Course 2022-2023 in Sustainable Finance
Lecture 8. Awareness of Climate Change Impacts

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March 2024

1The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.
Agenda

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- Lecture 2: ESG Scoring
- Lecture 3: Impact of ESG Investing on Asset Prices and Portfolio Returns
- Lecture 4: Sustainable Financial Products
- Lecture 5: Impact Investing
- Lecture 6: Engagement & Voting Policy
- Lecture 7: Extra-financial Accounting
- Lecture 8: Awareness of Climate Change Impacts
- Lecture 9: The Ecosystem of Climate Change
- Lecture 10: Economic Models & Climate Change
- Lecture 11: Climate Risk Measures
- Lecture 12: Transition Risk Modeling
- Lecture 13: Climate Portfolio Construction
- Lecture 14: Physical Risk Modeling
- Lecture 15: Climate Stress Testing & Risk Management
Figure 1: A glass house
The founding text of the greenhouse effect is a French scientific publication written by Joseph Fourier in 1824: *Remarques générales sur les températures du globe terrestre et des espaces planétaires*:

“The heat of the Earth’s surface derives from three sources, which must first be distinguished:

1. The Earth is heated by the sun’s rays, whose uneven distribution produces the diversity of climates.
2. The Earth’s temperature depends on the common temperature of planetary spaces, as it is exposed to the irradiation of the innumerable stars that surround the solar system on all sides.
3. The earth has retained within its mass some of the primitive heat it contained when the planets were formed.”

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2 General Remarks on Global and Planetary Temperatures.
In 1859, Tyndall showed that water vapor has a high heat absorption capacity. He went on to show that carbon dioxide and other gases could also absorb and radiate heat. Tyndall is the first scientist to prove that greenhouse gases exist and are responsible for the greenhouse effect.

Three years earlier, in 1856 and 1857, the American scientist Eunice Newton Foote had published two research papers with experiments showing that water vapor and carbon dioxide absorb heat from solar radiation.

Svante Arrhenius was the first scientist to calculate the effect of a change in atmospheric CO$_2$ on ground temperature (1896) ⇒ climate sensitivity.
Figure 2: Diagram showing how the greenhouse effect works

Figure 3: The pioneers of the greenhouse effect

Joseph Fourier (1766-1830)
Eunice Newton Foote (1819-1888)
John Tyndall (1820-1893)
Svante Arrhenius (1859-1927)
**Table 1: List of greenhouse gases**

<table>
<thead>
<tr>
<th>Greenhouse gas</th>
<th>Formula</th>
<th>Kyoto Protocol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water vapor</td>
<td>H$_2$O</td>
<td></td>
</tr>
<tr>
<td>Carbon dioxide</td>
<td>CO$_2$</td>
<td>✓</td>
</tr>
<tr>
<td>Methane</td>
<td>CH$_4$</td>
<td>✓</td>
</tr>
<tr>
<td>Nitrous oxide</td>
<td>N$_2$O</td>
<td>✓</td>
</tr>
<tr>
<td>Ozone</td>
<td>O$_3$</td>
<td></td>
</tr>
<tr>
<td><strong>Fluorinated or F-gases</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sulfur hexafluoride</td>
<td>SF$_6$</td>
<td>✓</td>
</tr>
<tr>
<td>Nitrogen trifluoride</td>
<td>NF$_3$</td>
<td>✓</td>
</tr>
<tr>
<td>Chlorofluorocarbons</td>
<td>CFCs (CFC-11, CFC-12, etc.)</td>
<td></td>
</tr>
<tr>
<td>Hydrofluorocarbons</td>
<td>HFCs (HFC-23, HFC-32, etc.)</td>
<td>✓</td>
</tr>
<tr>
<td>Hydrochlorofluorocarbons</td>
<td>HCFCs (HCFC-12, etc.)</td>
<td></td>
</tr>
<tr>
<td>Perfluorocarbons</td>
<td>PFCs (CF$_4$, C$_2$F$_6$, etc.)</td>
<td>✓</td>
</tr>
</tbody>
</table>
Callendar Effect

“By fuel combustion man has added about 150 000 million tons of carbon dioxide to the air during the past half century. The author estimates from the best available data that approximately three quarters of this has remained in the atmosphere. The radiation absorption coefficients of carbon dioxide and water vapour are used to show the effect of carbon dioxide on sky radiation. From this the increase in mean temperature, due to the artificial production of carbon dioxide, is estimated to be at the rate of 0.003°C per year at the present time. The temperature observations at zoo meteorological stations are used to show that world temperatures have actually increased at an average rate of 0.005°C per year during the past half century.” (Callendar, 1938, page 223).
Collected papers on global warming by David Archer and Raymond Pierrehumbert

- 1824
  On the Temperatures of the Terrestrial Sphere and Interplanetary Space (Fourier)
- 1861
  On the Absorption and Radiation of Heat by Gases and Vapours, and on the Physical Connexion of Radiation, Absorption, and Conduction (Tyndall)
- 1896
  On the Influence of Carbonic Acid in the Air upon the Temperature of the Ground (Arrhenius)
- 1938
  The Artificial Production of Carbon Dioxide and its Influence on Temperature (Callendar)
- 1956
  The Influence of the $15\mu$ Carbon-dioxide Band on the Atmospheric Infra-red Cooling Rate (Plass)
- 1957
  Carbon Dioxide Exchange Between Atmosphere and Ocean and the Question of an Increase of Atmospheric CO$_2$ during the Past Decades (Revelle and Suess)
- 1958
  Distribution of Matter in the Sea and Atmosphere: Changes in the Carbon Dioxide Content of the Atmosphere and Sea due to Fossil Fuel Combustion (Bolin and Eriksson)
- 1960
  The Concentration and Isotopic Abundances of Carbon Dioxide in the Atmosphere (Keeling)
Collected papers on global warming by David Archer and Raymond Pierrehumbert

- 1967
  Thermal Equilibrium of the Atmosphere with a Given Distribution of Relative Humidity (Manabe and Wetherald)

- 1969
  The Effect of Solar Radiation Variations on the Climate of the Earth (Budyko)
  A Global Climatic Model Based on the Energy Balance of the Earth-Atmosphere System (Sellers)

- 1970
  Is Carbon Dioxide from Fossil Fuel Changing Man’s Environment? (Keeling)

- 1972
  Man-Made Carbon Dioxide and the Greenhouse Effect (Sawyer)

- 1975
  The Effects of Doubling the CO₂ Concentration on the Climate of a General Circulation Model (Manabe and Wetherald)

- 1977
  Changes of Land Biota and Their Importance for the Carbon Cycle (Bolin)
  Neutralization of Fossil Fuel CO₂ by Marine Calcium Carbonate (Broecker and Takahashi)

- 1979
  Carbon Dioxide and Climate: A Scientific Assessment (Charney, Arakawa, Baker et al.)

- 1984
  Climate Sensitivity: Analysis of Feedback Mechanisms (Hansen, Lacis, Rind et al.)
Global warming

Figure 4: Keeling curve: monthly mean CO₂ concentration in Mauna Loa

Source: Keeling et al. (2001) and https://scrippsc02.ucsd.edu.
Charney Report (1979)

On July 23-27, 1979, Jule Charney formed a study group to “assess the scientific basis for projection of possible future climatic changes resulting from man-made releases of carbon dioxide into the atmosphere”. The study group had 13 members, including eminent scientists Akio Arakawa, Bert Bolin, Henry Stommel, etc. The report examined the results of five global climate models that simulate the climatic response to an increase in atmospheric CO2: three by Manabe and his colleagues at NOAA’s Geophysical Fluid Dynamics Laboratory and two by James Hansen and his colleagues at NASA’s Goddard Institute for Space Studies. The report estimated a climate sensitivity of $3^\circ C$, with an error of $\pm 1.5^\circ C$. 
Figure 5: The fathers of the concept of global warming

Guy Stewart Callendar (1898-1964)
Roger Revelle (1909-1991)
Charles David Keeling (1928-2005)
Wallace Broecker (1931-2019)
Testimony of James Hansen to the US Senate (1988)

“Mr. Chairman and committee members, thank you for the opportunity to present the results of my research on the greenhouse effect which has been carried out with my colleagues at the NASA Goddard Institute for Space Studies. I would like to draw three main conclusions. Number one, the earth is warmer in 1988 than at any time in the history of instrumental measurements. Number two, the global warming is now large enough that we can ascribe with a high degree of confidence a cause and effect relationship to the greenhouse effect. And number three, our computer climate simulations indicate that the greenhouse effect is already large enough to begin to affect the probability of extreme events such as summer heat waves.”
In 1988, the United Nations Environment Programme (UNEP) and the World Meteorological Organization (WMO) established the Intergovernmental Panel on Climate Change (IPCC) to provide policy makers with regular scientific assessments of climate change, its impacts and potential future risks, and to recommend options for adaptation and mitigation.
The Anthropocene is a proposed geological epoch that dates from the beginning of significant human impacts on Earth’s geology and ecosystems, including but not limited to human-induced climate change. The term Anthropocene was popularized by Paul Crutzen and Eugene Stoermer in 2000 (Crutzen and Stoermer, 2000).

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3 Paul Crutzen (1933-2021) was awarded the 1995 Nobel Prize in Chemistry for his work in atmospheric chemistry, and in particular for his efforts to study the formation and decomposition of atmospheric ozone.

4 Eugene Stoermer (1934-2012) was a professor of biology who specialized in the study of freshwater species and diatoms.
### Definition

**Anthropocene**

Geologic history of the climate

**Geographic factors of climate change**

- Anthropogenic factors
- Geologic history
- Radiation

**Scientific evidence of global warming**

- $72.1 \pm 0.2$ age (Ma)
- $\sim 139.8$
- $\sim 132.6$
- $\sim 113.0$
- $27.82$
- $23.03$
- $20.44$
- $13.82$
- $66.0$
- $61.6$
- $59.2$
- $56.0$
- $47.8$
- $41.2$
- $33.9$
- $0.0117$
- $0.774$
- $0.129$
- $3.600$
- $0.0082$
- $2.58$
- $1.80$
- $3.600$
- $0.0082$
- $2.58$
- $1.80$
- $3.600$
- $0.0082$
- $2.58$
- $1.80$

**The physics of climate change**

- Source: International Commission on Stratigraphy (2023), [www.stratigraphy.org/chart](http://www.stratigraphy.org/chart)

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**INTERNATIONAL CHRONOSTRATIGRAPHIC CHART**

**Source:** International Commission on Stratigraphy (2023), [www.stratigraphy.org](http://www.stratigraphy.org)
Table 2: Units of time

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>(in year)</th>
<th>Symbol</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kyr/kyr</td>
<td>Thousand/Kilo years</td>
<td>$10^3$</td>
<td>ka</td>
<td>Kiloannus</td>
</tr>
<tr>
<td>Myr/yr</td>
<td>Mega/Million years</td>
<td>$10^6$</td>
<td>Ma</td>
<td>Megaannus</td>
</tr>
<tr>
<td>Gyr/yr</td>
<td>Giga/Billion years</td>
<td>$10^9$</td>
<td>Ga</td>
<td>Gigaannus</td>
</tr>
</tbody>
</table>
To estimate temperatures during the Precambrian, scientists use indirect methods, including geochemical proxies (chemical properties of rocks and minerals), paleontological studies (type and distribution of fossils and sedimentary rocks), and general circulation models.

One of the most common methods is the clumped isotope thermometer.

The ratio of the two carbon isotopes $^{12}\text{C}$ and $^{13}\text{C}$ is:

$$\delta^{13}\text{C} = 1000 \times \left( \frac{^{13}\text{C}_{\text{sample}}}{^{12}\text{C}_{\text{sample}}} / \frac{^{13}\text{C}_{\text{standard}}}{^{12}\text{C}_{\text{standard}}} - 1 \right)$$

The unit of $\delta^{13}\text{C}$ is parts per thousand.

- Faint young Sun paradox
- Snowball Earth hypothesis
Figure 7: Carbon isotopic evolution of marine carbonate

Source: Shields and Veizer (2002, Figure 2, page 5) & https://earthref.org/ERDA/48.
Figure 8: Neoproterozoic carbon isotope data compilation

Source: Cox et al. (2016, Figure 2, page 90).
Geological history of the climate

Figure 9: Cosmic calendar

Temperature scales

- Three different scales are commonly used to measure temperature: Celsius, Kelvin, and Fahrenheit
- Their symbols are °C, K, and °F
- The relationships between the Celsius and Kelvin scales are $T_{\circ C} = T_{K} - 273.15$ and $T_{K} = T_{\circ C} + 273.15$.
- For Celsius and Fahrenheit, we have $T_{\circ C} = \frac{5}{9} (T_{\circ F} - 32)$ and $T_{\circ F} = \frac{9}{5} T_{\circ F} + 32$
- Absolute zero is $-273.15^\circ C$, $0 K$ and $-459.67^\circ F$, implying that $T \geq -273.15^\circ C$, $T \geq 0 K$, and $T \geq -459.67^\circ F$
- The melting point (at standard pressure) is obtained at temperatures of $0^\circ C$, $273.15 K$ and $32^\circ F$
- The boiling point of water corresponds to temperatures of $100^\circ C$, $373.15 K$ and $212^\circ F$
Figure 10: Earth temperature since 500 Myr BP (°C vs. 1960-1990 average)
Table 3: Recovered deep and very deep ice cores

<table>
<thead>
<tr>
<th></th>
<th>Greenland</th>
<th></th>
<th>Antarctica</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Site 2</td>
<td>1956</td>
<td>305 m</td>
<td>Byrd Station</td>
<td>1957-1958</td>
</tr>
<tr>
<td>Site 2</td>
<td>1957</td>
<td>411 m</td>
<td>Little America</td>
<td>1958-1959</td>
</tr>
<tr>
<td>Camp Century</td>
<td>1961-1966</td>
<td>1387 m</td>
<td>Byrd Station</td>
<td>1966-1968</td>
</tr>
<tr>
<td>Dye 3</td>
<td>1971</td>
<td>372 m</td>
<td>Vostock</td>
<td>1990-1998</td>
</tr>
<tr>
<td>Milcent</td>
<td>1973</td>
<td>398 m</td>
<td>Dome Fuji</td>
<td>1994-1997</td>
</tr>
<tr>
<td>Crete</td>
<td>1974</td>
<td>405 m</td>
<td>Vostock</td>
<td>2005-2007</td>
</tr>
<tr>
<td>Dye 3</td>
<td>1979-1981</td>
<td>2037 m</td>
<td>Dome Fuji</td>
<td>2003-2007</td>
</tr>
<tr>
<td>GRIP</td>
<td>1989-1992</td>
<td>3029 m</td>
<td>Dome C</td>
<td>1999-2005</td>
</tr>
<tr>
<td>GISP 2</td>
<td>1989-1993</td>
<td>3057 m</td>
<td>Kohnen Station</td>
<td>2001-2006</td>
</tr>
<tr>
<td>NGRIP</td>
<td>1996-2004</td>
<td>3090 m</td>
<td>WAIS</td>
<td>2006-2011</td>
</tr>
</tbody>
</table>
Palaeoclimate during the Phanerozoic

