Course 2023-2024 in Financial Risk Management
Tutorial Sessions

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[1] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

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Course 2023-2024 in Financial Risk Management
Tutorial Session 1

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Agenda

- **Tutorial Session 1: Market Risk**
- Tutorial Session 2: Credit Risk
- Tutorial Session 3: Counterparty Credit Risk and Collateral Risk
- Tutorial Session 4: Operational Risk & Asset Liability Management Risk
- Tutorial Session 5: Copulas, EVT & Stress Testing

Covariance matrix

Exercise

We consider a universe of three stocks A, B and C.

Covariance matrix

Question 1

The covariance matrix of stock returns is:

$$\Sigma = \begin{pmatrix} 4\% & & \\ 3\% & 5\% & \\ 2\% & -1\% & 6\% \end{pmatrix}$$

Covariance matrix

Question 1.a

Calculate the volatility of stock returns.

Covariance matrix

We have:

$$\sigma_A = \sqrt{\Sigma_{1,1}} = \sqrt{4\%} = 20\%$$

For the other stocks, we obtain $\sigma_B = 22.36\%$ and $\sigma_C = 24.49\%$.

Covariance matrix

Question 1.b

Deduce the correlation matrix.

Covariance matrix

The correlation is the covariance divided by the product of volatilities:

$$\rho\left(R_A, R_B\right) = \frac{\Sigma_{1,2}}{\sqrt{\Sigma_{1,1} \times \Sigma_{2,2}}} = \frac{3\%}{20\% \times 22.36\%} = 67.08\%$$

We obtain:

$$\rho = \left(\begin{array}{rrr} 100.00\% & & \\ 67.08\% & 100.00\% & \\ 40.82\% & -18.26\% & 100.00\% \end{array}\right)$$

Covariance matrix

Question 2

We assume that the volatilities are 10%, 20% and 30%. whereas the correlation matrix is equal to:

$$\rho = \begin{pmatrix} 100\% & & \\ 50\% & 100\% & \\ 25\% & 0\% & 100\% \end{pmatrix}$$

Covariance matrix

Question 2.a

Write the covariance matrix.

Covariance matrix

Using the formula $\Sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j$, it follows that:

$$\Sigma = \begin{pmatrix} 1.00\% & & \\ 1.00\% & 4.00\% & \\ 0.75\% & 0.00\% & 9.00\% \end{pmatrix}$$

Covariance matrix

Question 2.b

Calculate the volatility of the portfolio $(50\%, 50\%, 0)$.

Covariance matrix

We deduce that:

$$\begin{aligned}\sigma^2(w) &= 0.5^2 \times 1\% + 0.5^2 \times 4\% + 2 \times 0.5 \times 0.5 \times 1\% \\ &= 1.75\%\end{aligned}$$

and $\sigma(w) = 13.23\%$.

Covariance matrix

Question 2.c

Calculate the volatility of the portfolio $(60\%, -40\%, 0)$. Comment on this result.

Covariance matrix

It follows that:

$$\begin{aligned}\sigma^2(w) &= 0.6^2 \times 1\% + (-0.4)^2 \times 4\% + 2 \times 0.6 \times (-0.4) \times 1\% \\ &= 0.52\%\end{aligned}$$

and $\sigma(w) = 7.21\%$. This long/short portfolio has a lower volatility than the previous long-only portfolio, because part of the risk is hedged by the positive correlation between stocks A and B.

Covariance matrix

Question 2.d

We assume that the portfolio is long \$150 in stock A, long \$500 in stock B and short \$200 in stock C. Find the volatility of this long/short portfolio.

Covariance matrix

We have:

$$\begin{aligned}\sigma^2(w) &= 150^2 \times 1\% + 500^2 \times 4\% + (-200)^2 \times 9\% + \\ &\quad 2 \times 150 \times 500 \times 1\% + \\ &\quad 2 \times 150 \times (-200) \times 0.75\% + \\ &\quad 2 \times 500 \times (-200) \times 0\% \\ &= 14\,875\end{aligned}$$

The volatility is equal to \$121.96 and is measured in USD contrary to the two previous results which were expressed in %.

Covariance matrix

Question 3

We consider that the vector of stock returns follows a one-factor model:

$$R = \beta \mathcal{F} + \varepsilon$$

We assume that $\mathcal{F}$ and ε are independent. We note $\sigma_{\mathcal{F}}^2$ the variance of $\mathcal{F}$ and $D = \mathrm{diag}\left(\tilde{\sigma}_1^2, \tilde{\sigma}_2^2, \tilde{\sigma}_3^2\right)$ the covariance matrix of idiosyncratic risks ε_t. We use the following numerical values: $\sigma_{\mathcal{F}} = 50\%$, $\beta_1 = 0.9$, $\beta_2 = 1.3$, $\beta_3 = 0.1$, $\tilde{\sigma}_1 = 5\%$, $\tilde{\sigma}_2 = 5\%$ and $\tilde{\sigma}_3 = 15\%$.

Covariance matrix

Question 3.a

Calculate the volatility of stock returns.

Covariance matrix

We have:

$$\mathbb{E}[R] = \beta\mathbb{E}[\mathcal{F}] + \mathbb{E}[\varepsilon]$$

and:

$$R - \mathbb{E}[R] = \beta(\mathcal{F} - \mathbb{E}[\mathcal{F}]) + \varepsilon - \mathbb{E}[\varepsilon]$$

It follows that:

$$\begin{aligned}
\operatorname{cov}(R) &= \mathbb{E}\left[(R - \mathbb{E}[R])(R - \mathbb{E}[R])^\top\right] \\
&= \mathbb{E}\left[\beta(\mathcal{F} - \mathbb{E}[\mathcal{F}])(\mathcal{F} - \mathbb{E}[\mathcal{F}])\beta^\top\right] + \\
&\quad 2 \times \mathbb{E}\left[\beta(\mathcal{F} - \mathbb{E}[\mathcal{F}])(\varepsilon - \mathbb{E}[\varepsilon])^\top\right] + \\
&\quad \mathbb{E}\left[(\varepsilon - \mathbb{E}[\varepsilon])(\varepsilon - \mathbb{E}[\varepsilon])^\top\right] \\
&= \sigma_{\mathcal{F}}^2 \beta\beta^\top + D
\end{aligned}$$

Covariance matrix

We deduce that:

$$\sigma(R_i) = \sqrt{\sigma_{\mathcal{F}}^2 \beta_i^2 + \tilde{\sigma}_i^2}$$

We obtain $\sigma(R_A) = 18.68\%$, $\sigma(R_B) = 26.48\%$ and $\sigma(R_C) = 15.13\%$.

Covariance matrix

Question 3.b

Calculate the correlation between stock returns.

Covariance matrix

The correlation between stocks i and j is defined as follows:

$$\rho\left(R_i, R_j\right) = \frac{\sigma_{\mathcal{F}}^2 \beta_i \beta_j}{\sigma\left(R_i\right)\sigma\left(R_j\right)}$$

We obtain:

$$\rho = \left(\begin{array}{rrr} 100.00\% & & \\ 94.62\% & 100.00\% & \\ 12.73\% & 12.98\% & 100.00\% \end{array}\right)$$

Expected shortfall of an equity portfolio

Exercise

We consider an investment universe, which is composed of two stocks A and B. The current prices of the two stocks are respectively equal to \$100 and \$200. Their volatilities are equal to 25% and 20% whereas the cross-correlation is equal to -20%. The portfolio is long of 4 stocks A and 3 stocks B.

Expected shortfall of an equity portfolio

Question 1

Calculate the Gaussian expected shortfall at the 97.5% confidence level for a ten-day time horizon.

Expected shortfall of an equity portfolio

We have:

$$\begin{aligned} \Pi &= 4\left(P_{A,t+h} - P_{A,t}\right) + 3\left(P_{B,t+h} - P_{B,t}\right) \\ &= 4P_{A,t}R_{A,t+h} + 3P_{B,t}R_{B,t+h} \\ &= 400 \times R_{A,t+h} + 600 \times R_{B,t+h} \end{aligned}$$

where $R_{A,t+h}$ and $R_{B,t+h}$ are the stock returns for the period $[t, t+h]$. We deduce that the variance of the P&L is:

$$\begin{aligned} \sigma^2\left(\Pi\right) &= 400 \times (25\%)^2 + 600 \times (20\%)^2 + \\ & \quad 2 \times 400 \times 600 \times (-20\%) \times 25\% \times 20\% \\ &= 19\,600 \end{aligned}$$

Expected shortfall of an equity portfolio

We deduce that $\sigma(\Pi) = \$140$. We know that the one-year expected shortfall is a linear function of the volatility:

$$\begin{aligned} \mathrm{ES}_\alpha(w; \text{one year}) &= \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha} \times \sigma(\Pi) \\ &= 2.34 \times 140 \\ &= \$327.60 \end{aligned}$$

The 10-day expected shortfall is then equal to \$64.25:

$$\begin{aligned} \mathrm{ES}_\alpha(w; \text{ten days}) &= \sqrt{\frac{10}{260}} \times 327.60 \\ &= \$64.25 \end{aligned}$$

Expected shortfall of an equity portfolio

Question 2

The eight worst scenarios of daily stock returns among the last 250 historical scenarios are the following:

s	1	2	3	4	5	6	7	8
R_A	-3%	-4%	-3%	-5%	-6%	$+3\%$	$+1\%$	-1%
R_B	-4%	$+1\%$	-2%	-1%	$+2\%$	-7%	-3%	-2%

Calculate then the historical expected shortfall at the 97.5% confidence level for a ten-day time horizon.

Expected shortfall of an equity portfolio

We have:

$$\Pi_s = 400 \times R_{A,s} + 600 \times R_{B,s}$$

We deduce that the value Π_s of the daily P&L for each scenario s is:

s	1	2	3	4	5	6	7	8
Π_s	-36	-10	-24	-26	-12	-30	-14	-16
$\Pi_{s:250}$	-36	-30	-26	-24	-16	-14	-12	-10

Expected shortfall of an equity portfolio

The value-at-risk at the 97.5% confidence level correspond to the 6.25^{th} order statistic[3]. We deduce that the historical expected shortfall for a one-day time horizon is equal to:

$$\begin{aligned}
\mathrm{ES}_\alpha(w;\text{one day}) &= -\mathbb{E}\left[\Pi \mid \Pi \leq -\mathrm{VaR}_\alpha(\Pi)\right] \\
&= -\frac{1}{6}\sum_{s=1}^{6}\Pi_{s:250} \\
&= \frac{1}{6}(36+30+26+24+16+14) \\
&= 24.33
\end{aligned}$$

By considering the square-root-of-time rule, it follows that the 10-day expected shortfall is equal to \$76.95.

[3] We have $2.5\% \times 250 = 6.25$.

Value-at-risk of a long/short portfolio

Exercise

We consider a long/short portfolio composed of a long (buying) position in asset A and a short (selling) position in asset B. The long exposure is \$2 mn whereas the short exposure is \$1 mn. Using the historical prices of the last 250 trading days of assets A and B, we estimate that the asset volatilities σ_A and σ_B are both equal to 20% per year and that the correlation $\rho_{A,B}$ between asset returns is equal to 50%. In what follows, we ignore the mean effect.

Value-at-risk of a long/short portfolio

We note $S_{A,t}$ (resp. $S_{B,t}$) the price of stock A (resp. B) at time t. The portfolio value is:

$$P_t(w) = w_A S_{A,t} + w_B S_{B,t}$$

where w_A and w_B are the number of stocks A and B. We deduce that the P&L between t and $t+1$ is:

$$\begin{aligned}\Pi(w) &= P_{t+1} - P_t \\ &= w_A \left(S_{A,t+1} - S_{A,t}\right) + w_B \left(S_{B,t+1} - S_{B,t}\right) \\ &= w_A S_{A,t} R_{A,t+1} + w_B S_{B,t} R_{B,t+1} \\ &= W_{A,t} R_{A,t+1} + W_{B,t} R_{B,t+1}\end{aligned}$$

where $R_{A,t+1}$ and $R_{B,t+1}$ are the asset returns of A and B between t and $t+1$, and $W_{A,t}$ and $W_{B,t}$ are the nominal wealth invested in stocks A and B at time t.

Value-at-risk of a long/short portfolio

Question 1

Calculate the Gaussian VaR of the long/short portfolio for a one-day holding period and a 99% confidence level.

Value-at-risk of a long/short portfolio

We have $W_{A,t} = +2$ and $W_{B,t} = -1$. The P&L (expressed in USD million) has the following expression:

$$\Pi(w) = 2R_{A,t+1} - R_{B,t+1}$$

We have $\Pi(w) \sim \mathcal{N}\left(0, \sigma^2(\Pi)\right)$ with:

$$\begin{aligned}
\sigma(\Pi) &= \sqrt{(2\sigma_A)^2 + (-\sigma_B)^2 + 2\rho_{A,B} \times (2\sigma_A) \times (-\sigma_B)} \\
&= \sqrt{4 \times 0.20^2 + (-0.20)^2 - 4 \times 0.5 \times 0.20^2} \\
&= \sqrt{3} \times 20\% \\
&\simeq 34.64\%
\end{aligned}$$

Value-at-risk of a long/short portfolio

The annual volatility of the long/short portfolio is then equal to \$346 400. We consider the square-root-of-time rule to calculate the daily value-at-risk:

$$\begin{aligned} \mathrm{VaR}_{99\%}(w;\text{one day}) &= \frac{1}{\sqrt{260}} \times \Phi^{-1}(0.99) \times \sqrt{3} \times 20\% \\ &= 5.01\% \end{aligned}$$

The 99% value-at-risk is then equal to \$50 056.

Value-at-risk of a long/short portfolio

Question 2

How do you calculate the historical VaR? Using the historical returns of the last 250 trading days, the five worst scenarios of the 250 simulated daily P&L of the portfolio are $-58\,700$, $-56\,850$, $-54\,270$, $-52\,170$ and $-49\,231$. Calculate the historical VaR for a one-day holding period and a 99% confidence level.

Value-at-risk of a long/short portfolio

We use the historical data to calculate the scenarios of asset returns $(R_{A,t+1}, R_{B,t+1})$. We then deduce the empirical distribution of the P&L with the formula $\Pi(w) = 2R_{A,t+1} - R_{B,t+1}$. Finally, we calculate the empirical quantile. With 250 scenarios, the 1% decile is between the second and third worst cases:

$$\begin{aligned} \mathrm{VaR}_{99\%}(w; \text{one day}) &= -\left[-56\,850 + \frac{1}{2}(-54\,270 - (-56\,850))\right] \\ &= 55\,560 \end{aligned}$$

The probability to lose \$55 560 per day is equal to 1%. We notice that the difference between the historical VaR and the Gaussian VaR is equal to 11%.

Value-at-risk of a long/short portfolio

Question 3

We assume that the multiplication factor m_c is 3. Deduce the required capital if the bank uses an internal model based on the Gaussian value-at-risk. Same question when the bank uses the historical VaR. Compare these figures with those calculated with the standardized measurement method.

Value-at-risk of a long/short portfolio

If we assume that the average of the last 60 VaRs is equal to the current VaR, we obtain:

$$\mathcal{K}^{\mathrm{IMA}} = m_c \times \sqrt{10} \times \mathrm{VaR}_{99\%} (w; \text{one day})$$

$\mathcal{K}^{\mathrm{IMA}}$ is respectively equal to \$474 877 and \$527 088 for the Gaussian and historical VaRs. In the case of the standardized measurement method, we have:

$$\begin{aligned} \mathcal{K}^{\mathrm{Specific}} &= 2 \times 8\% + 1 \times 8\% \\ &= \$240\,000 \end{aligned}$$

and:

$$\begin{aligned} \mathcal{K}^{\mathrm{General}} &= |2 - 1| \times 8\% \\ &= \$80\,000 \end{aligned}$$

Value-at-risk of a long/short portfolio

We deduce that:

$$\begin{aligned}\mathcal{K}^{\mathrm{SMM}} &= \mathcal{K}^{\mathrm{Specific}} + \mathcal{K}^{\mathrm{General}} \\ &= \$320\,000\end{aligned}$$

The internal model-based approach does not achieve a reduction of the required capital with respect to the standardized measurement method. Moreover, we have to add the stressed VaR under Basel 2.5 and the IMA regulatory capital is at least multiplied by a factor of 2.

Value-at-risk of a long/short portfolio

Question 4

Show that the Gaussian VaR is multiplied by a factor equal to $\sqrt{7/3}$ if the correlation $\rho_{A,B}$ is equal to -50%. How do you explain this result?

Value-at-risk of a long/short portfolio

If $\rho_{A,B} = -0.50$, the volatility of the P&L becomes:

$$\begin{aligned}\sigma(\Pi) &= \sqrt{4 \times 0.20^2 + (-0.20)^2 - 4 \times (-0.5) \times 0.20^2} \\ &= \sqrt{7} \times 20\%\end{aligned}$$

We deduce that:

$$\frac{\mathrm{VaR}_\alpha(\rho_{A,B} = -50\%)}{\mathrm{VaR}_\alpha(\rho_{A,B} = +50\%)} = \frac{\sigma(\Pi; \rho_{A,B} = -50\%)}{\sigma(\Pi; \rho_{A,B} = +50\%)} = \sqrt{\frac{7}{3}} = 1.53$$

The value-at-risk increases because the hedging effect of the positive correlation vanishes. With a negative correlation, a long/short portfolio becomes more risky than a long-only portfolio.

Value-at-risk of a long/short portfolio

Question 5

The portfolio manager sells a call option on the stock A. The delta of the option is equal to 50%. What does the Gaussian value-at-risk of the long/short portfolio become if the nominal of the option is equal to \$2 mn? Same question when the nominal of the option is equal to \$4 mn. How do you explain this result?

Value-at-risk of a long/short portfolio

The P&L formula becomes:

$$\Pi(w) = W_{A,t}R_{A,t+1} + W_{B,t}R_{B,t+1} - \left(\mathcal{C}_{A,t+1} - \mathcal{C}_{A,t}\right)$$

where $\mathcal{C}_{A,t}$ is the call option price. We have:

$$\mathcal{C}_{A,t+1} - \mathcal{C}_{A,t} \simeq \boldsymbol{\Delta}_t \left(S_{A,t+1} - S_{A,t}\right)$$

where $\boldsymbol{\Delta}_t$ is the delta of the option. If the nominal of the option is USD 2 million, we obtain:

$$\begin{aligned}\Pi(w) &= 2R_A - R_B - 2 \times 0.5 \times R_A \\ &= R_A - R_B \qquad (1)\end{aligned}$$

and:

$$\begin{aligned}\sigma(\Pi) &= \sqrt{0.20^2 + (-0.20)^2 - 2 \times 0.5 \times 0.20^2} \\ &= 20\%\end{aligned}$$

Value-at-risk of a long/short portfolio

If the nominal of the option is USD 4 million, we obtain:

$$\begin{aligned} \Pi(w) &= 2R_A - R_B - 4 \times 0.5 \times R_A \\ &= -R_B \end{aligned} \qquad (2)$$

and $\sigma(\Pi) = 20\%$. In both cases, we have:

$$\begin{aligned} \mathrm{VaR}_{99\%}(w; \text{one day}) &= \frac{1}{\sqrt{260}} \times \Phi^{-1}(0.99) \times 20\% \\ &= \$28\,900 \end{aligned}$$

The value-at-risk of the long/short portfolio (1) is then equal to the value-at-risk of the short portfolio (2) because of two effects: the absolute exposure of the long/short portfolio is higher than the absolute exposure of the short portfolio, but a part of the risk of the long/short portfolio is hedged by the positive correlation between the two stocks.

Value-at-risk of a long/short portfolio

Question 6

The portfolio manager replaces the short position on the stock B by selling a call option on the stock B. The delta of the option is equal to 50%. Show that the Gaussian value-at-risk is minimum when the nominal is equal to four times the correlation $\rho_{A,B}$. Deduce then an expression of the lowest Gaussian VaR. Comment on these results.

Value-at-risk of a long/short portfolio

We have:

$$\Pi(w) = W_{A,t} R_{A,t+1} - (\mathcal{C}_{B,t+1} - \mathcal{C}_{B,t})$$

and:

$$\mathcal{C}_{B,t+1} - \mathcal{C}_{B,t} \simeq \boldsymbol{\Delta}_t (S_{B,t+1} - S_{B,t})$$

where $\boldsymbol{\Delta}_t$ is the delta of the option. We note x the nominal of the option expressed in USD million. We obtain:

$$\begin{aligned}
\Pi(w) &= 2R_A - x \times \boldsymbol{\Delta}_t \times R_B \\
&= 2R_A - \frac{x}{2} R_B
\end{aligned}$$

We have[4]:

$$\begin{aligned}
\sigma^2(\Pi) &= 4\sigma_A^2 + \frac{x^2\sigma_B^2}{4} + 2\rho_{A,B} \times (2\sigma_A) \times \left(-\frac{x}{2}\sigma_B\right) \\
&= \frac{\sigma_A^2}{4}\left(x^2 - 8\rho_{A,B}x + 16\right)
\end{aligned}$$

[4] Because $\sigma_A = \sigma_B = 20\%$.

Value-at-risk of a long/short portfolio

Minimizing the Gaussian value-at-risk is equivalent to minimizing the variance of the P&L. We deduce that the first-order condition is:

$$\frac{\partial \sigma^2(\Pi)}{\partial x} = \frac{\sigma_A^2}{4}\left(2x - 8\rho_{A,B}\right) = 0$$

We deduce that the minimum VaR is reached when the nominal of the option is $x = 4\rho_{A,B}$. We finally obtain:

$$\begin{aligned}\sigma(\Pi) &= \frac{\sigma_A}{2}\sqrt{16\rho_{A,B}^2 - 32\rho_{A,B}^2 + 16} \\ &= 2\sigma_A\sqrt{1-\rho_{A,B}^2}\end{aligned}$$

and:

$$\begin{aligned}\mathrm{VaR}_{99\%}(w;\text{one day}) &= \frac{1}{\sqrt{260}} \times 2.33 \times 2 \times 20\% \times \sqrt{1-\rho_{A,B}^2} \\ &\simeq 5.78\% \times \sqrt{1-\rho_{A,B}^2}\end{aligned}$$

Value-at-risk of a long/short portfolio

If $\rho_{A,B}$ is negative (resp. positive), the exposure x is negative meaning that we have to buy (resp. to sell) a call option on stock B in order to hedge a part of the risk related to stock A. If $\rho_{A,B}$ is equal to zero, the exposure x is equal to zero because a position on stock B adds systematically a supplementary risk to the portfolio.

Risk management of exotic options

Exercise

Let us consider a short position on an exotic option, whose its current value $\mathcal{C}_t$ is equal to \$6.78. We assume that the price S_t of the underlying asset is \$100 and the implied volatility Σ_t is equal to 20%.

Risk management of exotic options

Let $\mathcal{C}_t$ be the option price at time t. The P&L of the trader between t and $t+1$ is:

$$\Pi = -\left(\mathcal{C}_{t+1} - \mathcal{C}_t\right)$$

The formulation of the exercise suggests that there are two main risk factors: the price of the underlying asset S_t and the implied volatility Σ_t. We then obtain:

$$\Pi = C_t\left(S_t, \Sigma_t\right) - C_{t+1}\left(S_{t+1}, \Sigma_{t+1}\right)$$

Risk management of exotic options

Question 1

At time $t+1$, the value of the underlying asset is \$97 and the implied volatility remains constant. We find that the P&L of the trader between t and $t+1$ is equal to \$1.37. Can we explain the P&L by the sensitivities knowing that the estimates of delta $\boldsymbol{\Delta}_t$, gamma $\boldsymbol{\Gamma}_t$ and vega[a] $\boldsymbol{\upsilon}_t$ are respectively equal to 49%, 2% and 40%?

[a] measured in volatility points.

Risk management of exotic options

We have:

$$
\begin{aligned}
\Pi &= C_t\left(S_t, \Sigma_t\right) - C_{t+1}\left(S_{t+1}, \Sigma_{t+1}\right) \\
&\approx -\boldsymbol{\Delta}_t\left(S_{t+1} - S_t\right) - \frac{1}{2}\boldsymbol{\Gamma}_t\left(S_{t+1} - S_t\right)^2 - \boldsymbol{\upsilon}_t\left(\Sigma_{t+1} - \Sigma_t\right)
\end{aligned}
$$

Using the numerical values of $\boldsymbol{\Delta}_t$, $\boldsymbol{\Gamma}_t$ and $\boldsymbol{\upsilon}_t$, we obtain:

$$
\begin{aligned}
\Pi &\approx -0.49 \times (97 - 100) - \frac{1}{2} \times 0.02 \times (97 - 100)^2 \\
&= 1.47 - 0.09 \\
&= 1.38
\end{aligned}
$$

We explain the P&L by the sensitivities very well.

Risk management of exotic options

Question 2

At time $t+2$, the price of the underlying asset is \$97 while the implied volatility increases from 20% to 22%. The value of the option $\mathcal{C}_{t+2}$ is now equal to \$6.17. Can we explain the P&L by the sensitivities knowing that the estimates of delta $\boldsymbol{\Delta}_{t+1}$, gamma $\boldsymbol{\Gamma}_{t+1}$ and vega $\boldsymbol{\upsilon}_{t+1}$ are respectively equal to 43%, 2% and 38%?

Risk management of exotic options

We have:

$$\begin{aligned}
\Pi &= C_{t+1}\left(S_{t+1},\Sigma_{t+1}\right) - C_{t+2}\left(S_{t+2},\Sigma_{t+2}\right) \\
&\approx -\boldsymbol{\Delta}_{t+1}\left(S_{t+2}-S_{t+1}\right) - \frac{1}{2}\boldsymbol{\Gamma}_{t+1}\left(S_{t+2}-S_{t+1}\right)^2 - \\
&\quad \boldsymbol{\upsilon}_{t+1}\left(\Sigma_{t+2}-\Sigma_{t+1}\right)
\end{aligned}$$

Using the numerical values of $\boldsymbol{\Delta}_{t+1}$, $\boldsymbol{\Gamma}_{t+1}$ and $\boldsymbol{\upsilon}_{t+1}$, we obtain:

$$\begin{aligned}
\Pi &\approx -0.49 \times 0 - \frac{1}{2} \times 0.02 \times 0^2 - 0.38 \times (22 - 20) \\
&= -0.76
\end{aligned}$$

Risk management of exotic options

To compare this value with the true P&L, we have to calculate $\mathcal{C}_{t+1}$:

$$\begin{aligned}\mathcal{C}_{t+1} &= \mathcal{C}_t - (\mathcal{C}_t - \mathcal{C}_{t+1}) \\ &= 6.78 - 1.37 \\ &= 5.41\end{aligned}$$

We deduce that:

$$\begin{aligned}\Pi &= \mathcal{C}_{t+1} - \mathcal{C}_{t+2} \\ &= 5.41 - 6.17 \\ &= -0.76\end{aligned}$$

Again, the sensitivities explain the P&L very well.

Risk management of exotic options

Question 3

At time $t+3$, the price of the underlying asset is \$95 and the value of the implied volatility is 19%. We find that the P&L of the trader between $t+2$ and $t+3$ is equal to \$0.58. Can we explain the P&L by the sensitivities knowing that the estimates of delta $\boldsymbol{\Delta}_{t+2}$, gamma $\boldsymbol{\Gamma}_{t+2}$ and vega $\boldsymbol{\upsilon}_{t+2}$ are respectively equal to 44%, 1.8% and 38%.

Risk management of exotic options

We have:

$$\begin{aligned}\Pi &= C_{t+2}\left(S_{t+2},\Sigma_{t+2}\right) - C_{t+3}\left(S_{t+3},\Sigma_{t+3}\right) \\ &\approx -\boldsymbol{\Delta}_{t+2}\left(S_{t+3}-S_{t+2}\right) - \frac{1}{2}\boldsymbol{\Gamma}_{t+2}\left(S_{t+3}-S_{t+2}\right)^2 - \\ &\quad \boldsymbol{\upsilon}_{t+2}\left(\Sigma_{t+3}-\Sigma_{t+2}\right)\end{aligned}$$

Using the numerical values of $\boldsymbol{\Delta}_{t+2}$, $\boldsymbol{\Gamma}_{t+2}$ and $\boldsymbol{\upsilon}_{t+2}$, we obtain:

$$\begin{aligned}\Pi &\approx -0.44\times(95-97) - \frac{1}{2}\times 0.018\times(95-97)^2 - \\ &\quad 0.38\times(19-22) \\ &= 0.88 - 0.036 + 1.14 \\ &= 1.984\end{aligned}$$

The P&L approximated by the Greek coefficients largely overestimate the true value of the P&L.

Risk management of exotic options

Question 4

What can we conclude in terms of model risk?

Risk management of exotic options

We notice that the approximation using the Greek coefficients works very well when one risk factor remains constant:

- Between t and $t+1$, the price of the underlying asset changes, but not the implied volatility;
- Between $t+1$ and $t+2$, this is the implied volatility that changes whereas the price of the underlying asset is constant.

Therefore, we can assume that the bad approximation between $t+2$ and $t+3$ is due to the cross effect between S_t and Σ_t. In terms of model risk, the P&L is then exposed to the vanna risk, meaning that the Black-Scholes model is not appropriate to price and hedge this exotic option.

