5) &= -5 + 9 = 4 \\ \mathrm{MTM}_{A+B}(\mathcal{C}_6) &= -5 + 5 = 0 \\ \mathrm{MTM}_{A+B}(\mathcal{C}_7) &= 1 + 1 = 2 \end{aligned}$$

Only the contract $\mathcal{C}_6$ satisfies the relationship (3).

(b) We have (TR-GDR, pages 216-217):

$$\mathrm{EAD} = \sum_{i=1}^{I} \max(\mathrm{MTM}(\mathcal{C}_i), 0)$$

We then obtain:

$$\begin{aligned} \mathrm{EAD}_A &= 10 + 6 + 17 + 1 = 34 \\ \mathrm{EAD}_B &= 6 + 9 + 5 + 1 = 21 \end{aligned}$$

(c) We have (TR-GDR, page 217):

$$\mathrm{EAD} = \max\left(\sum_{i=1}^{I} \mathrm{MTM}(\mathcal{C}_i), 0\right)$$

We then obtain:

$$\begin{aligned} \mathrm{EAD}_A &= \max(10 - 5 + 6 + 17 - 5 - 5 + 1, 0) = \max(19, 0) = 19 \\ \mathrm{EAD}_B &= \max(-11 + 6 - 3 - 12 + 9 + 5 + 1, 0) = \max(-5, 0) = 0 \end{aligned}$$

(d) We have (TR-GDR, page 217):

$$\begin{aligned}
\text{EAD}_A &= \max(10-5+6,0)+17+1=29 \\
\text{EAD}_B &= \max(-11+6-3,0)+9+5+1=15
\end{aligned}$$

4. (a) It is obvious that we cannot use spot measures to compute the capital charge. This excludes formulas that are defined for a given future date t: potential future exposure, expected exposure and effective expected exposure. Only the peak exposure, the expected positive exposure and the effective expected positive exposure can be used to define the exposure-at-default. PE is a quantile risk measure whereas EPE and EEPE are weighted average risk measure. The Basle Committee has chosen the EEPE measure to define the exposure-at-default. We generally have:

$$\text{EAD} = 1.4 \times \text{EEPE}$$

We can justify this choice because:

i. PE may produce large and non-smoothed EAD;
ii. EPE is not necessarily an increasing function with respect to the time horizon h;
iii. EEPE is an increasing function with respect to the time horizon h.

(b) The cumulative distribution function of X is:

$$\begin{aligned}
\mathbf{F}(x) &= \Pr\{X \leq x\} \\
&= \int_0^x \frac{u^a}{a+1}\,\mathrm{d}u \\
&= x^{a+1}
\end{aligned}$$

We deduce that:

$$\begin{aligned}
\mathbf{F}_{[0,t]}(e) &= \Pr\{e(t) \leq e\} \\
&= \Pr\left\{\sigma\sqrt{t}X \leq e\right\} \\
&= \Pr\left\{X \leq \frac{e}{\sigma\sqrt{t}}\right\} \\
&= \left(\frac{e}{\sigma\sqrt{t}}\right)^{a+1}
\end{aligned}$$

and:

$$f_{[0,t]}(e) = \frac{(a+1)\,e^a}{\left(\sigma\sqrt{t}\right)^{a+1}}$$

It follows that:

$$\text{PFE}_\alpha(0;t) = \mathbf{F}_{[0,t]}^{-1}(\alpha) = \alpha^{1/(a+1)}\sigma\sqrt{t}$$

and:

$$\text{PE}_\alpha(0) = \alpha^{1/(a+1)}\sigma$$

because T is equal to 1. The expected exposure is:

$$\text{EE}(0;t) = \int_0^{\sigma\sqrt{t}} e\frac{(a+1)\,e^a}{\left(\sigma\sqrt{t}\right)^{a+1}}\,\mathrm{d}e = \frac{(a+1)\,\sigma\sqrt{t}}{a+2}$$

