Portfolio Allocation with Skewness Risk

Benjamin Bruder, Nazar Kostyuchyk and Thierry Roncalli*

*Lyxor Asset Management[1], France

EDHEC Business School

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[1] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Lyxor Asset Management.

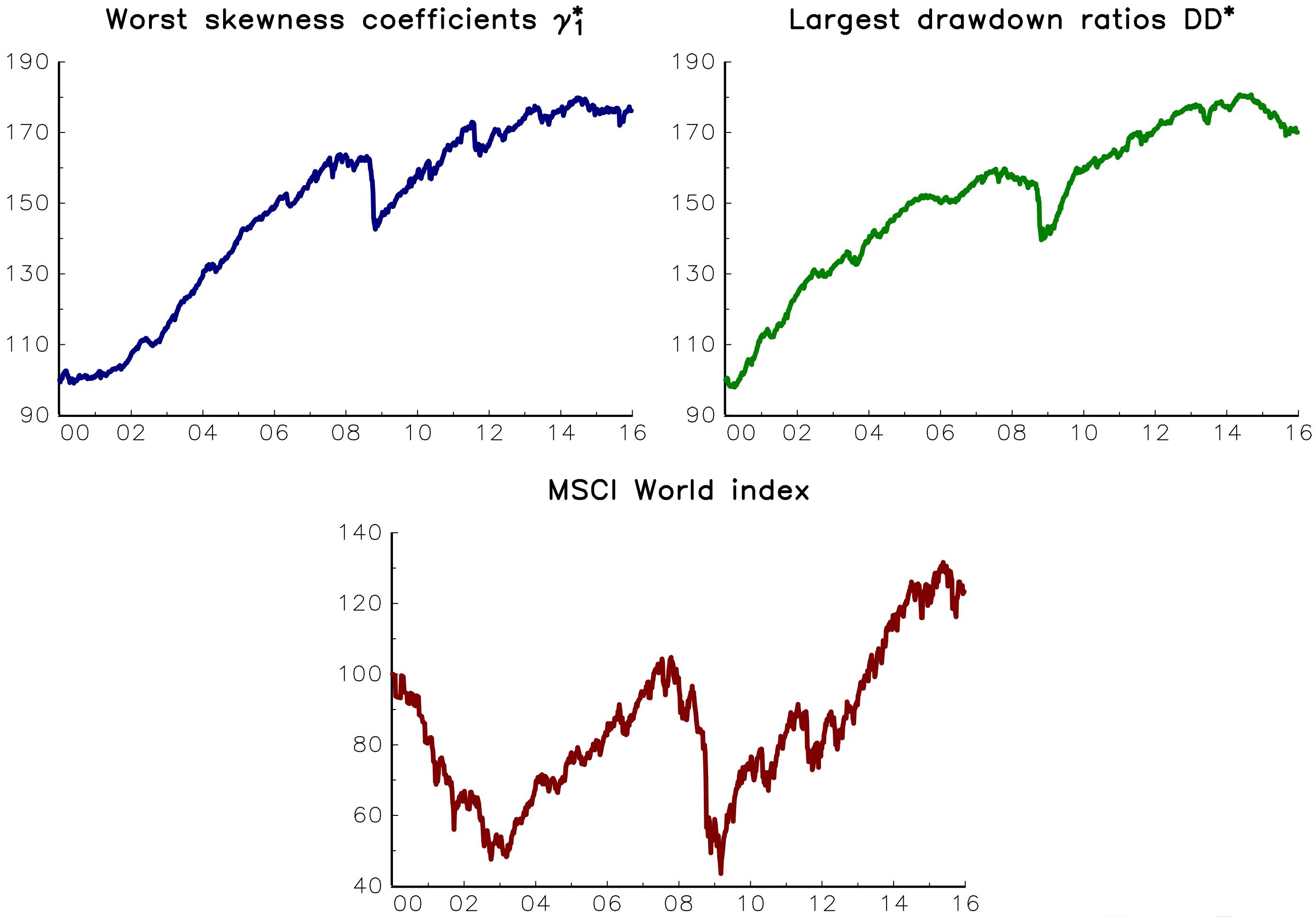
The skewness puzzle

- Alternative risk premia = extension of equity factor investing to other asset classes (in a long/short format)
- Alternative risk premia encompasses two different types of risk factor:
  - Skewness risk premia (= pure risk premia)
  - Market anomalies ($\neq$ risk premia)
- ARP (in particular skewness risk premia) are not all-weather strategies:
  - Extreme risks of ARP are high and may be correlated
  - Aggregation of skewness is not straightforward
  - Skewness diversification $\neq$ volatility diversification

$$\begin{aligned} \sigma(X+Y) &\leq \sigma(X)+\sigma(X) \\ \gamma_1(X+Y) &\nleq \gamma_1(X)+\gamma_1(X) \end{aligned}$$

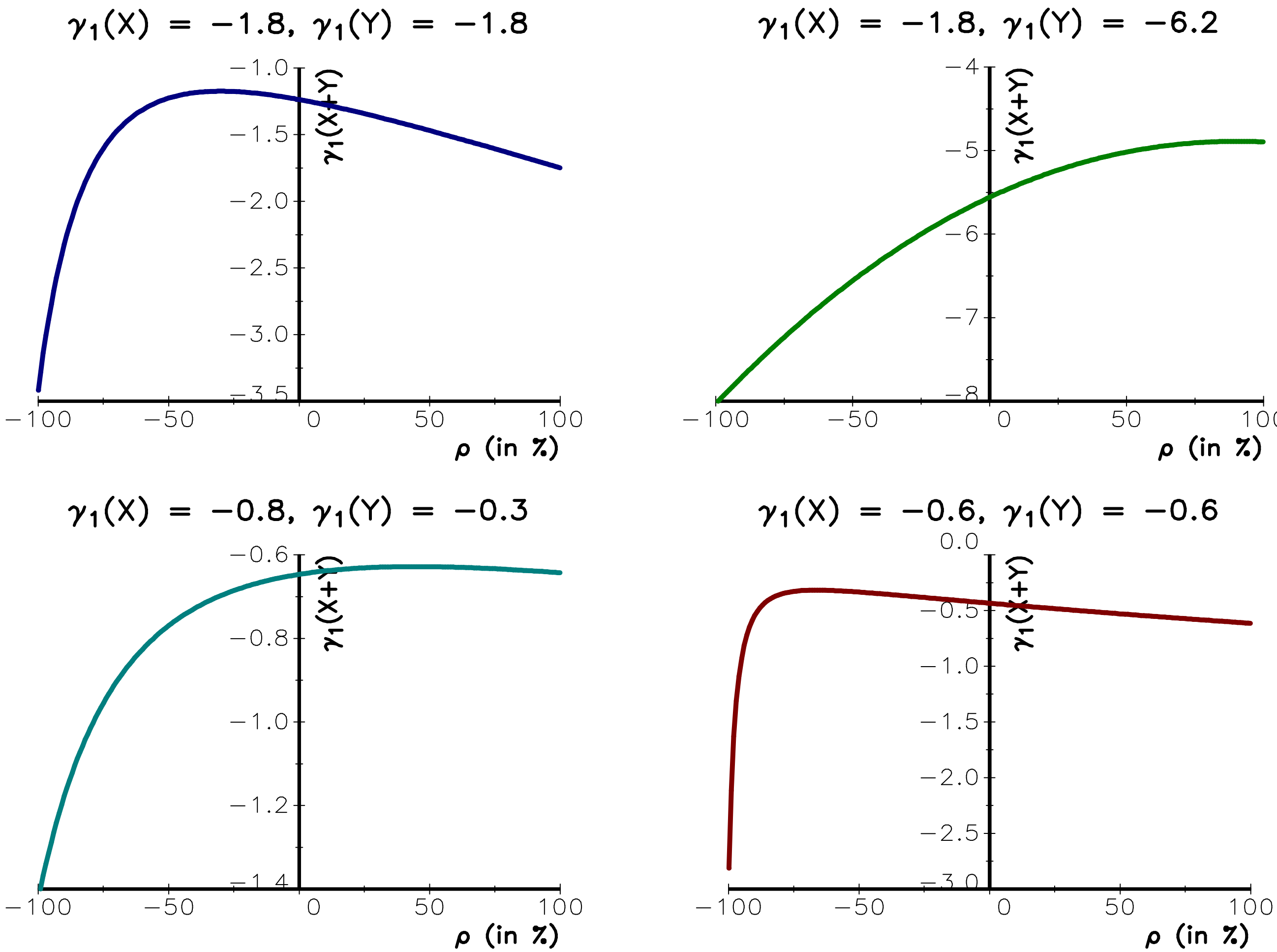
The skewness puzzle

Figure: Skewness aggregation of L/S alternative risk premia



The skewness puzzle

Figure: Skewness aggregation in the case of the bivariate log-normal distribution



Summary I

Recent trends in asset management

- Risk parity
- Equity factor investing
- Alternative risk premia

$\Rightarrow$ These 3 topics are related to the concept of diversification.

What is the issue?

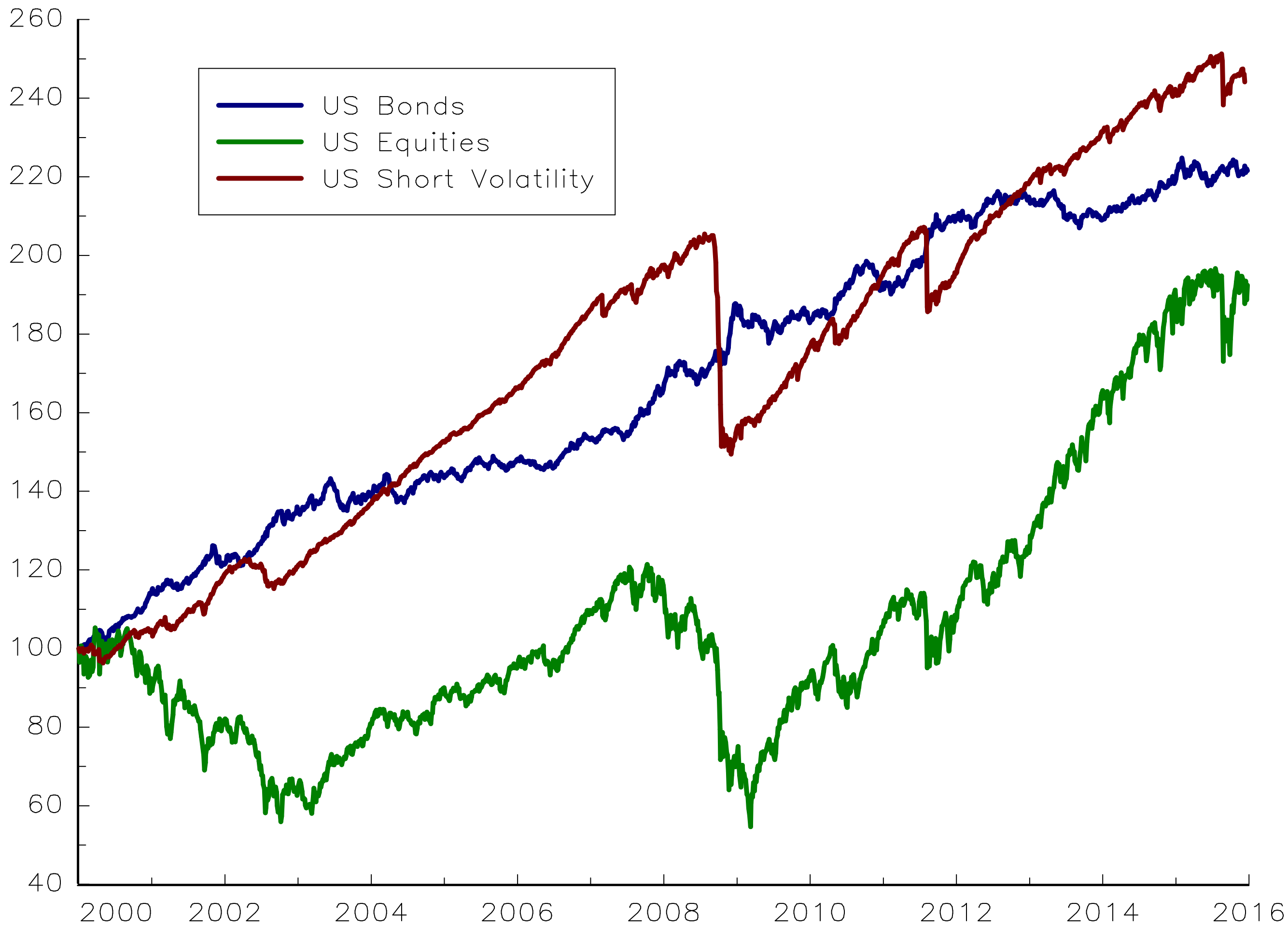
- Risk parity = volatility risk measure
- Equity factor investing = distressed risk (default/liquidity)
- Alternative risk premia = skewness risk premia + market anomalies

$\Rightarrow$ Skewness risk?