Figure 11: Greenland deep drilling sites
Palaeoclimate during the Phanerozoic

Figure 12: Antarctica deep drilling sites
Palaeoclimate during the Phanerozoic

Table 4: Isotopes of chemical elements

<table>
<thead>
<tr>
<th>Element</th>
<th>Stable isotopes</th>
<th>Unstable isotopes</th>
<th>Major isotope</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen</td>
<td>$^1$H and $^2$H</td>
<td>$^3$H − $^1$H</td>
<td>Protium (99.98%)</td>
</tr>
<tr>
<td>Carbon</td>
<td>$^{12}$C and $^{13}$C</td>
<td>$^{11}$C − $^{12}$C and $^{14}$C − $^{22}$C</td>
<td>Carbon-12 (98.90%)</td>
</tr>
<tr>
<td>Oxygen</td>
<td>$^{16}$O and $^{18}$O</td>
<td>$^{17}$O − $^{18}$O − $^{27}$O</td>
<td>Oxygen-16 (99.76%)</td>
</tr>
<tr>
<td>Uranium</td>
<td></td>
<td>$^{232}$U − $^{242}$U</td>
<td>Uranium-238 (99.27%)</td>
</tr>
</tbody>
</table>
Palaeoclimate during the Phanerozoic

- Analysis of trapped air bubbles in ice cores provides a direct record of the composition of the atmosphere at the time the ice formed.
- Dansgaard (1964) defined the relative deviation $\delta$ of the heavy isotope content as follows:

$$\delta = 1000 \times \left( \frac{R_{\text{sample}} - R_{\text{standard}}}{R_{\text{standard}}} \right)$$

where $R$ is the absolute content.
- $\delta$ is measured in $\%$.
- As the chemical formula of water is $\text{H}_2\text{O}$, climate reconstruction from ice cores is based on the analysis of hydrogen and oxygen.
  - In the case of hydrogen, the common isotope is $^1\text{H}$, while the heavy isotope is $^2\text{H}$ (also called the deuterium or D).
  - The ratio $R$ is then $\frac{^2\text{H}}{^1\text{H}}$ and the relative variation is written as $\delta^D$.
  - In the case of oxygen, the common isotope is $^{16}\text{O}$, while the heavy isotope is $^{18}\text{O}$.
  - The ratio $R$ is then $\frac{^{18}\text{O}}{^{16}\text{O}}$ and the relative variation is written as $\delta^{18}\text{O}$. 
Figure 13: Part of an ice core at WAIS Divide Field Camp

Source: Eli Duke, Antarctica: WAIS Divide Field Camp (Flickr),
www.flickr.com/photos/80547277@N00/9518403333.
The aim of ice core analysis is to estimate the temperature function $t \mapsto T(t)$ with respect to the time age $t$.

The raw analysis provides two measurements: the depth $d$ of the ice core drilling and the isotope ratio measure $\delta$.

We therefore observe the isotope function $d \mapsto \delta(d)$.

To obtain the temperature function, we proceed in two steps:

1. First, we transform the depth $d$ of the ice core drilling into the time age $t$:
   \[ t = \varphi_t(d) \]

2. We then estimate the temperature $T$ associated with the isotope ratio $\delta(d)$:
   \[ T = \varphi_T(\delta(d)) \]

Combining the two previous equations gives the desired parametric function $t \mapsto T(t)$.
Palaeoclimate during the Phanerozoic

Figure 14: Isotopic reconstruction of Vostok ice cores

Source: Petit et al. (1999) & Author’s calculations.
Figure 15: Temperature reconstruction of Vostok ice cores

Source: Petit et al. (1999).
Palaeoclimate during the Phanerozoic

**Figure 16:** Gas concentration of Vostok ice cores

Figure 17: Evolution of the atmospheric CO$_2$ during the last 420 million years

Source: Foster et al. (2017, Figure 1, page 3).
Temperature anomaly

We define the temperature anomaly at time $t$ as the difference between the temperature at time $t$ and the temperature for a reference period:

$$\Delta \mathcal{T}(t) = \mathcal{T}(t) - \mathcal{T}_{\text{Base}}$$

where $\mathcal{T}_{\text{Base}}$ is the reference temperature. It is generally the average of the temperature of the reference period:

$$\mathcal{T}_{\text{Base}} = \frac{\sum_{j \in \text{Base}} \mathcal{T}(j)}{n_{\text{Base}}}$$

For example, the reference temperature can be the average temperature of the 20th century (from 1901 to 2000) or the pre-industrial period.
**Figure 18:** Global average land-ocean temperature anomaly relative to 1961-1990 average

Source: Morice et al. (2021).
Figure 19: Average land-ocean temperature anomaly in the northern and southern hemispheres relative to the 1961-1990 average

Source: Morice et al. (2021).
**Temperature anomaly**

**Table 5:** Linear projection of land-ocean temperature anomaly (in °C)

<table>
<thead>
<tr>
<th>Year</th>
<th>Global</th>
<th>HadCRUT5</th>
<th>NOAAGlobalTemp v5.1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>HadCRUT5</td>
<td>Northern</td>
<td>Southern</td>
</tr>
<tr>
<td>2050</td>
<td>1.4336</td>
<td>1.9595</td>
<td>0.9078</td>
</tr>
<tr>
<td>2075</td>
<td>1.9288</td>
<td>2.6540</td>
<td>1.2035</td>
</tr>
<tr>
<td>2100</td>
<td>2.4239</td>
<td>3.3486</td>
<td>1.4992</td>
</tr>
<tr>
<td>Slope</td>
<td>0.0198</td>
<td>0.0278</td>
<td>0.0118</td>
</tr>
</tbody>
</table>

**Table 6:** Linear projection of land and ocean temperature anomalies (in °C)

<table>
<thead>
<tr>
<th>Year</th>
<th>Global</th>
<th>Northern</th>
<th>Southern</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Land</td>
<td>Ocean</td>
<td>Land</td>
</tr>
<tr>
<td>2050</td>
<td>2.4212</td>
<td>1.0238</td>
<td>2.8388</td>
</tr>
<tr>
<td>2075</td>
<td>3.2386</td>
<td>1.3243</td>
<td>3.8176</td>
</tr>
<tr>
<td>2100</td>
<td>4.0560</td>
<td>1.6247</td>
<td>4.7964</td>
</tr>
<tr>
<td>Slope</td>
<td>0.0327</td>
<td>0.0120</td>
<td>0.0392</td>
</tr>
</tbody>
</table>

Source: NOAAGlobalTemp v5.1 & Author’s calculations.
Temperature anomaly

Figure 20: Average temperature anomaly (land-ocean, land and ocean)

Source: Vose et al. (2021).
Figure 21: Projection of temperature anomaly by 2100 (in °C)

According to Friedlingstein et al. (2022), the global carbon budget has five main components:

1. Fossil fuel combustion and oxidation from all energy and industrial processes, including cement production and carbonation ($CE_{Industry}$)
2. Emissions from land-use change ($CE_{Land}$)
3. The growth rate of atmospheric CO$_2$ concentration ($CE_{AT}$)
4. The uptake of CO$_2$ by the oceans (ocean sink) ($CS_{Ocean}$)
5. The uptake of CO$_2$ by the land (land sink) ($CS_{Land}$)

From a theoretical point of view, we have the following identity:

$$CE_{AT} = (CE_{Industry} + CE_{Land}) - (CS_{Ocean} + CS_{Land})$$

Positive emissions - Negative emissions

The estimated budget imbalance is equal to:

$$CB_{Imbalance} = CE_{Industry} + CE_{Land} - (CE_{AT} + CS_{Ocean} + CS_{Land})$$
In 2021, the authors estimate the following figures expressed in gigatonnes of carbon: \( \mathcal{CE}^{*}_{\text{Industry}} = 10.13, \mathcal{CE}_{\text{Land}} = 1.08, \mathcal{CE}_{\text{AT}} = 5.23, \mathcal{CS}_{\text{Ocean}} = 2.88, \mathcal{CS}_{\text{Land}} = 3.45, \mathcal{CS}_{\text{Cement}} = 0.23, \) and \( \mathcal{CB}_{\text{Imbalance}} = -0.58 \)

Expressed in gigatonnes of \( \text{CO}_2 \) we obtain: \( \mathcal{CE}^{*}_{\text{Industry}} = 37.12, \mathcal{CE}_{\text{Land}} = 3.94, \mathcal{CE}_{\text{AT}} = 19.14, \mathcal{CS}_{\text{Ocean}} = 10.55, \mathcal{CS}_{\text{Land}} = 12.64, \mathcal{CS}_{\text{Cement}} = 0.84, \) and \( \mathcal{CB}_{\text{Imbalance}} = -2.12 \)

Anthropogenic \( \text{CO}_2 \) emissions are therefore 36.28 GtCO\(_2\) for industrial processes \( (\mathcal{CE}_{\text{Industry}}) \) and 3.94 GtCO\(_2\) for land-use change.

Since the total is 40.22 GtCO\(_2\), 26.23% and 31.43% of the total anthropogenic \( \text{CO}_2 \) emissions have been absorbed by oceans and land, respectively, while 47.60% remain in the atmosphere.
Figure 22: Annual CO₂ emissions (in GtCO₂)

Source: Friedlingstein et al. (2022) & Author’s calculations.
Anthropogenic GHG emissions

Figure 23: Cumulative CO₂ emissions and carbon sinks (in GtCO₂)

Source: Friedlingstein et al. (2022) & Author’s calculations.
Figure 24: Cumulative CO$_2$ budget imbalance in atmosphere (in GtCO$_2$)

Source: Friedlingstein et al. (2022) & Author’s calculations.
Anthropogenic GHG emissions

The airborne fraction is the ratio of the atmospheric CO$_2$ growth rate to total anthropogenic emissions. For the period 1750-2021, this ratio is equal to:

$$\text{AF} = \frac{\text{CE}_{\text{AT}}}{\text{CE}_{\text{Industry}} + \text{CE}_{\text{Land}}} = \frac{1076}{1717 + 742} = 43.8\%$$

This means that 43.8% of anthropogenic emissions have not been absorbed by natural carbon sinks.

“The observed stability of the airborne fraction over the 1960-2020 period indicates that the ocean and land CO$_2$ sinks have on average been removing about 55% of the anthropogenic emissions.” (Friedlingstein et al., 2022, page 4834).
Table 7: Breakdown of anthropogenic CO₂ emission by energy source (in %)

<table>
<thead>
<tr>
<th>Energy</th>
<th>1850</th>
<th>1900</th>
<th>1950</th>
<th>1975</th>
<th>2000</th>
<th>2020</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coal</td>
<td>100.00</td>
<td>95.85</td>
<td>65.10</td>
<td>34.74</td>
<td>36.18</td>
<td>40.11</td>
</tr>
<tr>
<td>Oil</td>
<td>0.00</td>
<td>3.55</td>
<td>26.79</td>
<td>47.64</td>
<td>40.42</td>
<td>33.23</td>
</tr>
<tr>
<td>Gas</td>
<td>0.00</td>
<td>0.60</td>
<td>6.09</td>
<td>13.17</td>
<td>18.61</td>
<td>20.38</td>
</tr>
<tr>
<td>Cement</td>
<td>0.00</td>
<td>0.00</td>
<td>1.13</td>
<td>1.99</td>
<td>2.87</td>
<td>4.33</td>
</tr>
<tr>
<td>Flaring</td>
<td>0.00</td>
<td>0.00</td>
<td>0.81</td>
<td>2.18</td>
<td>1.05</td>
<td>1.12</td>
</tr>
<tr>
<td>Other</td>
<td>0.00</td>
<td>0.00</td>
<td>0.08</td>
<td>0.29</td>
<td>0.86</td>
<td>0.83</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Energy</th>
<th>1850</th>
<th>1900</th>
<th>1950</th>
<th>1975</th>
<th>2000</th>
<th>2020</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coal</td>
<td>100.00</td>
<td>97.58</td>
<td>85.24</td>
<td>63.49</td>
<td>50.00</td>
<td>46.29</td>
</tr>
<tr>
<td>Oil</td>
<td>0.00</td>
<td>2.05</td>
<td>12.13</td>
<td>27.43</td>
<td>35.10</td>
<td>35.02</td>
</tr>
<tr>
<td>Gas</td>
<td>0.00</td>
<td>0.37</td>
<td>2.15</td>
<td>6.88</td>
<td>11.72</td>
<td>14.52</td>
</tr>
<tr>
<td>Cement</td>
<td>0.00</td>
<td>0.00</td>
<td>0.40</td>
<td>1.17</td>
<td>1.77</td>
<td>2.56</td>
</tr>
<tr>
<td>Flaring</td>
<td>0.00</td>
<td>0.00</td>
<td>0.03</td>
<td>0.87</td>
<td>1.03</td>
<td>1.06</td>
</tr>
<tr>
<td>Other</td>
<td>0.00</td>
<td>0.00</td>
<td>0.06</td>
<td>0.16</td>
<td>0.39</td>
<td>0.55</td>
</tr>
</tbody>
</table>

Source: Friedlingstein et al. (2022) & Author’s calculations.
Figure 25: Energy source breakdown of anthropogenic cumulative CO₂ (in %)

Source: Friedlingstein et al. (2022) & Author’s calculations.
Figure 26: Share of CO$_2$ emissions by region (in % of total)

Source: Friedlingstein et al. (2022).
Figure 27: Share of cumulative CO$_2$ emissions by region (in % of total)

Source: Friedlingstein et al. (2022).
Figure 28: Share of CO\textsubscript{2} emissions by country (in % of total)

Source: Friedlingstein et al. (2022).
Figure 29: Share of cumulative CO$_2$ emissions by country (in % of total)

Source: Friedlingstein et al. (2022).
The Kaya identity is defined as:

Anthropogenic CO₂ emissions = Population × \( \frac{GDP}{Population} \) × \( \frac{Energy}{GDP} \) × \( \frac{CO₂ \, emissions}{Energy} \)

Using the notations of Kaya and Yokobori (1997), this identity is generally expressed as:

\[ F = P \times \frac{G}{P} \times \frac{E}{G} \times \frac{F}{E} \]

Therefore, the key drivers of anthropogenic CO₂ emissions include four main factors:

- the population (demographics)
- the GDP per capita (economics)
- the energy intensity of the GDP (engineering)
- the carbon intensity (physics)
Figure 30: Key drivers of the Kaya identity

Source: Friedlingstein et al. (2022).
Anthropogenic GHG emissions

**Figure 31:** CO$_2$ emissions per capita (in tCO$_2$ per person)

Source: Friedlingstein *et al.* (2022).
Anthropogenic GHG emissions

Figure 32: GHG emissions (in GtCO$_2$e)

