Course 2023-2024 in Financial Risk Management
Tutorial Session 1

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[1] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Agenda

- **Tutorial Session 1: Market Risk**
- Tutorial Session 2: Credit Risk
- Tutorial Session 3: Counterparty Credit Risk and Collateral Risk
- Tutorial Session 4: Operational Risk & Asset Liability Management Risk
- Tutorial Session 5: Copulas, EVT & Stress Testing

Covariance matrix

Exercise

We consider a universe of three stocks A, B and C.

Covariance matrix

Question 1

The covariance matrix of stock returns is:

$$\Sigma = \begin{pmatrix} 4\% & & \\ 3\% & 5\% & \\ 2\% & -1\% & 6\% \end{pmatrix}$$

Covariance matrix

Question 1.a

Calculate the volatility of stock returns.

Covariance matrix

We have:

$$\sigma_A = \sqrt{\Sigma_{1,1}} = \sqrt{4\%} = 20\%$$

For the other stocks, we obtain $\sigma_B = 22.36\%$ and $\sigma_C = 24.49\%$.

Covariance matrix

Question 1.b

Deduce the correlation matrix.

Covariance matrix

The correlation is the covariance divided by the product of volatilities:

$$\rho(R_A, R_B) = \frac{\Sigma_{1,2}}{\sqrt{\Sigma_{1,1} \times \Sigma_{2,2}}} = \frac{3\%}{20\% \times 22.36\%} = 67.08\%$$

We obtain:

$$\rho = \begin{pmatrix} 100.00\% & & \\ 67.08\% & 100.00\% & \\ 40.82\% & -18.26\% & 100.00\% \end{pmatrix}$$

Covariance matrix

Question 2

We assume that the volatilities are 10%, 20% and 30%. whereas the correlation matrix is equal to:

$$\rho = \begin{pmatrix} 100\% & & \\ 50\% & 100\% & \\ 25\% & 0\% & 100\% \end{pmatrix}$$

Covariance matrix

Question 2.a

Write the covariance matrix.

Covariance matrix

Using the formula $\Sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j$, it follows that:

$$\Sigma = \begin{pmatrix} 1.00\% & & \\ 1.00\% & 4.00\% & \\ 0.75\% & 0.00\% & 9.00\% \end{pmatrix}$$

Covariance matrix

Question 2.b

Calculate the volatility of the portfolio $(50\%, 50\%, 0)$.

Covariance matrix

We deduce that:

$$\begin{aligned}\sigma^2(w) &= 0.5^2 \times 1\% + 0.5^2 \times 4\% + 2 \times 0.5 \times 0.5 \times 1\% \\ &= 1.75\%\end{aligned}$$

and $\sigma(w) = 13.23\%$.

Covariance matrix

Question 2.c

Calculate the volatility of the portfolio $(60\%, -40\%, 0)$. Comment on this result.

Covariance matrix

It follows that:

$$\begin{aligned}\sigma^2(w) &= 0.6^2 \times 1\% + (-0.4)^2 \times 4\% + 2 \times 0.6 \times (-0.4) \times 1\% \\ &= 0.52\%\end{aligned}$$

and $\sigma(w) = 7.21\%$. This long/short portfolio has a lower volatility than the previous long-only portfolio, because part of the risk is hedged by the positive correlation between stocks A and B.

Covariance matrix

Question 2.d

We assume that the portfolio is long \$150 in stock A, long \$500 in stock B and short \$200 in stock C. Find the volatility of this long/short portfolio.

Covariance matrix

We have:

$$\begin{aligned}\sigma^2(w) &= 150^2 \times 1\% + 500^2 \times 4\% + (-200)^2 \times 9\% + \\ & 2 \times 150 \times 500 \times 1\% + \\ & 2 \times 150 \times (-200) \times 0.75\% + \\ & 2 \times 500 \times (-200) \times 0\% \\ &= 14\,875\end{aligned}$$

The volatility is equal to \$121.96 and is measured in USD contrary to the two previous results which were expressed in %.

