Solutions of the Financial Risk Management Examination

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Remark 1 *The first five questions are corrected in TR-GDR*[1] *and in the document of exercise solutions, which is available in my web page*[2].

1 The Basle II regulation

2 Market risk

3 Credit risk

4 Counterparty credit risk

5 Operational risk

6 Value at risk of a long/short portfolio

The principal reference for this exercise is TR-GDR (pages 61-63). We note $P_A(t)$ (resp. $P_B(t)$) the value of the stock A (resp. B) at the date t. The portfolio value is:

$$P(t) = x_A \cdot P_A(t) + x_B \cdot P_B(t)$$

with x_A and x_B the number of stocks A and B. We deduce that the PnL between t and $t+1$ is:

$$\begin{aligned}
\mathrm{PnL}(t;t+1) &= P(t+1) - P(t) \\
&= x_A (P_A(t+1) - P_A(t)) + x_B (P_B(t+1) - P_B(t)) \\
&= x_A P_A(t) R_A(t;t+1) + x_B P_B(t) R_B(t;t+1)
\end{aligned}$$

with $R_A(t;t+1)$ and $R_B(t;t+1)$ the asset returns of A and B between the dates t and $t+1$.

1. We have $x_A = +1$, $x_B = -1$ and $P_A(t) = P_B(t) = 100$. It comes that:

   $$\mathrm{PnL}(t;t+1) = 100 \cdot (R_A - R_B)$$

   We have $R_A - R_B \sim \mathcal{N}(0, \sigma_{A-B})$ with:

   $$\begin{aligned}
   \sigma_{A-B} &= \sqrt{0.20^2 + (-0.20)^2 + 2 \cdot 0.5 \cdot 0.20 \cdot (-0.20)} \\
   &= 20\%
   \end{aligned}$$

[1] Thierry Roncalli, *La Gestion des Risques Financiers*, Economica, deuxième édition, 2009.
[2] The direct link is `www.thierry-roncalli.com/download/gdr-correction.pdf`.

The annual volatility of the long/short portfolio is then equal to 20%. To compute the value at risk for a time horizon of one day, we consider the square root rule (TR-GDR, page 74). We obtain[3]:

$$
\begin{aligned}
\mathrm{VaR}_{1D} &= \Phi^{-1}(0.99) \cdot 100 \cdot \sigma_{A-B} \cdot \frac{1}{\sqrt{260}} \\
&= 2.33 \cdot 100 \cdot 0.20 \cdot \frac{1}{\sqrt{260}} \\
&= 2.89
\end{aligned}
$$

The probability to lose 2.89 **euros per day is equal to 1%.**

2. We have $\mathrm{PnL}(t;t+1) = 100 \cdot (R_A - R_B)$. We use the historical data to calculate the scenarios of asset returns (R_A, R_B). We then deduce the empirical distribution of the PnL. Finally, we compute the corresponding empirical quantile. With 250 scenarios, the 1% decile is between the second and third worst cases:

$$
\begin{aligned}
\mathrm{VaR}_{1D} &= -\left[-3.09 + \frac{1}{2}\left(-2.72 - (-3.09)\right)\right] \\
&= 2.905
\end{aligned}
$$

The probability to lose 2.905 **euros per day is equal to 1%.** This result is very similar to the one calculated with the gaussian VaR.

3. The PnL formula becomes (TR-GDR, pages 91-95) :

$$
\begin{aligned}
\mathrm{PnL}(t;t+1) = \; & (P_A(t+1) - P_A(t)) - \\
& (P_B(t+1) - P_B(t)) - \\
& (C_A(t+1) - C_A(t))
\end{aligned}
$$

with $C_A(t)$ the call option price. We have:

$$
C_A(t+1) - C_A(t) \simeq \Delta \cdot (P_A(t+1) - P_A(t))
$$

where Δ is the delta of the option. We deduce that:

$$
\mathrm{PnL}(t;t+1) = 50 \cdot R_A - 100 \cdot R_B
$$

For the analytical VaR, we obtain:

$$
\begin{aligned}
\mathrm{VaR}_{1D} &= \Phi^{-1}(0.99) \cdot \sigma_{A/B} \cdot \frac{1}{\sqrt{260}} \\
&= 2.33 \cdot 17.32 \cdot \frac{1}{\sqrt{260}} \\
&= 2.50
\end{aligned}
$$

because:

$$
\begin{aligned}
\sigma_{A/B} &= \sqrt{(50 \cdot 0.20)^2 + (-100 \cdot 0.20)^2 + 2 \cdot 0.5 \cdot (50 \cdot 0.20) \times (-100 \cdot 0.20)} \\
&= 17.32
\end{aligned}
$$

The daily 99% VaR decreases from 2.89 **euros to** 2.50 **euros.**

[3] because $\Phi^{-1}(0.99) = -\Phi^{-1}(0.01)$.

