Course 2023-2024 in Financial Risk Management
Lecture 8. Model Risk

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[1] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

General information

1. **Overview**
   The objective of this course is to understand the theoretical and practical aspects of risk management
2. **Prerequisites**
   M1 Finance or equivalent
3. **ECTS**
   4
4. **Keywords**
   Finance, Risk Management, Applied Mathematics, Statistics
5. **Hours**
   Lectures: 36h, Training sessions: 15h, HomeWork: 30h
6. **Evaluation**
   There will be a final three-hour exam, which is made up of questions and exercises
7. **Course website**
   `http://www.thierry-roncalli.com/RiskManagement.html`

Objective of the course

The objective of the course is twofold:

1. knowing and understanding the financial regulation (banking and others) and the international standards (especially the Basel Accords)
2. being proficient in risk measurement, including the mathematical tools and risk models

Class schedule

Course sessions

- September 15 (6 hours, AM+PM)
- September 22 (6 hours, AM+PM)
- September 19 (6 hours, AM+PM)
- October 6 (6 hours, AM+PM)
- October 13 (6 hours, AM+PM)
- October 27 (6 hours, AM+PM)

Tutorial sessions

- October 20 (3 hours, AM)
- October 20 (3 hours, PM)
- November 10 (3 hours, AM)
- November 10 (3 hours, PM)
- November 17 (3 hours, PM)

Class times: Fridays 9:00am-12:00pm, 1:00pm–4:00pm, University of Evry, Room 209 IDF

Agenda

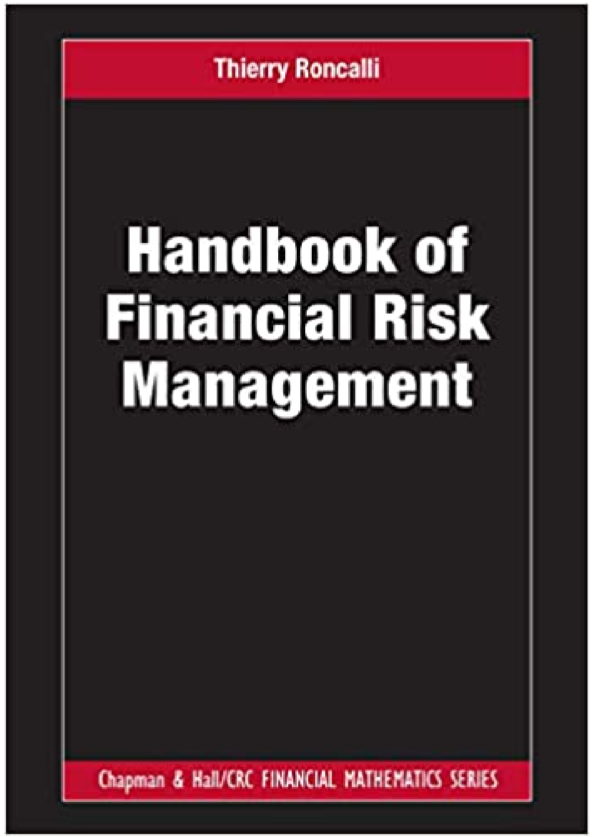
- Lecture 1: Introduction to Financial Risk Management
- Lecture 2: Market Risk
- Lecture 3: Credit Risk
- Lecture 4: Counterparty Credit Risk and Collateral Risk
- Lecture 5: Operational Risk
- Lecture 6: Liquidity Risk
- Lecture 7: Asset Liability Management Risk
- Lecture 8: Model Risk
- Lecture 9: Copulas and Extreme Value Theory
- Lecture 10: Monte Carlo Simulation Methods
- Lecture 11: Stress Testing and Scenario Analysis
- Lecture 12: Credit Scoring Models

Agenda

- Tutorial Session 1: Market Risk
- Tutorial Session 2: Credit Risk
- Tutorial Session 3: Counterparty Credit Risk and Collateral Risk
- Tutorial Session 4: Operational Risk & Asset Liability Management Risk
- Tutorial Session 5: Copulas, EVT & Stress Testing

Textbook

- Roncalli, T. (2020), *Handbook of Financial Risk Management*, Chapman & Hall/CRC Financial Mathematics Series.



Additional materials

- Slides, tutorial exercises and past exams can be downloaded at the following address:

  `http://www.thierry-roncalli.com/RiskManagement.html`

- Solutions of exercises can be found in the companion book, which can be downloaded at the following address:

  `http://www.thierry-roncalli.com/RiskManagementBook.html`

Agenda

Main issue

Easy

Computing an option price using a stochastic model

Hard

Computing an option price that corresponds to the cost of the hedging strategy

$$\text{Pricing} = \text{Hedging}$$

The Black-Scholes model

Black and Scholes (1973) assumed that the dynamics of the asset price $S(t)$ is given by a GBM:

$$\begin{cases} \mathrm{d}S(t) = \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t) \\ S(t_0) = S_0 \end{cases}$$

where:

- S_0 is the current price
- μ is the drift
- σ is the volatility of the diffusion
- $W(t)$ is a standard Brownian motion

The Black-Scholes model

Let $f(S(T))$ be the payoff of a contingent claim where T is the maturity of the derivative contract. The price V of the contingent claim is then equal to the cost of the hedging portfolio. We can show that:

$$\left\{ \begin{array}{l} \frac{1}{2}\sigma^2 S^2 \partial_S^2 V(t,S) + (\mu - \lambda(t)\sigma) S \partial_S V(t,S) + \partial_t V(t,S) - r(t) V(t,S) = 0 \\ V(T, S(T)) = f(S(T)) \end{array} \right.$$

This equation is called the fundamental pricing equation

The Black-Scholes model

The function $\lambda(t)$ is interpreted as the risk price of the Wiener process $W(t)$:

$$\lambda(t) = \frac{\mu - b(t)}{\sigma}$$

where $b(t)$ is the cost-of-carry
We have:

$$\begin{cases} \frac{1}{2}\sigma^2 S^2 \partial_S^2 V(t,S) + b(t) S \partial_S V(t,S) + \partial_t V(t,S) - r(t) V(t,S) = 0 \\ V(T, S(T)) = f(S(T)) \end{cases}$$

The current price of the derivatives contract is equal to $V(t_0, S_0)$

The Black-Scholes model

- Girsanov theorem with $g(t) = -\lambda(t)$:

$$\left\{ \begin{array}{l} \mathrm{d}S(t) = b(t) S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W^{\mathbb{Q}}(t) \\ S(t_0) = S_0 \end{array} \right.$$

- $W^{\mathbb{Q}}(t)$ is a Brownian motion under the probability $\mathbb{Q}$ defined by:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(-\int_0^t \lambda(s)\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t \lambda^2(s)\,\mathrm{d}s\right)$$

- Feynman-Kac formula with $h(t,x) = r(t)$ and $g(t,x) = 0$:

$$V_0 = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(t)\,\mathrm{d}t} f(S(T)) \,\middle|\, \mathcal{F}_0 \right]$$

- V_0 is called the martingale solution and $\mathbb{Q}$ is called the risk-neutral probability measure

Application to European options

We consider an European call option whose payoff at maturity is equal to:

$$\mathcal{C}(T) = (S(T) - K)^+$$

We assume that the interest rate $r(t)$ and the cost-of-carry parameter $b(t)$ are constant:

$$\begin{aligned}
\mathcal{C}_0 &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r\,\mathrm{d}t}(S(T) - K)^+ \middle| \mathcal{F}_0\right] \\
&= e^{-rT}\mathbb{E}\left[\left(S_0 e^{\left(b-\frac{1}{2}\sigma^2\right)T+\sigma W^{\mathbb{Q}}(T)} - K\right)^+\right] \\
&= e^{-rT}\int_{-d_2}^{\infty}\left(S_0 e^{\left(b-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}x} - K\right)\phi(x)\,\mathrm{d}x \\
&= S_0 e^{(b-r)T}\Phi(d_1) - Ke^{-rT}\Phi(d_2)
\end{aligned}$$

where:

$$\begin{aligned}
d_1 &= \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{S_0}{K} + bT\right) + \frac{1}{2}\sigma\sqrt{T} \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}$$

Application to European options

Let us now consider an European put option with the following payoff:

$$\mathcal{P}(T) = (K - S(T))^{+}$$

We have:

$$\mathcal{C}(T) - \mathcal{P}(T) = (S(T) - K)^{+} - (K - S(T))^{+} = S(T) - K$$

We deduce that:

$$\begin{aligned}
\mathcal{C}_0 - \mathcal{P}_0 &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r \,\mathrm{d}t} (S(T) - K) \middle| \mathcal{F}_0 \right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-rT} S(T) \middle| \mathcal{F}_0 \right] - Ke^{-rT} \\
&= S_0 e^{(b-r)T} - Ke^{-rT}
\end{aligned}$$

This equation is known as the put-call parity and we have:

$$\begin{aligned}
\mathcal{P}_0 &= \mathcal{C}_0 - S_0 e^{(b-r)T} + Ke^{-rT} \\
&= -S_0 e^{(b-r)T} \Phi(-d_1) + Ke^{-rT} \Phi(-d_2)
\end{aligned}$$

Application to European options

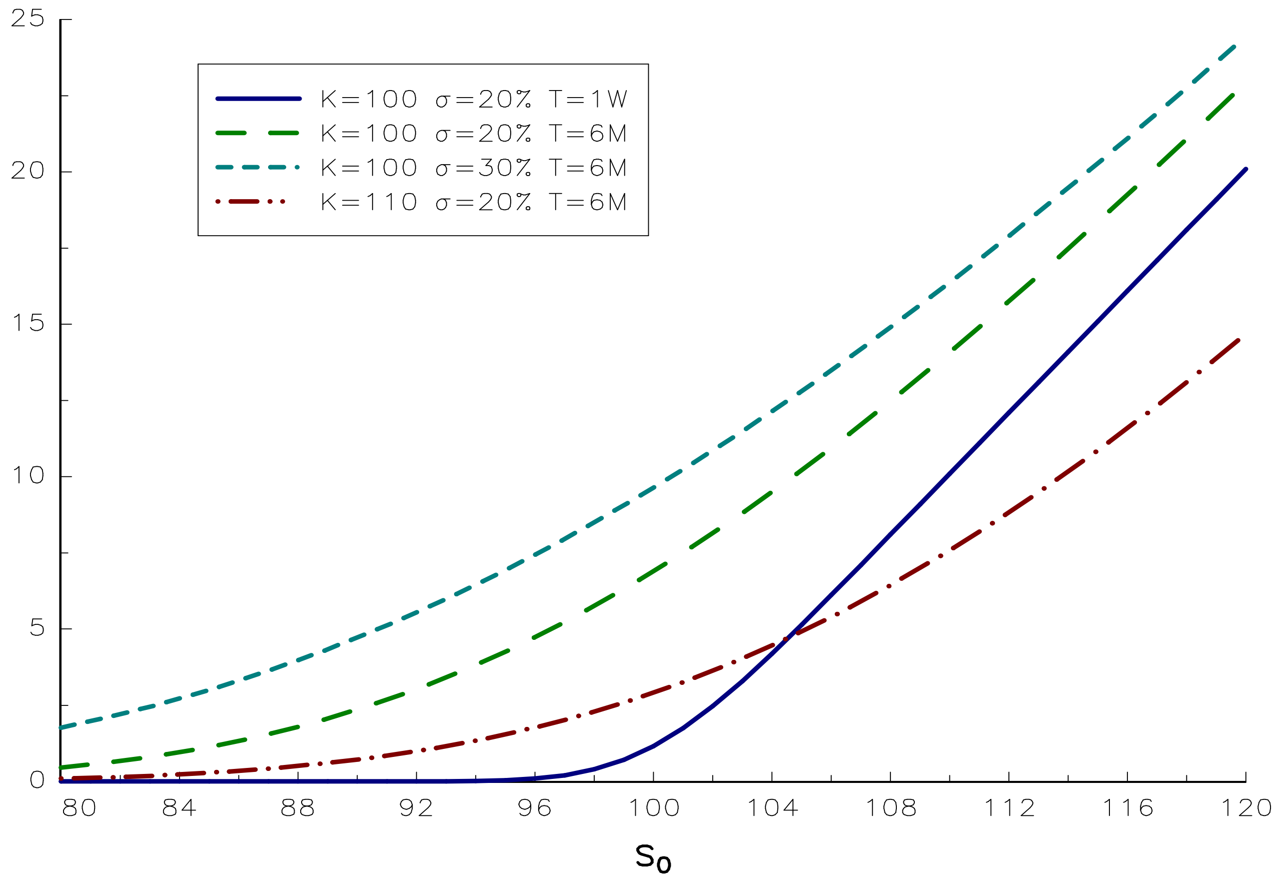


Figure: Price of the call option

Application to European options

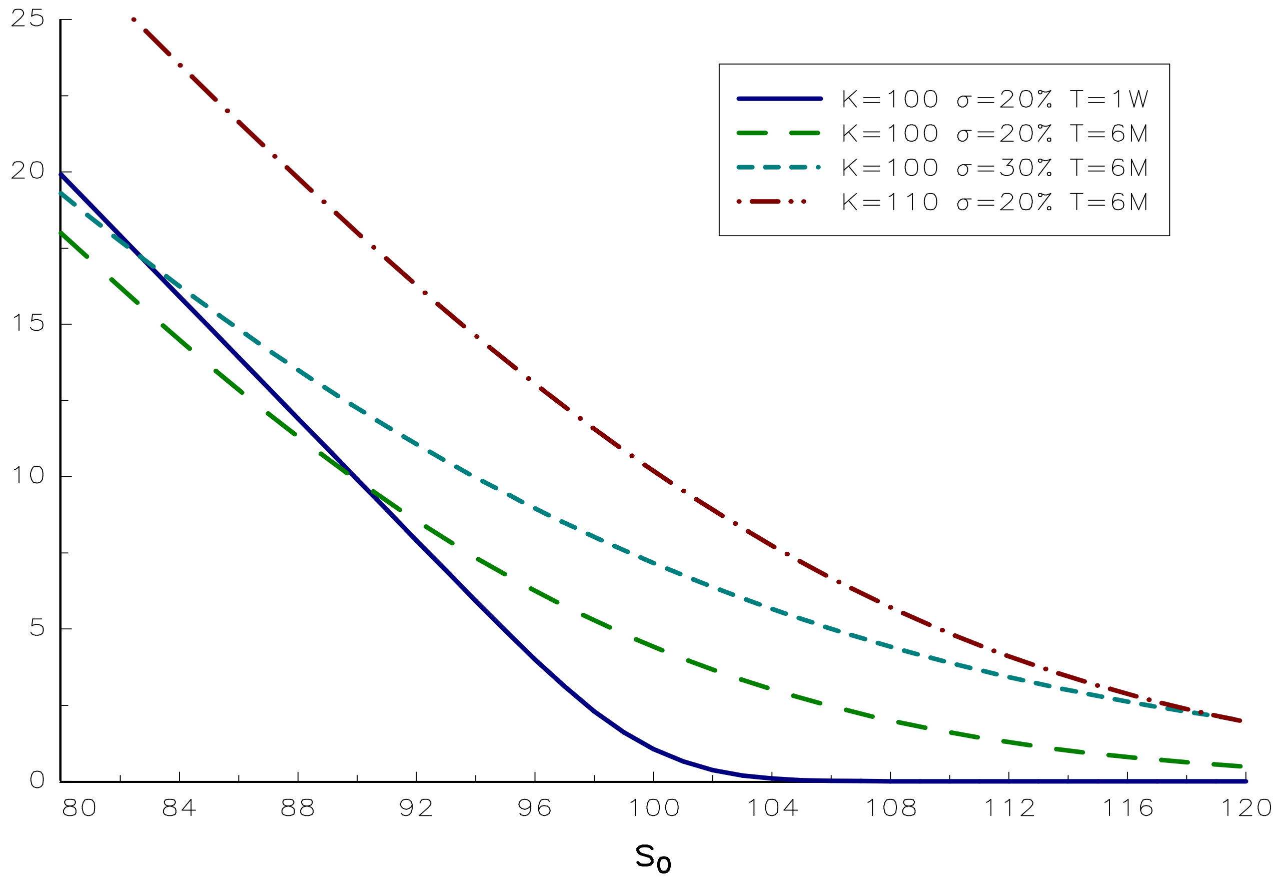


Figure: Price of the put option

Principle of dynamic hedging

- n assets that do not pay dividends or coupons during the period $[0, T]$
- For asset i, we have:

$$S_i(t) = S_i(0) + \int_0^t \mu_i(u)\,\mathrm{d}u + \int_0^t \sigma_i(u)\,\mathrm{d}W_i(u)$$

- We set up a trading portfolio $(\phi_1(t), \ldots, \phi_n(t))$ invested in the assets $(S_1(t), \ldots, S_n(t))$
- The value of this trading portfolio is:

$$X(t) = \sum_{i=1}^{n} \phi_i(t) S_i(t)$$

Self-financing strategy

- The portfolio is self-financing if the following conditions hold:

$$\begin{cases} \mathrm{d}X(t) - \sum_{i=1}^{n} \phi_i(t)\,\mathrm{d}S_i(t) = 0 \\ X(0) = 0 \end{cases}$$

  1. The first condition means that all trades are financed by selling or buying assets in the portfolio
  2. The second condition implies that we don't need money to set up the initial portfolio

- This implies that:

$$\begin{aligned} X(t) &= X_0 + \sum_{i=1}^{n} \int_0^t \phi_i(u)\,\mathrm{d}S_i(u) \\ &= \sum_{i=1}^{n} \phi_i(0)\,S_i(0) + \sum_{i=1}^{n} \int_0^t \phi_i(u)\,\mathrm{d}S_i(u) \end{aligned}$$

Principle of dynamic hedging

- We have:

$$\mathrm{d}S(t) = \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t)$$

- The risk-free asset $B(t)$ satisfies:

$$\mathrm{d}B(t) = rB(t)\,\mathrm{d}t$$

- We set up a trading portfolio $(\phi(t), \psi(t))$ invested in the stock $S(t)$ and the risk-free asset $B(t)$
- The value of this portfolio is:

$$V(t) = \phi(t)\,S(t) + \psi(t)\,B(t)$$

Application to the Black-Scholes model

- We form a strategy $X(t)$ in which we are long the call option $\mathcal{C}(t, S(t))$ and short the trading portfolio $V(t)$:

$$\begin{aligned} X(t) &= \mathcal{C}(t, S(t)) - V(t) \\ &= \mathcal{C}(t, S(t)) - \phi(t) S(t) - \psi(t) B(t) \end{aligned}$$

- We have:

$$\begin{aligned} \mathrm{d}X(t) &= \partial_S \mathcal{C}(t, S(t)) \, \mathrm{d}S(t) + \\ & \left(\partial_t \mathcal{C}(t, S(t)) + \frac{1}{2} \sigma^2 S^2(t) \, \partial_S^2 \mathcal{C}(t, S(t)) \right) \mathrm{d}t - \\ & \phi(t) \, \mathrm{d}S(t) - \psi(t) \, \mathrm{d}B(t) \end{aligned}$$

- By assuming that $\phi(t) = \partial_S \mathcal{C}(t, S(t))$, we obtain:

$$\mathrm{d}X(t) = \left(\partial_t \mathcal{C}(t, S(t)) + \frac{1}{2} \sigma^2 S^2(t) \, \partial_S^2 \mathcal{C}(t, S(t)) - r\psi(t) B(t) \right) \mathrm{d}t$$

Application to the Black-Scholes model

- $X(t)$ is self-financing if $\mathrm{d}X(t) = 0$ or:

$$\psi(t) = \frac{\partial_t \mathcal{C}(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t)\, \partial_S^2 \mathcal{C}(t, S(t))}{rB(t)}$$

- We deduce that:

$$\begin{aligned}
\mathcal{C}(t, S(t)) &= \phi(t) S(t) + \psi(t) B(t) \\
&= \partial_S \mathcal{C}(t, S(t)) S(t) + \\
&\quad \frac{\partial_t \mathcal{C}(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t)\, \partial_S^2 \mathcal{C}(t, S(t))}{rB(t)} B(t)
\end{aligned}$$

Application to the Black-Scholes model

- This implies that $\mathcal{C}(t, S(t))$ satisfies the following PDE:

$$\frac{1}{2}\sigma^2 S^2 \partial_S^2 \mathcal{C}(t, S) + rS\partial_S \mathcal{C}(t, S) + \partial_t \mathcal{C}(t, S) - r\mathcal{C}(t, S) = 0$$

- Since $X(t)$ is self-financing ($X(t) = 0$), we also deduce that the trading portfolio $V(t)$ is the replicating portfolio of the call option:

$$\begin{aligned} V(t) &= \phi(t) S(t) + \psi(t) B(t) \\ &= \mathcal{C}(t, S(t)) - X(t) \\ &= \mathcal{C}(t, S(t)) \end{aligned}$$

Application to the Black-Scholes model

- If we define the replicating cost as follows:

$$\begin{aligned} C(t) &= \int_0^t \phi(u)\,\mathrm{d}S(u) + \int_0^t \psi(u)\,\mathrm{d}B(u) \\ &= \int_0^t \left(\mu S(u)\phi(u) + rB(u)\psi(u)\right)\mathrm{d}u + \int_0^T \sigma S(u)\phi(u)\,\mathrm{d}W(u) \end{aligned}$$

we have:

$$\begin{aligned} C(t) &= \int_0^t \mu S(u)\,\partial_S \mathcal{C}(u, S(u))\,\mathrm{d}u + \int_0^T \sigma S(u)\,\partial_S \mathcal{C}(u, S(u))\,\mathrm{d}W(u) \\ & \int_0^t \left(\partial_t \mathcal{C}(u, S(u)) + \frac{1}{2}\sigma^2 S^2(u)\,\partial_S^2 \mathcal{C}(u, S(u))\right)\mathrm{d}u \\ &= \int_0^t \mathrm{d}\mathcal{C}(u, S(u)) = \mathcal{C}(t, S(t)) - \mathcal{C}(0, S_0) \end{aligned}$$

- We verify that the replicating cost is exactly equal to the P&L of the long exposure on the call option

Cost-of-carry

- Let us consider a stock that pays a continuous dividend δ, the self-financing portfolio is:

$$X(t) = \mathcal{C}(t, S(t)) - \phi(t) S(t) - \psi(t) B(t)$$

We deduce that the change in the value of this portfolio is:

$$\mathrm{d}X(t) = \mathrm{d}\mathcal{C}(t, S(t)) - \phi(t)\,\mathrm{d}S(t) - \psi(t)\,\mathrm{d}B(t) - \phi(t) \cdot \underbrace{\delta \cdot S(t)\,\mathrm{d}t}_{\text{dividend}}$$

- Using the same rationale than previously, we obtain $\phi(t) = \partial_S \mathcal{C}(t, S(t))$ and:

$$\psi(t) = \frac{\partial_t \mathcal{C}(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t)\,\partial_S^2 \mathcal{C}(t, S(t)) - \delta S(t)\,\partial_S \mathcal{C}(t, S(t))}{rB(t)}$$

Finally, we obtain the following PDE:

$$\frac{1}{2}\sigma^2 S^2 \partial_S^2 \mathcal{C}(t, S) + (r - \delta) S \partial_S \mathcal{C}(t, S) + \partial_t \mathcal{C}(t, S) - r\mathcal{C}(t, S) = 0$$

Cost-of-carry

- When the stock does not pay a dividend, the cost-of-carry parameter b is equal to the interest rate r
- When the stock pays a continuous dividend, the cost-of-carry parameter b is equal to $r - \delta$
- In the case of futures or forward contracts, the cost-of-carry is equal to zero
- For currency options, the cost-of-carry is the difference between the domestic interest rate r and the foreign interest rate $r^{\star}$

Table: Impact of the dividend on the option premium

	Put option				Call option			
S_0 / δ	0.00	0.02	0.05	0.07	0.00	0.02	0.05	0.07
90	1.28	1.44	1.73	1.94	13.50	12.67	11.48	10.72
100	4.42	4.83	5.50	5.97	6.89	6.31	5.50	5.00
110	10.19	10.87	11.91	12.63	2.91	2.59	2.16	1.90

Delta hedging

- The Black-Scholes model assumes that the replicating portfolio is rebalanced continuously
- In practice, it is rebalanced at some fixed dates t_i:

$$0 = t_0 < t_1 < \cdots < t_n = T$$

- At the initial date, we have:

$$X(t_0) = \mathcal{C}(t_0, S(t_0)) - V(t_0) = 0$$

where:

$$V(t_0) = \phi(t_0) \cdot S(t_0) + \psi(t_0) \cdot B(t_0)$$

- Because we have $\phi(t_0) = \boldsymbol{\Delta}(t_0)$ and $X(t_0) = 0$, we deduce that:

$$\psi(t_0) = \mathcal{C}(t_0, S(t_0)) - \boldsymbol{\Delta}(t_0) S(t_0)$$

Delta hedging

- At time t_1, the value of the replicating portfolio is then equal to:

$$V(t_1) = \boldsymbol{\Delta}(t_0) S(t_1) + (\mathcal{C}(t_0, S(t_0)) - \boldsymbol{\Delta}(t_0) S(t_0)) \cdot (1 + r(t_0)(t_1 - t_0))$$

- It follows that:

$$X(t_1) = \mathcal{C}(t_1, S(t_1)) - V(t_1)$$

- We are not sure that $X(t_1) = 0$ because it is not possible to hedge the jump $S(t_1) - S(t_0)$. We rebalance the portfolio and we have:

$$V(t_1) = \phi(t_1) \cdot S(t_1) + \psi(t_1) \cdot B(t_1)$$

- We deduce that:

$$\phi(t_1) = \boldsymbol{\Delta}(t_1)$$

  and:

$$\psi(t_1) = V(t_1) - \boldsymbol{\Delta}(t_1) S(t_1)$$

Delta hedging

- At time t_2, the value of the replicating portfolio is equal to:

$$V(t_2) = \boldsymbol{\Delta}(t_1) S(t_2) + (V(t_1) - \boldsymbol{\Delta}(t_1) S(t_1)) \cdot (1 + r(t_1)(t_2 - t_1))$$

- More generally, we have:

$$X(t_i) = \mathcal{C}(t_i, S(t_i)) - V(t_i)$$

and:

$$V(t_i) = \underbrace{\boldsymbol{\Delta}(t_{i-1}) S(t_i)}_{V_S(t_i)} + \underbrace{(V(t_{i-1}) - \boldsymbol{\Delta}(t_{i-1}) S(t_{i-1})) \cdot (1 + r(t_{i-1})(t_i - t_{i-1}))}_{V_B(t_i)}$$

where $V_S(t_i)$ is the component due to the delta exposure on the asset and $V_B(t_i)$ is the component due to the cash exposure on the risk-free bond

Delta hedging

- We notice that:

$$\begin{aligned} V_S(t_i) &= \boldsymbol{\Delta}(t_{i-1}) \cdot S(t_i) \\ &= \boldsymbol{\Delta}(t_{i-1}) \cdot S(t_{i-1}) \cdot (1 + R_S(t_{i-1}; t_i)) \end{aligned}$$

and:

$$\begin{aligned} V_B(t_i) &= (V(t_{i-1}) - \boldsymbol{\Delta}(t_{i-1}) \cdot S(t_{i-1})) \cdot (1 + r(t_{i-1}) \cdot (t_i - t_{i-1})) \\ &= (V(t_{i-1}) - \boldsymbol{\Delta}(t_{i-1}) \cdot S(t_{i-1})) \cdot (1 + R_B(t_{i-1}; t_i)) \end{aligned}$$

where $R_S(t_{i-1}; t_i)$ and $R_B(t_{i-1}; t_i)$ are the asset and bond returns between t_{i-1} and t_i

Delta hedging

- At the maturity, we obtain:

$$\begin{aligned} X(T) &= X(t_n) \\ &= (S(T) - K)^+ - V(t_n) \end{aligned}$$

- $\Pi(T) = -X(T)$ is the P&L of the delta hedging strategy. To measure its efficiency, we consider the ratio π defined as follows:

$$\pi = \frac{\Pi(T)}{\mathcal{C}(t_0, S(t_0))}$$

Delta hedging

Example #1

We consider the replication of 100 ATM call options. The current price of the asset is 100 and the maturity of the option is 20 weeks. We consider the following parameter: $b = r = 5\%$ and $\sigma = 20\%$. We rebalance the replicating portfolio every week.

