Course 2023-2024 in Sustainable Finance
Lecture 13. Exercise — Equity and Bond Portfolio Optimization with Green Preferences

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1The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.
Lecture 1: Introduction
Lecture 2: ESG Scoring
Lecture 3: Impact of ESG Investing on Asset Prices and Portfolio Returns
Lecture 4: Sustainable Financial Products
Lecture 5: Impact Investing
Lecture 6: Engagement & Voting Policy
Lecture 7: Extra-financial Accounting
Lecture 8: Awareness of Climate Change Impacts
Lecture 9: The Ecosystem of Climate Change
Lecture 10: Economic Models & Climate Change
Lecture 11: Climate Risk Measures
Lecture 12: Transition Risk Modeling
Lecture 13: Climate Portfolio Construction
Lecture 14: Physical Risk Modeling
Lecture 15: Climate Stress Testing & Risk Management
We consider an investment universe of 8 issuers. In the table below, we report the carbon emissions $CE_{i,j}$ (in ktCO$_2$e) of these companies and their revenues $Y_i$ (in $bn), and we indicate in the last row whether the company belongs to sector $\text{Sector}_1$ or $\text{Sector}_2$:

<table>
<thead>
<tr>
<th>Issuer</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
<th>#5</th>
<th>#6</th>
<th>#7</th>
<th>#8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$CE_{i,1}$</td>
<td>75</td>
<td>5000</td>
<td>720</td>
<td>50</td>
<td>2500</td>
<td>25</td>
<td>30000</td>
<td>5</td>
</tr>
<tr>
<td>$CE_{i,2}$</td>
<td>75</td>
<td>5000</td>
<td>1030</td>
<td>350</td>
<td>4500</td>
<td>5</td>
<td>2000</td>
<td>64</td>
</tr>
<tr>
<td>$CE_{i,3}$</td>
<td>24000</td>
<td>15000</td>
<td>1210</td>
<td>550</td>
<td>500</td>
<td>187</td>
<td>30000</td>
<td>199</td>
</tr>
<tr>
<td>$Y_i$</td>
<td>300</td>
<td>328</td>
<td>125</td>
<td>100</td>
<td>200</td>
<td>102</td>
<td>107</td>
<td>25</td>
</tr>
<tr>
<td>$\text{Sector}$</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

The benchmark $b$ of this investment universe is defined as:

$$b = (22\%, 19\%, 17\%, 13\%, 11\%, 8\%, 6\%, 4\%)$$

In what follows, we consider long-only portfolios.
Question 1

We want to compute the carbon intensity of the benchmark.
Question (a)

Compute the carbon intensities $CI_{i,j}$ of each company $i$ for the scopes 1, 2 and 3.
We have:

\[ CI_{i,j} = \frac{CE_{i,j}}{Y_i} \]

For instance, if we consider the 8\textsuperscript{th} issuer, we have\textsuperscript{2}:

\[ CI_{8,1} = \frac{CE_{8,1}}{Y_8} = \frac{5}{25} = 0.20 \text{ tCO}_2\text{e}/\$ \text{ mn} \]
\[ CI_{8,2} = \frac{CE_{8,2}}{Y_8} = \frac{64}{25} = 2.56 \text{ tCO}_2\text{e}/\$ \text{ mn} \]
\[ CI_{8,3} = \frac{CE_{8,3}}{Y_8} = \frac{199}{25} = 7.96 \text{ tCO}_2\text{e}/\$ \text{ mn} \]

\textsuperscript{2}Because 1 ktCO\textsubscript{2}e/$ bn = 1 tCO\textsubscript{2}e/$ mn.
Since we have:

<table>
<thead>
<tr>
<th>Issuer</th>
<th>#1</th>
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<th>#5</th>
<th>#6</th>
<th>#7</th>
<th>#8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$CE_{i,1}$</td>
<td>75</td>
<td>5000</td>
<td>720</td>
<td>50</td>
<td>2500</td>
<td>25</td>
<td>30000</td>
<td>5</td>
</tr>
<tr>
<td>$CE_{i,2}$</td>
<td>75</td>
<td>5000</td>
<td>1030</td>
<td>350</td>
<td>4500</td>
<td>5</td>
<td>2000</td>
<td>64</td>
</tr>
<tr>
<td>$CE_{i,3}$</td>
<td>24000</td>
<td>15000</td>
<td>1210</td>
<td>550</td>
<td>500</td>
<td>187</td>
<td>30000</td>
<td>199</td>
</tr>
</tbody>
</table>

we obtain:

<table>
<thead>
<tr>
<th>Issuer</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
<th>#5</th>
<th>#6</th>
<th>#7</th>
<th>#8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$CI_{i,1}$</td>
<td>0.25</td>
<td>15.24</td>
<td>5.76</td>
<td>0.50</td>
<td>12.50</td>
<td>0.25</td>
<td>280.37</td>
<td>0.20</td>
</tr>
<tr>
<td>$CI_{i,2}$</td>
<td>0.25</td>
<td>15.24</td>
<td>8.24</td>
<td>3.50</td>
<td>22.50</td>
<td>0.05</td>
<td>18.69</td>
<td>2.56</td>
</tr>
<tr>
<td>$CI_{i,3}$</td>
<td>80.00</td>
<td>45.73</td>
<td>9.68</td>
<td>5.50</td>
<td>2.50</td>
<td>1.83</td>
<td>280.37</td>
<td>7.96</td>
</tr>
</tbody>
</table>
Question (b)

Deduce the carbon intensities $C{I}_{i,j}$ of each company $i$ for the scopes $1 + 2$ and $1 + 2 + 3$. 
We have:

\[
CI_{i,1-2} = \frac{CE_{i,1} + CE_{i,2}}{Y_i} = CI_{i,1} + CI_{i,2}
\]

and:

\[
CI_{i,1-3} = CI_{i,1} + CI_{i,2} + CI_{i,3}
\]

We deduce that:

<table>
<thead>
<tr>
<th>Issuer</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
<th>#5</th>
<th>#6</th>
<th>#7</th>
<th>#8</th>
</tr>
</thead>
<tbody>
<tr>
<td>CI_{i,1}</td>
<td>0.25</td>
<td>15.24</td>
<td>5.76</td>
<td>0.50</td>
<td>12.50</td>
<td>0.25</td>
<td>280.37</td>
<td>0.20</td>
</tr>
<tr>
<td>CI_{i,1-2}</td>
<td>0.50</td>
<td>30.49</td>
<td>14.00</td>
<td>4.00</td>
<td>35.00</td>
<td>0.29</td>
<td>299.07</td>
<td>2.76</td>
</tr>
<tr>
<td>CI_{i,1-3}</td>
<td>80.50</td>
<td>76.22</td>
<td>23.68</td>
<td>9.50</td>
<td>37.50</td>
<td>2.12</td>
<td>579.44</td>
<td>10.72</td>
</tr>
</tbody>
</table>
Question (c)

Deduce the weighted average carbon intensity (WACI) of the benchmark if we consider the scope 1 + 2 + 3.
We have:

\[ CI(b) = \sum_{i=1}^{8} b_i CI_i \]

\[ = 0.22 \times 80.50 + 0.19 \times 76.2195 + 0.17 \times 23.68 + 0.13 \times 9.50 + \]

\[ + 0.11 \times 37.50 + 0.08 \times 2.1275 + 0.06 \times 579.4393 + 0.04 \times 10.72 \]

\[ = 76.9427 \text{ tCO}_2e/\$ \text{ mn} \]
Question (d)

We assume that the market capitalization of the benchmark portfolio is equal to $10$ tn and we invest $1$ bn.
Question (d).i

Deduce the market capitalization of each company (expressed in $ bn).
We have:

\[ b_i = \frac{MC_i}{\sum_{k=1}^{8} MC_k} \]

and \( \sum_{k=1}^{8} MC_k = $10 \text{ tn.} \) We deduce that:

\[ MC_i = 10 \times b_i \]

We obtain the following values of market capitalization expressed in $ bn:

<table>
<thead>
<tr>
<th>Issuer</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
<th>#5</th>
<th>#6</th>
<th>#7</th>
<th>#8</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC_i</td>
<td>2200</td>
<td>1900</td>
<td>1700</td>
<td>1300</td>
<td>1100</td>
<td>800</td>
<td>600</td>
<td>400</td>
</tr>
</tbody>
</table>
Question (d).ii

Compute the ownership ratio for each asset (expressed in bps).
Let $W$ be the wealth invested in the benchmark portfolio $b$. The wealth invested in asset $i$ is equal to $b_i W$. We deduce that the ownership ratio is equal to:

$$\varpi_i = \frac{b_i W}{MC_i} = \frac{b_i W}{b_i \sum_{k=1}^{n} MC_k} = \frac{W}{\sum_{k=1}^{n} MC_k}$$

When we invest in a capitalization-weighted portfolio, the ownership ratio is the same for all the assets. In our case, we have:

$$\varpi_i = \frac{1}{10 \times 1000} = 0.01\%$$

The ownership ratio is equal to 1 basis point.
Question (d).iii

Compute the carbon emissions of the benchmark portfolio\(^a\) if we invest $1 \text{ bn} and we consider the scope 1 + 2 + 3.

\(^a\)We assume that the float percentage is equal to 100% for all the 8 companies.
Using the financed emissions approach, the carbon emissions of our investment is equal to:

\[
CE \ (\$1 \ \text{bn}) = 0.01\% \times (75 + 75 + 24000) + 0.01\% \times (5000 + 5000 + 15000) + \ldots + 0.01\% \times (5 + 64 + 199) = 12.3045 \ \text{ktCO}_2\text{e}
\]
Question (d).iv

Compare the (exact) carbon intensity of the benchmark portfolio with the WACI value obtained in Question 1.(c).
We compute the revenues of our investment:

\[ Y \text{ ($1 bn)} = 0.01\% \sum_{i=1}^{8} Y_i = 0.1287 \text{ bn} \]

We deduce that the exact carbon intensity is equal to:

\[ CI \text{ ($1 bn)} = \frac{CE \text{ ($1 bn)}}{Y \text{ ($1 bn)}} = \frac{12.3045}{0.1287} = 95.6061 \text{ tCO}_2\text{e}/$ mn \]

We notice that the WACI of the benchmark underestimates the exact carbon intensity of our investment by 19.5%:

\[ 76.9427 < 95.6061 \]
**Question 2**

We want to manage an equity portfolio with respect to the previous investment universe and reduce the weighted average carbon intensity of the benchmark by the rate $R$. We assume that the volatility of the stocks is respectively equal to 22%, 20%, 25%, 18%, 40%, 23%, 13% and 29%. The correlation matrix between these stocks is given by:

$$\rho = \begin{pmatrix}
100 & 80 & 70 & 60 & 70 & 50 & 60 & 70 & 60 & 100 \\
80 & 100 & 75 & 65 & 50 & 60 & 70 & 75 & 65 & 70 \\
70 & 75 & 100 & 80 & 70 & 50 & 70 & 75 & 65 & 70 \\
60 & 65 & 80 & 100 & 85 & 70 & 80 & 80 & 65 & 60 \\
70 & 50 & 70 & 85 & 100 & 100 & 100 & 100 & 100 & 100 \\
50 & 60 & 70 & 80 & 100 & 60 & 100 & 100 & 100 & 100 \\
70 & 50 & 70 & 75 & 80 & 50 & 100 & 100 & 100 & 100 \\
60 & 65 & 70 & 75 & 65 & 70 & 60 & 100 & 100 & 100 \\
\end{pmatrix}$$
Question (a)

Compute the covariance matrix $\Sigma$. 
The covariance matrix $\Sigma = (\Sigma_{i,j})$ is defined by:

$$\Sigma_{i,j} = \rho_{i,j} \sigma_i \sigma_j$$

We obtain the following numerical values (expressed in bps):

$$\Sigma = \begin{pmatrix}
484.0 & 352.0 & 385.0 & 237.6 & 616.0 & 253.0 & 200.2 & 382.8 \\
352.0 & 400.0 & 375.0 & 234.0 & 400.0 & 276.0 & 130.0 & 377.0 \\
385.0 & 375.0 & 625.0 & 360.0 & 700.0 & 402.5 & 227.5 & 507.5 \\
237.6 & 234.0 & 360.0 & 324.0 & 612.0 & 331.2 & 175.5 & 391.5 \\
616.0 & 400.0 & 700.0 & 612.0 & 1600.0 & 552.0 & 416.0 & 754.0 \\
253.0 & 276.0 & 402.5 & 331.2 & 552.0 & 529.0 & 149.5 & 466.9 \\
200.2 & 130.0 & 227.5 & 175.5 & 416.0 & 149.5 & 169.0 & 226.2 \\
382.8 & 377.0 & 507.5 & 391.5 & 754.0 & 466.9 & 226.2 & 841.0
\end{pmatrix}$$
Question (b)