Covariance matrix

Question 3

We consider that the vector of stock returns follows a one-factor model:

$$R = \beta \mathcal{F} + \varepsilon$$

We assume that $\mathcal{F}$ and ε are independent. We note $\sigma_{\mathcal{F}}^2$ the variance of $\mathcal{F}$ and $D = \operatorname{diag}\left(\tilde{\sigma}_1^2, \tilde{\sigma}_2^2, \tilde{\sigma}_3^2\right)$ the covariance matrix of idiosyncratic risks ε_t. We use the following numerical values: $\sigma_{\mathcal{F}} = 50\%$, $\beta_1 = 0.9$, $\beta_2 = 1.3$, $\beta_3 = 0.1$, $\tilde{\sigma}_1 = 5\%$, $\tilde{\sigma}_2 = 5\%$ and $\tilde{\sigma}_3 = 15\%$.

Covariance matrix

Question 3.a

Calculate the volatility of stock returns.

Covariance matrix

We have:

$$\mathbb{E}\left[R\right] = \beta\mathbb{E}\left[\mathcal{F}\right] + \mathbb{E}\left[\varepsilon\right]$$

and:

$$R - \mathbb{E}\left[R\right] = \beta\left(\mathcal{F} - \mathbb{E}\left[\mathcal{F}\right]\right) + \varepsilon - \mathbb{E}\left[\varepsilon\right]$$

It follows that:

$$\begin{aligned}
\operatorname{cov}\left(R\right) &= \mathbb{E}\left[\left(R - \mathbb{E}\left[R\right]\right)\left(R - \mathbb{E}\left[R\right]\right)^{\top}\right] \\
&= \mathbb{E}\left[\beta\left(\mathcal{F} - \mathbb{E}\left[\mathcal{F}\right]\right)\left(\mathcal{F} - \mathbb{E}\left[\mathcal{F}\right]\right)\beta^{\top}\right] + \\
&\quad 2 \times \mathbb{E}\left[\beta\left(\mathcal{F} - \mathbb{E}\left[\mathcal{F}\right]\right)\left(\varepsilon - \mathbb{E}\left[\varepsilon\right]\right)^{\top}\right] + \\
&\quad \mathbb{E}\left[\left(\varepsilon - \mathbb{E}\left[\varepsilon\right]\right)\left(\varepsilon - \mathbb{E}\left[\varepsilon\right]\right)^{\top}\right] \\
&= \sigma_{\mathcal{F}}^{2}\beta\beta^{\top} + D
\end{aligned}$$

Covariance matrix

We deduce that:

$$\sigma\left(R_i\right)=\sqrt{\sigma_{\mathcal{F}}^2\beta_i^2+\tilde{\sigma}_i^2}$$

We obtain $\sigma\left(R_A\right)=18.68\%$, $\sigma\left(R_B\right)=26.48\%$ and $\sigma\left(R_C\right)=15.13\%$.

Covariance matrix

Question 3.b

Calculate the correlation between stock returns.

Covariance matrix

The correlation between stocks i and j is defined as follows:

$$\rho\left(R_i, R_j\right) = \frac{\sigma_{\mathcal{F}}^2 \beta_i \beta_j}{\sigma\left(R_i\right) \sigma\left(R_j\right)}$$

We obtain:

$$\rho = \left(\begin{array}{rrr} 100.00\% & & \\ 94.62\% & 100.00\% & \\ 12.73\% & 12.98\% & 100.00\% \end{array}\right)$$

Expected shortfall of an equity portfolio

Exercise

We consider an investment universe, which is composed of two stocks A and B. The current prices of the two stocks are respectively equal to \$100 and \$200. Their volatilities are equal to 25% and 20% whereas the cross-correlation is equal to -20%. The portfolio is long of 4 stocks A and 3 stocks B.

Expected shortfall of an equity portfolio

Question 1

Calculate the Gaussian expected shortfall at the 97.5% confidence level for a ten-day time horizon.

Expected shortfall of an equity portfolio

We have:

$$\begin{aligned}\Pi &= 4\left(P_{A,t+h} - P_{A,t}\right) + 3\left(P_{B,t+h} - P_{B,t}\right) \\ &= 4P_{A,t}R_{A,t+h} + 3P_{B,t}R_{B,t+h} \\ &= 400 \times R_{A,t+h} + 600 \times R_{B,t+h}\end{aligned}$$

where $R_{A,t+h}$ and $R_{B,t+h}$ are the stock returns for the period $[t, t+h]$. We deduce that the variance of the P&L is:

$$\begin{aligned}\sigma^2\left(\Pi\right) &= 400 \times (25\%)^2 + 600 \times (20\%)^2 + \\ & \quad 2 \times 400 \times 600 \times (-20\%) \times 25\% \times 20\% \\ &= 19\,600\end{aligned}$$