Delta hedging

- $T = 20/52$
- $K = 100$
- $\mathcal{C}(t_0, S(t_0)) = \$5.90$
- The replicating portfolio is rebalanced at times t_i:

$$t_i = \frac{i}{52}$$

Delta hedging

Table: An example of delta hedging strategy (negative P&L)

i	t_i	$S(t_i)$	$\Delta(t_{i-1})$	$V_S(t_i)$	$V_B(t_i)$	$V(t_i)$	$\mathcal{C}(t_i, S(t_i))$	$X(t_i)$	$\Pi(t_i)$
0	0.00	100.00	0.00	0.00	590.90	590.90	590.90	0.00	0.00
1	0.02	95.63	58.59	5603.15	−5273.36	329.79	350.22	20.43	−20.43
2	0.04	95.67	43.72	4182.80	−3854.96	327.84	336.15	8.31	−8.31
3	0.06	94.18	43.24	4072.36	−3812.62	259.75	260.57	0.82	−0.82
4	0.08	92.73	37.29	3457.72	−3255.16	202.55	196.22	−6.33	6.33
5	0.10	96.59	31.34	3027.23	−2706.31	320.93	326.47	5.54	−5.54
6	0.12	101.68	44.63	4537.99	−3993.73	544.26	582.71	38.45	−38.45
7	0.13	101.41	63.39	6428.19	−5906.72	521.47	545.64	24.17	−24.17
8	0.15	100.22	62.36	6249.97	−5808.29	441.68	453.62	11.94	−11.94
9	0.17	99.32	57.57	5718.25	−5333.51	384.74	382.58	−2.16	2.16
10	0.19	101.64	53.46	5433.52	−4929.49	504.03	495.99	−8.04	8.04
11	0.21	101.81	63.27	6441.30	−5932.22	509.08	483.87	−25.21	25.21
12	0.23	102.62	64.10	6578.19	−6022.97	555.22	513.53	−41.69	41.69
13	0.25	107.56	67.97	7311.26	−6426.42	884.84	876.68	−8.16	8.16
14	0.27	102.05	86.90	8867.94	−8470.05	397.89	424.07	26.18	−26.18
15	0.29	100.88	66.19	6677.01	−6362.67	314.34	321.76	7.41	−7.41
16	0.31	106.90	59.86	6399.37	−5730.15	669.21	756.02	86.80	−86.80
17	0.33	107.66	90.32	9723.75	−8994.54	729.22	806.47	77.25	−77.25
18	0.35	101.79	94.74	9643.97	−9480.00	163.96	276.24	112.27	−112.27
19	0.37	101.76	69.88	7111.04	−6955.85	155.19	228.08	72.89	−72.89
20	0.38	101.83	75.10	7647.28	−7494.04	153.24	183.00	29.76	**−29.76**

Delta hedging

Table: An example of delta hedging strategy (positive P&L)

i	t_i	$S(t_i)$	$\Delta(t_{i-1})$	$V_S(t_i)$	$V_B(t_i)$	$V(t_i)$	$\mathcal{C}(t_i, S(t_i))$	$X(t_i)$	$\Pi(t_i)$
0	0.00	100.00	0.00	0.00	590.90	590.90	590.90	0.00	0.00
1	0.02	98.50	58.59	5771.31	−5273.36	497.95	489.70	−8.25	8.25
2	0.04	97.00	53.45	5184.51	−4771.31	413.19	396.75	−16.44	16.44
3	0.06	95.47	47.89	4571.99	−4236.14	335.85	311.62	−24.24	24.24
4	0.08	98.17	41.87	4110.19	−3664.81	445.38	419.94	−25.44	25.44
5	0.10	100.48	51.10	5134.88	−4575.85	559.03	528.68	−30.35	30.35
6	0.12	102.92	59.19	6092.33	−5394.04	698.28	664.00	−34.29	34.29
7	0.13	105.50	67.69	7140.94	−6274.05	866.89	829.99	−36.90	36.90
8	0.15	101.81	76.13	7750.53	−7171.44	579.09	550.21	−28.88	28.88
9	0.17	100.65	63.86	6427.97	−5928.66	499.31	457.48	−41.83	41.83
10	0.19	98.86	59.15	5847.59	−5459.40	388.19	337.04	−51.15	51.15
11	0.21	99.26	50.91	5053.11	−4649.03	404.09	335.31	−68.78	68.78
12	0.23	101.78	52.25	5317.65	−4786.50	531.15	458.03	−73.12	73.12
13	0.25	99.28	64.14	6367.78	−6002.74	365.03	288.19	−76.84	76.84
14	0.27	99.19	51.19	5077.96	−4722.07	355.89	257.52	−98.36	98.36
15	0.29	95.53	49.97	4773.36	−4604.77	168.59	92.40	−76.18	76.18
16	0.31	98.02	26.47	2594.85	−2362.61	232.23	148.05	−84.19	84.19
17	0.33	97.03	39.61	3843.35	−3653.84	189.51	83.97	−105.54	105.54
18	0.35	96.64	29.34	2835.17	−2659.65	175.51	44.51	−131.01	131.01
19	0.37	95.01	21.11	2005.37	−1866.05	139.32	3.75	−135.56	135.56
20	0.38	93.67	3.62	338.73	−204.45	134.27	0.00	−134.27	**134.27**

Delta hedging

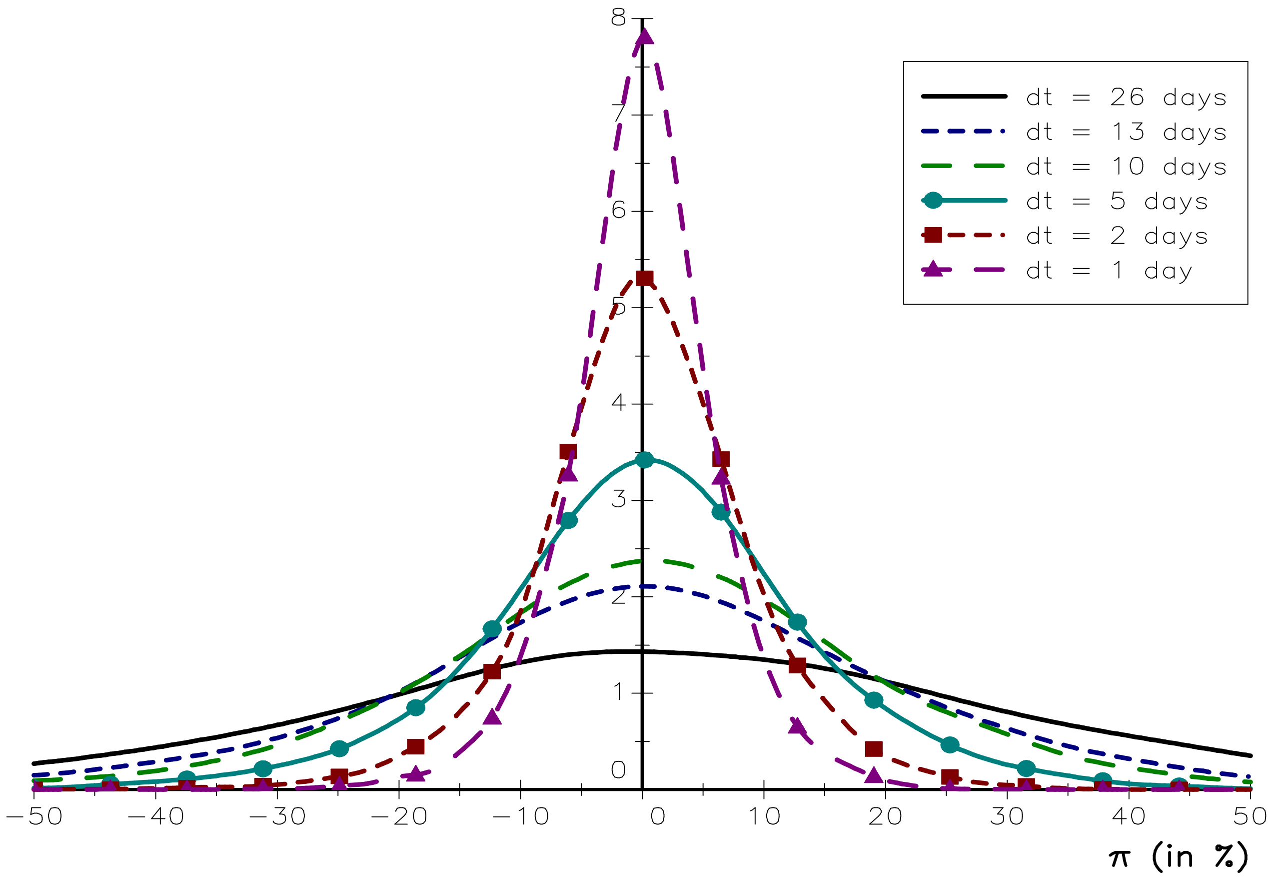


Figure: Probability density function of the hedging ratio π

Delta hedging

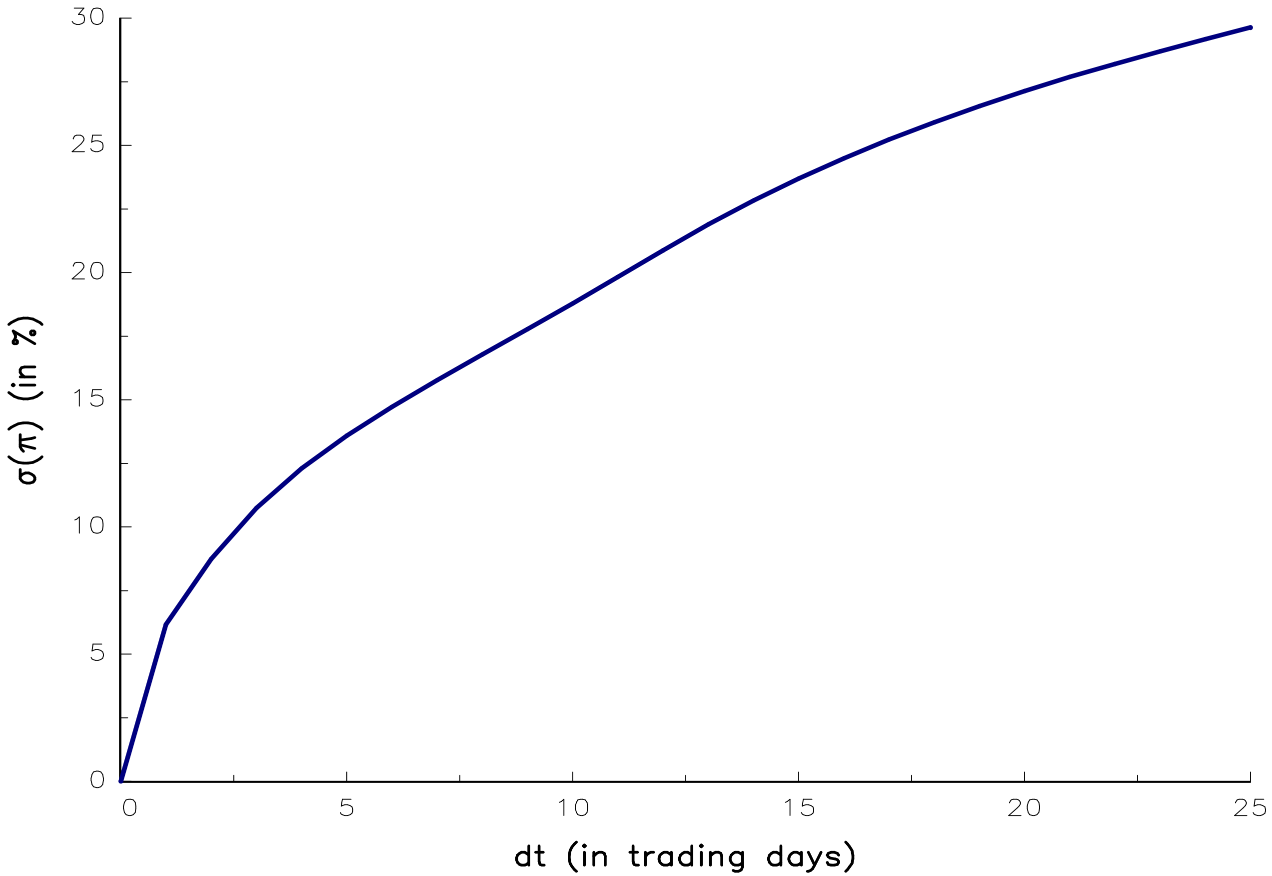


Figure: Relationship between the hedging efficiency $\sigma(\pi)$ and the hedging frequency

Delta hedging

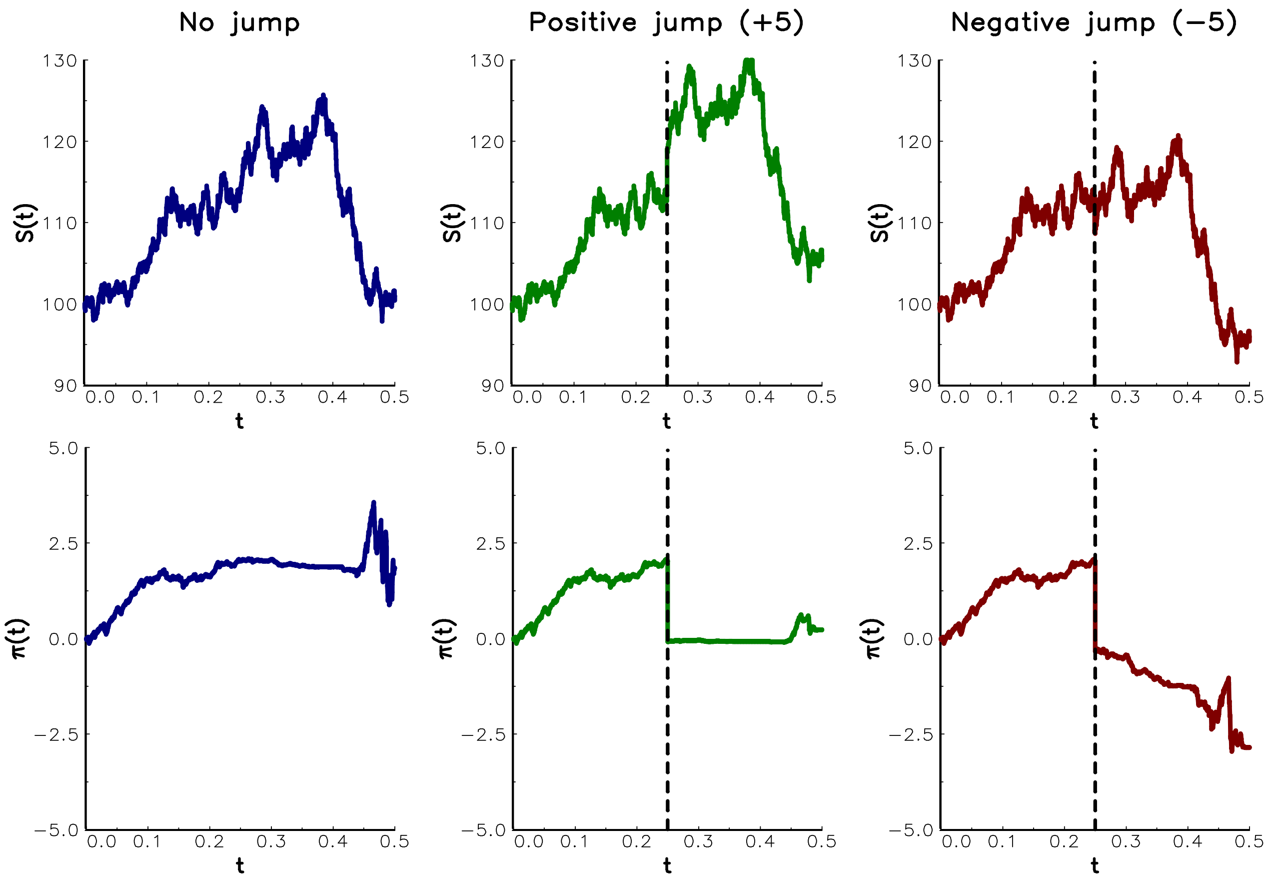


Figure: Impact of a jump on the hedging ratio $\pi(t)$

Delta hedging

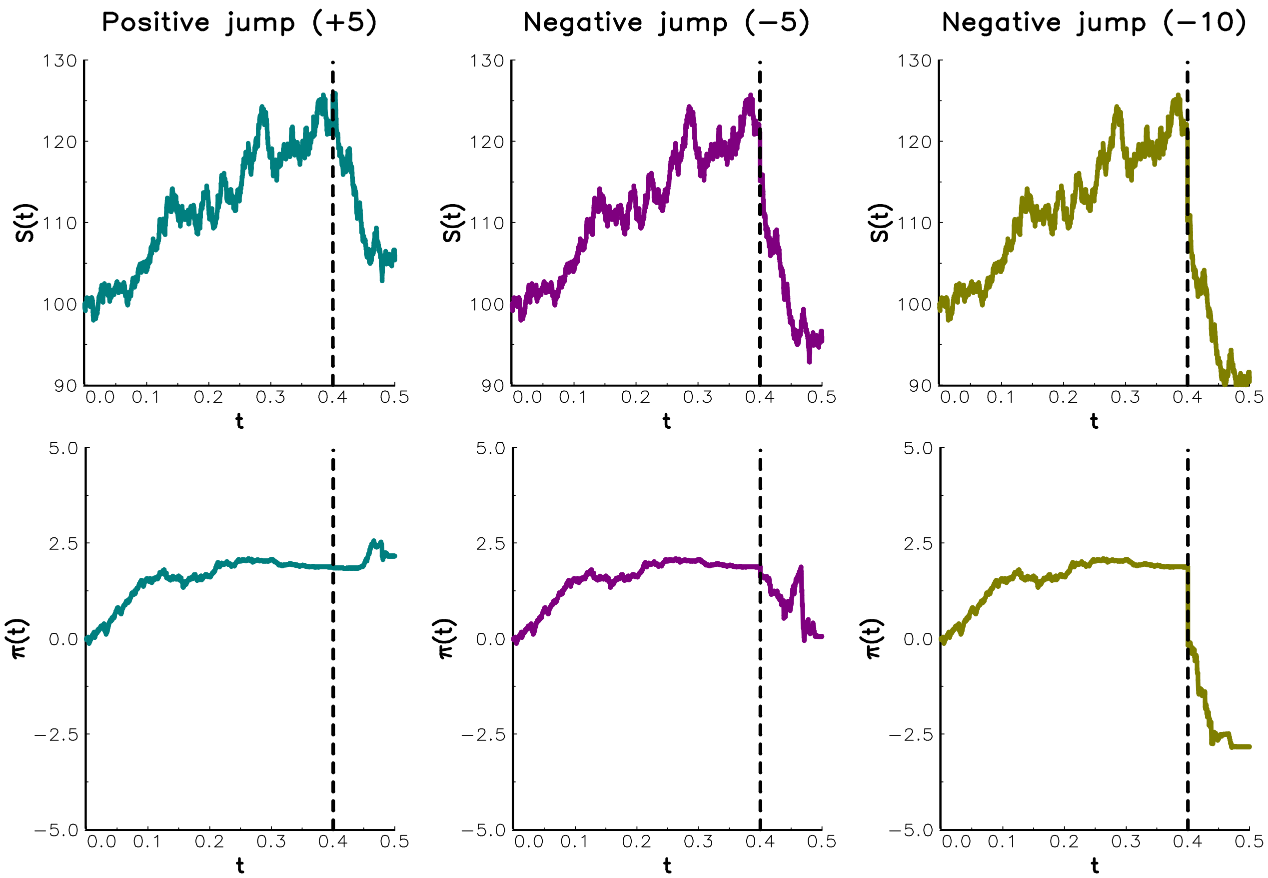


Figure: Impact of a jump on the hedging ratio $\pi(t)$

Greek sensitivities

We have

$$\boldsymbol{\Delta}(t) = \frac{\partial\, \mathcal{C}(t, S(t))}{\partial\, S(t)}$$

and:

$$\mathcal{C}(t + \mathrm{d}t, S(t + h)) - \mathcal{C}(t, S(t)) \approx \boldsymbol{\Delta}(t) \cdot (S(t + \mathrm{d}t) - S(t))$$

$\Rightarrow$ Taylor expansion to other orders and other parameters

Greek sensitivities

The delta-gamma-theta approximation is:

$$\begin{aligned}\mathcal{C}(t+\mathrm{d}t, S(t+h)) - \mathcal{C}(t, S(t)) &\approx \boldsymbol{\Delta}(t) \cdot (S(t+\mathrm{d}t) - S(t)) + \\ &\frac{1}{2}\boldsymbol{\Gamma}(t) \cdot (S(t+\mathrm{d}t) - S(t))^2 + \\ &\boldsymbol{\Theta}(t) \cdot ((t+\mathrm{d}t) - t)\end{aligned}$$

where:

$$\boldsymbol{\Gamma}(t) = \frac{\partial^2 \mathcal{C}(t, S(t))}{\partial S(t)^2} = \frac{\partial \boldsymbol{\Delta}(t)}{\partial S(t)}$$

and:

$$\boldsymbol{\Theta}(t) = \frac{\partial \mathcal{C}(t, S(t))}{\partial t} = -\frac{\partial \mathcal{C}(t, S(t))}{\partial T}$$

Greek sensitivities

- We have:

$$\boldsymbol{\Theta}(t) = \frac{\partial\, \mathcal{C}(t, S(t))}{\partial\, t} = -\frac{\partial\, \mathcal{C}(t, S(t))}{\partial\, T}$$

- We recall that the option price satisfies the PDE:

$$\frac{1}{2}\sigma^2 S^2 \boldsymbol{\Gamma} + bS\boldsymbol{\Delta} + \boldsymbol{\Theta} - r\mathcal{C} = 0$$

- We deduce that the theta of the option can be calculated as follows:

$$\boldsymbol{\Theta} = r\mathcal{C} - \frac{1}{2}\sigma^2 S^2 \boldsymbol{\Gamma} - bS\boldsymbol{\Delta}$$

Greek sensitivities

Example #2

We consider a call option, whose strike K is equal to 100. The risk-free rate and the cost-of-carry parameter are equal to 5%. For the volatility coefficient, we consider two cases: (a) $\sigma = 20\%$ and (b) $\sigma = 50\%$.