7 Parameter estimation for operational risk

1. (a) The density of the gaussian distribution $Y \sim \mathcal{N}(\mu, \sigma)$ is:

$$g(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right)$$

Let $X \sim \mathcal{LN}(\mu, \sigma)$. We have:

$$X = e^Y$$

It comes that:

$$f(x) = g(y)\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right|$$

with $y = \ln x$. We deduce that:

$$\begin{aligned} f(x) &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right) \cdot \frac{1}{x} \\ &= \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2\right) \end{aligned}$$

(b) The log-likelihood function of the sample $\{L_1, \ldots, L_n\}$ is:

$$\begin{aligned} \mathcal{L}(\mu, \sigma) &= \ln \prod_{i=1}^{n} f(L_i) \\ &= \sum_{i=1}^{n} \ln f(L_i) \\ &= -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln \sigma^2 - \sum_{i=1}^{n} \ln L_i - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\ln L_i - \mu}{\sigma}\right)^2 \end{aligned}$$

(c) We have:

$$\{\hat{\mu}, \hat{\sigma}\} = \arg\max \mathcal{L}(\mu, \sigma)$$

We notice that:

$$\begin{aligned} \max \mathcal{L}(\mu, \sigma) &= \max\left(-\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln \sigma^2 - \sum_{i=1}^{n} \ln L_i - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\ln L_i - \mu}{\sigma}\right)^2\right) \\ &= \max\left(-\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln \sigma^2 - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^2\right) \end{aligned}$$

with $x_i = \ln L_i$. We recognize the gaussian log-likelihood function. We deduce that:

$$\begin{aligned} \hat{\mu} &= \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}\sum_{i=1}^{n} \ln L_i \\ \hat{\sigma} &= \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\ln L_i - \frac{1}{n}\sum_{i=1}^{n}\ln L_i\right)^2} \end{aligned}$$

(d) Using Bayes' formula, we have:

$$\begin{aligned} \Pr\{X \le x \mid X \ge H\} &= \frac{\Pr\{H \le X \le x\}}{\Pr\{X \ge H\}} \\ &= \frac{\mathbf{F}(x) - \mathbf{F}(H)}{1 - \mathbf{F}(H)} \end{aligned}$$

with $\mathbf{F}$ the cdf of X. It comes that the conditional density is:

$$\begin{aligned} f_H(x) &= \partial_x \Pr\{X \leq x \mid X \geq H\} \\ &= \frac{f(x)}{1 - \mathbf{F}(H)} \\ &= \frac{1}{\left(1 - \Phi\left(\frac{\ln H - \mu}{\sigma}\right)\right)} \cdot \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right) \end{aligned}$$

It comes that the log-likelihood function of the sample $\{L_1, \ldots, L_n\}$ is:

$$\begin{aligned} \mathcal{L}(\mu, \sigma) &= \ln \prod_{i=1}^{n} f_H(L_i) \\ &= -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2 - \sum_{i=1}^{n}\ln L_i - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\ln L_i - \mu}{\sigma}\right)^2 - \\ &\quad n\ln\left(1 - \Phi\left(\frac{\ln H - \mu}{\sigma}\right)\right) \end{aligned}$$

2. (a) By definition, we have:

$$\Pr\{N_t = n\} = e^{-\lambda}\frac{\lambda^n}{n!}$$

We deduce that:

$$\begin{aligned} \mathbb{E}[N_t] &= \sum_{n=0}^{\infty} n \cdot \Pr\{N_t = n\} \\ &= \sum_{n=0}^{\infty} n e^{-\lambda}\frac{\lambda^n}{n!} \\ &= \lambda e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!} \\ &= \lambda \end{aligned}$$