Source: Jones et al. (2023) & Author’s calculations.
### Table 8: 2021 greenhouse gas emissions (in GtCO$_2$e)

<table>
<thead>
<tr>
<th>CE$_{Industry}$</th>
<th>CH$<em>4$ CE$</em>{Land}$</th>
<th>CE$_{Total}$</th>
<th>CO$<em>2$ CE$</em>{Land}$</th>
<th>CE$_{Total}$</th>
<th>N$<em>2$O CE$</em>{Land}$</th>
<th>CE$_{Total}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.23</td>
<td>3.95</td>
<td>10.18</td>
<td>37.11</td>
<td>4.00</td>
<td>41.12</td>
<td>0.79</td>
</tr>
<tr>
<td>(61.2%)</td>
<td>(38.8%)</td>
<td>(18.8%)</td>
<td>(90.3%)</td>
<td>(9.7%)</td>
<td>(75.8%)</td>
<td>(26.7%)</td>
</tr>
</tbody>
</table>

Source: Jones et al. (2023) & Author’s calculations.
Figure 33: Planetary boundaries

Source: Richardson et al. (2023, Figure 1, page 4).
### Table 9: Current status of planetary boundaries

<table>
<thead>
<tr>
<th>No</th>
<th>Earth process system</th>
<th>Control variable</th>
<th>$m_{1750}$</th>
<th>$m_{boundary}$</th>
<th>$m_{upper}$</th>
<th>$m_{2023}$</th>
<th>Crossed</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Climate change</td>
<td>Atmospheric CO$_2$ concentration (ppm)</td>
<td>280</td>
<td>350</td>
<td>450</td>
<td>417</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Atmospheric radiative forcing (W/m$^2$)</td>
<td></td>
<td>0</td>
<td>1.0</td>
<td>1.5</td>
<td>2.91</td>
</tr>
<tr>
<td>2</td>
<td>Biodiversity loss</td>
<td>Genetic diversity (E/MSY)</td>
<td>1</td>
<td>10</td>
<td>100</td>
<td>&gt; 100</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Functional integrity (% HANPP)</td>
<td>1.9</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>✓</td>
</tr>
<tr>
<td>3</td>
<td>Stratospheric ozone depletion</td>
<td>Stratospheric O$_3$ concentration (DU)</td>
<td>290</td>
<td>276</td>
<td>261</td>
<td>284.6</td>
<td>✓</td>
</tr>
<tr>
<td>4</td>
<td>Ocean acidification</td>
<td>Carbonate ion concentration ($\Omega_{arg}$)</td>
<td>3.44</td>
<td>2.752</td>
<td>2.75</td>
<td>2.8</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Phosphate flow (TgP/yr) – global</td>
<td>0</td>
<td>11</td>
<td>100</td>
<td>22.6</td>
<td>✓</td>
</tr>
<tr>
<td>5</td>
<td>Biogeochemical flows</td>
<td>Phosphate flow (TgP/yr) – regional</td>
<td>0</td>
<td>6.2</td>
<td>11.2</td>
<td>17.5</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Nitrogen flow (TgN/yr)</td>
<td>0</td>
<td>62</td>
<td>82</td>
<td>190</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Area of forested land (%) – global</td>
<td>100</td>
<td>75</td>
<td>54</td>
<td>60</td>
<td>✓</td>
</tr>
<tr>
<td>6</td>
<td>Land-use change</td>
<td>% area remaining – tropical</td>
<td>100</td>
<td>85</td>
<td>60</td>
<td>58.6</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td></td>
<td>% area remaining – temperate</td>
<td>100</td>
<td>50</td>
<td>30</td>
<td>41.1</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td></td>
<td>% area remaining – boreal</td>
<td>100</td>
<td>85</td>
<td>60</td>
<td>63.5</td>
<td>✓</td>
</tr>
<tr>
<td>7</td>
<td>Freshwater change</td>
<td>Blue water (%)</td>
<td>9.4</td>
<td>10.2</td>
<td>50</td>
<td>18.2</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Green water (%)</td>
<td>9.8</td>
<td>11.1</td>
<td>50</td>
<td>15.8</td>
<td>✓</td>
</tr>
<tr>
<td>8</td>
<td>Atmospheric aerosol loading</td>
<td>Inter-hemispheric difference (AOD)</td>
<td>0.03</td>
<td>0.10</td>
<td>0.25</td>
<td>0.076</td>
<td>✓</td>
</tr>
<tr>
<td>9</td>
<td>Novel entities</td>
<td>Synthetic chemicals (%)</td>
<td>0</td>
<td>0</td>
<td>NA</td>
<td>&gt; 0</td>
<td>✓</td>
</tr>
</tbody>
</table>

Source: Richardson et al. (2023, Table 1, pages 4-5).
“Temperature [...] is a measure of the energy contained in the movement of molecules. Therefore, to understand how the temperature is maintained, one must consider the energy balance that is formally stated in the first law of thermodynamics. The basic global energy balance of Earth is between energy coming from the Sun and energy returned to space by Earth’s radiative emission. The generation of energy in the interior of Earth has a negligible influence on its energy budget.” (Hartmann, 2016, page 25).

- Energy cannot be created or destroyed (law of conservation of energy)
- The total amount of energy in a closed system is conserved
- Energy and forcing are two interchangeable terms, meaning that $F_{\text{solar}} := \mathcal{E}_{\text{solar}}$, $F_{\text{thermal}} := \mathcal{E}_{\text{infrared}}$, etc.
Total solar irradiance

- The total amount of electromagnetic energy emitted by the Sun, also called the solar luminosity is $L_{\odot} = 3.828 \times 10^{26}$ watts
- Total solar irradiance (TSI) is defined as:
  \[
  S_d = \frac{L_{\odot}}{4\pi d^2}
  \]
  where $d$ is the distance of the sphere from the Sun in meters
- For the Earth, the distance is between 147.1 and 152.1 million kilometers
- Using a mean value of 149.6 million kilometers, we get:
  \[
  S_0 = \frac{3.828 \times 10^{26}}{4\pi (149.6 \times 10^9)^2} = 1372.11 \text{ W/m}^2
  \]
- A direct measurement by astrophysicists gives 1368 W/m$^2$
Planck radiation law and spectral density of electromagnetic radiation

In physics, Planck’s law describes the spectral distribution of electromagnetic radiation emitted by a black body in thermal equilibrium at a given temperature. The expression for the spectral density function is:

\[ B_\nu (\nu, T) = \frac{2\nu^3}{c^2} \frac{1}{\exp \left( \frac{h\nu}{kT} \right) - 1} \]

where \( h = 6.62607015 \times 10^{-34} \text{ J Hz}^{-1} \) is Planck’s constant, \( c = 299 792 458 \text{ m s}^{-1} \) is the speed of light in a vacuum, \( k = 1.380649 \times 10^{-23} \text{ J K}^{-1} \) is Boltzmann’s constant, \( T \) is the temperature measured in Kelvin, and \( \nu \) is the frequency in hertz.
Planck law can be written in terms of the wavelength $\lambda$:

$$\lambda = \frac{c}{\nu} \left( \sim \frac{\text{m s}^{-1}}{\text{Hz}} = \text{m} \right)$$

and we have:

- **Ultraviolet** wavelength: less than 380 nm
- Visible wavelength: 380 nm to 780 nm
- **Infrared** wavelength: greater than 780 nm
Planck law

Figure 34: Spectral density function $B_\lambda (\lambda, T)$ (in $10^{12}$ W/m² m⁻¹)
**Planck law**

**Figure 35**: Comparison of the radiation spectra of sunlight and the Earth’s surface (in $10^{12} \text{W/m}^2 \text{m}^{-1}$)

- **Sunlight**: $\mathcal{T} = 5525 \text{K}$
- **Earth surface**: $\mathcal{T} = 288 \text{K}$

![Radiation Spectra Comparison](image)
Stefan-Boltzmann law

The Stefan-Boltzmann law describes the relationship between the total amount of radiation $\mathcal{E}$ emitted by a body and its temperature $T$:

$$\mathcal{E} = \varepsilon \sigma T^4$$

where:

- $\varepsilon \in [0, 1]$ is the emissivity of the body
- $\sigma = 5.67 \times 10^{-8}$ W/m$^2$K$^{-4}$ is the Stefan-Boltzmann constant
- For an ideal black body, we have $\varepsilon = 1$
Effective temperature of stars

- The radius of the Sun $R_\odot$ is about 696,342 kilometers.
- The solar irradiance at the photosphere is equal to:

$$S_\odot = \frac{L_\odot}{4\pi R_\odot^2} = \frac{3.828 \times 10^{26}}{4\pi (696,342 \times 10^3)^2} = 62,822,741 \text{ W/m}^2$$

- If we assume that the sun is a perfect black body ($\varepsilon \approx 0.96$), we have:

$$\sigma T_\odot^4 = S_\odot \iff T_\odot = \sqrt[4]{\frac{S_\odot}{\sigma}}$$

- We have:

$$T_\odot = \sqrt[4]{\frac{62,822,741}{5.67 \times 10^{-8}}} = 5,769 \text{ K}$$
Effective temperature of stars

Remark

The previous analysis can be extended to other stars. Let $R_{\text{star}}$ be the stellar radius of the star. Since we have $L_{\text{star}} = 4\pi R_{\text{star}}^2 S_{\text{star}}$ and $S_{\text{star}} = \mathcal{E} = \sigma T^4$, we get:

$$T_{\text{star}} = \sqrt[4]{\frac{L_{\text{star}}}{4\pi R_{\text{star}}^2 \sigma}}$$

$T_{\text{star}}$ is defined as the temperature of a black body radiating the same amount of energy per unit area as the star. It may differ from the actual temperature of a star, which depends on its kinetic energy.
Incoming solar radiation

- The incoming solar radiation is equal to:

\[ F_{\text{solar}} = \frac{1}{4} (1 - \alpha_p) S_0 \]

where \( \alpha_p \) is the planetary albedo, which measures the amount of reflected sunlight.
- \( \alpha_p \) is equal to zero for a perfect black body.
- \( \alpha_p \) one for a perfect white body.

**Remark**

The ratio \( \frac{1}{4} \) comes from the fact that no point on the planet receives the sun’s energy continuously during a full day. On average, we can show that a point on the planet receives \( \frac{1}{4} \) of the solar energy, which is the ratio of the projected area of the sphere \( (\text{Area} = \pi r^2) \) divided by the surface area of the sphere \( (\text{Area} = 4\pi r^2) \).
Figure 36: Incoming solar radiation
In the case of the Earth, we have $\alpha_p \approx 0.29$ and:

$$F_{\text{solar}} = \frac{1}{4} (1 - 0.29) \times 1368 = 242.82 \text{ W/m}^2$$
Interpretation

Consider a room with a surface area of \( x \) square meters and receiving an energy \( \mathcal{E} \) expressed in watts. The radiation per square meter received by this room is equal to \( \mathcal{E}/x \). If the room receives the same equivalent solar radiation \( F_{\text{Solar}} \), the energy must be equal to:

\[
\mathcal{E} = x \cdot F_{\text{Solar}}
\]

Using a standard 200 watt lamp, we can calculate the number of lamps required to achieve the same equivalent solar radiation. The results are shown below:

<table>
<thead>
<tr>
<th>( x ) (in ( m^2 ))</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{E} ) (in watts)</td>
<td>242.8</td>
<td>1214</td>
<td>2428</td>
<td>4856</td>
<td>12141</td>
<td>24282</td>
</tr>
<tr>
<td># lights</td>
<td>1.2</td>
<td>9</td>
<td>12</td>
<td>24</td>
<td>61</td>
<td>121</td>
</tr>
</tbody>
</table>

For a room of 20 \( m^2 \), we need 24 lights.
Figure 37: Zero-order model

Incoming solar radiation

\[ \frac{1}{4} S_0 \]

Albedo

\[ \frac{1}{4} \alpha_p S_0 \]

Earth

\[ \frac{1}{4} (1 - \alpha_p) S_0 \]

Infrared radiation

\[ \sigma T_e^4 \]

Thierry Roncalli

Course 2022-2023 in Sustainable Finance
The Earth receives the incoming solar radiation $F_{\text{solar}}$, while the black body radiation is given by the Stefan-Boltzmann law.

We deduce that:

$$\sigma T_e^4 = \frac{1}{4} \left(1 - \alpha_p\right) S_0 \Leftrightarrow T_e = \sqrt[4]{\frac{(1 - \alpha_p) S_0}{4\sigma}}$$

The numerical calculation gives:

$$T_e = \sqrt[4]{\frac{(1 - 0.29) \times 1368}{4 \times 5.67 \times 10^{-8}}} = 255.81 \text{ K} = 255.81^{\circ}\text{C} - 273.15^{\circ}\text{C} = -17.34^{\circ}\text{C}$$

The effective temperature of the Earth is then close to $-17.34^{\circ}\text{C}$.
Without greenhouse gases, the surface temperature of the Earth should be equal to the effective temperature. However, we observe:

\[ T_s \approx +15^\circ C \gg T_e \approx -17^\circ C \]
“Since solar radiation is mostly visible and near infrared, and Earth emits primarily thermal infrared radiation, the atmosphere may affect solar and terrestrial radiation very differently. [...] Since the atmospheric layer absorbs all of the energy emitted by the surface below it and emits like a blackbody, the only radiation emitted to space is from the atmosphere in this model.” (Hartman, 2016, pages 32-32).
Figure 38: Zero-order model with greenhouse effect

\[
\frac{1}{4} (1 - \alpha_p) S_0
\]
Impact of the greenhouse effect

- The energy balance for the Earth’s surface is:

\[ F_{\text{solar}} + \sigma T_a^4 = \sigma T_s^4 \]

while the radiation balance for the atmosphere verifies:

\[ \sigma T_s^4 = 2\sigma T_a^4 \]

- It follows that:

\[ F_{\text{solar}} + \sigma T_a^4 = 2\sigma T_a^4 \]

or (we have \( \sigma T_e^4 = \frac{1}{4} (1 - \alpha_p) S_0 := F_{\text{solar}} \)):

\[ F_{\text{solar}} = \sigma T_a^4 = \sigma T_e^4 \]

- We conclude that:

\[ \begin{cases} T_a = T_e \\ T_s = \sqrt{2} T_e \end{cases} \]
Impact of the greenhouse effect

- Using the previous numerical values, we obtain:

\[
\begin{align*}
T_a &= 255.81 \text{ K} = -17.34 \degree \text{C} \\
T_s &= 304.22 \text{ K} = 31.07 \degree \text{C}
\end{align*}
\]

- We find that the surface temperature is warmer than the observed global mean surface temperature.