Write the optimization problem if the objective function is to minimize the tracking error risk under the constraint of carbon intensity reduction.
The tracking error variance of portfolio $w$ with respect to benchmark $b$ is equal to:

$$\sigma^2 (w \mid b) = (w - b)^\top \Sigma (w - b)$$

The carbon intensity constraint has the following expression:

$$\sum_{i=1}^{8} w_i CI_i \leq (1 - R) CI(b)$$

where $R$ is the reduction rate and $CI(b)$ is the carbon intensity of the benchmark. Let $CI^* = (1 - R) CI(b)$ be the target value of the carbon footprint. The optimization problem is then:

$$w^* = \arg \min \frac{1}{2} \sigma^2 (w \mid b)$$

s.t. \[
\begin{align*}
\sum_{i=1}^{8} w_i CI_i & \leq CI^* \\
\sum_{i=1}^{8} w_i &= 1 \\
0 & \leq w_i \leq 1
\end{align*}
\]

We add the second and third constraints in order to obtain a long-only portfolio.
Question (c)

Give the QP formulation of the optimization problem.
The objective function is equal to:

\[ f(w) = \frac{1}{2} \sigma^2 (w \mid b) = \frac{1}{2} (w - b)^\top \Sigma (w - b) = \frac{1}{2} w^\top \Sigma w - w^\top \Sigma b + \frac{1}{2} b^\top \Sigma b \]

while the matrix form of the carbon intensity constraint is:

\[ CI^\top w \leq CI^* \]

where \( CI = (CI_1, \ldots, CI_8) \) is the column vector of carbon intensities. Since \( b^\top \Sigma b \) is a constant and does not depend on \( w \), we can cast the previous optimization problem into a QP problem:

\[ w^* = \arg \min \frac{1}{2} w^\top Q w - w^\top R \]

subject to:

\[ \begin{cases} Aw = B \\ Cw \leq D \\ w^- \leq w \leq w^+ \end{cases} \]

We have \( Q = \Sigma, R = \Sigma b, A = \mathbf{1}_8^\top, B = 1, C = CI^\top, D = CI^*, w^- = \mathbf{0}_8 \) and \( w^+ = \mathbf{1}_8 \).
Question (d)

\( R \) is equal to 20%. Find the optimal portfolio if we target scope 1 + 2. What is the value of the tracking error volatility?
We have:

\[ CI(b) = 0.22 \times 0.50 + 0.19 \times 30.4878 + \ldots + 0.04 \times 2.76 \]
\[ = 30.7305 \text{ tCO}_2e/\$ \text{ mn} \]

We deduce that:

\[ CI^* = (1 - R) CI(b) = 0.80 \times 30.7305 = 24.5844 \text{ tCO}_2e/\$ \text{ mn} \]

Therefore, the inequality constraint of the QP problem is:

\[
\begin{pmatrix}
0.50 & 30.49 & 14.00 & 4.00 & 35.00 & 0.29 & 299.07 & 2.76 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2 \\
\vdots \\
w_7 \\
w_8
\end{pmatrix}
\leq 24.5844\]
We obtain the following optimal solution:

\[
\begin{pmatrix}
23.4961 \\
17.8129 \\
17.1278 \\
15.4643 \\
10.4037 \\
7.5903 \\
4.0946 \\
4.0104
\end{pmatrix}
\]

The minimum tracking error volatility \( \sigma (w^* \mid b) \) is equal to 15.37 bps.
Question (e)

Same question if $\mathcal{R}$ is equal to 30%, 50%, and 70%.
Table 1: Solution of the equity optimization problem (scope $SC_{1-2}$)

<table>
<thead>
<tr>
<th>$R$</th>
<th>0%</th>
<th>20%</th>
<th>30%</th>
<th>50%</th>
<th>70%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>22.0000</td>
<td>23.4961</td>
<td>24.2441</td>
<td>25.7402</td>
<td>30.4117</td>
</tr>
<tr>
<td>$w_2$</td>
<td>19.0000</td>
<td>17.8129</td>
<td>17.2194</td>
<td>16.0323</td>
<td>9.8310</td>
</tr>
<tr>
<td>$w_3$</td>
<td>17.0000</td>
<td>17.1278</td>
<td>17.1917</td>
<td>17.3194</td>
<td>17.8348</td>
</tr>
<tr>
<td>$w_4$</td>
<td>13.0000</td>
<td>15.4643</td>
<td>16.6964</td>
<td>19.1606</td>
<td>23.3934</td>
</tr>
<tr>
<td>$w_5$</td>
<td>11.0000</td>
<td>10.4037</td>
<td>10.1055</td>
<td>9.5091</td>
<td>7.1088</td>
</tr>
<tr>
<td>$w_6$</td>
<td>8.0000</td>
<td>7.5903</td>
<td>7.3854</td>
<td>6.9757</td>
<td>6.7329</td>
</tr>
<tr>
<td>$w_7$</td>
<td>6.0000</td>
<td>4.0946</td>
<td>3.1418</td>
<td>1.2364</td>
<td>0.0000</td>
</tr>
<tr>
<td>$w_8$</td>
<td>4.0000</td>
<td>4.0104</td>
<td>4.0157</td>
<td>4.0261</td>
<td>4.6874</td>
</tr>
</tbody>
</table>

$\mathbf{CI}(w) = \begin{bmatrix} 30.7305 \\ 24.5844 \\ 21.5114 \\ 15.3653 \\ 9.2192 \end{bmatrix}$