(b) We have:

$$\begin{aligned} \mathbb{E}\left[\prod_{i=0}^{m}(N_t - i)\right] &= \sum_{n=0}^{\infty}\prod_{i=0}^{m}(n - i)\, e^{-\lambda}\frac{\lambda^n}{n!} \\ &= \sum_{n=0}^{\infty}(n(n-1)\cdots(n-m))\, e^{-\lambda}\frac{\lambda^n}{n!} \end{aligned}$$

The term of the sum is equal to zero if $n = 0, 1, \ldots, m$. It comes that:

$$\begin{aligned} \mathbb{E}\left[\prod_{i=0}^{m}(N_t - i)\right] &= \sum_{n=m+1}^{\infty}(n(n-1)\cdots(n-m))\, e^{-\lambda}\frac{\lambda^n}{n!} \\ &= e^{-\lambda}\sum_{n=m+1}^{\infty}\frac{\lambda^n}{(n-m-1)!} \\ &= \lambda^{m+1}e^{-\lambda}\sum_{n=m+1}^{\infty}\frac{\lambda^{n-m-1}}{(n-m-1)!} \\ &= \lambda^{m+1}e^{-\lambda}\sum_{n'=0}^{\infty}\frac{\lambda^{n'}}{n'!} \end{aligned}$$

with $n' = n - (m+1)$. It comes that:

$$\begin{aligned}
\mathbb{E}\left[\prod_{i=0}^{m}(N_t - i)\right] &= \lambda^{m+1}e^{-\lambda}e^{\lambda} \\
&= \lambda^{m+1} \qquad (1)
\end{aligned}$$

We deduce that:

$$\begin{aligned}
\operatorname{var}(N_t) &= \mathbb{E}\left[N_t^2\right] - \mathbb{E}^2[N_t] \\
&= \mathbb{E}\left[N_t^2 - N_t\right] + \mathbb{E}[N_t] - \mathbb{E}^2[N_t] \\
&= \mathbb{E}[N_t(N_t - 1)] + \mathbb{E}[N_t] - \mathbb{E}^2[N_t]
\end{aligned}$$

Using the formula (1) with $m = 1$, we finally obtain:

$$\begin{aligned}
\operatorname{var}(N_t) &= \lambda^{1+1} + \lambda - \lambda^2 \\
&= \lambda
\end{aligned}$$

(c) The estimator based on the first moment is:

$$\hat{\lambda} = \frac{1}{T}\sum_{t=1}^{T} N_t$$

whereas the estimator based on the second moment is:

$$\hat{\lambda} = \frac{1}{T}\sum_{t=1}^{T}\left(N_t - \frac{1}{T}\sum_{t=1}^{T} N_t\right)^2$$

3. Let L be the random sum:

$$L = \sum_{i=0}^{N_t} L_i$$

where $L_i \sim \mathcal{LN}(\mu, \sigma)$, $L_i \perp L_j$ and $N_t \sim \mathcal{P}(\lambda)$.

(a) We have:

$$\begin{aligned}
\mathbb{E}[L] &= \mathbb{E}\left[\sum_{i=0}^{N_t} L_i\right] \\
&= \mathbb{E}[N_t]\,\mathbb{E}[L_i] \\
&= \lambda \exp\left(\mu + \frac{1}{2}\sigma^2\right)
\end{aligned}$$

(b) Because $\left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i=1}^{n} x_i^2 + \sum_{i \neq j} x_i x_j$, it comes that:

$$\begin{aligned}
\mathbb{E}\left[L^2\right] &= \mathbb{E}\left[\sum_{i=0}^{N_t} L_i^2 + \sum_{i \neq j}^{N_t}\sum^{N_t} L_i L_j\right] \\
&= \mathbb{E}[N_t]\,\mathbb{E}\left[L_i^2\right] + \mathbb{E}[N_t(N_t - 1)]\,\mathbb{E}[L_i L_j] \\
&= \mathbb{E}[N_t]\,\mathbb{E}\left[L_i^2\right] + \left(\mathbb{E}\left[N_t^2\right] - \mathbb{E}[N_t]\right)\mathbb{E}[L_i]\,\mathbb{E}[L_j]
\end{aligned}$$