- This is because the assumption that the atmosphere absorbs all the heat radiated from the surface is not true.
Another way to illustrate the greenhouse effect is to estimate the reflection parameter $\gamma_p$, which measures the net thermal radiation of the atmosphere with respect to the black body energy.

We deduce that the balance $E_{\text{net}}$ is:

$$E_{\text{net}} = F_{\text{solar}} - \sigma T_s^4 - \gamma_p \sigma T_s^4$$

Solving the equation $E_{\text{net}} = 0$ gives:

$$\gamma_p = 1 - \frac{F_{\text{solar}}}{\sigma T_s^4} = 1 - \frac{(1 - \alpha_p) S_0}{4 \sigma T_s^4}$$

Using a surface temperature of 15°C, we get $\gamma_p = 0.3788$.

$\Rightarrow$ Only 62% of the infrared radiation goes into space and 38% stays on the surface.
"A layer of atmosphere that is almost opaque for longwave radiation can be crudely approximated as a blackbody that absorbs all terrestrial radiation that is incident upon it and emits like a blackbody at its temperature. For an atmosphere with a large infrared optical depth, the radiative transfer process can be represented with a series of blackbodies arranged in vertical layers. Two layers centered at 0.5 km and 2.0 km altitudes provide a simple approximation for Earth’s atmosphere.” (Hartman, 2016, page 71).
Two-layer model of the atmosphere

Figure 39: Two-layer model

\[ \frac{1}{4} (1 - \alpha_p) S_0 \]

\[ \sigma T_1^4 \]

\[ \sigma T_2^4 \]

\[ \sigma T_3^4 \]

Layer 1

Layer 2

Surface

Earth
Two-layer model of the atmosphere

- We have:
  \[
  \begin{align*}
  F_{\text{solar}} + \sigma T_1^4 &= \sigma T_s^4 \\
  \sigma T_2^4 + \sigma T_4^4 &= 2\sigma T_1^4 \\
  \sigma T_1^4 &= 2\sigma T_2^4
  \end{align*}
  \]

- By replacing \( F_{\text{solar}} \) by \( \sigma T_e^4 \) and dividing the equations by \( \sigma \), we get:
  \[
  \begin{align*}
  T_s^4 &= 3 T_e^4 \\
  T_1^4 &= 2 T_e^4 \\
  T_2^4 &= 1 T_e^4
  \end{align*}
  \]

- The solution is then equal to:
  \[
  \begin{align*}
  T_s &= \sqrt[4]{3} T_e = 336.67 \text{ K} = 63.52^\circ \text{C} \\
  T_1 &= \sqrt[4]{2} T_e = 304.22 \text{ K} = 31.07^\circ \text{C} \\
  T_2 &= \sqrt[4]{1} T_e = 255.81 \text{ K} = -17.34^\circ \text{C}
  \end{align*}
  \]
Multi-layer model of the atmosphere

- Let $n$ be the total number of layers and $T_k$ be the temperature at layer $k$
- We have:
  \[
  \begin{cases}
  T_s = T_0 = \sqrt{n+1} T_e \\
  T_k = \sqrt{n+1} - k T_e \\
  \end{cases}
  \quad \text{for } k = 0, 1, \ldots, n
  \]

- The temperature decreases with the layer index:
  \[
  \frac{\partial T_k}{\partial k} = -\frac{1}{4} (n + 1 - k)^{-3/4} T_e \leq 0
  \]

**Table 10: Layers of the Earth’s atmosphere**

<table>
<thead>
<tr>
<th>Index</th>
<th>Layer</th>
<th>Altitude (in Km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Troposphere</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>Stratosphere</td>
<td>50</td>
</tr>
<tr>
<td>3</td>
<td>Mesosphere</td>
<td>80</td>
</tr>
<tr>
<td>4</td>
<td>Thermosphere</td>
<td>500</td>
</tr>
<tr>
<td>5</td>
<td>Exosphere</td>
<td>6200</td>
</tr>
</tbody>
</table>
Emissivity model of the atmosphere

Figure 40: One-layer model with atmospheric emissivity

\[ \frac{1}{4} (1 - \alpha_p) S_0 \]

\[ (1 - \varepsilon) \sigma T_s^4 \]

\[ \varepsilon \sigma T_a^4 \]

\[ \sigma T_s^4 \]

\[ \varepsilon \sigma T_a^4 \]
Emissivity model of the atmosphere

- The balance at the top of the atmosphere is:
  \[ F_{\text{solar}} - (1 - \varepsilon) \sigma T_s^4 - \varepsilon \sigma T_a^4 = 0 \]

- The balance of the atmosphere is:
  \[ \varepsilon \sigma T_s^4 - 2 \varepsilon \sigma T_a^4 = 0 \]

- The balance at the surface is:
  \[ F_{\text{solar}} + \varepsilon \sigma T_a^4 - \sigma T_s^4 = 0 \]

- The first equation is equivalent to:
  \[ (1 - \varepsilon) \sigma T_s^4 + \varepsilon \sigma T_a^4 = F_{\text{solar}} = \sigma T_e^4 \]

- Using the second equation, it follows that:
  \[ \sigma T_e^4 = (1 - \varepsilon) \sigma T_s^4 + \frac{1}{2} \varepsilon \sigma T_s^4 = \left(1 - \frac{1}{2} \varepsilon \right) T_s^4 \]
Emissivity model of the atmosphere

We conclude that:

\[ T_s = \sqrt[4]{\frac{2}{2 - \varepsilon}} T_e \]

and:

\[ T_a = \sqrt[4]{\frac{1}{2 - \varepsilon}} T_e \]

Therefore, for a given temperature \( T_s^* \), we can find the unique value of the emissivity:

\[ \varepsilon^* = 2 - 2 \left( \frac{T_e}{T_s^*} \right)^4 \]

Since \( T_s^* \approx 15^\circ C \), the emissivity of the atmosphere is 78%. This model then predicts an atmospheric temperature of \(-30.8^\circ C\).
Emissivity model of the atmosphere

Figure 41: Relationship between atmospheric emissivity and temperature

\[ T_s \] and \[ T_a \]
More generally, the Earth’s energy balance is the sum of net shortwave radiation and net longwave radiation:

\[
\mathcal{E}_{\text{net}} = \left( \mathcal{E}_{\text{down}}^{\text{short}} - \mathcal{E}_{\text{up}}^{\text{short}} \right) + \left( \mathcal{E}_{\text{down}}^{\text{long}} - \mathcal{E}_{\text{up}}^{\text{long}} \right)
\]

where \( \mathcal{E}_{\text{down}}^{\text{short/long}} \) is shortwave/longwave downward radiation and \( \mathcal{E}_{\text{up}}^{\text{short/long}} \) is shortwave/longwave upward radiation.

In the previous model, we have \( \mathcal{E}_{\text{short}}^{\text{net}} = \frac{1}{4} (1 - \alpha_p) S_0 \) and

\[
\mathcal{E}_{\text{long}}^{\text{net}} = \varepsilon \sigma T_a^4 - \sigma T_s^4
\]
Emissivity model of the atmosphere

Figure 42: Earth's Energy Budget
Specific heat capacity

Definition

The specific heat capacity $c$ of a substance is the heat capacity $C$ of the substance divided by the mass of the substance:

$$c = \frac{C}{M} = \frac{1}{M} \frac{\Delta \mathcal{E}}{\Delta T}$$

(1)

where:

- $\Delta \mathcal{E}$ is the amount of heat required to raise the temperature of the substance by $\Delta T$
- $M$ is the mass of the substance in kilograms (kg)
- $\Delta \mathcal{E}$ is the change in energy in joules (J)
- $\Delta T$ is the change in temperature in Kelvin (K)
- $c$ is the specific heat capacity in joules per kilogram per Kelvin ($J \text{ kg}^{-1} \text{ K}^{-1}$)
The specific heat capacity of water is 4186 J kg\(^{-1}\) K\(^{-1}\).

From Equation (1), we deduce that:

\[ \Delta E = Mc \Delta T \]  

(2)

In this case, the amount of energy required to raise the temperature of 1 m\(^3\) of water by 10°C is equal to:

\[ \Delta E = 10^3 \times 4186 \times 10 = 4186000 \text{ J} \]
Specific heat capacity

- The specific heat capacity of air is about $1000 \text{ J kg}^{-1} \text{ K}^{-1}$
- The mass of the atmosphere is $5.148 \times 10^{18} \text{ kg}$
- The amount of energy required to raise the temperature of the atmosphere by $1^\circ \text{C}$ is:

$$\Delta \varepsilon = (5.148 \times 10^{18} \text{ kg}) \times (1000 \text{ J kg}^{-1} \text{ K}^{-1}) \times 1 \text{ K}$$

$$= 5.148 \times 10^{21} \text{ J}$$
We can write Equation (1) as follows:

\[ \Delta T = \frac{\Delta E}{Mc} \]  

(3)

This equation gives the change in temperature for a change in energy. For example, adding 1000 joules of energy to one liter of water will increase its temperature by about 0.239°C:

\[ \Delta T = \frac{1000}{1 \times 4186} = 0.239 \]
Specific heat capacity

- Equations (1)–(3) can be modified by scaling the mass $M$ of the substance to standardize the required energy $\Delta \mathcal{E}$.
- A possible scaling factor can be the surface area:

$$m = \frac{M}{\text{Area}}$$

and we get:

$$\Delta \mathcal{E} = mc \Delta T$$  \hspace{1cm} (4)

- For the atmosphere, we have:

$$m = \frac{M}{\text{Area}} = \frac{5.148 \times 10^{18} \text{ kg}}{510.0645 \times 10^6 \times 10^6 \text{ m}^2} = 1.0093 \times 10^4 \text{ kg m}^{-2}$$

because the radius $r$ of the Earth is 6371 km and the surface of the Earth is approximately $\text{Area} = 4\pi r^2 = 510.0645 \text{ million km}^2$.
Specific heat capacity

- In some climate modeling textbooks, Equation (4) is expressed using other formulas for the mass of the atmosphere per unit area.
- For example, $m$ can be replaced by the product of height $h$ and density $\rho$, or by the ratio of pressure $p$ to gravitational acceleration $g$. However, all these quantities are equivalent because we have:

$$m = h\rho = (8.2 \times 10^3 \text{ m}) \times (1.225 \text{ kg m}^{-3}) = 1.0045 \times 10^3 \text{ kg m}^{-2}$$

and:

$$m = \frac{p}{g} = \frac{101325 \text{ Pa}}{9.81 \text{ m s}^{-2}} = \frac{101325 \text{ m}^{-1} \text{ kg s}^{-2}}{9.81 \text{ m s}^{-2}} = 1.0329 \times 10^4 \text{ kg m}^{-2}$$

where $h = 8.2 \text{ km}$ is the height of the atmosphere, $\rho = 1.225 \text{ kg m}^{-3}$ is the density of the atmosphere, $p = 101325 \text{ Pa}$ is the standard atmospheric pressure at sea level on Earth, and $g = 9.81 \text{ m s}^{-2}$ is the acceleration due to gravity at the Earth’s surface.

- Therefore, we obtain the following equivalent formulas:

$$mc \Delta T = h\rho c \Delta T = \frac{p}{g} c \Delta T = \Delta E$$
Radiative relaxation timescale

- We transform Equation (4) into a differential equation:

\[ mc \frac{dT}{dt} = \frac{dE}{dt} \]

- For a black body, we have:

\[ F_{solar} - \sigma T^4 = 0 \]

- We deduce that:

\[ E = F_{solar} - \sigma T^4 = \sigma T_e^4 - \sigma T^4 \]
Let us assume that $T = T_e + \Delta T$. It follows that:

$$mc \frac{d\Delta T}{dt} = -4\sigma T_e^3 \Delta T$$

because:

$$\frac{\partial}{\partial \Delta T} (\sigma T_e^4 - \sigma T^4) = -4\sigma T_e^3$$

Let $\tau_e$ be the radiative relaxation timescale defined as:

$$\tau_e = \frac{mc}{4\sigma T_e^3}$$

We have:

$$\begin{cases} \frac{d\Delta T}{dt} = -\frac{1}{\tau_e} \Delta T \\ \Delta T (0) = \Delta T_0 \end{cases}$$
Radiative relaxation timescale

- The solution of this ordinary differential equation is well-known and we get:

\[
\Delta T(t) = \exp \left( -\frac{t}{\tau_e} \right) \Delta T_0
\]

- \( \Delta T(t) \) gives the impulse response of an initial temperature shock of \( \Delta T_0 \)

- Because \( \tau_e > 0 \), we conclude that the system is stable:

\[
\lim_{t \to \infty} \Delta T(t) = 0
\]

- We note that the equation for \( \Delta T(t) \) describes an exponential survival function with parameter \( \tau_e^{-1} \)

- We deduce that the radiative relaxation timescale \( \tau_e \) is the mean lifetime
Numerical value of $\tau_e$

Using the previously obtained values for the atmosphere ($m = 1.0093 \times 10^4 \text{ kg m}^{-2}$, $c = 1000 \text{ J kg}^{-1} \text{ K}^{-1}$ and $T_e = 255.81 \text{ K}$), the radiative relaxation timescale is equal to 31 days:

$$
\tau_e = \frac{(1.0093 \times 10^4 \text{ kg m}^{-2}) \times (1000 \text{ J kg}^{-1} \text{ K}^{-1})}{4 \times (5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}) \times (255.81 \text{ K})^3}
= \frac{2658427 \text{ J W}^{-1}}{2658427 \text{ s}}
= \frac{24 \times 3600 \text{ s}}{24 \times 3600 \text{ s}}
= 30.77 \text{ days}
$$
Radiative relaxation timescale

**Figure 43:** Impulse response function for $\Delta T_0 = +1^\circ C$ and a black body

\[ \Delta T_0 = +1^\circ C \]
Radiative relaxation timescale

- For a gray body, we found that:
  \[ \sigma T_e^4 = (2 - \varepsilon) \sigma T_\alpha^4 = \left( \frac{2 - \varepsilon}{2} \right) \sigma T_s^4 \]

- We deduce that:
  \[
  mc \frac{dT_a}{dt} = \frac{d}{dt} \left( \sigma T_e^4 - (2 - \varepsilon) \sigma T_\alpha^4 \right)
  \]

and:

\[
mc \frac{dT_s}{dt} = \frac{d}{dt} \left( \sigma T_e^4 - \left( \frac{2 - \varepsilon}{2} \right) \sigma T_s^4 \right)
\]

- We obtain:
  \[
  \Delta T_a(t) = \exp \left( -\frac{t}{\tau_a} \right) \Delta T_0 \quad \text{and} \quad \Delta T_s(t) = \exp \left( -\frac{t}{\tau_s} \right) \Delta T_0
  \]

where:

\[
\tau_a = \frac{mc}{4(2 - \varepsilon) \sigma T_e^3} \quad \text{and} \quad \tau_s = \frac{mc}{2(2 - \varepsilon) \sigma T_e^3}
\]
Radiative relaxation timescale

Figure 44: Impulse response function for $\Delta T_0 = +1^\circ C$ and a gray body\(^5\)

\[ \Delta T (\text{in } ^\circ C) = a = 2.279 \text{ days} = e = 30.77 \text{ days} = s = 45.58 \text{ days} \]

\(^5\)We consider an emissivity value of 65%
“[…] Climate forcing is a change to the climate system that can be expected to change the climate. Examples would be doubling the CO₂, increasing the total solar irradiance (TSI) by 2%, introducing volcanic aerosols into the stratosphere, etc. Climate forcings are usually quantified in terms of how many W/m² they change the energy balance when imposed. For example, instantaneously doubling the CO₂ changes the energy balance at the top of the atmosphere by about 4 W/m². A feedback process is a response of the climate system to surface warming that then alters the energy balance in such a way as to change the temperature response to the forcing. A positive feedback makes the forced response bigger, and a negative feedback makes it smaller. Classic examples of positive feedbacks are ice-albedo feedback and water-vapor feedback. When it warms, ice melts, and this reduces Earth’s albedo and causes further warming. When it cools, ice grows, and this increases Earth’s albedo, causing further cooling […]” Hartmann (2016, page 294).
We assume that some extra energy $\Delta F$ is added to the system. $\Delta F$ is called the radiative forcing and is measured in W/m$^2$. The climate response is generally measured as the change in the surface temperature $\Delta T_s$.