Expected shortfall of an equity portfolio

We deduce that $\sigma(\Pi) = \$140$. We know that the one-year expected shortfall is a linear function of the volatility:

$$\begin{aligned}\mathrm{ES}_\alpha(w; \text{one year}) &= \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha} \times \sigma(\Pi) \\ &= 2.34 \times 140 \\ &= \$327.60\end{aligned}$$

The 10-day expected shortfall is then equal to \$64.25:

$$\begin{aligned}\mathrm{ES}_\alpha(w; \text{ten days}) &= \sqrt{\frac{10}{260}} \times 327.60 \\ &= \$64.25\end{aligned}$$

Expected shortfall of an equity portfolio

Question 2

The eight worst scenarios of daily stock returns among the last 250 historical scenarios are the following:

s	1	2	3	4	5	6	7	8
R_A	-3%	-4%	-3%	-5%	-6%	$+3\%$	$+1\%$	-1%
R_B	-4%	$+1\%$	-2%	-1%	$+2\%$	-7%	-3%	-2%

Calculate then the historical expected shortfall at the 97.5% confidence level for a ten-day time horizon.

Expected shortfall of an equity portfolio

We have:

$$\Pi_s = 400 \times R_{A,s} + 600 \times R_{B,s}$$

We deduce that the value Π_s of the daily P&L for each scenario s is:

s	1	2	3	4	5	6	7	8
Π_s	-36	-10	-24	-26	-12	-30	-14	-16
$\Pi_{s:250}$	-36	-30	-26	-24	-16	-14	-12	-10

Expected shortfall of an equity portfolio

The value-at-risk at the 97.5% confidence level correspond to the 6.25^{th} order statistic[2]. We deduce that the historical expected shortfall for a one-day time horizon is equal to:

$$\begin{aligned}
\mathrm{ES}_{\alpha}\left(w;\text{one day}\right) &= -\mathbb{E}\left[\Pi \mid \Pi \leq -\mathrm{VaR}_{\alpha}\left(\Pi\right)\right] \\
&= -\frac{1}{6}\sum_{s=1}^{6}\Pi_{s:250} \\
&= \frac{1}{6}\left(36+30+26+24+16+14\right) \\
&= 24.33
\end{aligned}$$

By considering the square-root-of-time rule, it follows that the 10-day expected shortfall is equal to \$76.95.

[2] We have $2.5\% \times 250 = 6.25$.

Value-at-risk of a long/short portfolio

Exercise

We consider a long/short portfolio composed of a long (buying) position in asset A and a short (selling) position in asset B. The long exposure is \$2 mn whereas the short exposure is \$1 mn. Using the historical prices of the last 250 trading days of assets A and B, we estimate that the asset volatilities σ_A and σ_B are both equal to 20% per year and that the correlation $\rho_{A,B}$ between asset returns is equal to 50%. In what follows, we ignore the mean effect.

Value-at-risk of a long/short portfolio

We note $S_{A,t}$ (resp. $S_{B,t}$) the price of stock A (resp. B) at time t. The portfolio value is:

$$P_t(w) = w_A S_{A,t} + w_B S_{B,t}$$

where w_A and w_B are the number of stocks A and B. We deduce that the P&L between t and $t+1$ is:

$$\begin{aligned}
\Pi(w) &= P_{t+1} - P_t \\
&= w_A \left(S_{A,t+1} - S_{A,t}\right) + w_B \left(S_{B,t+1} - S_{B,t}\right) \\
&= w_A S_{A,t} R_{A,t+1} + w_B S_{B,t} R_{B,t+1} \\
&= W_{A,t} R_{A,t+1} + W_{B,t} R_{B,t+1}
\end{aligned}$$

where $R_{A,t+1}$ and $R_{B,t+1}$ are the asset returns of A and B between t and $t+1$, and $W_{A,t}$ and $W_{B,t}$ are the nominal wealth invested in stocks A and B at time t.

Value-at-risk of a long/short portfolio

Question 1

Calculate the Gaussian VaR of the long/short portfolio for a one-day holding period and a 99% confidence level.