Skewness diversification $\neq$ volatility diversification

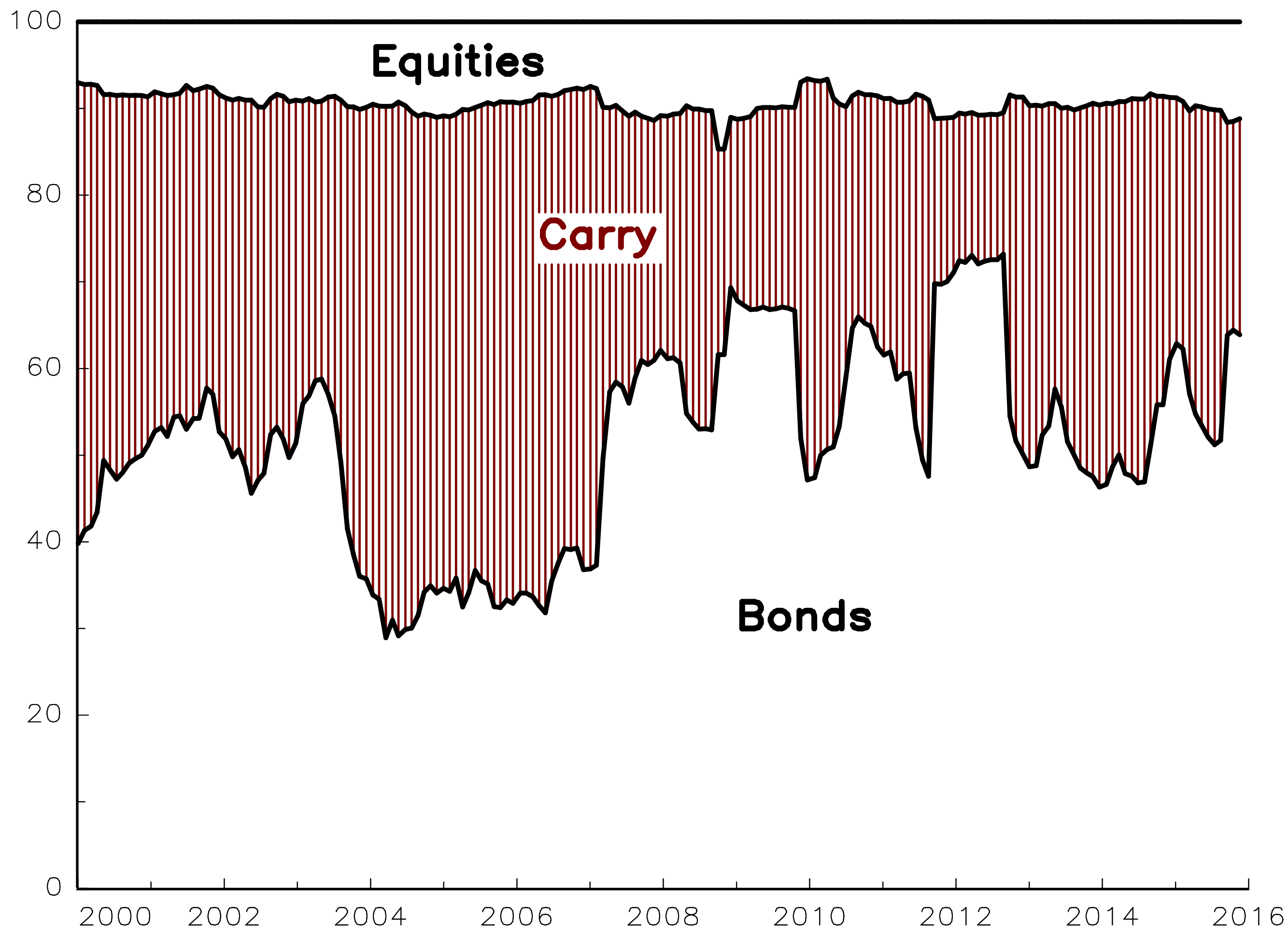
Summary II

Figure: Cumulative performance of US bonds, US equities and US short volatility



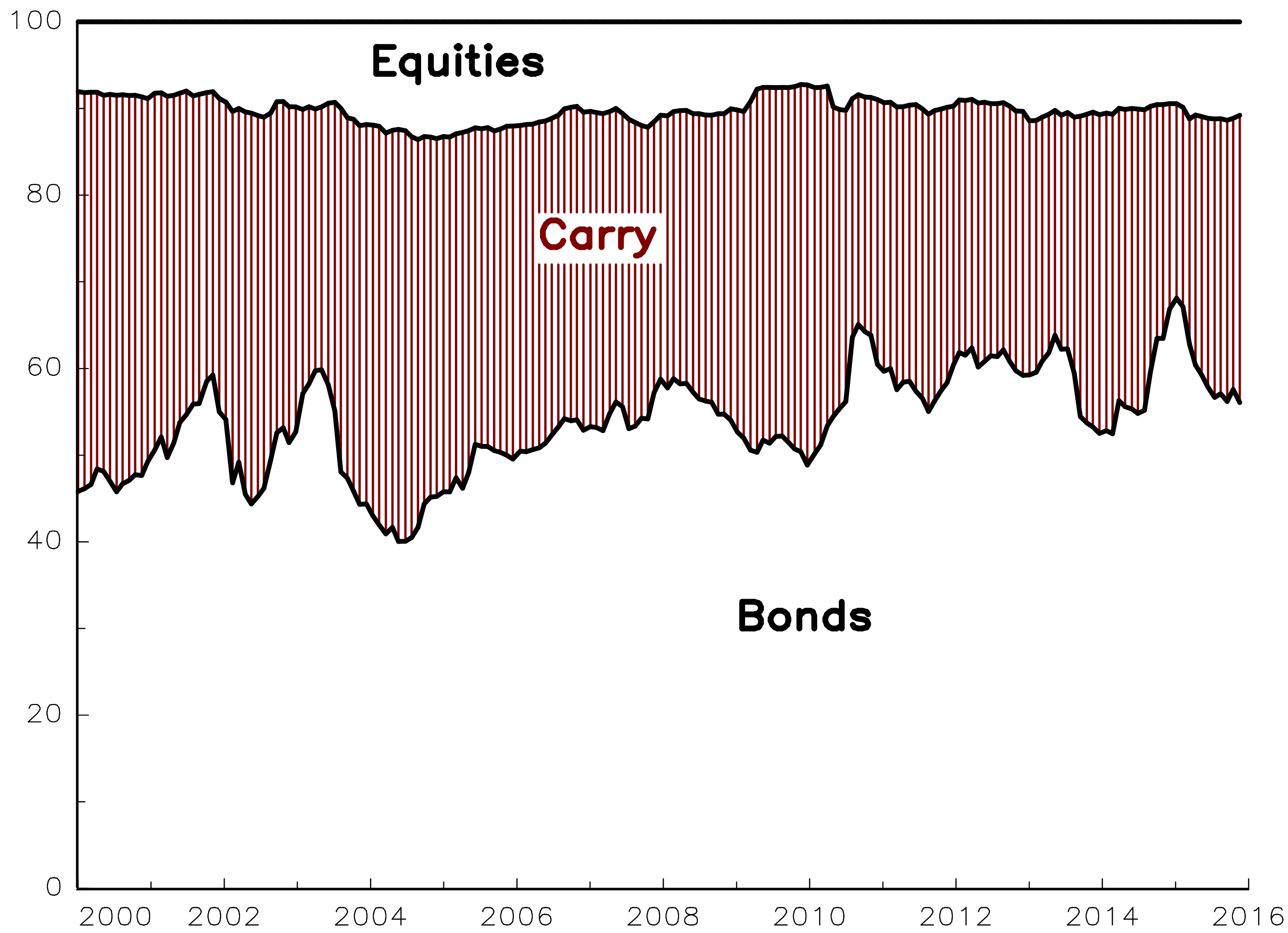
Summary III

Figure: Volatility-based ERC portfolio



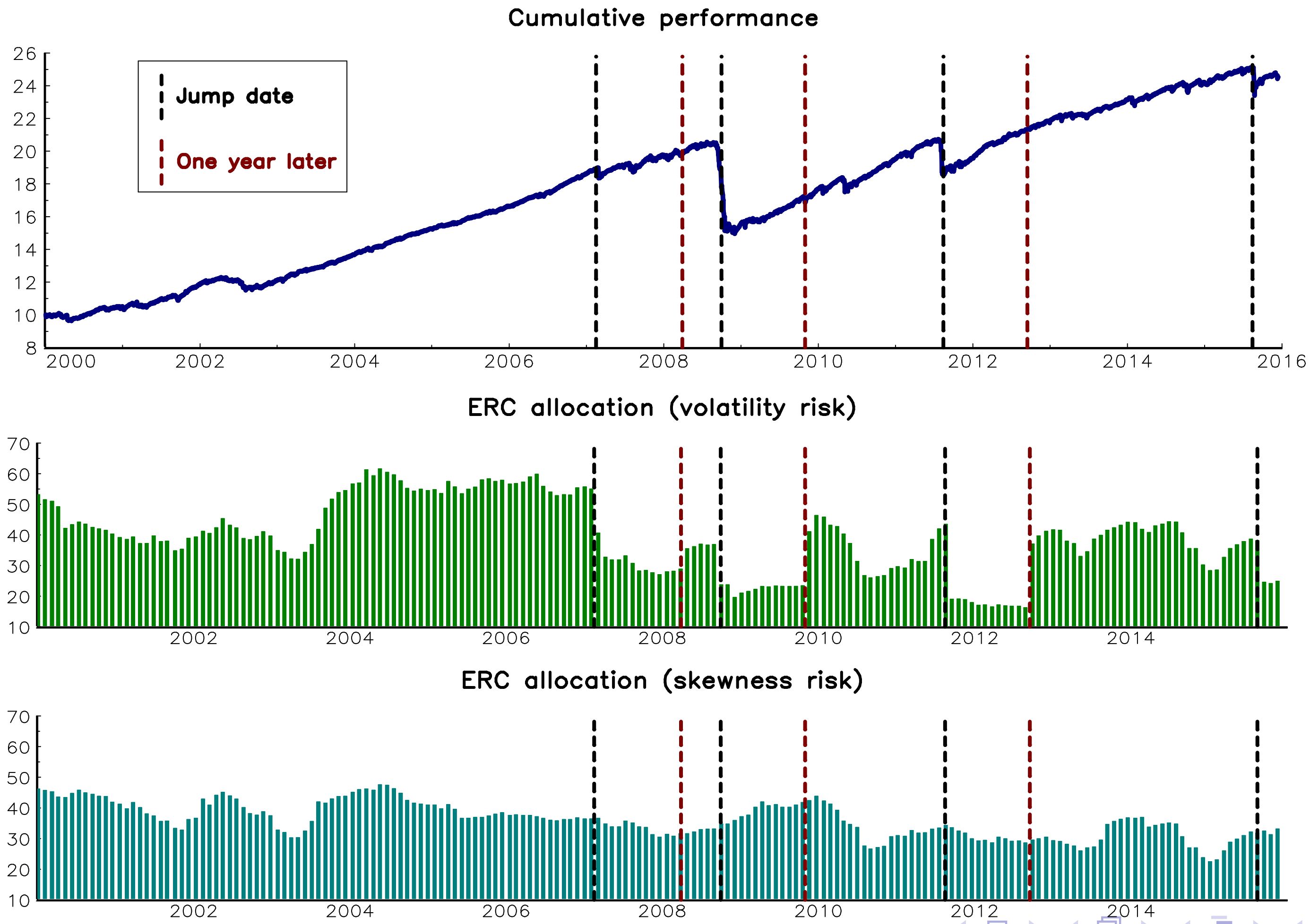
Summary IV

Figure: Skewness-based ERC portfolio



Summary V

Figure: Comparison of the carry allocation



Summary VI

- Factor investing
  $\Rightarrow$ The allocation in size and value risk factors is generally overestimated

- Alternative Risk premia
  $\Rightarrow$ The turnover issue
  $\Rightarrow$ How to allocate between skewness risk premia and market anomalies?
  $\Rightarrow$ Relevance of the trend-following strategy
  $\Rightarrow$ Momentum crashes?

- Skewness hedging vs volatility hedging
  $\Rightarrow$ It is difficult to diversify the skewness risk
  $\Rightarrow$ Volatility optimization leads to non-optimal portfolios
  $\Rightarrow$ Sizing the risk exposure is the only solution
  $\Rightarrow$ Mixing skewness risk premia and market anomalies
  $\Rightarrow$ The case of CTA strategies

The jump-diffusion representation

- n risky assets represented by the vector of prices $S_t = (S_{1,t}, \ldots, S_{n,t})$ with:

$$\begin{cases} \mathrm{d}S_t = \mathrm{diag}(S_t)\,\mathrm{d}L_t \\ \mathrm{d}L_t = \mu\,\mathrm{d}t + \Sigma^{1/2}\,\mathrm{d}W_t + \mathrm{d}Z_t \end{cases}$$

  where Z_t is a pure n-dimensional jump process.

- We assume that the jump process Z_t is a compound Poisson process:

$$Z_t = \sum_{i=1}^{N_t} Z_i$$

  where $N_t \sim \mathcal{P}(\lambda)$ and $Z_i \sim \mathcal{N}\left(\tilde{\mu}, \tilde{\Sigma}\right)$.

The characteristic function of asset returns $R_t = (R_{1,t}, \ldots, R_{n,t})$ for the holding period $\mathrm{d}t$ may be approximated by:

$$\mathbb{E}\left[e^{-iu.R_t}\right] \approx (1 - \lambda\,\mathrm{d}t) \cdot e^{\left(iu^\top \mu - \frac{1}{2}u^\top \Sigma u\right)\mathrm{d}t} + (\lambda\,\mathrm{d}t) \cdot e^{iu^\top(\mu\,\mathrm{d}t + \tilde{\mu}) - \frac{1}{2}u^\top\left(\Sigma\,\mathrm{d}t + \tilde{\Sigma}\right)u}$$

The Gaussian mixture representation

We consider a Gaussian mixture model with two regimes to define R_t:

1. The continuous component, which has the probability $(1-\lambda\,\mathrm{d}t)$ to occur, is driven by the Gaussian distribution $\mathcal{N}(\mu\,\mathrm{d}t, \Sigma\,\mathrm{d}t)$;
2. The jump component, which has the probability $\lambda\,\mathrm{d}t$ to occur, is driven by the Gaussian distribution $\mathcal{N}\left(\tilde{\mu}, \tilde{\Sigma}\right)$.