Greek sensitivities

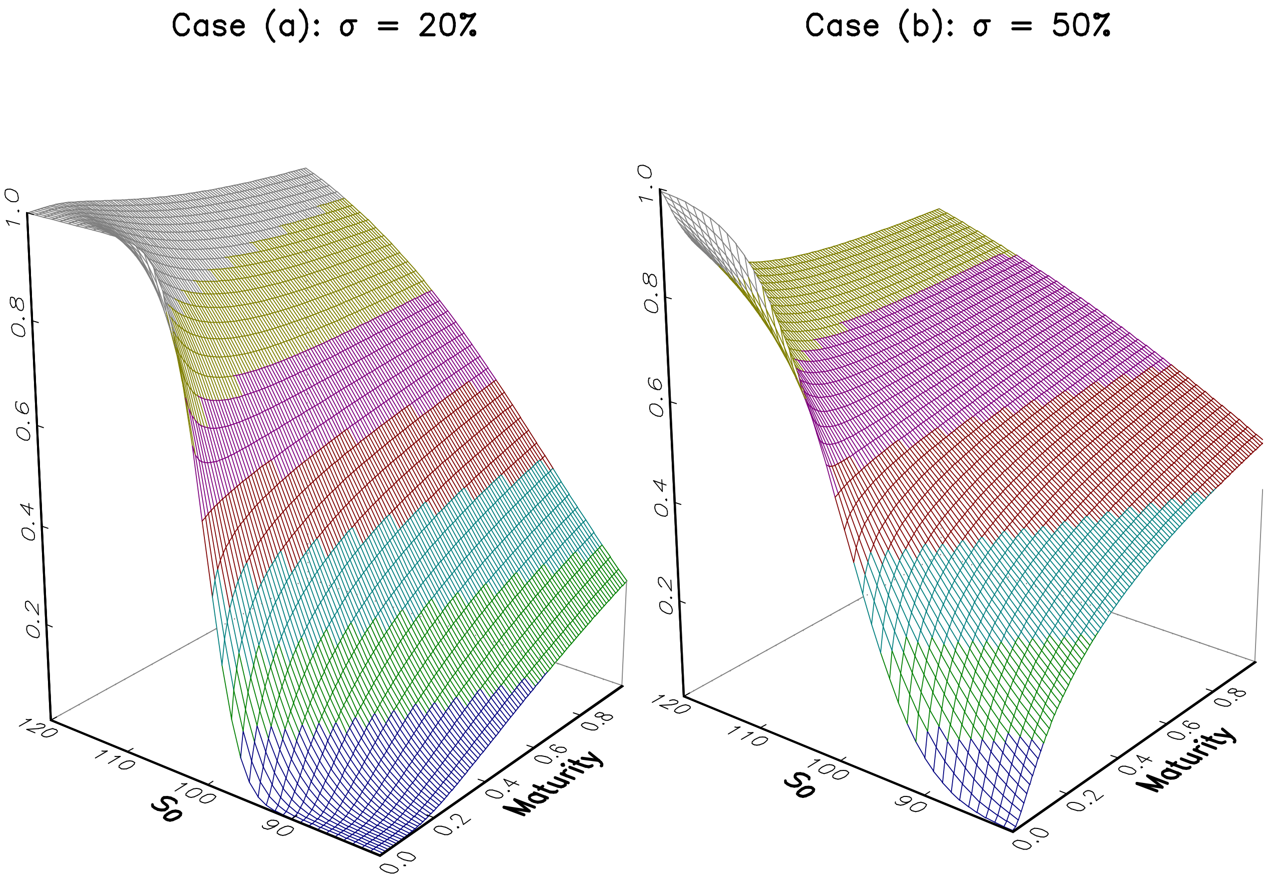


Figure: Delta coefficient of the call option

Greek sensitivities

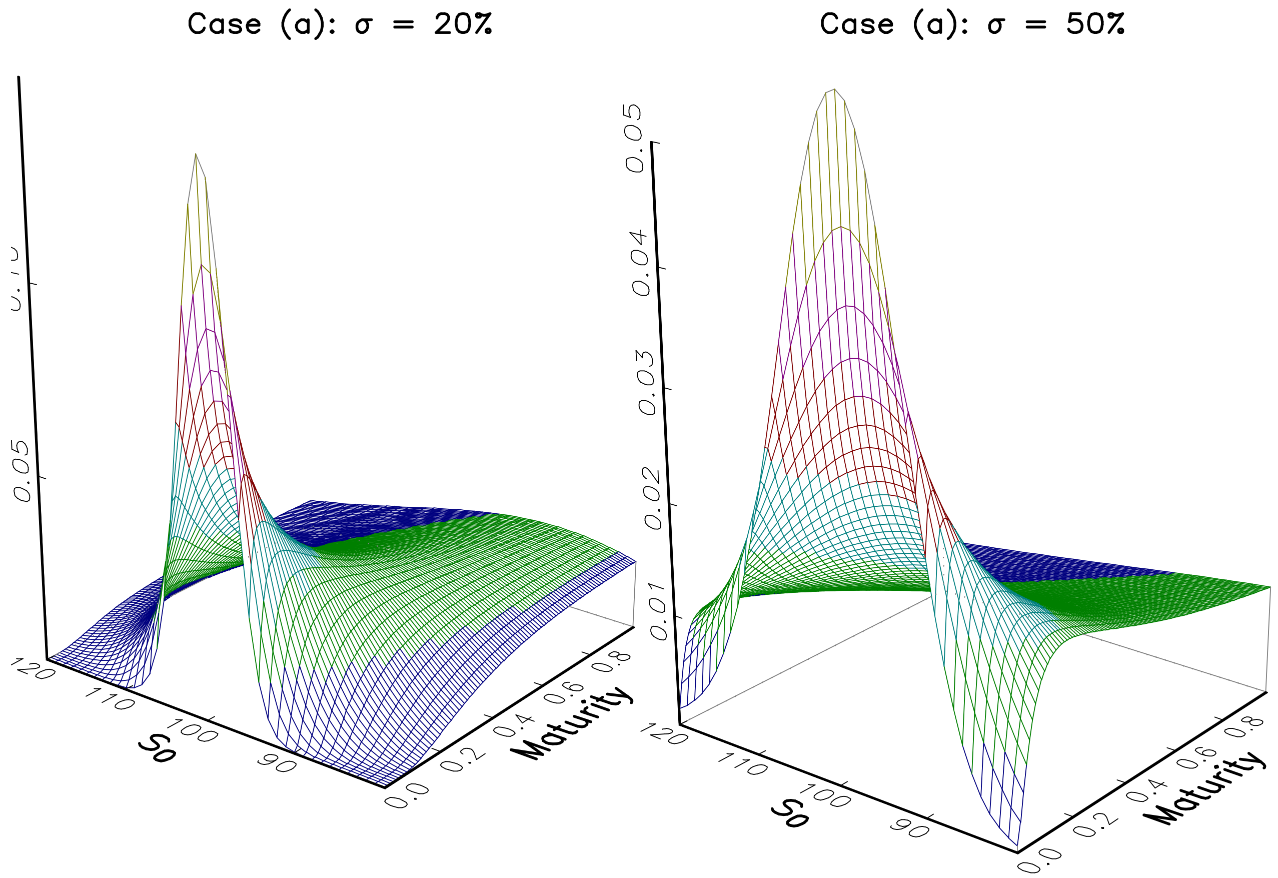


Figure: Gamma coefficient of the call option

Greek sensitivities

- **We assume a delta neutral hedging portfolio**
- The trader can face four configurations of residual risk:

		$\boldsymbol{\Gamma}$ $-$	$\boldsymbol{\Gamma}$ $+$
$\boldsymbol{\Theta}$	$-$		✓
$\boldsymbol{\Theta}$	$+$	✓	

- The configurations $(\boldsymbol{\Gamma} < 0, \boldsymbol{\Theta} < 0)$ and $(\boldsymbol{\Gamma} > 0, \boldsymbol{\Theta} > 0)$ are not realistic

Greek sensitivities

Two main configurations:

(a) a negative gamma exposure with a positive theta

(b) a positive gamma exposure with a negative theta

Two P&L profiles:

(a) If the gamma is negative, the best situation is obtained when the asset price does not move. Any changes in the asset price reduce the P&L, which can be negative if the gamma effect is more important than the theta effect. We also notice that the gain is bounded and the loss is unbounded in this configuration

(b) If the theta is negative, the loss is bounded and maximum when the asset price does not move. Any changes in the asset price increase the P&L because the gamma is positive. In this configuration, the gain is unbounded

Greek sensitivities

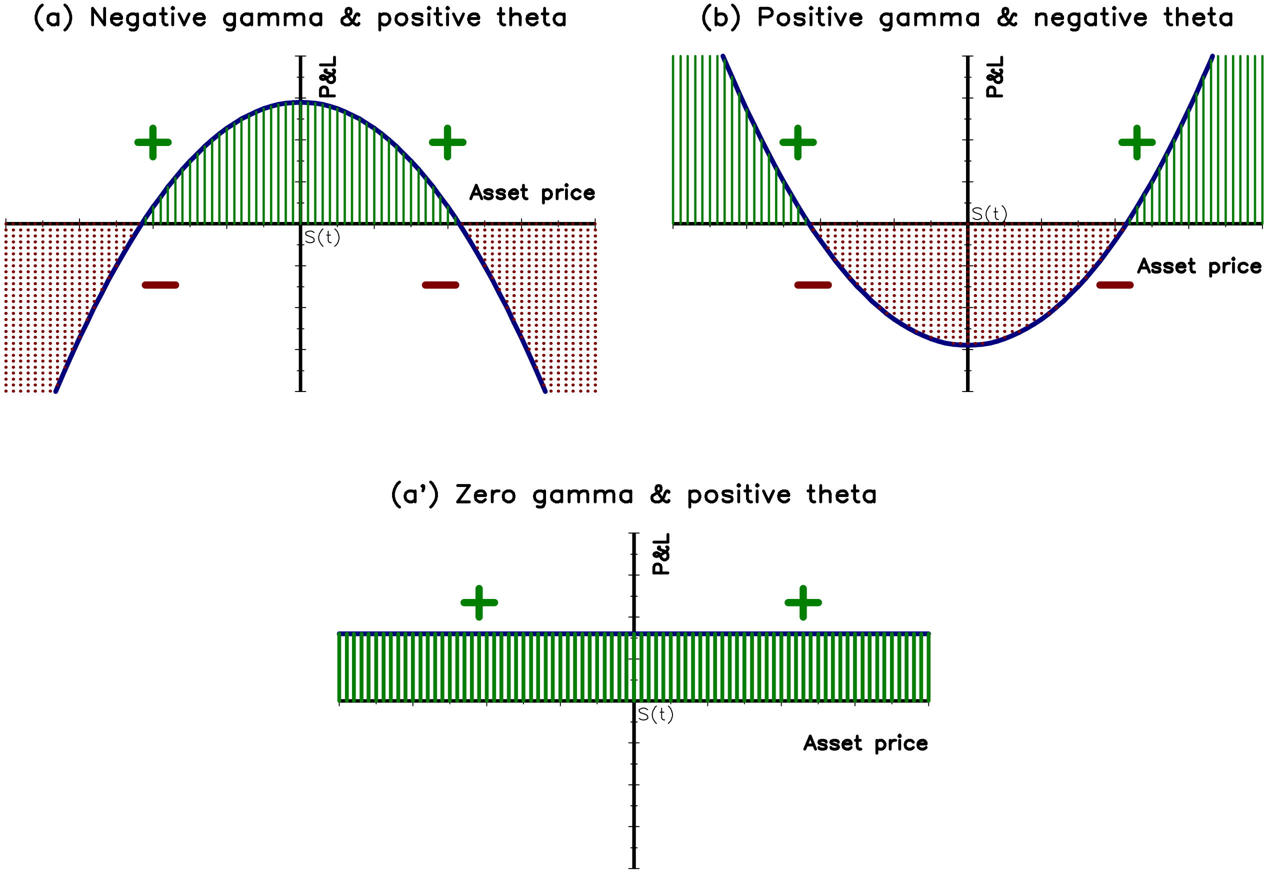


Figure: P&L of the delta neutral hedging portfolio

Greek sensitivities

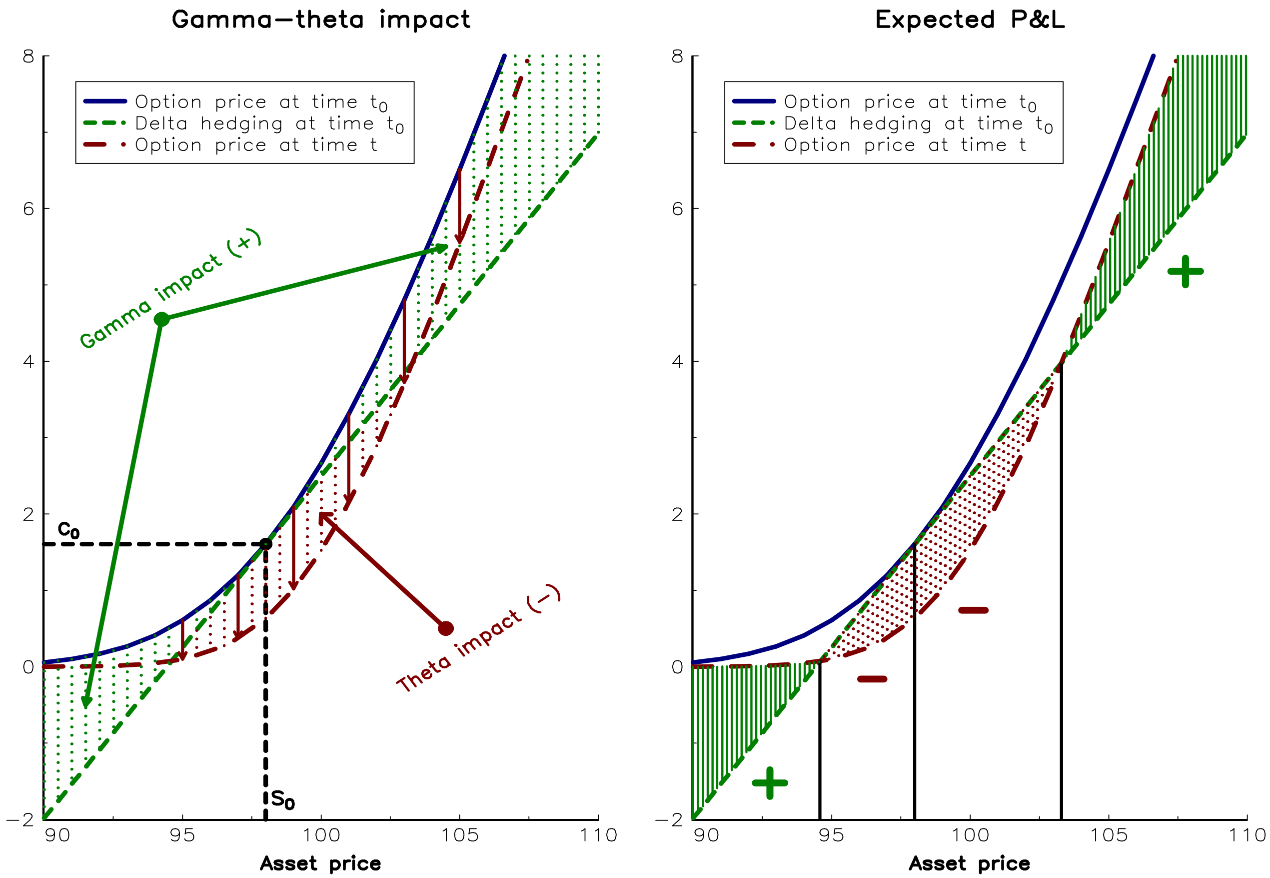


Figure: Illustration of the configuration ($\boldsymbol{\Gamma} > 0, \boldsymbol{\Theta} < 0$)

The implied volatility

Definition

- The implied volatility is the root of the following non-linear equation:

$$f_{\mathrm{BS}}\left(S_0, K, \sigma_{\mathrm{implied}}, T, b, r\right) = V\left(T, K\right)$$

  where f_{BS} is the Black-scholes formula and $V\left(T, K\right)$ is the market price of the option, whose maturity date is T and whose strike is K
- By convention, the implied volatility is denoted by Σ, and is a function of the parameters T and K:

$$\sigma_{\mathrm{implied}} = \Sigma\left(T, K\right)$$

The implied volatility

Example #3

We consider a call option, whose maturity is one year. The current price of the underlying asset is normalized and is equal to 100. Moreover, the risk-free rate and the cost-of-carry parameter are equal to 5%. Below, we report the market price of European call options of three assets for several strikes:

K	90	95	98	100	101	102	105	110
$\mathcal{C}_1(T,K)$	16.70	13.35	11.55	10.45	9.93	9.42	8.02	6.04
$\mathcal{C}_2(T,K)$	18.50	14.50	12.00	10.45	9.60	9.00	7.50	5.70
$\mathcal{C}_3(T,K)$	18.00	14.00	11.80	10.45	9.90	9.50	8.40	7.40

The implied volatility

Table: Implied volatility $\Sigma(T, K)$

K	90	95	98	100	101	102	105	110
$\Sigma_1(T, K)$	20.00	20.01	19.99	20.0	20.01	19.99	20.00	20.00
$\Sigma_2(T, K)$	26.18	23.41	21.24	20.0	19.14	18.90	18.69	19.14
$\Sigma_3(T, K)$	24.53	21.95	20.68	20.0	19.93	20.20	20.95	23.43

The implied volatility

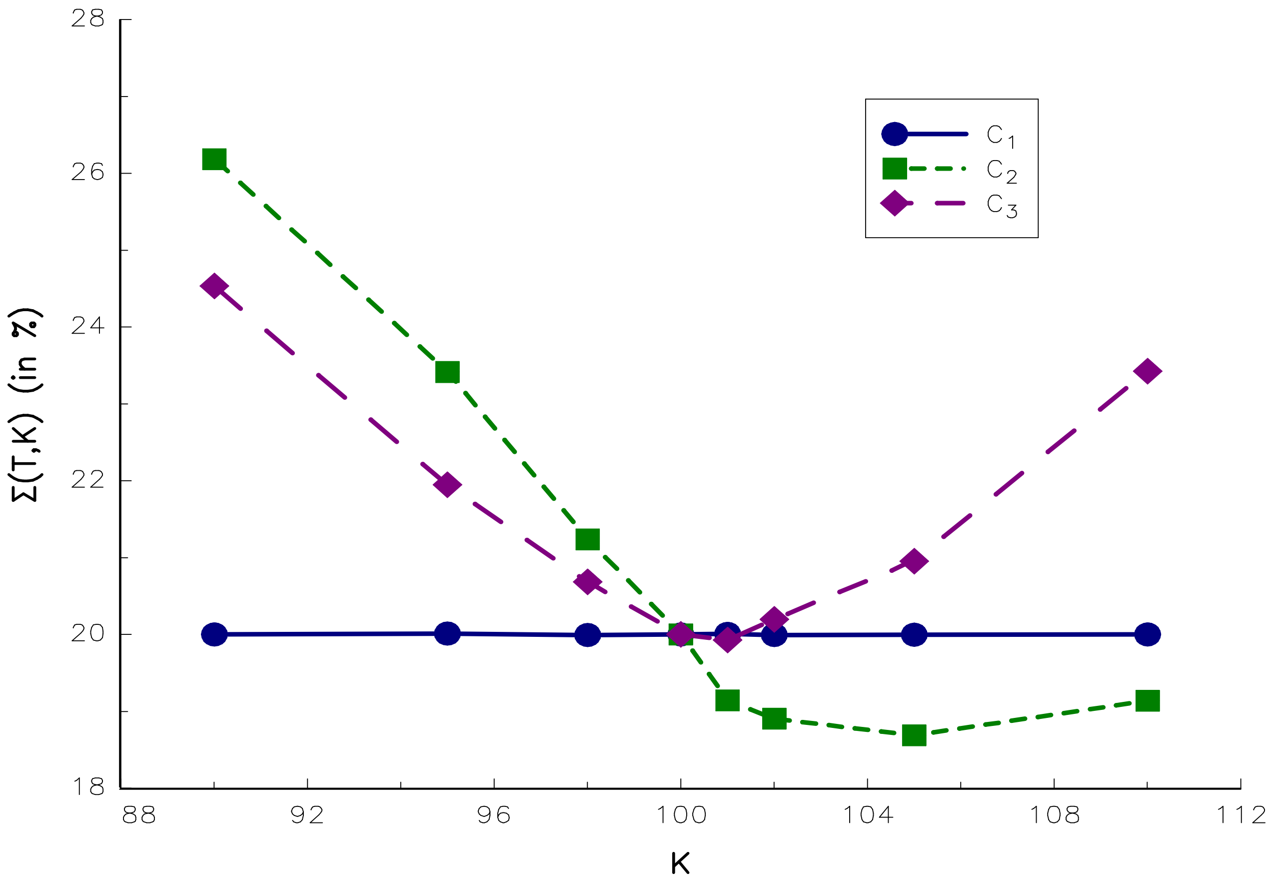


Figure: Volatility smile

The implied volatility

- When the curve of implied volatility is decreasing and increasing, the curve is called a volatility smile
- When the curve of implied volatility is just decreasing, it is called a volatility skew
- If we consider the maturity dimension, the term structure of implied volatility is known as the volatility surface

Relationship between the implied volatility and the risk-neutral density

- We have:

$$\begin{aligned}
\mathcal{C}_t(T,K) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r\,\mathrm{d}s}(S(T)-K)^+ \,\middle|\, \mathcal{F}_t\right] \\
&= e^{-r(T-t)}\int_{-\infty}^{\infty}(S-K)^+ q_t(T,S)\,\mathrm{d}S \\
&= e^{-r(T-t)}\int_{K}^{\infty}(S-K)\,q_t(T,S)\,\mathrm{d}S
\end{aligned}$$

  where $q_t(T,S)$ is the risk-neutral probability density function of $S(T)$ at time t

- By definition, the risk-neutral cumulative distribution function $\mathbb{Q}_t(T,S)$ is equal to:

$$\mathbb{Q}_t(T,S) = \int_{-\infty}^{S} q_t(T,x)\,\mathrm{d}x$$

Relationship between the implied volatility and the risk-neutral density

- We deduce that:

$$\begin{aligned} \frac{\partial \mathcal{C}_t(T,K)}{\partial K} &= -e^{-r(T-t)} \int_K^\infty q_t(T,S)\,\mathrm{d}S \\ &= -e^{-r(T-t)} \left(1 - \mathbb{Q}_t(T,K)\right) \end{aligned}$$

and:

$$\frac{\partial^2 \mathcal{C}_t(T,K)}{\partial K^2} = e^{-r(T-t)} q_t(T,K)$$

- It follows that:

$$\begin{aligned} \mathbb{Q}_t(T,K) &= \Pr\{S(T) \leq K \mid \mathcal{F}_t\} \\ &= 1 + e^{r(T-t)} \cdot \partial_K \mathcal{C}_t(T,K) \end{aligned}$$

Relationship between the implied volatility and the risk-neutral density

- We note $\Sigma_t(T,K)$ the volatility surface and $\mathcal{C}_t^\star(T,K,\Sigma)$ the Black-Scholes formula. It follows that:

$$\begin{aligned}\mathbb{Q}_t(T,K) &= 1 + e^{r(T-t)} \cdot \partial_K \mathcal{C}_t^\star(T,K,\Sigma_t(T,K)) + \\ &\quad e^{r(T-t)} \cdot \partial_\Sigma \mathcal{C}_t^\star(T,K,\Sigma_t(T,K)) \cdot \partial_K \Sigma_t(T,K)\end{aligned}$$

where:

$$\partial_K \mathcal{C}_t^\star(T,K,\Sigma) = -e^{-r(T-t)} \cdot \Phi(d_2)$$

and:

$$\partial_\Sigma \mathcal{C}_t^\star(T,K,\Sigma) = S(t) \cdot e^{(b-r)(T-t)} \cdot \sqrt{T-t} \cdot \phi\left(d_2 + \Sigma\sqrt{T-t}\right)$$

Relationship between the implied volatility and the risk-neutral density

- The risk-neutral probability density function is equal to:

$$q_t(T,K) = \partial_K \mathbb{Q}_t(T,K) = e^{r(T-t)} \cdot \partial_K^2 \mathcal{C}_t(T,K)$$

where:

$$\begin{array}{rcl}\partial_K^2 \mathcal{C}_t(T,K) & = & \partial_K^2 \mathcal{C}_t^\star(T,K,\Sigma_t) + 2 \cdot \partial_{K,\Sigma}^2 \mathcal{C}_t^\star(T,K,\Sigma_t) \cdot \partial_K \Sigma_t(T,K) + \\ & & \partial_\Sigma \mathcal{C}_t^\star(T,K,\Sigma_t) \cdot \partial_K^2 \Sigma_t(T,K) + \\ & & \partial_\Sigma^2 \mathcal{C}_t^\star(T,K,\Sigma_t) \cdot (\partial_K \Sigma_t(T,K))^2\end{array}$$

and:

$$\begin{array}{rcl}\partial_K^2 \mathcal{C}_t^\star(T,K,\Sigma) & = & e^{-r(T-t)} \dfrac{\phi(d_2)}{K\Sigma\sqrt{T-t}} \\ \partial_{K,\Sigma}^2 \mathcal{C}_t^\star(T,K,\Sigma) & = & e^{(b-r)(T-t)} \dfrac{S(t)\, d_1 \phi(d_1)}{\Sigma K} \\ \partial_\Sigma^2 \mathcal{C}_t^\star(T,K,\Sigma) & = & e^{(b-r)(T-t)} \dfrac{S(t)\, d_1 d_2 \sqrt{T-t}\phi(d_1)}{\Sigma}\end{array}$$

Relationship between the implied volatility and the risk-neutral density

Example #4

We assume that $S(t) = 100$, $T - t = 10$, $b = r = 5\%$ and:

$$\Sigma_t(T, K) = 0.25 + \ln\left(1 + 10^{-6}(K - 90)^2 + 10^{-6}(K - 180)^2\right)$$