**Definition of the climate sensitivity**

The climate sensitivity is defined as:

$$\phi := \frac{\Delta T_s}{\Delta F}$$

Using differential notation, we have:

$$\phi := \frac{dT_s}{dF}$$

implying that:

$$dT_s = \phi dF$$
Feedback mechanism

- We assume that the perturbation $dF$ depends on the temperature $T_s$ and some exogenous factors $x_i$:

$$dF = \frac{\partial F}{\partial T_s}dT_s + \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}dx_i$$

- We conclude that:

$$\left(1 - \phi \frac{\partial F}{\partial T_s}\right)dT_s = \phi \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}dx_i$$

- We then get a feedback mechanism, because the temperature dynamics depend on the factors $x_i$, but also on the temperature response.
Climate feedback

- For the one-layer model with emissivity, we have $\mathcal{E} = 0$ where:

$$
\mathcal{E} = F_{\text{solar}} - \left(\frac{2 - \varepsilon}{2}\right) \sigma T_s^4
$$

$$
= \frac{1}{4} (1 - \alpha_p) S_0 - \left(\frac{2 - \varepsilon}{2}\right) \sigma T_s^4
$$

- The first-order Taylor Series expansion of $\mathcal{E} = 0$ gives:

$$
\Delta \mathcal{E} = \frac{1}{4} (1 - \alpha_p) \Delta S_0 - \frac{1}{4} S_0 \Delta \alpha_p + \frac{1}{2} \sigma T_s^4 \Delta \varepsilon - 4 \left(\frac{2 - \varepsilon}{2}\right) \sigma T_s^3 \Delta T_s
$$

- Remember that each perturbation $\Delta y$ depends on the temperature $T_s$ and some exogenous factors $x_i$:

$$
\Delta y = \frac{\partial y}{\partial T_s} \Delta T_s + \sum_{i=1}^{n} \frac{\partial y}{\partial x_i} \Delta x_i
$$
Climate feedback

- We have:

\[
\Delta \mathcal{E} = \frac{1}{4} (1 - \alpha_p) \Delta S_0 - \frac{1}{4} S_0 \left( \frac{\partial \alpha_p}{\partial T_s} \Delta T_s + \sum_{i=1}^{n} \frac{\partial \alpha_p}{\partial x_i} \Delta x_i \right) + \frac{1}{2} \sigma T_s^4 \left( \frac{\partial \varepsilon}{\partial T_s} \Delta T_s + \sum_{i=1}^{n} \frac{\partial \varepsilon}{\partial x_i} \Delta x_i \right) - 4 \left( \frac{2 - \varepsilon}{2} \right) \sigma T_s^3 \Delta T_s
\]

- We deduce that:

\[
\Delta \mathcal{E} = \lambda \Delta T_s + \sum_{i=0}^{n} \Delta F_i
\]  
(5)

where \( \Delta F_0 := \Delta F_{\text{solar}} = \frac{1}{4} (1 - \alpha_p) \Delta S_0 \),

\[
\Delta F_i = \left( - \frac{1}{4} S_0 \frac{\partial \alpha_p}{\partial x_i} + \frac{1}{2} \sigma T_s^4 \frac{\partial \varepsilon}{\partial x_i} \right) \Delta x_i \quad \text{for } i \geq 1
\]

\[
\lambda = - \frac{1}{4} S_0 \frac{\partial \alpha_p}{\partial T_s} + \frac{1}{2} \sigma T_s^4 \frac{\partial \varepsilon}{\partial T_s} - 4 \left( \frac{2 - \varepsilon}{2} \right) \sigma T_s^3
\]
Since $\Delta E$ and $\Delta F_i$ are measured in $W/m^2$ and $\Delta T_s$ is measured in Kelvin, we deduce that $\lambda$ is measured in $W m^{-2} K^{-1}$.

We transform $\Delta E$ into $\Delta T_s$ by considering the heat capacity $c$ expressed in $W m^{-2} K^{-1}$ s:

$$
c \frac{d\Delta T_s}{dt} = \lambda \Delta T_s + \Delta F$$

where $\Delta F = \sum_{i=0}^{n} F_i$.

The climate feedback parameter $\lambda$ can be positive or negative, and we have the following mathematical properties:

- If $\lambda > 0$, the system is unstable;
- If $\lambda < 0$, the system is stable and the equilibrium is reached when:

$$
\Delta T_s = \Delta T_s^* = -\frac{\Delta F}{\lambda} = -\phi \Delta F
$$
We see that the climate feedback parameter can be decomposed into three components:

\[ \lambda = \lambda_0 + \lambda_{\alpha p} + \lambda_\varepsilon \]

where:

1. \( \lambda_0 \) is the Planck feedback or the black body response:

\[ \lambda_0 = -4 \left( \frac{2 - \varepsilon}{2} \right) \sigma T_s^3 \]

2. \( \lambda_{\alpha p} \) is the surface albedo feedback:

\[ \lambda_{\alpha p} = -\frac{1}{4} S_0 \frac{\partial \alpha_p}{\partial T_s} \]

3. \( \lambda_\varepsilon \) is the emissivity feedback:

\[ \lambda_\varepsilon = \frac{1}{2} \sigma T_s^4 \frac{\partial \varepsilon}{\partial T_s} \]
Planck feedback

Since $\varepsilon < 1$, the Planck feedback is negative, meaning that it stabilizes the climate and counteracts global warming. In fact, as the Earth warms, it emits more thermal radiation into space, and this increased longwave radiation acts as a natural cooling mechanism. Conversely, as the Earth cools, it emits less thermal radiation into space, and this decreased longwave radiation acts as a natural warming mechanism.

Earlier we found that $\varepsilon = 0.78$. So the estimate of $\lambda_0$ is:

$$
\lambda_0 = -4 \left( \frac{2 - 0.78}{2} \right) \times (5.67 \times 10^{-8}) \times (273.15 + 15)^3
$$

$$
= -3.310 \text{ W m}^{-2} \text{ K}^{-1}
$$

---

$^6$For comparison, the most recent estimate is $-3.22 \text{ W m}^{-2} \text{ K}^{-1}$, while the 90% confidence interval is from $-3.4$ to $-3.0 \text{ W m}^{-2} \text{ K}^{-1}$ (IPCC, 2021, Chapter 7, page 968)
The sign of the surface albedo feedback depends on the sign of \( \frac{\partial \alpha_p}{\partial T_s} \).

Two main factors play a role in determining \( \alpha_p \):

"The albedo of the planet for solar radiation is primarily determined by the clouds and surface, with the main variable component of the latter being the ice/snow cover." (Hansen et al., 1984, page 166).

- If the Earth were completely covered in ice, its albedo \( \alpha_p \) would be about 84%, meaning it would reflect most of the sunlight that hits it.
- If the Earth were covered by a dark green forest canopy, the albedo would be about 14%.
- Cloud albedo feedback occurs because changes in cloud cover, cloud altitude or cloud properties affect the amount of reflected shortwave radiation. This feedback can be either positive or negative.
Albedo feedback

- According to IPCC AR6, the value of the global surface albedo feedback is estimated to be $0.35 \text{ W m}^{-2} \text{ K}^{-1}$, with a 90% confidence interval from $0.10$ to $0.60 \text{ W m}^{-2} \text{ K}^{-1}$.
- This means that:

$$\frac{\partial \alpha_p}{\partial T_s} = - \frac{4 \lambda \alpha_p}{S_0} = -\frac{4 \times 0.35}{1368} = -1.023 \times 10^{-3}$$
Ice-albedo feedback modeling (Sellers model)

- If $\alpha_p = f_{\text{albedo}}(T_s)$, then $\lambda_{\alpha_p} = -\frac{1}{4} S_0 f'_{\text{albedo}}(T_s)$
- Sellers (1969) suggested:

$$\alpha_p = \begin{cases} 
    b(\phi) - 0.009T_s & \text{if } T_s \leq 283.16 \text{ K} \\
    b(\phi) - 2.548 & \text{if } T_s \geq 283.16 \text{ K}
\end{cases}$$

where $b(\phi)$ is an estimated coefficient that depends on the latitude $\phi$.

- On average we have $\overline{b(\phi)} = 2.8811$
- We deduce that:

$$\lambda_{\alpha_p} = \frac{1}{4} \times 1\,368 \times 0.009 = 3.078 \text{ W m}^{-2} \text{ K}^{-1}$$

- It is obvious that this positive feedback has been overestimated. The reason is that snow and sea ice cover about 10% of the Earth’s surface. Therefore, we get $\lambda_{\alpha_p} \approx 0.3 \text{ W m}^{-2} \text{ K}^{-1}$
Ice-albedo feedback modeling (Sellers model)
Ice-albedo feedback modeling (Budyko model)

- Budyko (1969) assumed that:

\[
\alpha_p = \begin{cases} 
\alpha_{\text{cold}} & \text{if } T_s \leq T_{\text{cold}} \\
\alpha_{\text{warm}} + (\alpha_{\text{cold}} - \alpha_{\text{warm}}) \left( \frac{T_{\text{warm}} - T_s}{T_{\text{warm}} - T_{\text{cold}}} \right)^\eta & \text{if } T_{\text{cold}} \leq T_s \leq T_{\text{warm}} \\
\alpha_{\text{warm}} & \text{if } T_s \geq T_{\text{warm}}
\end{cases}
\]

where \( \eta \geq 1 \)

- It follows that:

\[
\lambda_{\alpha_p} (T_s) = \frac{1}{4} \eta S_0 (\alpha_{\text{cold}} - \alpha_{\text{warm}}) \left( \frac{T_{\text{warm}} - T_s}{T_{\text{warm}} - T_{\text{cold}}} \right)^{\eta-1} \eta \cdot 1 \{ T_{\text{cold}} \leq T_s \leq T_{\text{warm}} \}
\]

- Using the values \( \alpha_{\text{cold}} = 0.7, T_{\text{cold}} = 260 \, \text{K}, \alpha_{\text{warm}} = 0.3, \)
\( T_{\text{warm}} = 295 \, \text{K} \) and \( \eta = 2 \), we get
\( \lambda_{\alpha_p} (282 \, \text{K}) = 2.90 \, \text{W m}^{-2} \text{K}^{-1} \),
\( \lambda_{\alpha_p} (288 \, \text{K}) = 1.56 \, \text{W m}^{-2} \text{K}^{-1} \) and
\( \lambda_{\alpha_p} (293 \, \text{K}) = 0.45 \, \text{W m}^{-2} \text{K}^{-1} \)
Emissivity feedback

- The emissivity feedback is the sum of several components:

\[ \lambda_\varepsilon = \lambda_{\text{water vapor}} + \lambda_{\text{lapse rate}} + \lambda_{\text{cloud longwave}} + \cdots \]
The water vapor feedback, also known as the specific humidity feedback, is the most important positive and destabilizing feedback. It can be described as follows:

“[...] As the temperature increases, the amount of water vapor in saturated air increases. Since water vapor is the principal greenhouse gas, increasing water vapor content will increase the greenhouse effect of the atmosphere and raise the surface temperature even further.” (Hartmann, 2016, page 297).

According to IPCC AR6, the value $\lambda_{\text{water vapor}}$ of the water vapor feedback is assessed to be $1.85 \text{ W m}^{-2} \text{ K}^{-1}$. 
The lapse rate mechanism describes the relationship between temperature and altitude in the atmosphere:

$$\Gamma = -\frac{\partial T}{\partial z} \in [4 \text{ K km}^{-1}, 10 \text{ K km}^{-1}]$$

where $z$ is the altitude in kilometers

- On average, the lapse rate is about $6.5^\circ\text{C}$ per kilometer
- According to IPCC AR6, the average value of $\lambda_{\text{lapse rate}}$ is $-0.50 \text{ W m}^{-2} \text{ K}^{-1}$
The case of cloud feedbacks is more complicated because it involves several mechanisms:

1. high cloud altitude
2. tropical high cloud amount
3. subtropical marine low cloud
4. land cloud
5. midlatitude cloud amount
6. extratropical cloud optical depth
7. Arctic cloud

According to IPCC AR6, the value $\lambda_{\text{cloud}}$ of the net cloud feedback is estimated to be $0.42 \text{ W m}^{-2} \text{K}^{-1}$

One of the difficulties is to decompose the cloud feedback into shortwave and longwave feedbacks, since the global surface albedo feedback already includes shortwave cloud mechanisms. Assuming that $2/3$ of the cloud feedback is longwave radiation, we get:

$$\lambda_e \approx 1.85 - 0.50 + \frac{2}{3} \times 0.42 = 1.63 \text{ W m}^{-2} \text{K}^{-1}$$
Total feedback (deterministic analysis)

- We have:

\[
\lambda = \lambda_0 + \lambda_{\alpha p} + \lambda_{\varepsilon} \\
= -3.31 + 0.35 + 1.63 \\
= -1.33 \text{ W m}^{-2} \text{ K}^{-1}
\]

- This value is obtained with a simple one-layer model with emissivity
Total feedback (stochastic analysis)

- We assume that \( \tilde{\lambda} \sim N(\mu_\lambda, \sigma_\lambda^2) \)
- As before, we decompose the feedback as a sum of individual feedbacks:
  \[
  \tilde{\lambda} = \sum_{i=1}^{n} \tilde{\lambda}_i
  \]
  where \( \tilde{\lambda}_i \sim N(\mu_i, \sigma_i^2) \)
- Assuming that the individual feedbacks are independent, we have:
  \[
  \begin{align*}
  \mu_\lambda &= \sum_{i=1}^{n} \mu_i \\
  \sigma_\lambda &= \sqrt{\sum_{i=1}^{n} \sigma_i^2}
  \end{align*}
  \]
Total feedback (stochastic analysis)