The multivariate density function of R_t is:

$$\begin{aligned} f(y) &= \frac{1-\lambda\,\mathrm{d}t}{(2\pi)^{n/2}\left|\Sigma\,\mathrm{d}t\right|^{1/2}} e^{-\frac{1}{2}(y-\mu\,\mathrm{d}t)^\top(\Sigma\,\mathrm{d}t)^{-1}(y-\mu\,\mathrm{d}t)} + \\ & \quad \frac{\lambda\,\mathrm{d}t}{(2\pi)^{n/2}\left|\Sigma\,\mathrm{d}t+\tilde{\Sigma}\right|^{1/2}} e^{-\frac{1}{2}(y-(\mu\,\mathrm{d}t+\tilde{\mu}))^\top\left(\Sigma\,\mathrm{d}t+\tilde{\Sigma}\right)^{-1}(y-(\mu\,\mathrm{d}t+\tilde{\mu}))} \end{aligned}$$

The characteristic function of R_t is equal to:

$$\mathbb{E}\left[e^{-iu.R_t}\right] = (1-\lambda\,\mathrm{d}t)\cdot e^{\left(iu^\top\mu-\frac{1}{2}u^\top\Sigma u\right)\mathrm{d}t} + (\lambda\,\mathrm{d}t)\cdot e^{iu^\top(\mu\,\mathrm{d}t+\tilde{\mu})-\frac{1}{2}u^\top\left(\Sigma\,\mathrm{d}t+\tilde{\Sigma}\right)u}$$

Distribution function of the portfolio's return

Let $x = (x_1, \ldots, x_n)$ be the vector of weights in the portfolio. We have:

$$R(x) = Y = B_1 \cdot Y_1 + B_2 \cdot Y_2$$

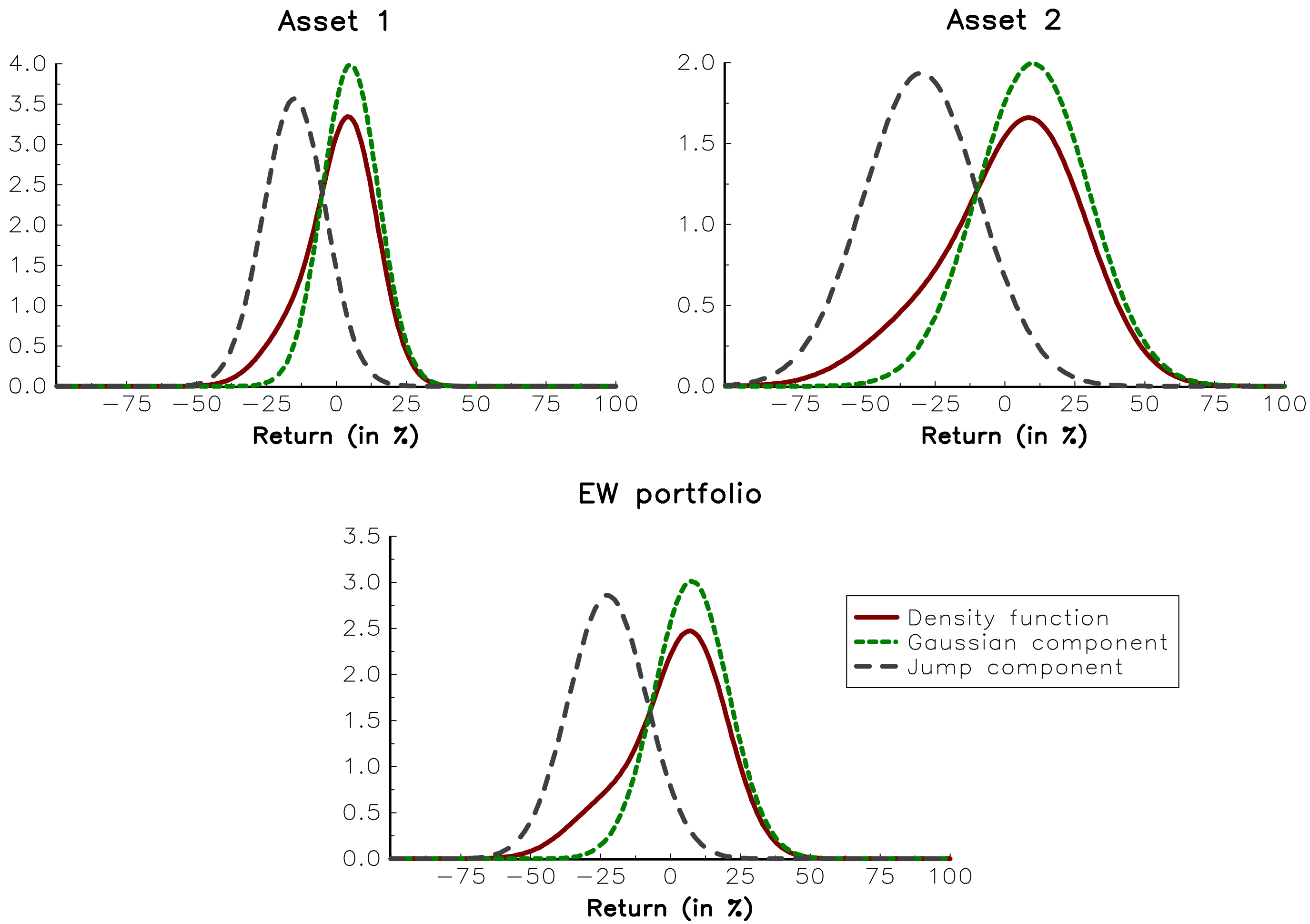
where:

- $B_1 \sim \mathcal{B}(\pi_1)$, $B_2 = 1 - B_1 \sim \mathcal{B}(\pi_2)$, $\pi_1 = 1 - \lambda$ and $\pi_2 = \lambda$ $(\underline{\mathcal{H} : \mathrm{d}t = 1})$;
- $Y_1 \sim \mathcal{N}(\mu_1(x), \sigma_1^2(x))$, $\mu_1(x) = x^\top \mu$ and $\sigma_1^2(x) = x^\top \Sigma x$;
- $Y_2 \sim \mathcal{N}(\mu_2(x), \sigma_2^2(x))$, $\mu_2(x) = x^\top (\mu + \tilde{\mu})$ and $\sigma_2^2(x) = x^\top (\Sigma + \tilde{\Sigma}) x$.

$\Rightarrow$ The portfolio's return $R(x)$ has the following density function:

$$\begin{aligned} f(y) &= \pi_1 f_1(y) + \pi_2 f_2(y) \\ &= (1-\lambda) \frac{1}{\sigma_1(x)} \phi\left(\frac{y - \mu_1(x)}{\sigma_1(x)}\right) + \lambda \frac{1}{\sigma_2(x)} \phi\left(\frac{y - \mu_2(x)}{\sigma_2(x)}\right) \end{aligned}$$

Distribution function of the portfolio's return



Parameters: $\mu_1 = 5\%$, $\sigma_1 = 10\%$, $\tilde{\mu}_1 = -20\%$, $\tilde{\sigma}_1 = 5\%$, $\mu_2 = 10\%$, $\sigma_2 = 20\%$, $\tilde{\mu}_2 = -40\%$, $\tilde{\sigma}_2 = 5\%$, $\rho = 50\%$, $\tilde{\rho} = 60\%$ and $\lambda = 0.20$.

Relationship between jump risk and skewness risk

The skewness of $R(x)$ is equal to:

$$\gamma_1 = \frac{(\lambda - \lambda^2)\left((1-2\lambda)\left(x^\top \tilde{\mu}\right)^3 + 3\left(x^\top \tilde{\mu}\right)\left(x^\top \tilde{\Sigma} x\right)\right)}{\left(x^\top \Sigma x + \lambda x^\top \tilde{\Sigma} x + (\lambda - \lambda^2)\left(x^\top \tilde{\mu}\right)^2\right)^{3/2}}$$

The portfolio exhibits skewness, except for some limit cases:

$$\gamma_1 = 0 \Leftrightarrow x^\top \tilde{\mu} = 0 \text{ or } \lambda = 0 \text{ or } \lambda = 1$$

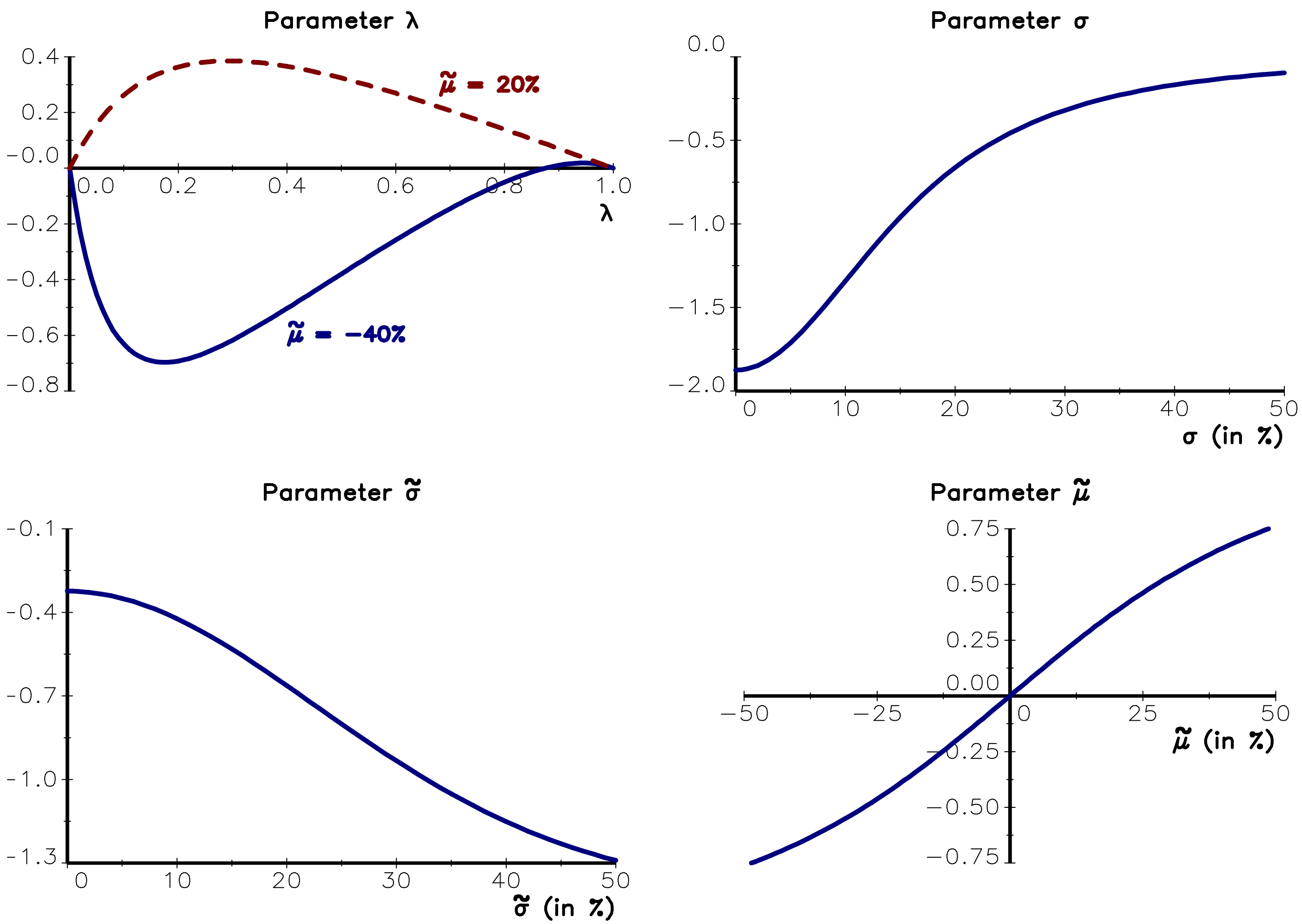
We have:

- If $x^\top \tilde{\mu} > 0$, then $\gamma_1 > 0$;
- If $x^\top \tilde{\mu} < 0$, then $\gamma_1 < 0$ in most cases.

$\Rightarrow$ We retrieve the result of Hamdan *et al.* (2016):

Skewness risk is maximum when volatility risk is minimum

Relationship between jump risk and skewness risk



Parameters: $\sigma = 20\%$, $\tilde{\mu} = -40\%$, $\tilde{\sigma} = 20\%$ and $\lambda = 25\%$.

Estimation of the parameters

With the notations $\mu_1 = \mu$, $\Sigma_1 = \Sigma$, $\mu_2 = \mu + \tilde{\mu}$, $\Sigma_2 = \Sigma + \tilde{\Sigma}$, the log-likelihood function becomes:

$$\ell(\theta) = \sum_{t=1}^{T} \ln \sum_{j=1}^{2} \pi_j \phi_n(R_t; \mu_j, \Sigma_j)$$

$\Rightarrow$ The method of maximum likelihood is not suitable (weak identification)
$\Rightarrow$ Parameters are estimated using the EM algorithm

Posterior probability to have a jump at time t

We have:

$$\begin{aligned} \pi_{2,t} &= \Pr\{B_2 = 1 \mid R_t\} \\ &= \frac{\pi_2 \phi_n(R_t; \mu_2, \Sigma_2)}{\sum_{s=1}^{2} \pi_s \phi_n(R_t; \mu_s, \Sigma_s)} \end{aligned}$$

The expected shortfall risk measure

Definition of the expected shortfall

$$\mathrm{ES}_{\alpha}(x) = \mathbb{E}\left[L(x) \mid L(x) \geq \mathrm{VaR}_{\alpha}(x)\right]$$

where $L(x) = -R(x)$ is the portfolio's loss.

We obtain:

$$\mathrm{ES}_{\alpha}(x) = (1-\lambda)\cdot\varphi(\mathrm{VaR}_{\alpha}(x), \mu_1(x), \sigma_1(x)) + \lambda\cdot\varphi(\mathrm{VaR}_{\alpha}(x), \mu_2(x), \sigma_2(x))$$

where the function $\varphi(a,b,c)$ is defined by:

$$\varphi(a,b,c) = \frac{c}{1-\alpha}\phi\left(\frac{a+b}{c}\right) - \frac{b}{1-\alpha}\Phi\left(-\frac{a+b}{c}\right)$$

Here, the value-at-risk $\mathrm{VaR}_{\alpha}(x)$ is the root of the following equation:

$$(1-\lambda)\cdot\Phi\left(\frac{\mathrm{VaR}_{\alpha}(x)+\mu_1(x)}{\sigma_1(x)}\right) + \lambda\cdot\Phi\left(\frac{\mathrm{VaR}_{\alpha}(x)+\mu_2(x)}{\sigma_2(x)}\right) = \alpha$$

Analytical expression of risk contributions

We obtain a complicated expression of the risk contribution:

$$\mathcal{RC}_i(x) = x_i \frac{\partial \operatorname{ES}_\alpha(x)}{\partial x_i} = \ldots$$

But it is an analytical formula!

$\Rightarrow$ **No numerical issues for implementing the model**

Euler decomposition

We have:

$$\sum_{i=1}^{n} \mathcal{RC}_i(x) = \operatorname{ES}_\alpha(x)$$

$\Rightarrow$ Comparison with the value-at-risk based on the Cornish-Fisher expansion

Risk decomposition

Example

We consider three assets, whose expected returns are equal to 10%, 15% and 20%. Their volatilities are equal to 20%, 25% and 30% while the correlation matrix of asset returns is provided by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & \\ 0.50 & 1.00 & \\ 0.20 & 0.40 & 1.00 \end{pmatrix}$$

For the jumps, we assume that $\tilde{\mu}_i = -10\%$, $\tilde{\sigma}_i = 20\%$ and $\tilde{\rho}_{i,j} = 50\%$. Moreover, the intensity λ of jumps is equal to 0.25, meaning that we observe a jump every four years on average.