Relationship between the implied volatility and the risk-neutral density

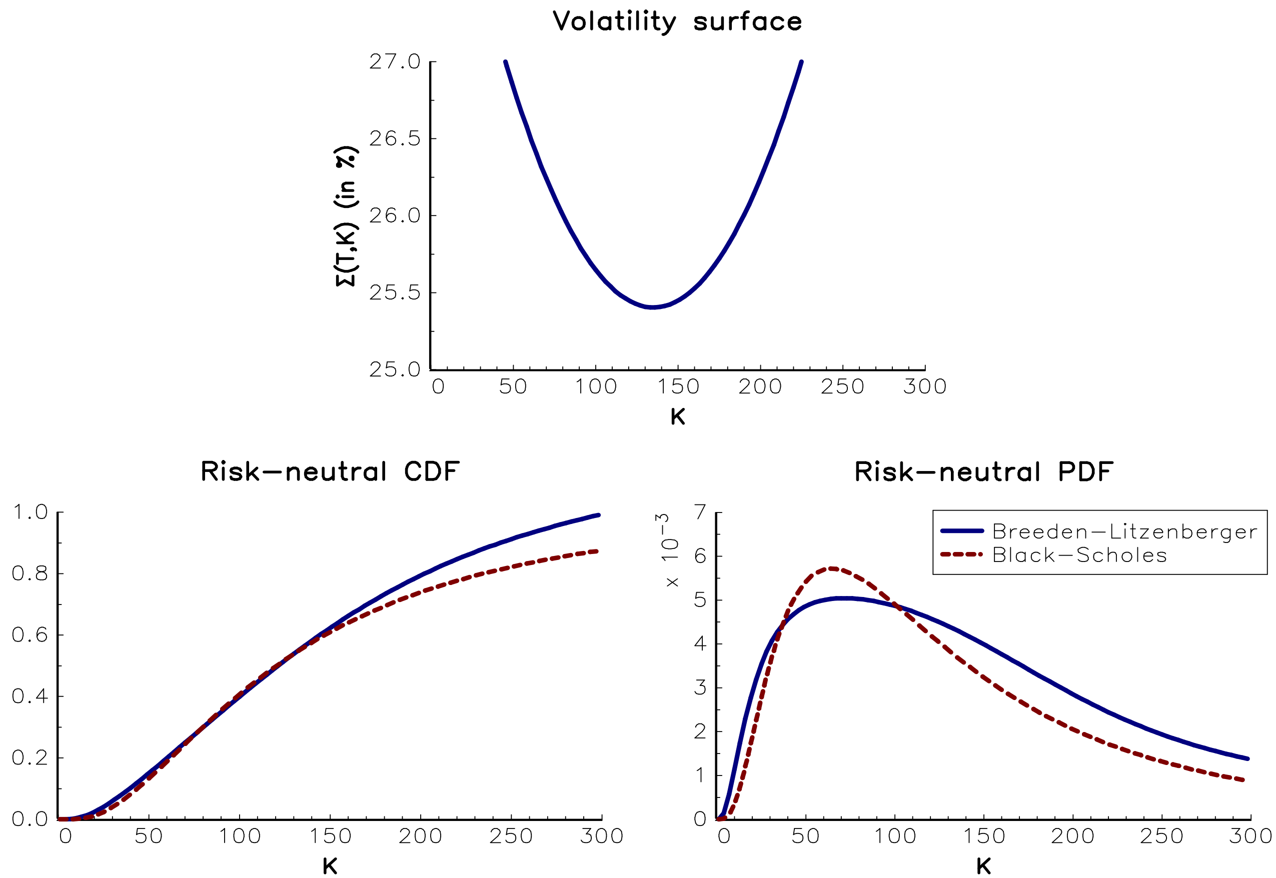


Figure: Risk-neutral probability density function

Robustness of the Black-Scholes formula

We can show that:

$$V(T) - f(S(T)) = \frac{1}{2}\int_0^T e^{r(T-t)}\mathbf{\Gamma}(t)\left(\Sigma^2(T,K) - \sigma^2(t)\right)S^2(t)\,\mathrm{d}t$$

where $f(S(T))$ is the payoff of the option. We obtain the following results:

- if $\mathbf{\Gamma}(t) \geq 0$, a positive P&L is achieved by overestimating the realized volatility:

$$\Sigma(T,K) \geq \sigma(t) \Longrightarrow V(T) \geq f(S(T))$$

- if $\mathbf{\Gamma}(t) \leq 0$, a positive P&L is achieved by underestimating the realized volatility:

$$\Sigma(T,K) \leq \sigma(t) \Longrightarrow V(T) \geq f(S(T))$$

- the variance of the hedging error is an increasing function of the absolute value of the gamma coefficient:

$$|\mathbf{\Gamma}(t)| \nearrow \Longrightarrow \mathrm{var}\left(V(T) - f(S(T))\right) \nearrow$$

Robustness of the Black-Scholes formula

Example #5

We consider the replication of 100 ATM call options. The current price of the asset is 100 and the maturity of the option is 6 months (or 130 trading days). We consider the following parameters: $b = r = 5\%$. We rebalance the delta hedging portfolio every trading day. Moreover, we assume that the option is priced and hedged with a 20% implied volatility.

Relationship between the implied volatility and the risk-neutral density

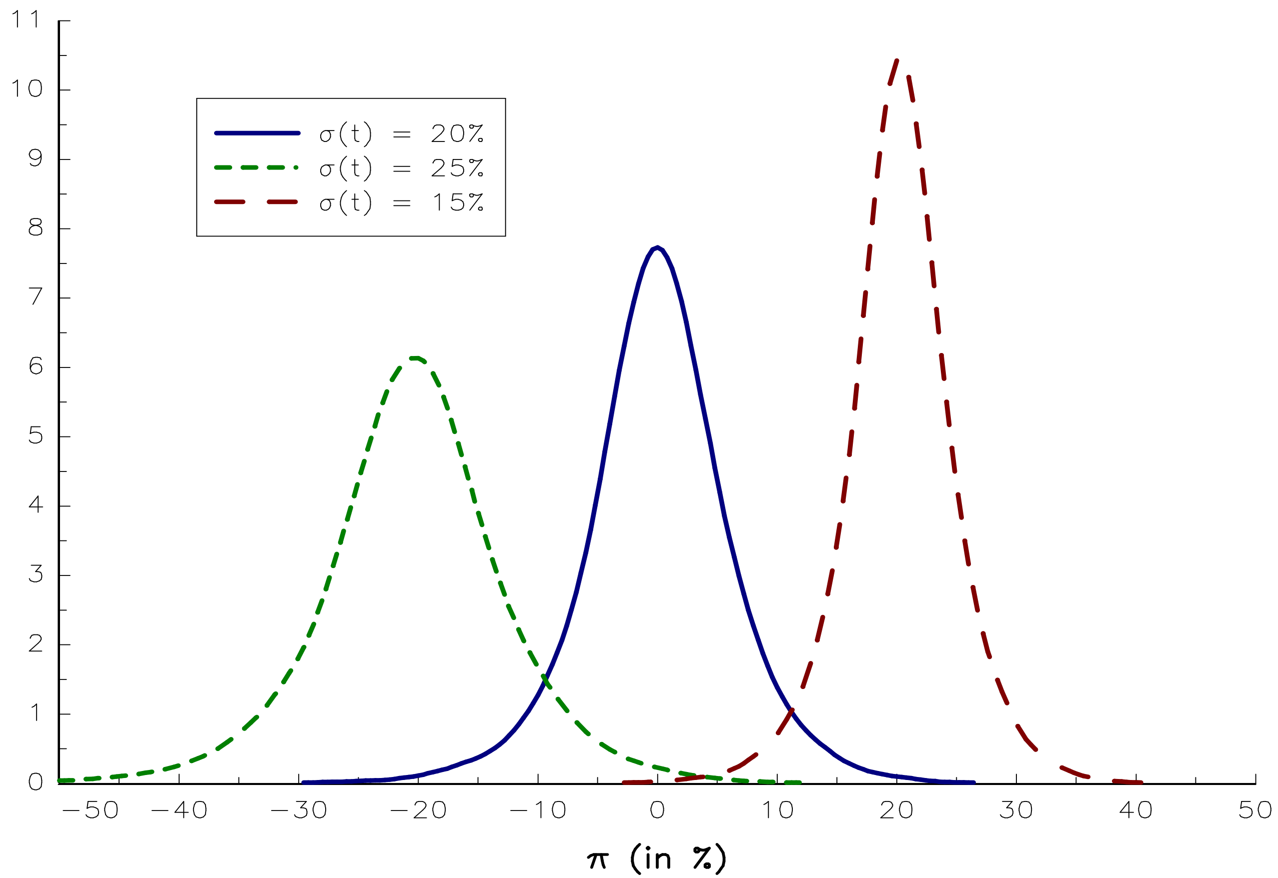


Figure: Hedging error when the implied volatility is 20%

Robustness of the Black-Scholes formula

We obtain:

$$\Pr\left\{\pi > 0 \mid \Sigma = 20\%, \sigma = 15\%\right\} = 99.04\%$$

and:

$$\Pr\left\{\pi > 0 \mid \Sigma = 20\%, \sigma = 25\%\right\} = 0.09\%$$

Vasicek model

- The instantaneous interest rate follows an Ornstein-Uhlenbeck process:

$$\begin{cases} \mathrm{d}r(t) = a(b - r(t))\,\mathrm{d}t + \sigma\,\mathrm{d}W(t) \\ r(t_0) = r_0 \end{cases}$$

- The value $V(t,r)$ of a zero-coupon bond satisfies the PDE:

$$\frac{1}{2}\sigma^2\frac{\partial^2 V(t,r)}{\partial r^2} + (a(b - r(t)) - \lambda(t)\sigma)\frac{\partial V(t,r)}{\partial r} + \frac{\partial V(t,r)}{\partial t} - r(t)V(t,r) = 0$$

with $V(T,r) = 1$

- The Feynman-Kac formula implies:

$$V(0,r_0) = \mathbb{E}^{\mathbb{Q}}\left[\left. e^{-\int_0^T r(t)\,\mathrm{d}t} \right| \mathcal{F}_0\right]$$

where $\mathrm{d}r(t) = (a(b - r(t)) - \lambda(t)\sigma)\,\mathrm{d}t + \sigma\,\mathrm{d}W^{\mathbb{Q}}(t)$

Vasicek model

By assuming that $\lambda(t) = \lambda$, we have:

$$V(0, r_0) = \exp\left(-r_0\beta - \left(b' - \frac{\sigma^2}{2a^2}\right)(T - \beta) - \frac{\sigma^2\beta^2}{4a}\right)$$

where $b' = b - \dfrac{\lambda\sigma}{a}$ and $\beta = \dfrac{1 - e^{-aT}}{a}$

Vasicek model

We recall that the zero-coupon rate is defined by:

$$B(t,T) = e^{-(T-t)R(t,T)}$$

We deduce that:

$$\begin{aligned} R(t,T) &= -\frac{1}{T-t}\ln B(t,T) \\ &= \left(b' - \frac{\sigma^2}{2a^2}\right) + \left(r_t - b' + \frac{\sigma^2}{2a^2}\right)\frac{\beta}{T-t} + \frac{\sigma^2\beta^2}{4a(T-t)} \end{aligned}$$

Vasicek model

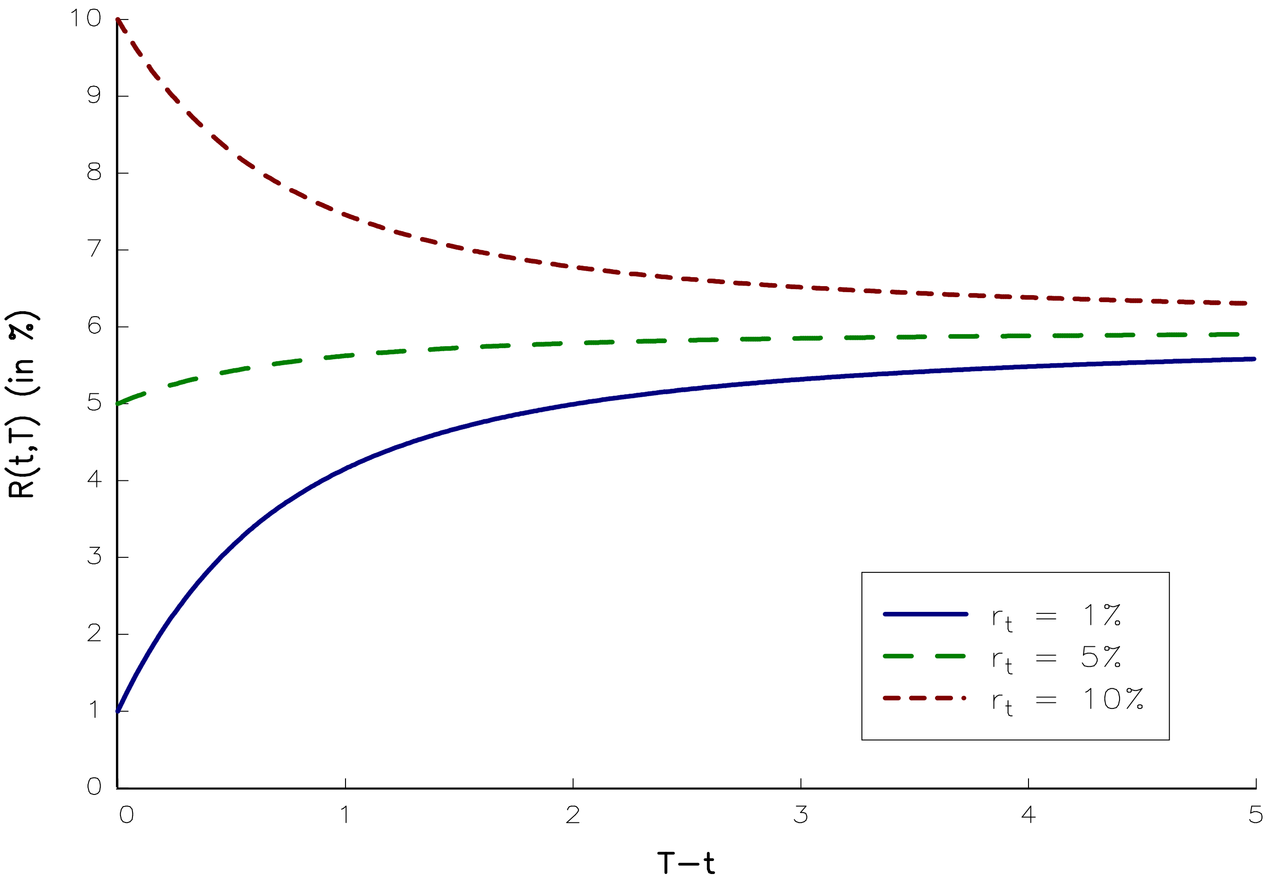


Figure: Vasicek model ($a = 2.5$, $b = 6\%$ and $\sigma = 5\%$)

Vasicek model

- Let $F(t, T_1, T_2)$ be the forward rate at time t for the period $[T_1, T_2]$:

$$B(t, T_2) = e^{-(T_2 - T_1)F(t, T_1, T_2)} B(t, T_1)$$

- We deduce that the expression of $F(t, T_1, T_2)$ is:

$$F(t, T_1, T_2) = -\frac{1}{(T_2 - T_1)} \ln \frac{B(t, T_2)}{B(t, T_1)}$$

- It follows that the instantaneous forward rate is given by this equation:

$$f(t, T) = F(t, T, T) = -\frac{\partial \ln B(t, T)}{\partial T}$$

Vasicek model

Another expression of the price of the zero-coupon bond is:

$$B(t, r_t) = \exp\left(-(T-t) R_\infty - (r_t - R_\infty)\left(\frac{1 - e^{-a(T-t)}}{a}\right) - \frac{\sigma^2 \left(1 - e^{-a(T-t)}\right)^2}{4a^3}\right)$$

where:

$$R_\infty = \lim_{T \to \infty} R(t, T) = b' - \frac{\sigma^2}{2a^2}$$

Therefore, the instantaneous forward rate in the Vasicek model is:

$$f(t, T) = R_\infty + (r_t - R_\infty) e^{-a(T-t)} + \frac{\sigma^2 \left(1 - e^{-a(T-t)}\right) e^{-a(T-t)}}{2a^2}$$

Vasicek model

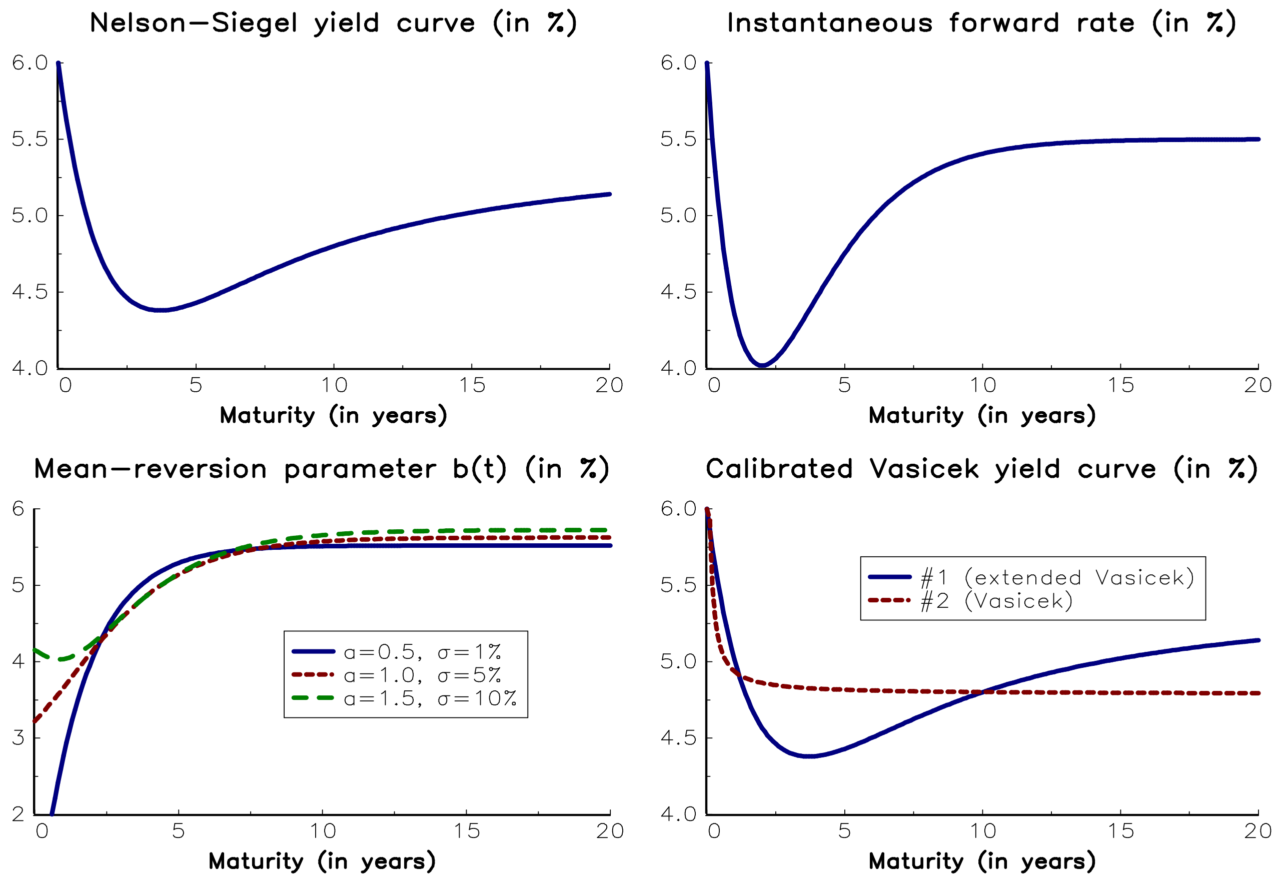


Figure: Calibration of the Vasicek model

Caps, floors and swaptions

- Future dates $T_0, T_1, \ldots, T_n$
- The payoff of a caplet is $(T_i - T_{i-1}) \left(F(T_{i-1}, T_{i-1}, T_i) - K\right)^+$, where K is the strike of the caplet and $F(T_{i-1}, T_{i-1}, T_i)$ is the forward rate at the future date T_{i-1}
- $\delta_{i-1} = T_i - T_{i-1}$ is the tenor of the caplet
- T_{i-1} is the resetting date (or the fixing date) of the forward rate
- T_i is the maturity date of the caplet
- A cap is a portfolio of successive caplets:

$$\mathrm{Cap}(t) = \sum_{i=1}^{n} \mathrm{Caplet}(t, T_{i-1}, T_i)$$

Caps, floors and swaptions

A floor is a portfolio of successive floorlets:

$$\mathrm{Floor}\left(t\right)=\sum_{i=1}^{n}\mathrm{Floorlet}\left(t,T_{i-1},T_{i}\right)$$

where the payoff of the floorlet is $\left(T_{i}-T_{i-1}\right)\left(K-F\left(T_{i-1},T_{i-1},T_{i}\right)\right)^{+}$

Caps, floors and swaptions

A par swap rate is the fixed rate of an interest rate swap:

$$\mathrm{Sw}\left(t\right)=\frac{B\left(t,T_0\right)-B\left(t,T_n\right)}{\sum_{i=1}^{n}\left(T_i-T_{i-1}\right)\cdot B\left(t,T_i\right)}$$

The payoff of a payer swaption is:

$$\left(\mathrm{Sw}\left(T_0\right)-K\right)^{+}\sum_{i=1}^{n}\left(T_i-T_{i-1}\right)B\left(T_0,T_i\right)$$

where $\mathrm{Sw}\left(T_0\right)$ is the forward swap rate

Caps, floors and swaptions

Generally, caps, floors and swaptions are written on the Libor rate, which is defined as a simple forward rate:

$$L(t, T_{i-1}, T_i) = \frac{1}{T_i - T_{i-1}} \left(\frac{B(t, T_{i-1})}{B(t, T_i)} - 1 \right)$$

We have:

$$\mathrm{Caplet}(t, T_{i-1}, T_i) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^{T_i} r(s)\,\mathrm{d}s} \delta_{i-1} \left(L(T_{i-1}, T_{i-1}, T_i) - K \right)^+ \middle| \mathcal{F}_t \right]$$

and:

$$\mathrm{Swaption}(t) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^{T_n} r(s)\,\mathrm{d}s} \left(\mathrm{Sw}(T_0) - K \right)^+ \sum_{i=1}^{n} \delta_{i-1} B(T_0, T_i) \middle| \mathcal{F}_t \right]$$

$\Rightarrow$ **the discount factor is stochastic** and is not independent from the forward rate $L(T_{i-1}, T_{i-1}, T_i)$ or the forward swap rate $\mathrm{Sw}(T_0)$

Change of numéraire and equivalent martingale measure

- The price of the contingent claim, whose payoff is $V(T) = f(S(T))$ at time T, is given by:

$$V(0) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(s)\,\mathrm{d}s} \cdot V(T) \middle| \mathcal{F}_0 \right]$$

- $\mathbb{Q}$ is the risk-neutral probability measure
- We can rewrite this equation as follows:

$$\frac{V(0)}{M(0)} = \mathbb{E}^{\mathbb{Q}}\left[\frac{V(T)}{M(T)} \middle| \mathcal{F}_0 \right]$$

  where $M(0) = 1$ and:

$$M(t) = \exp\left(\int_0^t r(s)\,\mathrm{d}s\right)$$

- Under the probability measure $\mathbb{Q}$, we know that $\tilde{V}(t) = V(t)/M(t)$ is an $\mathcal{F}_t$-martingale

Change of numéraire and equivalent martingale measure

- The money market account $M(t)$ is then the numéraire when the martingale measure is the risk-neutral probability measure, but other numraires can be used in order to simplify pricing problems:

  "The use of the risk-neutral probability measure has proved to be very powerful for computing the prices of contingent claims [...] We show here that many other probability measures can be defined in the same way to solve different asset-pricing problems, in particular option pricing. Moreover, these probability measure changes are in fact associated with numéraire changes" (Geman et al., 1995, page 443).