We deduce that:

$$\text{EEE}(0;t) = \frac{(a+1)\,\sigma\sqrt{t}}{a+2}$$

and:

$$\text{EEPE}(0;h) = \frac{1}{h}\int_0^h \frac{(a+1)\,\sigma\sqrt{t}}{a+2}\,\mathrm{d}t = \frac{2\,(a+1)\,\sigma\sqrt{h}}{3\,(a+2)}$$

From a regulatory point of view, the exposure-at-default is calculated with $h = 1$. We finally obtain:

$$\begin{aligned} \text{EAD} &= 1.4 \times \text{EEPE}\left(0;1\right) \\ &\simeq \frac{\left(a+1\right)\sigma}{a+2} \end{aligned}$$

(c) This product can not be an amortizing swap because EE is an increasing function with respect to t. It is more like an option profile.

9 Risk contribution in the Basle II model

1. The loss L follows a Gaussian probability distribution:

$$L \sim \mathcal{N}\left(0, \sqrt{x^\top \Sigma x}\right)$$

We deduce that:

$$\text{VaR}\left(x;\alpha\right) = \Phi^{-1}\left(\alpha\right)\sqrt{x^\top \Sigma x}$$

2. We have:

$$\begin{aligned} \frac{\partial \, \text{VaR}\left(x,\alpha\right)}{\partial \, x} &= \frac{\partial}{\partial \, x}\left(\Phi^{-1}\left(\alpha\right)\left(x^\top \Sigma x\right)^{\frac{1}{2}}\right) \\ &= \Phi^{-1}\left(\alpha\right)\frac{1}{2}\left(x^\top \Sigma x\right)^{-\frac{1}{2}}\left(2\Sigma x\right) \\ &= \Phi^{-1}\left(\alpha\right)\frac{\Sigma x}{\sqrt{x^\top \Sigma x}} \end{aligned}$$

The marginal value-at-risk of the i^{th} loan is then (TR-GDR, page 497):

$$\mathcal{MR}_i = \frac{\partial \, \text{VaR}\left(\alpha\right)}{\partial \, x_i} = \Phi^{-1}\left(\alpha\right)\frac{\left(\Sigma x\right)_i}{\sqrt{x^\top \Sigma x}}$$

The risk contribution of the i^{th} loan is the product of the exposure by the marginal risk:

$$\begin{aligned} \mathcal{RC}_i &= x_i \times \mathcal{MR}_i \\ &= \Phi^{-1}\left(\alpha\right)\frac{x_i \times \left(\Sigma x\right)_i}{\sqrt{x^\top \Sigma x}} \end{aligned}$$

3. We consider the random vector $Y = (e, L)$. By construction, Y is a Gaussian random vector (TR-GDR, page 497) with:

$$\begin{pmatrix} e \\ L \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & \Sigma x \\ x^\top \Sigma & x^\top \Sigma x \end{pmatrix}\right)$$

The conditional distribution function of e given that $L = \ell$ is Gaussian and we have:

$$\mathbb{E}\left[e \mid L = \ell\right] = \mathbf{0} + \Sigma x \left(x^\top \Sigma x\right)^{-1}\left(\ell - 0\right)$$

We obtain:

$$\begin{aligned} \mathbb{E}\left[e \mid L = \mathbf{F}^{-1}\left(\alpha\right)\right] &= \Sigma x \left(x^\top \Sigma x\right)^{-1}\Phi^{-1}\left(\alpha\right)\sqrt{x^\top \Sigma x} \\ &= \Phi^{-1}\left(\alpha\right)\frac{\Sigma x}{\sqrt{x^\top \Sigma x}} \\ &= \frac{\partial \, \text{VaR}\left(x;\alpha\right)}{\partial \, x} \end{aligned}$$

The marginal VaR of the i^{th} loan is then equal to the conditional mean of the individual loss e_i given that the portfolio loss is exactly equal to the value-at-risk.

4. We have to make the following assumptions (TR-GDR, page 179):

   (i) The loss given default is independent from the default time;

   (ii) The portfolio is infinitely-grained.