**Table 11**: Parameters $\mu_i$ and $\sigma_i$ of feedback parameters

<table>
<thead>
<tr>
<th>Feedback mechanism</th>
<th>$\mu_i$</th>
<th>$\sigma_i$</th>
<th>Shortwave</th>
</tr>
</thead>
<tbody>
<tr>
<td>High-cloud altitude</td>
<td>+0.20</td>
<td>0.10</td>
<td></td>
</tr>
<tr>
<td>Tropical marine low cloud</td>
<td>+0.25</td>
<td>0.16</td>
<td></td>
</tr>
<tr>
<td>Tropical anvil cloud area</td>
<td>−0.20</td>
<td>0.20</td>
<td></td>
</tr>
<tr>
<td>Land cloud amount</td>
<td>+0.08</td>
<td>0.08</td>
<td></td>
</tr>
<tr>
<td>Middle-latitude marine low-cloud amount</td>
<td>+0.12</td>
<td>0.12</td>
<td></td>
</tr>
<tr>
<td>High-latitude low-cloud optical depth</td>
<td>+0.00</td>
<td>0.10</td>
<td>✓</td>
</tr>
<tr>
<td>Planck feedback</td>
<td>−3.20</td>
<td>0.10</td>
<td>✓</td>
</tr>
<tr>
<td>Water vapor + lapse rate</td>
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<td>0.15</td>
<td></td>
</tr>
<tr>
<td>Surface albedo</td>
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<td>0.15</td>
<td>✓</td>
</tr>
<tr>
<td>Total cloud</td>
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<td></td>
</tr>
<tr>
<td>Stratospheric</td>
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<td>0.10</td>
<td></td>
</tr>
<tr>
<td>Atmospheric composition changes</td>
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<td>0.15</td>
<td></td>
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<tr>
<td>Climate feedback parameter</td>
<td>−1.30</td>
<td>0.44</td>
<td></td>
</tr>
</tbody>
</table>

Source: Sherwood et al. (2020, Table 1, page 18).
Figure 45: Probability density function of individual cloud feedbacks
Using the parameters of the individual cloud feedbacks, the distribution of the total cloud feedback is Gaussian with:

\[
\mu_{\text{total cloud}} = 0.20 + 0.25 - 0.20 + 0.08 + 0.12 + 0.00
= 0.45 \text{ W m}^{-2} \text{ K}^{-1}
\]

and:

\[
\sigma_{\text{total cloud}} = \sqrt{0.10^2 + 0.16^2 + 0.20^2 + 0.08^2 + 0.12^2 + 0.10^2}
= 0.3262 \text{ W m}^{-2} \text{ K}^{-1}
\]
Total feedback (positive feedback)

Figure 46: Probability density function of positive feedbacks
The aggregate positive feedback is the sum of the five main positive feedback components:

\[ \mu_{\text{positive}} = 1.15 + 0.30 + 0.45 + 0.00 + 0.00 = 1.90 \text{ W m}^{-2} \text{ K}^{-1} \]

and:

\[ \sigma_{\text{positive}} = \sqrt{0.15^2 + 0.15^2 + 0.33^2 + 0.10^2 + 0.15^2} = 0.4317 \text{ W m}^{-2} \text{ K}^{-1} \]
Total feedback

Figure 47: Comparison of $\tilde{\lambda}_{\text{Planck}}$ and $\tilde{\lambda}_{\text{positive}}$
Total feedback

- If we aggregate the positive feedback with the Planck feedback, the climate feedback parameter is \( \tilde{\lambda} \sim \mathcal{N}(\mu_\lambda, \sigma_\lambda^2) \) where:

  \[
  \mu_\lambda = -1.30 \text{ W m}^{-2} \text{ K}^{-1}
  \]
  \[
  \sigma_\lambda = 0.44 \text{ W m}^{-2} \text{ K}^{-1}
  \]

- The probability that the climate feedback is positive is small:

  \[
  \Pr \left\{ \tilde{\lambda} \geq 0 \right\} = \Pr \left\{ \frac{\tilde{\lambda} - \mu_\lambda}{\sigma_\lambda} \geq -\frac{\mu_\lambda}{\sigma_\lambda} \right\} = 1 - \Phi \left( -\frac{1.30}{0.44} \right) = 0.16\% 
  \]
Total feedback

**Figure 48:** Probability density function of the climate feedback parameter
Equilibrium climate sensitivity (ECS)

Definition of ECS

We recall that the definition of equilibrium is:

$$\Delta T^* = -\frac{\Delta F}{\lambda} = -\phi \Delta F$$

Earth’s equilibrium climate sensitivity (ECS) is the long-term global mean surface temperature change due to a specific value of radiative forcing corresponding to a doubling of CO$_2$ in the atmosphere:

$$\text{ECS} := \Delta T_{2\times CO_2} = -\frac{\Delta F_{2\times CO_2}}{\lambda}$$

where ECS (or $\Delta T_{2\times CO_2}$) is the equilibrium climate sensitivity, $\lambda$ is the climate sensitivity parameter and $\Delta F_{2\times CO_2}$ is the radiative forcing resulting from a doubling of the atmospheric carbon dioxide concentration.
We assume $\Delta F = 4 \text{ W/m}^2$ and $\lambda = -1.33 \text{ W m}^{-2} \text{ K}^{-1}$

The equilibrium warming is then equal to $3^\circ C$:

$$\Delta T_s^* = -\frac{4 \text{ W/m}^2}{-1.33 \text{ W m}^{-2} \text{ K}^{-1}} = 3.0075 \text{ K}$$

Remember that the dynamics of the temperature change is given by the ordinary differential equation:

$$c \frac{d\Delta T_s}{dt} = \lambda \Delta T_s + \Delta F$$

We assume that the solution has the following form:

$$\Delta T_s (t) = e^{At} B + C$$

We deduce that:

$$\frac{d\Delta T_s}{dt} = Ae^{At} B = A (\Delta T_s (t) - C)$$
Stochastic equilibrium temperature modeling

- The identification of the parameters results in:

\[
\begin{aligned}
A &= \frac{\lambda}{c} \\
-AC &= \frac{\Delta F}{c} \\
B + C &= \Delta T_s (0)
\end{aligned}
\]

- The solutions are then:

\[
A = c^{-1} \lambda, \\
C = -\frac{\Delta F}{A c} = -\frac{\Delta F}{\lambda} = \Delta T_s^*
\]

and:

\[
B = \Delta T_s (0) - C = \Delta T_s (0) - \Delta T_s^*
\]

- Finally, we conclude that:

\[
\Delta T_s (t) = \exp \left(-\frac{t}{\tau}\right) (\Delta T_s (0) - \Delta T_s^*) + \Delta T_s^*
\]

where:

\[
\tau = -\frac{c}{\lambda}
\]
Stochastic equilibrium temperature modeling

- Again, we obtain that the equation for $\Delta T_s (t)$ describes an exponential survival function with parameter $\tau^{-1}$
- Using a specific heat capacity of $c = 4 \times 10^8 \text{ J m}^{-2} \text{ K}^{-1}$, we get:

$$\tau = -\frac{c}{\lambda} = -\frac{4 \times 10^8 \text{ J m}^{-2} \text{ K}^{-1}}{-1.33 \text{ W m}^{-2} \text{ K}^{-1}} = \frac{4 \times 10^8 \text{ W m}^{-2} \text{ K}^{-1} \Delta T_s(0)}{1.33 \text{ W m}^{-2} \text{ K}^{-1}} = 3.0075 \times 10^8 \text{ s}$$

- The relaxation time $\tau$ of the climate system is then equal to 3480.92 days or 9.53 years
- Due to the exponential distribution, $\tau$ is also the mean lifetime
- Since we have:

$$\Delta T_s (t) - \Delta T_s^* = \exp \left( -\frac{t}{\tau} \right) \left( \Delta T_s(0) - \Delta T_s^* \right)$$

the half-life $t_{1/2}$ is defined as the solution of the equation

$$\exp \left( -\frac{t_{1/2}}{\tau} \right) = \frac{1}{2} \quad \text{or} \quad t_{1/2} = \tau \ln (2) = 6.61 \text{ years}$$

- The equilibrium is $\lim_{t \to \infty} \Delta T_s (t) = \Delta T_s^* = 3.0075 \degree \text{C}$.
Stochastic equilibrium temperature modeling

**Figure 49:** Surface temperature dynamics after a radiative forcing of $4 \text{ W/m}^2$ ($\lambda = -1.33 \text{ W m}^{-2} \text{ K}^{-1}$)
The previous analysis assumes that the climate feedback parameter is certain. In fact, it is stochastic, which means that the equilibrium temperature and the temperature dynamics are stochastic.
Stochastic equilibrium temperature modeling

- If $\lambda \sim \mathcal{N} (\mu_\lambda, \sigma_\lambda^2)$, the equilibrium temperature is equal to:

$$\Delta \tilde{T}_s^* = -\frac{\Delta F}{\lambda \tilde{\lambda}} = \frac{1}{\xi}$$

where:

$$\xi = -\frac{\lambda}{\Delta F} \sim \mathcal{N} (\mu_\xi, \sigma_\xi^2) \equiv \mathcal{N} \left(-\frac{\mu_\lambda}{\Delta F}, \frac{\sigma_\lambda^2}{\Delta F^2}\right)$$

- We deduce that $\Delta \tilde{T}_s^*$ follows a reciprocal normal distribution:

$$f(x) = \frac{1}{\sigma_\xi x^2 \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{x^{-1} - \mu_\xi}{\sigma_\xi}\right)^2}$$

- Using the previous calibration of the climate feedback parameter $\lambda \sim \mathcal{N} (-1.30, 0.44^2)$, the distribution of $\Delta \tilde{T}_s^*$ is right skewed and has an excess of kurtosis.
Figure 50: Probability density function of the equilibrium temperature 
($\Delta F = 4 \text{ W/m}^2$, $\mu_\lambda = -1.30 \text{ W m}^{-2} \text{ K}^{-1}$ and $\sigma_\lambda = 0.44 \text{ W m}^{-2} \text{ K}^{-1}$)
The cumulative distribution function and the exceedance probability are equal to:

<table>
<thead>
<tr>
<th>Temperature $\theta$</th>
<th>2°C</th>
<th>3°C</th>
<th>4°C</th>
<th>5°C</th>
<th>7°C</th>
<th>10°C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Pr \left{ \Delta \tilde{T}_s^* \geq \theta \right}$</td>
<td>94.26%</td>
<td>52.86%</td>
<td>24.61%</td>
<td>12.63%</td>
<td>4.73%</td>
<td>1.88%</td>
</tr>
</tbody>
</table>

The probability of observing an equilibrium temperature greater than 4°C and 7°C is close to 25% and 5%.
The mean lifetime follows a reciprocal normal distribution:

$$\tilde{\tau} = -\frac{c}{\lambda} \sim R\mathcal{N}\left(-\frac{\mu_\lambda}{c}, \frac{\sigma^2_\lambda}{c}\right)$$
Figure 51: Probability density function of the relaxation time 
($\mu_\lambda = -1.30 \, \text{W m}^{-2} \, \text{K}^{-1}$ and $\sigma_\lambda = 0.44 \, \text{W m}^{-2} \, \text{K}^{-1}$)
Figure 52: Monte Carlo simulation of the surface temperature dynamics after a radiative forcing of $4 \text{ W/m}^2$ ($\mu_\lambda = -1.30 \text{ W m}^{-2} \text{ K}^{-1}$ and $\sigma_\lambda = 0.44 \text{ W m}^{-2} \text{ K}^{-1}$)
According to IPCC AR6, the total anthropogenic ERF\(^7\) over the industrial era (1750-2019) is \(2.72 \text{ W/m}^2\).

The contribution of anthropogenic greenhouse gas emissions is \(3.84 \text{ W/m}^2\) and the combined effect of all radiative feedbacks (including Planck feedback) is estimated to be \(-1.16 \text{ W/m}^2\).

The best estimate of the ECS is \(3^\circ\text{C}\).

The likely range is \(2.5^\circ\text{C}\) to \(4^\circ\text{C}\) (with a probability of 66%), and the very likely range is \(2^\circ\text{C}\) to \(5^\circ\text{C}\) (with a probability of 90%).

---

\(^7\)Effective radiative forcing (ERF) is a measure of the change in radiative flux at the top of the atmosphere and does not include the Planck feedback.
Equilibrium climate sensitivity estimation

Figure 53: Anthropogenic effective radiative forcing (ERF) from 1750 to 2019 by contributing forcing agents

Source: IPCC (2021, Figure 7.6, Chapter 7, page 959).
Equilibrium climate sensitivity estimation

Table 12: Effective radiative forcing from 1750 to 2019

<table>
<thead>
<tr>
<th>Forcing agent</th>
<th>1800</th>
<th>1850</th>
<th>1900</th>
<th>1950</th>
<th>2000</th>
<th>2010</th>
<th>2019</th>
</tr>
</thead>
<tbody>
<tr>
<td>CO₂</td>
<td>0.070</td>
<td>0.140</td>
<td>0.346</td>
<td>0.648</td>
<td>1.561</td>
<td>1.854</td>
<td>2.156</td>
</tr>
<tr>
<td>CH₄</td>
<td>0.025</td>
<td>0.049</td>
<td>0.119</td>
<td>0.245</td>
<td>0.509</td>
<td>0.518</td>
<td>0.544</td>
</tr>
<tr>
<td>N₂O</td>
<td>0.004</td>
<td>0.007</td>
<td>0.032</td>
<td>0.069</td>
<td>0.157</td>
<td>0.181</td>
<td>0.208</td>
</tr>
<tr>
<td>Other GHG</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.010</td>
<td>0.375</td>
<td>0.392</td>
<td>0.408</td>
</tr>
<tr>
<td>O₃</td>
<td>0.015</td>
<td>0.030</td>
<td>0.081</td>
<td>0.167</td>
<td>0.399</td>
<td>0.443</td>
<td>0.474</td>
</tr>
<tr>
<td>H₂O (stratospheric)</td>
<td>0.002</td>
<td>0.005</td>
<td>0.011</td>
<td>0.022</td>
<td>0.047</td>
<td>0.048</td>
<td>0.050</td>
</tr>
<tr>
<td>Contrails</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.005</td>
<td>0.039</td>
<td>0.044</td>
<td>0.058</td>
</tr>
<tr>
<td>Aerosol</td>
<td>−0.018</td>
<td>−0.078</td>
<td>−0.346</td>
<td>−0.708</td>
<td>−1.221</td>
<td>−1.266</td>
<td>−1.058</td>
</tr>
<tr>
<td>Black carbon on snow</td>
<td>0.002</td>
<td>0.006</td>
<td>0.020</td>
<td>0.032</td>
<td>0.069</td>
<td>0.085</td>
<td>0.080</td>
</tr>
<tr>
<td>Land use</td>
<td>−0.011</td>
<td>−0.031</td>
<td>−0.084</td>
<td>−0.144</td>
<td>−0.194</td>
<td>−0.197</td>
<td>−0.200</td>
</tr>
<tr>
<td>Total (anthropogenic)</td>
<td>0.089</td>
<td>0.128</td>
<td>0.179</td>
<td>0.346</td>
<td>1.739</td>
<td>2.103</td>
<td>2.720</td>
</tr>
<tr>
<td>Volcanic</td>
<td>0.183</td>
<td>0.194</td>
<td>0.198</td>
<td>0.182</td>
<td>0.175</td>
<td>0.137</td>
<td>0.140</td>
</tr>
<tr>
<td>Solar</td>
<td>−0.043</td>
<td>0.008</td>
<td>−0.037</td>
<td>0.057</td>
<td>0.110</td>
<td>−0.008</td>
<td>−0.022</td>
</tr>
<tr>
<td>Total (natural)</td>
<td>0.140</td>
<td>0.202</td>
<td>0.160</td>
<td>0.239</td>
<td>0.285</td>
<td>0.129</td>
<td>0.118</td>
</tr>
<tr>
<td>Total</td>
<td>0.229</td>
<td>0.330</td>
<td>0.339</td>
<td>0.585</td>
<td>2.025</td>
<td>2.232</td>
<td>2.838</td>
</tr>
</tbody>
</table>

Source: IPCC (2021, Figure 7.6, Chapter 7, page 959).
Equilibrium climate sensitivity estimation

Figure 54: Contribution to effective radiative forcing (ERF) (a) and global mean surface air temperature (GSAT) change (b) from component emissions between 1750 to 2019 based on CMIP6 models.