Value-at-risk of a long/short portfolio

We have $W_{A,t} = +2$ and $W_{B,t} = -1$. The P&L (expressed in USD million) has the following expression:

$$\Pi(w) = 2R_{A,t+1} - R_{B,t+1}$$

We have $\Pi(w) \sim \mathcal{N}\left(0, \sigma^2(\Pi)\right)$ with:

$$\begin{aligned}
\sigma(\Pi) &= \sqrt{(2\sigma_A)^2 + (-\sigma_B)^2 + 2\rho_{A,B} \times (2\sigma_A) \times (-\sigma_B)} \\
&= \sqrt{4 \times 0.20^2 + (-0.20)^2 - 4 \times 0.5 \times 0.20^2} \\
&= \sqrt{3} \times 20\% \\
&\simeq 34.64\%
\end{aligned}$$

Value-at-risk of a long/short portfolio

The annual volatility of the long/short portfolio is then equal to \$346 400. We consider the square-root-of-time rule to calculate the daily value-at-risk:

$$\begin{aligned}
\mathrm{VaR}_{99\%}\left(w;\text{one day}\right) &= \frac{1}{\sqrt{260}} \times \Phi^{-1}\left(0.99\right) \times \sqrt{3} \times 20\% \\
&= 5.01\%
\end{aligned}$$

The 99% value-at-risk is then equal to \$50 056.

Value-at-risk of a long/short portfolio

Question 2

How do you calculate the historical VaR? Using the historical returns of the last 250 trading days, the five worst scenarios of the 250 simulated daily P&L of the portfolio are $-58\,700$, $-56\,850$, $-54\,270$, $-52\,170$ and $-49\,231$. Calculate the historical VaR for a one-day holding period and a 99% confidence level.

Value-at-risk of a long/short portfolio

We use the historical data to calculate the scenarios of asset returns $(R_{A,t+1}, R_{B,t+1})$. We then deduce the empirical distribution of the P&L with the formula $\Pi(w) = 2R_{A,t+1} - R_{B,t+1}$. Finally, we calculate the empirical quantile. With 250 scenarios, the 1% decile is between the second and third worst cases:

$$\begin{aligned}\mathrm{VaR}_{99\%}(w;\text{one day}) &= -\left[-56\,850 + \frac{1}{2}(-54\,270 - (-56\,850))\right] \\ &= 55\,560\end{aligned}$$

The probability to lose \$55 560 per day is equal to 1%. We notice that the difference between the historical VaR and the Gaussian VaR is equal to 11%.

Value-at-risk of a long/short portfolio

Question 3

We assume that the multiplication factor m_c is 3. Deduce the required capital if the bank uses an internal model based on the Gaussian value-at-risk. Same question when the bank uses the historical VaR. Compare these figures with those calculated with the standardized measurement method.

Value-at-risk of a long/short portfolio

If we assume that the average of the last 60 VaRs is equal to the current VaR, we obtain:

$$\mathcal{K}^{\mathrm{IMA}} = m_c \times \sqrt{10} \times \mathrm{VaR}_{99\%}\left(w; \text{one day}\right)$$

$\mathcal{K}^{\mathrm{IMA}}$ is respectively equal to \$474 877 and \$527 088 for the Gaussian and historical VaRs. In the case of the standardized measurement method, we have:

$$\begin{aligned}\mathcal{K}^{\mathrm{Specific}} &= 2 \times 8\% + 1 \times 8\% \\ &= \$240\,000\end{aligned}$$

and:

$$\begin{aligned}\mathcal{K}^{\mathrm{General}} &= \left|2 - 1\right| \times 8\% \\ &= \$80\,000\end{aligned}$$

Value-at-risk of a long/short portfolio

We deduce that:

$$\begin{aligned}\mathcal{K}^{\text{SMM}} &= \mathcal{K}^{\text{Specific}} + \mathcal{K}^{\text{General}} \\ &= \$320\,000\end{aligned}$$

The internal model-based approach does not achieve a reduction of the required capital with respect to the standardized measurement method. Moreover, we have to add the stressed VaR under Basel 2.5 and the IMA regulatory capital is at least multiplied by a factor of 2.

Value-at-risk of a long/short portfolio

Question 4

Show that the Gaussian VaR is multiplied by a factor equal to $\sqrt{7/3}$ if the correlation $\rho_{A,B}$ is equal to -50%. How do you explain this result?