$\sigma(w | b) = \begin{bmatrix} 0.00 \\ 15.37 \\ 23.05 \\ 38.42 \\ 72.45 \end{bmatrix}$
In Table 1, we report the optimal solution $w^*$ (expressed in %) of the optimization problem for different values of $R$. We also indicate the carbon intensity of the portfolio (in tCO$_2$/mn) and the tracking error volatility (in bps). For instance, if $R$ is set to 50%, the weights of assets #1, #3, #4 and #8 increase whereas the weights of assets #2, #5, #6 and #7 decrease. The carbon intensity of this portfolio is equal to 15.3653 tCO$_2$/mn. The tracking error volatility is below 40 bps, which is relatively low.
Question (f)

We target scope 1 + 2 + 3. Find the optimal portfolio if $R$ is equal to 20%, 30%, 50% and 70%. Give the value of the tracking error volatility for each optimized portfolio.
In this case, the inequality constraint $Cw \leq D$ is defined by:

$$C = c^T x_{1-3} = \begin{pmatrix} 80.5000 \\ 76.2195 \\ 23.6800 \\ 9.5000 \\ 37.5000 \\ 2.1275 \\ 579.4393 \\ 10.7200 \end{pmatrix}^T$$

and:

$$D = (1 - \mathcal{R}) \times 76.9427$$
We obtain the results given in Table 2.

**Table 2**: Solution of the equity optimization problem (scope $\mathcal{SC}_{1-3}$)

<table>
<thead>
<tr>
<th>$\mathcal{R}$</th>
<th>0%</th>
<th>20%</th>
<th>30%</th>
<th>50%</th>
<th>70%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>22.0000</td>
<td>23.9666</td>
<td>24.9499</td>
<td>26.4870</td>
<td>13.6749</td>
</tr>
<tr>
<td>$w_2$</td>
<td>19.0000</td>
<td>17.4410</td>
<td>16.6615</td>
<td>8.8001</td>
<td>0.0000</td>
</tr>
<tr>
<td>$w_3$</td>
<td>17.0000</td>
<td>17.1988</td>
<td>17.2981</td>
<td>19.4253</td>
<td>24.1464</td>
</tr>
<tr>
<td>$w_4$</td>
<td>13.0000</td>
<td>16.5034</td>
<td>18.2552</td>
<td>25.8926</td>
<td>41.0535</td>
</tr>
<tr>
<td>$w_5$</td>
<td>11.0000</td>
<td>10.2049</td>
<td>9.8073</td>
<td>7.1330</td>
<td>3.5676</td>
</tr>
<tr>
<td>$w_6$</td>
<td>8.0000</td>
<td>7.4169</td>
<td>7.1254</td>
<td>7.0659</td>
<td>8.8851</td>
</tr>
<tr>
<td>$w_7$</td>
<td>6.0000</td>
<td>3.2641</td>
<td>1.8961</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>$w_8$</td>
<td>4.0000</td>
<td>4.0043</td>
<td>4.0065</td>
<td>5.1961</td>
<td>8.6725</td>
</tr>
<tr>
<td>$\mathcal{CI}(w)$</td>
<td>76.9427</td>
<td>61.5541</td>
<td>53.8599</td>
<td>38.4713</td>
<td>23.0828</td>
</tr>
<tr>
<td>$\sigma(w \mid b)$</td>
<td>0.00</td>
<td>21.99</td>
<td>32.99</td>
<td>104.81</td>
<td>414.48</td>
</tr>
</tbody>
</table>
Question (g)

Compare the optimal solutions obtained in Questions 2.(e) and 2.(f).
Figure 1: Impact of the scope on the tracking error volatility
Figure 2: Impact of the scope on the portfolio allocation (in %)
In Figure 1, we report the relationship between the reduction rate $\mathcal{R}$ and the tracking error volatility $\sigma(w | b)$. The choice of the scope has little impact when $\mathcal{R} \leq 45\%$. Then, we notice a high increase when we consider the scope $1 + 2 + 3$. The portfolio’s weights are given in Figure 2. For assets #1 and #3, the behavior is divergent when we compare scopes $1 + 2$ and $1 + 2 + 3$. 

Question 3

We want to manage a bond portfolio with respect to the previous investment universe and reduce the weighted average carbon intensity of the benchmark by the rate $R$. We use the scope $1 + 2 + 3$. In the table below, we report the modified duration $MD_i$ and the duration-times-spread factor $DTS_i$ of each corporate bond $i$:

<table>
<thead>
<tr>
<th>Asset</th>
<th>#1</th>
<th>#2</th>
<th>#3</th>
<th>#4</th>
<th>#5</th>
<th>#6</th>
<th>#7</th>
<th>#8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$MD_i$ (in years)</td>
<td>3.56</td>
<td>7.48</td>
<td>6.54</td>
<td>10.23</td>
<td>2.40</td>
<td>2.30</td>
<td>9.12</td>
<td>7.96</td>
</tr>
<tr>
<td>$DTS_i$ (in bps)</td>
<td>103</td>
<td>155</td>
<td>75</td>
<td>796</td>
<td>89</td>
<td>45</td>
<td>320</td>
<td>245</td>
</tr>
<tr>
<td>Sector</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
Question 3 (Cont’d)

We remind that the active risk can be calculated using three functions. For the active share, we have:

\[
\mathcal{R}_{AS} (w \mid b) = \sigma^2_{AS} (w \mid b) = \sum_{i=1}^{n} (w_i - b_i)^2
\]

We also consider the MD-based tracking error risk:

\[
\mathcal{R}_{MD} (w \mid b) = \sigma^2_{MD} (w \mid b) = \sum_{j=1}^{n_{sector}} \left( \sum_{i \in \text{Sector}_j} (w_i - b_i) \text{MD}_i \right)^2
\]

and the DTS-based tracking error risk:

\[
\mathcal{R}_{DTS} (w \mid b) = \sigma^2_{DTS} (w \mid b) = \sum_{j=1}^{n_{sector}} \left( \sum_{i \in \text{Sector}_j} (w_i - b_i) \text{DTS}_i \right)^2
\]
Finally, we define the synthetic risk measure as a combination of AS, MD and DTS active risks:

\[ \mathcal{R} (w \mid b) = \varphi_{AS} \mathcal{R}_{AS} (w \mid b) + \varphi_{MD} \mathcal{R}_{MD} (w \mid b) + \varphi_{DTS} \mathcal{R}_{DTS} (w \mid b) \]

where \( \varphi_{AS} \geq 0, \varphi_{MD} \geq 0 \) and \( \varphi_{DTS} \geq 0 \) indicate the weight of each risk. In what follows, we use the following numerical values: \( \varphi_{AS} = 100, \varphi_{MD} = 25 \) and \( \varphi_{DTS} = 1 \). The reduction rate \( \mathcal{R} \) of the weighted average carbon intensity is set to 50% for the scope 1 + 2 + 3.
Question (a)