Change of numéraire and equivalent martingale measure

Let us consider another numéraire $N(t) > 0$ and the associated probability measure given by the Radon-Nikodym derivative:

$$\frac{\mathrm{d}\mathbb{Q}^{\star}}{\mathrm{d}\mathbb{Q}} = \frac{N(T)/N(0)}{M(T)/M(0)} = e^{-\int_0^T r(s)\,\mathrm{d}s} \cdot \frac{N(T)}{N(0)}$$

We have:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^{\star}}\left[\left.\frac{V(T)}{N(T)}\right|\mathcal{F}_0\right] &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{V(T)}{N(T)} \cdot \frac{\mathrm{d}\mathbb{Q}^{\star}}{\mathrm{d}\mathbb{Q}}\right|\mathcal{F}_t\right] \\
&= \frac{M(0)}{N(0)} \cdot \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{V(T)}{M(T)}\right|\mathcal{F}_0\right] \\
&= \frac{M(0)}{N(0)} \cdot V(0)
\end{aligned}$$

Change of numéraire and equivalent martingale measure

- We deduce that:

$$\frac{V(0)}{N(0)} = \mathbb{E}^{\mathbb{Q}^\star}\left[\left.\frac{V(T)}{N(T)}\right| \mathcal{F}_0\right]$$

- We have changed the numéraire $(M(t) \to N(t))$ and the probability measure $(\mathbb{Q} \to \mathbb{Q}^\star)$
- More generally, we have:

$$V(t) = N(t) \cdot \mathbb{E}^{\mathbb{Q}^\star}\left[\left.\frac{V(T)}{N(T)}\right| \mathcal{F}_t\right]$$

- Thanks to Girsanov theorem, we notice that $e^{-\int_0^t r(s)\,\mathrm{d}s} N(t)$ is an $\mathcal{F}_t$-martingale

Change of numéraire and equivalent martingale measure

- The forward numéraire is the zero-coupon bond price of maturity T:

$$N(t) = B(t, T)$$

- The probability measure is called the forward probability and is denoted by $\mathbb{Q}^{\star}(T)$
- By noticing that $N(T) = B(T, T) = 1$, we have:

$$V(t) = B(t, T)\,\mathbb{E}^{\mathbb{Q}^{\star}(T)}\left[\left. V(T) \right| \mathcal{F}_t\right]$$

Change of numéraire and equivalent martingale measure

- In the case of a caplet, we obtain:

$$\begin{aligned}
\mathrm{Caplet}\left(t, T_{i-1}, T_i\right) &= \delta_{i-1}\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{M\left(t\right)}{M\left(T_i\right)}\left(L\left(T_{i-1}, T_{i-1}, T_i\right) - K\right)^{+}\right| \mathcal{F}_t\right] \\
&= \delta_{i-1}\mathbb{E}^{\mathbb{Q}^{\star}\left(T_i\right)}\left[\left.\frac{N\left(t\right)}{N\left(T_i\right)}\left(L\left(T_{i-1}, T_{i-1}, T_i\right) - K\right)^{+}\right| \mathcal{F}_t\right] \\
&= \delta_{i-1}B\left(t, T_i\right)\mathbb{E}^{\mathbb{Q}^{\star}\left(T_i\right)}\left[\left.\left(L\left(T_{i-1}, T_{i-1}, T_i\right) - K\right)^{+}\right| \mathcal{F}_t\right]
\end{aligned}$$

  where $L\left(t, T_{i-1}, T_i\right)$ is an $\mathcal{F}_t$-martingale under the forward probability measure $\mathbb{Q}^{\star}\left(T_i\right)$

- The general formula of the caplet price is:

$$\mathrm{Caplet}\left(t, T_{i-1}, T_i\right) = B\left(t, T_i\right)\mathbb{E}^{\mathbb{Q}^{\star}\left(T_i\right)}\left[\left.\left(\frac{1}{B\left(T_{i-1}, T_i\right)} - \left(1 + \delta_{i-1}K\right)\right)^{+}\right| \mathcal{F}_t\right]$$

Change of numéraire and equivalent martingale measure

If we use the standard Black model, we obtain:

$$\mathrm{Caplet}\left(t, T_{i-1}, T_i\right) = \delta_{i-1} B\left(t, T_i\right)\left(L\left(t, T_{i-1}, T_i\right)\Phi\left(d_1\right) - K\Phi\left(d_2\right)\right)$$

where σ_{i-1} is the volatility of the Libor rate $L\left(t, T_{i-1}, T_i\right)$,

$$d_1 = \frac{1}{\sigma_{i-1}\sqrt{T_{i-1} - t}} \ln \frac{L\left(t, T_{i-1}, T_i\right)}{K} + \frac{1}{2}\sigma_{i-1}\sqrt{T_{i-1} - t}$$

and:

$$d_2 = d_1 - \sigma_{i-1}\sqrt{T_{i-1} - t}$$

Change of numéraire and equivalent martingale measure

- The annuity numéraire is equal to:

$$N(t) = \sum_{i=1}^{n} (T_i - T_{i-1}) B(t, T_i)$$

- While the forward swap rate is a martingale under the annuity probability measure $\mathbb{Q}^{\star}$, the price of the swaption is:

$$\begin{aligned}
\mathrm{Swaption}(t) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{M(t)}{M(T_n)} (\mathrm{Sw}(T_0) - K)^+ \sum_{i=1}^{n} \delta_{i-1} B(T_0, T_i) \right| \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}^{\star}}\left[\left.\frac{N(t)}{N(T_0)} (\mathrm{Sw}(T_0) - K)^+ \sum_{i=1}^{n} \delta_{i-1} B(T_0, T_i) \right| \mathcal{F}_t\right] \\
&= N(t)\, \mathbb{E}^{\mathbb{Q}^{\star}}\left[\left.(\mathrm{Sw}(T_0) - K)^+ \right| \mathcal{F}_t\right] \\
&= N(t)\, \mathbb{E}^{\mathbb{Q}^{\star}}\left[\left.\left(\frac{1 - B(T_0, T_n)}{N(T_0)} - K\right)^+ \right| \mathcal{F}_t\right]
\end{aligned}$$

The HJM model

- Under the risk-neutral probability measure $\mathbb{Q}$, the dynamics of the instantaneous forward rate for the maturity T is given by:

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)\,\mathrm{d}s + \int_0^t \sigma(s,T)\,\mathrm{d}W^{\mathbb{Q}}(s)$$

  where $f(0,T)$ is the current forward rate

- Therefore, the stochastic differential equation is:

$$\mathrm{d}f(t,T) = \alpha(t,T)\,\mathrm{d}t + \sigma(t,T)\,\mathrm{d}W^{\mathbb{Q}}(t)$$

The HJM model

- We can show that:

$$\alpha(t, T) = \sigma(t, T) \int_t^T \sigma(t, u) \,\mathrm{d}u$$

This equation is known as the '*drift restriction*' and is necessary to ensure no-arbitrage opportunities

- We have:

$$\mathrm{d}B(t, T) = r(t) B(t, T) \,\mathrm{d}t + b(t, T) B(t, T) \,\mathrm{d}W^{\mathbb{Q}}(t)$$

where $b(t, T) = -\int_t^T \sigma(t, u) \,\mathrm{d}u$

The HJM model

- The drift restriction implies that the dynamics of the instantaneous forward rate $f(t,T)$ is given by:

$$\mathrm{d}f(t,T) = \left(\sigma(t,T)\int_t^T \sigma(t,u)\,\mathrm{d}u\right)\mathrm{d}t + \sigma(t,T)\,\mathrm{d}W^{\mathbb{Q}}(t)$$

- If we are interested in the instantaneous spot rate $r(t)$, we obtain:

$$\begin{aligned} r(t) &= f(t,t) \\ &= r(0) + \int_0^t \left(\sigma(s,t)\int_s^t \sigma(s,u)\,\mathrm{d}u\right)\mathrm{d}s + \int_0^t \sigma(s,t)\,\mathrm{d}W^{\mathbb{Q}}(s) \end{aligned}$$

Market models

1. Libor market model (LMM)
   - Under the forward probability measure $\mathbb{Q}^{\star}(T_{i+1})$, the Libor rate $L_i(t) = L(t, T_i, T_{i+1})$ is a martingale:

$$\mathrm{d}L_i(t) = \gamma_i(t) L_i(t) \,\mathrm{d}W_i^{\mathbb{Q}^{\star}(T_{i+1})}(t)$$

   - We can use the Black formula to price caplets and floorlets where the volatility σ_i is defined by:

$$\sigma_i^2 = \frac{1}{T_i - t}\int_t^{T_i} \gamma_i^2(s)\,\mathrm{d}s$$

2. Swap market model (SMM)
   - We have:

$$\mathrm{d}\,\mathrm{Sw}(t) = \eta(t)\,\mathrm{Sw}(t)\,\mathrm{d}W^{\mathbb{Q}^{\star}}(t)$$

The uncertain volatility model (UVM)

We recall that:

$$V(T) - f(S(T)) = \frac{1}{2}\int_0^T e^{r(T-t)}\boldsymbol{\Gamma}(t)\left(\Sigma^2(T,K) - \sigma^2(t)\right)S^2(t)\,\mathrm{d}t$$

If we assume that $\sigma(t) \in [\sigma^-, \sigma^+]$, we obtain a simple rule for achieving a positive P&L:

- if $\boldsymbol{\Gamma}(t) \geq 0$, we have to hedge the portfolio by considering an implied volatility that is equal to the upper bound σ^+;
- if $\boldsymbol{\Gamma}(t) \leq 0$, we set the implied volatility to the lower bound σ^-.

$\Rightarrow$ This rule is valid if the gamma of the option is always positive or negative, that is when the payoff is convex

How to extend this rule when the gamma can change its sign during the life of the option?

The uncertain volatility model (UVM)

- We assume that:

$$\mathrm{d}S(t) = r(t)\, S(t)\, \mathrm{d}t + \sigma(t)\, S(t)\, \mathrm{d}W^{\mathbb{Q}}(t)$$

where:

$$\sigma^{-} \leq \sigma(t) \leq \sigma^{+}$$

- Let $V(t, S(t))$ be the option price, whose payoff is $f(S(T))$. $V(t, S(t))$ is bounded:

$$V^{-}(t, S(t)) \leq V(t, S(t)) \leq V^{+}(t, S(t))$$

where $V^{-}(t, S(t)) = \inf_{\mathbb{Q}(\sigma)} \mathbb{E}^{\mathbb{Q}(\sigma)}\left[\exp\left(-\int_t^T r(s)\, \mathrm{d}s\right) f(S(T))\right]$, $V^{+}(t, S(t)) = \sup_{\mathbb{Q}(\sigma)} \mathbb{E}^{\mathbb{Q}(\sigma)}\left[\exp\left(-\int_t^T r(s)\, \mathrm{d}s\right) f(S(T))\right]$ and $\mathbb{Q}(\sigma)$ denotes all the probability measures

- We can then show that V^{-} and V^{+} satisfy the HJB equation:

$$\sup_{\sigma^{-}\leq\sigma(t)\leq\sigma^{+}} / \inf \left(\frac{1}{2}\sigma^{2}(t)\, S^{2} \frac{\partial^{2} V(t, S)}{\partial S^{2}} + b(t)\, S \frac{\partial V(t, S)}{\partial S}\right) + \frac{\partial V(t, S)}{\partial t} - r(t)\, V(t,$$

The uncertain volatility model (UVM)

- Solving the HJB equation is equivalent to solve the modified Black-Scholes PDE:

$$\left\{ \begin{array}{l} \frac{1}{2}\sigma^2 \left(\mathbf{\Gamma}\left(t,S\right)\right) S^2 \partial_S^2 V\left(t,S\right) + b\left(t\right) S \partial_S V\left(t,S\right) + \partial_t V\left(t,S\right) - r\left(t\right) V\left(t,S\right) = \\ V\left(T,S\left(T\right)\right) = f\left(S\left(T\right)\right) \end{array} \right.$$

where:

$$\sigma\left(x\right) = \left\{ \begin{array}{ll} \sigma^+ & \text{if } x \geq 0 \\ \sigma^- & \text{if } x < 0 \end{array} \right. \qquad \text{for } V\left(t,S\left(t\right)\right) = V^+\left(t,S\left(t\right)\right)$$

and:

$$\sigma\left(x\right) = \left\{ \begin{array}{ll} \sigma^- & \text{if } x > 0 \\ \sigma^+ & \text{if } x \leq 0 \end{array} \right. \qquad \text{for } V\left(t,S\left(t\right)\right) = V^-\left(t,S\left(t\right)\right)$$

The uncertain volatility model (UVM)

- Let u_i^m be the numerical solution of $V(t_m, S_i)$. At each iteration m, we approximate the gamma coefficient by the central difference method:

  $$\boldsymbol{\Gamma}(t_m, S_i) \simeq \frac{u_{i+1}^m - 2u_i^m + u_{i+1}^m}{h^2}$$

  By assuming that:

  $$\operatorname{sign}(\boldsymbol{\Gamma}(t_m, S_i)) \approx \operatorname{sign}(\boldsymbol{\Gamma}(t_{m+1}, S_i))$$

  we can compute the values taken by $\sigma(\boldsymbol{\Gamma}(t, S))$ and solve the PDE for the next iteration $m + 1$

The uncertain volatility model (UVM)

- If we consider the European call option, we have $\boldsymbol{\Gamma}(t, S) > 0$, meaning that:

$$V^{+}(t, S(t)) = C_{\mathrm{BS}}\left(t, S(t), \sigma^{+}\right)$$

and:

$$V^{-}(t, S(t)) = C_{\mathrm{BS}}\left(t, S(t), \sigma^{-}\right)$$

where $C_{\mathrm{BS}}(t, S, \sigma)$ is the Black-Scholes price at time t when the underlying price is equal to S and the implied volatility is equal to Σ. Then, the worst-case scenario occurs when the volatility $\sigma(t)$ reaches the upper bound σ^{+}

The uncertain volatility model (UVM)

Example #6

We consider a double KOC barrier option:

$$f_{\mathrm{Barrier}}\left(S\left(T\right)\right)=\mathbb{1}\left\{S\left(t\right)\in\left[L,H\right],t\in\mathcal{T}\right\}\cdot f_{\mathrm{Vanilla}}\left(S\left(T\right)\right)$$

with the following parameters: $K=100$, $L=80$, $H=120$, $T=1$, $b=5\%$ and $r=5\%$. We assume that the volatility $\sigma\left(t\right)$ lies in the range of 15% and 25%. We assume a continuous barrier $\mathcal{T}=\left[0,1\right]$ and a window barrier $\mathcal{T}=\left[0.25,0.75\right]$.

The uncertain volatility model (UVM)

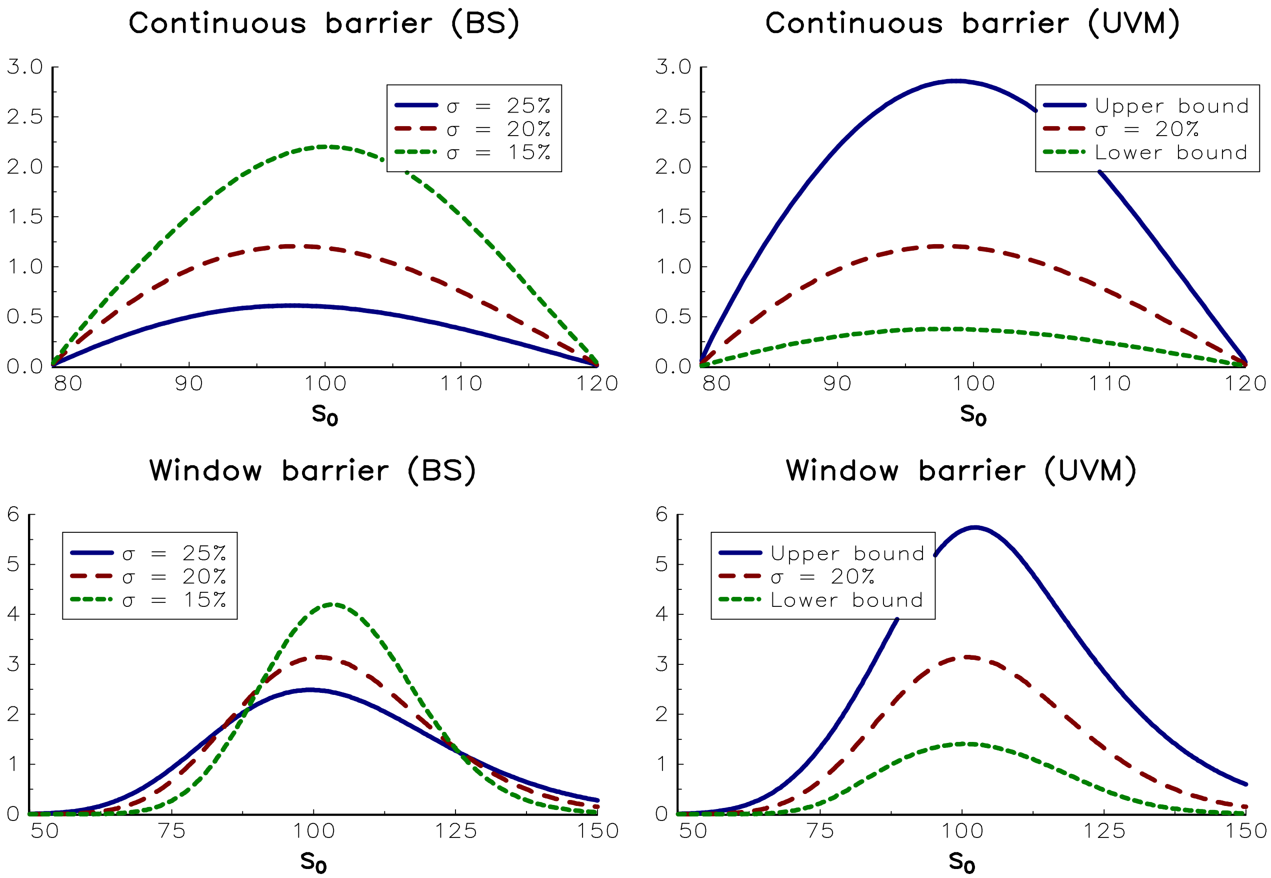


Figure: Comparing BS and UVM prices of the double KOC barrier option

The shifted log-normal model

This model assumes that the asset price $S(t)$ is a linear transformation of a log-normal random variable $X(t)$:

$$S(t) = \alpha(t) + \beta(t) X(t)$$

where $\beta(t) \geq 0$. Then, the payoff of the European call option is:

$$\begin{aligned} f(S(T)) &= (S(T) - K)^{+} \\ &= (\alpha(T) + \beta(T) X(T) - K)^{+} \\ &= \beta(T) \left(X(T) - \frac{K - \alpha(T)}{\beta(T)} \right)^{+} \end{aligned}$$

This type of approach is interesting because the pricing of options can then be done using the Black-Scholes formula:

$$\mathcal{C}(0, S_0) = \beta(T) C_{\mathrm{BS}} \left(X_0, \frac{K - \alpha(T)}{\beta(T)}, \sigma_X, T, b_X, r \right)$$

where b_X and σ_X are the drift and diffusion coefficients of $X(t)$ under the risk-neutral probability measure $\mathbb{Q}$

The fixed-strike parametrization

Let us suppose that:

$$S(t) = \alpha + \beta \exp\left(\left(b^{\mathbb{Q}}(t) - \frac{1}{2}\sigma^2\right) t + \sigma W^{\mathbb{Q}}(t)\right)$$

We have $S_0 = \alpha + \beta$ meaning that:

$$S(t) = \alpha + (S_0 - \alpha) \exp\left(\left(b^{\mathbb{Q}}(t) - \frac{1}{2}\sigma^2\right) t + \sigma W^{\mathbb{Q}}(t)\right)$$

Let b the cost-of-carry parameter of the asset. Under the risk-neutral probability measure, the martingale condition is:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-bt} S(t) \mid \mathcal{F}_0\right] = S_0$$

The fixed-strike parametrization

Since we have $\mathbb{E}^{\mathbb{Q}}\left[S(t)\right] = \alpha + (S_0 - \alpha)\, e^{b^{\mathbb{Q}}(t)t}$, we deduce that the no-arbitrage condition implies that:

$$\alpha + (S_0 - \alpha)\, e^{b^{\mathbb{Q}}(t)t} = S_0 e^{bt} \Leftrightarrow b^{\mathbb{Q}}(t) = \frac{1}{t} \ln\left(\frac{S_0 e^{bt} - \alpha}{S_0 - \alpha}\right)$$

The payoff of the European call option is:

$$f(S(T)) = (S(T) - K)^{+} = ((S(T) - \alpha) - (K - \alpha))^{+}$$

We deduce that the price of the option is given by:

$$\mathcal{C}(0, S_0) = C_{\mathrm{BS}}\left(S_0 - \alpha, K - \alpha, \sigma, T, b^{\mathbb{Q}}(T), r\right)$$

The fixed-strike parametrization

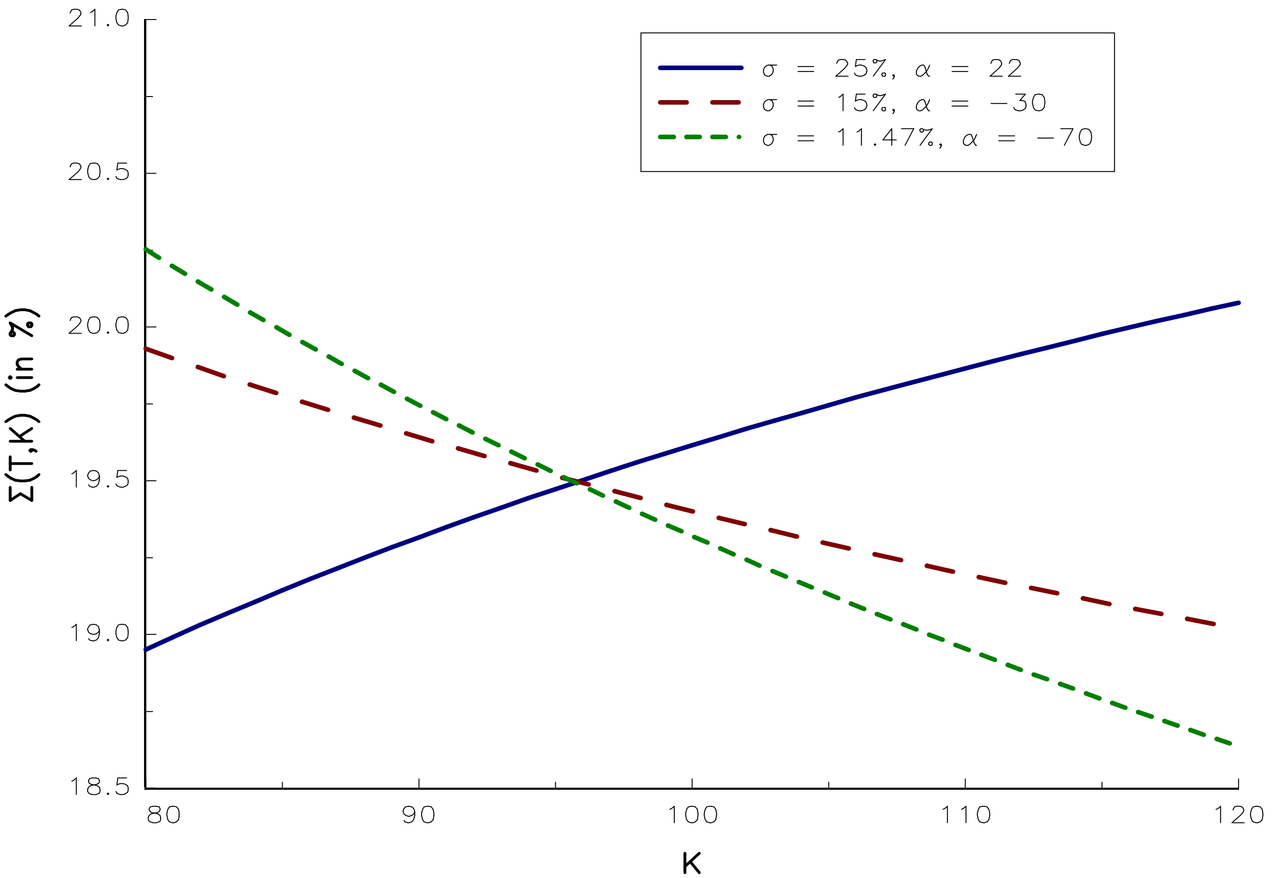


Figure: Volatility skew generated by the SLN model (fixed-strike parametrization)

The floating-strike parametrization

Let us now suppose that:

$$S(t) = \alpha e^{\varphi t} + \beta e^{\left(b - \frac{1}{2}\sigma^2\right)t + \sigma W^{\mathbb{Q}}(t)}$$

We have $S_0 = \alpha + \beta$ and $\mathbb{E}^{\mathbb{Q}}[S(t)] = \alpha e^{\varphi t} + \beta e^{bt}$. We deduce that the stochastic process $e^{-bt}S(t)$ is a $\mathcal{F}_t$-martingale if it is equal to:

$$S(t) = \alpha e^{bt} + (S_0 - \alpha)\, e^{\left(b - \frac{1}{2}\sigma^2\right)t + \sigma W^{\mathbb{Q}}(t)}$$

The payoff of the European call option becomes:

$$f(S(T)) = (S(T) - K)^+ = \left(\left(S(T) - \alpha e^{bT}\right) - \left(K - \alpha e^{bT}\right)\right)^+$$

It follows that the option price is equal to:

$$\mathcal{C}(0, S_0) = C_{\mathrm{BS}}\left(S_0 - \alpha, K - \alpha e^{bT}, \sigma, T, b, r\right)$$

The floating-strike parametrization

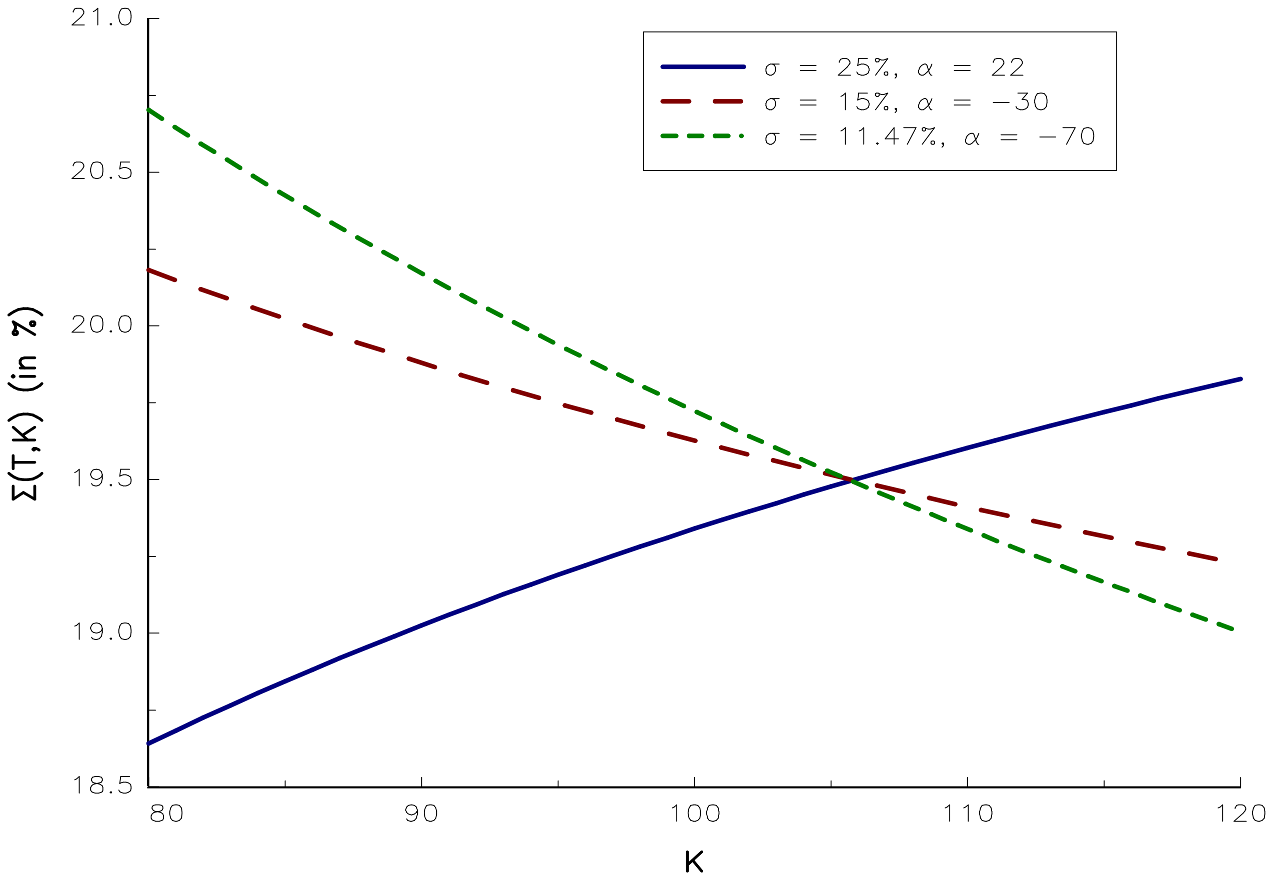


Figure: Volatility skew generated by the SLN model (floating-strike parametrization)