Source: IPCC (2021, Figure 6.12, Chapter 6, page 854).
Ratio distribution

Normal ratio distribution

- We assume that $X \sim \mathcal{N} \left( \mu_x, \sigma_x^2 \right)$ and $Y \sim \mathcal{N} \left( \mu_y, \sigma_y^2 \right)$
- $Z = X / Y \sim \mathcal{NRD} \left( \mu_x, \sigma_x^2, \mu_y, \sigma_y^2 \right)$
- The CDF of $Z$ is:

$$F_z(z) = \Phi_2 \left( -\frac{\mu_y z - \mu_x}{\sigma_x \sigma_y a(z)}, -\frac{\mu_y}{\sigma_y}, \rho_z \right) + \Phi_2 \left( \frac{\mu_y z - \mu_x}{\sigma_x \sigma_y a(z)}, \frac{\mu_y}{\sigma_y}, \rho_z \right)$$

- The PDF of $Z$ is:

$$f_z(z) = \frac{b(z)}{\sigma_x \sigma_y \sqrt{2\pi a^3(z)}} \left( 2\Phi \left( \frac{b(z)}{a(z)} \right) - 1 \right) \exp \left( \frac{b^2(z) - ca^2(z)}{2a^2(z)} \right) + \frac{1}{\sigma_x \sigma_y a^2(z)} \exp \left( -\frac{c}{2} \right)$$

- $a(z) = \sqrt{\frac{1}{\sigma_x^2} z^2 + \frac{1}{\sigma_y^2}}$, $b(z) = \frac{\mu_x}{\sigma_x^2} z + \frac{\mu_y}{\sigma_y^2}$, $c = \frac{\mu_x^2}{\sigma_x^2} + \frac{\mu_y^2}{\sigma_y^2}$ and

$$\rho_z = \frac{z}{\sigma_x a(z)}$$
Equilibrium climate sensitivity estimation

We now have all the information we need to calculate the equilibrium climate sensitivity.
Equilibrium climate sensitivity estimation

- Using the parameters from Sherwood *et al.* (2020)
  \[ \lambda \sim \mathcal{N}(-1.30, 0.44^2) \text{ and } \Delta F_{2\times CO_2} \sim \mathcal{N}(4.00, 0.30^2) \], we get:

  \[
  \text{ECS} = -\frac{\Delta F_{2\times CO_2}}{\lambda} \\
  = -\frac{\mathcal{N}(4.00, 0.30^2)}{\mathcal{N}(-1.30, 0.44^2)} \\
  = \mathcal{NRD}(-4.00, 0.30^2, -1.30, 0.44^2)
  \]

  where \( \mathcal{NRD} \) is the normal ratio distribution

- The exceedance probability is equal to:

<table>
<thead>
<tr>
<th>Temperature ( \theta )</th>
<th>2(^\circ)C</th>
<th>3(^\circ)C</th>
<th>4(^\circ)C</th>
<th>5(^\circ)C</th>
<th>7(^\circ)C</th>
<th>10(^\circ)C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pr {ECS \geq \theta}</td>
<td>93.24%</td>
<td>52.79%</td>
<td>24.92%</td>
<td>12.85%</td>
<td>4.81%</td>
<td>1.91%</td>
</tr>
</tbody>
</table>

- We have a probability about 25\% to observe a global warming greater than 4\(^\circ\)C
Equilibrium climate sensitivity estimation

Figure 55: Probability density function of the equilibrium climate sensitivity ($\Delta F = +4.00 \text{ W/m}^2$, $\sigma (\Delta F) = 0.30 \text{ W/m}^2$, $\mu_\lambda = -1.30 \text{ W m}^{-2} \text{ K}^{-1}$ and $\sigma_\lambda = 0.44 \text{ W m}^{-2} \text{ K}^{-1}$)
An example of unstable equilibrium

- In the Budyko model, we have at equilibrium:

\[
\mathcal{E} = \frac{1}{4} (1 - \alpha_p \left( T_s \right)) S_0 - \left( \frac{2 - \varepsilon}{2} \right) \sigma T_s^4 = 0
\]

where:

\[
\alpha_p \left( T_s \right) = \begin{cases} 
\alpha_{\text{cold}} & \text{if } T_s \leq T_{\text{cold}} \\
\alpha_{\text{warm}} + (\alpha_{\text{cold}} - \alpha_{\text{warm}}) \left( \frac{T_{\text{warm}} - T_s}{T_{\text{warm}} - T_{\text{cold}}} \right)^\eta & \text{if } T_{\text{cold}} \leq T_s \leq T_{\text{warm}} \\
\alpha_{\text{warm}} & \text{if } T_s \geq T_{\text{warm}}
\end{cases}
\]

- The two forcing functions are

\[
\begin{cases} 
F_{\text{solar}} \left( T_s \right) = \frac{1}{4} \left( 1 - \alpha_p \left( T_s \right) \right) S_0 \\
F_{\text{blackbody}} \left( T_s \right) = \left( \frac{2 - \varepsilon}{2} \right) \sigma T_s^4
\end{cases}
\]
An example of unstable equilibrium

**Figure 56:** Equilibrium states of the Budyko ice-albedo model
An example of unstable equilibrium

There are three equilibrium states:

1. $T_1^* = 233.38 \, \text{K}$
2. $T_2^* = 268.43 \, \text{K}$
3. $T_3^* = 288.13 \, \text{K}$
An example of unstable equilibrium

- The total feedback is equal to:

\[ \lambda(T_s) = \lambda_0(T_s) + \lambda_{\alpha_p}(T_s) \]

where:

\[ \lambda_0(T_s) = -4 \left( \frac{2 - \varepsilon}{2} \right) \sigma T_s^3 \]

and:

\[ \lambda_{\alpha_p}(T_s) = \frac{1}{4} \eta S_0 (\alpha_{\text{cold}} - \alpha_{\text{warm}}) \left( \frac{T_{\text{warm}} - T_s}{T_{\text{warm}} - T_{\text{cold}}} \right)^{\eta - 1} \cdot \mathbb{1} \{ T_{\text{cold}} \leq T_s \leq T_{\text{warm}} \} \]

<table>
<thead>
<tr>
<th>State</th>
<th>Temperature</th>
<th>$\lambda_0(T_s)$</th>
<th>$\lambda_{\alpha_p}(T_s)$</th>
<th>$\lambda(T_s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1^*$</td>
<td>233.38 K</td>
<td>$-39.77^\circ$C</td>
<td>$-1.76$</td>
<td>$0.00$</td>
</tr>
<tr>
<td>$T_2^*$</td>
<td>268.43 K</td>
<td>$-4.72^\circ$C</td>
<td>$-2.68$</td>
<td>$6.75$</td>
</tr>
<tr>
<td>$T_3^*$</td>
<td>288.13 K</td>
<td>$14.98^\circ$C</td>
<td>$-3.31$</td>
<td>$0.45$</td>
</tr>
</tbody>
</table>
An example of unstable equilibrium

- $T_1^*$ and $T_3^*$ are two stable equilibria because $\lambda$ is negative, but $T_2^*$ is an unstable equilibrium.

- To illustrate this instability, we consider the dynamics of the temperature:

$$c \frac{dT_s}{dt} = F_{\text{solar}}(T_s) - F_{\text{blackbody}}(T_s) + \Delta F$$

where $c = 4 \times 10^8 \text{ J m}^{-2} \text{ K}^{-1}$ is the heat capacity and $\Delta F$ is the perturbation.

- The fourth panel shows a small perturbation $\Delta F = \pm 0.1 \text{ W/m}^2$. 

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An example of unstable equilibrium

Figure 57: Equilibrium states of the Budyko ice-albedo model
The concept of a tipping point emerged from stability analysis and bifurcation.

The term was popularized in 2000 by Malcolm Gladwell in his book “The Tipping Point: How Little Things Can Make a Big Difference”, which explored the concept in sociological change.

In climate change, it was popularized in the late 2000s by Lenton et al. (2008).

In common parlance, a tipping point is a change.
“The term tipping point commonly refers to a critical threshold at which a tiny perturbation can qualitatively alter the state or development of a system.” (Lenton et al., 2008, page 1786).

“A climate tipping point occurs when a small change in forcing triggers a strongly nonlinear response in the internal dynamics of part of the climate system, qualitatively changing its future state.” (Lenton et al., 2011, page 201).
“Tipping points refer to critical thresholds in a system that, when exceeded, can lead to a significant change in the state of the system, often with an understanding that the change is irreversible.” (IPCC, 2018, page 262).

A tipping point is “a hypothesized critical threshold when global or regional climate changes from one stable state to another stable state. The tipping point event may be irreversible.” (IPCC, 2021, page 1463).
Bifurcation theory

Figure 58: Stable equilibria
Bifurcation theory

Figure 59: Unstable equilibrium
Figure 60: Irreversible tipping point
Bifurcation theory

- We consider the dynamical system of the form:

\[
\frac{dx}{dt} = f(x, \mu)
\]

where \( \mu \) is the parameter set
- If \( f(x, \mu) > 0 \), then \( x \) increases with time, while if \( f(x, \mu) < 0 \), then \( x \) decreases with time
- A fixed point \( x^* \) is then a value of \( x \) such that the system does not change with time, i.e. \( f(x^*, \mu) = 0 \)
- Let \( x = x^* + \Delta x \) where \( \Delta x \) is small. We have:

\[
\frac{dx}{dt} = f(x^* + \Delta x, \mu) \\
= f(x^*, \mu) + \frac{\partial f(x^*, \mu)}{\partial x} \Delta x + O(\Delta x^2) \\
= f(x^*, \mu) + \lambda \Delta x + O(\Delta x^2)
\]

where \( \lambda \) is the feedback of the system
Bifurcation theory

- Since we have:
  \[
  \frac{dx}{dt} = \frac{d(x^* + \Delta x)}{dt} = \frac{d\Delta x}{dt}
  \]
  and:
  \[
  f(x^* + \Delta x, \mu) \approx f(x^*, \mu) + \lambda \Delta x = \lambda \Delta x
  \]
- We deduce that:
  \[
  \frac{d\Delta x}{dt} = \lambda \Delta x
  \]
- If \( \lambda > 0 \), then any small perturbation of the fixed point grows exponentially (the fixed point is unstable)
- If \( \lambda < 0 \), then any small perturbation of the fixed point decays exponentially (the fixed point is stable)
- The relaxation timescale is equal to:
  \[
  \tau = \frac{1}{|\partial_x f(x^*, \mu)|}
  \]
Bifurcation theory

Example #1

We consider the dynamical system

$$\frac{dx}{dt} = x^2 + \mu$$
Bifurcation theory

- We have $\partial_x f(x, \mu) = 2x$
- The fixed points are solutions of the equation $x^2 = -\mu$
- If $\mu > 0$, there are no fixed points
- If $\mu = 0$, the fixed point is $x^* = 0$ and is not stable
- If $\mu < 0$, there are two fixed points. $x_1^* = -\sqrt{-\mu}$ is stable while $x_2^* = \sqrt{-\mu}$ is unstable
The stability analysis evaluates the behavior of $f (x^* + \varepsilon, \mu)$ and checks if the point $x^* + \Delta x$ converges to the point $x^*$

- We can consider a second approach to stability where we evaluate the behavior of $f (x^*, \mu + \varepsilon)$, i.e. we apply the perturbation not directly to $x$ but to the control parameter.

- We say that the value $\mu^*$ is a bifurcation value if $f (x, \mu^*)$ is not structurally stable.
Bifurcation theory

- In the previous example, the dynamical system exhibits a bifurcation that occurs at $\mu = 0$.
- This type of bifurcation is called a saddle-node or fold bifurcation, because fixed points are created or destroyed.
Scientific evidence of global warming
From the Holocene to the Anthropocene?
The physics of climate change
Energy balance models
Climate sensitivity and feedback
Tipping points

Bifurcation theory

Figure 61: Bifurcation diagram

Saddle-node

Transcritical

Supercritical pitchfork

Subcritical pitchfork

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Bifurcation theory

- A transcritical bifurcation occurs when fixed points exchange stability for a critical value of $\mu$.

- For example, the system $\frac{dx(t)}{dt} = \mu x - x^2$ has a transcritical bifurcation at $\mu = 0$. 