Compute the modified duration $MD(b)$ and the duration-times-spread factor $DTS(b)$ of the benchmark.
We have:

\[
MD(b) = \sum_{i=1}^{n} b_i \, MD_i \\
= 0.22 \times 3.56 + 0.19 \times 7.48 + \ldots + 0.04 \times 7.96 \\
= 5.96 \text{ years}
\]

and:

\[
DTS(b) = \sum_{i=1}^{n} b_i \, DTS_i \\
= 0.22 \times 103 + 0.19 \times 155 + \ldots + 0.04 \times 155 \\
= 210.73 \text{ bps}
\]
Question (b)

Let $w_{ew}$ be the equally-weighted portfolio. Compute $\text{MD}(w_{ew})$, $\text{DTS}(w_{ew})$, $\sigma_{AS}(w_{ew} \mid b)$, $\sigma_{MD}(w_{ew} \mid b)$ and $\sigma_{DTS}(w_{ew} \mid b)$.

$^a$Precise the corresponding unit (years, bps or %) for each metric.
We have:

\[
\begin{align*}
\text{MD} (w_{ew}) &= 6.20 \text{ years} \\
\text{DTS} (w_{ew}) &= 228.50 \text{ bps} \\
\sigma_{AS} (w_{ew} | b) &= 17.03\% \\
\sigma_{MD} (w_{ew} | b) &= 1.00 \text{ years} \\
\sigma_{DTS} (w_{ew} | b) &= 36.19 \text{ bps}
\end{align*}
\]
Question (c)

We consider the following optimization problem:

\[
  w^* = \arg\min \frac{1}{2} \mathcal{R}_{AS}(w \mid b)
\]

\[
\begin{align*}
    \sum_{i=1}^{n} w_i &= 1 \\
    \text{MD}(w) &= \text{MD}(b) \\
    \text{DTS}(w) &= \text{DTS}(b) \\
    \mathcal{CI}(w) &\leq (1 - \mathcal{R}) \mathcal{CI}(b) \\
    0 &\leq w_i \leq 1
  \end{align*}
\]

Give the analytical value of the objective function. Find the optimal portfolio \( w^* \). Compute \( \text{MD}(w^*) \), \( \text{DTS}(w^*) \), \( \sigma_{AS}(w^* \mid b) \), \( \sigma_{MD}(w^* \mid b) \) and \( \sigma_{DTS}(w^* \mid b) \).
We have:

\[ R_{AS} (w | b) = (w_1 - 0.22)^2 + (w_2 - 0.19)^2 + (w_3 - 0.17)^2 + (w_4 - 0.13)^2 + (w_5 - 0.11)^2 + (w_6 - 0.08)^2 + (w_7 - 0.06)^2 + (w_8 - 0.04)^2 \]

The objective function is then:

\[ f (w) = \frac{1}{2} R_{AS} (w | b) \]

The optimal solution is equal to:

\[ w^* = (17.30\%, 17.41\%, 20.95\%, 14.41\%, 10.02\%, 11.09\%, 0\%, 8.81\%) \]

The risk metrics are:

\[
\begin{align*}
\text{MD} (w^*) &= 5.96 \text{ years} \\
\text{DTS} (w^*) &= 210.73 \text{ bps} \\
\sigma_{AS} (w^* | b) &= 10.57\% \\
\sigma_{MD} (w^* | b) &= 0.43 \text{ years} \\
\sigma_{DTS} (w^* | b) &= 15.21 \text{ bps}
\end{align*}
\]
**Question (d)**

We consider the following optimization problem:

\[
\begin{align*}
\nonumber w^* &= \arg \min \frac{\varphi_{AS}}{2} \mathcal{R}_{AS} (w \mid b) + \frac{\varphi_{MD}}{2} \mathcal{R}_{MD} (w \mid b) \\
\text{s.t.} & \begin{cases}
\sum_{i=1}^{n} w_i = 1 \\
\text{DTS}(w) = \text{DTS}(b) \\
\text{CI}(w) \leq (1 - \mathcal{R}) \text{CI}(b) \\
0 \leq w_i \leq 1
\end{cases}
\end{align*}
\]

Give the analytical value of the objective function. Find the optimal portfolio \( w^* \). Compute \( \text{MD}(w^*) \), \( \text{DTS}(w^*) \), \( \sigma_{AS} (w^* \mid b) \), \( \sigma_{MD} (w^* \mid b) \), and \( \sigma_{DTS} (w^* \mid b) \).
We have\(^3\):

\[
R_{MD} (w | b) = \left( \sum_{i=1,3,4,6} (w_i - b_i) MD_i \right)^2 + \left( \sum_{i=2,5,7,8} (w_i - b_i) MD_i \right)^2
\]

\[
= \left( \sum_{i=1,3,4,6} w_i MD_i - \sum_{i=1,3,4,6} b_i MD_i \right)^2 + \left( \sum_{i=2,5,7,8} w_i MD_i - \sum_{i=2,5,7,8} b_i MD_i \right)^2
\]

\[
= (3.56w_1 + 6.54w_3 + 10.23w_4 + 2.30w_6 - 3.4089)^2 + (7.48w_2 + 2.40w_5 + 9.12w_7 + 7.96w_8 - 2.5508)^2
\]

The objective function is then:

\[
f (w) = \frac{\varphi_{AS}}{2} R_{AS} (w | b) + \frac{\varphi_{MD}}{2} R_{MD} (w | b)
\]

\(^3\)We verify that 3.4089 + 2.5508 = 5.9597 years.
The optimal solution is equal to:

\[ w^* = (16.31\%, 18.44\%, 17.70\%, 13.82\%, 11.67\%, 11.18\%, 0\%, 10.88\%) \]