Risk decomposition

Table: Expected shortfall decomposition (without jumps)

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	20.00	8.90	1.78	6.26
2	20.00	18.22	3.64	12.80
3	60.00	38.40	23.04	80.94
$\mathrm{ES}_\alpha(x)$			28.46	

Table: Expected shortfall decomposition (with jumps)

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	20.00	20.39	4.08	10.96
2	20.00	27.31	5.46	14.67
3	60.00	46.13	27.68	74.37
$\mathrm{ES}_\alpha(x)$			37.22	

Risk budgeting portfolios

The RB portfolio is defined by the following non-linear system:

$$\begin{cases} \mathcal{RC}_i(x) = b_i \mathcal{R}(x) \\ b_i > 0 \\ x_i \geq 0 \\ \sum_{i=1}^n b_i = 1 \\ \sum_{i=1}^n x_i = 1 \end{cases}$$

where b_i is the ex-ante risk budget of asset i expressed in relative terms.

Numerical solution of the RB portfolio

$$y^\star = \arg\min \mathrm{ES}_\alpha(y) - \sum_{i=1}^n b_i \ln y_i \quad \text{u.c.} \quad y \geq \mathbf{0}$$

The RB portfolio corresponds to the normalized portfolio:

$$x_i^\star = \frac{y_i^\star}{\sum_{j=1}^n y_j^\star}$$

Existence and uniqueness

We have to impose the following restriction:

$$\mathcal{R}(x) = \mathrm{ES}_\alpha(x) \geq 0$$

The issue comes from the homogeneity property $\mathcal{R}(\delta x) = \delta \mathcal{R}(x)$ where δ is a positive scalar.

Theorem

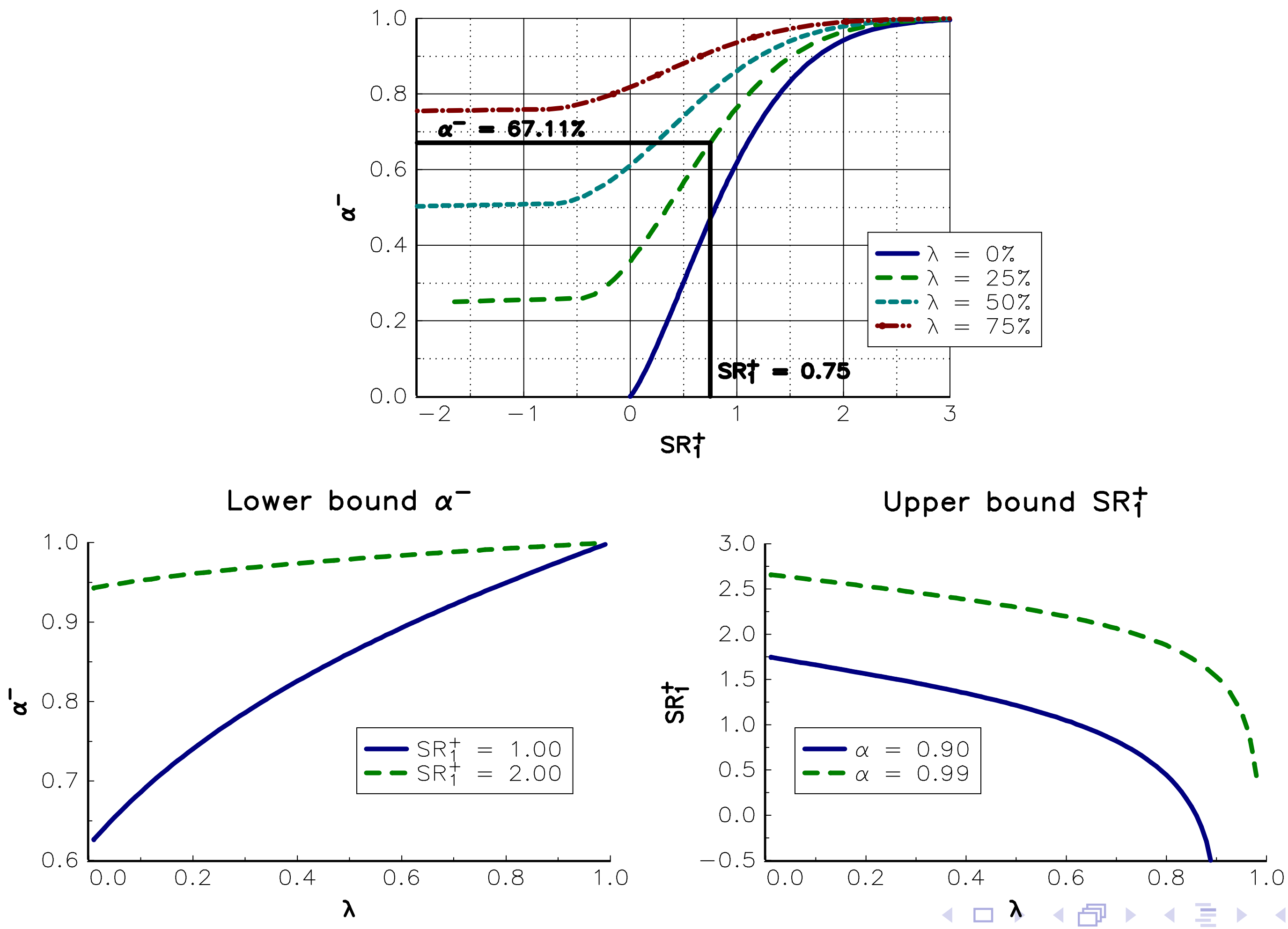
If $\alpha \geq \max(\alpha^-, \lambda)$, the RB portfolio exists and is unique, where α^- be the root of the equation below:

$$\frac{1-\lambda}{1-\alpha^-}\phi\left(\Phi^{-1}\left(\frac{\alpha^- - \lambda}{1-\lambda}\right)\right) + \lambda\Phi^{-1}\left(\frac{\alpha^- - \lambda}{1-\lambda}\right) = (1+\lambda)\mathrm{SR}_1^+$$

and SR_1^+ is the maximum Sharpe ratio under the first regime.

Existence and uniqueness

Figure: Relationship between λ, SR_1^+ and α^-



An example

Example

We have $\mu_1 = 3\%$, $\mu_2 = 8\%$, $\mu_3 = 12\%$, $\sigma_1 = 8\%$, $\sigma_2 = 20\%$, $\sigma_3 = 30\%$ and:

$$\rho = \begin{pmatrix} 1.00 & & \\ 0.50 & 1.00 & \\ 0.20 & 0.40 & 1.00 \end{pmatrix}$$

For the jumps, we have $\tilde{\mu}_1 = -15\%$, $\tilde{\mu}_2 = -40\%$, $\tilde{\mu}_3 = 0\%$, $\tilde{\sigma}_1 = 15\%$, $\tilde{\sigma}_2 = 20\%$, $\tilde{\sigma}_3 = 10\%$ and:

$$\tilde{\rho} = \begin{pmatrix} 1.00 & & \\ 0.50 & 1.00 & \\ 0.00 & 0.00 & 1.00 \end{pmatrix}$$

The intensity λ of jumps is equal to 0.25.

ERC portfolio

Table: ERC portfolio (Gaussian risk measure)

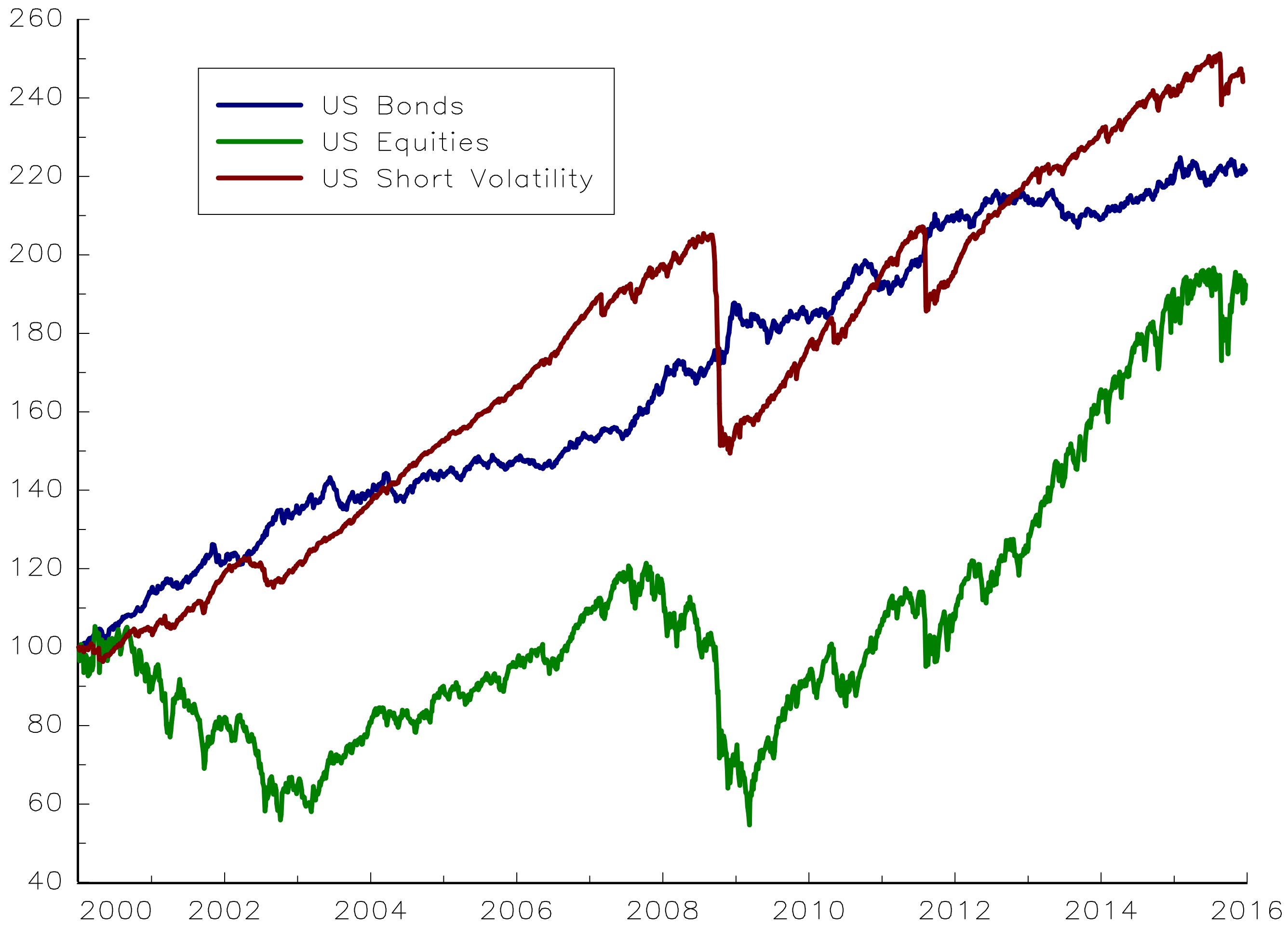
Asset	volatility risk measure				95% expected shortfall			
	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	60.94	5.96	3.63	33.33	60.85	9.24	5.62	33.33
2	22.20	16.35	3.63	33.33	21.96	25.60	5.62	33.33
3	16.87	21.52	3.63	33.33	17.19	32.72	5.62	33.33
	$\sigma(x)$		10.89		$\mathrm{ES}_\alpha(x)$		16.87	

Table: ERC portfolio (95% expected shortfall with jumps)

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1	44.70	24.70	11.04	33.33
2	19.87	55.56	11.04	33.33
3	35.42	31.17	11.04	33.33
$\mathrm{ES}_\alpha(x)$			33.12	

Statistics

Figure: Cumulative performance of US bonds, US equities and US short volatility



Statistics

Table: Worst returns (in %)

Asset	Daily	Weekly	Monthly	Annually	Maximum
Bonds	−1.67	−2.81	−4.40	−3.41	−6.03
Equities	−9.03	−18.29	−29.67	−49.69	−55.25
Carry	−6.82	−11.04	−23.43	−23.37	−27.30

Table: Skewness coefficients

Asset	Daily	Weekly	Monthly	Annually	Volatility
Bonds	−0.12	−0.17	0.07	0.22	4.17
Equities	0.01	−0.44	−0.81	−0.57	18.38
Carry	−7.24	−5.77	−6.32	−2.23	5.50

EM estimates of the mixture model

$$f(y) = (1-\pi)\,\phi_1\left(y;\mu_1\,\mathrm{d}t,\sigma_1^2\,\mathrm{d}t\right) + \pi\,\phi_1\left(y;\mu_2\,\mathrm{d}t,\sigma_2^2\,\mathrm{d}t\right)$$