The forward parametrization

If we consider the forward price $F(t)$ instead of the spot price $S(t)$, the two models coincide because we have $b = 0$:

$$\mathrm{d}F(t) = \sigma (F(t) - \alpha)\,\mathrm{d}W^{\mathbb{Q}}(t)$$

and the price of the option is given by the Black formula:

$$\mathcal{C}(0, S_0) = C_{\mathrm{Black}}(F_0 - \alpha, K - \alpha, \sigma, T, r)$$

The forward parametrization

We can prove the following results:

- monotonicity in strike:

$$\operatorname{sign}\left(\frac{\partial\,\Sigma\left(T,K\right)}{\partial\,K}\right)=\operatorname{sign}\alpha$$

- upper and lower bounds:

$$\begin{cases} \Sigma\left(T,K\right)<\sigma & \text{if } \alpha>0 \\ \Sigma\left(T,K\right)>\sigma & \text{if } \alpha<0 \end{cases}$$

- sharpness of bound:

$$\lim_{K\rightarrow\infty}\Sigma\left(T,K\right)=\sigma$$

- short-expiry behavior:

$$\lim_{T\rightarrow 0}\Sigma\left(T,K\right)=\begin{cases} \dfrac{\sigma\ln\left(F_0/K\right)}{\ln\left(\left(F_0-\alpha\right)/\left(K-\alpha\right)\right)} & \text{if } K\neq F_0 \\ \sigma\left(1-\alpha F_0^{-1}\right) & \text{if } K=F_0 \end{cases}$$

The forward parametrization

Table: Error of the SLN implied volatility formula (in bps)

K	$(\alpha = 22, \sigma = 25\%)$			$(\alpha = -70, \sigma = 12\%)$		
	1M	1Y	5Y	1M	1Y	5Y
80	1.0	11.1	57.0	-0.9	-12.9	-66.0
90	0.7	10.6	54.1	-1.0	-11.9	-61.4
100	0.9	10.2	51.6	-1.1	-11.3	-57.3
110	1.0	9.7	49.6	-0.8	-10.8	-53.8
120	0.7	9.3	47.7	-0.6	-10.3	-51.3

Mixture of SLN distributions

- One limitation of the SLN model is that it only produces a volatility skew, and not a volatility smile
- The (risk-neutral) probability density function $f(x)$ of the asset price density is given by the mixture of known basic densities:

$$f(x) = \sum_{j=1}^{m} p_j f_j(x)$$

where f_j is the j^{th} basic density, $p_j > 0$ and $\sum_{j=1}^{m} p_j = 1$

- Let $G(S(T))$ be the payoff of an European option. We have:

$$\mathcal{C}(0, S_0) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT} G(S(T)) \middle| \mathcal{F}_0 \right] = \ldots = \sum_{j=1}^{m} p_j \mathbb{E}^{\mathbb{Q}_j}\left[e^{-rT} G(S(T)) \middle| \mathcal{F}_0 \right]$$

Mixture of SLN distributions

- If we consider a mixture of two shifted log-normal models, the price of the European call option is equal to:

$$\begin{aligned}\mathcal{C}(0, S_0) &= p \cdot C_{\mathrm{SLN}}(S_0, K, \sigma_1, T, b, r, \alpha_1) + \\ &\quad (1-p) \cdot C_{\mathrm{SLN}}(S_0, K, \sigma_2, T, b, r, \alpha_2)\end{aligned}$$

  where C_{SLN} is the formula of the SLN model
- The model has five parameters: σ_1, σ_2, α_1, α_2 and p

Mixture of SLN distributions

Example #7

We consider a calibration set of five options, whose strike and implied volatilities are equal to:

K_j	80	90	100	110	120
$\Sigma(1, K_j)$	21%	19%	18.25%	18.5%	19%

The current value of the asset price is equal to 100, the maturity of options is one year, the cost-of-carry parameter is set to 0 and the interest rate is 5%

The parameters are estimated by minimizing the weighted least squares:

$$\min \sum_{j=1}^{n} w_j \left(\hat{C}_j - C_{\mathrm{SLN}}\left(S_0, K_j, \sigma_1, \sigma_2, T_j, b, r, \alpha_1, \alpha_2, p\right)\right)^2$$

where $\hat{C}_j = C_{\mathrm{BS}}\left(S_0, K_j, \Sigma\left(T_j, K_j\right), T_j, b, r\right)$ and w_j is the weight of the j^{th} option

Mixture of SLN distributions

We consider three parameterizations:

- (#1) the weights w_j are uniform, and we impose that $\alpha_1 = \alpha_2$ and $p = 50\%$
- (#2) the weights w_j are uniform, and p is set to 25%
- (#3) the weights w_j are inversely proportional to option prices $\hat{C}_j$, and p is set to 50%

Table: Calibrated parameters of the mixed SLN model

Model	#1	#2	#3
σ_1	16.5%	8.2%	10.2%
σ_2	7.3%	17.2%	21.7%
α_1	-53.3	-289.7	-145.2
α_1	-53.3	19.6	47.4
p	50.0%	25.0%	50.0%

Mixture of SLN distributions

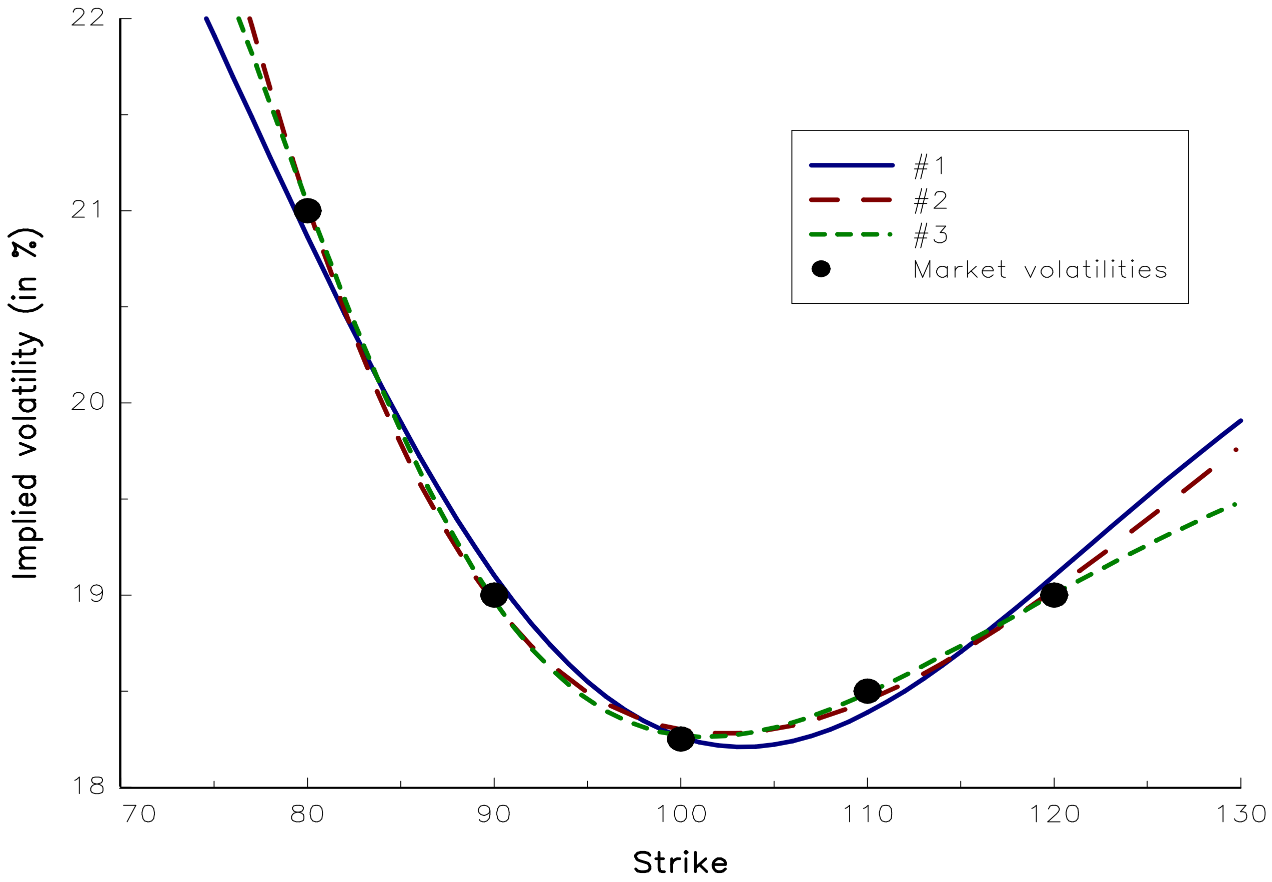


Figure: Implied volatility (in %) of calibrated mixed SLN models

Local volatility model

- We assume that:

$$\mathrm{d}S(t) = bS(t)\,\mathrm{d}t + \sigma(t, S(t))\,S(t)\,\mathrm{d}W^{\mathbb{Q}}(t)$$

- $\sigma(t, S) \Rightarrow \Sigma(T, K)$
- $\Sigma(T, K) \Rightarrow \sigma(t, S)$ (Dupire model)
- Relationship with the Breeden-Litzenberger model

The Fokker-Planck equation

The risk-neutral probability density function $q_t(T, S)$ of the asset price $S(T)$ satisfies the forward Chapman-Kolmogorov equation:

$$\frac{\partial\, q_t(T, S)}{\partial\, T} = -\frac{\partial\, [bSq_t(T, S)]}{\partial\, S} + \frac{1}{2}\frac{\partial^2\, \left[\sigma^2(T, S)\, S^2 q_t(T, S)\right]}{\partial\, S^2}$$

The initial condition is:

$$q_t(t, S) = \mathbb{1}\{S = S_t\}$$

where S_t is the value of $S(t)$ that is known at time t

The Breeden-Litzenberger formulas

We have:

$$\begin{aligned}
\mathcal{C}_t(T,K) &= e^{-r(T-t)}\int_K^\infty (S-K)\,q_t(T,S)\,\mathrm{d}S \\
\frac{\partial\,\mathcal{C}_t(T,K)}{\partial\,K} &= -e^{-r(T-t)}\int_K^\infty q_t(T,S)\,\mathrm{d}S \\
\frac{\partial^2\,\mathcal{C}_t(T,K)}{\partial\,K^2} &= e^{-r(T-t)}q_t(T,K) \\
\frac{\partial\,\mathcal{C}_t(T,K)}{\partial\,T} &= -r\mathcal{C}_t(T,K)+e^{-r(T-t)}\int_K^\infty (S-K)\,\frac{\partial\,q_t(T,S)}{\partial\,T}\,\mathrm{d}S
\end{aligned}$$

Derivation of the forward equation

We deduce that:

$$\frac{\partial\,\mathcal{C}_t\left(T,K\right)}{\partial\,T}=-r\mathcal{C}_t\left(T,K\right)+e^{-r(T-t)}\mathcal{I}$$

where:

$$\mathcal{I}=\frac{1}{2}\sigma^2\left(T,K\right)K^2q_t\left(T,K\right)+be^{r(T-t)}\left(\mathcal{C}_t\left(T,K\right)-K\frac{\partial\,\mathcal{C}_t\left(T,K\right)}{\partial\,K}\right)$$

It follows that:

$$\begin{aligned}\frac{\partial\,\mathcal{C}_t\left(T,K\right)}{\partial\,T} &= -r\mathcal{C}_t\left(T,K\right)+\frac{1}{2}\sigma^2\left(T,K\right)K^2\frac{\partial^2\,\mathcal{C}_t\left(T,K\right)}{\partial\,K^2}+\\ & \quad b\left(\mathcal{C}_t\left(T,K\right)-K\frac{\partial\,\mathcal{C}_t\left(T,K\right)}{\partial\,K}\right)\end{aligned}$$

Derivation of the forward equation

We conclude that:

$$\frac{1}{2}\sigma^2(T,K)K^2\frac{\partial^2\mathcal{C}_t(T,K)}{\partial K^2} - bK\frac{\partial\mathcal{C}_t(T,K)}{\partial K} - \frac{\partial\mathcal{C}_t(T,K)}{\partial T} + (b-r)\mathcal{C}_t(T,K) = 0$$

Differences between backward and forward PDE approaches

The backward PDE is:

$$\left\{ \begin{array}{l} \frac{1}{2}\sigma^2 (t, S) S^2 \partial_S^2 V (t, S) + bS\partial_S V (t, S) + \partial_t V (t, S) - rV (t, S) = 0 \\ V (T, S (T)) = f (T, S (T), K) \end{array} \right.$$

The forward PDE is:

$$\left\{ \begin{array}{l} \frac{1}{2}\sigma^2 (T, K) K^2 \partial_K^2 V (T, K) - bK\partial_K V (T, K) - \partial_T V (T, K) + (b - r) V (T, K) = \\ V (t, K) = f (t, S_t, K) \end{array} \right.$$

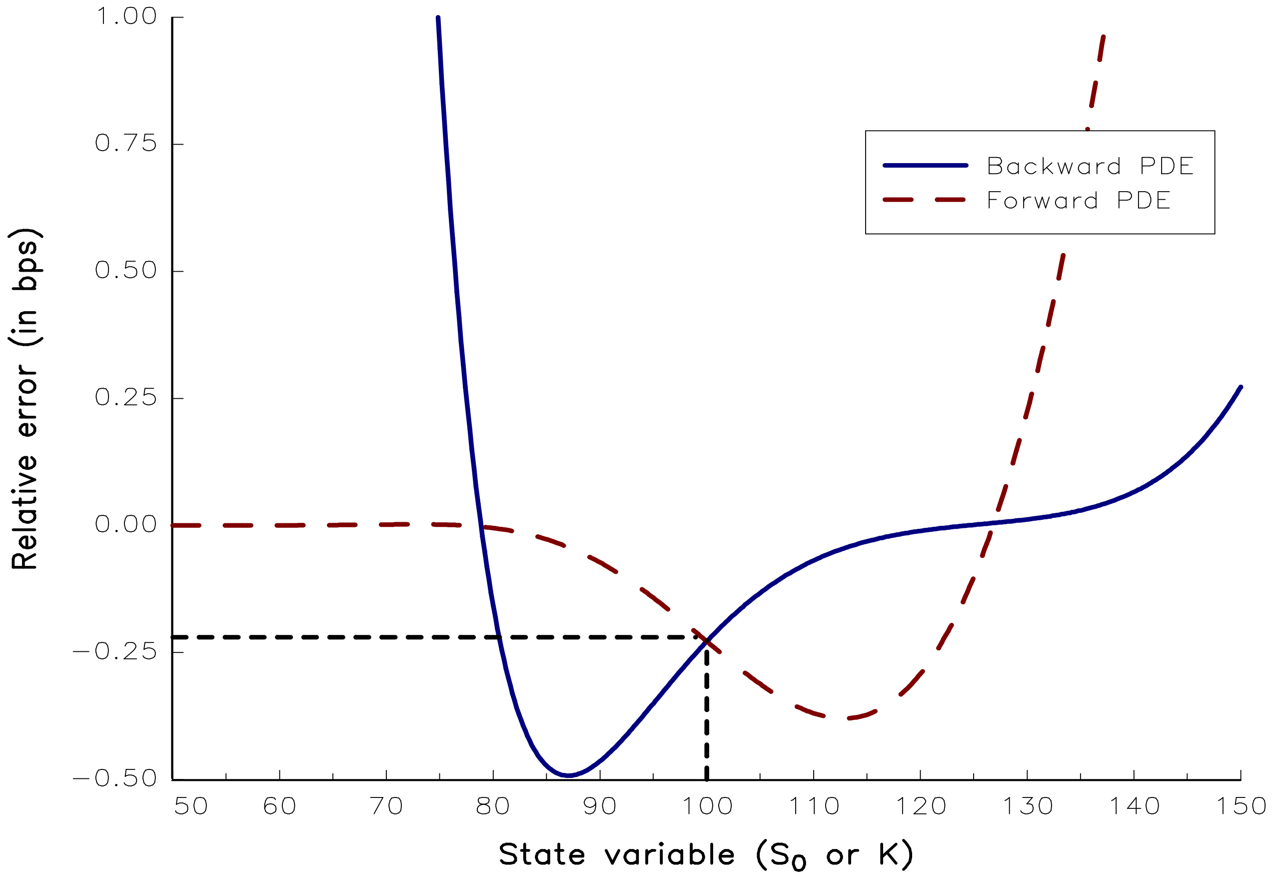
Differences between backward and forward PDE approaches

- In the backward formulation, the state variables are t and S, whereas the fixed variables are T and K
- In the forward formulation, the state variables are T and K, whereas the fixed variables are the current time t and the current asset price S_t
- The backward PDE approach suggests that we can hedge the option using a dynamic portfolio of the underlying asset
- The forward PDE approach suggests that we can hedge the option using a static portfolio of call and put options

Differences between backward and forward PDE approaches

- We consider the pricing of an European call option with the following parameters: $S_0 = 100$, $K = 100$, $\sigma(t, S) = 20\%$, $T = 0.5$, $b = 2\%$ and $r = 5\%$
- In the case of the backward PDE, we consider the usual boundary conditions:
$$\left\{ \begin{array}{l} \mathcal{C}(t, S) = 0 \\ \partial_S \mathcal{C}(t, +\infty) = 1 \end{array} \right.$$
- For the forward PDE, the boundary conditions are:
$$\left\{ \begin{array}{l} \partial_K \mathcal{C}(T, 0) = -1 \\ \mathcal{C}(T, +\infty) = 0 \end{array} \right.$$
- The backward and forward PDEs are solved using the Crank-Nicholson scheme

Differences between backward and forward PDE approaches



Duality between local volatility and implied volatility

We have:

$$\sigma^2(T,K) = 2\frac{bK\partial_K\mathcal{C}(T,K) + \partial_T\mathcal{C}(T,K) - (b-r)\mathcal{C}(T,K)}{K^2\partial_K^2\mathcal{C}(T,K)}$$

Duality between local volatility and implied volatility

We can show:

$$\sigma(T,K)=\sqrt{\frac{A(T,K)}{B(T,K)}}$$

where:

$$\begin{aligned} A(T,K) &= \Sigma^2(T,K)+2bKT\Sigma(T,K)\,\partial_K\Sigma(T,K)+ \\ & \quad 2T\Sigma(T,K)\,\partial_T\Sigma(T,K) \end{aligned}$$

and:

$$\begin{aligned} B(T,K) &= 1+2K\sqrt{T}d_1\partial_K\Sigma(T,K)+K^2T\Sigma(T,K)\,\partial_K^2\Sigma(T,K)+ \\ & \quad K^2Td_1d_2\left(\partial_K\Sigma(T,K)\right)^2 \end{aligned}$$

Duality between local volatility and implied volatility

Example #8

We assume that the implied volatility is equal to:

$$\Sigma(T, K) = \Sigma_0 + \alpha (S_0 - K)^2$$

where $\Sigma_0 = 20\%$, $\alpha = 1$ bp, $S_0 = 100$ and $b = 5\%$.

Duality between local volatility and implied volatility

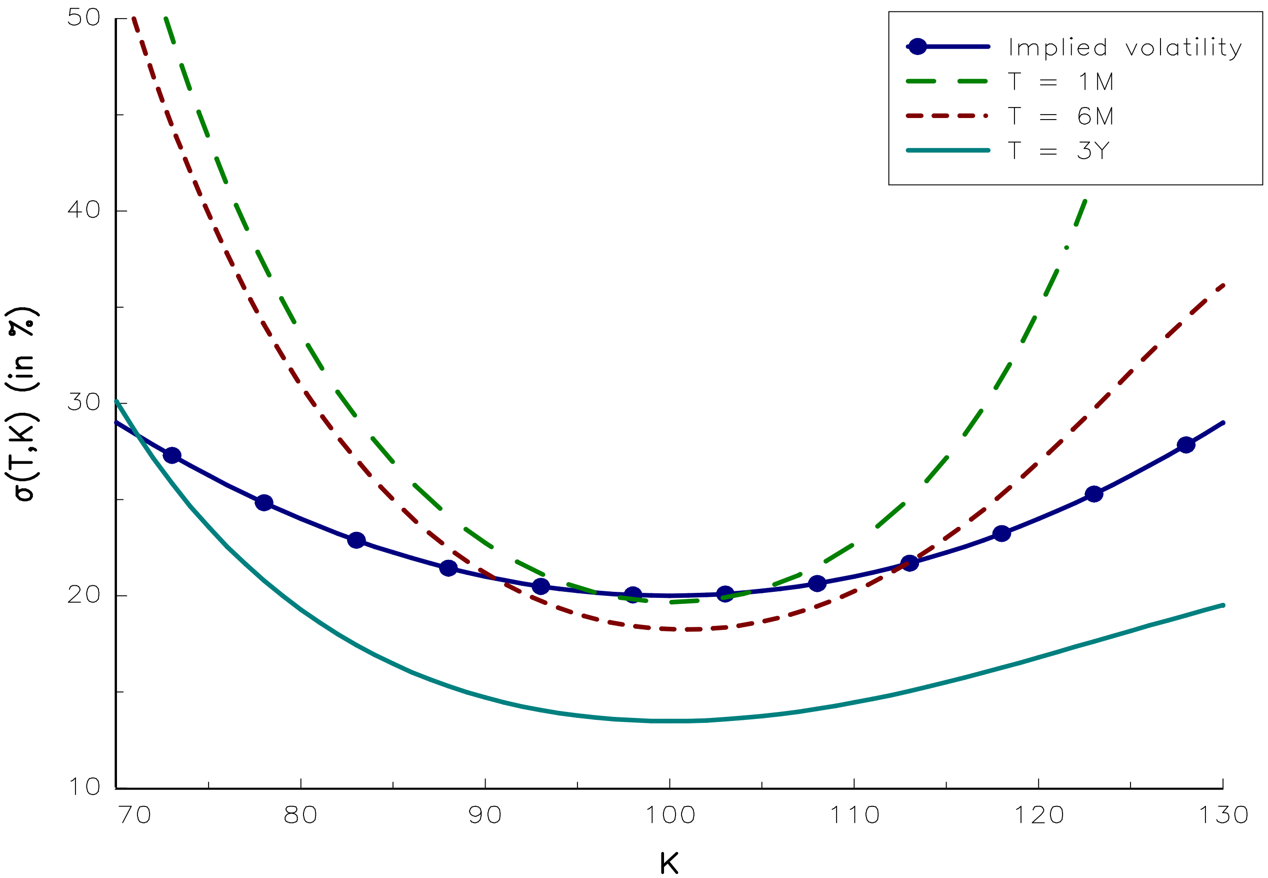


Figure: Calibrated local volatility $\sigma(T, S)$ (in %)

Duality between local volatility and implied volatility

If there is no skew, the local volatility function does not depend on the strike:

$$\sigma^2(T) = \Sigma^2(T) + 2T\Sigma(T)\frac{\partial\,\Sigma(T)}{\partial\,T}$$

We notice that:

$$\sigma^2(T) = \Sigma^2(T) + 2T\Sigma(T)\frac{\partial\,\Sigma(T)}{\partial\,T} = \frac{\partial\,T\Sigma^2(T)}{\partial\,T}$$

or:

$$\Sigma^2(T) = \frac{1}{T}\int_0^T \sigma^2(t)\,\mathrm{d}t$$

The implied variance is then the time series average of the local variance

Duality between local volatility and implied volatility

- Let x be the log-moneyness:

$$x = \varphi(T, K) = \ln \frac{S_0}{K} + bT$$

- We introduce the functions $\tilde{\Sigma}$ and $\tilde{\sigma}$ such that $\Sigma(T, K) = \tilde{\Sigma}(T, \varphi(T, K))$ and $\sigma(T, K) = \tilde{\sigma}(T, \varphi(T, K))$
- We can show that the implied volatility is the harmonic mean of the local volatility:

$$\frac{1}{\tilde{\Sigma}(0, x)} = \int_0^1 \frac{\mathrm{d}y}{\tilde{\sigma}(0, xy)}$$