The risk metrics are:

\[
\begin{align*}
\text{MD} (w^*) &= 5.93 \text{ years} \\
\text{DTS} (w^*) &= 210.73 \text{ bps} \\
\sigma_{AS} (w^* | b) &= 11.30\% \\
\sigma_{MD} (w^* | b) &= 0.03 \text{ years} \\
\sigma_{DTS} (w^* | b) &= 3.70 \text{ bps}
\end{align*}
\]
Question (e)

We consider the following optimization problem:

$$w^* = \arg \min \frac{1}{2} \mathcal{R}(w \mid b)$$

$$\text{s.t.} \begin{cases} \sum_{i=1}^{n} w_i = 1 \\ CI(w) \leq (1 - R) CI(b) \\ 0 \leq w_i \leq 1 \end{cases}$$

Give the analytical value of the objective function. Find the optimal portfolio $w^*$. Compute $MD(w^*)$, $DTS(w^*)$, $\sigma_{AS}(w^* \mid b)$, $\sigma_{MD}(w^* \mid b)$ and $\sigma_{DTS}(w^* \mid b)$. 
We have\(^4\):

\[
\mathcal{R}_{\text{DTS}}(w \mid b) = \left( \sum_{i=1,3,4,6} (w_i - b_i) DTS_i \right)^2 + \left( \sum_{i=2,5,7,8} (w_i - b_i) DTS_i \right)^2 \\
= (103w_1 + 75w_3 + 796w_4 + 45w_6 - 142.49)^2 + \\
(155w_2 + 89w_5 + 320w_7 + 245w_8 - 68.24)^2
\]

The objective function is then:

\[
f(w) = \frac{\varphi_{\text{AS}}}{2} \mathcal{R}_{\text{AS}}(w \mid b) + \frac{\varphi_{\text{MD}}}{2} \mathcal{R}_{\text{MD}}(w \mid b) + \frac{\varphi_{\text{DTS}}}{2} \mathcal{R}_{\text{DTS}}(w \mid b)
\]

\(^4\)We verify that 142.49 + 68.24 = 210.73 bps.
The optimal solution is equal to:

\[ w^* = (16.98\%, 17.21\%, 18.26\%, 13.45\%, 12.10\%, 9.46\%, 0\%, 12.55\%) \]

The risk metrics are:

\[
\begin{align*}
\text{MD} (w^*) &= 5.97 \text{ years} \\
\text{DTS} (w^*) &= 210.68 \text{ bps} \\
\sigma_{AS} (w^* | b) &= 11.94\% \\
\sigma_{MD} (w^* | b) &= 0.03 \text{ years} \\
\sigma_{DTS} (w^* | b) &= 0.06 \text{ bps}
\end{align*}
\]
Question (f)

Comment on the results obtained in Questions 3.(c), 3.(d) and 3.(e).
Table 3: Solution of the bond optimization problem (scope $SC_{1-3}$)

<table>
<thead>
<tr>
<th>Problem</th>
<th>Benchmark</th>
<th>3.(c)</th>
<th>3.(d)</th>
<th>3.(e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>22.0000</td>
<td>17.3049</td>
<td>16.3102</td>
<td>16.9797</td>
</tr>
<tr>
<td>$w_2$</td>
<td>19.0000</td>
<td>17.4119</td>
<td>18.4420</td>
<td>17.2101</td>
</tr>
<tr>
<td>$w_3$</td>
<td>17.0000</td>
<td>20.9523</td>
<td>17.6993</td>
<td>18.2582</td>
</tr>
<tr>
<td>$w_4$</td>
<td>13.0000</td>
<td>14.4113</td>
<td>13.8195</td>
<td>13.4494</td>
</tr>
<tr>
<td>$w_5$</td>
<td>11.0000</td>
<td>10.0239</td>
<td>11.6729</td>
<td>12.1008</td>
</tr>
<tr>
<td>$w_6$</td>
<td>8.0000</td>
<td>11.0881</td>
<td>11.1792</td>
<td>9.4553</td>
</tr>
<tr>
<td>$w_7$</td>
<td>6.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>$w_8$</td>
<td>4.0000</td>
<td>8.8075</td>
<td>10.8769</td>
<td>12.5464</td>
</tr>
<tr>
<td>$MD(w)$</td>
<td>5.9597</td>
<td>5.9597</td>
<td>5.9344</td>
<td>5.9683</td>
</tr>
<tr>
<td>$DTS(w)$</td>
<td>210.7300</td>
<td>210.7300</td>
<td>210.7300</td>
<td>210.6791</td>
</tr>
<tr>
<td>$\sigma_{AS}(w</td>
<td>b)$</td>
<td>0.0000</td>
<td>10.5726</td>
<td>11.3004</td>
</tr>
<tr>
<td>$\sigma_{MD}(w</td>
<td>b)$</td>
<td>0.0000</td>
<td>0.4338</td>
<td>0.0254</td>
</tr>
<tr>
<td>$\sigma_{DTS}(w</td>
<td>b)$</td>
<td>0.0000</td>
<td>15.2056</td>
<td>3.7018</td>
</tr>
<tr>
<td>$CI(w)$</td>
<td>76.9427</td>
<td>38.4713</td>
<td>38.4713</td>
<td>38.4713</td>
</tr>
</tbody>
</table>
Question (g)

How to find the previous solution of Question 3.(e) using a QP solver?
The goal is to write the objective function into a quadratic function:

\[
f(w) = \frac{\varphi_{\text{AS}}}{2} \mathcal{R}_{\text{AS}}(w | b) + \frac{\varphi_{\text{MD}}}{2} \mathcal{R}_{\text{MD}}(w | b) + \frac{\varphi_{\text{DTS}}}{2} \mathcal{R}_{\text{DTS}}(w | b)
\]

\[
= \frac{1}{2} w^\top Q(b) w - w^\top R(b) + c(b)
\]

where:

\[
\mathcal{R}_{\text{AS}}(w | b) = (w_1 - 0.22)^2 + (w_2 - 0.19)^2 + (w_3 - 0.17)^2 + (w_4 - 0.13)^2 + (w_5 - 0.11)^2 + (w_6 - 0.08)^2 + (w_7 - 0.06)^2 + (w_8 - 0.04)^2
\]

\[
\mathcal{R}_{\text{MD}}(w | b) = (3.56w_1 + 6.54w_3 + 10.23w_4 + 2.30w_6 - 3.4089)^2 + (7.48w_2 + 2.40w_5 + 9.12w_7 + 7.96w_8 - 2.5508)^2
\]