We have:

$$\begin{aligned}
\mathbb{E}[N_t] &= \lambda \\
\mathbb{E}\left[N_t^2\right] &= \operatorname{var}(N_t) + \mathbb{E}^2[N_t] = \lambda + \lambda^2 \\
\mathbb{E}\left[L_i^2\right] &= \operatorname{var}(L_i) + \mathbb{E}^2[L_i] = e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right) + \left(e^{\mu+\frac{1}{2}\sigma^2}\right)^2 = e^{2\mu+2\sigma^2} \\
\mathbb{E}[L_i]\,\mathbb{E}[L_j] &= e^{\mu+\frac{1}{2}\sigma^2}e^{\mu+\frac{1}{2}\sigma^2} = e^{2\mu+\sigma^2}
\end{aligned}$$

We deduce that:

$$\begin{aligned}
\mathbb{E}\left[L^2\right] &= \lambda\mathbb{E}\left[L_i^2\right] + \left(\lambda+\lambda^2-\lambda\right)\mathbb{E}[L_i]\,\mathbb{E}[L_j] \\
&= \lambda\mathbb{E}\left[L_i^2\right] + \lambda^2\mathbb{E}[L_i]\,\mathbb{E}[L_j] \\
&= \lambda e^{2\mu+2\sigma^2} + \lambda^2 e^{2\mu+\sigma^2}
\end{aligned}$$

and:

$$\begin{aligned}
\operatorname{var}(L) &= \mathbb{E}\left[L^2\right] - \mathbb{E}^2[L] \\
&= \lambda e^{2\mu+2\sigma^2} + \lambda^2 e^{2\mu+\sigma^2} - \lambda^2\left(e^{\mu+\frac{1}{2}\sigma^2}\right)^2 \\
&= \lambda e^{2\mu+2\sigma^2}
\end{aligned}$$

(c) We have:

$$\begin{cases} \mathbb{E}[L] = \lambda e^{\mu+\frac{1}{2}\sigma^2} \\ \operatorname{var}(L) = \lambda e^{2\mu+2\sigma^2} \end{cases}$$

We deduce that:

$$\frac{\operatorname{var}(L)}{\mathbb{E}^2[L]} = \frac{\lambda e^{2\mu+2\sigma^2}}{\lambda^2 e^{2\mu+\sigma^2}} = \frac{e^{\sigma^2}}{\lambda}$$

It comes that:

$$\sigma^2 = \ln\lambda + \ln\operatorname{var}(L) - \ln\mathbb{E}^2[L]$$

and:

$$\begin{aligned}
\mu &= \ln\mathbb{E}[L] - \ln\lambda - \frac{1}{2}\sigma^2 \\
&= \ln\mathbb{E}[L] + \ln\mathbb{E}^2[L] - \frac{3}{2}\ln\lambda - \ln\operatorname{var}(L)
\end{aligned}$$

Let $\hat{\lambda}$ be an estimated value of λ. We finally obtain:

$$\hat{\mu} = \ln m + \ln m^2 - \frac{3}{2}\ln\hat{\lambda} - \ln V$$

and

$$\hat{\sigma} = \sqrt{\ln\hat{\lambda} + \ln V - \ln m^2}$$

where m and V are the empirical mean and variance of aggregated losses.

8 Copula functions

1. (a) If P is a lower triangular matrix, then the matrix Σ may be decomposed as:

$$\Sigma = PP^\top$$

This is the Cholesky decomposition. In our case, we have:

$$P = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$$

We verify that:

$$\begin{aligned} PP^\top &= \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} 1 & \rho \\ 0 & \sqrt{1-\rho^2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \\ &= \Sigma \end{aligned}$$

The copula of U is the copula of the gaussian standardized vector $X = (X_1, X_2)$ with correlation ρ. Let n_1 and n_2 be two random drawing of $\mathcal{N}(0,1)$. Using the Cholesky decomposition[4], we can simulate X by the following way:

$$\begin{aligned} x_1 &= n_1 \\ x_2 &= \rho n_1 + \sqrt{1-\rho^2} n_2 \end{aligned}$$

We deduce that we can simulate U with the relations:

$$\begin{aligned} u_1 &= \Phi(x_1) = \Phi(n_1) \\ u_2 &= \Phi(x_2) = \Phi\left(\rho n_1 + \sqrt{1-\rho^2} n_2\right) \end{aligned}$$