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**Bifurcation theory**

**Figure 62: Bifurcation diagram**

- **Saddle-node**
  - Stable
  - Unstable

- **Transcritical**

- **Supercritical pitchfork**

- **Subcritical pitchfork**
A pitchfork bifurcation occurs when the system is unchanged and exhibits symmetry when $x = -x$. There are two forms of pitchfork bifurcation:

1. In a supercritical pitchfork bifurcation, a stable fixed point becomes unstable at a critical value of $\mu$. A canonical example is:

$$\frac{dx}{dt} = \mu x - x^3$$

2. In a subcritical pitchfork bifurcation, an unstable fixed point becomes stable at a critical value of $\mu$. A canonical example is:

$$\frac{dx}{dt} = \mu x + x^3 - x^5$$
Bifurcation theory

Figure 63: Bifurcation diagram

Saddle-node

Transcritical

Super-critical pitchfork

Sub-critical pitchfork

Stable

Unstable
Definition

In bifurcation theory, hysteresis refers to a phenomenon where the behavior of the system depends on the history of its past states.
Hysteresis

- Consider the previous subcritical pitchfork bifurcation and suppose the system is at equilibrium $x^* = 0$
- If we increase $\mu$, the system jumps to one of the stable branches
- If we decrease $\mu$, the system does not return to its past equilibrium, but stays on the stable branch
- However, if we decrease $\mu$ even more, the system jumps to its past equilibrium $x^* = 0$ when $\mu$ reaches $\mu^* = -0.25$
- The path of the system has formed a hysteresis loop
Bifurcation theory

**Figure 64: Bifurcation diagram**

- **Saddle-node**
- **Transcritical**
- **Supercritical pitchfork**
- **Subcritical pitchfork**

- Stable
- Unstable
The Rössler model is a system of ordinary differential equations that exhibits chaotic dynamics:

\[
\begin{align*}
\frac{dx(t)}{dt} &= - (y(t) + z(t)) \\
\frac{dy(t)}{dt} &= x(t) + ay(t) \\
\frac{dz(t)}{dt} &= b + x(t)z(t) - cz(t)
\end{align*}
\]

where \(a\), \(b\) and \(c\) are three parameters that control the behavior of the system.

- The Rössler attractor is a three-dimensional surface described by \((x(t), y(t), z(t))\).
- The attractor is very sensitive to the initial conditions \((x(0), y(0), z(0))\) and the set of parameters \((a, b, c)\).
Chaos theory

Figure 65: Rössler attractor
We use the default values $S_0 = 1368 \text{ W/m}^2$, $\varepsilon = 78\%$, $\eta = 3$, $\alpha_{\text{cold}} = 0.7$, $T_{\text{cold}} = 260 \text{ K}$, $\alpha_{\text{warm}} = 0.3$ and $T_{\text{warm}} = 295 \text{ K}$

We have three fixed points: $T_1^* = 233.38 \text{ K}$, $T_2^* = 268.43 \text{ K}$, and $T_3^* = 288.13 \text{ K}$

$T_1^*$ and $T_3^*$ are stable

$T_3^*$ is unstable
Application to the Budyko ice-albedo model

- To perform the bifurcation analysis of the Budyko ice-albedo model, we consider the range $T_s \in [200 \, \text{K}, 300 \, \text{K}]$ and divide the range into ten intervals.

- Using the bisection algorithm, we solve the equation for each interval of $T_s$ and each value of the parameter of interest:

$$\frac{1}{4} (1 - \alpha_p ((T_s))) S_0 - \left( \frac{2 - \varepsilon}{2} \right) \sigma T_s^4 = 0$$

- We collect all fixed points $\{T_1^*, T_2^*, \ldots\}$

- For each fixed point we calculate the feedback:

$$\lambda (T_s) = \frac{1}{4} \eta S_0 (\alpha_{\text{cold}} - \alpha_{\text{warm}}) \frac{(T_{\text{warm}} - T_s)^{\eta-1}}{(T_{\text{warm}} - T_{\text{cold}})^{\eta}}.$$

$$\mathbb{1} \{T_{\text{cold}} \leq T_s \leq T_{\text{warm}}\} - 4 \left( \frac{2 - \varepsilon}{2} \right) \sigma T_s^3$$
Application to the Budyko ice-albedo model

Figure 66: Bifurcation of the Budyko ice-albedo model
The relaxation timescale is equal to:

\[ \tau = \frac{c}{|\lambda(T_s)|} \]

where \( c = 4 \times 10^8 \text{ J m}^{-2} \text{ K}^{-1} \) is the heat capacity
Application to the Budyko ice-albedo model

Figure 67: Relaxation timescale of the Budyko ice-albedo model (in years)
Figure 68: Bifurcation and overshooting tipping points

(a) Slow/fast tipping onset elements

(b) AMOC overshooting

Source: Ritchie et al. (2021, Figures 1c & 3c, pages 518 & 520.)
Climate tipping elements

Figure 69: Geographical distribution of global and regional tipping elements

Source: Armstrong McKay et al. (2022).
### Table 13: Threshold, timescale, and impact estimates for the global and regional tipping elements

<table>
<thead>
<tr>
<th>Category</th>
<th>#</th>
<th>Climate tipping element</th>
<th>Tipping point</th>
<th>Threshold</th>
<th>Timescale</th>
<th>Maximum impact</th>
</tr>
</thead>
<tbody>
<tr>
<td>Global</td>
<td>1</td>
<td>Greenland ice sheet collapse</td>
<td>1.5°C (0.8-3.0)</td>
<td>10 kyr (1-15)</td>
<td>0.13°C (0.5-3.0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>West Antarctic ice sheet collapse</td>
<td>1.5°C (1.0-3.0)</td>
<td>2 kyr (0.5-13)</td>
<td>0.05°C (1.0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>Labrador-Imringer seas collapse</td>
<td>1.8°C (1.1-3.8)</td>
<td>10 yr (5-50)</td>
<td>-0.50°C (-3.0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>East Antarctic subglacial basins collapse</td>
<td>3.0°C (2.0-6.0)</td>
<td>2 kyr (0.5-10)</td>
<td>0.05°C (0.4-2.0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>Amazon rainforest dieback</td>
<td>3.5°C (2.0-6.0)</td>
<td>100 yr (50-200)</td>
<td>0.20°C (0.6-1.2)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>Boreal permafrost collapse</td>
<td>4.0°C (3.0-6.0)</td>
<td>50 yr (10-300)</td>
<td>0.40°C (0.6-1.2)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>AMOC collapse</td>
<td>4.0°C (1.4-8.0)</td>
<td>50 yr (15-300)</td>
<td>-0.50°C (-4/-10)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>Arctic winter sea ice collapse</td>
<td>6.3°C (4.5-8.7)</td>
<td>20 yr (10-100)</td>
<td>0.60°C (0.6-1.2)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>East Antarctic ice sheet collapse</td>
<td>7.5°C (5.0-10.0)</td>
<td>&gt; 10 kyr</td>
<td>0.60°C (2.0)</td>
<td></td>
</tr>
<tr>
<td>Regional</td>
<td>10</td>
<td>Low-latitude coral reefs die-off</td>
<td>1.5°C (1.0-2.0)</td>
<td>10 yr</td>
<td>0.04°C</td>
<td></td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>Boreal permafrost abrupt thaw</td>
<td>1.5°C (1.0-2.3)</td>
<td>200 yr (100-300)</td>
<td>0.04°C</td>
<td></td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>Barents sea ice abrupt loss</td>
<td>1.6°C (1.5-1.7)</td>
<td>25 yr</td>
<td>0.08°C</td>
<td></td>
</tr>
<tr>
<td></td>
<td>13</td>
<td>Mountain glaciers loss</td>
<td>2.0°C (1.5-3.0)</td>
<td>200 yr (50-1000)</td>
<td>0.08°C</td>
<td></td>
</tr>
<tr>
<td></td>
<td>14</td>
<td>Sahel and West African monsoon greening</td>
<td>2.8°C (2.0-3.5)</td>
<td>50 yr (10-500)</td>
<td>0.08°C</td>
<td></td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>Boreal forest (southern) die-off</td>
<td>4.0°C (1.4-5.0)</td>
<td>100 yr (50+)</td>
<td>-0.18°C (-0.5/-2)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>Boreal forest (northern) expansion</td>
<td>4.0°C (1.5-7.2)</td>
<td>100 yr (40+)</td>
<td>0.14°C (0.5-1.0)</td>
<td></td>
</tr>
</tbody>
</table>

Source: Armstrong McKay *et al.* (2022, Table 1, page 3).
1. Greenland ice sheet

The Greenland ice sheet is the second largest ice sheet in the world. It covers 80% of the surface of Greenland. Its melting would increase sea level rise, possibly up to 7.4 meters, accelerate ocean acidification, and have a potentially positive feedback effect on climate change.
2. West Antarctic ice sheet

The West Antarctic ice sheet is a large ice sheet in Antarctica. It sits on a bedrock that is mostly below sea level and has formed a deep subglacial basin due to the weight of the ice sheet, which can be up to 4 kilometers thick in places. Its collapse could raise global sea levels, possibly up to 3 meters.
3. Labrador-Irminger seas

The Labrador and Irminger seas are located in the subpolar North Atlantic, between Canada, Greenland and Iceland. These seas are characterized by cold and salty waters that generate deep convection. This deep convection and the regulation of ocean salinity influence the circulation of the Atlantic meridional overturning circulation (AMOC). The collapse of the deep convection system in the Labrador and Irminger seas would affect the overall circulation in the North Subpolar Gyre.
4. East Antarctic subglacial basins

East Antarctic subglacial basins are large, ice-filled depressions in the bedrock adjacent to the East Antarctic ice sheet. They also serve as reservoirs for meltwater. Certain subglacial basins, such as Wilkes, Aurora, and Recovery, are more susceptible to a situation called marine ice sheet instability (MISI). As the ice shelf at the edge of the ice sheet retreats, warm ocean water can flow into the deeper basin, further destabilizing the ice shelf. This process can create a self-perpetuating cycle of ice melt and ice shelf retreat.
5. Amazon rainforest

The Amazon rainforest, also known as the Amazon jungle or Amazonia, is the largest rainforest in the world. It spans nine countries: Brazil, Peru, Colombia, Ecuador, Bolivia, Venezuela, Guyana, Suriname, and French Guiana. It contains the largest and most biodiverse area of tropical rainforest in the world. The Amazon rainforest acts as a massive carbon sink, absorbing and storing vast amounts of carbon dioxide through the process of photosynthesis. If the forest were to be subject to widespread deforestation or degradation, the stored carbon could be released back into the atmosphere, contributing to increased greenhouse gas concentrations. The Amazon also plays a critical role in the Earth’s water cycle, influencing regional and global weather patterns. Its dense vegetation effectively captures rainwater and slowly releases it into streams and rivers, helping to maintain stable water levels, prevent flooding, and provide a steady source of fresh water.
6. Boreal permafrost

Boreal permafrost is a permanently frozen layer of soil and rock that underlies much of the world’s boreal forest. It is found in Siberia, Alaska, Northern Canada, and the Tibetan Plateau. Permafrost forms when the soil temperature remains below 0°C for at least two consecutive years. Boreal permafrost contains large amounts of organic carbon stored in the form of dead plant material that could not decompose due to the cold temperatures. It is also one of the largest reservoirs of methane. Rising temperatures may cause the boreal permafrost to thaw, releasing large amounts of carbon and methane into the atmosphere.
7. AMOC

The Atlantic meridional overturning circulation (AMOC) is a large, complex system of ocean currents that transports warm water from the tropics to the North Atlantic and cold water from the North Atlantic to the subtropics. It is also known as the Gulf Stream system. A weakening of the AMOC could have complex and regionally specific effects on temperatures. On a global scale, it could result in less warm water reaching higher latitudes, leading to cooler sea surface temperatures in the North Atlantic and warmer temperatures in the Southern Hemisphere.
8. Arctic winter sea ice

Arctic winter sea ice is the maximum extent of sea ice that forms in the Arctic Ocean during the winter months. It helps to regulate global temperatures by reflecting sunlight back into space. This albedo reflection helps to cool the Arctic. As sea ice melts, more sunlight is absorbed by the ocean, causing a further warming trend\(^a\). However, the impact of the albedo effect remains controversial.

\(^a\)Moreover, the warming Arctic has the potential to release methane from permafrost.
9. East Antarctic ice sheet

The East Antarctic ice sheet is the largest and thickest ice sheet on Earth. A complete collapse would raise the global sea levels by 50 meters. However, the East Antarctic ice sheet is generally considered to be more stable than the West Antarctic ice sheet, due to its higher elevation and more remote location.
10. Low-latitude coral reefs

Low-latitude coral reefs occur in the Atlantic, Indian, and Pacific Oceans, most notably in the Philippines, Indonesia, and Australia. They require warm, sunny weather and unpolluted water. Therefore, coral reefs can be affected by climate change, although their impact on climate change is more limited.
11. Boreal permafrost

We have already seen that the boreal permafrost is a global tipping element, but it is also a regional tipping element. In fact, an abrupt thaw of the boreal permafrost would have devastating consequences for the region, affecting infrastructure (roads, buildings, transportation), the environment (flooding, forests, vegetation), and living conditions and health.
12. Barents sea ice

Barents sea ice is found in the Barents sea, an arm of the Arctic Ocean between Norway and Russia. The sea ice forms during the winter months and melts during the summer months. This regional tipping element is strongly related to two global tipping elements: Labrador-Irminger seas and AMOC.
13. Mountain glaciers

Mountain glaciers are large masses of ice that form on mountains at high altitudes. They are formed from compacted snow that has accumulated over many years. The melting of mountain glaciers would have a major regional impact on human life.
14. Sahel and West African monsoon

The West African monsoon is a seasonal wind pattern that affects the Sahel, bringing moisture from the Atlantic Ocean during the rainy season and drying out the region during the dry season. It is responsible for the region’s agriculture and supports the livelihoods of millions of people. Changes in rainfall can affect vegetation, agriculture and people.
15. Boreal forest (southern)

The boreal forest, also known as the taiga, is a biome that surrounds the Arctic region. Countries with significant areas of boreal forest include Canada, Russia, Sweden, Norway and Finland. The southern edge of the boreal forest is the boundary between the boreal forest and temperate forests or grasslands. The risk could be an abrupt die-off.
16. Boreal forest (northern)

The northern edge of the boreal forest is typically found at higher latitudes, closer to the Arctic Circle. The change could be an abrupt expansion into a tundra forest characterised by treeless landscapes and permafrost.
Cascading tipping points and climate domino effects

Figure 70: Double fold bifurcation

\[
\begin{align*}
( -\sqrt{\frac{4}{27}}, +\sqrt{\frac{1}{3}} ) & \quad \text{Stable} \\
( \sqrt{\frac{4}{27}}, -\sqrt{\frac{1}{3}} ) & \quad \text{Unstable}
\end{align*}
\]

Source: Klose et al. (2020, Figure 1, page 3).
Cascading tipping points and climate domino effects

Figure 71: Convergence to the equilibrium

\[ x_i(t) = \begin{cases} 1 & \text{if } t_i = 1 \text{ year} \\ 10 & \text{if } t_i = 10 \text{ years} \\ 50 & \text{if } t_i = 50 \text{ years} \end{cases} \]
Figure 72: Master-slave bifurcation

Source: Klose et al. (2020, Figure 2, page 6).
Figure 73: Equilibria of the slave subsystem

Stable lower branch

Stable upper branch

Unstable branch
Cascading tipping points and climate domino effects

Figure 74: Interactions between climate tipping elements and their roles in tipping cascades

Source: Wunderling et al. (2021, Figure 1, page 603).