   In this case, we have:

$$\mathbb{E}\left[e_i \mid L = \mathbf{F}^{-1}(\alpha)\right] = \mathbb{E}\left[\mathrm{LGD}_i\right] \times \mathbb{E}\left[D_i \mid L = \mathbf{F}^{-1}(\alpha)\right]$$

5. We have:

$$\begin{aligned} p_i &= \Pr\left\{\tau_i \leq M_i\right\} \\ &= \Pr\left\{Z_i \leq B_i\right\} \\ &= \Phi\left(B_i\right) \end{aligned}$$

   We then deduce:

$$\begin{aligned} \Pr\left\{\tau_i \leq M_i \mid X = x\right\} &= \Pr\left\{Z_i \leq B_i \mid X = x\right\} \\ &= \Pr\left\{\sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_i \leq B_i \mid X = x\right\} \\ &= \Pr\left\{\varepsilon_i \leq \frac{B_i - \sqrt{\rho}X}{\sqrt{1-\rho}} \mid X = x\right\} \\ &= \Phi\left(\frac{B_i - \sqrt{\rho}x}{\sqrt{1-\rho}}\right) \\ &= \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho}x}{\sqrt{1-\rho}}\right) \end{aligned}$$

6. In the Basle II model, we have:

$$\begin{aligned} L &= g(X) \\ &= \sum_{i=1}^{n} x_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho}X}{\sqrt{1-\rho}}\right) \end{aligned}$$

   with $g'(x) < 0$. We deduce that:

$$\begin{aligned} \mathrm{VaR}(x;\alpha) = \mathbf{F}^{-1}(\alpha) &\Leftrightarrow \Pr\left\{g(X) \leq \mathrm{VaR}(x;\alpha)\right\} = \alpha \\ &\Leftrightarrow \Pr\left\{X \geq g^{-1}\left(\mathrm{VaR}(x;\alpha)\right)\right\} = \alpha \\ &\Leftrightarrow \Pr\left\{X \leq g^{-1}\left(\mathrm{VaR}(x;\alpha)\right)\right\} = 1-\alpha \\ &\Leftrightarrow g^{-1}\left(\mathrm{VaR}(x;\alpha)\right) = \Phi^{-1}(1-\alpha) \end{aligned}$$

   It follows that:

$$\mathbb{E}\left[e_i \mid L = \mathbf{F}^{-1}(\alpha)\right] = \mathbb{E}\left[\mathrm{LGD}_i\right] \times \mathbb{E}\left[D_i \mid X = \Phi^{-1}(1-\alpha)\right]$$

7. We have (TR-GDR, page 182):

$$\begin{aligned} \mathcal{RC}_i &= x_i \times \mathcal{MR}_i \\ &= x_i \times \mathbb{E}\left[e_i \mid L = \mathbf{F}^{-1}(\alpha)\right] \\ &= x_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times \mathbb{E}\left[D_i \mid X = \Phi^{-1}(1-\alpha)\right] \\ &= x_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho}\Phi^{-1}(1-\alpha)}{\sqrt{1-\rho}}\right) \\ &= x_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times \Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right) \end{aligned}$$

8. (a) We note:

$$p_i(X)=\Phi\left(\frac{\Phi^{-1}(p_i)-\sqrt{\rho}X}{\sqrt{1-\rho}}\right)$$

We have:

$$\begin{array}{rcl}\mathrm{ES}(x;\alpha) & = & \mathbb{E}\left[L \mid L\geq \mathrm{VaR}(x;\alpha)\right] \\ & = & \mathbb{E}\left[L \mid g(X)\geq \mathrm{VaR}(x;\alpha)\right] \\ & = & \mathbb{E}\left[L \mid X\leq g^{-1}(\mathrm{VaR}(x;\alpha))\right] \\ & = & \mathbb{E}\left[\sum_{i=1}^{n} x_i\times \mathbb{E}\left[\mathrm{LGD}_i\right]\times p_i(X) \mid X\leq \Phi^{-1}(1-\alpha)\right] \\ & = & \sum_{i=1}^{n} x_i\times \mathbb{E}\left[\mathrm{LGD}_i\right]\times \mathbb{E}\left[p_i(X) \mid X\leq \Phi^{-1}(1-\alpha)\right] \\ & = & \sum_{i=1}^{n} x_i\times \mathbb{E}\left[\mathrm{LGD}_i\right]\times \mathbb{E}\left[D_i \mid X\leq \Phi^{-1}(1-\alpha)\right]\end{array}$$

(b) It follows that:

$$\begin{array}{rcl}\mathbb{E}\left[p_i(X) \mid X\leq \Phi^{-1}(1-\alpha)\right] & = & \mathbb{E}\left[\Phi\left(\frac{\Phi^{-1}(p_i)-\sqrt{\rho}X}{\sqrt{1-\rho}}\right) \mid X\leq \Phi^{-1}(1-\alpha)\right] \\ & = & \int_{-\infty}^{\Phi^{-1}(1-\alpha)} \Phi\left(\frac{\Phi^{-1}(p_i)}{\sqrt{1-\rho}}+\frac{-\sqrt{\rho}}{\sqrt{1-\rho}}x\right)\frac{\phi(x)}{\Phi\left(\Phi^{-1}(1-\alpha)\right)}\,\mathrm{d}x \\ & = & \frac{\Phi_2\left(\Phi^{-1}(1-\alpha),\Phi^{-1}(p_i);\sqrt{\rho}\right)}{1-\alpha} \\ & = & \frac{\mathbf{C}\left(1-\alpha,p_i;\sqrt{\rho}\right)}{1-\alpha}\end{array}$$

where $\mathbf{C}$ is the Gaussian copula. We deduce that:

$$\mathcal{RC}_i=x_i\times \mathbb{E}\left[\mathrm{LGD}_i\right]\times\frac{\mathbf{C}\left(1-\alpha,p_i;\sqrt{\rho}\right)}{1-\alpha}$$

(c) If $\rho=0$, we have:

$$\begin{array}{rcl}\Phi\left(\frac{\Phi^{-1}(p_i)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right) & = & \Phi\left(\Phi^{-1}(p_i)\right) \\ & = & p_i\end{array}$$

and:

$$\begin{array}{rcl}\frac{\mathbf{C}\left(1-\alpha,p_i;\sqrt{\rho}\right)}{1-\alpha} & = & \frac{(1-\alpha)\,p_i}{1-\alpha} \\ & = & p_i\end{array}$$

The risk contribution is the same for the value-at-risk and the expected shortfall:

$$\mathcal{RC}_i=x_i\times \mathbb{E}\left[\mathrm{LGD}_i\right]\times p_i$$

It depends on the exposure-at-default x_i, the expected loss-given-default $\mathbb{E}\left[\mathrm{LGD}_i\right]$ and the unconditional probability of default p_i. If $\rho=1$ and $\alpha>50\%$, we have:

$$\begin{array}{rcl}\Phi\left(\frac{\Phi^{-1}(p_i)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right) & = & \lim_{\rho\rightarrow 1}\Phi\left(\frac{\Phi^{-1}(p_i)+\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right) \\ & = & 1\end{array}$$

If $\rho = 1$ and α is high ($\alpha > 1 - \sup_i p_i$), we have:

$$\begin{aligned}
\frac{\mathbf{C}\left(1-\alpha, p_i; \sqrt{\rho}\right)}{1-\alpha} &= \frac{\min\left(1-\alpha; p_i\right)}{1-\alpha} \\
&= 1
\end{aligned}$$

In this case, the risk contribution is the same for the value-at-risk and the expected shortfall:

$$\mathcal{RC}_i = x_i \times \mathbb{E}\left[\mathrm{LGD}_i\right]$$

However, it does not depend on the unconditional probability of default p_i.