Figure: Bonds

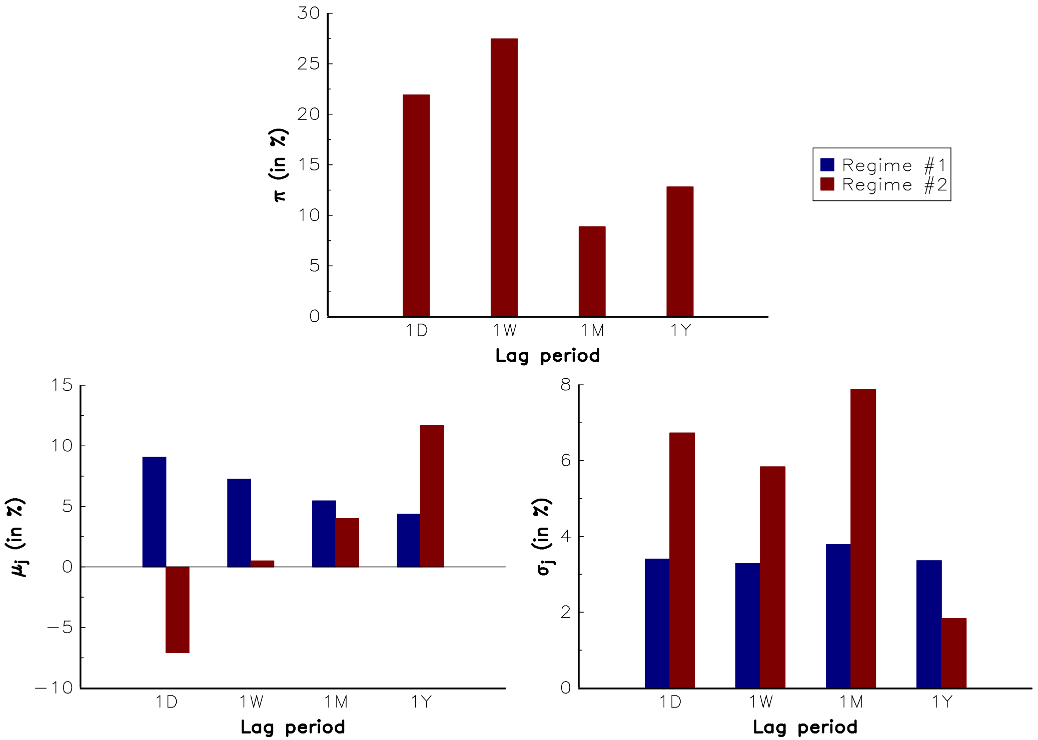
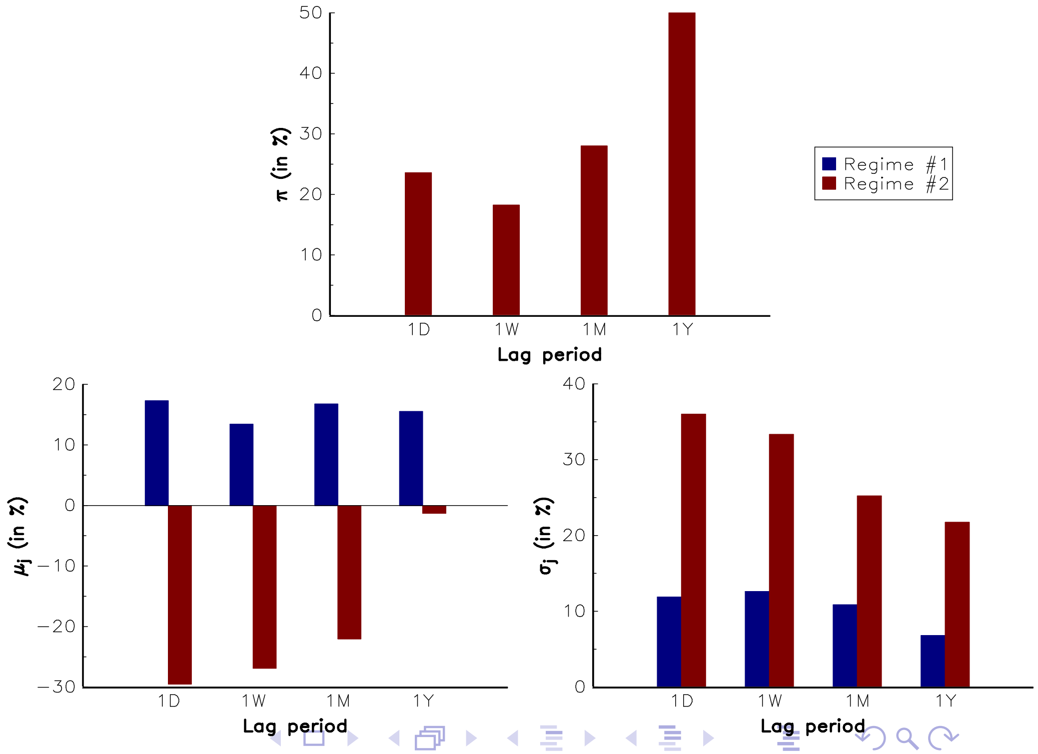
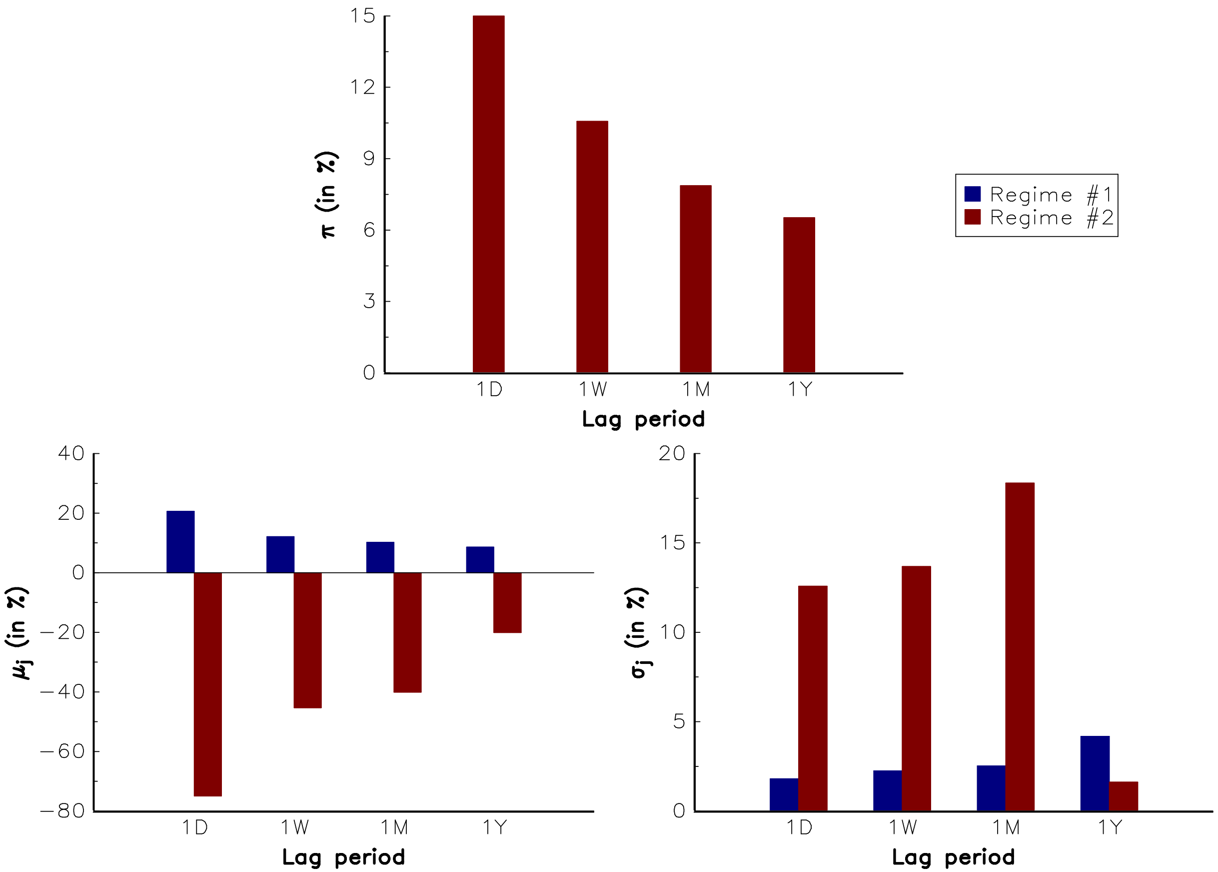


Figure: Equities



EM estimates of the mixture model

Figure: Carry



Main results:

- We obtain a two-volatility regime!
- Daily and annually lag periods are not adapted.

How to obtain a mixture model with jumps?

The probability density function is:

$$f(y) = (1-\pi)\phi_3(y; \mu\,\mathrm{d}t, \Sigma\,\mathrm{d}t) + \pi\phi_3\left(y; \mu\,\mathrm{d}t + \tilde{\mu}, \Sigma\,\mathrm{d}t + \tilde{\Sigma}\right)$$

$\Rightarrow \pi$ can not be estimated by EM.

The two-step estimation procedure

1. We calibrate the probability π according to the expected drawdown:

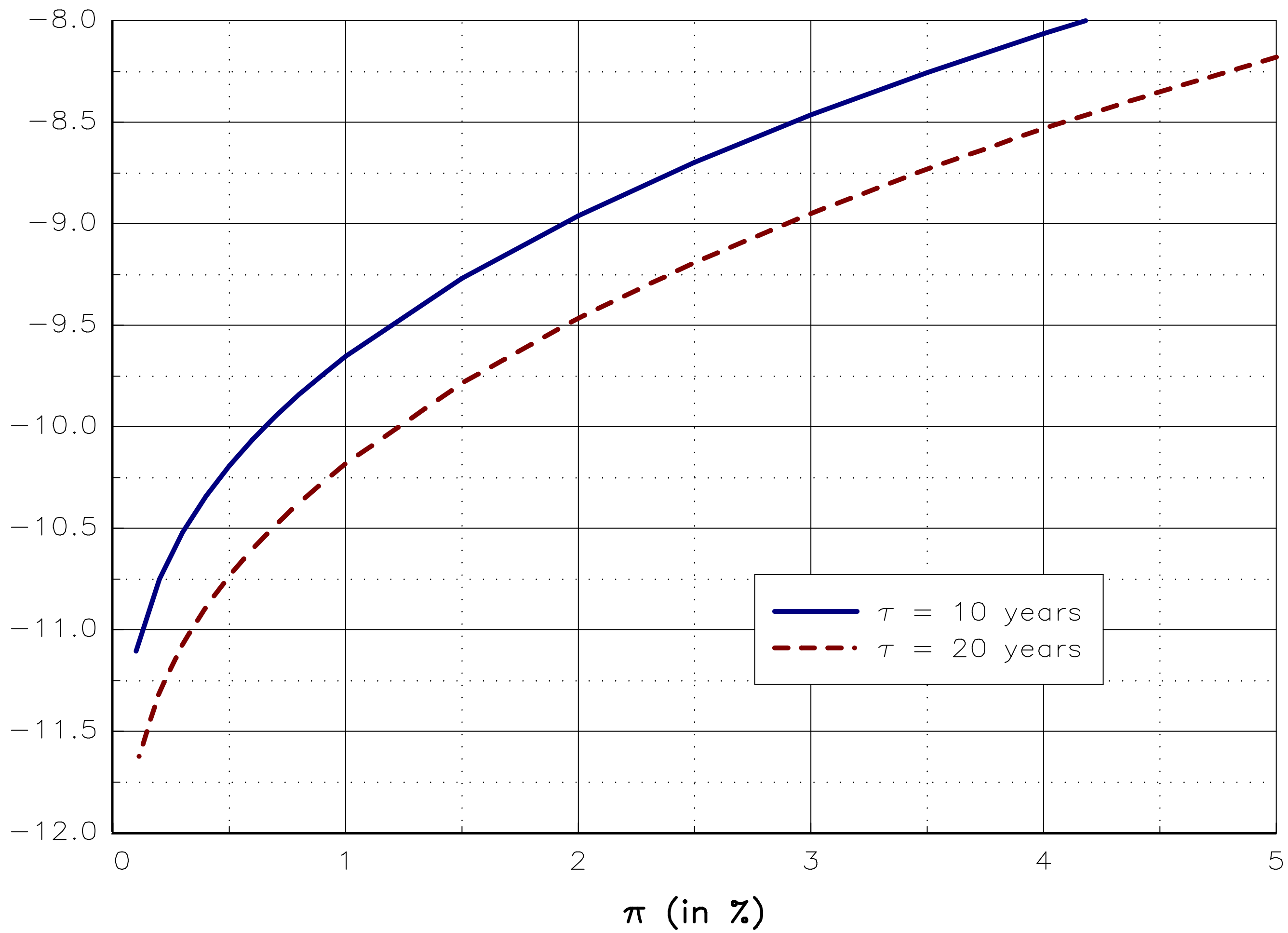
$$\mathbb{E}\left[\mathrm{DD}_i(\tau)\right] = \tilde{\mu}_i + \Phi^{-1}\left(\frac{\mathrm{d}t}{\tau}\right)\tilde{\sigma}_i$$

where τ is the given return period.

2. We calibrate the parameters μ, σ, $\tilde{\mu}$ and $\tilde{\Sigma}$ by CML.

Calibration of the jump probability

Figure: Expected weekly drawdown (in %) of the carry risk premium



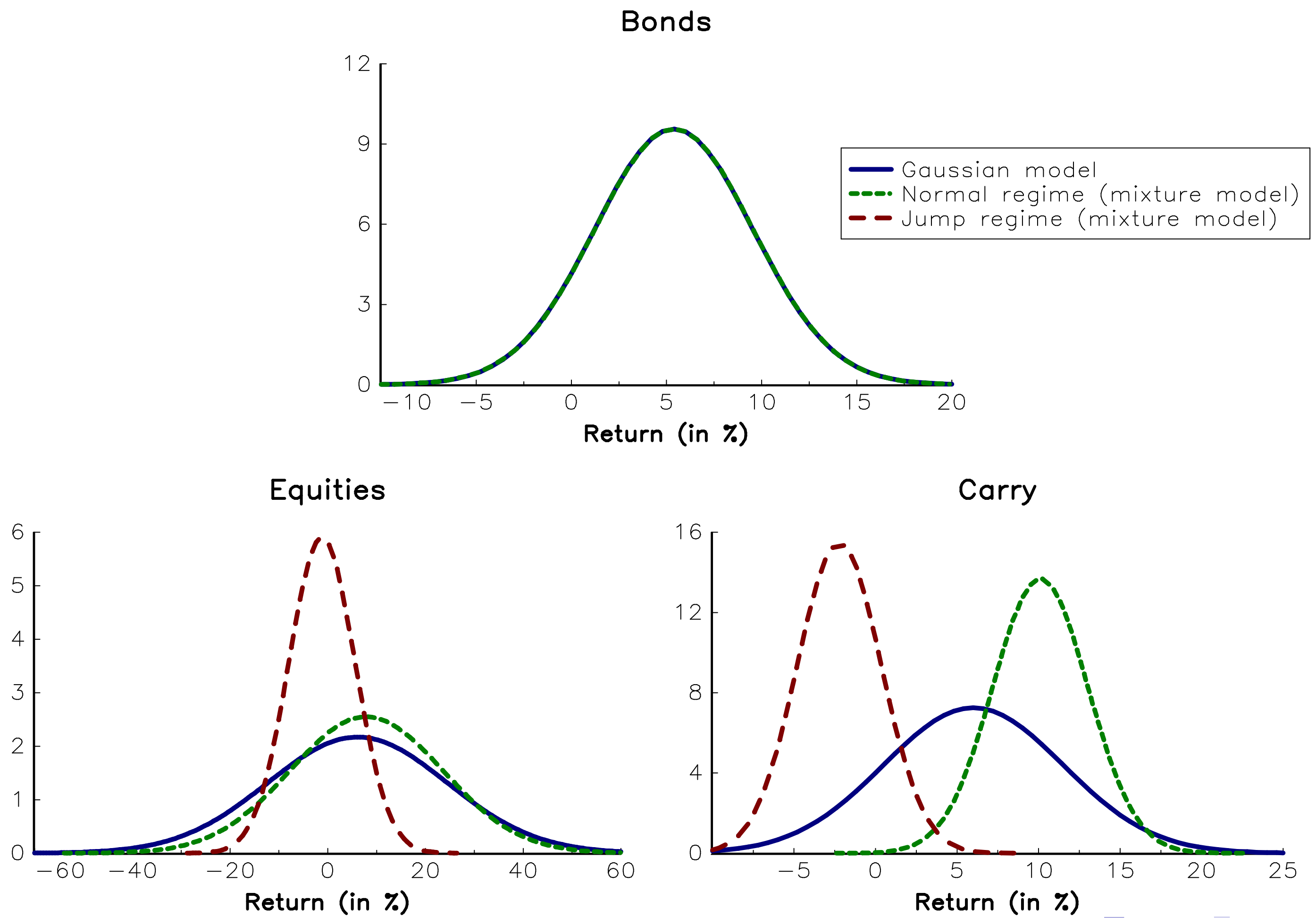
Calibration of the other parameters

Table: Estimation of the constrained mixture model when $\pi = 0.5\%$ (weekly model)