Course 2023-2024 in Financial Risk Management
Tutorial Session 2

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[5]The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Agenda

- Tutorial Session 1: Market Risk
- **Tutorial Session 2: Credit Risk**
- Tutorial Session 3: Counterparty Credit Risk and Collateral Risk
- Tutorial Session 4: Operational Risk & Asset Liability Management Risk
- Tutorial Session 5: Copulas, EVT & Stress Testing

Single and multi-name credit default swaps

Question 1

We assume that the default time $\boldsymbol{\tau}$ follows an exponential distribution with parameter λ. Write the cumulative distribution function $\mathbf{F}$, the survival function $\mathbf{S}$ and the density function f of the random variable $\boldsymbol{\tau}$. How do we simulate this default time?

Single and multi-name credit default swaps

We have $\mathbf{F}(t) = 1 - e^{-\lambda t}$, $\mathbf{S}(t) = e^{-\lambda t}$ and $f(t) = \lambda e^{-\lambda t}$. We know that $\mathbf{S}(\boldsymbol{\tau}) \sim \mathcal{U}_{[0,1]}$. Indeed, we have:

$$\begin{aligned}
\Pr\{U \leq u\} &= \Pr\{\mathbf{S}(\boldsymbol{\tau}) \leq u\} \\
&= \Pr\left\{\boldsymbol{\tau} \geq \mathbf{S}^{-1}(u)\right\} \\
&= \mathbf{S}\left(\mathbf{S}^{-1}(u)\right) \\
&= u
\end{aligned}$$

It follows that $\boldsymbol{\tau} = \mathbf{S}^{-1}(U)$ with $U \sim \mathcal{U}_{[0,1]}$. Let u be a uniform random variate. Simulating τ is then equivalent to transform u into t:

$$t = -\frac{1}{\lambda} \ln u$$

Single and multi-name credit default swaps

Question 2

We consider a CDS 3M with two-year maturity and \$1 mn notional principal. The recovery rate $\mathcal{R}$ is equal to 40% whereas the spread $\mathcal{s}$ is equal to 150 bps at the inception date. We assume that the protection leg is paid at the default time.

Single and multi-name credit default swaps

Question 2.a

Give the cash flow chart. What is the P&L of the protection seller A if the reference entity does not default? What is the PnL of the protection buyer B if the reference entity defaults in one year and two months?

Single and multi-name credit default swaps

The premium leg is paid quarterly. The coupon payment is then equal to:

$$\begin{aligned}\mathcal{PL}(t_m) &= \Delta t_m \times \mathcal{s} \times N \\ &= \frac{1}{4} \times 150 \times 10^{-4} \times 10^{6} \\ &= \$3\,750\end{aligned}$$

In case of default, the default leg paid by protection seller is equal to:

$$\begin{aligned}\mathcal{DL} &= (1 - \mathcal{R}) \times N \\ &= (1 - 40\%) \times 10^{6} \\ &= \$600\,000\end{aligned}$$

Single and multi-name credit default swaps

The corresponding cash flow chart is given in Figure 1. If the reference entity does not default, the P&L of the protection seller is the sum of premium interests:

$$\Pi^{\text{seller}} = 8 \times 3\,750 = \$30\,000$$

If the reference entity defaults in one year and two months, the P&L of the protection buyer is[6]:

$$\begin{aligned}\Pi^{\text{buyer}} &= (1 - \mathcal{R}) \times N - \sum_{t_m < \boldsymbol{\tau}} \Delta t_m \times \mathcal{s} \times N \\ &= (1 - 40\%) \times 10^6 - \left(4 + \frac{2}{3}\right) \times 3\,750 \\ &= \$582\,500\end{aligned}$$

[6] We include the accrued premium.

Single and multi-name credit default swaps

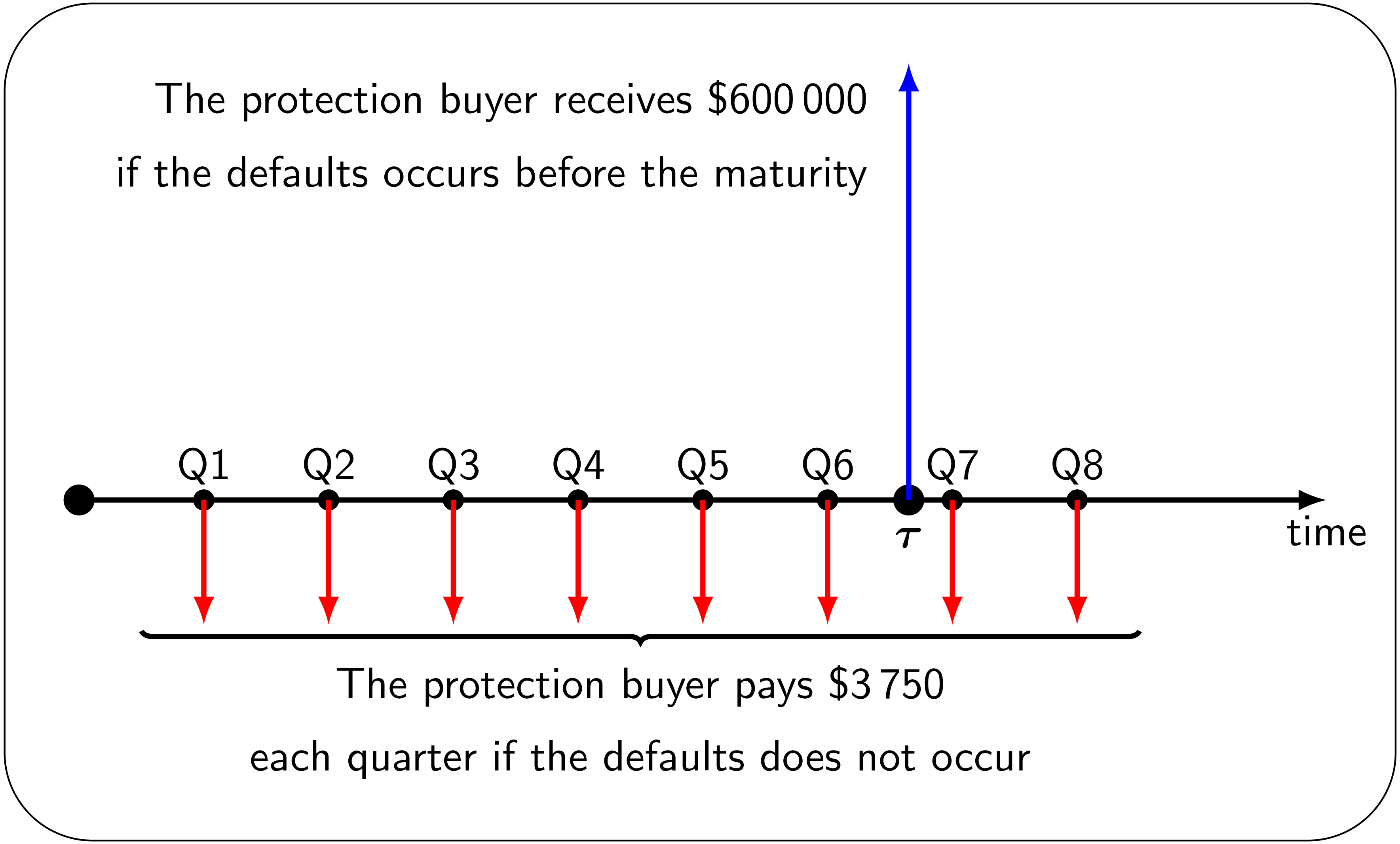

Figure: Cash flow chart of the CDS contract

Single and multi-name credit default swaps

Question 2.b

What is the relationship between $\mathcal{s}$, $\mathcal{R}$ and λ? What is the implied one-year default probability at the inception date?

Single and multi-name credit default swaps

Using the credit triangle relationship, we have:

$$\mathcal{s} \simeq (1 - \mathcal{R}) \times \lambda$$

We deduce that[7]:

$$\begin{aligned} \mathrm{PD} &\simeq \lambda \\ &\simeq \frac{\mathcal{s}}{1 - \mathcal{R}} \\ &= \frac{150 \times 10^{-4}}{1 - 40\%} \\ &= 2.50\% \end{aligned}$$

[7] We recall that the one-year default probability is approximately equal to λ:

$$\begin{aligned} \mathrm{PD} &= 1 - \mathbf{S}(1) \\ &= 1 - e^{-\lambda} \\ &\simeq 1 - (1 - \lambda) \\ &\simeq \lambda \end{aligned}$$

Single and multi-name credit default swaps

Question 2.c

Seven months later, the CDS spread has increased and is equal to 450 bps. Estimate the new default probability. The protection buyer B decides to realize his P&L. For that, he reassigns the CDS contract to the counterparty C. Explain the offsetting mechanism if the risky PV01 is equal to 1.189.

Single and multi-name credit default swaps

We denote by s' the new CDS spread. The default probability becomes:

$$\begin{aligned}\mathrm{PD} &= \frac{s'}{1-\mathcal{R}} \\ &= \frac{450 \times 10^{-4}}{1-40\%} \\ &= 7.50\%\end{aligned}$$

The protection buyer is short credit and benefits from the increase of the default probability. His mark-to-market is therefore equal to:

$$\begin{aligned}\Pi^{\mathrm{buyer}} &= N \times (s' - s) \times \mathrm{RPV}_{01} \\ &= 10^{6} \times (450 - 150) \times 10^{-4} \times 1.189 \\ &= \$35\,671\end{aligned}$$

The offsetting mechanism is then the following: the protection buyer B transfers the agreement to C, who becomes the new protection buyer; C continues to pay a premium of 150 bps to the protection seller A; in return, C pays a cash adjustment of \$35 671 to B.

Single and multi-name credit default swaps

Question 3

We consider the following CDS spread curves for three reference entities:

Maturity	#1	#2	#3
6M	130 bps	1 280 bps	30 bps
1Y	135 bps	970 bps	35 bps
3Y	140 bps	750 bps	50 bps
5Y	150 bps	600 bps	80 bps

Single and multi-name credit default swaps

Question 3.a

Define the notion of credit curve. Comment the previous spread curves.

Single and multi-name credit default swaps

For a given date t, the credit curve is the relationship between the maturity T and the spread $\mathcal{S}_t(T)$. The credit curve of the reference entity #1 is almost flat. For the entity #2, the spread is very high in the short-term, meaning that there is a significative probability that the entity defaults. However, if the entity survive, the market anticipates that it will improve its financial position in the long-run. This explains that the credit curve #2 is decreasing. For reference entity #3, we obtain opposite conclusions. The company is actually very strong, but there are some uncertainties in the future[8]. The credit curve is then increasing.

[8] An example is a company whose has a monopoly because of a strong technology, but faces a hard competition because technology is evolving fast in its domain (e.g. Blackberry at the end of 2000s).

Single and multi-name credit default swaps

Question 3.b

Using the Merton Model, we estimate that the one-year default probability is equal to 2.5% for #1, 5% for #2 and 2% for #3 at a five-year horizon time. Which arbitrage position could we consider about the reference entity #2?

Single and multi-name credit default swaps

If we consider a standard recovery rate (40%), the implied default probability is 2.50% for #1, 10% for #2 and 1.33% for #3. We can consider a short credit position in #2. In this case, we sell the 5Y protection on #2 because the model tells us that the market default probability is over-estimated. In place of this directional bet, we could consider a relative value strategy: selling the 5Y protection on #2 and buying the 5Y protection on #3.

Single and multi-name credit default swaps

Question 4

We consider a basket of n single-name CDS.

Single and multi-name credit default swaps

Question 4.a

What is a first-to-default (FtD), a second-to-default (StD) and a last-to-default (LtD)?

Single and multi-name credit default swaps

Let $\boldsymbol{\tau}_{k:n}$ be the k^{th} default among the basket. FtD, StD and LtD are three CDS products, whose credit event is related to the default times $\boldsymbol{\tau}_{1:n}$, $\boldsymbol{\tau}_{2:n}$ and $\boldsymbol{\tau}_{n:n}$.

Single and multi-name credit default swaps

Question 4.b

Define the notion of default correlation˙What is its impact on three previous spreads?

Single and multi-name credit default swaps

The default correlation ρ measures the dependence between two default times $\boldsymbol{\tau}_i$ and $\boldsymbol{\tau}_j$. The spread of the FtD (resp. LtD) is a decreasing (resp. increasing) function with respect to ρ.

Single and multi-name credit default swaps

Question 4.c

We assume that $n = 3$. Show the following relationship:

$$s_1^{\mathrm{CDS}} + s_2^{\mathrm{CDS}} + s_3^{\mathrm{CDS}} = s^{\mathrm{FtD}} + s^{\mathrm{StD}} + s^{\mathrm{LtD}}$$

where s_i^{CDS} is the CDS spread of the i^{th} reference entity.

Single and multi-name credit default swaps

To fully hedge the credit portfolio of the 3 entities, we can buy the 3 CDS. Another solution is to buy the FtD plus the StD and the LtD (or the third-to-default). Because these two hedging strategies are equivalent, we deduce that:

$$\mathcal{s}_1^{\mathrm{CDS}} + \mathcal{s}_2^{\mathrm{CDS}} + \mathcal{s}_3^{\mathrm{CDS}} = \mathcal{s}^{\mathrm{FtD}} + \mathcal{s}^{\mathrm{StD}} + \mathcal{s}^{\mathrm{LtD}}$$

Single and multi-name credit default swaps

Question 4.d

Many professionals and academics believe that the subprime crisis is due to the use of the Normal copula. Using the results of the previous question, what could you conclude?

Single and multi-name credit default swaps

We notice that the default correlation does not affect the value of the CDS basket, but only the price distribution between FtD, StD and LtD. We obtain a similar result for CDO[9]. In the case of the subprime crisis, all the CDO tranches have suffered, meaning that the price of the underlying basket has dropped. The reasons were the underestimation of default probabilities.

[9] The junior, mezzanine and senior tranches can be viewed as FtD, StD and LtD.

Risk contribution in the Basel II model

Question 1

We note L the portfolio loss of n credit and w_i the exposure at default of the i^{th} credit. We have:

$$L(w) = w^\top \varepsilon = \sum_{i=1}^{n} w_i \times \varepsilon_i \qquad (3)$$

where ε_i is the unit loss of the i^{th} credit. Let $\mathbf{F}$ be the cumulative distribution function of $L(w)$.

Risk contribution in the Basel II model

Question 1.a

We assume that $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \sim \mathcal{N}(\mathbf{0}, \Sigma)$. Compute the value-at-risk $\mathrm{VaR}_\alpha(w)$ of the portfolio when the confidence level is equal to α.

Risk contribution in the Basel II model

The portfolio loss L follows a Gaussian probability distribution:

$$L(w) \sim \mathcal{N}\left(0, \sqrt{w^\top \Sigma w}\right)$$

We deduce that:

$$\mathrm{VaR}_\alpha(w) = \Phi^{-1}(\alpha)\sqrt{w^\top \Sigma w}$$

Risk contribution in the Basel II model

Question 1.b

Deduce the marginal value-at-risk of the i^{th} credit. Define then the risk contribution $\mathcal{RC}_i$ of the i^{th} credit.

Risk contribution in the Basel II model

We have:

$$\begin{aligned}\frac{\partial \operatorname{VaR}_\alpha(w)}{\partial w} &= \frac{\partial}{\partial w}\left(\Phi^{-1}(\alpha)\left(w^\top \Sigma w\right)^{\frac{1}{2}}\right) \\ &= \Phi^{-1}(\alpha)\frac{1}{2}\left(w^\top \Sigma w\right)^{-\frac{1}{2}}(2\Sigma w) \\ &= \Phi^{-1}(\alpha)\frac{\Sigma w}{\sqrt{w^\top \Sigma w}}\end{aligned}$$

The marginal value-at-risk of the i^{th} credit is then:

$$\mathcal{MR}_i = \frac{\partial \operatorname{VaR}_\alpha(w)}{\partial w_i} = \Phi^{-1}(\alpha)\frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}}$$

The risk contribution of the i^{th} credit is the product of the exposure by the marginal risk:

$$\begin{aligned}\mathcal{RC}_i &= w_i \times \mathcal{MR}_i \\ &= \Phi^{-1}(\alpha)\frac{w_i \times (\Sigma w)_i}{\sqrt{x^\top \Sigma x}}\end{aligned}$$

Risk contribution in the Basel II model

Question 1.c

Check that the marginal value-at-risk is equal to:

$$\frac{\partial \ \mathrm{VaR}_{\alpha}\left(w\right)}{\partial \ w_i} = \mathbb{E}\left[\varepsilon_i \mid L\left(w\right) = \mathbf{F}^{-1}\left(\alpha\right)\right]$$

Comment on this result.

Risk contribution in the Basel II model

By construction, the random vector $(\varepsilon, L(w))$ is Gaussian with:

$$\begin{pmatrix} \varepsilon \\ L(w) \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & \Sigma w \\ w^\top \Sigma & w^\top \Sigma w \end{pmatrix}\right)$$

We deduce that the conditional distribution function of ε given that $L(w) = \ell$ is Gaussian and we have:

$$\mathbb{E}[\varepsilon \mid L(w) = \ell] = \mathbf{0} + \Sigma w \left(w^\top \Sigma w\right)^{-1} (\ell - 0)$$

We finally obtain:

$$\begin{aligned} \mathbb{E}\left[\varepsilon \mid L(w) = \mathbf{F}^{-1}(\alpha)\right] &= \Sigma w \left(w^\top \Sigma w\right)^{-1} \Phi^{-1}(\alpha) \sqrt{w^\top \Sigma w} \\ &= \Phi^{-1}(\alpha) \frac{\Sigma w}{\sqrt{w^\top \Sigma w}} \\ &= \frac{\partial \operatorname{VaR}_\alpha(w)}{\partial w} \end{aligned}$$

The marginal VaR of the i^{th} credit is then equal to the conditional mean of the individual loss ε_i given that the portfolio loss is exactly equal to the value at risk

Risk contribution in the Basel II model

Question 2

We consider the Basel II model of credit risk and the value-at-risk risk measure. The expression of the portfolio loss is given by:

$$L = \sum_{i=1}^{n} \mathrm{EAD}_i \times \mathrm{LGD}_i \times \mathbb{1}\left\{\boldsymbol{\tau}_i < M_i\right\} \tag{4}$$

Risk contribution in the Basel II model

Question 2.a

Define the different parameters EAD_i, LGD_i, $\boldsymbol{\tau}_i$ and M_i. Show that Model (4) can be written as Model (3) by identifying w_i and ε_i.

Risk contribution in the Basel II model

EAD_i is the exposure at default, LGD_i is the loss given default, $\boldsymbol{\tau}_i$ is the default time and T_i is the maturity of the credit i. We have:

$$\left\{ \begin{array}{l} w_i = \mathrm{EAD}_i \\ \varepsilon_i = \mathrm{LGD}_i \times \mathbb{1}\left\{\boldsymbol{\tau}_i < T_i\right\} \end{array} \right.$$

The exposure at default is not random, which is not the case of the loss given default.

Risk contribution in the Basel II model

Question 2.b

What are the necessary assumptions $(\mathcal{H})$ to obtain this result:

$$\mathbb{E}\left[\varepsilon_i \mid L = \mathbf{F}^{-1}(\alpha)\right] = \mathbb{E}\left[\mathrm{LGD}_i\right] \times \mathbb{E}\left[D_i \mid L = \mathbf{F}^{-1}(\alpha)\right]$$

with $D_i = \mathbb{1}\left\{\boldsymbol{\tau}_i < M_i\right\}$.

Risk contribution in the Basel II model

We have to make the following assumptions:

(i) the loss given default LGD_i is independent from the default time $\boldsymbol{\tau}_i$;

(ii) the portfolio is infinitely fine-grained meaning that there is no exposure concentration:

$$\frac{\mathrm{EAD}_i}{\sum_{i=1}^{n} \mathrm{EAD}_i} \simeq 0$$

(iii) the default times depend on a common risk factor X and the relationship is monotonic (increasing or decreasing).

In this case, we have:

$$\mathbb{E}\left[\varepsilon_i \mid L = \mathbf{F}^{-1}(\alpha)\right] = \mathbb{E}\left[\mathrm{LGD}_i\right] \times \mathbb{E}\left[D_i \mid L = \mathbf{F}^{-1}(\alpha)\right]$$

with $D_i = \mathbb{1}\{\boldsymbol{\tau}_i < T_i\}$.

Risk contribution in the Basel II model

Question 2.c

Deduce the risk contribution $\mathcal{RC}_i$ of the i^{th} credit and the value-at-risk of the credit portfolio.

Risk contribution in the Basel II model

It follows that:

$$\begin{aligned}\mathcal{RC}_i &= w_i \times \mathcal{MR}_i \\ &= \mathrm{EAD}_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times \mathbb{E}\left[D_i \mid L = \mathbf{F}^{-1}(\alpha)\right]\end{aligned}$$

The expression of the value-at-risk is then:

$$\begin{aligned}\mathrm{VaR}_\alpha(w) &= \sum_{i=1}^{n} \mathcal{RC}_i \\ &= \sum_{i=1}^{n} \mathrm{EAD}_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times \mathbb{E}\left[D_i \mid L = \mathbf{F}^{-1}(\alpha)\right]\end{aligned}$$

Risk contribution in the Basel II model

Question 2.d

We assume that the credit i defaults before the maturity M_i if a latent variable Z_i goes below a barrier B_i:

$$\boldsymbol{\tau}_i \leq M_i \Leftrightarrow Z_i \leq B_i$$

We consider that $Z_i = \sqrt{\rho} X + \sqrt{1-\rho} \varepsilon_i$ where Z_i, X and ε_i are three independent Gaussian variables $\mathcal{N}(0,1)$. X is the factor (or the systematic risk) and ε_i is the idiosyncratic risk.

Risk contribution in the Basel II model

Question 2.d (i)

Interpret the parameter ρ.

Risk contribution in the Basel II model

We have

$$\begin{aligned}\mathbb{E}\left[Z_i Z_j\right] &= \mathbb{E}\left[\left(\sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_i\right)\left(\sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_j\right)\right] \\ &= \rho\end{aligned}$$

ρ is the constant correlation between assets Z_i and Z_j.

Risk contribution in the Basel II model

Question 2.d (ii)

Calculate the unconditional default probability:

$$p_i = \Pr\{\boldsymbol{\tau}_i \leq M_i\}$$

Risk contribution in the Basel II model

We have:

$$\begin{aligned} p_i &= \Pr\{\tau_i \leq T_i\} \\ &= \Pr\{Z_i \leq B_i\} \\ &= \Phi(B_i) \end{aligned}$$

Risk contribution in the Basel II model

Question 2.d (iii)

Calculate the conditional default probability:

$$p_i(x) = \Pr\{\boldsymbol{\tau}_i \leq M_i \mid X = x\}$$

Risk contribution in the Basel II model

It follows that:

$$
\begin{aligned}
p_i(x) &= \Pr\{Z_i \leq B_i \mid X = x\} \\
&= \Pr\left\{\sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_i \leq B_i \mid X = x\right\} \\
&= \Pr\left\{\varepsilon_i \leq \frac{B_i - \sqrt{\rho}X}{\sqrt{1-\rho}} \middle| X = x\right\} \\
&= \Phi\left(\frac{B_i - \sqrt{\rho}x}{\sqrt{1-\rho}}\right) \\
&= \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho}x}{\sqrt{1-\rho}}\right)
\end{aligned}
$$

Risk contribution in the Basel II model

Question 2.e

Show that, under the previous assumptions $(\mathcal{H})$, the risk contribution $\mathcal{RC}_i$ of the i^{th} credit is:

$$\mathcal{RC}_i = \mathrm{EAD}_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times \Phi\left(\frac{\Phi^{-1}\left(p_i\right) + \sqrt{\rho}\Phi^{-1}\left(\alpha\right)}{\sqrt{1-\rho}}\right) \tag{5}$$

when the risk measure is the value-at-risk.

Risk contribution in the Basel II model

Under the assumptions $(\mathcal{H})$, we know that:

$$\begin{aligned} L &= \sum_{i=1}^{n} \mathrm{EAD}_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times p_i(X) \\ &= \sum_{i=1}^{n} \mathrm{EAD}_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho} X}{\sqrt{1-\rho}}\right) \\ &= g(X) \end{aligned}$$

with $g'(x) < 0$. We deduce that:

$$\begin{aligned} \mathrm{VaR}_\alpha(w) = \mathbf{F}^{-1}(\alpha) \quad &\Leftrightarrow \quad \Pr\{g(X) \le \mathrm{VaR}_\alpha(w)\} = \alpha \\ &\Leftrightarrow \quad \Pr\left\{X \ge g^{-1}(\mathrm{VaR}_\alpha(w))\right\} = \alpha \\ &\Leftrightarrow \quad \Pr\left\{X \le g^{-1}(\mathrm{VaR}_\alpha(w))\right\} = 1-\alpha \\ &\Leftrightarrow \quad g^{-1}(\mathrm{VaR}_\alpha(w)) = \Phi^{-1}(1-\alpha) \\ &\Leftrightarrow \quad \mathrm{VaR}_\alpha(w) = g\left(\Phi^{-1}(1-\alpha)\right) \end{aligned}$$

Risk contribution in the Basel II model

It follows that:

$$\begin{aligned}\mathrm{VaR}_{\alpha}(w) &= g\left(\Phi^{-1}(1-\alpha)\right) \\ &= \sum_{i=1}^{n} \mathrm{EAD}_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times p_i\left(\Phi^{-1}(1-\alpha)\right)\end{aligned}$$

The risk contribution $\mathcal{RC}_i$ of the ith credit is then:

$$\begin{aligned}\mathcal{RC}_i &= \mathrm{EAD}_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times p_i\left(\Phi^{-1}(1-\alpha)\right) \\ &= \mathrm{EAD}_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho}\Phi^{-1}(1-\alpha)}{\sqrt{1-\rho}}\right) \\ &= \mathrm{EAD}_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times \Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)\end{aligned}$$

Risk contribution in the Basel II model

Question 3

We now assume that the risk measure is the expected shortfall:

$$\mathrm{ES}_{\alpha}\left(w\right) = \mathbb{E}\left[L \mid L \geq \mathrm{VaR}_{\alpha}\left(w\right)\right]$$

Risk contribution in the Basel II model

Question 3.a

In the case of the Basel II framework, show that we have:

$$\mathrm{ES}_{\alpha}(w) = \sum_{i=1}^{n} \mathrm{EAD}_i \times \mathbb{E}[\mathrm{LGD}_i] \times \mathbb{E}\left[p_i(X) \mid X \leq \Phi^{-1}(1-\alpha)\right]$$

Risk contribution in the Basel II model

We note Ω the event $X \leq g^{-1}\left(\mathrm{VaR}_\alpha\left(w\right)\right)$ or equivalently $X \leq \Phi^{-1}\left(1-\alpha\right)$. We have:

$$
\begin{aligned}
\mathrm{ES}_\alpha\left(w\right) &= \mathbb{E}\left[L \mid L \geq \mathrm{VaR}_\alpha\left(w\right)\right] \\
&= \mathbb{E}\left[L \mid g\left(X\right) \geq \mathrm{VaR}_\alpha\left(w\right)\right] \\
&= \mathbb{E}\left[L \mid X \leq g^{-1}\left(\mathrm{VaR}_\alpha\left(w\right)\right)\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n} \mathrm{EAD}_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times p_i\left(X\right) \mid \Omega\right] \\
&= \sum_{i=1}^{n} \mathrm{EAD}_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times \mathbb{E}\left[p_i\left(X\right) \mid \Omega\right]
\end{aligned}
$$

Risk contribution in the Basel II model

Question 3.b

By using the following result:

$$\int_{-\infty}^{c} \Phi(a + bx)\phi(x)\,\mathrm{d}x = \Phi_2\left(c, \frac{a}{\sqrt{1+b^2}}; \frac{-b}{\sqrt{1+b^2}}\right)$$

where $\Phi_2(x, y; \rho)$ is the cdf of the bivariate Gaussian distribution with correlation ρ on the space $[-\infty, x] \times [-\infty, y]$, deduce that the risk contribution $\mathcal{RC}_i$ of the i^{th} credit in the Basel II model is:

$$\mathcal{RC}_i = \mathrm{EAD}_i \times \mathbb{E}[\mathrm{LGD}_i] \times \frac{\mathbf{C}\left(1-\alpha, p_i; \sqrt{\rho}\right)}{1-\alpha} \qquad (6)$$

when the risk measure is the expected shortfall. Here $\mathbf{C}(u_1, u_2; \theta)$ is the Normal copula with parameter θ.

Risk contribution in the Basel II model

It follows that:

$$\begin{aligned}
\mathbb{E}\left[p_i(X) \mid \Omega\right] &= \mathbb{E}\left[\Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho}X}{\sqrt{1-\rho}}\right) \middle| \Omega\right] \\
&= \int_{-\infty}^{\Phi^{-1}(1-\alpha)} \Phi\left(\frac{\Phi^{-1}(p_i)}{\sqrt{1-\rho}} + \frac{-\sqrt{\rho}}{\sqrt{1-\rho}}x\right) \times \\
&\qquad \frac{\phi(x)}{\Phi\left(\Phi^{-1}(1-\alpha)\right)} \,\mathrm{d}x \\
&= \frac{\Phi_2\left(\Phi^{-1}(1-\alpha), \Phi^{-1}(p_i); \sqrt{\rho}\right)}{1-\alpha} \\
&= \frac{\mathbf{C}\left(1-\alpha, p_i; \sqrt{\rho}\right)}{1-\alpha}
\end{aligned}$$

where $\mathbf{C}$ is the Gaussian copula. We deduce that:

$$\mathcal{RC}_i = \mathrm{EAD}_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times \frac{\mathbf{C}\left(1-\alpha, p_i; \sqrt{\rho}\right)}{1-\alpha}$$

Risk contribution in the Basel II model

Question 3.c

What become the results (5) and (6) if the correlation ρ is equal to zero? Same question if $\rho = 1$.