\[
\mathcal{R}_{\text{DTS}}(w | b) = (103w_1 + 75w_3 + 796w_4 + 45w_6 - 142.49)^2 + (155w_2 + 89w_5 + 320w_7 + 245w_8 - 68.24)^2
\]
We use the analytical approach which is described in Section 11.1.2 on pages 332-339. Moreover, we rearrange the universe such that the first fourth assets belong to the first sector and the last fourth assets belong to the second sector. In this case, we have:

\[
W = \begin{pmatrix}
W_1, W_3, W_4, W_6, & W_2, W_5, W_7, W_8 \\
\underline{\text{Sector}_1} & \underline{\text{Sector}_2}
\end{pmatrix}
\]
The matrix $Q(b)$ is block-diagonal:

$$Q(b) = \begin{pmatrix} Q_1 & 0_{4,4} \\ 0_{4,4} & Q_2 \end{pmatrix}$$

where the matrices $Q_1$ and $Q_2$ are equal to:

$$Q_1 = \begin{pmatrix}
11025.8400 & 8307.0600 & 82898.4700 & 4839.7000 \\
8307.0600 & 6794.2900 & 61372.6050 & 3751.0500 \\
82898.4700 & 61372.6050 & 636332.3225 & 36408.2250 \\
4839.7000 & 3751.0500 & 36408.2250 & 2257.2500 \\
\end{pmatrix}$$

and:

$$Q_2 = \begin{pmatrix}
25523.7600 & 14243.8000 & 51305.4400 & 39463.5200 \\
14243.8000 & 8165.0000 & 29027.2000 & 22282.6000 \\
51305.4400 & 29027.2000 & 104579.3600 & 80214.8800 \\
39463.5200 & 22282.6000 & 80214.8800 & 61709.0400 \\
\end{pmatrix}$$
The vector $R(b)$ is defined as follows:

$$R(b) = \begin{pmatrix}
15001.8621 \\
11261.1051 \\
114306.8662 \\
6616.0617 \\
11073.1996 \\
6237.4080 \\
22424.3824 \\
17230.4092
\end{pmatrix}$$

Finally, the value of $c(b)$ is equal to:

$$c(b) = 12714.3386$$
Using a QP solver, we obtain the following numerical solution:

\[
\begin{pmatrix}
w_1 \\
w_3 \\
w_4 \\
w_6 \\
w_2 \\
w_5 \\
w_7 \\
w_8 \\
\end{pmatrix} = \begin{pmatrix}
16.9796 \\
18.2582 \\
13.4494 \\
9.4553 \\
17.2102 \\
12.1009 \\
0.0000 \\
12.5464 \\
\end{pmatrix} \times 10^{-2}
\]

We observe some small differences (after the fifth digit) because the QP solver is more efficient than a traditional nonlinear solver.
Question 4

We consider a variant of Question 3 and assume that the synthetic risk measure is:

\[ D(w \mid b) = \varphi_{AS} D_{AS}(w \mid b) + \varphi_{MD} D_{MD}(w \mid b) + \varphi_{DTS} D_{DTS}(w \mid b) \]

where:

\[ D_{AS}(w \mid b) = \frac{1}{2} \sum_{i=1}^{n} |w_i - b_i| \]

\[ D_{MD}(w \mid b) = \sum_{j=1}^{n_{sector}} \left| \sum_{i \in \text{Sector}_j} (w_i - b_i) \cdot MD_i \right| \]

\[ D_{DTS}(w \mid b) = \sum_{j=1}^{n_{sector}} \left| \sum_{i \in \text{Sector}_j} (w_i - b_i) \cdot DTS_i \right| \]
Question (a)

Define the corresponding optimization problem when the objective is to minimize the active risk and reduce the carbon intensity of the benchmark by $\mathcal{R}$. 
The optimization problem is:

\[ w^* = \arg \min_{w} D(w \mid b) \]

s.t. \[
\begin{align*}
\mathbf{1}_8^T w &= 1 \\
\mathbf{C}\mathbf{I}^T w &\leq (1 - \mathbf{R}) \mathbf{C}\mathbf{I}(b) \\
\mathbf{0}_8 &\leq w \leq \mathbf{1}_8
\end{align*}
\]
### Question (b)

Give the LP formulation of the optimization problem.
We use the absolute value trick and obtain the following optimization problem:

\[ w^* = \arg \min \frac{1}{2} \varphi_{AS} \sum_{i=1}^{8} \tau_{i,w} + \varphi_{MD} \sum_{j=1}^{2} \tau_{j,MD} + \varphi_{DTS} \sum_{j=1}^{2} \tau_{j,DTS} \]

subject to:

\[
\begin{align*}
1^\top_8 w &= 1 \\
0_8 &\leq w &\leq 1_8 \\
CI^\top w &\leq (1 - R) CI (b) \\
|w_i - b_i| &\leq \tau_{i,w} \\
\sum_{i \in \text{sector}_j} (w_i - b_i) \text{MD}_i &\leq \tau_{j,MD} \\
\sum_{i \in \text{sector}_j} (w_i - b_i) \text{DTS}_i &\leq \tau_{j,DTS} \\
\tau_{i,w} &\geq 0, \tau_{j,MD} \geq 0, \tau_{j,DTS} \geq 0
\end{align*}
\]
We can now formulate this problem as a standard LP problem:

\[ x^* = \arg \min c^\top x \]

subject to:

\[
\begin{align*}
Ax &= B \\
Cx &\leq D \\
x^- &\leq x \leq x^+
\end{align*}
\]

where \( x \) is the \( 20 \times 1 \) vector defined as follows:

\[
x = \begin{pmatrix}
w \\
\tau_w \\
\tau_{MD} \\
\tau_{DTS}
\end{pmatrix}
\]
The $20 \times 1$ vector $c$ is equal to:

$$
c = \begin{pmatrix}
    0_8 \\
    \frac{1}{2} \varphi_{AS} 1_8 \\
    \varphi_{MD} 1_2 \\
    \varphi_{DTS} 1_2
\end{pmatrix}
$$