Regime	Asset	μ_i	σ_i	$\rho_{i,j}$		
Normal	Bonds	5.38	4.17	100.00		
	Equities	7.89	15.64	-36.80	100.00	
	Carry	10.10	2.91	-25.17	57.43	100.00
Regime	Asset	$\tilde{\mu}_i$	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$		
Jump	Bonds	0.00	0.00	100.00		
	Equities	-1.20	6.76	0.00	100.00	
	Carry	-2.23	2.57	0.00	60.45	100.00

Impact of the jump component

Figure: PDF of asset returns (weekly model)



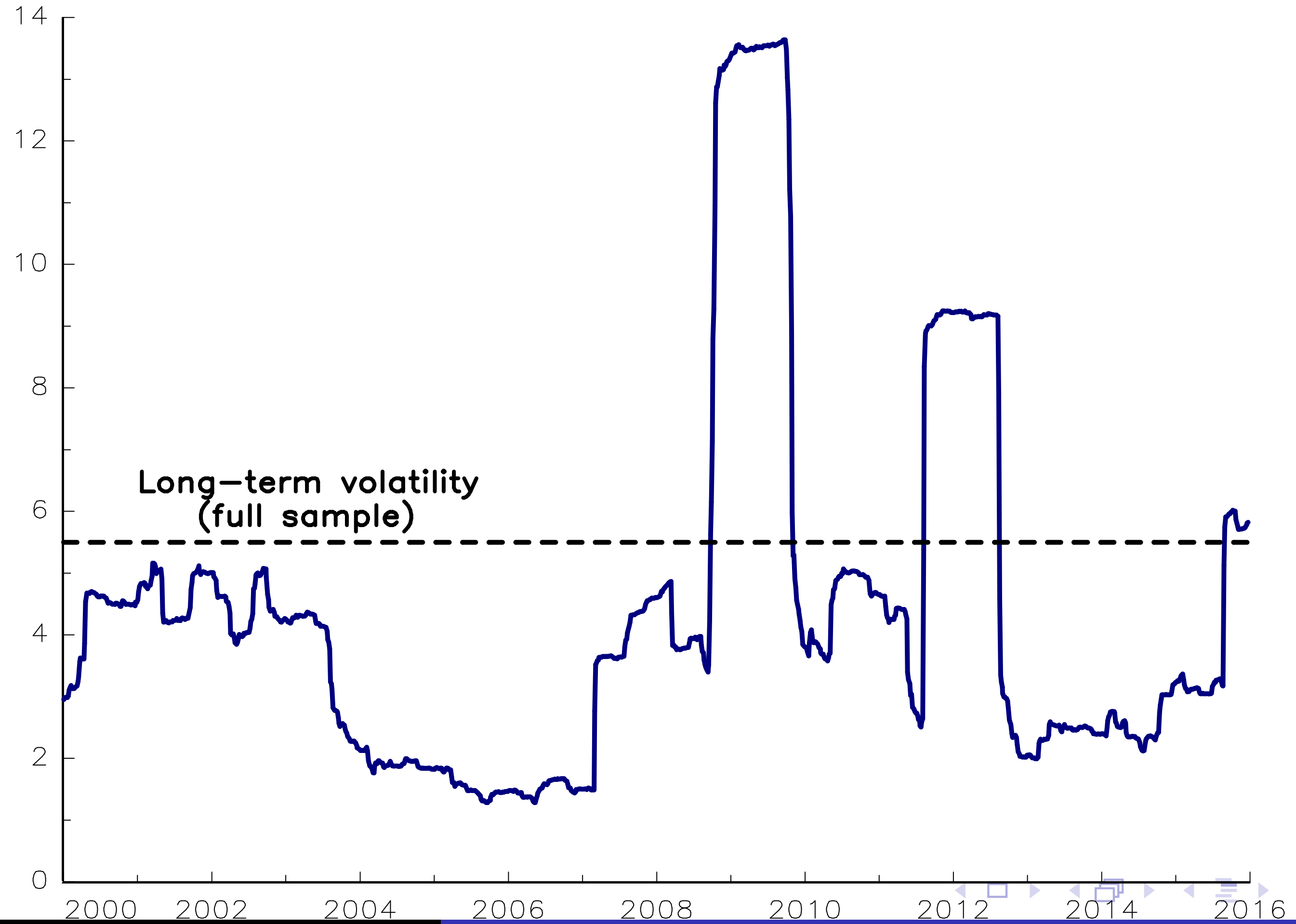
Comparing in-sample ERC portfolios

Table: Weights (in %) of the ERC portfolio

Asset	Weekly model			Monthly model		
		Jump model			Jump model	
	Gaussian	Normal	Mixture	Gaussian	Normal	Mixture
Bonds	59.17	46.46	52.71	61.36	44.36	62.22
Equities	10.43	9.18	10.36	12.09	10.06	12.25
carry	30.40	44.36	36.93	26.55	45.58	25.53

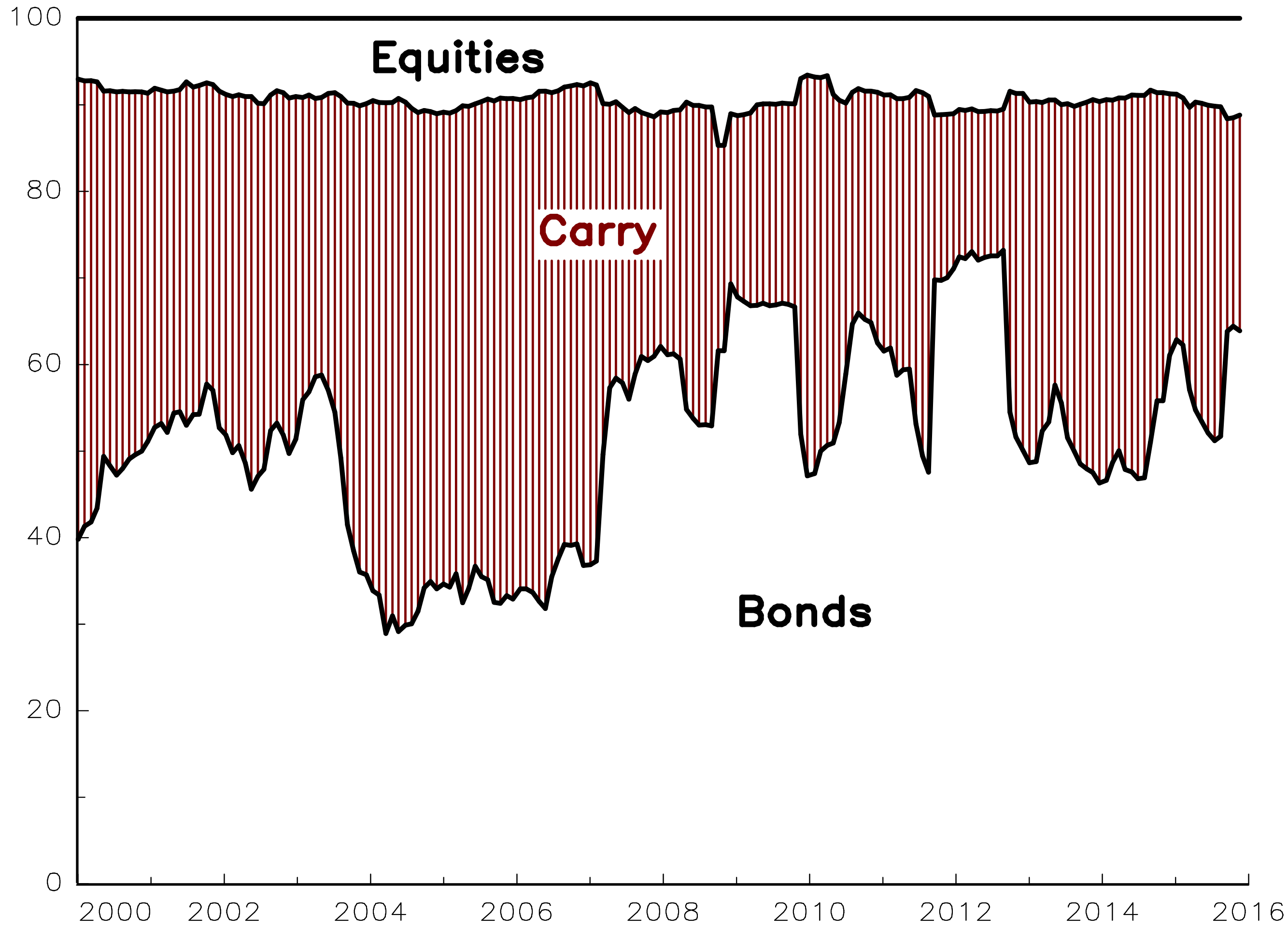
How explaining these results?

Figure: One-year rolling volatility (in %) of the carry risk premium (weekly model)



Volatility-based ERC portfolio

Figure: Dynamics of the ERC weights (Gaussian model)



How to produce an out-of-sample skewness-based risk parity portfolio?

Main assumption

We assume that the parameters π, $\tilde{\mu}$ and $\tilde{\Sigma}$ are given once and for all, and are set to the previous estimates.

Alternative approach: we can calibrate π, $\tilde{\mu}$ and $\tilde{\Sigma}$ using stress scenarios.

Bad solution

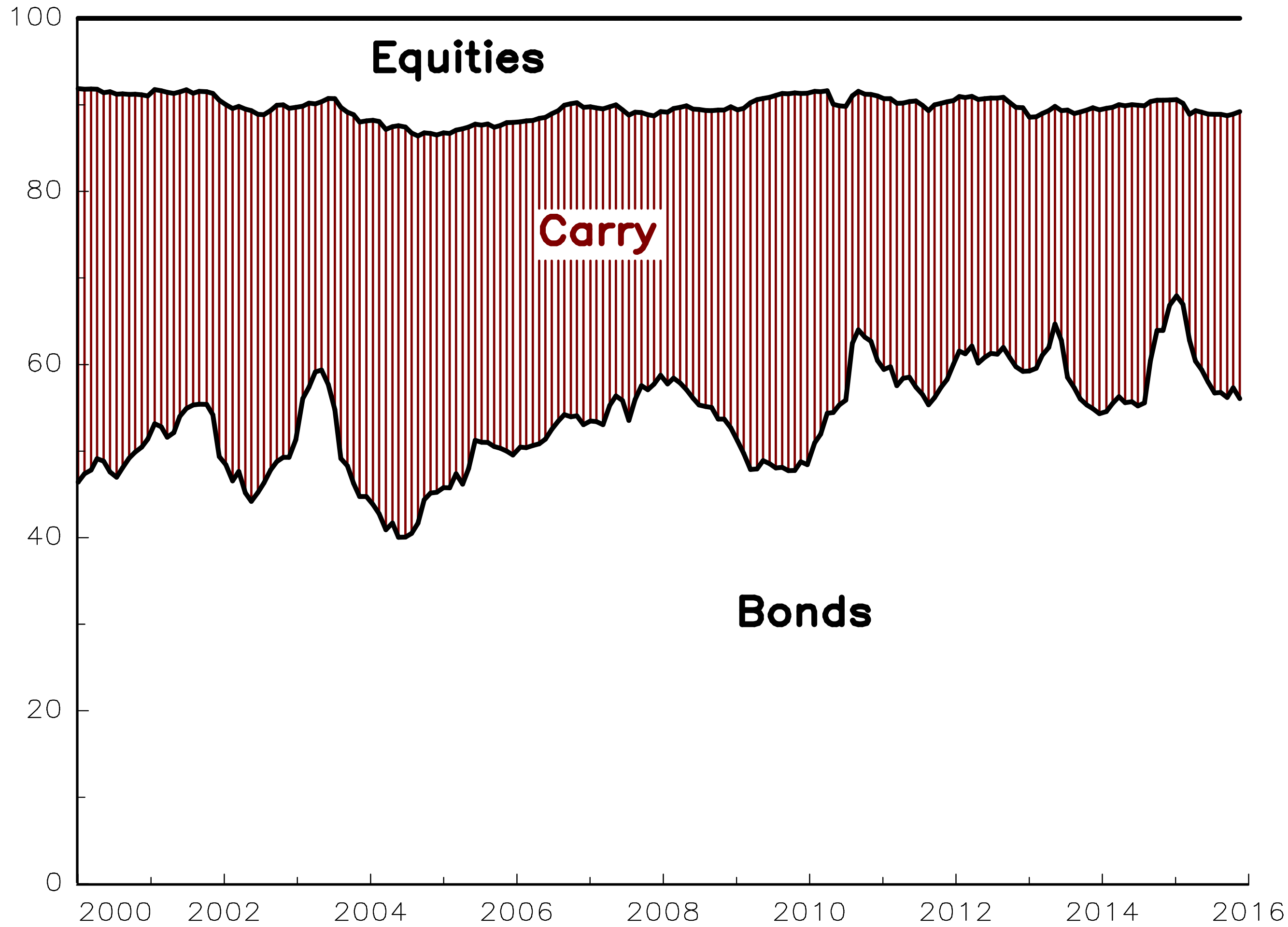
We apply the ML method with the data of the rolling window:

$$\left(\hat{\mu}_t, \hat{\Sigma}_t\right) = \arg\max_{(\mu_t, \Sigma_t)} \sum_{s=1}^{n_{\mathrm{rw}}} \ln \left(\begin{array}{c} (1-\pi)\,\phi_3\left(R_{t-s}, \mu_t\,\mathrm{d}t, \Sigma_t\,\mathrm{d}t\right) + \\ \pi\phi_3\left(R_{t-s}, \mu_t\,\mathrm{d}t + \tilde{\mu}, \Sigma_t\,\mathrm{d}t + \tilde{\Sigma}\right) \end{array} \right)$$

where n_{rw} is the length of the rolling window.