Risk contribution in the Basel II model

If $\rho = 0$, we have:

$$\begin{aligned}\Phi\left(\frac{\Phi^{-1}(p_i)+\sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right) &= \Phi\left(\Phi^{-1}(p_i)\right) \\ &= p_i\end{aligned}$$

and:

$$\begin{aligned}\frac{\mathbf{C}\left(1-\alpha, p_i; \sqrt{\rho}\right)}{1-\alpha} &= \frac{(1-\alpha)\,p_i}{1-\alpha} \\ &= p_i\end{aligned}$$

The risk contribution is the same for the value-at-risk and the expected shortfall:

$$\begin{aligned}\mathcal{RC}_i &= \mathrm{EAD}_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times p_i \\ &= \mathbb{E}\left[L_i\right]\end{aligned}$$

It corresponds to the expected loss of the credit.

Risk contribution in the Basel II model

If $\rho = 1$ and $\alpha > 50\%$, we have:

$$\begin{aligned}
\Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right) &= \lim_{\rho \to 1} \Phi\left(\frac{\Phi^{-1}(p_i) + \Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right) \\
&= 1
\end{aligned}$$

If $\rho = 1$ and α is high $(\alpha > 1 - \sup_i p_i)$, we have:

$$\begin{aligned}
\frac{\mathbf{C}\left(1-\alpha, p_i; \sqrt{\rho}\right)}{1-\alpha} &= \frac{\min\left(1-\alpha; p_i\right)}{1-\alpha} \\
&= 1
\end{aligned}$$

In this case, the risk contribution is the same for the value-at-risk and the expected shortfall:

$$\mathcal{RC}_i = \mathrm{EAD}_i \times \mathbb{E}\left[\mathrm{LGD}_i\right]$$

However, it does not depend on the unconditional probability of default p_i.

Risk contribution in the Basel II model

Question 4

The risk contributions (5) and (6) were obtained considering the assumptions $(\mathcal{H})$ and the default model defined in Question 2(d). What are the implications in terms of Pillar 2?

Risk contribution in the Basel II model

Pillar 2 concerns the non-compliance of assumptions $(\mathcal{H})$. In particular, we have to understand the impact on the credit risk measure if the portfolio is not infinitely fine-grained or if asset correlations are not constant.

Modeling loss given default

Question 1

What is the difference between the recovery rate and the loss given default?

Modeling loss given default

The loss given default is equal to:

$$\mathrm{LGD} = 1 - \boldsymbol{\mathcal{R}} + c$$

where c is the recovery (or litigation) cost. Consider for example a \$200 credit and suppose that the borrower defaults. If we recover \$140 and the litigation cost is \$20, we obtain $\boldsymbol{\mathcal{R}} = 70\%$ and $\mathrm{LGD} = 40\%$, but not $\mathrm{LGD} = 30\%$.

Modeling loss given default

Question 2

We consider a bank that grants 250 000 credits per year. The average amount of a credit is equal to \$50 000. We estimate that the average default probability is equal to 1% and the average recovery rate is equal to 65%. The total annual cost of the litigation department is equal to \$12.5 mn. Give an estimation of the loss given default?

Modeling loss given default

The amounts outstanding of credit is:

$$\begin{aligned} \mathrm{EAD} &= 250\,000 \times 50\,000 \\ &= \$12.5 \text{ bn} \end{aligned}$$

The annual loss after recovery is equal to:

$$\begin{aligned} L &= \mathrm{EAD} \times (1 - \mathcal{R}) \times \mathrm{PD} + C \\ &= 43.75 + 12.5 \\ &= \$56.25 \text{ mn} \end{aligned}$$

where C is the litigation cost.

Modeling loss given default

We deduce that:

$$\begin{aligned} \mathrm{LGD} &= \frac{L}{\mathrm{EAD} \times \mathrm{PD}} \\ &= \frac{54}{12.5 \times 10^3 \times 1\%} \\ &= 45\% \end{aligned}$$

This figure is larger than 35%, which is the loss given default without taking into account the recovery cost.

Modeling loss given default

Question 3

The probability density function of the beta probability distribution $\mathcal{B}(a, b)$ is:

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{\mathbf{B}(a, b)}$$

where $\mathbf{B}(a, b) = \int_0^1 u^{a-1}(1-u)^{b-1}\,\mathrm{d}u$.

Modeling loss given default

Question 3.a

Why is the beta probability distribution a good candidate to model the loss given default? Which parameter pair (a, b) correspond to the uniform probability distribution?

Modeling loss given default

The Beta distribution allows to obtain all the forms of LGD (bell curve, inverted-U shaped curve, etc.). The uniform distribution corresponds to the case $\alpha = 1$ and $\beta = 1$. Indeed, we have:

$$
\begin{aligned}
f(x) &= \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,\mathrm{d}u} \\
&= 1
\end{aligned}
$$

Modeling loss given default

Question 3.b

Let us consider a sample $(x_1, \ldots, x_n)$ of n losses in case of default. Write the log-likelihood function. Deduce the first-order conditions of the maximum likelihood estimator.

Modeling loss given default

We have:

$$\begin{aligned}\boldsymbol{\ell}(\alpha,\beta) &= \sum_{i=1}^{n} \ln f(x_i) \\ &= -n \ln \mathbf{B}(\alpha,\beta) + (\alpha-1)\sum_{i=1}^{n}\ln x_i + (\beta-1)\sum_{i=1}^{n}\ln(1-x_i)\end{aligned}$$

The first-order conditions are:

$$\frac{\partial\,\boldsymbol{\ell}(\alpha,\beta)}{\partial\,\alpha} = -n\frac{\partial_\alpha \mathbf{B}(\alpha,\beta)}{\mathbf{B}(\alpha,\beta)} + \sum_{i=1}^{n}\ln x_i = 0$$

and:

$$\frac{\partial\,\boldsymbol{\ell}(\alpha,\beta)}{\partial\,\beta} = -n\frac{\partial_\beta \mathbf{B}(\alpha,\beta)}{\mathbf{B}(\alpha,\beta)} + \sum_{i=1}^{n}\ln(1-x_i) = 0$$

Modeling loss given default

Question 3.c

We recall that the first two moments of the beta probability distribution are:

$$\begin{aligned}\mathbb{E}[X] &= \frac{a}{a+b} \\ \sigma^2(X) &= \frac{ab}{(a+b)^2(a+b+1)}\end{aligned}$$

Find the method of moments estimator.

Modeling loss given default

Let μ_{LGD} and σ_{LGD} be the mean and standard deviation of the LGD parameter. The method of moments consists in estimating α and β such that:

$$\frac{\alpha}{\alpha+\beta} = \mu_{\mathrm{LGD}}$$

and:

$$\frac{\alpha\beta}{\left(\alpha+\beta\right)^2\left(\alpha+\beta+1\right)} = \sigma^2_{\mathrm{LGD}}$$

We have:

$$\beta = \alpha\frac{\left(1-\mu_{\mathrm{LGD}}\right)}{\mu_{\mathrm{LGD}}}$$

and:

$$\left(\alpha+\beta\right)^2\left(\alpha+\beta+1\right)\sigma^2_{\mathrm{LGD}} = \alpha\beta$$

Modeling loss given default

It follows that:

$$
\begin{aligned}
(\alpha+\beta)^2 &= \left(\alpha + \alpha\frac{(1-\mu_{\mathrm{LGD}})}{\mu_{\mathrm{LGD}}}\right)^2 \\
&= \frac{\alpha^2}{\mu_{\mathrm{LGD}}^2}
\end{aligned}
$$

and:

$$
\alpha\beta = \frac{\alpha^2}{\mu_{\mathrm{LGD}}^2}\left(\alpha + \alpha\frac{(1-\mu_{\mathrm{LGD}})}{\mu_{\mathrm{LGD}}} + 1\right)\sigma_{\mathrm{LGD}}^2 = \alpha^2\frac{(1-\mu_{\mathrm{LGD}})}{\mu_{\mathrm{LGD}}}
$$

We deduce that:

$$
\alpha\left(1 + \frac{(1-\mu_{\mathrm{LGD}})}{\mu_{\mathrm{LGD}}}\right) = \frac{(1-\mu_{\mathrm{LGD}})\,\mu_{\mathrm{LGD}}}{\sigma_{\mathrm{LGD}}^2} - 1
$$

Modeling loss given default

We finally obtain:

$$\hat{\alpha}_{\mathrm{MM}} = \frac{\mu_{\mathrm{LGD}}^2 \left(1 - \mu_{\mathrm{LGD}}\right)}{\sigma_{\mathrm{LGD}}^2} - \mu_{\mathrm{LGD}} \tag{7}$$

$$\hat{\beta}_{\mathrm{MM}} = \frac{\mu_{\mathrm{LGD}} \left(1 - \mu_{\mathrm{LGD}}\right)^2}{\sigma_{\mathrm{LGD}}^2} - \left(1 - \mu_{\mathrm{LGD}}\right) \tag{8}$$

Modeling loss given default

Question 4

We consider a risk class $\mathcal{C}$ corresponding to a customer/product segmentation specific to retail banking. A statistical analysis of 1 000 loss data available for this risk class gives the following results:

LGD_k	0%	25%	50%	75%	100%
n_k	100	100	600	100	100

where n_k is the number of data corresponding to LGD_k.

Modeling loss given default

Question 4.a

We consider a portfolio of 100 homogeneous credits, which belong to the risk class $\mathcal{C}$. The notional is \$10 000 whereas the annual default probability is equal to 1%. Calculate the expected loss of this credit portfolio with a one-year horizon time if we use the previous empirical distribution to model the LGD parameter.

Modeling loss given default

The mean of the loss given default is equal to:

$$\begin{aligned} \mu_{\mathrm{LGD}} &= \frac{100 \times 0\% + 100 \times 25\% + 600 \times 50\% + \ldots}{1000} \\ &= 50\% \end{aligned}$$

The expression of the expected loss is:

$$\mathrm{EL} = \sum_{i=1}^{100} \mathrm{EAD}_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times \mathrm{PD}_i$$

where PD_i is the default probability of credit i. We finally obtain:

$$\begin{aligned} \mathrm{EL} &= \sum_{i=1}^{100} 10\,000 \times 50\% \times 1\% \\ &= \$5\,000 \end{aligned}$$

Modeling loss given default

Question 4.b

We assume that the LGD parameter follows a beta distribution $\mathcal{B}(a, b)$. Calibrate the parameters a and b with the method of moments.

Modeling loss given default

We have $\mu_{\mathrm{LGD}} = 50\%$ and:

$$\begin{aligned}
\sigma_{\mathrm{LGD}} &= \sqrt{\frac{100 \times (0 - 0.5)^2 + 100 \times (0.25 - 0.5)^2 + \ldots}{1000}} \\
&= \sqrt{\frac{2 \times 0.5^2 + 2 \times 0.25^2}{10}} \\
&= \sqrt{\frac{0.625}{10}} \\
&= 25\%
\end{aligned}$$

Using Equations (7) and (8), we deduce that:

$$\begin{aligned}
\hat{\alpha}_{\mathrm{MM}} &= \frac{0.5^2 \times (1 - 0.5)}{0.25^2} - 0.5 = 1.5 \\
\hat{\beta}_{\mathrm{MM}} &= \frac{0.5 \times (1 - 0.5)^2}{0.25^2} - (1 - 0.5) = 1.5
\end{aligned}$$

Modeling loss given default

Question 4.c

We assume that the Basel II model is valid. We consider the portfolio described in Question 4(a) and calculate the unexpected loss. What is the impact if we use a uniform probability distribution instead of the calibrated beta probability distribution? Why does this result hold even if we consider different factors to model the default time?

Modeling loss given default

The previous portfolio is homogeneous and infinitely fine-grained. In this case, we know that the unexpected loss depends on the mean of the loss given default and not on the entire probability distribution. Because the expected value of the calibrated Beta distribution is 50%, there is no difference with the uniform distribution, which has also a mean equal to 50%. This result holds for the Basel model with one factor, and remains true when they are more factors.

Course 2023-2024 in Financial Risk Management
Tutorial Session 3

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[10] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Agenda

- Tutorial Session 1: Market Risk
- Tutorial Session 2: Credit Risk
- **Tutorial Session 3: Counterparty Credit Risk and Collateral Risk**
- Tutorial Session 4: Operational Risk & Asset Liability Management Risk
- Tutorial Session 5: Copulas, EVT & Stress Testing

Impact of netting agreements in counterparty credit risk

Question 1

The table below gives the current mark-to-market of 7 OTC contracts between Bank A and Bank B:

	Equity			Fixed income		FX	
	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$
A	$+10$	-5	$+6$	$+17$	-5	-5	$+1$
B	-11	$+6$	-3	-12	$+9$	$+5$	$+1$

The table should be read as follows: Bank A has a mark-to-market equal to 10 for the contract $\mathcal{C}_1$ whereas Bank B has a mark-to-market equal to -11 for the same contract, Bank A has a mark-to-market equal to -5 for the contract $\mathcal{C}_2$ whereas Bank B has a mark-to-market equal to $+6$ for the same contract, etc.

Impact of netting agreements in counterparty credit risk

Question 1.a

Explain why there are differences between the MtM values of a same OTC contract.

Impact of netting agreements in counterparty credit risk

Let $\mathrm{MtM}_A(\mathcal{C})$ and $\mathrm{MTM}_B(\mathcal{C})$ be the MtM values of Bank A and Bank B for the contract $\mathcal{C}$. We must theoretically verify that:

$$\begin{aligned} \mathrm{MtM}_{A+B}(\mathcal{C}) &= \mathrm{MTM}_A(\mathcal{C}) + \mathrm{MTM}_B(\mathcal{C}) \\ &= 0 \end{aligned} \tag{9}$$

In the case of listed products, the previous relationship is verified. In the case of OTC products, there are no market prices, forcing the bank to use pricing models for the valuation. The MTM value is then a mark-to-model price. Because the two banks do not use the same model with the same parameters, we note a discrepancy between the two mark-to-market prices:

$$\mathrm{MTM}_A(\mathcal{C}) + \mathrm{MTM}_B(\mathcal{C}) \neq 0$$

Impact of netting agreements in counterparty credit risk

For instance, we obtain:

$$\begin{aligned}
\mathrm{MTM}_{A+B}(\mathcal{C}_1) &= 10 - 11 = -1 \\
\mathrm{MTM}_{A+B}(\mathcal{C}_2) &= -5 + 6 = 1 \\
\mathrm{MTM}_{A+B}(\mathcal{C}_3) &= 6 - 3 = 3 \\
\mathrm{MTM}_{A+B}(\mathcal{C}_4) &= 17 - 12 = 5 \\
\mathrm{MTM}_{A+B}(\mathcal{C}_5) &= -5 + 9 = 4 \\
\mathrm{MTM}_{A+B}(\mathcal{C}_6) &= -5 + 5 = 0 \\
\mathrm{MTM}_{A+B}(\mathcal{C}_7) &= 1 + 1 = 2
\end{aligned}$$

Only the contract $\mathcal{C}_6$ satisfies the relationship (9).

Impact of netting agreements in counterparty credit risk

Question 1.b

Calculate the exposure at default of Bank A.

Impact of netting agreements in counterparty credit risk

We have:

$$\mathrm{EAD} = \sum_{i=1}^{7} \max\left(\mathrm{MTM}\left(\mathcal{C}_i\right), 0\right)$$

We deduce that:

$$\begin{aligned}
\mathrm{EAD}_A &= 10 + 6 + 17 + 1 = 34 \\
\mathrm{EAD}_B &= 6 + 9 + 5 + 1 = 21
\end{aligned}$$

Impact of netting agreements in counterparty credit risk

Question 1.c

Same question if there is a global netting agreement.

Impact of netting agreements in counterparty credit risk

If there is a global netting agreement, the exposure at default becomes:

$$\mathrm{EAD} = \max\left(\sum_{i=1}^{7} \mathrm{MTM}\left(\mathcal{C}_i\right), 0\right)$$

Using the numerical values, we obtain:

$$\begin{aligned}\mathrm{EAD}_A &= \max\left(10 - 5 + 6 + 17 - 5 - 5 + 1, 0\right) \\ &= \max\left(19, 0\right) \\ &= 19\end{aligned}$$

and:

$$\begin{aligned}\mathrm{EAD}_B &= \max\left(-11 + 6 - 3 - 12 + 9 + 5 + 1, 0\right) \\ &= \max\left(-5, 0\right) \\ &= 0\end{aligned}$$

Impact of netting agreements in counterparty credit risk

Question 1.d

Same question if the netting agreement only concerns equity products.

Impact of netting agreements in counterparty credit risk

If the netting agreement only concerns equity contracts, we have:

$$\mathrm{EAD} = \max\left(\sum_{i=1}^{3} \mathrm{MTM}\left(\mathcal{C}_i\right), 0\right) + \sum_{i=4}^{7} \max\left(\mathrm{MTM}\left(\mathcal{C}_i\right), 0\right)$$

It follows that:

$$\begin{aligned}
\mathrm{EAD}_A &= \max(10 - 5 + 6, 0) + 17 + 1 = 29 \\
\mathrm{EAD}_B &= \max(-11 + 6 - 3, 0) + 9 + 5 + 1 = 15
\end{aligned}$$

Impact of netting agreements in counterparty credit risk

Question 2

In the following, we measure the impact of netting agreements on the exposure at default.

Impact of netting agreements in counterparty credit risk

Question 2.a

We consider a first OTC contract $\mathcal{C}_1$ between Bank A and Bank B. The mark-to-market $\mathrm{MtM}_1(t)$ of Bank A for the contract $\mathcal{C}_1$ is defined as follows:

$$\mathrm{MtM}_1(t) = x_1 + \sigma_1 W_1(t)$$

where $W_1(t)$ is a Brownian motion. Calculate the potential future exposure of Bank A.

Impact of netting agreements in counterparty credit risk

The potential future exposure $e_1(t)$ is defined as follows:

$$e_1(t) = \max(x_1 + \sigma_1 W_1(t), 0)$$

We deduce that:

$$\begin{aligned}\mathbb{E}[e_1(t)] &= \int_{-\infty}^{\infty} \max(x, 0) f(x) \, \mathrm{d}x \\ &= \int_0^{\infty} x f(x) \, \mathrm{d}x\end{aligned}$$

where $f(x)$ is the density function of $\mathrm{MtM}_1(t)$. As we have $\mathrm{MtM}_1(t) \sim \mathcal{N}\left(x_1, \sigma_1^2 t\right)$, we deduce that:

$$\mathbb{E}[e_1(t)] = \int_0^{\infty} \frac{x}{\sigma_1\sqrt{2\pi t}} \exp\left(-\frac{1}{2}\left(\frac{x - x_1}{\sigma_1\sqrt{t}}\right)^2\right) \mathrm{d}x$$

Impact of netting agreements in counterparty credit risk

With the change of variable $y = \sigma_1^{-1} t^{-1/2} (x - x_1)$, we obtain:

$$\begin{aligned}
\mathbb{E}\left[e_1(t)\right] &= \int_{\frac{-x_1}{\sigma_1\sqrt{t}}}^{\infty} \frac{x_1 + \sigma_1\sqrt{t}y}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right)\,\mathrm{d}y \\
&= x_1 \int_{\frac{-x_1}{\sigma_1\sqrt{t}}}^{\infty} \phi(y)\,\mathrm{d}y + \sigma_1\sqrt{t}\int_{\frac{-x_1}{\sigma_1\sqrt{t}}}^{\infty} y\phi(y)\,\mathrm{d}y \\
&= x_1\Phi\left(\frac{x_1}{\sigma_1\sqrt{t}}\right) + \sigma_1\sqrt{t}\Big[-\phi(y)\Big]_{\frac{-x_1}{\sigma_1\sqrt{t}}}^{\infty} \\
&= x_1\Phi\left(\frac{x_1}{\sigma_1\sqrt{t}}\right) + \sigma_1\sqrt{t}\phi\left(\frac{x_1}{\sigma_1\sqrt{t}}\right)
\end{aligned}$$

because $\phi(-x) = \phi(x)$ and $\Phi(-x) = 1 - \Phi(x)$.

Impact of netting agreements in counterparty credit risk

Question 2.b

We consider a second OTC contract between Bank A and Bank B. The mark-to-market is also given by the following expression:

$$\mathrm{MtM}_2(t) = x_2 + \sigma_2 W_2(t)$$

where $W_2(t)$ is a second Brownian motion that is correlated with $W_1(t)$. Let ρ be this correlation such that $\mathbb{E}[W_1(t) W_2(t)] = \rho t$. Calculate the expected exposure of bank A if there is no netting agreement.

Impact of netting agreements in counterparty credit risk

When there is no netting agreement, we have:

$$e(t) = e_1(t) + e_2(t)$$

We deduce that:

$$\begin{aligned}
\mathbb{E}\left[e(t)\right] &= \mathbb{E}\left[e_1(t)\right] + \mathbb{E}\left[e_2(t)\right] \\
&= x_1 \Phi\left(\frac{x_1}{\sigma_1\sqrt{t}}\right) + \sigma_1\sqrt{t}\phi\left(\frac{x_1}{\sigma_1\sqrt{t}}\right) + \\
&\quad x_2 \Phi\left(\frac{x_2}{\sigma_2\sqrt{t}}\right) + \sigma_2\sqrt{t}\phi\left(\frac{x_2}{\sigma_2\sqrt{t}}\right)
\end{aligned}$$

Impact of netting agreements in counterparty credit risk

Question 2.c

Same question when there is a global netting agreement between Bank A and Bank B.

Impact of netting agreements in counterparty credit risk

In the case of a netting agreement, the potential future exposure becomes:

$$\begin{aligned} e(t) &= \max\left(\mathrm{MtM}_1(t) + \mathrm{MtM}_2(t), 0\right) \\ &= \max\left(\mathrm{MtM}_{1+2}(t), 0\right) \\ &= \max\left(x_1 + x_2 + \sigma_1 W_1(t) + \sigma_2 W_2(t), 0\right) \end{aligned}$$

We deduce that:

$$\mathrm{MtM}_{1+2}(t) \sim \mathcal{N}\left(x_1 + x_2, \left(\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2\right) t\right)$$

Using results of Question 2(a), we finally obtain:

$$\begin{aligned} \mathbb{E}\left[e(t)\right] = {} & (x_1 + x_2)\,\Phi\left(\frac{x_1 + x_2}{\sqrt{\left(\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2\right) t}}\right) + \\ & \sqrt{\left(\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2\right) t}\,\phi\left(\frac{x_1 + x_2}{\sqrt{\left(\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2\right) t}}\right) \end{aligned}$$

Impact of netting agreements in counterparty credit risk

Question 2.d

Comment on these results.

Impact of netting agreements in counterparty credit risk

We have represented the expected exposure $\mathbb{E}\left[e\left(t\right)\right]$ in Figure 2 when $x_1 = x_2 = 0$ and $\sigma_1 = \sigma_2$. We note that it is an increasing function of the time t and the volatility σ. We also observe that the netting agreement may have a big impact, especially when the correlation is low or negative.

Impact of netting agreements in counterparty credit risk

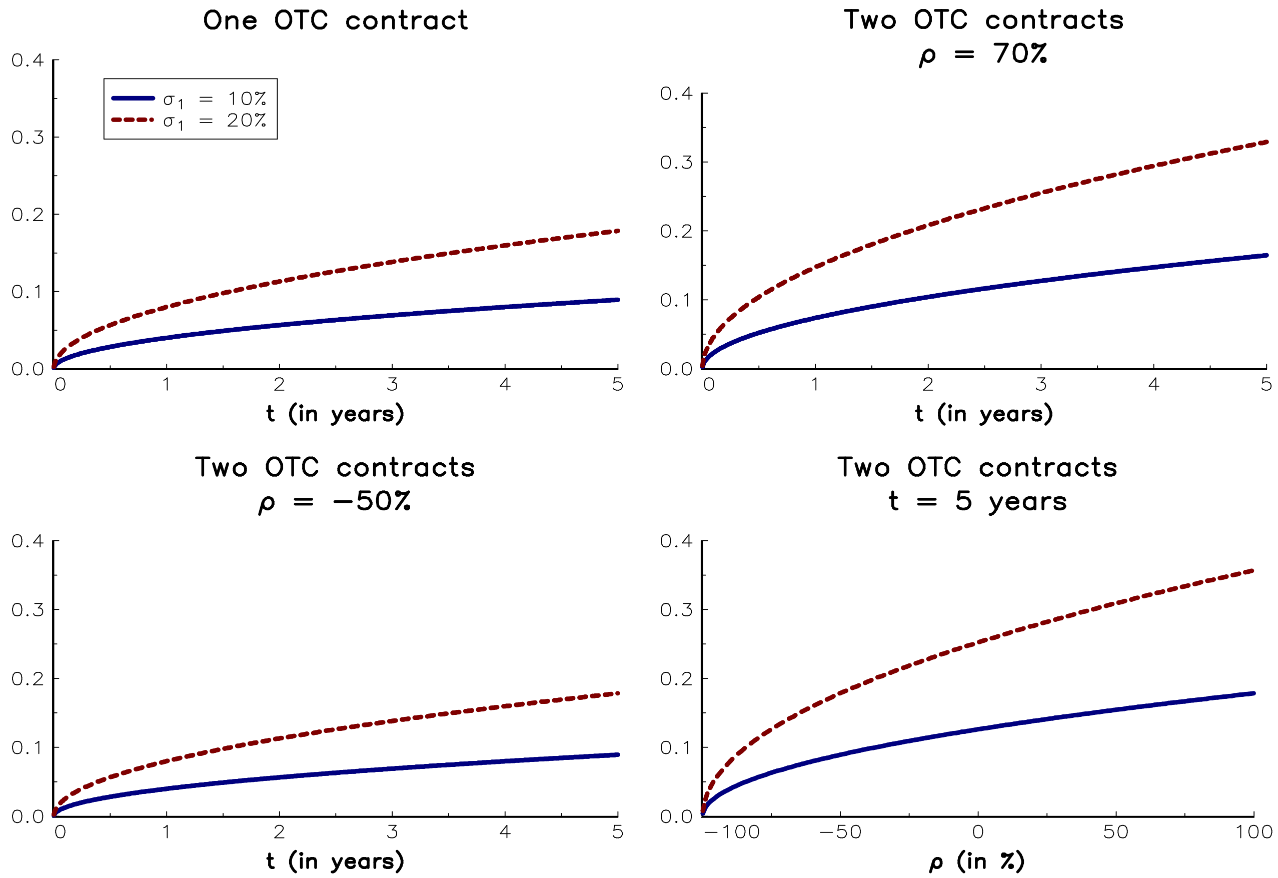

Figure: Expected exposure $\mathbb{E}[e(t)]$ when there is a netting agreement

Calculation of the CCR capital charge

We denote by $e(t)$ the potential future exposure of an OTC contract with maturity T. The current date is set to $t = 0$. Let N and σ be the notional and the volatility of the underlying contract. We assume that $e(t) = N\sigma\sqrt{t}X$ with $0 \leq X \leq 1$, $\Pr\{X \leq x\} = x^{\gamma}$ and $\gamma > 0$.

Calculation of the CCR capital charge

Question 1

Calculate the peak exposure $\mathrm{PE}_{\alpha}(t)$, the expected exposure $\mathrm{EE}(t)$ and the effective expected positive exposure $\mathrm{EEPE}(0; t)$.

Calculation of the CCR capital charge

We have:

$$\begin{aligned}
\mathbf{F}_{[0,t]}(x) &= \Pr\{e(t) \leq x\} \\
&= \Pr\left\{N\sigma\sqrt{t}U \leq x\right\} \\
&= \Pr\left\{U \leq \frac{x}{N\sigma\sqrt{t}}\right\} \\
&= \left(\frac{x}{N\sigma\sqrt{t}}\right)^{\gamma}
\end{aligned}$$

with $x \in \left[0, N\sigma\sqrt{t}\right]$. We deduce that:

$$\begin{aligned}
\mathrm{PE}_{\alpha}(t) &= \mathbf{F}_{[0,t]}^{-1}(\alpha) \\
&= N\sigma\sqrt{t}\alpha^{1/\gamma}
\end{aligned}$$

Calculation of the CCR capital charge

For the expected exposure, we obtain:

$$\begin{aligned}
\mathrm{EE}(t) &= \mathbb{E}[e(t)] \\
&= \int_0^{N\sigma\sqrt{t}} x \frac{\gamma}{\left(N\sigma\sqrt{t}\right)^{\gamma}} x^{\gamma-1}\,\mathrm{d}x \\
&= \frac{\gamma}{\left(N\sigma\sqrt{t}\right)^{\gamma}} \left[\frac{x^{\gamma+1}}{\gamma+1}\right]_0^{N\sigma\sqrt{t}} \\
&= \frac{\gamma}{\gamma+1} N\sigma\sqrt{t}
\end{aligned}$$

Calculation of the CCR capital charge

We deduce that:

$$\mathrm{EEE}\left(t\right)=\frac{\gamma}{\gamma+1}N\sigma\sqrt{t}$$

and:

$$\begin{aligned}\mathrm{EEPE}\left(0;t\right) &= \frac{1}{t}\int_0^t \mathrm{EEE}\left(s\right)\,\mathrm{d}s \\ &= \frac{1}{t}\int_0^t \frac{\gamma}{\gamma+1}N\sigma\sqrt{s}\,\mathrm{d}s \\ &= \frac{\gamma}{\gamma+1}N\sigma\frac{1}{t}\left[\frac{2}{3}s^{3/2}\right]_0^t \\ &= \frac{2\gamma}{3\left(\gamma+1\right)}N\sigma\sqrt{t}\end{aligned}$$

Calculation of the CCR capital charge

Question 2

The bank manages the credit risk with the foundation IRB approach and the counterparty credit risk with an internal model. We consider an OTC contract with the following parameters: N is equal to \$3 mn, the maturity T is one year, the volatility σ is set to 20% and γ is estimated at 2.