(b) We have:

$$\begin{aligned} \mathbf{C}(u_1, u_2) &= \Phi\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho\right) \\ &= \int_{-\infty}^{\Phi^{-1}(u_1)} \Phi\left(\frac{\Phi^{-1}(u_2) - \rho x}{\sqrt{1-\rho^2}}\right) \phi(x)\, \mathrm{d}x \\ &= \int_0^{u_1} \Phi\left(\frac{\Phi^{-1}(u_2) - \rho \Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right) \mathrm{d}u \end{aligned}$$

because $x = \Phi^{-1}(u)$, $\mathrm{d}u = \phi(x)\,\mathrm{d}x$, $\Phi(-\infty) = 0$ and $\Phi\left(\Phi^{-1}(u_1)\right) = u_1$. The conditional copula function $\mathbf{C}_{2|1}$ is then equal to:

$$\begin{aligned} \mathbf{C}_{2|1}(u_1, u_2) &= \frac{\partial\, \mathbf{C}(u_1, u_2)}{\partial\, u_1} \\ &= \Phi\left(\frac{\Phi^{-1}(u_2) - \rho \Phi^{-1}(u_1)}{\sqrt{1-\rho^2}}\right) \end{aligned}$$

(c) We know that $\Pr\{U_1 \leq u_1\} = u_1$ and $\mathbf{C}_{2|1}(u_1, u_2) = \Pr\{U_2 \leq u_2 \mid U_1 = u_1\}$. Because $\mathbf{C}(U_1, 1)$ and $\mathbf{C}_{2|1}(u_1, U_2)$ are two independent uniform random variables, we obtain the following algorithm:

i. we simulate two independent uniform variates v_1 and v_2;
ii. set u_1 equal to v_1;
iii. set u_2 equal to $K_{u_1}^{-1}(v_2)$ where $K_{u_1}(u_2) = \mathbf{C}_{2|1}(u_1, u_2)$.

[4] We have $\mathcal{N}(\mu, \Sigma) = \mu + P\mathcal{N}(\mathbf{0}, I)$.

If we apply this algorithm to the Normal copula, we obtain:

$$\Phi\left(\frac{\Phi^{-1}(u_2)-\rho\Phi^{-1}(u_1)}{\sqrt{1-\rho^2}}\right)=v_2$$

It comes that:

$$\Phi^{-1}(u_2)=\rho\Phi^{-1}(u_1)+\sqrt{1-\rho^2}\Phi^{-1}(v_2)$$

We deduce that:

$$\begin{aligned} u_1 &= v_1 \\ u_2 &= \Phi\left(\rho\Phi^{-1}(v_1)+\sqrt{1-\rho^2}\Phi^{-1}(v_2)\right) \end{aligned}$$

We see that this algorithm is a special case of the Cholesky algorithm if we take $n_1=\Phi^{-1}(v_1)$ and $n_2=\Phi^{-1}(v_2)$. Whereas n_1 and n_2 are directly simulated in the Cholesky algorithm with a gaussian random generator, they are simulated using the inverse transform in the conditional distribution method.

(d) We know that:

$$1-e^{-\lambda_1\tau_1}\sim\mathcal{U}_{[0,1]}$$

We deduce that:

$$\begin{aligned} \tau_1 &= -\lambda_1^{-1}\ln(1-u_1) \\ \tau_2 &= -\lambda_2^{-1}\ln(1-u_2) \end{aligned}$$

2. (a) If $\rho=1$, the Normal copula becomes the upper Frechet copula. It means that τ_1 and τ_2 are comonotonic. Because $U_1=U_2$, we have:

$$1-e^{-\lambda_1\tau_1}=1-e^{-\lambda_2\tau_2}$$

or:

$$\lambda_1\tau_1=\lambda_2\tau_2$$

We have $\mathbb{E}[\tau_1]=\sigma(\tau_1)=1/\lambda_1$ and:

$$\begin{aligned} \mathbb{E}[\tau_1\tau_2] &= \mathbb{E}\left[\frac{\lambda_2}{\lambda_1}\tau_2\tau_2\right] \\ &= \frac{\lambda_2}{\lambda_1}\left(\operatorname{var}(\tau_2)+\mathbb{E}^2[\tau_2]\right) \\ &= \frac{\lambda_2}{\lambda_1}\left(\frac{1}{\lambda_2^2}+\frac{1}{\lambda_2^2}\right) \\ &= \frac{2}{\lambda_1\lambda_2} \end{aligned}$$