Out-of-sample skewness-based ERC portfolio

Figure: Dynamics of the ERC weights (mixture model, ML method)



Introducing the thresholding approach

Ait-Sahalia and Jacod, 2012

We observe a jump at time t when the absolute return of R_t is larger than a given level $r^\star$:

$$J_t = 1 \Leftrightarrow |R_t - v| \geq r^\star$$

Using a sample of asset returns, we can then create two sub-samples:

- a sample of asset returns without jump that satisfy $|R_t - v| < r^\star$;
- a sample of asset returns with jumps that satisfy $|R_t - v| \geq r^\star$;

Introducing the filtering approach

Filtering approach to detect jump

- We calculate the posterior probability of the jump regime:

$$\hat{\pi}_t = \frac{\pi \phi_n \left(R_t, \mu \,\mathrm{d}t + \tilde{\mu}, \Sigma \,\mathrm{d}t + \tilde{\Sigma} \right)}{(1-\pi)\,\phi_n \left(R_t, \mu \,\mathrm{d}t, \Sigma \,\mathrm{d}t \right) + \pi \phi_n \left(R_t, \mu \,\mathrm{d}t + \tilde{\mu}, \Sigma \,\mathrm{d}t + \tilde{\Sigma} \right)}$$

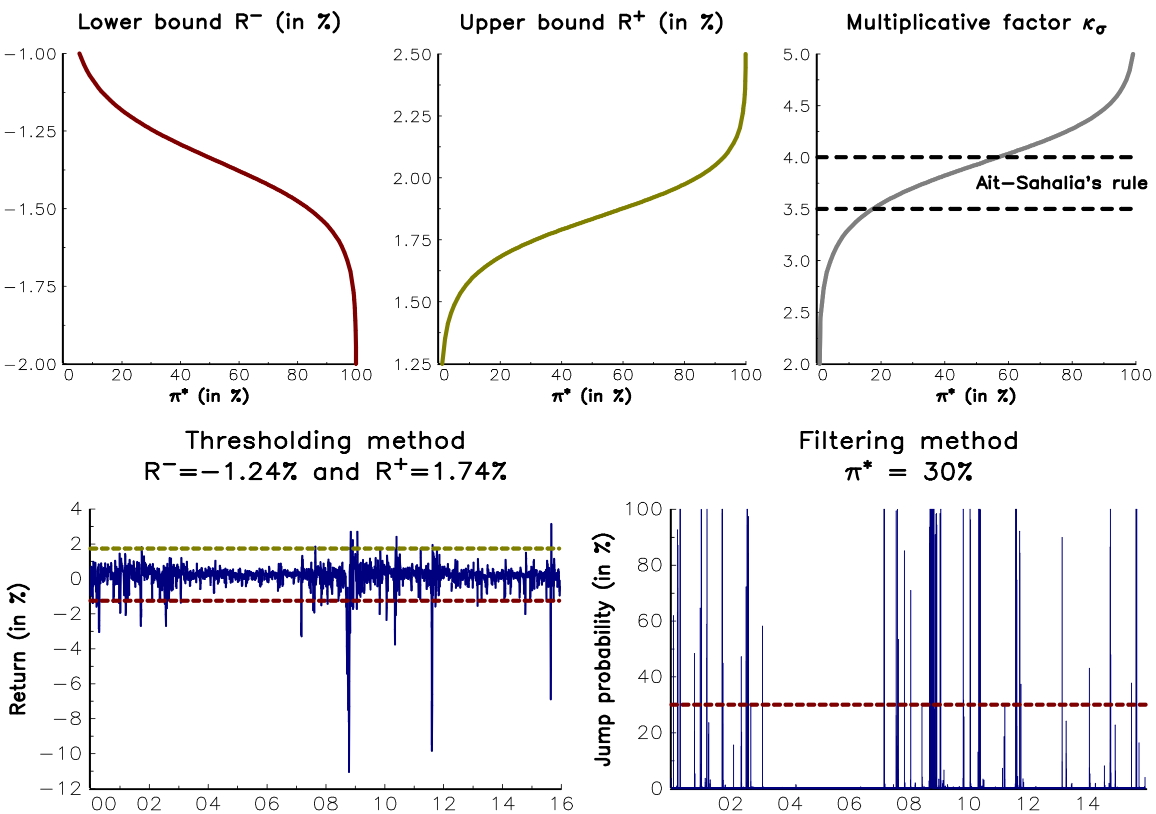
- We will say that we observe a jump at time t when the posterior probability is larger than a threshold $\pi^\star$:

$$J_t = 1 \Leftrightarrow \hat{\pi}_t \geq \pi^\star$$

Equivalence between the two approaches ($n = 1$)

$$\hat{\pi}_t \geq \pi^\star \Leftrightarrow R_t \leq R^- \text{ or } R_t \geq R^+$$

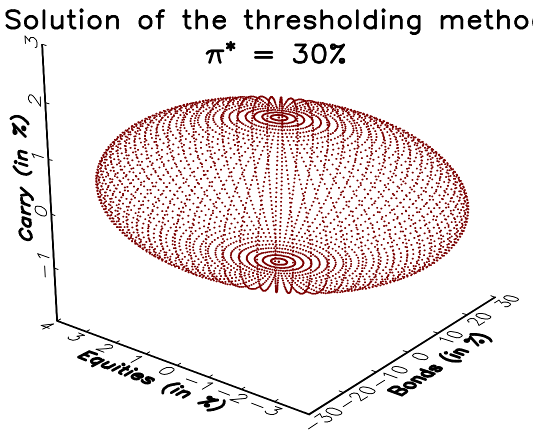
Figure: Detecting the jumps of the carry risk premium (weekly model)

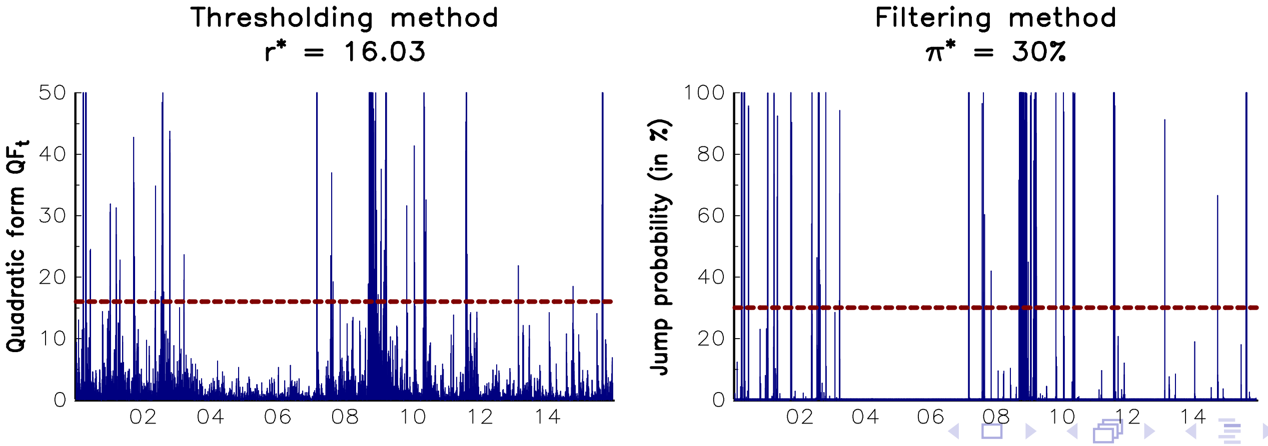


Non-equivalence between the two approaches ($n > 1$)

$$\hat{\pi}_t \geq \pi^\star \not\Leftrightarrow R_{i,t} \leq R_i^- \text{ and } R_{i,t} \geq R_i^+$$

$$\hat{\pi}_t \geq \pi^\star \Leftrightarrow (R_t - v)^\top Q (R_t - v) \geq r^\star$$

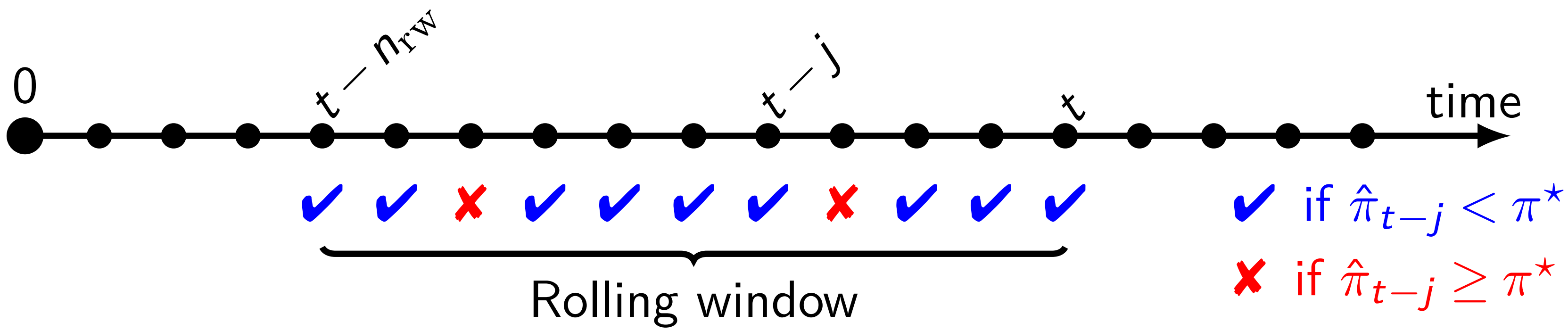




The out-of-sample filtering approach

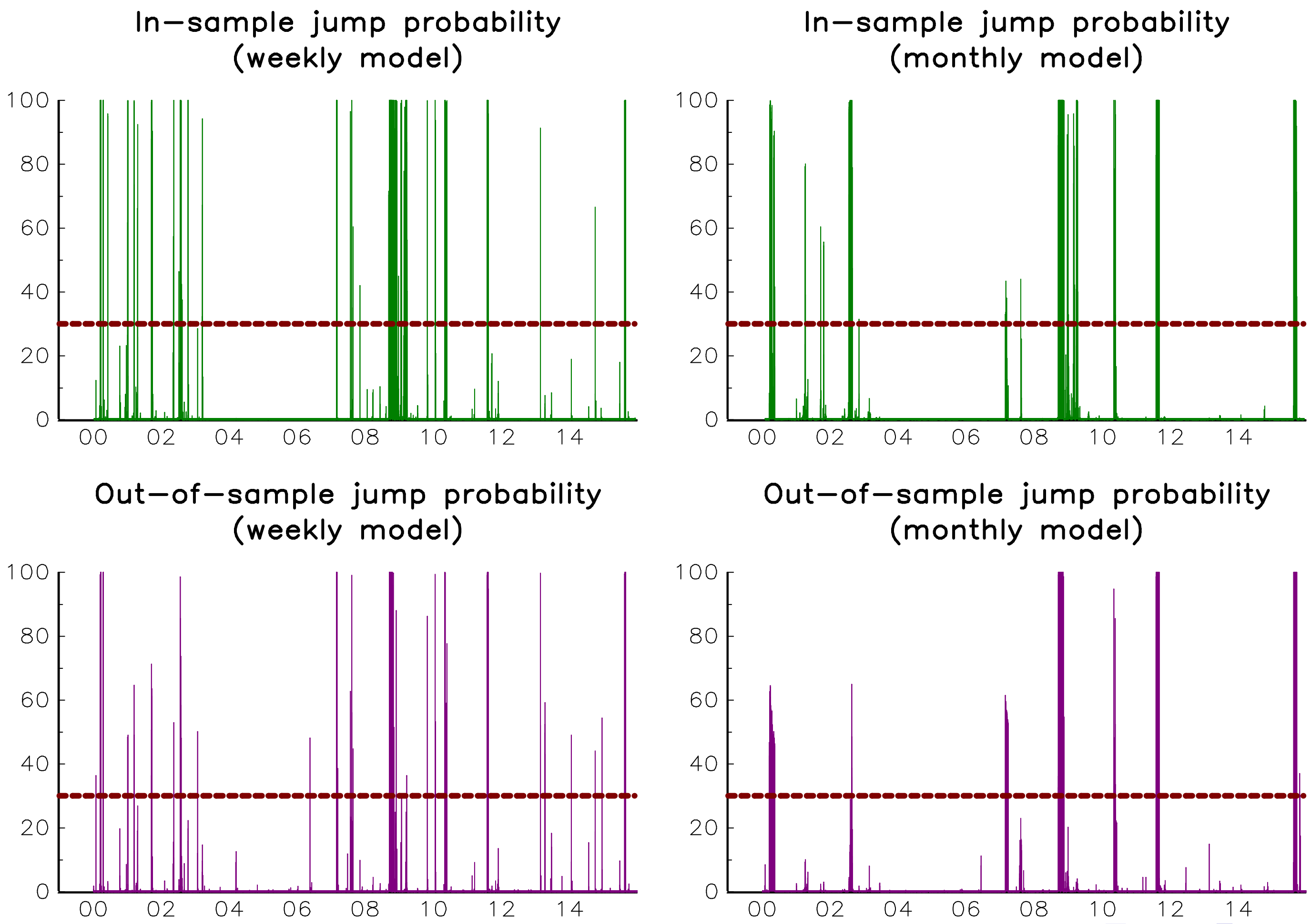
Filtering algorithm

- Given $\hat{\mu}_{t-1}$ and $\hat{\Sigma}_{t-1}$, we estimate the jump probability for each dates of the rolling window which ends at time t;
- Given the previous jump probabilities, we estimate the parameters $\hat{\mu}_t$ and $\hat{\Sigma}_t$ by deleting the dates of the rolling window that correspond to a jump;
- We iterate the algorithm until today.