Calculation of the CCR capital charge

Question 2.a

Calculate the exposure at default EAD knowing that the bank uses the regulatory value for the parameter α.

Calculation of the CCR capital charge

When the bank uses an internal model, the regulatory exposure at default is:

$$\mathrm{EAD} = \alpha \times \mathrm{EEPE}\,(0;1)$$

Using the standard value $\alpha = 1.4$, we obtain:

$$\begin{aligned}\mathrm{EAD} &= 1.4 \times \frac{4}{9} \times 3 \times 10^{6} \times 0.20 \\ &= \$373\,333\end{aligned}$$

Calculation of the CCR capital charge

Question 2.b

The default probability of the counterparty is estimated at 1%. Calculate then the capital charge for counterparty credit risk of this OTC contract[a].

[a] We will take a value of 70% for the LGD parameter and a value of 20% for the default correlation. We can also use the approximations $-1.06 \approx -1$ and $\Phi(-1) \approx 16\%$.

Calculation of the CCR capital charge

While the bank uses the FIRB approach, the required capital is:

$$\mathcal{K} = \mathrm{EAD} \times \mathbb{E}\left[\mathrm{LGD}\right] \times \left(\Phi\left(\frac{\Phi^{-1}\left(\mathrm{PD}\right) + \sqrt{\rho}\Phi^{-1}\left(99.9\%\right)}{\sqrt{1-\rho}}\right) - \mathrm{PD}\right)$$

When ρ is equal to 20%, we have:

$$\begin{aligned}\frac{\Phi^{-1}\left(\mathrm{PD}\right) + \sqrt{\rho}\Phi^{-1}\left(99.9\%\right)}{\sqrt{1-\rho}} &= \frac{-2.33 + \sqrt{0.20} \times 3.09}{\sqrt{1-0.20}} \\ &= -1.06\end{aligned}$$

By using the approximations $-1.06 \simeq 1$ and $\Phi\left(-1\right) \simeq 0.16$, we obtain:

$$\begin{aligned}\mathcal{K} &= 373\,333 \times 0.70 \times \left(0.16 - 0.01\right) \\ &= \$39\,200\end{aligned}$$

The required capital of this OTC product for counterparty credit risk is then equal to \$39 200.

Calculation of CVA and DVA measures

We consider an OTC contract with maturity T between Bank A and Bank B. We denote by $\mathrm{MtM}(t)$ the risk-free mark-to-market of Bank A. The current date is set to $t = 0$ and we assume that:

$$\mathrm{MtM}(t) = N \cdot \sigma \cdot \sqrt{t} \cdot X$$

where N is the notional of the OTC contract, σ is the volatility of the underlying asset and X is a random variable, which is defined on the support $[-1, 1]$ and whose density function is:

$$f(x) = \frac{1}{2}$$

Calculation of CVA and DVA measures

Question 1

Define the concept of positive exposure $e^{+}(t)$. Show that the cumulative distribution function $\mathbf{F}_{[0,t]}$ of $e^{+}(t)$ has the following expression:

$$\mathbf{F}_{[0,t]}(x) = \mathbb{1}\left\{0 \leq x \leq \sigma\sqrt{t}\right\} \cdot \left(\frac{1}{2} + \frac{x}{2 \cdot N \cdot \sigma \cdot \sqrt{t}}\right)$$

where $\mathbf{F}_{[0,t]}(x) = 0$ if $x \leq 0$ and $\mathbf{F}_{[0,t]}(x) = 1$ if $x \geq \sigma\sqrt{t}$.

Calculation of CVA and DVA measures

The positive exposure $e^{+}(t)$ is the maximum between zero and the mark-to-market value:

$$\begin{aligned} e^{+}(t) &= \max\left(0, \mathrm{MtM}(t)\right) \\ &= \max\left(0, N\sigma\sqrt{t}X\right) \end{aligned}$$

We have:

$$\begin{aligned} \mathbf{F}_{[0,t]}(x) &= \Pr\left\{e^{+}(t) \leq x\right\} \\ &= \Pr\left\{\max\left(0, N\sigma\sqrt{t}X\right) \leq x\right\} \end{aligned}$$

We notice that:

$$\max\left(0, N\sigma\sqrt{t}X\right) = \begin{cases} 0 & \text{if } X \leq 0 \\ N\sigma\sqrt{t}X & \text{otherwise} \end{cases}$$

Calculation of CVA and DVA measures

By assuming that $x \in \left[0, N\sigma\sqrt{t}\right]$, we deduce that:

$$\begin{aligned}
\mathbf{F}_{[0,t]}(x) &= \Pr\left\{e^{+}(t) \le x, X \le 0\right\} + \Pr\left\{e^{+}(t) \le x, X > 0\right\} \\
&= \Pr\{0 \le x, X \le 0\} + \Pr\left\{N\sigma\sqrt{t}X \le x, X > 0\right\} \\
&= \frac{1}{2} + \frac{1}{2}\Pr\left\{N\sigma\sqrt{t}U \le x\right\} \\
&= \frac{1}{2} + \frac{1}{2}\Pr\left\{U \le \frac{x}{N\sigma\sqrt{t}}\right\}
\end{aligned}$$

where U is the standard uniform random variable. We finally obtain the following expression:

$$\mathbf{F}_{[0,t]}(x) = \frac{1}{2} + \frac{x}{2N\sigma\sqrt{t}}$$

If $x \le 0$ or $x \ge N\sigma\sqrt{t}$, it is easy to show that $\mathbf{F}_{[0,t]}(x) = 0$ and $\mathbf{F}_{[0,t]}(x) = 1$.

Calculation of CVA and DVA measures

Question 2

Deduce the value of the expected positive exposure $\mathrm{EpE}(t)$.

Calculation of CVA and DVA measures

The expected positive exposure $\mathrm{EpE}(t)$ is defined as follows:

$$\mathrm{EpE}(t) = \mathbb{E}\left[e^{+}(t)\right]$$

Using the expression of $\mathbf{F}_{[0,t]}(x)$, it follows that the density function of $e^{+}(t)$ is equal to:

$$\begin{aligned} f_{[0,t]}(x) &= \frac{\partial\, \mathbf{F}_{[0,t]}(x)}{\partial\, x} \\ &= \frac{1}{2N\sigma\sqrt{t}} \end{aligned}$$

Calculation of CVA and DVA measures

We deduce that:

$$\begin{aligned}
\mathrm{EpE}(t) &= \int_0^{N\sigma\sqrt{t}} x f_{[0,t]}(x)\,\mathrm{d}x \\
&= \int_0^{N\sigma\sqrt{t}} \frac{x}{2N\sigma\sqrt{t}}\,\mathrm{d}x \\
&= \left[\frac{x^2}{4N\sigma\sqrt{t}}\right]_0^{N\sigma\sqrt{t}} \\
&= \frac{N\sigma\sqrt{t}}{4}
\end{aligned}$$

Calculation of CVA and DVA measures

Question 3

We note $\boldsymbol{\mathcal{R}}_B$ the fixed and constant recovery rate of Bank B. Give the mathematical expression of the CVA.

Calculation of CVA and DVA measures

By definition, we have:

$$\mathrm{CVA} = (1 - \mathcal{R}_B) \times \int_0^T -B_0(t) \, \mathrm{EpE}(t) \, \mathrm{d}\mathbf{S}_B(t)$$

Calculation of CVA and DVA measures

Question 4

By using the definition of the lower incomplete gamma function $\gamma(s, x)$, show that the CVA is equal to:

$$\mathrm{CVA} = \frac{N \cdot (1 - \mathcal{R}_B) \cdot \sigma \cdot \gamma\left(\frac{3}{2}, \lambda_B T\right)}{4\sqrt{\lambda_B}}$$

when the default time of Bank B is exponential with parameter λ_B and interest rates are equal to zero.

Calculation of CVA and DVA measures

The interest rates are equal to zero meaning that $B_0(t) = 1$. Moreover, we have $\mathbf{S}_B(t) = e^{-\lambda_B t}$. We deduce that:

$$\begin{aligned}
\mathrm{CVA} &= (1 - \mathcal{R}_B) \times \int_0^T \frac{N\sigma\sqrt{t}}{4} \lambda_B e^{-\lambda_B t} \,\mathrm{d}t \\
&= \frac{N\lambda_B (1 - \mathcal{R}_B)\sigma}{4} \int_0^T \sqrt{t} e^{-\lambda_B t} \,\mathrm{d}t
\end{aligned}$$

The definition of the incomplete gamma function is:

$$\gamma(s, x) = \int_0^x t^{s-1} e^{-t} \,\mathrm{d}t$$

Calculation of CVA and DVA measures

By considering the change of variable $y = \lambda_B t$, we obtain:

$$\begin{aligned}
\int_0^T \sqrt{t} e^{-\lambda_B t} \,\mathrm{d}t &= \int_0^{\lambda_B T} \sqrt{\frac{y}{\lambda_B}} e^{-y} \frac{\mathrm{d}y}{\lambda_B} \\
&= \frac{1}{\lambda_B^{3/2}} \int_0^{\lambda_B T} y^{3/2-1} e^{-y} \,\mathrm{d}y \\
&= \frac{\gamma\left(\frac{3}{2}, \lambda_B T\right)}{\lambda_B^{3/2}}
\end{aligned}$$

It follows that:

$$\mathrm{CVA} = \frac{N\left(1 - \mathcal{R}_B\right) \sigma \gamma\left(\frac{3}{2}, \lambda_B T\right)}{4\sqrt{\lambda_B}}$$

Calculation of CVA and DVA measures

Question 5

Comment on this result.

Calculation of CVA and DVA measures

The CVA is proportional to the notional N of the OTC contract, the loss given default $(1-\mathcal{R}_B)$ of the counterparty and the volatility σ of the underlying asset. It is an increasing function of the maturity T because we have $\gamma\left(\frac{3}{2},\lambda_B T_2\right) > \gamma\left(\frac{3}{2},\lambda_B T_1\right)$ when $T_2 > T_1$. If the maturity is not very large (less than 10 years), the CVA is an increasing function of the default intensity λ_B.

Calculation of CVA and DVA measures

The limit cases are[11]:

$$\lim_{\lambda_B \to \infty} \mathrm{CVA} = \lim_{\lambda_B \to \infty} \frac{N (1 - \mathcal{R}_B) \sigma \gamma \left(\frac{3}{2}, \lambda_B T\right)}{4\sqrt{\lambda_B}} = 0$$

and:

$$\lim_{T \to \infty} \mathrm{CVA} = \frac{N (1 - \mathcal{R}_B) \sigma \Gamma \left(\frac{3}{2}\right)}{4\sqrt{\lambda_B}}$$

When the counterparty has a high default intensity, meaning that the default is imminent, the CVA is equal to zero because the mark-to-market value is close to zero. When the maturity is large, the CVA is a decreasing function of the intensity λ_B. Indeed, the probability to observe a large mark-to-market in the future increases when the default time is very far from the current date. We have illustrated these properties in Figure **??** with the following numerical values: $N = \$1$ mn, $\mathcal{R}_B = 40\%$ and $\sigma = 30\%$.

[11] We have $\lim_{x \to \infty} \gamma(s, x) = \Gamma(s)$.

Calculation of CVA and DVA measures

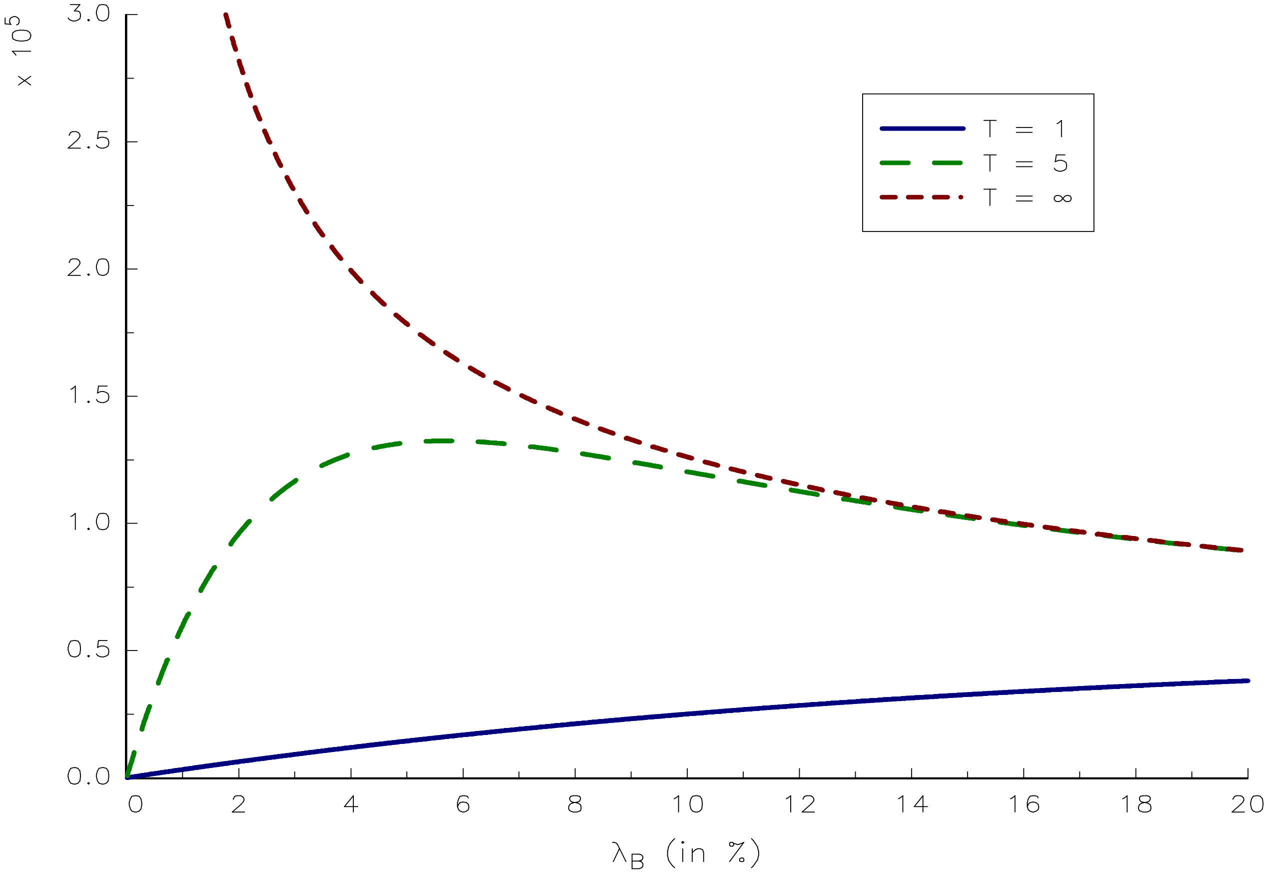

Figure: Evolution of the CVA with respect to maturity T and intensity λ_B

Calculation of CVA and DVA measures

Question 6

By assuming that the default time of Bank A is exponential with parameter λ_A, deduce the value of the DVA without additional computations.

Calculation of CVA and DVA measures

We notice that the mark-to-market is perfectly symmetric about 0. We deduce that the expected negative exposure $\mathrm{EnE}(t)$ is equal to the expected positive exposure $\mathrm{EpE}(t)$. It follows that the DVA is equal to:

$$\mathrm{DVA} = \frac{N\left(1 - \boldsymbol{\mathcal{R}}_A\right) \sigma \gamma \left(\frac{3}{2}, \lambda_A T\right)}{4\sqrt{\lambda_A}}$$

Course 2023-2024 in Financial Risk Management
Tutorial Session 5

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[12] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Agenda

- Tutorial Session 1: Market Risk
- Tutorial Session 2: Credit Risk
- Tutorial Session 3: Counterparty Credit Risk and Collateral Risk
- Tutorial Session 4: Operational Risk & Asset Liability Management Risk
- **Tutorial Session 5: Copulas, EVT & Stress Testing**

The bivariate Pareto copula

Exercise

We consider the bivariate Pareto distribution:

$$\begin{aligned}\mathbf{F}(x_1, x_2) = {} & 1 - \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha} - \left(\frac{\theta_2 + x_2}{\theta_2}\right)^{-\alpha} + \\ & \left(\frac{\theta_1 + x_1}{\theta_1} + \frac{\theta_2 + x_2}{\theta_2} - 1\right)^{-\alpha}\end{aligned}$$

where $x_1 \geq 0$, $x_2 \geq 0$, $\theta_1 > 0$, $\theta_2 > 0$ and $\alpha > 0$.

The bivariate Pareto copula

Question 1

Show that the marginal functions of $\mathbf{F}(x_1, x_2)$ correspond to univariate Pareto distributions.

The bivariate Pareto copula

We have:

$$\begin{aligned}\mathbf{F}_1(x_1) &= \Pr\{X_1 \leq x_1\} \\ &= \Pr\{X_1 \leq x_1, X_2 \leq \infty\} \\ &= \mathbf{F}(x_1, \infty)\end{aligned}$$

We deduce that:

$$\begin{aligned}\mathbf{F}_1(x_1) &= 1 - \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha} - \left(\frac{\theta_2 + \infty}{\theta_2}\right)^{-\alpha} + \\ &\quad \left(\frac{\theta_1 + x_1}{\theta_1} + \frac{\theta_2 + \infty}{\theta_2} - 1\right)^{-\alpha} \\ &= 1 - \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha}\end{aligned}$$

We conclude that $\mathbf{F}_1$ (and $\mathbf{F}_2$) is a Pareto distribution.

The bivariate Pareto copula

Question 2

Find the copula function associated to the bivariate Pareto distribution.

The bivariate Pareto copula

We have:

$$\mathbf{C}(u_1, u_2) = \mathbf{F}\left(\mathbf{F}_1^{-1}(u_1), \mathbf{F}_2^{-1}(u_2)\right)$$

It follows that:

$$\begin{aligned} 1 - \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha} &= u_1 \\ \Leftrightarrow \quad \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha} &= 1 - u_1 \\ \Leftrightarrow \quad \frac{\theta_1 + x_1}{\theta_1} &= (1 - u_1)^{-1/\alpha} \end{aligned}$$

We deduce that:

$$\begin{aligned} \mathbf{C}(u_1, u_2) &= 1 - (1 - u_1) - (1 - u_2) + \\ & \quad \left((1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1\right)^{-\alpha} \\ &= u_1 + u_2 - 1 + \left((1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1\right)^{-\alpha} \end{aligned}$$

The bivariate Pareto copula

Question 3

Deduce the copula density function.

The bivariate Pareto copula

We have:

$$\begin{aligned}
\frac{\partial\, \mathbf{C}\,(u_1, u_2)}{\partial\, u_1} &= 1 - \alpha \left((1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1 \right)^{-\alpha-1} \times \\
&\quad \left(-\frac{1}{\alpha} \right) (1-u_1)^{-1/\alpha-1} \times (-1) \\
&= 1 - \left((1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1 \right)^{-\alpha-1} \times \\
&\quad (1-u_1)^{-1/\alpha-1}
\end{aligned}$$

The bivariate Pareto copula

We deduce that the probability density function of the copula is:

$$\begin{aligned} c(u_1, u_2) &= \frac{\partial^2 \mathbf{C}(u_1, u_2)}{\partial\, u_1\, \partial\, u_2} \\ &= -(-\alpha - 1)\left((1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1\right)^{-\alpha-2} \times \\ &\quad \left(-\frac{1}{\alpha}\right)(1-u_2)^{-1/\alpha-1} \times (-1) \times (1-u_1)^{-1/\alpha-1} \\ &= \left(\frac{\alpha+1}{\alpha}\right)\left((1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1\right)^{-\alpha-2} \times \\ &\quad (1-u_1-u_2+u_1 u_2)^{-1/\alpha-1} \end{aligned}$$

The bivariate Pareto copula

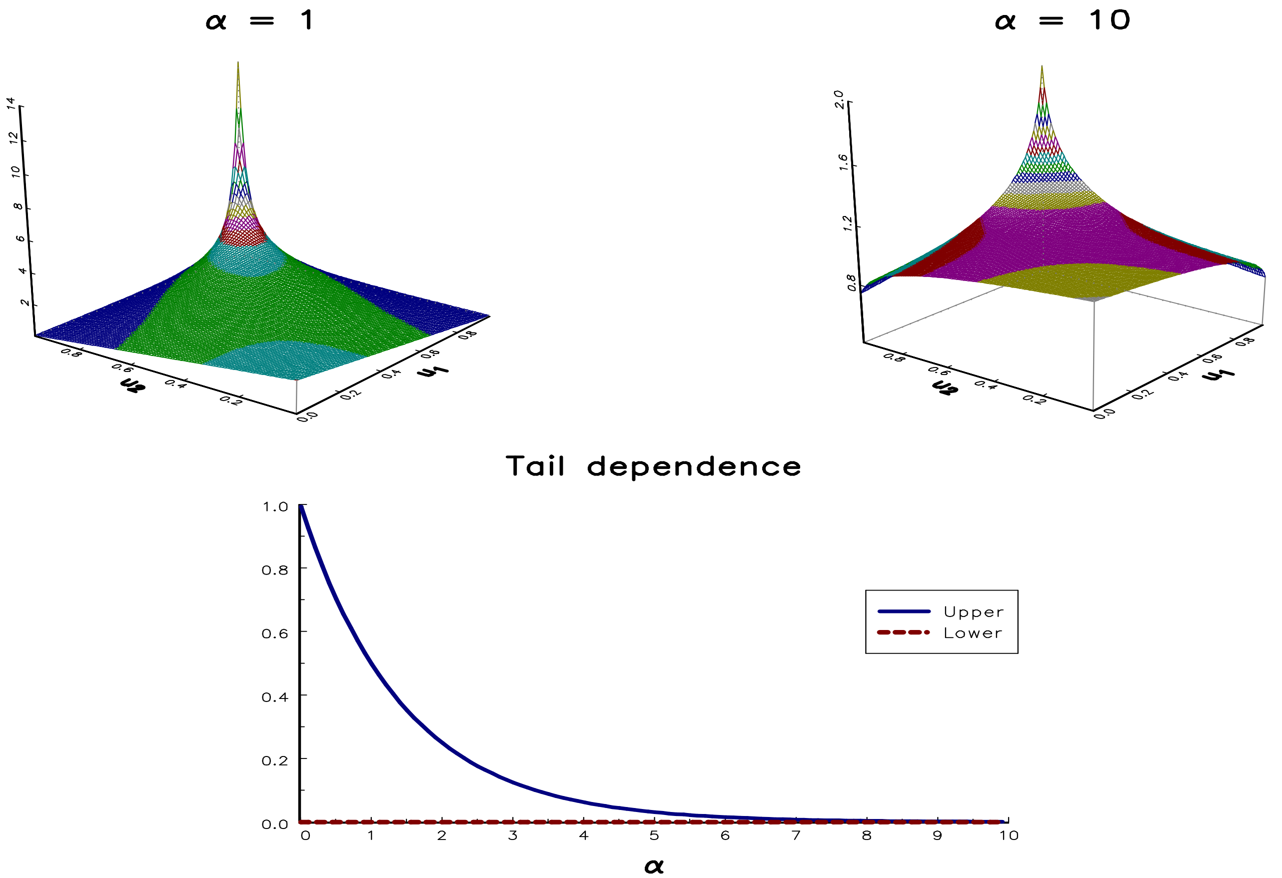
Remark

Another expression of $c(u_1, u_2)$ *is:*

$$\begin{aligned} c(u_1, u_2) &= \left(\frac{\alpha+1}{\alpha}\right)\left((1-u_1)(1-u_2)\right)^{1/\alpha} \times \\ & \left((1-u_1)^{1/\alpha} + (1-u_2)^{1/\alpha} - (1-u_1)^{1/\alpha}(1-u_2)^{1/\alpha}\right)^{-\alpha-2} \end{aligned}$$

The bivariate Pareto copula

In this Figure, we have reported the density of the Pareto copula when α is equal to 1 and 10.



The bivariate Pareto copula

Question 4

Show that the bivariate Pareto copula function has no lower tail dependence, but an upper tail dependence.

The bivariate Pareto copula

We have:

$$\begin{aligned}
\lambda^{-} &= \lim_{u \to 0^+} \frac{\mathbf{C}(u,u)}{u} \\
&= 2 \lim_{u \to 0^+} \frac{\partial\, \mathbf{C}(u,u)}{\partial\, u_1} \\
&= 2 \lim_{u \to 0^+} 1 - \left((1-u)^{-1/\alpha} + (1-u)^{-1/\alpha} - 1 \right)^{-\alpha-1} (1-u)^{-1/\alpha-1} \\
&= 2 \lim_{u \to 0^+} (1-1) \\
&= 0
\end{aligned}$$

The bivariate Pareto copula

We have:

$$\begin{aligned}
\lambda^{+} &= \lim_{u \to 1^{-}} \frac{1 - 2u + \mathbf{C}(u, u)}{1 - u} \\
&= \lim_{u \to 1^{-}} \frac{\left((1-u)^{-1/\alpha} + (1-u)^{-1/\alpha} - 1\right)^{-\alpha}}{1 - u} \\
&= \lim_{u \to 1^{-}} \left(1 + 1 - (1-u)^{1/\alpha}\right)^{-\alpha} \\
&= 2^{-\alpha}
\end{aligned}$$

The tail dependence coefficients λ^{-} and λ^{+} are given with respect to the parameter α in previous Figure. We deduce that the bivariate Pareto copula function has no lower tail dependence ($\lambda^{-} = 0$), but an upper tail dependence ($\lambda^{+} = 2^{-\alpha}$).

The bivariate Pareto copula

Question 5

Do you think that the bivariate Pareto copula family can reach the copula functions $\mathbf{C}^-$, $\mathbf{C}^\perp$ and $\mathbf{C}^+$? Justify your answer.

The bivariate Pareto copula

The bivariate Pareto copula family cannot reach $\mathbf{C}^{-}$ because λ^{-} is never equal to 1. We notice that:

$$\lim_{\alpha \to \infty} \lambda^{+} = 0$$

and

$$\lim_{\alpha \to 0} \lambda^{+} = 1$$

This implies that the bivariate Pareto copula may reach $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$ for these two limit cases: $\alpha \to \infty$ and $\alpha \to 0$. In fact, $\alpha \to 0$ does not correspond to the copula $\mathbf{C}^{+}$ because λ^{-} is always equal to 0.

The bivariate Pareto copula

Question 6

Let X_1 and X_2 be two Pareto-distributed random variables, whose parameters are (α_1, θ_1) and (α_2, θ_2).

The bivariate Pareto copula

Question 6.a

Show that the linear correlation between X_1 and X_2 is equal to 1 if and only if the parameters α_1 and α_2 are equal.

The bivariate Pareto copula

We note $U_1 = \mathbf{F}_1(X_1)$ and $U_2 = \mathbf{F}_2(X_2)$. X_1 and X_2 are comonotonic if and only if:

$$U_2 = U_1$$

We deduce that:

$$\begin{aligned} 1 - \left(\frac{\theta_2 + X_2}{\theta_2}\right)^{-\alpha_2} &= 1 - \left(\frac{\theta_1 + X_1}{\theta_1}\right)^{-\alpha_1} \\ \Leftrightarrow \quad \left(\frac{\theta_2 + X_2}{\theta_2}\right)^{-\alpha_2} &= \left(\frac{\theta_1 + X_1}{\theta_1}\right)^{-\alpha_1} \\ \Leftrightarrow \quad X_2 &= \theta_2\left(\left(\frac{\theta_1 + X_1}{\theta_1}\right)^{\alpha_1/\alpha_2} - 1\right) \end{aligned}$$

We know that $\rho\langle X_1, X_2\rangle = 1$ if and only if there is an increasing linear relationship between X_1 and X_2. This implies that:

$$\frac{\alpha_1}{\alpha_2} = 1$$

The bivariate Pareto copula

Question 6.b

Show that the linear correlation between X_1 and X_2 can never reached the lower bound -1.