The equality constraint is defined by $A = \begin{pmatrix} 1_8^T & 0_8^T & 0_2^T & 0_2^T \end{pmatrix}$ and $B = 1$. The bounds are $x^- = 0_{20}$ and $x^+ = \infty \cdot 1_{20}$. 
For the inequality constraint, we have:\(^5\):

\[
C \mathbf{x} \leq D \iff \begin{pmatrix} I_{8} & -I_{8} & 0_{8,2} & 0_{8,2} \\ -I_{8} & -I_{8} & 0_{8,2} & 0_{8,2} \\ C_{MD} & 0_{2,8} & -l_2 & 0_{2,2} \\ -C_{MD} & 0_{2,8} & -l_2 & 0_{2,2} \\ C_{DTS} & 0_{2,8} & 0_{2,2} & -l_2 \\ -C_{DTS} & 0_{2,8} & 0_{2,2} & -l_2 \\ CI^{\top} & 0_{1,8} & 0 & 0 \end{pmatrix} \begin{pmatrix} b \\ -b \\ MD^* \\ -MD^* \\ DTS^* \\ -DTS^* \\ (1 - R) CI(b) \end{pmatrix} \leq \begin{pmatrix} b \\ -b \\ MD^* \\ -MD^* \\ DTS^* \\ -DTS^* \\ (1 - R) CI(b) \end{pmatrix}
\]

where:

\[
C_{MD} = \begin{pmatrix} 3.56 & 0.00 & 6.54 & 10.23 & 0.00 & 2.30 & 0.00 & 0.00 \\ 0.00 & 7.48 & 0.00 & 0.00 & 2.40 & 0.00 & 9.12 & 7.96 \end{pmatrix}
\]

and:

\[
C_{DTS} = \begin{pmatrix} 103 & 0 & 75 & 796 & 0 & 45 & 0 & 0 \\ 0 & 155 & 0 & 0 & 89 & 0 & 320 & 245 \end{pmatrix}
\]

The 2 \times 1 vectors \(MD^*\) and \(DTS^*\) are respectively equal to \((3.4089, 2.5508)\) and \((142.49, 68.24)\).

\(^5C\) is a 25 \times 8 matrix and \(D\) is a 25 \times 1 vector.
Question (c)

Find the optimal portfolio when $R$ is set to 50%. Compare the solution with this obtained in Question 3.(e).
We obtain the following solution:

\[
\begin{align*}
\mathbf{w}^* &= (18.7360, 15.8657, 17.8575, 13.2589, 11, 9.4622, 0, 13.8196) \times 10^{-2} \\
\tau^*_w &= (3.2640, 3.1343, 0.8575, 0.2589, 0, 1.4622, 6, 9.8196) \times 10^{-2} \\
\tau_{MD} &= (0, 0) \\
\tau_{DTS} &= (0, 0)
\end{align*}
\]
### Table 4: Solution of the bond optimization problem (scope $\mathcal{SC}_{1-3}$)

<table>
<thead>
<tr>
<th>Problem</th>
<th>Benchmark</th>
<th>3.(e)</th>
<th>4.(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$</td>
<td>22.0000</td>
<td>16.9796</td>
<td>18.7360</td>
</tr>
<tr>
<td>$w_2$</td>
<td>19.0000</td>
<td>17.2102</td>
<td>15.8657</td>
</tr>
<tr>
<td>$w_3$</td>
<td>17.0000</td>
<td>18.2582</td>
<td>17.8575</td>
</tr>
<tr>
<td>$w_4$</td>
<td>13.0000</td>
<td>13.4494</td>
<td>13.2589</td>
</tr>
<tr>
<td>$w_5$</td>
<td>11.0000</td>
<td>12.1009</td>
<td>11.0000</td>
</tr>
<tr>
<td>$w_6$</td>
<td>8.0000</td>
<td>9.4553</td>
<td>9.4622</td>
</tr>
<tr>
<td>$w_7$</td>
<td>6.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>$w_8$</td>
<td>4.0000</td>
<td>12.5464</td>
<td>13.8196</td>
</tr>
</tbody>
</table>

- $\text{MD}(w)$: 5.9597 5.9683 5.9597
- $\text{DTS}(w)$: 210.7300 210.6791 210.7300
- $\sigma_{\text{AS}}(w \mid b)$: 0.0000 11.9400 12.4837
- $\sigma_{\text{MD}}(w \mid b)$: 0.0000 0.0308 0.0000
- $\sigma_{\text{DTS}}(w \mid b)$: 0.0000 0.0561 0.0000
- $\mathcal{D}_{\text{AS}}(w \mid b)$: 0.0000 25.6203 24.7964
- $\mathcal{D}_{\text{MD}}(w \mid b)$: 0.0000 0.0426 0.0000
- $\mathcal{D}_{\text{DTS}}(w \mid b)$: 0.0000 0.0608 0.0000
- $\mathcal{CI}(w)$: 76.9427 38.4713 38.4713
In Table 4, we compare the two solutions\(^6\). They are very close. In fact, we notice that the LP solution matches perfectly the MD and DTS constraints, but has a higher AS risk \(\sigma_{AS}(w \mid b)\). If we note the two solutions \(w^*(L_1)\) and \(w^*(L_2)\), we have:

\[
\begin{align*}
\mathcal{R}(w^*(L_2) \mid b) &= 1.4524 < \mathcal{R}(w^*(L_1) \mid b) = 1.5584 \\
\mathcal{D}(w^*(L_2) \mid b) &= 13.9366 > \mathcal{D}(w^*(L_1) \mid b) = 12.3982
\end{align*}
\]

There is a trade-off between the \(L_1\)- and \(L_2\)-norm risk measures. This is why we cannot say that one solution dominates the other.

---

\(^6\)The units are the following: \(\%\) for the weights \(w_i\), and the active share metrics \(\sigma_{AS}(w \mid b)\) and \(\mathcal{D}_{AS}(w \mid b)\); years for the modified duration metrics \(MD(w)\), \(\sigma_{MD}(w \mid b)\) and \(\mathcal{D}_{MD}(w \mid b)\); bps for the duration-times-spread metrics \(DTS(w)\), \(\sigma_{DTS}(w \mid b)\) and \(\mathcal{D}_{DTS}(w \mid b)\); \(t\text{CO}_2e/\$\) mn for the carbon intensity \(DTS(w)\).