We deduce that:

$$\begin{aligned} \rho\langle\tau_1,\tau_2\rangle &= \frac{\mathbb{E}[\tau_1\tau_2]-\mathbb{E}[\tau_1]\,\mathbb{E}[\tau_2]}{\sigma(\tau_1)\,\sigma(\tau_2)} \\ &= \frac{\frac{2}{\lambda_1\lambda_2}-\frac{1}{\lambda_1\lambda_2}}{\frac{1}{\lambda_1\lambda_2}} \\ &= +1 \end{aligned}$$

(b) If $\rho = -1$, the Normal copula becomes the lower Frechet copula. It means that τ_1 and τ_2 are counter-monotonic. Because $U_1 = 1 - U_2$, we have:

$$1 - e^{-\lambda_1 \tau_1} = 1 - \left(1 - e^{-\lambda_2 \tau_2}\right)$$

or:

$$\tau_1 = -\frac{\ln\left(1 - e^{-\lambda_2 \tau_2}\right)}{\lambda_1}$$

The correlation is equal to -1 if τ_1 is a decreasing linear function of τ_2. The function $f(t) = -\lambda_1^{-1} \ln\left(1 - e^{-\lambda_2 t}\right)$ is a decreasing function, but not a linear function. We deduce that:

$$\rho \langle \tau_1, \tau_2 \rangle > -1$$

9 Credit spreads

1. We have (TR-GDR, page 427) :

$$\begin{aligned} \mathbf{F}(t) &= 1 - e^{-\lambda t} \\ \mathbf{S}(t) &= e^{-\lambda t} \\ f(t) &= \lambda e^{-\lambda t} \end{aligned}$$

Let $U = \mathbf{S}(\tau)$. We have $U \in [0,1]$ and:

$$\begin{aligned} \Pr\{U \leq u\} &= \Pr\{\mathbf{S}(\tau) \leq u\} \\ &= \Pr\left\{\tau \leq \mathbf{S}^{-1}(u)\right\} \\ &= \mathbf{S}\left(\mathbf{S}^{-1}(u)\right) \\ &= u \end{aligned}$$

We deduce that $\mathbf{S}(\tau) \sim \mathcal{U}_{[0,1]}$ (TR-GDR, page 428). It comes that $\tau = \mathbf{S}^{-1}(U)$ with $U \sim \mathcal{U}_{[0,1]}$. Let u be a uniform random variate. Simulating τ is equivalent to transform u into t:

$$t = -\frac{1}{\lambda} \ln u$$

2. We have (TR-GDR, pages 409-411) :

$$\begin{aligned} P_- &= \frac{1}{4} \cdot s \cdot N \\ P_+ &= (1 - R) \cdot N \end{aligned}$$

with s the spread, N the notional, P_- the premium leg and P_+ the protection leg. The quarterly payment of the premium leg explains the factor 1/4 in the formula of P_-. We deduce the flow chart given in Figure 2.

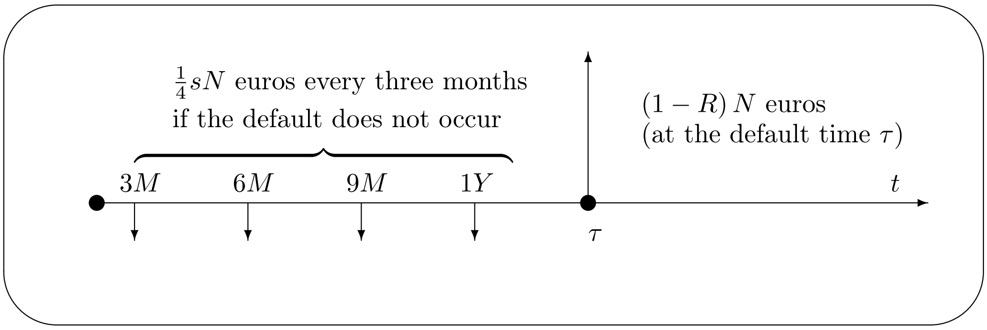
3. The ATM margin (or spread) is the value of s such that the CDS price is zero (TR-GDR, page 410):

$$P(t) = \mathbb{E}[P_- - P_+] = 0$$

We have the following triangle relation (TR-GDR, page 410):