The bivariate Pareto copula

X_1 and X_2 are countermonotonic if and only if:

$$U_2 = 1 - U_1$$

We deduce that:

$$\begin{aligned}
& \left(\frac{\theta_2 + X_2}{\theta_2}\right)^{-\alpha_2} = 1 - \left(\frac{\theta_1 + X_1}{\theta_1}\right)^{-\alpha_1} \\
\Leftrightarrow \quad & \left(\frac{\theta_2 + X_2}{\theta_2}\right)^{-\alpha_2} = 1 - \left(\frac{\theta_1 + X_1}{\theta_1}\right)^{-\alpha_1} \\
\Leftrightarrow \quad & X_2 = \theta_2 \left(\left(1 - \left(\frac{\theta_1 + X_1}{\theta_1}\right)^{-\alpha_1}\right)^{1/\alpha_2} - 1\right)
\end{aligned}$$

It is not possible to obtain a decreasing linear function between X_1 and X_2. This implies that $\rho \langle X_1, X_2 \rangle > -1$.

The bivariate Pareto copula

Question 6.c

Build a new bivariate Pareto distribution by assuming that the marginal distributions are $\mathcal{P}(\alpha_1, \theta_1)$ and $\mathcal{P}(\alpha_2, \theta_2)$ and the dependence is a bivariate Pareto copula function with parameter α. What is the relevance of this approach for building bivariate Pareto distributions?

The bivariate Pareto copula

We have:

$$\begin{aligned}\mathbf{F}'(x_1, x_2) &= \mathbf{C}\left(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)\right) \\ &= 1 - \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha_1} - \left(\frac{\theta_2 + x_2}{\theta_2}\right)^{-\alpha_2} + \\ &\quad \left(\left(\frac{\theta_1 + x_1}{\theta_1}\right)^{\alpha_1/\alpha} + \left(\frac{\theta_2 + x_2}{\theta_2}\right)^{\alpha_2/\alpha} - 1\right)^{-\alpha}\end{aligned}$$

The traditional bivariate Pareto distribution $\mathbf{F}(x_1, x_2)$ is a special case of $\mathbf{F}'(x_1, x_2)$ when:

$$\alpha_1 = \alpha_2 = \alpha$$

Using $\mathbf{F}'$ instead of $\mathbf{F}$, we can control the tail dependence, but also the univariate tail index of the two margins.

Calculation of correlation bounds

Question 1

Give the mathematical definition of the copula functions $\mathbf{C}^{-}$, $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$. What is the probabilistic interpretation of these copulas?

Calculation of correlation bounds

We have:

$$\begin{aligned}
\mathbf{C}^{-}(u_1, u_2) &= \max(u_1 + u_2 - 1, 0) \\
\mathbf{C}^{\perp}(u_1, u_2) &= u_1 u_2 \\
\mathbf{C}^{+}(u_1, u_2) &= \min(u_1, u_2)
\end{aligned}$$

Let X_1 and X_2 be two random variables. We have:

(i) $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^{-}$ if and only if there exists a non-increasing function f such that we have $X_2 = f(X_1)$;

(ii) $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^{\perp}$ if and only if X_1 and X_2 are independent;

(iii) $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^{+}$ if and only if there exists a non-decreasing function f such that we have $X_2 = f(X_1)$.

Calculation of correlation bounds

Question 2

We note $\boldsymbol{\tau}$ and LGD the default time and the loss given default of a counterparty. We assume that $\boldsymbol{\tau} \sim \mathcal{E}(\lambda)$ and $\mathrm{LGD} \sim \mathcal{U}_{[0,1]}$.

Calculation of correlation bounds

We note $U_1 = 1 - \exp(-\lambda \boldsymbol{\tau})$ and $U_2 = \mathrm{LGD}$.

Calculation of correlation bounds

Question 2.a

Show that the dependence between $\boldsymbol{\tau}$ and LGD is maximum when the following equality holds:

$$\mathrm{LGD} + e^{-\lambda \boldsymbol{\tau}} - 1 = 0$$

Calculation of correlation bounds

The dependence between $\boldsymbol{\tau}$ and LGD is maximum when we have $\mathbf{C}\langle\boldsymbol{\tau},\mathrm{LGD}\rangle=\mathbf{C}^{+}$. Since we have $U_1=U_2$, we conclude that:

$$\mathrm{LGD}+e^{-\lambda\boldsymbol{\tau}}-1=0$$

Calculation of correlation bounds

Question 2.b

Show that the linear correlation $\rho(\boldsymbol{\tau}, \mathrm{LGD})$ verifies the following inequality:

$$\left|\rho\left\langle\boldsymbol{\tau}, \mathrm{LGD}\right\rangle\right| \leq \frac{\sqrt{3}}{2}$$

Calculation of correlation bounds

We know that:

$$\rho \left\langle \boldsymbol{\tau}, \mathrm{LGD} \right\rangle \in \left[\rho_{\min} \left\langle \boldsymbol{\tau}, \mathrm{LGD} \right\rangle , \rho_{\max} \left\langle \boldsymbol{\tau}, \mathrm{LGD} \right\rangle \right]$$

where $\rho_{\min} \left\langle \boldsymbol{\tau}, \mathrm{LGD} \right\rangle$ (resp. $\rho_{\max} \left\langle \boldsymbol{\tau}, \mathrm{LGD} \right\rangle$) is the linear correlation corresponding to the copula $\mathbf{C}^{-}$ (resp. $\mathbf{C}^{+}$). It comes that:

$$\mathbb{E} \left[\boldsymbol{\tau} \right] = \sigma \left(\boldsymbol{\tau} \right) = \frac{1}{\lambda}$$

and:

$$\begin{aligned} \mathbb{E} \left[\mathrm{LGD} \right] &= \frac{1}{2} \\ \sigma \left(\mathrm{LGD} \right) &= \sqrt{\frac{1}{12}} \end{aligned}$$

Calculation of correlation bounds

In the case $\mathbf{C}\langle\boldsymbol{\tau}, \mathrm{LGD}\rangle = \mathbf{C}^{-}$, we have $U_1 = 1 - U_2$. It follows that $\mathrm{LGD} = e^{-\lambda\boldsymbol{\tau}}$. We have:

$$\begin{aligned}
\mathbb{E}\left[\boldsymbol{\tau}\,\mathrm{LGD}\right] = \mathbb{E}\left[\boldsymbol{\tau} e^{-\lambda\boldsymbol{\tau}}\right] &= \int_0^\infty t e^{-\lambda t} \lambda e^{-\lambda t}\,\mathrm{d}t \\
&= \int_0^\infty t\lambda e^{-2\lambda t}\,\mathrm{d}t \\
&= \left[-\frac{t e^{-2\lambda t}}{2}\right]_0^\infty + \frac{1}{2}\int_0^\infty e^{-2\lambda t}\,\mathrm{d}t \\
&= 0 + \frac{1}{2}\left[-\frac{e^{-2\lambda t}}{2\lambda}\right]_0^\infty \\
&= \frac{1}{4\lambda}
\end{aligned}$$

We deduce that:

$$\rho_{\min}\langle\boldsymbol{\tau}, \mathrm{LGD}\rangle = \left(\frac{1}{4\lambda} - \frac{1}{2\lambda}\right) \Big/ \left(\frac{1}{\lambda}\sqrt{\frac{1}{12}}\right) = -\frac{\sqrt{3}}{2}$$

Calculation of correlation bounds

In the case $\mathbf{C}\langle\boldsymbol{\tau},\mathrm{LGD}\rangle = \mathbf{C}^{+}$, we have $\mathrm{LGD} = 1 - e^{-\lambda\tau}$. We have:

$$\begin{aligned}
\mathbb{E}\left[\boldsymbol{\tau}\,\mathrm{LGD}\right] = \mathbb{E}\left[\boldsymbol{\tau}\left(1-e^{-\lambda\boldsymbol{\tau}}\right)\right] &= \int_0^\infty t\left(1-e^{-\lambda t}\right)\lambda e^{-\lambda t}\,\mathrm{d}t \\
&= \int_0^\infty t\lambda e^{-\lambda t}\,\mathrm{d}t - \int_0^\infty t\lambda e^{-2\lambda t}\,\mathrm{d}t \\
&= \left(\left[-te^{-\lambda t}\right]_0^\infty + \int_0^\infty e^{-\lambda t}\,\mathrm{d}t\right) - \frac{1}{4\lambda} \\
&= 0 + \left[-\frac{e^{-\lambda t}}{\lambda}\right]_0^\infty - \frac{1}{4\lambda} \\
&= \frac{3}{4\lambda}
\end{aligned}$$

We deduce that:

$$\rho_{\max}\langle\boldsymbol{\tau},\mathrm{LGD}\rangle = \left(\frac{3}{4\lambda}-\frac{1}{2\lambda}\right)\Big/\left(\frac{1}{\lambda}\sqrt{\frac{1}{12}}\right) = \frac{\sqrt{3}}{2}$$

Calculation of correlation bounds

We finally obtain the following result:

$$\left|\rho\left\langle\boldsymbol{\tau},\mathrm{LGD}\right\rangle\right|\leq\frac{\sqrt{3}}{2}$$

Calculation of correlation bounds

Question 2.c

Comment on these results.

Calculation of correlation bounds

We notice that $|\rho\left\langle \boldsymbol{\tau}, \mathrm{LGD}\right\rangle|$ is lower than 86.6%, implying that the bounds -1 and $+1$ can not be reached.

Calculation of correlation bounds

Question 3

We consider two exponential default times $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ with parameters λ_1 and λ_2.

Calculation of correlation bounds

Question 3.a

We assume that the dependence function between $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ is $\mathbf{C}^+$. Demonstrate that the following relation is true:

$$\boldsymbol{\tau}_1 = \frac{\lambda_2}{\lambda_1}\boldsymbol{\tau}_2$$

Calculation of correlation bounds

If the copula function of $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ is the Fréchet upper bound copula, $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are comonotone. We deduce that:

$$U_1 = U_2 \Longleftrightarrow 1 - e^{-\lambda_1 \boldsymbol{\tau}_1} = 1 - e^{-\lambda_2 \boldsymbol{\tau}_2}$$

and:

$$\boldsymbol{\tau}_1 = \frac{\lambda_2}{\lambda_1} \boldsymbol{\tau}_2$$

Calculation of correlation bounds

Question 3.b

Show that there exists a function f such that $\boldsymbol{\tau}_2 = f\left(\boldsymbol{\tau}_2\right)$ when the dependence function is $\mathbf{C}^-$.

Calculation of correlation bounds

We have $U_1 = 1 - U_2$. It follows that $\mathbf{S}_1(\boldsymbol{\tau}_1) = 1 - \mathbf{S}_2(\boldsymbol{\tau}_2)$. We deduce that:

$$e^{-\lambda_1 \boldsymbol{\tau}_1} = 1 - e^{-\lambda_2 \boldsymbol{\tau}_2}$$

and:

$$\boldsymbol{\tau}_1 = \frac{-\ln\left(1 - e^{-\lambda_2 \boldsymbol{\tau}_2}\right)}{\lambda_1}$$

There exists then a function f such that $\boldsymbol{\tau}_1 = f(\boldsymbol{\tau}_2)$ with:

$$f(t) = \frac{-\ln\left(1 - e^{-\lambda_2 t}\right)}{\lambda_1}$$

Calculation of correlation bounds

Question 3.c

Show that the lower and upper bounds of the linear correlation satisfy the following relationship:

$$-1 < \rho \langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle \leq 1$$

Calculation of correlation bounds

Using Question 2(b), we known that $\rho \in [\rho_{\min}, \rho_{\max}]$ where $\rho_{\min}$ and $\rho_{\max}$ are the correlations of $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ when the copula function is respectively $\mathbf{C}^-$ and $\mathbf{C}^+$. We also know that $\rho = 1$ (resp. $\rho = -1$) if there exists a linear and increasing (resp. decreasing) function f such that $\boldsymbol{\tau}_1 = f(\boldsymbol{\tau}_2)$. When the copula is $\mathbf{C}^+$, we have $f(t) = \frac{\lambda_2}{\lambda_1} t$ and $f'(t) = \frac{\lambda_2}{\lambda_1} > 0$. As it is a linear and increasing function, we deduce that $\rho_{\max} = 1$. When the copula is $\mathbf{C}^-$, we have:

$$f(t) = \frac{-\ln\left(1 - e^{-\lambda_2 t}\right)}{\lambda_1}$$

and:

$$f'(t) = -\frac{\lambda_2 e^{-\lambda_2 t} \ln\left(1 - e^{-\lambda_2 t}\right)}{\lambda_1 \left(1 - e^{-\lambda_2 t}\right)} < 0$$

The function $f(t)$ is decreasing, but it is not linear. We deduce that $\rho_{\min} \neq -1$ and:

$$-1 < \rho \leq 1$$

Calculation of correlation bounds

Question 3.d

In the more general case, show that the linear correlation of a random vector (X_1, X_2) can not be equal to -1 if the support of the random variables X_1 and X_2 is $[0, +\infty]$.

Calculation of correlation bounds

When the copula is $\mathbf{C}^{-}$, we know that there exists a decreasing function f such that $X_2 = f(X_1)$. We also know that the linear correlation reaches the lower bound -1 if the function f is linear:

$$X_2 = a + bX_1$$

This implies that $b < 0$. When X_1 takes the value $+\infty$, we obtain:

$$X_2 = a + b \times \infty$$

As the lower bound of X_2 is equal to zero 0, we deduce that $a = +\infty$. This means that the function $f(x) = a + bx$ does not exist. We conclude that the lower bound $\rho = -1$ can not be reached.

Calculation of correlation bounds

Question 4

We assume that (X_1, X_2) is a Gaussian random vector where $X_1 \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right)$, $X_2 \sim \mathcal{N}\left(\mu_2, \sigma_2^2\right)$ and ρ is the linear correlation between X_1 and X_2. We note $\theta = (\mu_1, \sigma_1, \mu_2, \sigma_2, \rho)$ the set of parameters.

Calculation of correlation bounds

Question 4.a

Find the probability distribution of $X_1 + X_2$.

Calculation of correlation bounds

$X_1 + X_2$ is a Gaussian random variable because it is a linear combination of the Gaussian random vector (X_1, X_2). We have:

$$\mathbb{E}[X_1 + X_2] = \mu_1 + \mu_2$$

and:

$$\operatorname{var}(X_1 + X_2) = \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2$$

We deduce that:

$$X_1 + X_2 \sim \mathcal{N}\left(\mu_1 + \mu_2, \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2\right)$$

Calculation of correlation bounds

Question 4.b

Then show that the covariance between $Y_1 = e^{X_1}$ and $Y_2 = e^{X_2}$ is equal to:

$$\operatorname{cov}(Y_1, Y_2) = e^{\mu_1 + \frac{1}{2}\sigma_1^2} e^{\mu_2 + \frac{1}{2}\sigma_2^2} \left(e^{\rho\sigma_1\sigma_2} - 1\right)$$

Calculation of correlation bounds

We have:

$$\begin{aligned}
\operatorname{cov}(Y_1, Y_2) &= \mathbb{E}[Y_1 Y_2] - \mathbb{E}[Y_2]\,\mathbb{E}[Y_2] \\
&= \mathbb{E}\left[e^{X_1+X_2}\right] - \mathbb{E}[Y_2]\,\mathbb{E}[Y_2]
\end{aligned}$$

We know that $e^{X_1+X_2}$ is a lognormal random variable. We deduce that:

$$\begin{aligned}
\mathbb{E}\left[e^{X_1+X_2}\right] &= \exp\left(\mathbb{E}[X_1+X_2] + \frac{1}{2}\operatorname{var}(X_1+X_2)\right) \\
&= \exp\left(\mu_1+\mu_2+\frac{1}{2}\left(\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2\right)\right) \\
&= e^{\mu_1+\frac{1}{2}\sigma_1^2}e^{\mu_2+\frac{1}{2}\sigma_2^2}e^{\rho\sigma_1\sigma_2}
\end{aligned}$$

We finally obtain:

$$\operatorname{cov}(Y_1, Y_2) = e^{\mu_1+\frac{1}{2}\sigma_1^2}e^{\mu_2+\frac{1}{2}\sigma_2^2}\left(e^{\rho\sigma_1\sigma_2}-1\right)$$

Calculation of correlation bounds

Question 4.c

Deduce the correlation between Y_1 and Y_2.

Calculation of correlation bounds

We have:

$$
\begin{aligned}
\rho \left\langle Y_1, Y_2 \right\rangle &= \frac{e^{\mu_1 + \frac{1}{2}\sigma_1^2} e^{\mu_2 + \frac{1}{2}\sigma_2^2} \left(e^{\rho \sigma_1 \sigma_2} - 1\right)}{\sqrt{e^{2\mu_1 + \sigma_1^2} \left(e^{\sigma_1^2} - 1\right)} \sqrt{e^{2\mu_2 + \sigma_2^2} \left(e^{\sigma_2^2} - 1\right)}} \\
&= \frac{e^{\rho \sigma_1 \sigma_2} - 1}{\sqrt{e^{\sigma_1^2} - 1} \sqrt{e^{\sigma_2^2} - 1}}
\end{aligned}
$$

Calculation of correlation bounds

Question 4.d

For which values of θ does the equality $\rho \langle Y_1, Y_2 \rangle = +1$ hold? Same question when $\rho \langle Y_1, Y_2 \rangle = -1$.

Calculation of correlation bounds

$\rho\langle Y_1, Y_2\rangle$ is an increasing function with respect to ρ. We deduce that:

$$\rho\langle Y_1, Y_2\rangle = 1 \Longleftrightarrow \rho = 1 \text{ and } \sigma_1 = \sigma_2$$

The lower bound of $\rho\langle Y_1, Y_2\rangle$ is reached if ρ is equal to -1. In this case, we have:

$$\rho\langle Y_1, Y_2\rangle = \frac{e^{-\sigma_1\sigma_2} - 1}{\sqrt{e^{\sigma_1^2} - 1}\sqrt{e^{\sigma_2^2} - 1}} > -1$$

It follows that $\rho\langle Y_1, Y_2\rangle \neq -1$.

Calculation of correlation bounds

Question 4.e

We consider the bivariate Black-Scholes model:

$$\begin{cases} \mathrm{d}S_1(t) = \mu_1 S_1(t)\,\mathrm{d}t + \sigma_1 S_1(t)\,\mathrm{d}W_1(t) \\ \mathrm{d}S_2(t) = \mu_2 S_2(t)\,\mathrm{d}t + \sigma_2 S_2(t)\,\mathrm{d}W_2(t) \end{cases}$$

with $\mathbb{E}[W_1(t)\,W_2(t)] = \rho t$. Deduce the linear correlation between $S_1(t)$ and $S_2(t)$. Find the limit case $\lim_{t\to\infty} \rho\langle S_1(t), S_2(t)\rangle$.

Calculation of correlation bounds

It is obvious that:

$$\rho \langle S_1(t), S_2(t) \rangle = \frac{e^{\rho \sigma_1 \sigma_2 t} - 1}{\sqrt{e^{\sigma_1^2 t} - 1}\sqrt{e^{\sigma_2^2 t} - 1}}$$

In the case $\sigma_1 = \sigma_2$ and $\rho = 1$, we have $\rho \langle S_1(t), S_2(t) \rangle = 1$. Otherwise, we obtain:

$$\lim_{t \to \infty} \rho \langle S_1(t), S_2(t) \rangle = 0$$

Calculation of correlation bounds

Question 4.f

Comment on these results.

Calculation of correlation bounds

In the case of lognormal random variables, the linear correlation does not necessarily range between -1 and $+1$.

Extreme value theory in the bivariate case

Question 1

What is an extreme value (EV) copula $\mathbf{C}$?

Extreme value theory in the bivariate case

An extreme value copula $\mathbf{C}$ satisfies the following relationship:

$$\mathbf{C}\left(u_1^t, u_2^t\right) = \mathbf{C}^t\left(u_1, u_2\right)$$

for all $t > 0$.

Extreme value theory in the bivariate case

Question 2

Show that $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$ are EV copulas. Why $\mathbf{C}^{-}$ can not be an EV copula?

Extreme value theory in the bivariate case

The product copula $\mathbf{C}^{\perp}$ is an EV copula because we have:

$$
\begin{aligned}
\mathbf{C}^{\perp}\left(u_1^t, u_2^t\right) &= u_1^t u_2^t \\
&= \left(u_1 u_2\right)^t \\
&= \left[\mathbf{C}^{\perp}\left(u_1, u_2\right)\right]^t
\end{aligned}
$$

Extreme value theory in the bivariate case

For the copula $\mathbf{C}^{+}$, we obtain:

$$\begin{aligned} \mathbf{C}^{+}\left(u_1^t, u_2^t\right) &= \min\left(u_1^t, u_2^t\right) \\ &= \left\{ \begin{array}{ll} u_1^t & \text{if } u_1 \leq u_2 \\ u_2^t & \text{otherwise} \end{array} \right. \\ &= \left(\min\left(u_1, u_2\right)\right)^t \\ &= \left[\mathbf{C}^{+}\left(u_1, u_2\right)\right]^t \end{aligned}$$

Extreme value theory in the bivariate case

However, the EV property does not hold for the Fréchet lower bound copula $\mathbf{C}^{-}$:

$$\mathbf{C}^{-}\left(u_1^t, u_2^t\right) = \max\left(u_1^t + u_2^t - 1, 0\right) \neq \max\left(u_1 + u_2 - 1, 0\right)^t$$

Indeed, we have $\mathbf{C}^{-}(0.5, 0.8) = \max(0.5 + 0.8 - 1, 0) = 0.3$ and:

$$\begin{array}{rcl} \mathbf{C}^{-}\left(0.5^2, 0.8^2\right) & = & \max(0.25 + 0.64 - 1, 0) \\ & = & 0 \\ & \neq & 0.3^2 \end{array}$$

Extreme value theory in the bivariate case

Question 3

We define the Gumbel-Hougaard copula as follows:

$$\mathbf{C}(u_1, u_2) = \exp\left(-\left[(-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}\right]^{1/\theta}\right)$$

with $\theta \geq 1$. Verify that it is an EV copula.

Extreme value theory in the bivariate case

We have:

$$\begin{aligned}
\mathbf{C}\left(u_1^t, u_2^t\right) &= \exp\left(-\left[\left(-\ln u_1^t\right)^\theta + \left(-\ln u_2^t\right)^\theta\right]^{1/\theta}\right) \\
&= \exp\left(-\left[\left(-t \ln u_1\right)^\theta + \left(-t \ln u_2\right)^\theta\right]^{1/\theta}\right) \\
&= \exp\left(-t\left[\left(-\ln u_1\right)^\theta + \left(-\ln u_2\right)^\theta\right]^{1/\theta}\right) \\
&= \left(e^{-\left[\left(-\ln u_1\right)^\theta + \left(-\ln u_2\right)^\theta\right]^{1/\theta}}\right)^t \\
&= \mathbf{C}^t\left(u_1, u_2\right)
\end{aligned}$$

Extreme value theory in the bivariate case

Question 4

What is the definition of the upper tail dependence λ? What is its usefulness in multivariate extreme value theory?

Extreme value theory in the bivariate case

The upper tail dependence λ is defined as follows:

$$\lambda = \lim_{u \to 1^+} \frac{1 - 2u + \mathbf{C}(u_1, u_2)}{1 - u}$$

It measures the probability to have an extreme in one direction knowing that we have already an extreme in the other direction. If λ is equal to 0, extremes are independent and the EV copula is the product copula $\mathbf{C}^{\perp}$. If λ is equal to 1, extremes are comonotonic and the EV copula is the Fréchet upper bound copula $\mathbf{C}^{+}$. Moreover, the upper tail dependence of the copula between the random variables is equal to the upper tail dependence of the copula between the extremes.

Extreme value theory in the bivariate case

Question 5

Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$. If $g'(x_0) \neq 0$, L'Hospital's rule states that:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

Deduce that the upper tail dependence λ of the Gumbel-Hougaard copula is $2 - 2^{1/\theta}$. What is the correlation of two extremes when $\theta = 1$?

Extreme value theory in the bivariate case

Using L'Hospital's rule, we have:

$$\begin{aligned}
\lambda &= \lim_{u \to 1^+} \frac{1 - 2u + e^{-\left[(-\ln u)^\theta + (-\ln u)^\theta\right]^{1/\theta}}}{1-u} \\
&= \lim_{u \to 1^+} \frac{1 - 2u + e^{-\left[2(-\ln u)^\theta\right]^{1/\theta}}}{1-u} \\
&= \lim_{u \to 1^+} \frac{1 - 2u + u^{2^{1/\theta}}}{1-u} \\
&= \lim_{u \to 1^+} \frac{0 - 2 + 2^{1/\theta} u^{2^{1/\theta}-1}}{-1} \\
&= \lim_{u \to 1^+} 2 - 2^{1/\theta} u^{2^{1/\theta}-1} \\
&= 2 - 2^{1/\theta}
\end{aligned}$$

Extreme value theory in the bivariate case

If θ is equal to 1, we obtain $\lambda = 0$. It comes that the EV copula is the product copula. Extremes are then not correlated. This result is not surprising because the Gumbel-Houggard copula is equal to the product copula when $\theta = 1$:

$$e^{-\left[(-\ln u_1)^1 + (-\ln u_2)^1\right]^1} = u_1 u_2 = \mathbf{C}^{\perp}(u_1, u_2)$$

Extreme value theory in the bivariate case

Question 6

We define the Marshall-Olkin copula as follows:

$$\mathbf{C}(u_1, u_2) = u_1^{1-\theta_1} u_2^{1-\theta_2} \min\left(u_1^{\theta_1}, u_2^{\theta_2}\right)$$

with $\{\theta_1, \theta_2\} \in [0,1]^2$.

Extreme value theory in the bivariate case

Question 6.a

Verify that it is an EV copula.

Extreme value theory in the bivariate case

We have:

$$
\begin{aligned}
\mathbf{C}\left(u_1^t, u_2^t\right) &= u_1^{t(1-\theta_1)} u_2^{t(1-\theta_2)} \min\left(u_1^{t\theta_1}, u_2^{t\theta_2}\right) \\
&= \left(u_1^{1-\theta_1}\right)^t \left(u_2^{1-\theta_2}\right)^t \left(\min\left(u_1^{\theta_1}, u_2^{\theta_2}\right)\right)^t \\
&= \left(u_1^{1-\theta_1} u_2^{1-\theta_2} \min\left(u_1^{\theta_1}, u_2^{\theta_2}\right)\right)^t \\
&= \mathbf{C}^t\left(u_1, u_2\right)
\end{aligned}
$$

Extreme value theory in the bivariate case

Question 6.b

Find the upper tail dependence λ of the Marshall-Olkin copula.

Extreme value theory in the bivariate case

If $\theta_1 > \theta_2$, we obtain:

$$\begin{aligned}
\lambda &= \lim_{u \to 1^+} \frac{1 - 2u + u^{1-\theta_1} u^{1-\theta_2} \min\left(u^{\theta_1}, u^{\theta_2}\right)}{1-u} \\
&= \lim_{u \to 1^+} \frac{1 - 2u + u^{1-\theta_1} u^{1-\theta_2} u^{\theta_1}}{1-u} \\
&= \lim_{u \to 1^+} \frac{1 - 2u + u^{2-\theta_2}}{1-u} \\
&= \lim_{u \to 1^+} \frac{0 - 2 + (2-\theta_2) u^{1-\theta_2}}{-1} \\
&= \lim_{u \to 1^+} 2 - 2u^{1-\theta_2} + \theta_2 u^{1-\theta_2} \\
&= \theta_2
\end{aligned}$$

If $\theta_2 > \theta_1$, we have $\lambda = \theta_1$. We deduce that the upper tail dependence of the Marshall-Olkin copula is $\min(\theta_1, \theta_2)$.

Extreme value theory in the bivariate case

Question 6.c

What is the correlation of two extremes when $\min(\theta_1, \theta_2) = 0$?

Extreme value theory in the bivariate case

If $\theta_1 = 0$ or $\theta_2 = 0$, we obtain $\lambda = 0$. It comes that the copula of the extremes is the product copula. Extremes are then not correlated.

Extreme value theory in the bivariate case

Question 6.d

In which case are two extremes perfectly correlated?

Extreme value theory in the bivariate case

Two extremes are perfectly correlated when we have $\theta_1 = \theta_2 = 1$. In this case, we obtain:

$$\mathbf{C}(u_1, u_2) = \min(u_1, u_2) = \mathbf{C}^+(u_1, u_2)$$

Maximum domain of attraction in the bivariate case

Question 1

We consider the following distributions of probability:

Distribution		$\mathbf{F}(x)$
Exponential	$\mathcal{E}(\lambda)$	$1 - e^{-\lambda x}$
Uniform	$\mathcal{U}_{[0,1]}$	x
Pareto	$\mathcal{P}(\alpha, \theta)$	$1 - \left(\frac{\theta + x}{\theta}\right)^{-\alpha}$

Maximum domain of attraction in the bivariate case

Question 1

For each distribution, we give the normalization parameters a_n and b_n of the Fisher-Tippet theorem and the corresponding limit distribution distribution $\mathbf{G}(x)$:

Distribution	a_n	b_n	$\mathbf{G}(x)$
Exponential	λ^{-1}	$\lambda^{-1} \ln n$	$\mathbf{\Lambda}(x) = e^{-e^{-x}}$
Uniform	n^{-1}	$1 - n^{-1}$	$\mathbf{\Psi}_1(x-1) = e^{x-1}$
Pareto	$\theta\alpha^{-1}n^{1/\alpha}$	$\theta n^{1/\alpha} - \theta$	$\mathbf{\Phi}_\alpha\left(1+\frac{x}{\alpha}\right) = e^{-\left(1+\frac{x}{\alpha}\right)^{-\alpha}}$

We note $\mathbf{G}(x_1, x_2)$ the asymptotic distribution of the bivariate random vector $(X_{1,n:n}, X_{2,n:n})$ where $X_{1,i}$ (resp. $X_{2,i}$) are *iid* random variables.