Value-at-risk of a long/short portfolio

If $\rho_{A,B} = -0.50$, the volatility of the P&L becomes:

$$\begin{aligned}\sigma(\Pi) &= \sqrt{4 \times 0.20^2 + (-0.20)^2 - 4 \times (-0.5) \times 0.20^2} \\ &= \sqrt{7} \times 20\%\end{aligned}$$

We deduce that:

$$\frac{\mathrm{VaR}_\alpha\left(\rho_{A,B} = -50\%\right)}{\mathrm{VaR}_\alpha\left(\rho_{A,B} = +50\%\right)} = \frac{\sigma\left(\Pi; \rho_{A,B} = -50\%\right)}{\sigma\left(\Pi; \rho_{A,B} = +50\%\right)} = \sqrt{\frac{7}{3}} = 1.53$$

The value-at-risk increases because the hedging effect of the positive correlation vanishes. With a negative correlation, a long/short portfolio becomes more risky than a long-only portfolio.

Value-at-risk of a long/short portfolio

Question 5

The portfolio manager sells a call option on the stock A. The delta of the option is equal to 50%. What does the Gaussian value-at-risk of the long/short portfolio become if the nominal of the option is equal to $2 mn? Same question when the nominal of the option is equal to $4 mn. How do you explain this result?

Value-at-risk of a long/short portfolio

The P&L formula becomes:

$$\Pi(w) = W_{A,t} R_{A,t+1} + W_{B,t} R_{B,t+1} - \left(\mathcal{C}_{A,t+1} - \mathcal{C}_{A,t}\right)$$

where $\mathcal{C}_{A,t}$ is the call option price. We have:

$$\mathcal{C}_{A,t+1} - \mathcal{C}_{A,t} \simeq \boldsymbol{\Delta}_t \left(S_{A,t+1} - S_{A,t}\right)$$

where $\boldsymbol{\Delta}_t$ is the delta of the option. If the nominal of the option is USD 2 million, we obtain:

$$\begin{aligned} \Pi(w) &= 2R_A - R_B - 2 \times 0.5 \times R_A \\ &= R_A - R_B \end{aligned} \quad (1)$$

and:

$$\begin{aligned} \sigma(\Pi) &= \sqrt{0.20^2 + (-0.20)^2 - 2 \times 0.5 \times 0.20^2} \\ &= 20\% \end{aligned}$$

Value-at-risk of a long/short portfolio

If the nominal of the option is USD 4 million, we obtain:

$$\begin{aligned} \Pi(w) &= 2R_A - R_B - 4 \times 0.5 \times R_A \\ &= -R_B \end{aligned} \qquad (2)$$

and $\sigma(\Pi) = 20\%$. In both cases, we have:

$$\begin{aligned} \mathrm{VaR}_{99\%}(w; \text{one day}) &= \frac{1}{\sqrt{260}} \times \Phi^{-1}(0.99) \times 20\% \\ &= \$28\,900 \end{aligned}$$

The value-at-risk of the long/short portfolio (1) is then equal to the value-at-risk of the short portfolio (2) because of two effects: the absolute exposure of the long/short portfolio is higher than the absolute exposure of the short portfolio, but a part of the risk of the long/short portfolio is hedged by the positive correlation between the two stocks.

Value-at-risk of a long/short portfolio

Question 6

The portfolio manager replaces the short position on the stock B by selling a call option on the stock B. The delta of the option is equal to 50%. Show that the Gaussian value-at-risk is minimum when the nominal is equal to four times the correlation $\rho_{A,B}$. Deduce then an expression of the lowest Gaussian VaR. Comment on these results.

Value-at-risk of a long/short portfolio

We have:

$$\Pi(w) = W_{A,t} R_{A,t+1} - \left(\mathcal{C}_{B,t+1} - \mathcal{C}_{B,t}\right)$$

and:

$$\mathcal{C}_{B,t+1} - \mathcal{C}_{B,t} \simeq \boldsymbol{\Delta}_t \left(S_{B,t+1} - S_{B,t}\right)$$

where $\boldsymbol{\Delta}_t$ is the delta of the option. We note x the nominal of the option expressed in USD million. We obtain:

$$\begin{aligned}\Pi(w) &= 2R_A - x \times \boldsymbol{\Delta}_t \times R_B \\ &= 2R_A - \frac{x}{2} R_B\end{aligned}$$

We have[3]:

$$\begin{aligned}\sigma^2(\Pi) &= 4\sigma_A^2 + \frac{x^2\sigma_B^2}{4} + 2\rho_{A,B} \times (2\sigma_A) \times \left(-\frac{x}{2}\sigma_B\right) \\ &= \frac{\sigma_A^2}{4}\left(x^2 - 8\rho_{A,B}x + 16\right)\end{aligned}$$

[3] Because $\sigma_A = \sigma_B = 20\%$.