- It follows that:

$$\frac{\partial \tilde{\Sigma}(0, 0)}{\partial x} = \frac{1}{2} \frac{\partial \tilde{\sigma}(0, 0)}{\partial x}$$

- The ATM slope of the implied volatility near expiry is equal to one half the slope of the local volatility

Dupire model in practice

- Time interpolation (e.g., linear interpolation of the total implied variance $\upsilon(T,K) = T\Sigma^2(T,K)$)
- Non-parametric interpolation (e.g., cubic spline interpolation)
- Parametric interpolation (e.g., SVI parametrization)

Dupire model in practice

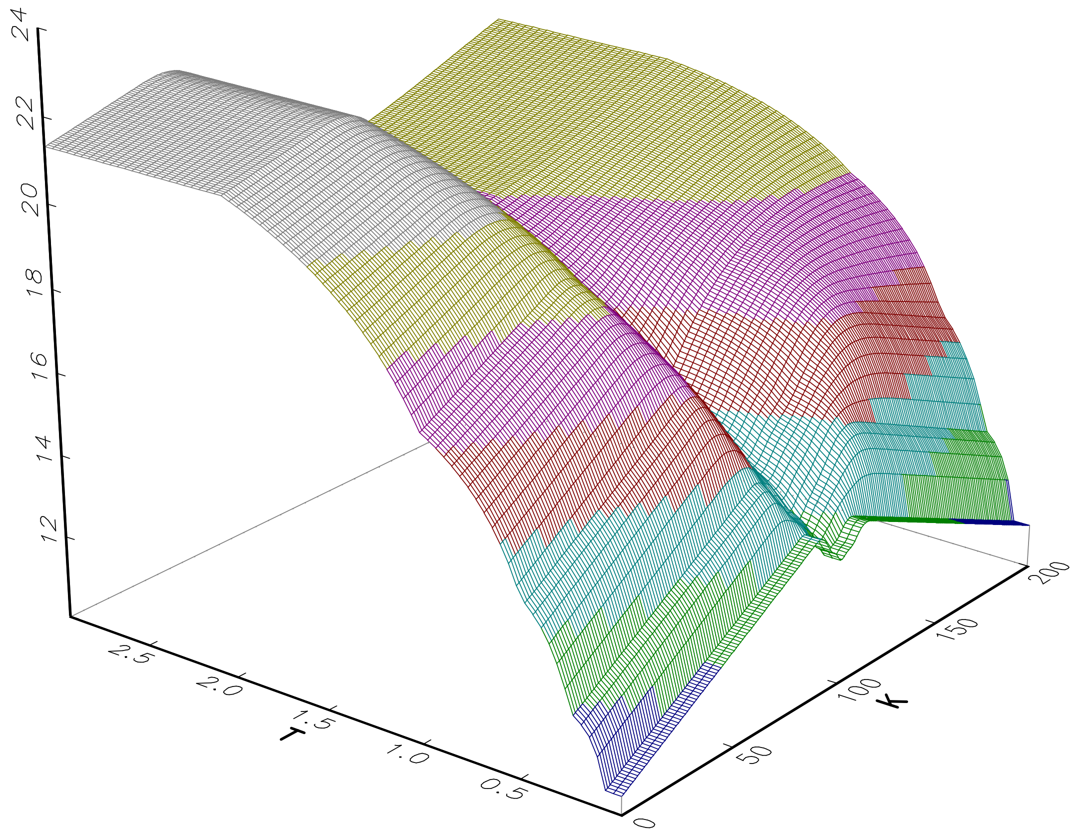


Figure: Implied volatility surface $\Sigma(T, K)$ (in %)

Dupire model in practice

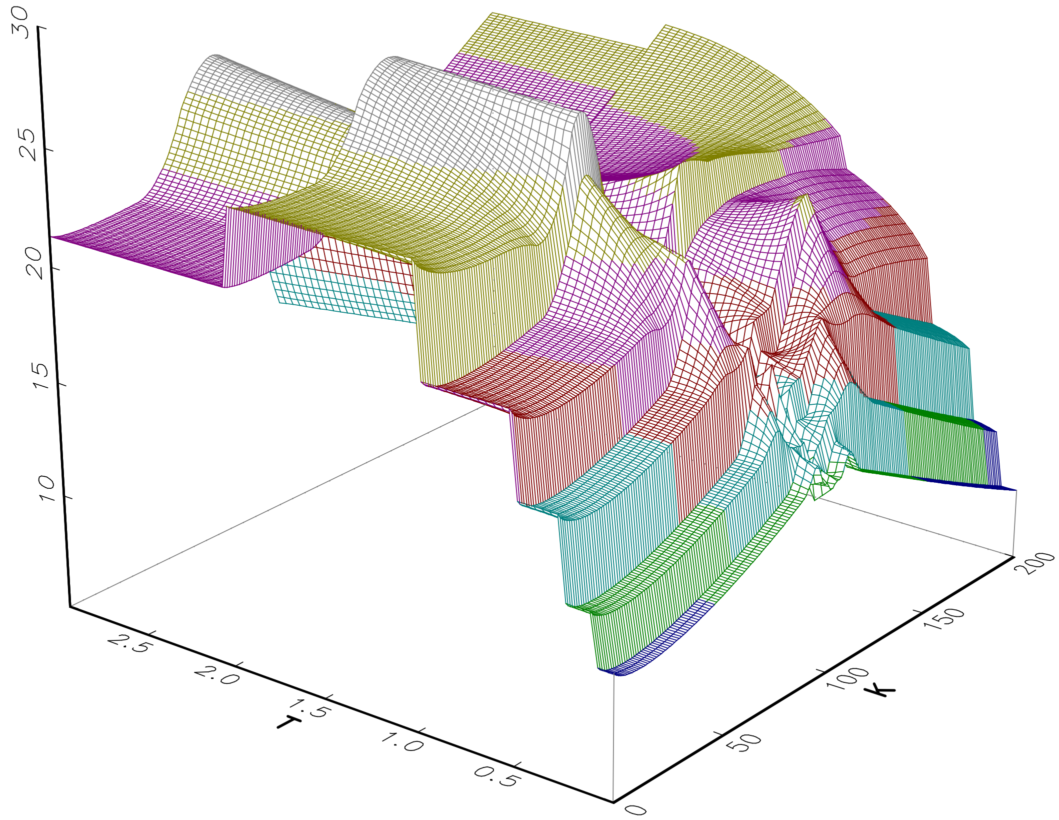


Figure: Local volatility surface $\sigma(T, K)$ (in %)

Hedging coefficients

- The delta of the option is:

$$\boldsymbol{\Delta} \approx \frac{V(T, K, S_t + \varepsilon) - V(T, K, S_t - \varepsilon)}{2\varepsilon}$$

- The gamma of the option is:

$$\boldsymbol{\Gamma} \approx \frac{V(T, K, S_t + \varepsilon) - 2V(T, K, S_t) + V(T, K, S_t - \varepsilon)}{\varepsilon^2}$$

- The vega is the sensitivity of the price to a parallel shift of $\Sigma(T, K, S_t)$:

$$\boldsymbol{\upsilon} = \frac{V'(T, K, S_t) - V(T, K, S_t)}{\varepsilon'}$$

  where $V'(T, K, S_t)$ is the option price obtained when the implied volatility surface is $\Sigma(T, K, S_t) + \varepsilon'$

Hedging coefficients

"Market smiles and skews are usually managed by using local volatility models $a\ la$ *Dupire. We discover that the dynamics of the market smile predicted by local vol models is opposite of observed market behavior: when the price of the underlying decreases, local vol models predict that the smile shifts to higher prices; when the price increases, these models predict that the smile shifts to lower prices. Due to this contradiction between model and market, delta and vega hedges derived from the model can be unstable and may perform worse than naive Black-Scholes' hedges" (Hagan et al., 2002, page 84).*

Application to exotic options

- We assume that S_0, $b = 5\%$ and r
- The option parameters are $K = 100$, $L = 90$ and $H = 115$
- The maturity is set to one year

Table: Barrier option pricing with the local volatility model

Option	Payoff	LV	BS-PDE		BS-RR	
			Σ_1	Σ_2	Σ_1	Σ_2
Call	$(S(T) - K)^+$	8.85	8.96	8.78	8.96	8.78
Put	$(K - S(T))^+$	3.97	4.08	3.90	4.08	3.90
DOC	$\mathbb{1}\{S(t) > L\} \cdot (S(T) - K)^+$	7.98	8.14	8.05	8.11	8.02
DOP	$\mathbb{1}\{S(t) > L\} \cdot (K - S(T))^+$	0.26	0.27	0.28	0.25	0.27
UOC	$\mathbb{1}\{S(t) < H\} \cdot (S(T) - K)^+$	0.99	0.88	0.94	0.83	0.89
UOP	$\mathbb{1}\{S(t) < H\} \cdot (K - S(T))^+$	3.81	3.90	3.75	3.89	3.74
KOC	$\mathbb{1}\{S(t) \in [L, H]\} \cdot (S(T) - K)^+$	0.65	0.56	0.64	0.52	0.59
KOP	$\mathbb{1}\{S(t) \in [L, H]\} \cdot (K - S(T))^+$	0.20	0.20	0.22	0.19	0.21
BCC	$\mathbb{1}\{S(T) \geq K\}$	0.58	0.56	0.57	0.56	0.57
BCP	$\mathbb{1}\{S(T) \leq K\}$	0.37	0.39	0.38	0.39	0.38

Stochastic volatility models

We assume that the joint dynamics of the spot price $S(t)$ and the stochastic volatility $\sigma(t)$ is:

$$\left\{ \begin{array}{l} \mathrm{d}S(t) = \mu(t) S(t) \,\mathrm{d}t + \sigma(t) S(t) \,\mathrm{d}W_1(t) \\ \mathrm{d}\sigma(t) = \zeta(\sigma(t)) \,\mathrm{d}t + \xi(\sigma(t)) \,\mathrm{d}W_2(t) \end{array} \right.$$

where $\mathbb{E}[W_1(t) W_2(t)] = \rho t$

Stochastic volatility models

The fundamental pricing equation is:

$$\begin{aligned}\frac{1}{2}\sigma^2 S^2 \partial_S^2 V(t,S,\sigma) + \rho\sigma S\xi(\sigma)\,\partial_{S,\sigma}^2 V(t,S,\sigma) + \frac{1}{2}\xi^2(\sigma)\,\partial_\sigma^2 V(t,S,\sigma) & \\ +(\mu-\lambda_S\sigma)\,S\partial_S V(t,S,\sigma) + (\zeta(\sigma)-\lambda_\sigma\xi(\sigma))\,\partial_\sigma V(t,S,\sigma) & \\ +\partial_t V(t,S,\sigma) - rV(t,S,\sigma) & = 0\end{aligned}$$

where $V(t,S,\sigma)$ is the price of the contingent claim, $V(T,S(T)) = f(S(T))$ and $f(S(T))$ is the option payoff

Stochastic volatility models

- The market price of the spot risk $W_1(t)$ is:

$$\lambda_S(t) = \frac{\mu(t) - b(t)}{\sigma(t)}$$

- We introduce the function $\zeta'(y)$:

$$\zeta'(\sigma(t)) = \zeta(\sigma(t)) - \lambda_\sigma(t)\,\xi(\sigma(t))$$

- The PDE becomes:

$$\begin{aligned}\frac{1}{2}\sigma^2 S^2 \partial_S^2 V(t,S,\sigma) + \rho\sigma S\xi(\sigma)\,\partial_{S,\sigma}^2 V(t,S,\sigma) + \frac{1}{2}\xi^2(\sigma)\,\partial_\sigma^2 V(t,S,\sigma) \\ + bS\partial_S V(t,S,\sigma) + \zeta'(\sigma)\,\partial_\sigma V(t,S,\sigma) + \partial_t V(t,S,\sigma) - rV(t,S,\sigma) &= 0\end{aligned}$$

Stochastic volatility models

- Using the Girsanov theorem, we deduce that the risk-neutral dynamics is:

$$\left\{ \begin{array}{l} \mathrm{d}S(t) = b(t)\, S(t)\, \mathrm{d}t + \sigma(t)\, S(t)\, \mathrm{d}W_1^{\mathbb{Q}}(t) \\ \mathrm{d}\sigma(t) = \zeta'(\sigma(t))\, \mathrm{d}t + \xi(\sigma(t))\, \mathrm{d}W_2^{\mathbb{Q}}(t) \end{array} \right.$$

- The martingale solution is then equal to:

$$V_0 = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r(t)\,\mathrm{d}t} f(S(T)) \middle| \mathcal{F}_0 \right]$$

Hedging portfolio

In the case of the Black-Scholes model, delta and vega sensitivities are equal to:

$$\boldsymbol{\Delta}_{\mathrm{BS}} = \frac{\partial \, V_{\mathrm{BS}} \left(S_0, K, \Sigma, T \right)}{\partial \, S_0}$$

and:

$$\boldsymbol{\upsilon}_{\mathrm{BS}} = \frac{\partial \, V_{\mathrm{BS}} \left(S_0, K, \Sigma, T \right)}{\partial \, \Sigma}$$

Hedging portfolio

In the case of the stochastic volatility model, we have:

$$\boldsymbol{\Delta}_{\mathrm{SV}} = \frac{\partial\, V_{\mathrm{SV}}\left(S_0, K, \sigma_0, T\right)}{\partial\, S_0}$$

If we assume that $V_{\mathrm{SV}}\left(S_0, K, \sigma_0, T\right) = V_{\mathrm{BS}}\left(S_0, K, \Sigma_{\mathrm{SV}}\left(T, S_0\right), T\right)$, we obtain:

$$\begin{aligned}\boldsymbol{\Delta}_{\mathrm{SV}} &= \frac{\partial\, V_{\mathrm{BS}}\left(S_0, K, \Sigma_{\mathrm{SV}}, T\right)}{\partial\, S_0} + \frac{\partial\, V_{\mathrm{BS}}\left(S_0, K, \Sigma_{\mathrm{SV}}, T\right)}{\partial\, \Sigma_{\mathrm{SV}}} \cdot \frac{\partial\, \Sigma_{\mathrm{SV}}\left(T, S_0\right)}{\partial\, S_0} \\ &= \boldsymbol{\Delta}_{\mathrm{BS}} + \boldsymbol{\upsilon}_{\mathrm{BS}} \cdot \frac{\partial\, \Sigma_{\mathrm{SV}}\left(T, S_0\right)}{\partial\, S_0}\end{aligned}$$

$\Rightarrow$ The delta of the SV model depends on the BS vega

Generally, we have $\partial_{S_0}\Sigma_{\mathrm{SV}}\left(T, S_0\right) \geq 0$ implying that $\boldsymbol{\Delta}_{\mathrm{SV}} \geq \boldsymbol{\Delta}_{\mathrm{BS}}$

Hedging portfolio

- The natural hedging portfolio should consist in two long/short exposures since we have two risk factors $S(t)$ and $\sigma(t)$
- We can define the vega sensitivity as follows:

$$\boldsymbol{\upsilon}_{\mathrm{SV}} = \frac{\partial\, V_{\mathrm{SV}}\left(S_0, K, \sigma_0, T\right)}{\partial\, \sigma_0}$$

  However, this definition has no interest since the stochastic volatility $\sigma(t)$ cannot be directly or even indirectly trade
- This is why most of traders prefer to use a BS vega:

$$\boldsymbol{\upsilon}_{\mathrm{SV}} = \frac{\partial\, V_{\mathrm{BS}}\left(S_0, K, \Sigma_{\mathrm{SV}}\left(T, S_0\right), T\right)}{\partial\, \Sigma_{\mathrm{SV}}}$$

  The vega is calculated with respect to the implied volatility $\Sigma_{\mathrm{SV}}(T, S_0)$ deduced from the stochastic volatility model

Heston model

We have:

$$\left\{\begin{array}{l}\mathrm{d}S(t)=\mu S(t)\,\mathrm{d}t+\sqrt{v(t)}S(t)\,\mathrm{d}W_1(t)\\ \mathrm{d}v(t)=\kappa(\theta-v(t))\,\mathrm{d}t+\xi\sqrt{v(t)}\,\mathrm{d}W_2(t)\end{array}\right.$$

where $S(0)=S_0$, $v(0)=v_0$ and $W(t)=(W_1(t),W_2(t))$ is a two-dimensional Wiener process with $\mathbb{E}[W_1(t)\,W_2(t)]=\rho\,t$

Heston model

- The stochastic variance $v(t)$ follows a CIR process: θ is the long-run variance, κ is the mean-reverting parameter and ξ is the volatility of the variance (also called the vovol parameter)
- We have $\sigma(t) = \sqrt{v(t)}$ and:

$$\mathrm{d}\sigma(t) = \left(\left(\frac{\kappa\theta}{2} - \frac{\xi^2}{8}\right)\frac{1}{\sigma(t)} - \frac{1}{2}\kappa\sigma(t)\right)\mathrm{d}t + \frac{1}{2}\xi\,\mathrm{d}W_2(t)$$

  The stochastic volatility is then an Ornstein-Uhlenbeck process if we impose $\theta = \xi^2/(4\kappa)$

Heston model

The PDE is:

$$\begin{aligned} \frac{1}{2} v S^2 \partial_S^2 V + \rho \xi v S \partial_{S,v}^2 V + \frac{1}{2} \xi^2 v \partial_v^2 V & \\ + bS \partial_S V + \left(\kappa \left(\theta - v\left(t\right)\right) - \lambda v\right) \partial_v V + \partial_t V - rV &= 0 \end{aligned}$$

The risk-neutral dynamics is:

$$\left\{ \begin{array}{l} \mathrm{d}S\left(t\right) = bS\left(t\right)\,\mathrm{d}t + \sqrt{v\left(t\right)} S\left(t\right)\,\mathrm{d}W_1^{\mathbb{Q}}\left(t\right) \\ \mathrm{d}v\left(t\right) = \left(\kappa\left(\theta - v\left(t\right)\right) - \lambda v\left(t\right)\right)\mathrm{d}t + \xi \sqrt{v\left(t\right)}\,\mathrm{d}W_2^{\mathbb{Q}}\left(t\right) \end{array} \right.$$

Heston model

The closed-form solutions of European call and put options are:

$$\begin{aligned}\mathcal{C}_0 &= S_0 e^{(b-r)T} P_1 - K e^{-rT} P_2 \\ \mathcal{P}_0 &= S_0 e^{(b-r)T} \left(P_1 - 1\right) - K e^{-rT} \left(P_2 - 1\right)\end{aligned}$$

where the probabilities P_1 and P_2 satisfy:

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left(\frac{e^{-i\phi \ln K} \varphi_j \left(S_0, v_0, T, \phi\right)}{i\phi}\right) \mathrm{d}\phi$$

$$\varphi_j \left(S_0, v_0, T, \phi\right) = \exp\left(C_j \left(T, \phi\right) + D_j \left(T, \phi\right) v_0 + i\phi \ln S_0\right)$$

$$C_j \left(T, \phi\right) = ib\phi T + \frac{a_j}{\xi^2} \left(\left(b_j - i\rho\xi\phi + d_j\right) T - 2 \ln\left(\frac{1 - g_j e^{d_j T}}{1 - g_j}\right)\right)$$

$$D_j \left(T, \phi\right) = \frac{b_j - i\rho\xi\phi + d_j}{\xi^2} \left(\frac{1 - e^{d_j T}}{1 - g_j e^{d_j T}}\right)$$

$$g_j = \frac{b_j - i\rho\xi\phi + d_j}{b_j - i\rho\xi\phi - d_j}$$

$$d_j = \sqrt{\left(i\rho\xi\phi - b_j\right)^2 - \xi^2 \left(2iu_j\phi - \phi^2\right)}$$

where $a_1 = a_2 = \kappa\theta$, $b_1 = \kappa + \lambda - \rho\xi$, $b_2 = \kappa + \lambda$, $u_1 = 1/2$ and $u_2 = -1/2$

Heston model

Example # 9

The parameters are equal to $S_0 = 100$, $b = r = 5\%$, $v_0 = \theta = 4\%$, $\kappa = 0.5$, $\xi = 0.9$ and $\lambda = 0$. We consider the pricing of the European call option, whose maturity is three months.

Heston model

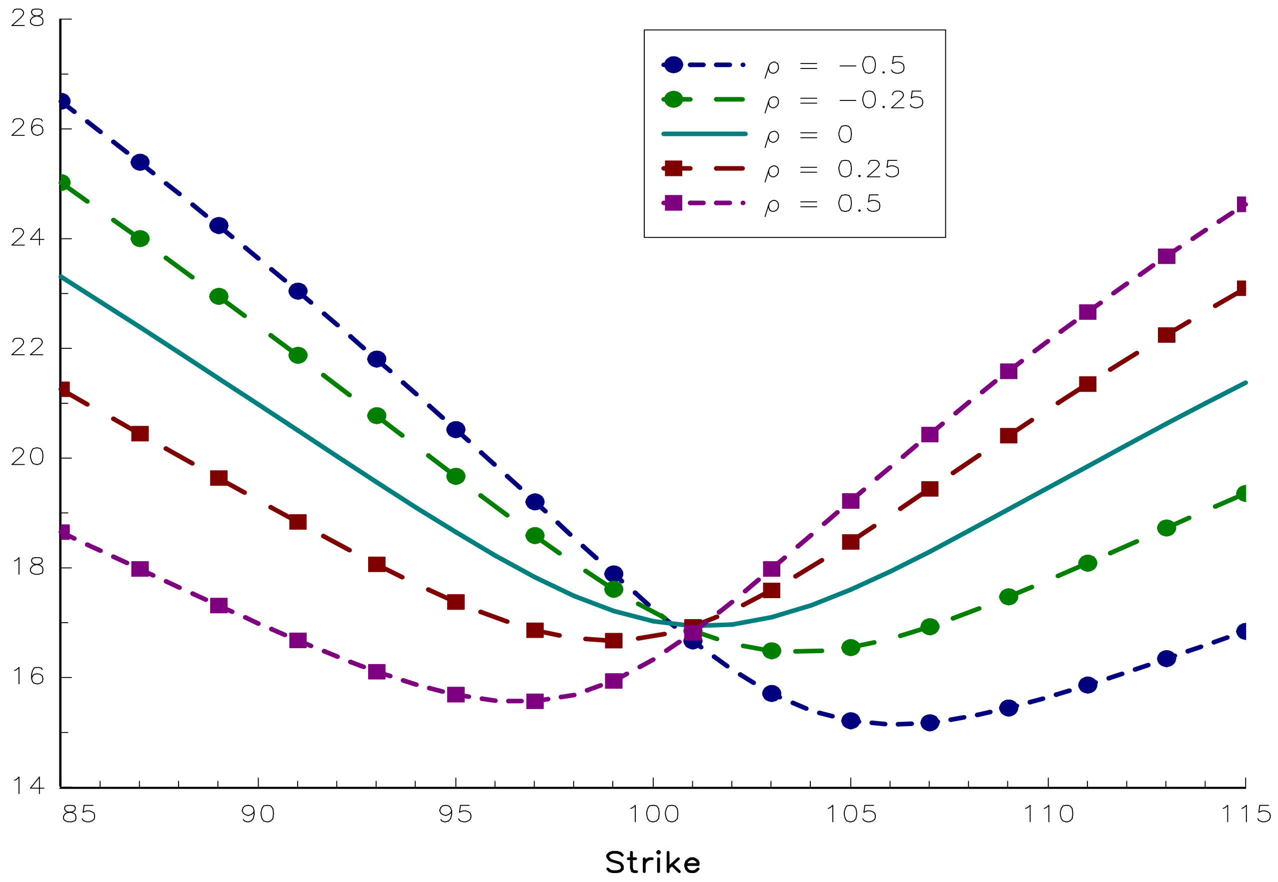


Figure: Implied volatility of the Heston model (in %)

Heston model

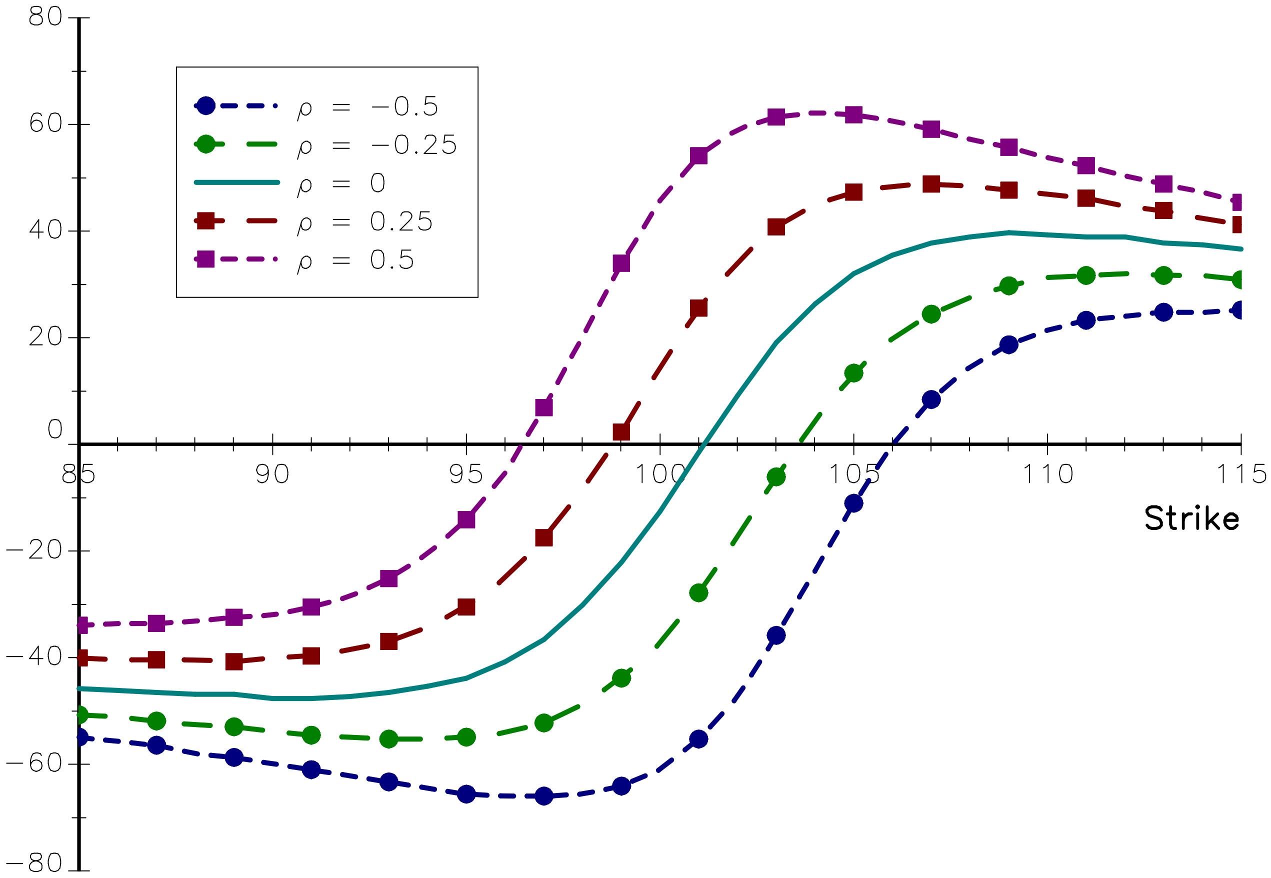


Figure: Skew $\omega(T,K) = \dfrac{\partial\,\Sigma(T,K)}{\partial\,K}$ of the Heston model (in bps)