Maximum domain of attraction in the bivariate case

Let $(X_{1,}X_2)$ be a bivariate random variable whose probability distribution is:

$$\mathbf{F}(x_1, x_2) = \mathbf{C}_{\langle X_1, X_2 \rangle}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2))$$

We know that the corresponding EV probability distribution is:

$$\mathbf{G}(x_1, x_2) = \mathbf{C}^{\star}_{\langle X_1, X_2 \rangle}(\mathbf{G}_1(x_1), \mathbf{G}_2(x_2))$$

where $\mathbf{G}_1$ and $\mathbf{G}_2$ are the two univariate EV probability distributions and $\mathbf{C}^{\star}_{\langle X_1, X_2 \rangle}$ is the EV copula associated to $\mathbf{C}_{\langle X_1, X_2 \rangle}$.

Maximum domain of attraction in the bivariate case

Question 1.a

What is the expression of $\mathbf{G}(x_1, x_2)$ when $X_{1,i}$ and $X_{2,i}$ are independent, $X_{1,i} \sim \mathcal{E}(\lambda)$ and $X_{2,i} \sim \mathcal{U}_{[0,1]}$?

Maximum domain of attraction in the bivariate case

We deduce that:

$$\begin{aligned} \mathbf{G}(x_1, x_2) &= \mathbf{C}^{\perp}(\mathbf{G}_1(x_1), \mathbf{G}_2(x_2)) \\ &= \mathbf{\Lambda}(x_1)\,\mathbf{\Psi}_1(x_2 - 1) \\ &= \exp\left(-e^{-x_1} + x_2 - 1\right) \end{aligned}$$

Maximum domain of attraction in the bivariate case

Question 1.b

Same question when $X_{1,i} \sim \mathcal{E}(\lambda)$ and $X_{2,i} \sim \mathcal{P}(\theta, \alpha)$.

Maximum domain of attraction in the bivariate case

We have:

$$\begin{aligned} \mathbf{G}(x_1, x_2) &= \mathbf{\Lambda}(x_1)\,\mathbf{\Phi}_\alpha\left(1 + \frac{x_2}{\alpha}\right) \\ &= \exp\left(-e^{-x_1} - \left(1 + \frac{x_2}{\alpha}\right)^{-\alpha}\right) \end{aligned}$$

Maximum domain of attraction in the bivariate case

Question 1.c

Same question when $X_{1,i} \sim \mathcal{U}_{[0,1]}$ and $X_{2,i} \sim \mathcal{P}(\theta, \alpha)$.

Maximum domain of attraction in the bivariate case

We have:

$$\begin{aligned}\mathbf{G}(x_1, x_2) &= \boldsymbol{\Psi}_1(x_1 - 1)\,\boldsymbol{\Phi}_\alpha\left(1 + \frac{x_2}{\alpha}\right) \\ &= \exp\left(x_1 - 1 - \left(1 + \frac{x_2}{\alpha}\right)^{-\alpha}\right)\end{aligned}$$

Maximum domain of attraction in the bivariate case

Question 2

What becomes the previous results when the dependence function between $X_{1,i}$ and $X_{2,i}$ is the Normal copula with parameter $\rho < 1$?

Maximum domain of attraction in the bivariate case

We know that the upper tail dependence is equal to zero for the Normal copula when $\rho < 1$. We deduce that the EV copula is the product copula. We then obtain the same results as previously.

Maximum domain of attraction in the bivariate case

Question 3

Same question when the parameter of the Normal copula is equal to one.

Maximum domain of attraction in the bivariate case

When the parameter ρ is equal to 1, the Normal copula is the Frchet upper bound copula $\mathbf{C}^{+}$, which is an EV copula. We deduce the following results:

$$\begin{aligned}\mathbf{G}(x_1, x_2) &= \min\left(\mathbf{\Lambda}(x_1), \mathbf{\Psi}_1(x_2 - 1)\right) \\ &= \min\left(\exp\left(-e^{-x_1}\right), \exp(x_2 - 1)\right) \qquad \text{(a)}\end{aligned}$$

$$\begin{aligned}\mathbf{G}(x_1, x_2) &= \min\left(\mathbf{\Lambda}(x_1), \mathbf{\Phi}_\alpha\left(1 + \frac{x_2}{\alpha}\right)\right) \\ &= \min\left(\exp\left(-e^{-x_1}\right), \exp\left(-\left(1 + \frac{x_2}{\alpha}\right)^{-\alpha}\right)\right) \qquad \text{(b)}\end{aligned}$$

$$\begin{aligned}\mathbf{G}(x_1, x_2) &= \min\left(\mathbf{\Psi}_1(x_1 - 1), \mathbf{\Phi}_\alpha\left(1 + \frac{x_2}{\alpha}\right)\right) \\ &= \min\left(\exp(x_2 - 1), \exp\left(-\left(1 + \frac{x_2}{\alpha}\right)^{-\alpha}\right)\right) \qquad \text{(c)}\end{aligned}$$

Maximum domain of attraction in the bivariate case

Question 4

Find the expression of $\mathbf{G}(x_1, x_2)$ when the dependence function is the Gumbel-Hougaard copula.

Maximum domain of attraction in the bivariate case

In the previous exercise, we have shown that the Gumbel-Houggard copula is an EV copula.

Maximum domain of attraction in the bivariate case

We deduce that:

$$\begin{aligned} \mathbf{G}(x_1, x_2) &= e^{-\left[(-\ln \mathbf{\Lambda}(x_1))^{\theta} + (-\ln \mathbf{\Psi}_1(x_2 - 1))^{\theta}\right]^{1/\theta}} \\ &= \exp\left(-\left[e^{-\theta x_1} + (1 - x_2)^{\theta}\right]^{1/\theta}\right) \qquad \text{(a)} \end{aligned}$$

$$\begin{aligned} \mathbf{G}(x_1, x_2) &= e^{-\left[(-\ln \mathbf{\Lambda}(x_1))^{\theta} + \left(-\ln \mathbf{\Phi}_{\alpha}\left(1 + \frac{x_2}{\alpha}\right)\right)^{\theta}\right]^{1/\theta}} \\ &= \exp\left(-\left[e^{-\theta x_1} + \left(1 + \frac{x_2}{\alpha}\right)^{-\alpha\theta}\right]^{1/\theta}\right) \qquad \text{(b)} \end{aligned}$$

$$\begin{aligned} \mathbf{G}(x_1, x_2) &= e^{-\left[(-\ln \mathbf{\Psi}_1(x_1 - 1))^{\theta} + \left(-\ln \mathbf{\Phi}_{\alpha}\left(1 + \frac{x_2}{\alpha}\right)\right)^{\theta}\right]^{1/\theta}} \\ &= \exp\left(-\left[(1 - x_1)^{\theta} + \left(1 + \frac{x_2}{\alpha}\right)^{-\alpha\theta}\right]^{1/\theta}\right) \qquad \text{(c)} \end{aligned}$$

Simulation of the bivariate Normal copula

Exercise

Let $X = (X_1, X_2)$ be a standard Gaussian vector with correlation ρ. We note $U_1 = \Phi(X_1)$ and $U_2 = \Phi(X_2)$.

Simulation of the bivariate Normal copula

Question 1

We note Σ the matrix defined as follows:

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

Calculate the Cholesky decomposition of Σ. Deduce an algorithm to simulate X.

Simulation of the bivariate Normal copula

P is a lower triangular matrix such that we have $\Sigma = PP^\top$. We know that:

$$P = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$$

We verify that:

$$\begin{aligned} PP^\top &= \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} 1 & \rho \\ 0 & \sqrt{1-\rho^2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \end{aligned}$$

Simulation of the bivariate Normal copula

We deduce that:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$$

where N_1 and N_2 are two independent standardized Gaussian random variables. Let n_1 and n_2 be two independent random variates, whose probability distribution is $\mathcal{N}(0,1)$. Using the Cholesky decomposition, we deduce that can simulate X in the following way:

$$\left\{ \begin{array}{l} x_1 \leftarrow n_1 \\ x_2 \leftarrow \rho n_1 + \sqrt{1-\rho^2} n_2 \end{array} \right.$$

Simulation of the bivariate Normal copula

Question 2

Show that the copula of (X_1, X_2) is the same that the copula of the random vector (U_1, U_2).

Simulation of the bivariate Normal copula

We have

$$\begin{aligned}\mathbf{C}\left\langle X_1, X_2\right\rangle &= \mathbf{C}\left\langle \Phi\left(X_1\right), \Phi\left(X_2\right)\right\rangle \\ &= \mathbf{C}\left\langle U_1, U_2\right\rangle\end{aligned}$$

because the function $\Phi(x)$ is non-decreasing. The copula of $U = (U_1, U_2)$ is then the copula of $X = (X_1, X_2)$.

Simulation of the bivariate Normal copula

Question 3

Deduce an algorithm to simulate the Normal copula with parameter ρ.

Simulation of the bivariate Normal copula

We deduce that we can simulate U with the following algorithm:

$$\left\{ \begin{array}{l} u_1 \leftarrow \Phi(x_1) = \Phi(n_1) \\ u_2 \leftarrow \Phi(x_2) = \Phi\left(\rho n_1 + \sqrt{1-\rho^2} n_2\right) \end{array} \right.$$

Simulation of the bivariate Normal copula

Question 4

Calculate the conditional distribution of X_2 knowing that $X_1 = x$. Then show that:

$$\Phi_2(x_1, x_2; \rho) = \int_{-\infty}^{x_1} \Phi\left(\frac{x_2 - \rho x}{\sqrt{1-\rho^2}}\right) \phi(x)\, \mathrm{d}x$$

Simulation of the bivariate Normal copula

Let X_3 be a Gaussian random variable, which is independent from X_1 and X_2. Using the Cholesky decomposition, we know that:

$$X_2 = \rho X_1 + \sqrt{1-\rho^2} X_3$$

It follows that:

$$\begin{aligned}
\Pr\left\{ X_2 \leq x_2 \mid X_1 = x \right\} &= \Pr\left\{ \rho X_1 + \sqrt{1-\rho^2} X_3 \leq x_2 \,\middle|\, X_1 = x \right\} \\
&= \Pr\left\{ X_3 \leq \frac{x_2 - \rho x}{\sqrt{1-\rho^2}} \right\} \\
&= \Phi\left(\frac{x_2 - \rho x}{\sqrt{1-\rho^2}} \right)
\end{aligned}$$

Simulation of the bivariate Normal copula

Then we deduce that:

$$\begin{aligned}
\Phi_2\left(x_1, x_2; \rho\right) &= \Pr\left\{X_1 \leq x_1, X_2 \leq x_2\right\} \\
&= \Pr\left\{X_1 \leq x_1, X_3 \leq \frac{x_2 - \rho X_1}{\sqrt{1-\rho^2}}\right\} \\
&= \mathbb{E}\left[\Pr\left\{X_1 \leq x_1, X_3 \leq \frac{x_2 - \rho X_1}{\sqrt{1-\rho^2}} \right| X_1\right\}\right] \\
&= \int_{-\infty}^{x_1} \Phi\left(\frac{x_2 - \rho x}{\sqrt{1-\rho^2}}\right) \phi\left(x\right) \,\mathrm{d}x
\end{aligned}$$

Simulation of the bivariate Normal copula

Question 5

Deduce an expression of the Normal copula.

Simulation of the bivariate Normal copula

Using the relationships $u_1 = \Phi(x_1)$, $u_2 = \Phi(x_2)$ and $\Phi_2(x_1, x_2; \rho) = \mathbf{C}(\Phi(x_1), \Phi(x_2); \rho)$, we obtain:

$$\begin{aligned} \mathbf{C}(u_1, u_2; \rho) &= \int_{-\infty}^{\Phi^{-1}(u_1)} \Phi\left(\frac{\Phi^{-1}(u_2) - \rho x}{\sqrt{1-\rho^2}}\right) \phi(x) \,\mathrm{d}x \\ &= \int_0^{u_1} \Phi\left(\frac{\Phi^{-1}(u_2) - \rho \Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right) \mathrm{d}u \end{aligned}$$

Simulation of the bivariate Normal copula

Question 6

Calculate the conditional copula function $\mathbf{C}_{2|1}$. Deduce an algorithm to simulate the Normal copula with parameter ρ.

Simulation of the bivariate Normal copula

We have:

$$\begin{aligned}\mathbf{C}_{2|1}(u_2 \mid u_1) &= \partial_{u_1}\mathbf{C}(u_1, u_2) \\ &= \Phi\left(\frac{\Phi^{-1}(u_2) - \rho\Phi^{-1}(u_1)}{\sqrt{1-\rho^2}}\right)\end{aligned}$$

Let v_1 and v_2 be two independent uniform random variates. The simulation algorithm corresponds to the following steps:

$$\left\{\begin{array}{l} u_1 = v_1 \\ \mathbf{C}_{2|1}(u_1, u_2) = v_2 \end{array}\right.$$

We deduce that:

$$\left\{\begin{array}{l} u_1 \leftarrow v_1 \\ u_2 \leftarrow \Phi\left(\rho\Phi^{-1}(v_1) + \sqrt{1-\rho^2}\Phi^{-1}(v_2)\right) \end{array}\right.$$

Simulation of the bivariate Normal copula

Question 7

Show that this algorithm is equivalent to the Cholesky algorithm found in Question 3.

Simulation of the bivariate Normal copula

We obtain the same algorithm, because we have the following correspondence:

$$\begin{cases} v_1 = \Phi(n_1) \\ v_2 = \Phi(n_2) \end{cases}$$

The algorithm described in Question 6 is then a special case of the Cholesky algorithm if we take $n_1 = \Phi^{-1}(v_1)$ and $n_2 = \Phi^{-1}(v_2)$. Whereas n_1 and n_2 are directly simulated in the Cholesky algorithm with a Gaussian random generator, they are simulated using the inverse transform in the conditional distribution method.

Construction of a stress scenario with the GEV distribution

Question 1

We note a_n and b_n the normalization constraints and $\mathbf{G}$ the limit distribution of the Fisher-Tippet theorem.

Construction of a stress scenario with the GEV distribution

We recall that:

$$\begin{aligned}\Pr\left\{\frac{X_{n:n}-b_n}{a_n}\leq x\right\} &= \Pr\left\{X_{n:n}\leq a_n x+b_n\right\} \\ &= \mathbf{F}^n\left(a_n x+b_n\right)\end{aligned}$$

and:

$$\mathbf{G}\left(x\right)=\lim_{n\rightarrow\infty}\mathbf{F}^n\left(a_n x+b_n\right)$$

Construction of a stress scenario with the GEV distribution

Question 1.a

Find the limit distribution $\mathbf{G}$ when $X \sim \mathcal{E}(\lambda)$, $a_n = \lambda^{-1}$ and $b_n = \lambda^{-1} \ln n$.

Construction of a stress scenario with the GEV distribution

We have:

$$\begin{aligned}\mathbf{F}^{n}\left(a_{n}x+b_{n}\right) &= \left(1-e^{-\lambda\left(\lambda^{-1}x+\lambda^{-1}\ln n\right)}\right)^{n} \\ &= \left(1-\frac{1}{n}e^{-x}\right)^{n}\end{aligned}$$

We deduce that:

$$\mathbf{G}\left(x\right)=\lim_{n\rightarrow\infty}\left(1-\frac{1}{n}e^{-x}\right)^{n}=e^{-e^{-x}}=\mathbf{\Lambda}\left(x\right)$$

Construction of a stress scenario with the GEV distribution

Question 1.b

Same question when $X \sim \mathcal{U}_{[0,1]}$, $a_n = n^{-1}$ and $b_n = 1 - n^{-1}$.

Construction of a stress scenario with the GEV distribution

We have:

$$\begin{aligned}\mathbf{F}^{n}\left(a_{n}x+b_{n}\right) &= \left(n^{-1}x+1-n^{-1}\right)^{n} \\ &= \left(1+\frac{1}{n}\left(x-1\right)\right)^{n}\end{aligned}$$

We deduce that:

$$\mathbf{G}\left(x\right)=\lim_{n\rightarrow\infty}\left(1+\frac{1}{n}\left(x-1\right)\right)^{n}=e^{x-1}=\mathbf{\Psi}_{1}\left(x-1\right)$$

Construction of a stress scenario with the GEV distribution

Question 1.c

Same question when X is a Pareto distribution:

$$\mathbf{F}(x) = 1 - \left(\frac{\theta + x}{\theta}\right)^{-\alpha},$$

$a_n = \theta\alpha^{-1}n^{1/\alpha}$ and $b_n = \theta n^{1/\alpha} - \theta$.

Construction of a stress scenario with the GEV distribution

We have:

$$\begin{aligned}
\mathbf{F}^{n}\left(a_n x + b_n\right) &= \left(1 - \left(\frac{\theta}{\theta + \theta\alpha^{-1} n^{1/\alpha} x + \theta n^{1/\alpha} - \theta}\right)^{\alpha}\right)^{n} \\
&= \left(1 - \left(\frac{1}{\alpha^{-1} n^{1/\alpha} x + n^{1/\alpha}}\right)^{\alpha}\right)^{n} \\
&= \left(1 - \frac{1}{n}\left(1 + \frac{x}{\alpha}\right)^{-\alpha}\right)^{n}
\end{aligned}$$

We deduce that:

$$\mathbf{G}\left(x\right) = \lim_{n\to\infty}\left(1 - \frac{1}{n}\left(1 + \frac{x}{\alpha}\right)^{-\alpha}\right)^{n} = e^{-\left(1+\frac{x}{\alpha}\right)^{-\alpha}} = \mathbf{\Phi}_{\alpha}\left(1 + \frac{x}{\alpha}\right)$$

Construction of a stress scenario with the GEV distribution

Question 2

We denote by **G** the GEV probability distribution:

$$\mathbf{G}(x) = \exp\left\{-\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/\xi}\right\}$$

What is the interest of this probability distribution? Write the log-likelihood function associated to the sample $\{x_1, \ldots, x_T\}$.

Construction of a stress scenario with the GEV distribution

The GEV distribution encompasses the three EV probability distributions. This is an interesting property, because we have not to choose between the three EV distributions. We have:

$$g(x) = \frac{1}{\sigma}\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\left(\frac{1+\xi}{\xi}\right)} \exp\left\{-\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\right\}$$

We deduce that:

$$\begin{aligned}\ell &= -\frac{n}{2}\ln\sigma^2 - \left(\frac{1+\xi}{\xi}\right)\sum_{i=1}^{n}\ln\left(1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right) - \\ &\quad \sum_{i=1}^{n}\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\end{aligned}$$

Construction of a stress scenario with the GEV distribution

Question 3

Show that for $\xi \to 0$, the distribution **G** tends toward the Gumbel distribution:

$$\mathbf{\Lambda}(x) = \exp\left(-\exp\left(-\left(\frac{x-\mu}{\sigma}\right)\right)\right)$$

Construction of a stress scenario with the GEV distribution

We notice that:

$$\lim_{\xi \to 0} (1 + \xi x)^{-1/\xi} = e^{-x}$$

Then we obtain:

$$\begin{aligned} \lim_{\xi \to 0} \mathbf{G}(x) &= \lim_{\xi \to 0} \exp \left\{ - \left[1 + \xi \left(\frac{x - \mu}{\sigma} \right) \right]^{-1/\xi} \right\} \\ &= \exp \left\{ - \lim_{\xi \to 0} \left[1 + \xi \left(\frac{x - \mu}{\sigma} \right) \right]^{-1/\xi} \right\} \\ &= \exp \left(- \exp \left(- \left(\frac{x - \mu}{\sigma} \right) \right) \right) \end{aligned}$$

Construction of a stress scenario with the GEV distribution

Question 4

We consider the minimum value of daily returns of a portfolio for a period of n trading days. We then estimate the GEV parameters associated to the sample of the opposite of the minimum values. We assume that ξ is equal to 1.

Construction of a stress scenario with the GEV distribution

Question 4.a

Show that we can approximate the portfolio loss (in %) associated to the return period $\mathcal{T}$ with the following expression:

$$r(\mathcal{T}) \simeq -\left(\hat{\mu} + \left(\frac{\mathcal{T}}{n} - 1\right)\hat{\sigma}\right)$$

where $\hat{\mu}$ and $\hat{\sigma}$ are the ML estimates of GEV parameters.

Construction of a stress scenario with the GEV distribution

We have:

$$\mathbf{G}^{-1}(\alpha) = \mu - \sigma\xi^{-1}\left[1 - (-\ln\alpha)^{-\xi}\right]$$

When the parameter ξ is equal to 1, we obtain:

$$\mathbf{G}^{-1}(\alpha) = \mu - \sigma\left(1 - (-\ln\alpha)^{-1}\right)$$

By definition, we have $\mathcal{T} = (1-\alpha)^{-1} n$. The return period $\mathcal{T}$ is then associate to the confidence level $\alpha = 1 - n/\mathcal{T}$. We deduce that:

$$\begin{aligned} R(\mathcal{T}) &\approx -\mathbf{G}^{-1}(1 - n/\mathfrak{t}) \\ &= -\left(\mu - \sigma\left(1 - (-\ln(1 - n/\mathcal{T}))^{-1}\right)\right) \\ &= -\left(\mu + \left(\frac{\mathcal{T}}{n} - 1\right)\sigma\right) \end{aligned}$$

We then replace μ and σ by their ML estimates $\hat{\mu}$ and $\hat{\sigma}$.

Construction of a stress scenario with the GEV distribution

Question 4.b

We set n equal to 21 trading days. We obtain the following results for two portfolios:

Portfolio	$\hat{\mu}$	$\hat{\sigma}$	ξ
#1	1%	3%	1
#2	10%	2%	1

Calculate the stress scenario for each portfolio when the return period is equal to one year. Comment on these results.

Construction of a stress scenario with the GEV distribution

For Portfolio #1, we obtain:

$$R(1\mathrm{Y}) = -\left(1\% + \left(\frac{252}{21} - 1\right) \times 3\%\right) = -34\%$$

For Portfolio #2, the stress scenario is equal to:

$$R(1\mathrm{Y}) = -\left(10\% + \left(\frac{252}{21} - 1\right) \times 2\%\right) = -32\%$$

We conclude that Portfolio #1 is more risky than Portfolio #2 if we consider a stress scenario analysis.

Course 2023-2024 in Financial Risk Management
Tutorial Session 5

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[13] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Agenda

- Tutorial Session 1: Market Risk
- Tutorial Session 2: Credit Risk
- Tutorial Session 3: Counterparty Credit Risk and Collateral Risk
- Tutorial Session 4: Operational Risk & Asset Liability Management Risk
- **Tutorial Session 5: Copulas, EVT & Stress Testing**

The bivariate Pareto copula

Exercise

We consider the bivariate Pareto distribution:

$$\begin{aligned}\mathbf{F}(x_1, x_2) &= 1 - \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha} - \left(\frac{\theta_2 + x_2}{\theta_2}\right)^{-\alpha} + \\ &\quad \left(\frac{\theta_1 + x_1}{\theta_1} + \frac{\theta_2 + x_2}{\theta_2} - 1\right)^{-\alpha}\end{aligned}$$

where $x_1 \geq 0$, $x_2 \geq 0$, $\theta_1 > 0$, $\theta_2 > 0$ and $\alpha > 0$.

The bivariate Pareto copula

Question 1

Show that the marginal functions of $\mathbf{F}(x_1, x_2)$ correspond to univariate Pareto distributions.

The bivariate Pareto copula

We have:

$$\begin{aligned} \mathbf{F}_1(x_1) &= \Pr\{X_1 \leq x_1\} \\ &= \Pr\{X_1 \leq x_1, X_2 \leq \infty\} \\ &= \mathbf{F}(x_1, \infty) \end{aligned}$$

We deduce that:

$$\begin{aligned} \mathbf{F}_1(x_1) &= 1 - \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha} - \left(\frac{\theta_2 + \infty}{\theta_2}\right)^{-\alpha} + \\ &\quad \left(\frac{\theta_1 + x_1}{\theta_1} + \frac{\theta_2 + \infty}{\theta_2} - 1\right)^{-\alpha} \\ &= 1 - \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha} \end{aligned}$$

We conclude that $\mathbf{F}_1$ (and $\mathbf{F}_2$) is a Pareto distribution.

The bivariate Pareto copula

Question 2

Find the copula function associated to the bivariate Pareto distribution.

The bivariate Pareto copula

We have:

$$\mathbf{C}(u_1, u_2) = \mathbf{F}\left(\mathbf{F}_1^{-1}(u_1), \mathbf{F}_2^{-1}(u_2)\right)$$

It follows that:

$$\begin{aligned} 1 - \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha} &= u_1 \\ \Leftrightarrow \quad \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha} &= 1 - u_1 \\ \Leftrightarrow \quad \frac{\theta_1 + x_1}{\theta_1} &= (1 - u_1)^{-1/\alpha} \end{aligned}$$

We deduce that:

$$\begin{aligned} \mathbf{C}(u_1, u_2) &= 1 - (1 - u_1) - (1 - u_2) + \\ & \quad \left((1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1\right)^{-\alpha} \\ &= u_1 + u_2 - 1 + \left((1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1\right)^{-\alpha} \end{aligned}$$

The bivariate Pareto copula

Question 3

Deduce the copula density function.

The bivariate Pareto copula

We have:

$$
\begin{aligned}
\frac{\partial\, \mathbf{C}\,(u_1, u_2)}{\partial\, u_1} &= 1 - \alpha \left((1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1 \right)^{-\alpha-1} \times \\
&\quad \left(-\frac{1}{\alpha} \right) (1-u_1)^{-1/\alpha-1} \times (-1) \\
&= 1 - \left((1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1 \right)^{-\alpha-1} \times \\
&\quad (1-u_1)^{-1/\alpha-1}
\end{aligned}
$$

The bivariate Pareto copula

We deduce that the probability density function of the copula is:

$$\begin{aligned}
c(u_1, u_2) &= \frac{\partial^2 \mathbf{C}(u_1, u_2)}{\partial u_1 \, \partial u_2} \\
&= -(-\alpha - 1)\left((1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1\right)^{-\alpha-2} \times \\
&\quad \left(-\frac{1}{\alpha}\right)(1-u_2)^{-1/\alpha-1} \times (-1) \times (1-u_1)^{-1/\alpha-1} \\
&= \left(\frac{\alpha+1}{\alpha}\right)\left((1-u_1)^{-1/\alpha} + (1-u_2)^{-1/\alpha} - 1\right)^{-\alpha-2} \times \\
&\quad (1 - u_1 - u_2 + u_1 u_2)^{-1/\alpha-1}
\end{aligned}$$

The bivariate Pareto copula

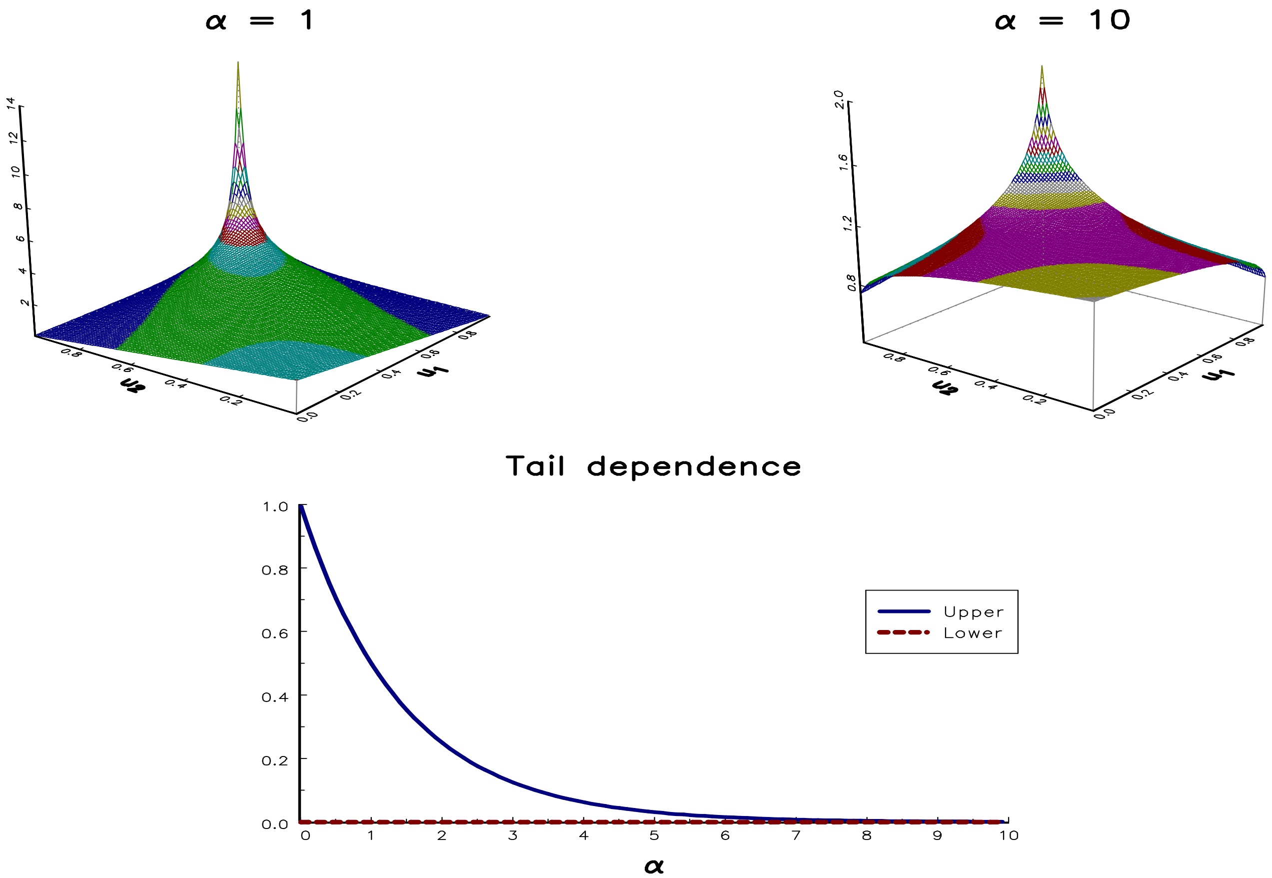
Remark

Another expression of $c(u_1, u_2)$ is:

$$\begin{aligned} c(u_1, u_2) &= \left(\frac{\alpha+1}{\alpha}\right)\left((1-u_1)(1-u_2)\right)^{1/\alpha} \times \\ &\left((1-u_1)^{1/\alpha} + (1-u_2)^{1/\alpha} - (1-u_1)^{1/\alpha}(1-u_2)^{1/\alpha}\right)^{-\alpha-2} \end{aligned}$$

The bivariate Pareto copula

In this Figure, we have reported the density of the Pareto copula when α is equal to 1 and 10.