Value-at-risk of a long/short portfolio

Minimizing the Gaussian value-at-risk is equivalent to minimizing the variance of the P&L. We deduce that the first-order condition is:

$$\frac{\partial \sigma^2(\Pi)}{\partial x} = \frac{\sigma_A^2}{4}\left(2x - 8\rho_{A,B}\right) = 0$$

We deduce that the minimum VaR is reached when the nominal of the option is $x = 4\rho_{A,B}$. We finally obtain:

$$\begin{aligned}\sigma(\Pi) &= \frac{\sigma_A}{2}\sqrt{16\rho_{A,B}^2 - 32\rho_{A,B}^2 + 16} \\ &= 2\sigma_A\sqrt{1-\rho_{A,B}^2}\end{aligned}$$

and:

$$\begin{aligned}\mathrm{VaR}_{99\%}(w;\text{one day}) &= \frac{1}{\sqrt{260}} \times 2.33 \times 2 \times 20\% \times \sqrt{1-\rho_{A,B}^2} \\ &\simeq 5.78\% \times \sqrt{1-\rho_{A,B}^2}\end{aligned}$$

Value-at-risk of a long/short portfolio

If $\rho_{A,B}$ is negative (resp. positive), the exposure x is negative meaning that we have to buy (resp. to sell) a call option on stock B in order to hedge a part of the risk related to stock A. If $\rho_{A,B}$ is equal to zero, the exposure x is equal to zero because a position on stock B adds systematically a supplementary risk to the portfolio.

Risk management of exotic options

Exercise

Let us consider a short position on an exotic option, whose its current value $\mathcal{C}_t$ is equal to \$6.78. We assume that the price S_t of the underlying asset is \$100 and the implied volatility Σ_t is equal to 20%.

Risk management of exotic options

Let $\mathcal{C}_t$ be the option price at time t. The P&L of the trader between t and $t+1$ is:

$$\Pi = -\left(\mathcal{C}_{t+1} - \mathcal{C}_t\right)$$

The formulation of the exercise suggests that there are two main risk factors: the price of the underlying asset S_t and the implied volatility Σ_t. We then obtain:

$$\Pi = \mathcal{C}_t\left(S_t, \Sigma_t\right) - \mathcal{C}_{t+1}\left(S_{t+1}, \Sigma_{t+1}\right)$$

Risk management of exotic options

Question 1

At time $t+1$, the value of the underlying asset is \$97 and the implied volatility remains constant. We find that the P&L of the trader between t and $t+1$ is equal to \$1.37. Can we explain the P&L by the sensitivities knowing that the estimates of delta $\boldsymbol{\Delta}_t$, gamma $\boldsymbol{\Gamma}_t$ and vega[a] $\boldsymbol{\upsilon}_t$ are respectively equal to 49%, 2% and 40%?

[a] measured in volatility points.

Risk management of exotic options

We have:

$$\begin{aligned}\Pi &= C_t\left(S_t,\Sigma_t\right)-C_{t+1}\left(S_{t+1},\Sigma_{t+1}\right)\\ &\approx -\boldsymbol{\Delta}_t\left(S_{t+1}-S_t\right)-\frac{1}{2}\boldsymbol{\Gamma}_t\left(S_{t+1}-S_t\right)^2-\boldsymbol{\upsilon}_t\left(\Sigma_{t+1}-\Sigma_t\right)\end{aligned}$$

Using the numerical values of $\boldsymbol{\Delta}_t$, $\boldsymbol{\Gamma}_t$ and $\boldsymbol{\upsilon}_t$, we obtain:

$$\begin{aligned}\Pi &\approx -0.49\times(97-100)-\frac{1}{2}\times 0.02\times(97-100)^2\\ &= 1.47-0.09\\ &= 1.38\end{aligned}$$

We explain the P&L by the sensitivities very well.

Risk management of exotic options

Question 2

At time $t+2$, the price of the underlying asset is \$97 while the implied volatility increases from 20% to 22%. The value of the option $\mathcal{C}_{t+2}$ is now equal to \$6.17. Can we explain the P&L by the sensitivities knowing that the estimates of delta $\boldsymbol{\Delta}_{t+1}$, gamma $\boldsymbol{\Gamma}_{t+1}$ and vega $\boldsymbol{\upsilon}_{t+1}$ are respectively equal to 43%, 2% and 38%?