$$s \simeq \lambda \times (1 - R)$$

Figure 1: Flow chart from the viewpoint of the protection buyer



4. Let PD be the annual default probability. We have

$$\begin{aligned}
\text{PD} &= 1 - \mathbf{S}(1) \\
&= 1 - e^{-\lambda} \\
&\simeq 1 - (1 - \lambda) \\
&\simeq \lambda
\end{aligned}$$

because λ is generally small ($\lambda \leq 10\%$). We deduce that:

$$\text{PD} \simeq \frac{s}{1-R}$$

5. We have:

$$\text{PD} = \frac{2\%}{1 - 25\%} = 267 \text{ bps}$$

10 Extreme value theory and stress-testing

1. See TR-GDR, page 121-129.

2. We have (TR-GDR, pages 131-133):

$$\begin{aligned}
\mathbf{G}_n(x) &= \Pr\{\max(X_1, \ldots, X_n) \leq x\} \\
&= \Pr\{X_1 \leq x, \ldots, X_n \leq x\} \\
&= \prod_{i=1}^{n} \Pr\{X_i \leq x\} \\
&= \Phi\left(\frac{x-\mu}{\sigma}\right)^n
\end{aligned}$$

3. See TR-GDR, page 139.

4. (a) An extreme value (EV) copula $\mathbf{C}$ satisfies the following relation:

$$\mathbf{C}\left(u_1^t, u_2^t\right) = \mathbf{C}^t(u_1, u_2)$$

for all $t > 0$.

(b) The product copula is an EV copula because:

$$\begin{aligned}
\mathbf{C}^{\perp}\left(u_1^t, u_2^t\right) &= u_1^t u_2^t \\
&= (u_1 u_2)^t \\
&= \left[\mathbf{C}^{\perp}(u_1, u_2)\right]^t
\end{aligned}$$

(c) We have:

$$
\begin{aligned}
\mathbf{C}\left(u_1^t, u_2^t\right) &= u_1^{t(1-\theta_1)} u_2^{t(1-\theta_2)} \min\left(u_1^{t\theta_1}, u_2^{t\theta_2}\right) \\
&= \left(u_1^{1-\theta_1}\right)^t \left(u_2^{1-\theta_2}\right)^t \left(\min\left(u_1^{\theta_1}, u_2^{\theta_2}\right)\right)^t \\
&= \left(u_1^{1-\theta_1} u_2^{1-\theta_2} \min\left(u_1^{\theta_1}, u_2^{\theta_2}\right)\right)^t \\
&= \mathbf{C}^t\left(u_1, u_2\right)
\end{aligned}
$$

(d) The upper tail dependence λ is defined as follows:

$$
\lambda = \lim_{u \to 1^+} \frac{1 - 2u + \mathbf{C}\left(u_1, u_2\right)}{1 - u}
$$

It indicates the probability to have an extreme in one direction knowing that we have already an extreme in the other direction. If $\lambda = 0$, extremes are independent and the copula of extreme values is the product copula. If $\lambda = 1$, extremes are comonotonic and the copula of extreme values is the upper Fréchet copula. Moreover, the upper tail dependence of the copula between the random variables is equal to the upper tail dependence of the copula between the extremes.

(e) If $\theta_1 > \theta_2$, we obtain using L'Hospital's rule:

$$
\begin{aligned}
\lambda &= \lim_{u \to 1^+} \frac{1 - 2u + u^{1-\theta_1} u^{1-\theta_2} \min\left(u^{\theta_1}, u^{\theta_2}\right)}{1 - u} \\
&= \lim_{u \to 1^+} \frac{1 - 2u + u^{1-\theta_1} u^{1-\theta_2} u^{\theta_1}}{1 - u} \\
&= \lim_{u \to 1^+} \frac{1 - 2u + u^{2-\theta_2}}{1 - u} \\
&= \lim_{u \to 1^+} \frac{0 - 2 + \left(2 - \theta_2\right) u^{1-\theta_2}}{-1} \\
&= \lim_{u \to 1^+} 2 - 2u^{1-\theta_2} + \theta_2 u^{1-\theta_2} \\
&= \theta_2
\end{aligned}
$$

If $\theta_2 > \theta_1$, $\lambda = \theta_1$. We deduce that the upper tail dependence of the the Marshall-Olkin copula is $\min\left(\theta_1, \theta_2\right)$.

(f) If $\theta_1 = 0$ or $\theta_2 = 0$, $\lambda = 0$. It comes that the copula of the extremes is the product copula. Extremes are then not correlated.