The bivariate Pareto copula

Question 4

Show that the bivariate Pareto copula function has no lower tail dependence, but an upper tail dependence.

The bivariate Pareto copula

We have:

$$
\begin{aligned}
\lambda^{-} &= \lim_{u \to 0^{+}} \frac{\mathbf{C}(u,u)}{u} \\
&= 2 \lim_{u \to 0^{+}} \frac{\partial\, \mathbf{C}(u,u)}{\partial\, u_1} \\
&= 2 \lim_{u \to 0^{+}} 1 - \left((1-u)^{-1/\alpha} + (1-u)^{-1/\alpha} - 1 \right)^{-\alpha-1} (1-u)^{-1/\alpha-1} \\
&= 2 \lim_{u \to 0^{+}} (1-1) \\
&= 0
\end{aligned}
$$

The bivariate Pareto copula

We have:

$$
\begin{aligned}
\lambda^{+} &= \lim_{u \to 1^{-}} \frac{1 - 2u + \mathbf{C}(u, u)}{1 - u} \\
&= \lim_{u \to 1^{-}} \frac{\left((1-u)^{-1/\alpha} + (1-u)^{-1/\alpha} - 1\right)^{-\alpha}}{1 - u} \\
&= \lim_{u \to 1^{-}} \left(1 + 1 - (1-u)^{1/\alpha}\right)^{-\alpha} \\
&= 2^{-\alpha}
\end{aligned}
$$

The tail dependence coefficients λ^{-} and λ^{+} are given with respect to the parameter α in previous Figure. We deduce that the bivariate Pareto copula function has no lower tail dependence ($\lambda^{-} = 0$), but an upper tail dependence ($\lambda^{+} = 2^{-\alpha}$).

The bivariate Pareto copula

Question 5

Do you think that the bivariate Pareto copula family can reach the copula functions $\mathbf{C}^{-}$, $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$? Justify your answer.

The bivariate Pareto copula

The bivariate Pareto copula family cannot reach $\mathbf{C}^-$ because λ^- is never equal to 1. We notice that:

$$\lim_{\alpha \to \infty} \lambda^+ = 0$$

and

$$\lim_{\alpha \to 0} \lambda^+ = 1$$

This implies that the bivariate Pareto copula may reach $\mathbf{C}^\perp$ and $\mathbf{C}^+$ for these two limit cases: $\alpha \to \infty$ and $\alpha \to 0$. In fact, $\alpha \to 0$ does not correspond to the copula $\mathbf{C}^+$ because λ^- is always equal to 0.

The bivariate Pareto copula

Question 6

Let X_1 and X_2 be two Pareto-distributed random variables, whose parameters are (α_1, θ_1) and (α_2, θ_2).

The bivariate Pareto copula

Question 6.a

Show that the linear correlation between X_1 and X_2 is equal to 1 if and only if the parameters α_1 and α_2 are equal.

The bivariate Pareto copula

We note $U_1 = \mathbf{F}_1(X_1)$ and $U_2 = \mathbf{F}_2(X_2)$. X_1 and X_2 are comonotonic if and only if:

$$U_2 = U_1$$

We deduce that:

$$\begin{aligned} 1 - \left(\frac{\theta_2 + X_2}{\theta_2}\right)^{-\alpha_2} &= 1 - \left(\frac{\theta_1 + X_1}{\theta_1}\right)^{-\alpha_1} \\ \Leftrightarrow \quad \left(\frac{\theta_2 + X_2}{\theta_2}\right)^{-\alpha_2} &= \left(\frac{\theta_1 + X_1}{\theta_1}\right)^{-\alpha_1} \\ \Leftrightarrow \quad X_2 &= \theta_2 \left(\left(\frac{\theta_1 + X_1}{\theta_1}\right)^{\alpha_1/\alpha_2} - 1\right) \end{aligned}$$

We know that $\rho\langle X_1, X_2\rangle = 1$ if and only if there is an increasing linear relationship between X_1 and X_2. This implies that:

$$\frac{\alpha_1}{\alpha_2} = 1$$

The bivariate Pareto copula

Question 6.b

Show that the linear correlation between X_1 and X_2 can never reached the lower bound -1.

The bivariate Pareto copula

X_1 and X_2 are countermonotonic if and only if:

$$U_2 = 1 - U_1$$

We deduce that:

$$\begin{aligned}
& \left(\frac{\theta_2 + X_2}{\theta_2}\right)^{-\alpha_2} = 1 - \left(\frac{\theta_1 + X_1}{\theta_1}\right)^{-\alpha_1} \\
\Leftrightarrow \quad & \left(\frac{\theta_2 + X_2}{\theta_2}\right)^{-\alpha_2} = 1 - \left(\frac{\theta_1 + X_1}{\theta_1}\right)^{-\alpha_1} \\
\Leftrightarrow \quad & X_2 = \theta_2 \left(\left(1 - \left(\frac{\theta_1 + X_1}{\theta_1}\right)^{-\alpha_1}\right)^{1/\alpha_2} - 1\right)
\end{aligned}$$

It is not possible to obtain a decreasing linear function between X_1 and X_2. This implies that $\rho \langle X_1, X_2 \rangle > -1$.

The bivariate Pareto copula

Question 6.c

Build a new bivariate Pareto distribution by assuming that the marginal distributions are $\mathcal{P}(\alpha_1, \theta_1)$ and $\mathcal{P}(\alpha_2, \theta_2)$ and the dependence is a bivariate Pareto copula function with parameter α. What is the relevance of this approach for building bivariate Pareto distributions?

The bivariate Pareto copula

We have:

$$\begin{aligned}\mathbf{F}'(x_1, x_2) &= \mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)) \\ &= 1 - \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha_1} - \left(\frac{\theta_2 + x_2}{\theta_2}\right)^{-\alpha_2} + \\ & \quad \left(\left(\frac{\theta_1 + x_1}{\theta_1}\right)^{\alpha_1/\alpha} + \left(\frac{\theta_2 + x_2}{\theta_2}\right)^{\alpha_2/\alpha} - 1\right)^{-\alpha}\end{aligned}$$

The traditional bivariate Pareto distribution $\mathbf{F}(x_1, x_2)$ is a special case of $\mathbf{F}'(x_1, x_2)$ when:

$$\alpha_1 = \alpha_2 = \alpha$$

Using $\mathbf{F}'$ instead of $\mathbf{F}$, we can control the tail dependence, but also the univariate tail index of the two margins.

Calculation of correlation bounds

Question 1

Give the mathematical definition of the copula functions $\mathbf{C}^{-}$, $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$. What is the probabilistic interpretation of these copulas?

Calculation of correlation bounds

We have:

$$\begin{aligned}\mathbf{C}^{-}(u_1,u_2) &= \max(u_1+u_2-1,0)\\ \mathbf{C}^{\perp}(u_1,u_2) &= u_1u_2\\ \mathbf{C}^{+}(u_1,u_2) &= \min(u_1,u_2)\end{aligned}$$

Let X_1 and X_2 be two random variables. We have:

(i) $\mathbf{C}\langle X_1,X_2\rangle = \mathbf{C}^{-}$ if and only if there exists a non-increasing function f such that we have $X_2 = f(X_1)$;

(ii) $\mathbf{C}\langle X_1,X_2\rangle = \mathbf{C}^{\perp}$ if and only if X_1 and X_2 are independent;

(iii) $\mathbf{C}\langle X_1,X_2\rangle = \mathbf{C}^{+}$ if and only if there exists a non-decreasing function f such that we have $X_2 = f(X_1)$.

Calculation of correlation bounds

Question 2

We note $\boldsymbol{\tau}$ and LGD the default time and the loss given default of a counterparty. We assume that $\boldsymbol{\tau} \sim \mathcal{E}(\lambda)$ and $\mathrm{LGD} \sim \mathcal{U}_{[0,1]}$.

Calculation of correlation bounds

We note $U_1 = 1 - \exp(-\lambda \boldsymbol{\tau})$ and $U_2 = \mathrm{LGD}$.

Calculation of correlation bounds

Question 2.a

Show that the dependence between $\boldsymbol{\tau}$ and LGD is maximum when the following equality holds:

$$\mathrm{LGD} + e^{-\lambda \boldsymbol{\tau}} - 1 = 0$$

Calculation of correlation bounds

The dependence between $\boldsymbol{\tau}$ and LGD is maximum when we have $\mathbf{C}\left\langle \boldsymbol{\tau}, \mathrm{LGD} \right\rangle = \mathbf{C}^{+}$. Since we have $U_1 = U_2$, we conclude that:

$$\mathrm{LGD} + e^{-\lambda \boldsymbol{\tau}} - 1 = 0$$

Calculation of correlation bounds

Question 2.b

Show that the linear correlation $\rho(\boldsymbol{\tau}, \mathrm{LGD})$ verifies the following inequality:

$$\left|\rho\left\langle\boldsymbol{\tau}, \mathrm{LGD}\right\rangle\right| \leq \frac{\sqrt{3}}{2}$$

Calculation of correlation bounds

We know that:

$$\rho\left\langle \boldsymbol{\tau}, \mathrm{LGD}\right\rangle \in \left[\rho_{\min}\left\langle \boldsymbol{\tau}, \mathrm{LGD}\right\rangle, \rho_{\max}\left\langle \boldsymbol{\tau}, \mathrm{LGD}\right\rangle\right]$$

where $\rho_{\min}\left\langle \boldsymbol{\tau}, \mathrm{LGD}\right\rangle$ (resp. $\rho_{\max}\left\langle \boldsymbol{\tau}, \mathrm{LGD}\right\rangle$) is the linear correlation corresponding to the copula $\mathbf{C}^{-}$ (resp. $\mathbf{C}^{+}$). It comes that:

$$\mathbb{E}\left[\boldsymbol{\tau}\right] = \sigma\left(\boldsymbol{\tau}\right) = \frac{1}{\lambda}$$

and:

$$\begin{aligned} \mathbb{E}\left[\mathrm{LGD}\right] &= \frac{1}{2} \\ \sigma\left(\mathrm{LGD}\right) &= \sqrt{\frac{1}{12}} \end{aligned}$$

Calculation of correlation bounds

In the case $\mathbf{C}\langle\boldsymbol{\tau},\mathrm{LGD}\rangle=\mathbf{C}^{-}$, we have $U_1=1-U_2$. It follows that $\mathrm{LGD}=e^{-\lambda\boldsymbol{\tau}}$. We have:

$$\begin{aligned}
\mathbb{E}\left[\boldsymbol{\tau}\,\mathrm{LGD}\right]=\mathbb{E}\left[\boldsymbol{\tau}e^{-\lambda\boldsymbol{\tau}}\right] &= \int_0^\infty te^{-\lambda t}\lambda e^{-\lambda t}\,\mathrm{d}t \\
&= \int_0^\infty t\lambda e^{-2\lambda t}\,\mathrm{d}t \\
&= \left[-\frac{te^{-2\lambda t}}{2}\right]_0^\infty+\frac{1}{2}\int_0^\infty e^{-2\lambda t}\,\mathrm{d}t \\
&= 0+\frac{1}{2}\left[-\frac{e^{-2\lambda t}}{2\lambda}\right]_0^\infty \\
&= \frac{1}{4\lambda}
\end{aligned}$$

We deduce that:

$$\rho_{\min}\langle\boldsymbol{\tau},\mathrm{LGD}\rangle=\left(\frac{1}{4\lambda}-\frac{1}{2\lambda}\right)\Big/\left(\frac{1}{\lambda}\sqrt{\frac{1}{12}}\right)=-\frac{\sqrt{3}}{2}$$

Calculation of correlation bounds

In the case $\mathbf{C}\langle\boldsymbol{\tau},\mathrm{LGD}\rangle=\mathbf{C}^{+}$, we have $\mathrm{LGD}=1-e^{-\lambda\tau}$. We have:

$$\begin{aligned}
\mathbb{E}\left[\boldsymbol{\tau}\,\mathrm{LGD}\right]=\mathbb{E}\left[\boldsymbol{\tau}\left(1-e^{-\lambda\boldsymbol{\tau}}\right)\right] &= \int_0^\infty t\left(1-e^{-\lambda t}\right)\lambda e^{-\lambda t}\,\mathrm{d}t \\
&= \int_0^\infty t\lambda e^{-\lambda t}\,\mathrm{d}t-\int_0^\infty t\lambda e^{-2\lambda t}\,\mathrm{d}t \\
&= \left(\left[-te^{-\lambda t}\right]_0^\infty+\int_0^\infty e^{-\lambda t}\,\mathrm{d}t\right)-\frac{1}{4\lambda} \\
&= 0+\left[-\frac{e^{-\lambda t}}{\lambda}\right]_0^\infty-\frac{1}{4\lambda} \\
&= \frac{3}{4\lambda}
\end{aligned}$$

We deduce that:

$$\rho_{\max}\langle\boldsymbol{\tau},\mathrm{LGD}\rangle=\left(\frac{3}{4\lambda}-\frac{1}{2\lambda}\right)\Big/\left(\frac{1}{\lambda}\sqrt{\frac{1}{12}}\right)=\frac{\sqrt{3}}{2}$$

Calculation of correlation bounds

We finally obtain the following result:

$$\left|\rho\left\langle \boldsymbol{\tau}, \mathrm{LGD}\right\rangle\right| \leq \frac{\sqrt{3}}{2}$$

Calculation of correlation bounds

Question 2.c

Comment on these results.

Calculation of correlation bounds

We notice that $|\rho \langle \boldsymbol{\tau}, \mathrm{LGD} \rangle|$ is lower than 86.6%, implying that the bounds -1 and $+1$ can not be reached.

Calculation of correlation bounds

Question 3

We consider two exponential default times $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ with parameters λ_1 and λ_2.

Calculation of correlation bounds

Question 3.a

We assume that the dependence function between $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ is $\mathbf{C}^+$. Demonstrate that the following relation is true:

$$\boldsymbol{\tau}_1 = \frac{\lambda_2}{\lambda_1}\boldsymbol{\tau}_2$$

Calculation of correlation bounds

If the copula function of $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ is the Fréchet upper bound copula, $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are comonotone. We deduce that:

$$U_1 = U_2 \Longleftrightarrow 1 - e^{-\lambda_1 \boldsymbol{\tau}_1} = 1 - e^{-\lambda_2 \boldsymbol{\tau}_2}$$

and:

$$\boldsymbol{\tau}_1 = \frac{\lambda_2}{\lambda_1} \boldsymbol{\tau}_2$$

Calculation of correlation bounds

Question 3.b

Show that there exists a function f such that $\boldsymbol{\tau}_2 = f(\boldsymbol{\tau}_2)$ when the dependence function is $\mathbf{C}^{-}$.

Calculation of correlation bounds

We have $U_1 = 1 - U_2$. It follows that $\mathbf{S}_1(\boldsymbol{\tau}_1) = 1 - \mathbf{S}_2(\boldsymbol{\tau}_2)$. We deduce that:

$$e^{-\lambda_1 \boldsymbol{\tau}_1} = 1 - e^{-\lambda_2 \boldsymbol{\tau}_2}$$

and:

$$\boldsymbol{\tau}_1 = \frac{-\ln\left(1 - e^{-\lambda_2 \boldsymbol{\tau}_2}\right)}{\lambda_1}$$

There exists then a function f such that $\boldsymbol{\tau}_1 = f(\boldsymbol{\tau}_2)$ with:

$$f(t) = \frac{-\ln\left(1 - e^{-\lambda_2 t}\right)}{\lambda_1}$$

Calculation of correlation bounds

Question 3.c

Show that the lower and upper bounds of the linear correlation satisfy the following relationship:

$$-1 < \rho \langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle \leq 1$$

Calculation of correlation bounds

Using Question 2(b), we known that $\rho \in [\rho_{\min}, \rho_{\max}]$ where $\rho_{\min}$ and $\rho_{\max}$ are the correlations of $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ when the copula function is respectively $\mathbf{C}^-$ and $\mathbf{C}^+$. We also know that $\rho = 1$ (resp. $\rho = -1$) if there exists a linear and increasing (resp. decreasing) function f such that $\boldsymbol{\tau}_1 = f(\boldsymbol{\tau}_2)$. When the copula is $\mathbf{C}^+$, we have $f(t) = \frac{\lambda_2}{\lambda_1} t$ and $f'(t) = \frac{\lambda_2}{\lambda_1} > 0$. As it is a linear and increasing function, we deduce that $\rho_{\max} = 1$. When the copula is $\mathbf{C}^-$, we have:

$$f(t) = \frac{-\ln\left(1 - e^{-\lambda_2 t}\right)}{\lambda_1}$$

and:

$$f'(t) = -\frac{\lambda_2 e^{-\lambda_2 t} \ln\left(1 - e^{-\lambda_2 t}\right)}{\lambda_1 \left(1 - e^{-\lambda_2 t}\right)} < 0$$

The function $f(t)$ is decreasing, but it is not linear. We deduce that $\rho_{\min} \neq -1$ and:

$$-1 < \rho \leq 1$$

Calculation of correlation bounds

Question 3.d

In the more general case, show that the linear correlation of a random vector (X_1, X_2) can not be equal to -1 if the support of the random variables X_1 and X_2 is $[0, +\infty]$.

Calculation of correlation bounds

When the copula is $\mathbf{C}^{-}$, we know that there exists a decreasing function f such that $X_2 = f(X_1)$. We also know that the linear correlation reaches the lower bound -1 if the function f is linear:

$$X_2 = a + bX_1$$

This implies that $b < 0$. When X_1 takes the value $+\infty$, we obtain:

$$X_2 = a + b \times \infty$$

As the lower bound of X_2 is equal to zero 0, we deduce that $a = +\infty$. This means that the function $f(x) = a + bx$ does not exist. We conclude that the lower bound $\rho = -1$ can not be reached.

Calculation of correlation bounds

Question 4

We assume that (X_1, X_2) is a Gaussian random vector where $X_1 \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right)$, $X_2 \sim \mathcal{N}\left(\mu_2, \sigma_2^2\right)$ and ρ is the linear correlation between X_1 and X_2. We note $\theta = (\mu_1, \sigma_1, \mu_2, \sigma_2, \rho)$ the set of parameters.

Calculation of correlation bounds

Question 4.a

Find the probability distribution of $X_1 + X_2$.

Calculation of correlation bounds

$X_1 + X_2$ is a Gaussian random variable because it is a linear combination of the Gaussian random vector (X_1, X_2). We have:

$$\mathbb{E}[X_1 + X_2] = \mu_1 + \mu_2$$

and:

$$\operatorname{var}(X_1 + X_2) = \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2$$

We deduce that:

$$X_1 + X_2 \sim \mathcal{N}\left(\mu_1 + \mu_2, \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2\right)$$

Calculation of correlation bounds

Question 4.b

Then show that the covariance between $Y_1 = e^{X_1}$ and $Y_2 = e^{X_2}$ is equal to:

$$\operatorname{cov}\left(Y_1, Y_2\right) = e^{\mu_1 + \frac{1}{2}\sigma_1^2} e^{\mu_2 + \frac{1}{2}\sigma_2^2} \left(e^{\rho\sigma_1\sigma_2} - 1\right)$$

Calculation of correlation bounds

We have:

$$\begin{aligned}\operatorname{cov}(Y_1, Y_2) &= \mathbb{E}[Y_1 Y_2] - \mathbb{E}[Y_2]\,\mathbb{E}[Y_2] \\ &= \mathbb{E}\left[e^{X_1+X_2}\right] - \mathbb{E}[Y_2]\,\mathbb{E}[Y_2]\end{aligned}$$

We know that $e^{X_1+X_2}$ is a lognormal random variable. We deduce that:

$$\begin{aligned}\mathbb{E}\left[e^{X_1+X_2}\right] &= \exp\left(\mathbb{E}[X_1+X_2] + \frac{1}{2}\operatorname{var}(X_1+X_2)\right) \\ &= \exp\left(\mu_1+\mu_2+\frac{1}{2}\left(\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2\right)\right) \\ &= e^{\mu_1+\frac{1}{2}\sigma_1^2}e^{\mu_2+\frac{1}{2}\sigma_2^2}e^{\rho\sigma_1\sigma_2}\end{aligned}$$

We finally obtain:

$$\operatorname{cov}(Y_1, Y_2) = e^{\mu_1+\frac{1}{2}\sigma_1^2}e^{\mu_2+\frac{1}{2}\sigma_2^2}\left(e^{\rho\sigma_1\sigma_2}-1\right)$$

Calculation of correlation bounds

Question 4.c

Deduce the correlation between Y_1 and Y_2.

Calculation of correlation bounds

We have:

$$\begin{aligned}\rho\left\langle Y_1, Y_2\right\rangle &= \frac{e^{\mu_1+\frac{1}{2}\sigma_1^2}e^{\mu_2+\frac{1}{2}\sigma_2^2}\left(e^{\rho\sigma_1\sigma_2}-1\right)}{\sqrt{e^{2\mu_1+\sigma_1^2}\left(e^{\sigma_1^2}-1\right)}\sqrt{e^{2\mu_2+\sigma_2^2}\left(e^{\sigma_2^2}-1\right)}} \\ &= \frac{e^{\rho\sigma_1\sigma_2}-1}{\sqrt{e^{\sigma_1^2}-1}\sqrt{e^{\sigma_2^2}-1}}\end{aligned}$$

Calculation of correlation bounds

Question 4.d

For which values of θ does the equality $\rho \langle Y_1, Y_2 \rangle = +1$ hold? Same question when $\rho \langle Y_1, Y_2 \rangle = -1$.

Calculation of correlation bounds

$\rho \langle Y_1, Y_2 \rangle$ is an increasing function with respect to ρ. We deduce that:

$$\rho \langle Y_1, Y_2 \rangle = 1 \Longleftrightarrow \rho = 1 \text{ and } \sigma_1 = \sigma_2$$

The lower bound of $\rho \langle Y_1, Y_2 \rangle$ is reached if ρ is equal to -1. In this case, we have:

$$\rho \langle Y_1, Y_2 \rangle = \frac{e^{-\sigma_1 \sigma_2} - 1}{\sqrt{e^{\sigma_1^2} - 1}\sqrt{e^{\sigma_2^2} - 1}} > -1$$

It follows that $\rho \langle Y_1, Y_2 \rangle \neq -1$.

Calculation of correlation bounds

Question 4.e

We consider the bivariate Black-Scholes model:

$$\begin{cases} \mathrm{d}S_1(t) = \mu_1 S_1(t)\,\mathrm{d}t + \sigma_1 S_1(t)\,\mathrm{d}W_1(t) \\ \mathrm{d}S_2(t) = \mu_2 S_2(t)\,\mathrm{d}t + \sigma_2 S_2(t)\,\mathrm{d}W_2(t) \end{cases}$$

with $\mathbb{E}[W_1(t)\,W_2(t)] = \rho t$. Deduce the linear correlation between $S_1(t)$ and $S_2(t)$. Find the limit case $\lim_{t\to\infty} \rho\langle S_1(t), S_2(t)\rangle$.

Calculation of correlation bounds

It is obvious that:

$$\rho\left\langle S_1(t), S_2(t)\right\rangle = \frac{e^{\rho\sigma_1\sigma_2 t}-1}{\sqrt{e^{\sigma_1^2 t}-1}\sqrt{e^{\sigma_2^2 t}-1}}$$

In the case $\sigma_1 = \sigma_2$ and $\rho = 1$, we have $\rho\left\langle S_1(t), S_2(t)\right\rangle = 1$. Otherwise, we obtain:

$$\lim_{t\to\infty}\rho\left\langle S_1(t), S_2(t)\right\rangle = 0$$

Calculation of correlation bounds

Question 4.f

Comment on these results.

Calculation of correlation bounds

In the case of lognormal random variables, the linear correlation does not necessarily range between -1 and $+1$.

Extreme value theory in the bivariate case

Question 1

What is an extreme value (EV) copula $\mathbf{C}$?

Extreme value theory in the bivariate case

An extreme value copula $\mathbf{C}$ satisfies the following relationship:

$$\mathbf{C}\left(u_1^t, u_2^t\right) = \mathbf{C}^t\left(u_1, u_2\right)$$

for all $t > 0$.

Extreme value theory in the bivariate case

Question 2

Show that $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$ are EV copulas. Why $\mathbf{C}^{-}$ can not be an EV copula?

Extreme value theory in the bivariate case

The product copula $\mathbf{C}^{\perp}$ is an EV copula because we have:

$$\begin{aligned}
\mathbf{C}^{\perp}\left(u_1^t, u_2^t\right) &= u_1^t u_2^t \\
&= \left(u_1 u_2\right)^t \\
&= \left[\mathbf{C}^{\perp}\left(u_1, u_2\right)\right]^t
\end{aligned}$$

Extreme value theory in the bivariate case

For the copula $\mathbf{C}^{+}$, we obtain:

$$\begin{aligned}
\mathbf{C}^{+}\left(u_1^t, u_2^t\right) &= \min\left(u_1^t, u_2^t\right) \\
&= \left\{ \begin{array}{ll} u_1^t & \text{if } u_1 \leq u_2 \\ u_2^t & \text{otherwise} \end{array} \right. \\
&= \left(\min\left(u_1, u_2\right)\right)^t \\
&= \left[\mathbf{C}^{+}\left(u_1, u_2\right)\right]^t
\end{aligned}$$

Extreme value theory in the bivariate case

However, the EV property does not hold for the Fréchet lower bound copula $\mathbf{C}^-$:

$$\mathbf{C}^-\left(u_1^t, u_2^t\right) = \max\left(u_1^t + u_2^t - 1, 0\right) \neq \max\left(u_1 + u_2 - 1, 0\right)^t$$

Indeed, we have $\mathbf{C}^-(0.5, 0.8) = \max(0.5 + 0.8 - 1, 0) = 0.3$ and:

$$\begin{aligned} \mathbf{C}^-\left(0.5^2, 0.8^2\right) &= \max(0.25 + 0.64 - 1, 0) \\ &= 0 \\ &\neq 0.3^2 \end{aligned}$$

Extreme value theory in the bivariate case

Question 3

We define the Gumbel-Hougaard copula as follows:

$$\mathbf{C}(u_1, u_2) = \exp\left(-\left[(-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}\right]^{1/\theta}\right)$$

with $\theta \geq 1$. Verify that it is an EV copula.

Extreme value theory in the bivariate case

We have:

$$\begin{aligned}
\mathbf{C}\left(u_1^t, u_2^t\right) &= \exp\left(-\left[\left(-\ln u_1^t\right)^{\theta} + \left(-\ln u_2^t\right)^{\theta}\right]^{1/\theta}\right) \\
&= \exp\left(-\left[\left(-t\ln u_1\right)^{\theta} + \left(-t\ln u_2\right)^{\theta}\right]^{1/\theta}\right) \\
&= \exp\left(-t\left[\left(-\ln u_1\right)^{\theta} + \left(-\ln u_2\right)^{\theta}\right]^{1/\theta}\right) \\
&= \left(e^{-\left[\left(-\ln u_1\right)^{\theta} + \left(-\ln u_2\right)^{\theta}\right]^{1/\theta}}\right)^t \\
&= \mathbf{C}^t\left(u_1, u_2\right)
\end{aligned}$$

Extreme value theory in the bivariate case

Question 4

What is the definition of the upper tail dependence λ? What is its usefulness in multivariate extreme value theory?