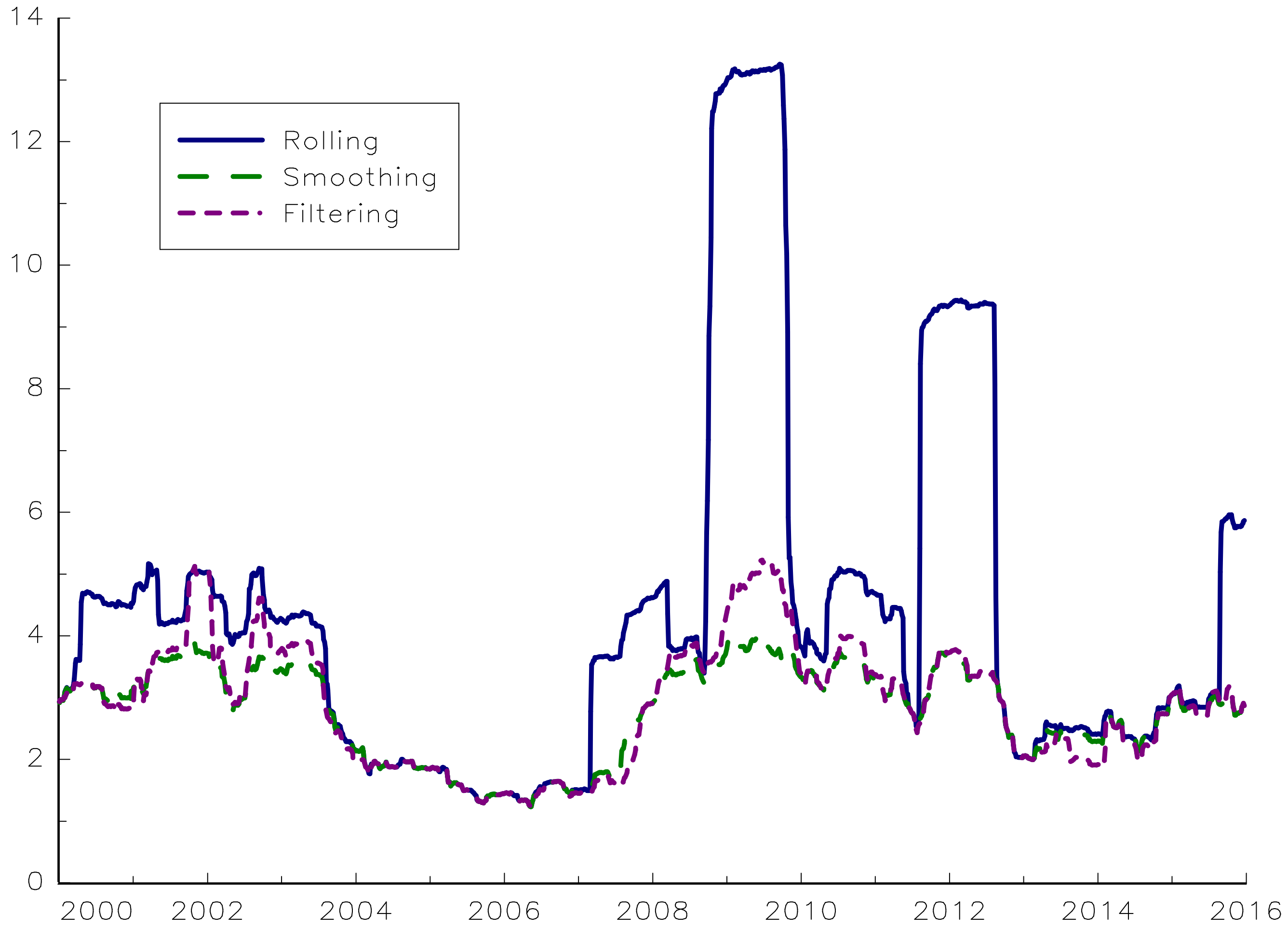
Estimating out-of sample jump probabilities

Figure: Jump probabilities (in %)



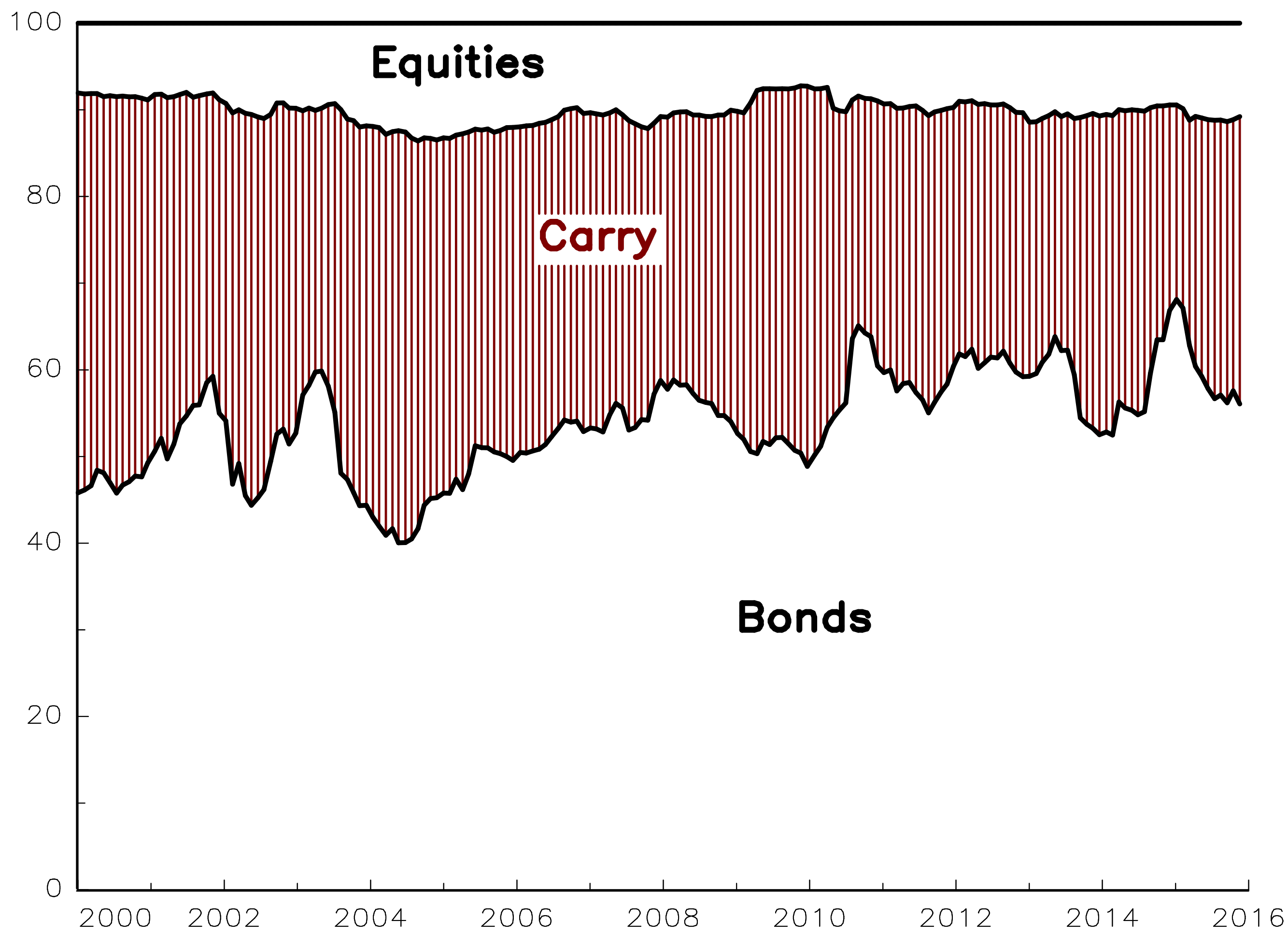
Using the filtering approach to estimate the volatility

Figure: Estimated volatility (in %) of the carry risk premium (weekly model)



The final backtest

Figure: Dynamics of the ERC weights (mixture model, filtering method)



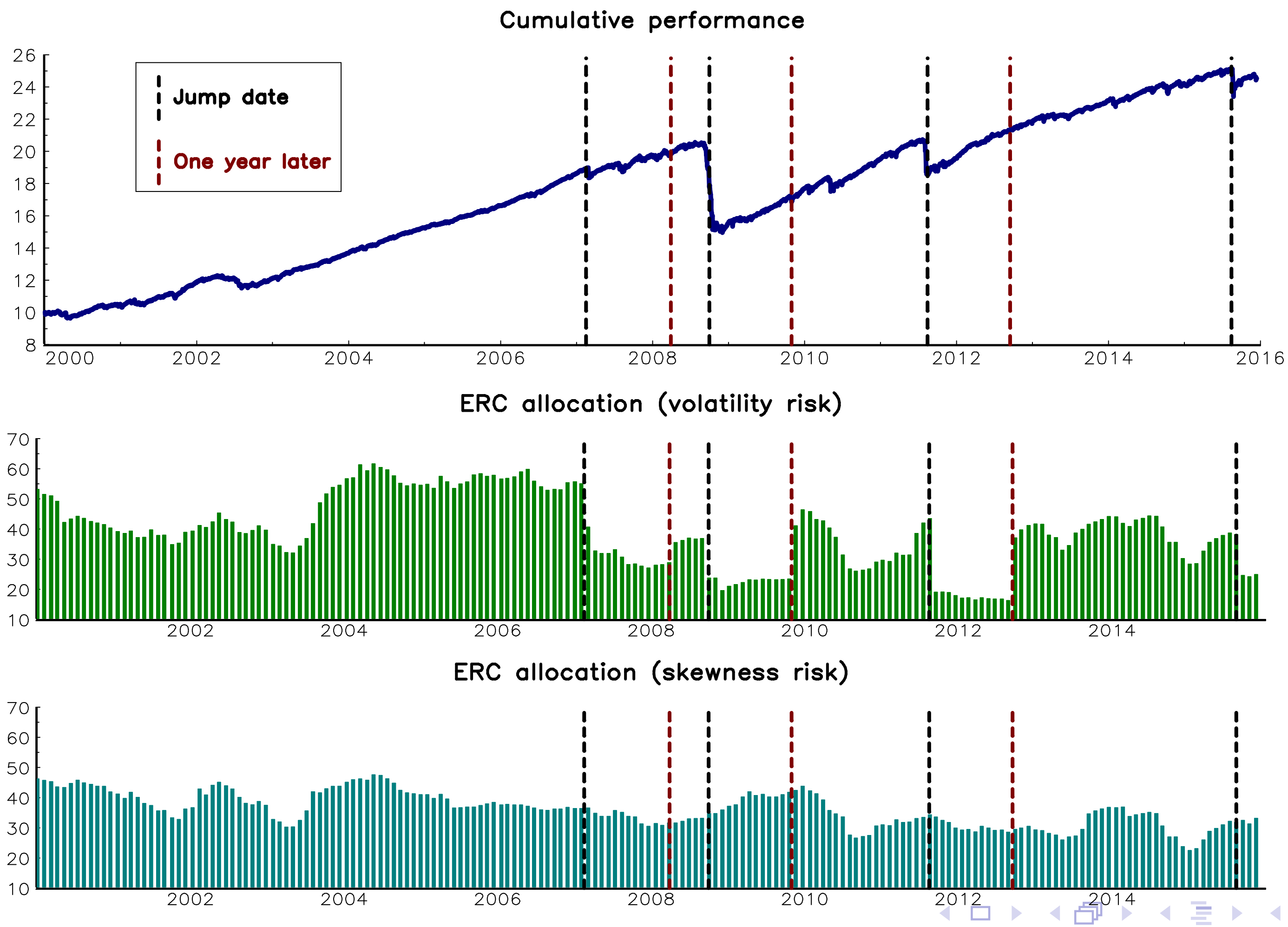
Analysis of the results

- The turnover of the skewness-based ERC allocation is 40% lower than the turnover of the volatility-based ERC allocation
- In terms of historical performance, volatility and Sharpe ratio, the two portfolios are equivalent, but:

$$\text{Before 2008} \neq \text{After 2008}$$

Analysis of the results

Figure: Comparison of the carry allocation (weekly model)



Factor investing

- Size: distressed risk due to the liquidity risk of small cap stocks
- Value: distressed risk due to the default risk of value stocks
- Low beta: no skewness risk
- Momentum: skewness risk
- Quality: limited skewness risk, but correlated with skewness risk of Momentum

$\Rightarrow$ Value (and Size) allocation is generally overestimated

Skewness risk premia

- The equity/bond/volatility asset mix policy = an emblematic illustration
- Other issues:
  - Portfolio of cross-asset carry risk premia
  - Introduction of less liquid strategies
- The case of equity cross-section momentum risk premium
  - Momentum crashes (Daniel and Moskowitz, 2016)
  - Volatility-based risk parity strategies are not adapted
  - Winners-minus-losers strategy $\neq$ winners-minus-market strategy

Volatility hedging versus skewness hedging

Table: Volatility and skewness risks of risk-based portfolios (weekly model)

Portfolio	MV	MV	ERC	MES
Model	Gaussian (full sample)	Jump model Normal	Jump model Mixture	Jump model Mixture
Bonds	63.26%	36.05%	52.71%	100.00%
Equities	2.23%	0.00%	10.36%	0.00%
Carry	34.51%	63.95%	36.93%	0.00%
$\sigma(x)$	2.62%	2.33%	2.75%	4.17%
γ_1	-2.75	-19.81	-6.17	0.00

The arithmetics of skewness

$$-(36.05\% \times 0.17 + 0\% \times 0.44 + 63.95\% \times 5.77) = -19.81 \quad ???$$

The case of CTA strategies

Belief / Misconceptions

- CTA is the right strategy for hedging the skewness risk
- Equity risk = skewness risk

CTA strategy and the equity market

- Equity risk = volatility risk, not skewness risk
- CTA is a good strategy for hedging the volatility risk of the equity market (e.g. 2008)
- It is not obvious that CTA is the right strategy for hedging skewness risks (e.g. 2011, January 2015, etc.)