Risk management of exotic options

We have:

$$\begin{aligned}\Pi &= C_{t+1}\left(S_{t+1},\Sigma_{t+1}\right) - C_{t+2}\left(S_{t+2},\Sigma_{t+2}\right) \\ &\approx -\boldsymbol{\Delta}_{t+1}\left(S_{t+2}-S_{t+1}\right) - \frac{1}{2}\boldsymbol{\Gamma}_{t+1}\left(S_{t+2}-S_{t+1}\right)^2 - \\ &\quad \boldsymbol{\upsilon}_{t+1}\left(\Sigma_{t+2}-\Sigma_{t+1}\right)\end{aligned}$$

Using the numerical values of $\boldsymbol{\Delta}_{t+1}$, $\boldsymbol{\Gamma}_{t+1}$ and $\boldsymbol{\upsilon}_{t+1}$, we obtain:

$$\begin{aligned}\Pi &\approx -0.49\times 0 - \frac{1}{2}\times 0.02\times 0^2 - 0.38\times(22-20) \\ &= -0.76\end{aligned}$$

Risk management of exotic options

To compare this value with the true P&L, we have to calculate $\mathcal{C}_{t+1}$:

$$\begin{aligned} \mathcal{C}_{t+1} &= \mathcal{C}_t - (\mathcal{C}_t - \mathcal{C}_{t+1}) \\ &= 6.78 - 1.37 \\ &= 5.41 \end{aligned}$$

We deduce that:

$$\begin{aligned} \Pi &= \mathcal{C}_{t+1} - \mathcal{C}_{t+2} \\ &= 5.41 - 6.17 \\ &= -0.76 \end{aligned}$$

Again, the sensitivities explain the P&L very well.

Risk management of exotic options

Question 3

At time $t+3$, the price of the underlying asset is \$95 and the value of the implied volatility is 19%. We find that the P&L of the trader between $t+2$ and $t+3$ is equal to \$0.58. Can we explain the P&L by the sensitivities knowing that the estimates of delta $\boldsymbol{\Delta}_{t+2}$, gamma $\boldsymbol{\Gamma}_{t+2}$ and vega $\boldsymbol{\upsilon}_{t+2}$ are respectively equal to 44%, 1.8% and 38%.

Risk management of exotic options

We have:

$$\begin{aligned} \Pi &= \mathcal{C}_{t+2}\left(S_{t+2},\Sigma_{t+2}\right)-\mathcal{C}_{t+3}\left(S_{t+3},\Sigma_{t+3}\right) \\ &\approx -\boldsymbol{\Delta}_{t+2}\left(S_{t+3}-S_{t+2}\right)-\frac{1}{2}\boldsymbol{\Gamma}_{t+2}\left(S_{t+3}-S_{t+2}\right)^2- \\ &\quad \boldsymbol{\upsilon}_{t+2}\left(\Sigma_{t+3}-\Sigma_{t+2}\right) \end{aligned}$$

Using the numerical values of $\boldsymbol{\Delta}_{t+2}$, $\boldsymbol{\Gamma}_{t+2}$ and $\boldsymbol{\upsilon}_{t+2}$, we obtain:

$$\begin{aligned} \Pi &\approx -0.44\times(95-97)-\frac{1}{2}\times 0.018\times(95-97)^2- \\ &\quad 0.38\times(19-22) \\ &= 0.88-0.036+1.14 \\ &= 1.984 \end{aligned}$$

The P&L approximated by the Greek coefficients largely overestimate the true value of the P&L.

Risk management of exotic options

Question 4

What can we conclude in terms of model risk?

Risk management of exotic options

We notice that the approximation using the Greek coefficients works very well when one risk factor remains constant:

- Between t and $t+1$, the price of the underlying asset changes, but not the implied volatility;
- Between $t+1$ and $t+2$, this is the implied volatility that changes whereas the price of the underlying asset is constant.

Therefore, we can assume that the bad approximation between $t+2$ and $t+3$ is due to the cross effect between S_t and Σ_t. In terms of model risk, the P&L is then exposed to the vanna risk, meaning that the Black-Scholes model is not appropriate to price and hedge this exotic option.