Extreme value theory in the bivariate case

The upper tail dependence λ is defined as follows:

$$\lambda = \lim_{u \to 1^+} \frac{1 - 2u + \mathbf{C}(u_1, u_2)}{1 - u}$$

It measures the probability to have an extreme in one direction knowing that we have already an extreme in the other direction. If λ is equal to 0, extremes are independent and the EV copula is the product copula $\mathbf{C}^\perp$. If λ is equal to 1, extremes are comonotonic and the EV copula is the Fréchet upper bound copula $\mathbf{C}^+$. Moreover, the upper tail dependence of the copula between the random variables is equal to the upper tail dependence of the copula between the extremes.

Extreme value theory in the bivariate case

Question 5

Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x\rightarrow x_0} f(x) = \lim_{x\rightarrow x_0} g(x) = 0$. If $g'(x_0) \neq 0$, L'Hospital's rule states that:

$$\lim_{x\rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x\rightarrow x_0} \frac{f'(x)}{g'(x)}$$

Deduce that the upper tail dependence λ of the Gumbel-Hougaard copula is $2 - 2^{1/\theta}$. What is the correlation of two extremes when $\theta = 1$?

Extreme value theory in the bivariate case

Using L'Hospital's rule, we have:

$$\begin{aligned}
\lambda &= \lim_{u \to 1^+} \frac{1 - 2u + e^{-\left[(-\ln u)^\theta + (-\ln u)^\theta\right]^{1/\theta}}}{1-u} \\
&= \lim_{u \to 1^+} \frac{1 - 2u + e^{-\left[2(-\ln u)^\theta\right]^{1/\theta}}}{1-u} \\
&= \lim_{u \to 1^+} \frac{1 - 2u + u^{2^{1/\theta}}}{1-u} \\
&= \lim_{u \to 1^+} \frac{0 - 2 + 2^{1/\theta} u^{2^{1/\theta}-1}}{-1} \\
&= \lim_{u \to 1^+} 2 - 2^{1/\theta} u^{2^{1/\theta}-1} \\
&= 2 - 2^{1/\theta}
\end{aligned}$$

Extreme value theory in the bivariate case

If θ is equal to 1, we obtain $\lambda = 0$. It comes that the EV copula is the product copula. Extremes are then not correlated. This result is not surprising because the Gumbel-Houggard copula is equal to the product copula when $\theta = 1$:

$$e^{-\left[(-\ln u_1)^1+(-\ln u_2)^1\right]^1} = u_1 u_2 = \mathbf{C}^{\perp}(u_1, u_2)$$

Extreme value theory in the bivariate case

Question 6

We define the Marshall-Olkin copula as follows:

$$\mathbf{C}(u_1, u_2) = u_1^{1-\theta_1} u_2^{1-\theta_2} \min\left(u_1^{\theta_1}, u_2^{\theta_2}\right)$$

with $\{\theta_1, \theta_2\} \in [0,1]^2$.

Extreme value theory in the bivariate case

Question 6.a

Verify that it is an EV copula.

Extreme value theory in the bivariate case

We have:

$$\begin{aligned}
\mathbf{C}\left(u_1^t, u_2^t\right) &= u_1^{t(1-\theta_1)} u_2^{t(1-\theta_2)} \min\left(u_1^{t\theta_1}, u_2^{t\theta_2}\right) \\
&= \left(u_1^{1-\theta_1}\right)^t \left(u_2^{1-\theta_2}\right)^t \left(\min\left(u_1^{\theta_1}, u_2^{\theta_2}\right)\right)^t \\
&= \left(u_1^{1-\theta_1} u_2^{1-\theta_2} \min\left(u_1^{\theta_1}, u_2^{\theta_2}\right)\right)^t \\
&= \mathbf{C}^t\left(u_1, u_2\right)
\end{aligned}$$

Extreme value theory in the bivariate case

Question 6.b

Find the upper tail dependence λ of the Marshall-Olkin copula.

Extreme value theory in the bivariate case

If $\theta_1 > \theta_2$, we obtain:

$$\begin{aligned}
\lambda &= \lim_{u \to 1^+} \frac{1 - 2u + u^{1-\theta_1} u^{1-\theta_2} \min\left(u^{\theta_1}, u^{\theta_2}\right)}{1-u} \\
&= \lim_{u \to 1^+} \frac{1 - 2u + u^{1-\theta_1} u^{1-\theta_2} u^{\theta_1}}{1-u} \\
&= \lim_{u \to 1^+} \frac{1 - 2u + u^{2-\theta_2}}{1-u} \\
&= \lim_{u \to 1^+} \frac{0 - 2 + (2-\theta_2) u^{1-\theta_2}}{-1} \\
&= \lim_{u \to 1^+} 2 - 2u^{1-\theta_2} + \theta_2 u^{1-\theta_2} \\
&= \theta_2
\end{aligned}$$

If $\theta_2 > \theta_1$, we have $\lambda = \theta_1$. We deduce that the upper tail dependence of the Marshall-Olkin copula is $\min(\theta_1, \theta_2)$.

Extreme value theory in the bivariate case

Question 6.c

What is the correlation of two extremes when $\min(\theta_1, \theta_2) = 0$?

Extreme value theory in the bivariate case

If $\theta_1 = 0$ or $\theta_2 = 0$, we obtain $\lambda = 0$. It comes that the copula of the extremes is the product copula. Extremes are then not correlated.

Extreme value theory in the bivariate case

Question 6.d

In which case are two extremes perfectly correlated?

Extreme value theory in the bivariate case

Two extremes are perfectly correlated when we have $\theta_1 = \theta_2 = 1$. In this case, we obtain:

$$\mathbf{C}(u_1, u_2) = \min(u_1, u_2) = \mathbf{C}^+(u_1, u_2)$$

Maximum domain of attraction in the bivariate case

Question 1

We consider the following distributions of probability:

Distribution		$\mathbf{F}(x)$
Exponential	$\mathcal{E}(\lambda)$	$1 - e^{-\lambda x}$
Uniform	$\mathcal{U}_{[0,1]}$	x
Pareto	$\mathcal{P}(\alpha, \theta)$	$1 - \left(\frac{\theta + x}{\theta}\right)^{-\alpha}$

Maximum domain of attraction in the bivariate case

Question 1

For each distribution, we give the normalization parameters a_n and b_n of the Fisher-Tippet theorem and the corresponding limit distribution distribution $\mathbf{G}(x)$:

Distribution	a_n	b_n	$\mathbf{G}(x)$
Exponential	λ^{-1}	$\lambda^{-1} \ln n$	$\boldsymbol{\Lambda}(x) = e^{-e^{-x}}$
Uniform	n^{-1}	$1 - n^{-1}$	$\boldsymbol{\Psi}_1(x-1) = e^{x-1}$
Pareto	$\theta \alpha^{-1} n^{1/\alpha}$	$\theta n^{1/\alpha} - \theta$	$\boldsymbol{\Phi}_\alpha\left(1 + \frac{x}{\alpha}\right) = e^{-\left(1+\frac{x}{\alpha}\right)^{-\alpha}}$

We note $\mathbf{G}(x_1, x_2)$ the asymptotic distribution of the bivariate random vector $(X_{1,n:n}, X_{2,n:n})$ where $X_{1,i}$ (resp. $X_{2,i}$) are *iid* random variables.

Maximum domain of attraction in the bivariate case

Let $(X_{1,} X_2)$ be a bivariate random variable whose probability distribution is:

$$\mathbf{F}(x_1, x_2) = \mathbf{C}_{\langle X_1, X_2 \rangle}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2))$$

We know that the corresponding EV probability distribution is:

$$\mathbf{G}(x_1, x_2) = \mathbf{C}^{\star}_{\langle X_1, X_2 \rangle}(\mathbf{G}_1(x_1), \mathbf{G}_2(x_2))$$

where $\mathbf{G}_1$ and $\mathbf{G}_2$ are the two univariate EV probability distributions and $\mathbf{C}^{\star}_{\langle X_1, X_2 \rangle}$ is the EV copula associated to $\mathbf{C}_{\langle X_1, X_2 \rangle}$.

Maximum domain of attraction in the bivariate case

Question 1.a

What is the expression of $\mathbf{G}(x_1, x_2)$ when $X_{1,i}$ and $X_{2,i}$ are independent, $X_{1,i} \sim \mathcal{E}(\lambda)$ and $X_{2,i} \sim \mathcal{U}_{[0,1]}$?

Maximum domain of attraction in the bivariate case

We deduce that:

$$\begin{aligned}\mathbf{G}(x_1, x_2) &= \mathbf{C}^{\perp}(\mathbf{G}_1(x_1), \mathbf{G}_2(x_2)) \\ &= \mathbf{\Lambda}(x_1)\,\mathbf{\Psi}_1(x_2 - 1) \\ &= \exp\left(-e^{-x_1} + x_2 - 1\right)\end{aligned}$$

Maximum domain of attraction in the bivariate case

Question 1.b

Same question when $X_{1,i} \sim \mathcal{E}(\lambda)$ and $X_{2,i} \sim \mathcal{P}(\theta, \alpha)$.

Maximum domain of attraction in the bivariate case

We have:

$$\begin{aligned}\mathbf{G}(x_1, x_2) &= \mathbf{\Lambda}(x_1)\,\mathbf{\Phi}_\alpha\left(1 + \frac{x_2}{\alpha}\right) \\ &= \exp\left(-e^{-x_1} - \left(1 + \frac{x_2}{\alpha}\right)^{-\alpha}\right)\end{aligned}$$

Maximum domain of attraction in the bivariate case

Question 1.c

Same question when $X_{1,i} \sim \mathcal{U}_{[0,1]}$ and $X_{2,i} \sim \mathcal{P}(\theta, \alpha)$.

Maximum domain of attraction in the bivariate case

We have:

$$\begin{aligned}\mathbf{G}(x_1, x_2) &= \boldsymbol{\Psi}_1(x_1 - 1)\,\boldsymbol{\Phi}_\alpha\left(1 + \frac{x_2}{\alpha}\right) \\ &= \exp\left(x_1 - 1 - \left(1 + \frac{x_2}{\alpha}\right)^{-\alpha}\right)\end{aligned}$$

Maximum domain of attraction in the bivariate case

Question 2

What becomes the previous results when the dependence function between $X_{1,i}$ and $X_{2,i}$ is the Normal copula with parameter $\rho < 1$?

Maximum domain of attraction in the bivariate case

We know that the upper tail dependence is equal to zero for the Normal copula when $\rho < 1$. We deduce that the EV copula is the product copula. We then obtain the same results as previously.

Maximum domain of attraction in the bivariate case

Question 3

Same question when the parameter of the Normal copula is equal to one.

Maximum domain of attraction in the bivariate case

When the parameter ρ is equal to 1, the Normal copula is the Frchet upper bound copula $\mathbf{C}^{+}$, which is an EV copula. We deduce the following results:

$$\begin{aligned}\mathbf{G}(x_1, x_2) &= \min\left(\boldsymbol{\Lambda}(x_1), \boldsymbol{\Psi}_1(x_2 - 1)\right) \\ &= \min\left(\exp\left(-e^{-x_1}\right), \exp(x_2 - 1)\right) \qquad \text{(a)}\end{aligned}$$

$$\begin{aligned}\mathbf{G}(x_1, x_2) &= \min\left(\boldsymbol{\Lambda}(x_1), \boldsymbol{\Phi}_\alpha\left(1 + \frac{x_2}{\alpha}\right)\right) \\ &= \min\left(\exp\left(-e^{-x_1}\right), \exp\left(-\left(1 + \frac{x_2}{\alpha}\right)^{-\alpha}\right)\right) \qquad \text{(b)}\end{aligned}$$

$$\begin{aligned}\mathbf{G}(x_1, x_2) &= \min\left(\boldsymbol{\Psi}_1(x_1 - 1), \boldsymbol{\Phi}_\alpha\left(1 + \frac{x_2}{\alpha}\right)\right) \\ &= \min\left(\exp(x_2 - 1), \exp\left(-\left(1 + \frac{x_2}{\alpha}\right)^{-\alpha}\right)\right) \qquad \text{(c)}\end{aligned}$$

Maximum domain of attraction in the bivariate case

Question 4

Find the expression of $\mathbf{G}(x_1, x_2)$ when the dependence function is the Gumbel-Hougaard copula.

Maximum domain of attraction in the bivariate case

In the previous exercise, we have shown that the Gumbel-Houggard copula is an EV copula.

Maximum domain of attraction in the bivariate case

We deduce that:

$$\begin{aligned}\mathbf{G}(x_1,x_2) &= e^{-\left[(-\ln \boldsymbol{\Lambda}(x_1))^{\theta}+(-\ln \boldsymbol{\Psi}_1(x_2-1))^{\theta}\right]^{1/\theta}} \\ &= \exp\left(-\left[e^{-\theta x_1}+(1-x_2)^{\theta}\right]^{1/\theta}\right) \qquad \text{(a)}\end{aligned}$$

$$\begin{aligned}\mathbf{G}(x_1,x_2) &= e^{-\left[(-\ln \boldsymbol{\Lambda}(x_1))^{\theta}+\left(-\ln \boldsymbol{\Phi}_{\alpha}\left(1+\frac{x_2}{\alpha}\right)\right)^{\theta}\right]^{1/\theta}} \\ &= \exp\left(-\left[e^{-\theta x_1}+\left(1+\frac{x_2}{\alpha}\right)^{-\alpha\theta}\right]^{1/\theta}\right) \qquad \text{(b)}\end{aligned}$$

$$\begin{aligned}\mathbf{G}(x_1,x_2) &= e^{-\left[(-\ln \boldsymbol{\Psi}_1(x_1-1))^{\theta}+\left(-\ln \boldsymbol{\Phi}_{\alpha}\left(1+\frac{x_2}{\alpha}\right)\right)^{\theta}\right]^{1/\theta}} \\ &= \exp\left(-\left[(1-x_1)^{\theta}+\left(1+\frac{x_2}{\alpha}\right)^{-\alpha\theta}\right]^{1/\theta}\right) \qquad \text{(c)}\end{aligned}$$

Simulation of the bivariate Normal copula

Exercise

Let $X = (X_1, X_2)$ be a standard Gaussian vector with correlation ρ. We note $U_1 = \Phi(X_1)$ and $U_2 = \Phi(X_2)$.

Simulation of the bivariate Normal copula

Question 1

We note Σ the matrix defined as follows:

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

Calculate the Cholesky decomposition of Σ. Deduce an algorithm to simulate X.

Simulation of the bivariate Normal copula

P is a lower triangular matrix such that we have $\Sigma = PP^\top$. We know that:

$$P = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$$

We verify that:

$$\begin{aligned} PP^\top &= \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} 1 & \rho \\ 0 & \sqrt{1-\rho^2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \end{aligned}$$

Simulation of the bivariate Normal copula

We deduce that:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$$

where N_1 and N_2 are two independent standardized Gaussian random variables. Let n_1 and n_2 be two independent random variates, whose probability distribution is $\mathcal{N}(0,1)$. Using the Cholesky decomposition, we deduce that can simulate X in the following way:

$$\left\{ \begin{array}{l} x_1 \leftarrow n_1 \\ x_2 \leftarrow \rho n_1 + \sqrt{1-\rho^2} n_2 \end{array} \right.$$

Simulation of the bivariate Normal copula

Question 2

Show that the copula of (X_1, X_2) is the same that the copula of the random vector (U_1, U_2).

Simulation of the bivariate Normal copula

We have

$$\begin{aligned} \mathbf{C}\left\langle X_1, X_2 \right\rangle &= \mathbf{C}\left\langle \Phi\left(X_1\right), \Phi\left(X_2\right) \right\rangle \\ &= \mathbf{C}\left\langle U_1, U_2 \right\rangle \end{aligned}$$

because the function $\Phi\left(x\right)$ is non-decreasing. The copula of $U = (U_1, U_2)$ is then the copula of $X = (X_1, X_2)$.

Simulation of the bivariate Normal copula

Question 3

Deduce an algorithm to simulate the Normal copula with parameter ρ.

Simulation of the bivariate Normal copula

We deduce that we can simulate U with the following algorithm:

$$\left\{ \begin{array}{l} u_1 \leftarrow \Phi(x_1) = \Phi(n_1) \\ u_2 \leftarrow \Phi(x_2) = \Phi\left(\rho n_1 + \sqrt{1-\rho^2} n_2\right) \end{array} \right.$$

Simulation of the bivariate Normal copula

Question 4

Calculate the conditional distribution of X_2 knowing that $X_1 = x$. Then show that:

$$\Phi_2\left(x_1, x_2; \rho\right) = \int_{-\infty}^{x_1} \Phi\left(\frac{x_2 - \rho x}{\sqrt{1-\rho^2}}\right) \phi\left(x\right) \mathrm{d}x$$

Simulation of the bivariate Normal copula

Let X_3 be a Gaussian random variable, which is independent from X_1 and X_2. Using the Cholesky decomposition, we know that:

$$X_2 = \rho X_1 + \sqrt{1-\rho^2} X_3$$

It follows that:

$$\begin{aligned} \Pr \left\{ X_2 \leq x_2 \mid X_1 = x \right\} &= \Pr \left\{ \rho X_1 + \sqrt{1-\rho^2} X_3 \leq x_2 \,\middle|\, X_1 = x \right\} \\ &= \Pr \left\{ X_3 \leq \frac{x_2 - \rho x}{\sqrt{1-\rho^2}} \right\} \\ &= \Phi \left(\frac{x_2 - \rho x}{\sqrt{1-\rho^2}} \right) \end{aligned}$$

Simulation of the bivariate Normal copula

Then we deduce that:

$$\begin{aligned}
\Phi_2(x_1, x_2; \rho) &= \Pr\{X_1 \leq x_1, X_2 \leq x_2\} \\
&= \Pr\left\{X_1 \leq x_1, X_3 \leq \frac{x_2 - \rho X_1}{\sqrt{1-\rho^2}}\right\} \\
&= \mathbb{E}\left[\Pr\left\{X_1 \leq x_1, X_3 \leq \frac{x_2 - \rho X_1}{\sqrt{1-\rho^2}} \,\middle|\, X_1\right\}\right] \\
&= \int_{-\infty}^{x_1} \Phi\left(\frac{x_2 - \rho x}{\sqrt{1-\rho^2}}\right) \phi(x)\,\mathrm{d}x
\end{aligned}$$

Simulation of the bivariate Normal copula

Question 5

Deduce an expression of the Normal copula.

Simulation of the bivariate Normal copula

Using the relationships $u_1 = \Phi(x_1)$, $u_2 = \Phi(x_2)$ and $\Phi_2(x_1, x_2; \rho) = \mathbf{C}(\Phi(x_1), \Phi(x_2); \rho)$, we obtain:

$$\begin{aligned}
\mathbf{C}(u_1, u_2; \rho) &= \int_{-\infty}^{\Phi^{-1}(u_1)} \Phi\left(\frac{\Phi^{-1}(u_2) - \rho x}{\sqrt{1-\rho^2}}\right) \phi(x)\, \mathrm{d}x \\
&= \int_0^{u_1} \Phi\left(\frac{\Phi^{-1}(u_2) - \rho \Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right) \mathrm{d}u
\end{aligned}$$

Simulation of the bivariate Normal copula

Question 6

Calculate the conditional copula function $\mathbf{C}_{2|1}$. Deduce an algorithm to simulate the Normal copula with parameter ρ.

Simulation of the bivariate Normal copula

We have:

$$\begin{aligned}\mathbf{C}_{2|1}\left(u_2 \mid u_1\right) &= \partial_{u_1} \mathbf{C}\left(u_1, u_2\right) \\ &= \Phi\left(\frac{\Phi^{-1}\left(u_2\right)-\rho\Phi^{-1}\left(u_1\right)}{\sqrt{1-\rho^2}}\right)\end{aligned}$$

Let v_1 and v_2 be two independent uniform random variates. The simulation algorithm corresponds to the following steps:

$$\left\{\begin{array}{l} u_1 = v_1 \\ \mathbf{C}_{2|1}\left(u_1, u_2\right) = v_2 \end{array}\right.$$

We deduce that:

$$\left\{\begin{array}{l} u_1 \leftarrow v_1 \\ u_2 \leftarrow \Phi\left(\rho\Phi^{-1}\left(v_1\right)+\sqrt{1-\rho^2}\Phi^{-1}\left(v_2\right)\right) \end{array}\right.$$

Simulation of the bivariate Normal copula

Question 7

Show that this algorithm is equivalent to the Cholesky algorithm found in Question 3.

Simulation of the bivariate Normal copula

We obtain the same algorithm, because we have the following correspondence:

$$\begin{cases} v_1 = \Phi(n_1) \\ v_2 = \Phi(n_2) \end{cases}$$

The algorithm described in Question 6 is then a special case of the Cholesky algorithm if we take $n_1 = \Phi^{-1}(v_1)$ and $n_2 = \Phi^{-1}(v_2)$. Whereas n_1 and n_2 are directly simulated in the Cholesky algorithm with a Gaussian random generator, they are simulated using the inverse transform in the conditional distribution method.

Construction of a stress scenario with the GEV distribution

Question 1

We note a_n and b_n the normalization constraints and $\mathbf{G}$ the limit distribution of the Fisher-Tippet theorem.

Construction of a stress scenario with the GEV distribution

We recall that:

$$\begin{aligned}\Pr\left\{\frac{X_{n:n}-b_n}{a_n}\leq x\right\} &= \Pr\left\{X_{n:n}\leq a_n x+b_n\right\} \\ &= \mathbf{F}^n\left(a_n x+b_n\right)\end{aligned}$$

and:

$$\mathbf{G}\left(x\right)=\lim_{n\rightarrow\infty}\mathbf{F}^n\left(a_n x+b_n\right)$$

Construction of a stress scenario with the GEV distribution

Question 1.a

Find the limit distribution **G** when $X \sim \mathcal{E}(\lambda)$, $a_n = \lambda^{-1}$ and $b_n = \lambda^{-1} \ln n$.

Construction of a stress scenario with the GEV distribution

We have:

$$\begin{aligned} \mathbf{F}^n \left(a_n x + b_n \right) &= \left(1 - e^{-\lambda \left(\lambda^{-1} x + \lambda^{-1} \ln n \right)} \right)^n \\ &= \left(1 - \frac{1}{n} e^{-x} \right)^n \end{aligned}$$

We deduce that:

$$\mathbf{G} \left(x \right) = \lim_{n \to \infty} \left(1 - \frac{1}{n} e^{-x} \right)^n = e^{-e^{-x}} = \mathbf{\Lambda} \left(x \right)$$

Construction of a stress scenario with the GEV distribution

Question 1.b

Same question when $X \sim \mathcal{U}_{[0,1]}$, $a_n = n^{-1}$ and $b_n = 1 - n^{-1}$.

Construction of a stress scenario with the GEV distribution

We have:

$$\begin{aligned}\mathbf{F}^{n}\left(a_{n}x+b_{n}\right) &= \left(n^{-1}x+1-n^{-1}\right)^{n} \\ &= \left(1+\frac{1}{n}\left(x-1\right)\right)^{n}\end{aligned}$$

We deduce that:

$$\mathbf{G}\left(x\right)=\lim_{n\rightarrow\infty}\left(1+\frac{1}{n}\left(x-1\right)\right)^{n}=e^{x-1}=\boldsymbol{\Psi}_{1}\left(x-1\right)$$

Construction of a stress scenario with the GEV distribution

Question 1.c

Same question when X is a Pareto distribution:

$$\mathbf{F}(x) = 1 - \left(\frac{\theta + x}{\theta}\right)^{-\alpha},$$

$a_n = \theta\alpha^{-1}n^{1/\alpha}$ and $b_n = \theta n^{1/\alpha} - \theta$.

Construction of a stress scenario with the GEV distribution

We have:

$$\begin{aligned}
\mathbf{F}^{n}\left(a_{n}x+b_{n}\right) &= \left(1-\left(\frac{\theta}{\theta+\theta\alpha^{-1}n^{1/\alpha}x+\theta n^{1/\alpha}-\theta}\right)^{\alpha}\right)^{n} \\
&= \left(1-\left(\frac{1}{\alpha^{-1}n^{1/\alpha}x+n^{1/\alpha}}\right)^{\alpha}\right)^{n} \\
&= \left(1-\frac{1}{n}\left(1+\frac{x}{\alpha}\right)^{-\alpha}\right)^{n}
\end{aligned}$$

We deduce that:

$$\mathbf{G}\left(x\right)=\lim_{n\rightarrow\infty}\left(1-\frac{1}{n}\left(1+\frac{x}{\alpha}\right)^{-\alpha}\right)^{n}=e^{-\left(1+\frac{x}{\alpha}\right)^{-\alpha}}=\mathbf{\Phi}_{\alpha}\left(1+\frac{x}{\alpha}\right)$$

Construction of a stress scenario with the GEV distribution

Question 2

We denote by **G** the GEV probability distribution:

$$\mathbf{G}(x) = \exp\left\{-\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/\xi}\right\}$$

What is the interest of this probability distribution? Write the log-likelihood function associated to the sample $\{x_1, \ldots, x_T\}$.

Construction of a stress scenario with the GEV distribution

The GEV distribution encompasses the three EV probability distributions. This is an interesting property, because we have not to choose between the three EV distributions. We have:

$$g(x) = \frac{1}{\sigma}\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\left(\frac{1+\xi}{\xi}\right)} \exp\left\{-\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}}\right\}$$

We deduce that:

$$\begin{aligned} \ell &= -\frac{n}{2}\ln\sigma^2 - \left(\frac{1+\xi}{\xi}\right)\sum_{i=1}^{n}\ln\left(1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right) - \\ & \quad \sum_{i=1}^{n}\left[1+\xi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{-\frac{1}{\xi}} \end{aligned}$$

Construction of a stress scenario with the GEV distribution

Question 3

Show that for $\xi \to 0$, the distribution $\mathbf{G}$ tends toward the Gumbel distribution:

$$\mathbf{\Lambda}(x) = \exp\left(-\exp\left(-\left(\frac{x-\mu}{\sigma}\right)\right)\right)$$

Construction of a stress scenario with the GEV distribution

We notice that:

$$\lim_{\xi \to 0} (1 + \xi x)^{-1/\xi} = e^{-x}$$

Then we obtain:

$$\begin{aligned} \lim_{\xi \to 0} \mathbf{G}(x) &= \lim_{\xi \to 0} \exp \left\{ - \left[1 + \xi \left(\frac{x - \mu}{\sigma} \right) \right]^{-1/\xi} \right\} \\ &= \exp \left\{ - \lim_{\xi \to 0} \left[1 + \xi \left(\frac{x - \mu}{\sigma} \right) \right]^{-1/\xi} \right\} \\ &= \exp \left(- \exp \left(- \left(\frac{x - \mu}{\sigma} \right) \right) \right) \end{aligned}$$

Construction of a stress scenario with the GEV distribution

Question 4

We consider the minimum value of daily returns of a portfolio for a period of n trading days. We then estimate the GEV parameters associated to the sample of the opposite of the minimum values. We assume that ξ is equal to 1.

Construction of a stress scenario with the GEV distribution

Question 4.a

Show that we can approximate the portfolio loss (in %) associated to the return period $\mathcal{T}$ with the following expression:

$$r(\mathcal{T}) \simeq -\left(\hat{\mu} + \left(\frac{\mathcal{T}}{n} - 1\right)\hat{\sigma}\right)$$

where $\hat{\mu}$ and $\hat{\sigma}$ are the ML estimates of GEV parameters.

Construction of a stress scenario with the GEV distribution

We have:

$$\mathbf{G}^{-1}(\alpha) = \mu - \sigma\xi^{-1}\left[1 - (-\ln\alpha)^{-\xi}\right]$$

When the parameter ξ is equal to 1, we obtain:

$$\mathbf{G}^{-1}(\alpha) = \mu - \sigma\left(1 - (-\ln\alpha)^{-1}\right)$$

By definition, we have $\mathcal{T} = (1-\alpha)^{-1} n$. The return period $\mathcal{T}$ is then associate to the confidence level $\alpha = 1 - n/\mathcal{T}$. We deduce that:

$$\begin{aligned} R(\mathcal{T}) &\approx -\mathbf{G}^{-1}(1 - n/\mathfrak{t}) \\ &= -\left(\mu - \sigma\left(1 - (-\ln(1 - n/\mathcal{T}))^{-1}\right)\right) \\ &= -\left(\mu + \left(\frac{\mathcal{T}}{n} - 1\right)\sigma\right) \end{aligned}$$

We then replace μ and σ by their ML estimates $\hat{\mu}$ and $\hat{\sigma}$.

Construction of a stress scenario with the GEV distribution

Question 4.b

We set n equal to 21 trading days. We obtain the following results for two portfolios:

Portfolio	$\hat{\mu}$	$\hat{\sigma}$	ξ
#1	1%	3%	1
#2	10%	2%	1

Calculate the stress scenario for each portfolio when the return period is equal to one year. Comment on these results.

Construction of a stress scenario with the GEV distribution

For Portfolio #1, we obtain:

$$R\left(1\text{Y}\right) = -\left(1\% + \left(\frac{252}{21} - 1\right) \times 3\%\right) = -34\%$$

For Portfolio #2, the stress scenario is equal to:

$$R\left(1\text{Y}\right) = -\left(10\% + \left(\frac{252}{21} - 1\right) \times 2\%\right) = -32\%$$

We conclude that Portfolio #1 is more risky than Portfolio #2 if we consider a stress scenario analysis.