SABR model

The dynamics of the forward rate $F(t)$ is given by:

$$\begin{cases} \mathrm{d}F(t) = \alpha(t) F(t)^{\beta} \, \mathrm{d}W_1^{\mathbb{Q}}(t) \\ \mathrm{d}\alpha(t) = \nu\alpha(t) \, \mathrm{d}W_2^{\mathbb{Q}}(t) \end{cases}$$

where $\mathbb{E}\left[W_1^{\mathbb{Q}}(t) \, W_2^{\mathbb{Q}}(t)\right] = \rho \, t$

The model has 4 parameters:

1. α the current value of $\alpha(t)$
2. β the exponent of the forward rate
3. ν the log-normal volatility of $\alpha(t)$
4. ρ the correlation between the two Brownian motions

SABR model

The implied Black volatility is:

$$\Sigma_B(T,K) = \frac{\alpha}{(F_0K)^{(1-\beta)/2}\left(1+\frac{(1-\beta)^2}{24}\ln^2\frac{F_0}{K}+\frac{(1-\beta)^4}{1920}\ln^4\frac{F_0}{K}\right)}\left(\frac{z}{\chi(z)}\right)\cdot$$
$$\left(1+\left(\frac{(1-\beta)^2\alpha^2}{24(F_0K)^{1-\beta}}+\frac{\rho\alpha\nu\beta}{4(F_0K)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right)T\right)$$

where:

$$z = \nu\alpha^{-1}(F_0K)^{(1-\beta)/2}\ln\frac{F_0}{K}$$

and:

$$\chi(z) = \ln\left(\sqrt{1-2\rho z+z^2}+z-\rho\right)-\ln(1-\rho)$$

SABR model

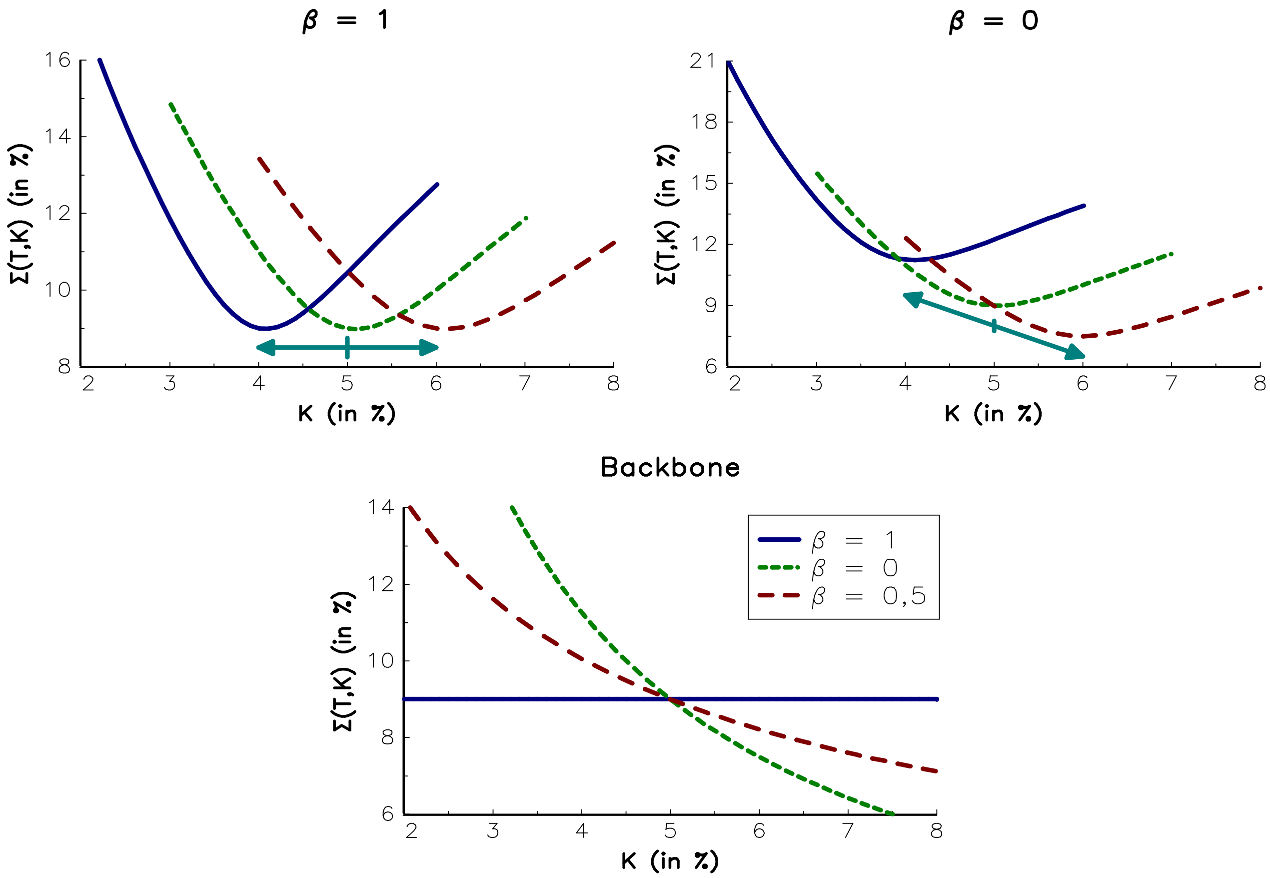


Figure: Impact of the parameter β

SABR model

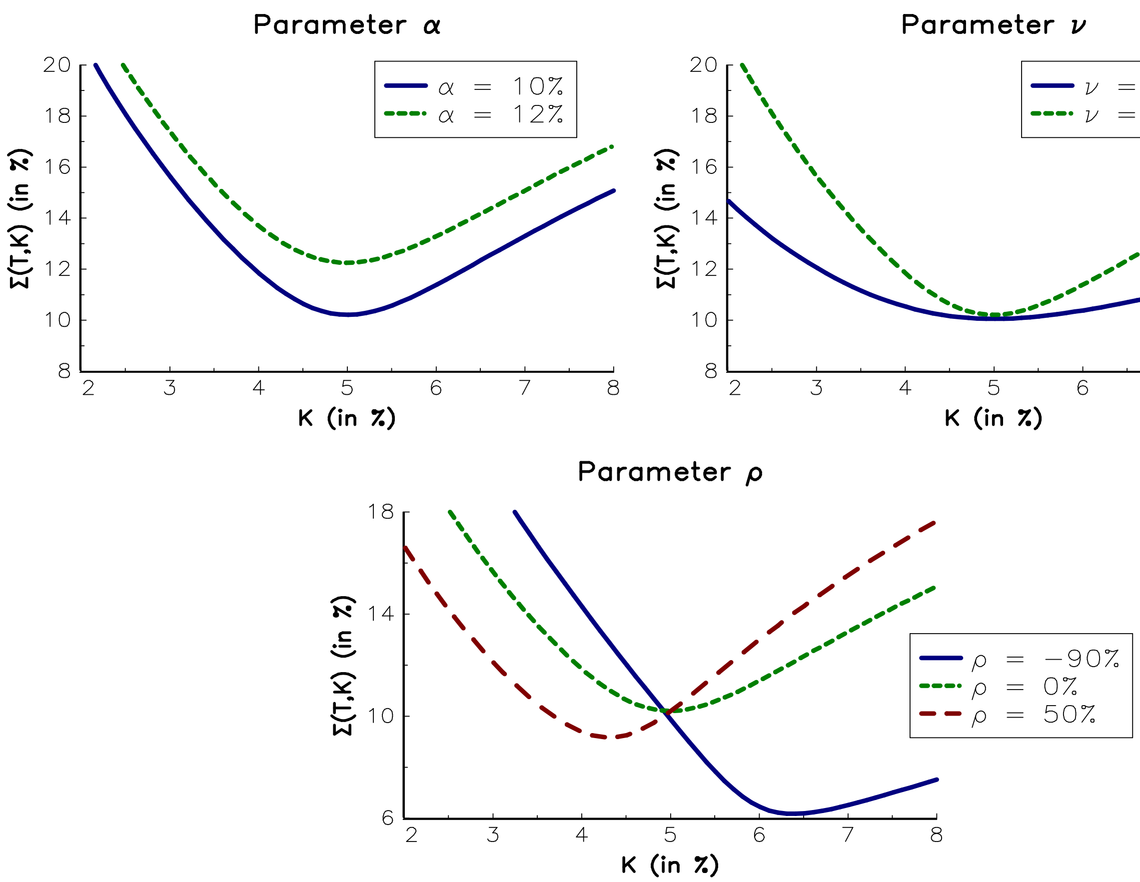


Figure: Impact of the parameters α, ν and ρ

SABR model

The parameters β and ρ impact the slope of the smile in a similar way. Then, they cannot be jointly identifiable. For example, let us consider the following smile when F_0 is equal to 5%: $\Sigma_B(1, 3\%) = 13\%$, $\Sigma_B(1, 4\%) = 10\%$, $\Sigma_B(1, 5\%) = 9\%$ and $\Sigma_B(1, 7\%) = 10\%$. If we calibrate this smile for different values of β, we obtain the following solutions:

β	α	ν	ρ
0.0	0.0044	0.3203	0.2106
0.5	0.0197	0.3244	0.0248
1.0	0.0878	0.3388	-0.1552

SABR model

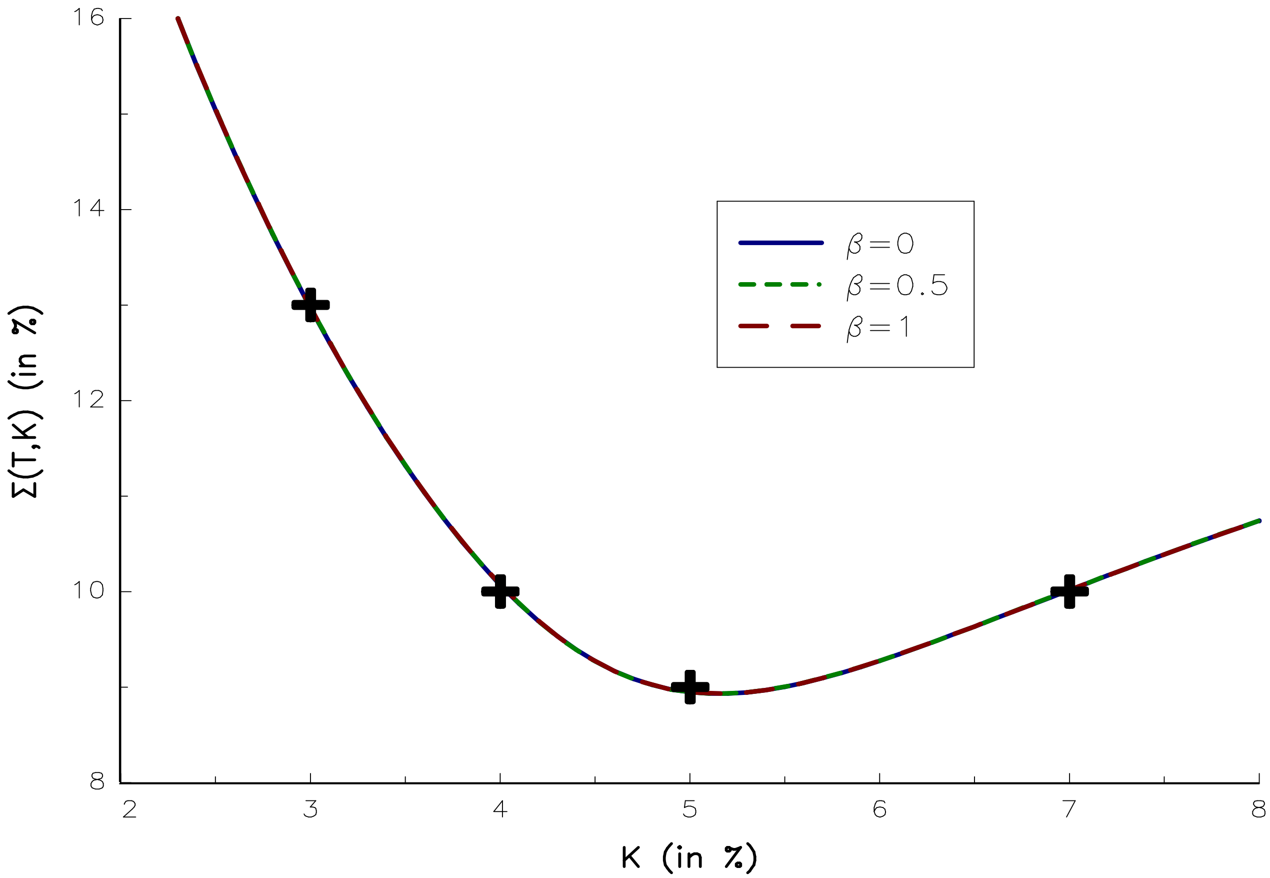


Figure: Implied volatility for different parameter sets (β, ρ)

SABR model

There are two approaches for estimating β:

1. β can be chosen from prior beliefs ($\beta = 0$ for the normal model, $\beta = 0.5$ for the CIR model and $\beta = 1$ for the log-normal model)
2. β can be statistically estimated by considering the dynamics of the forward rate. Indeed, we have

$$\Sigma_t \left(T, F_t \right) \simeq \frac{\alpha}{F_t^{1-\beta}}$$

We can consider the following linear regression:

$$\ln \Sigma_t \left(T, F_t \right) = \ln \alpha + \left(\beta - 1 \right) \ln F_t + u_t$$

SABR model

Table: Calibration of the parameter β in the SABR model

Rate	Level		Difference		Empirical quantile of $\hat{\beta}_{t,t+h}$				
	$\hat{\beta}$	R_c^2	$\hat{\beta}$	R_c^2	10%	25%	50%	75%	90%
1y1y	−0.06	0.91	0.59	0.15	−2.01	−0.14	0.71	1.00	2.17
1y5y	−0.29	0.87	0.32	0.27	−1.80	−0.28	0.73	1.11	2.76
1y10y	−0.37	0.80	0.34	0.22	−2.04	−0.23	0.71	1.11	2.69
5y1y	0.42	0.29	0.35	0.22	−1.58	−0.31	0.71	1.00	2.38
5y5y	−0.01	0.73	0.23	0.28	−2.12	−0.36	0.61	1.00	2.52
5y10y	−0.10	0.69	0.27	0.23	−1.99	−0.30	0.70	1.05	2.58
10y1y	0.96	0.00	0.28	0.20	−1.88	−0.20	0.80	1.07	2.43
10y5y	−0.10	0.65	0.28	0.20	−2.02	−0.29	0.73	1.02	2.76
10y10y	−0.47	0.73	0.27	0.20	−1.71	−0.24	0.85	1.07	2.93

SABR model

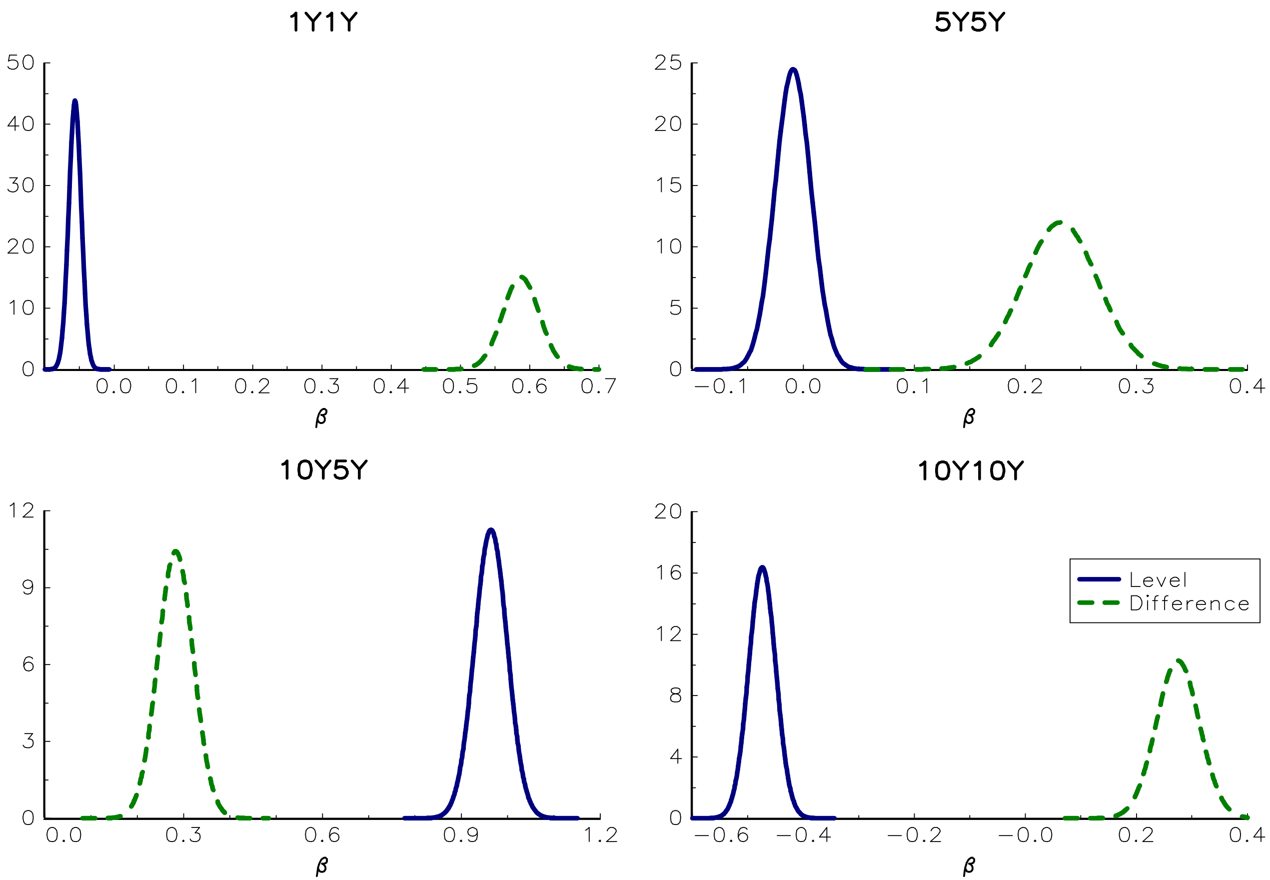


Figure: Probability density function of the estimate $\hat{\beta}$ (SABR model)

SABR model

- Once we have set the value of β, we estimate the parameters (α, ν, ρ) by fitting the observed implied volatilities
- However, we have seen that α is highly related to the ATM volatility. Indeed, we have:

$$\Sigma_B(T, F_0) = \frac{\alpha}{F_0^{1-\beta}} \left(1 + \left(\frac{(1-\beta)^2 \alpha^2}{24 F_0^{2-2\beta}} + \frac{\rho \alpha \nu \beta}{4 F_0^{1-\beta}} + \frac{2 - 3\rho^2}{24} \nu^2\right) T\right)$$

- We deduce that:

$$\alpha^3 \left(\frac{(1-\beta)^2 T}{24 F_0^{2-2\beta}}\right) + \alpha^2 \left(\frac{\rho \nu \beta T}{4 F_0^{1-\beta}}\right) + \alpha \left(1 + \frac{2 - 3\rho^2}{24} \nu^2 T\right) - \Sigma_B(T, F_0) F_0^{1-\beta} = 0$$

- Let $\alpha = g_\alpha(\Sigma_B(T, F_0), \nu, \rho)$ be the positive root of the cubic equation. Therefore, imposing that the smile passes through the ATM volatility $\Sigma_B(T, F_0)$ allows to reduce the calibration to two parameters (ν, ρ)

SABR model

Example #10

We consider the following smile:

K (in %)	2.8	3.0	3.5	3.7	4.0	4.5	5.0	7.0
$\Sigma(T, K)$ (in %)	13.2	12.8	12.0	11.6	11.0	10.0	9.0	10.0

The maturity T is equal to one year and the forward rate F_0 is set to 5%

SABR model

If we consider a stochastic log-normal model ($\beta = 1$), we obtain the following results:

Calibration	α (in %)	β	ν	ρ (in %)	RSS	Σ_{ATM} (in %)
#1	9.466	1.00	0.279	-23.70	0.630	9.51
#2	8.944	1.00	0.322	-22.90	1.222	9.00

SABR model

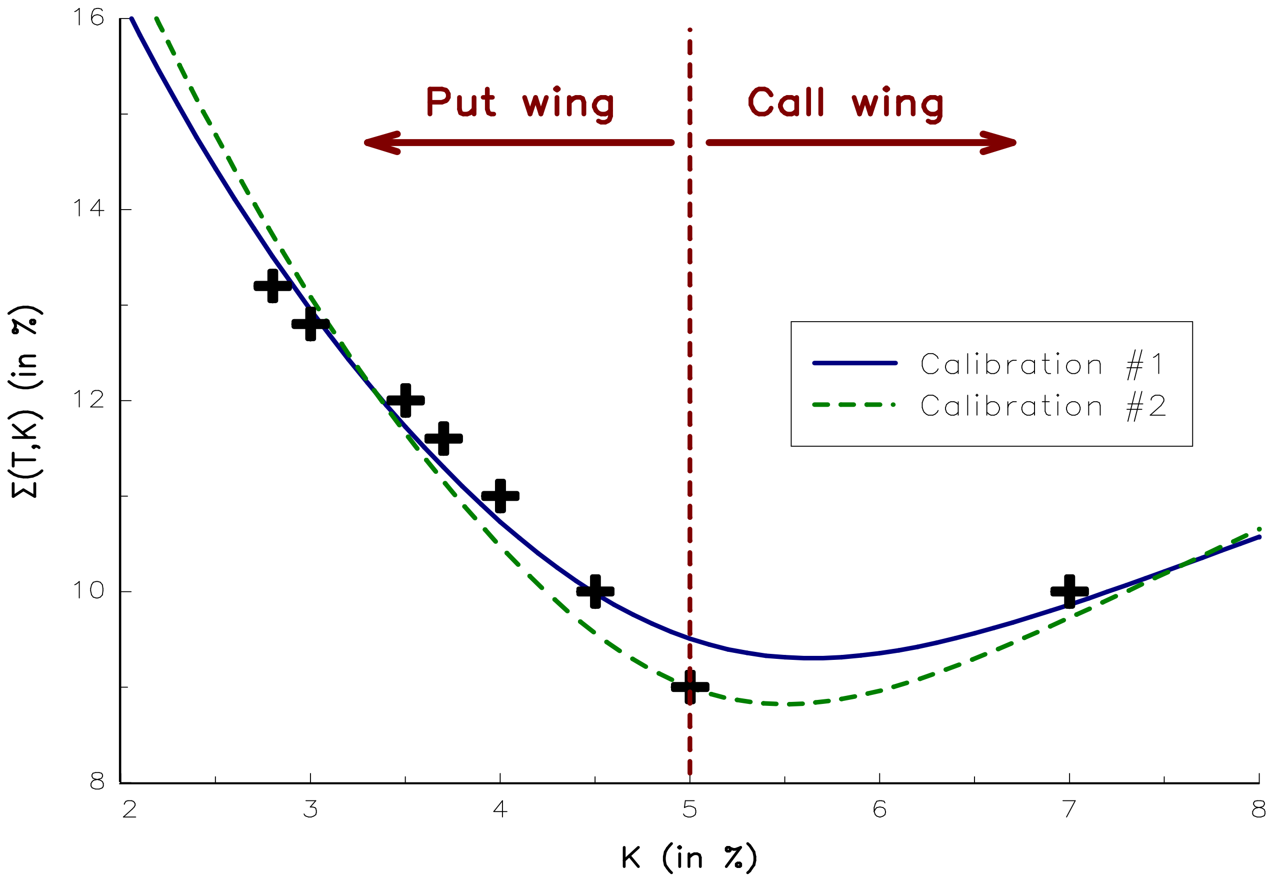


Figure: Calibration of the SABR model

SABR model

- The sensitivities correspond to the following formulas:

$$\boldsymbol{\Delta} = \frac{\partial \mathcal{C}_B}{\partial F_0} + \frac{\partial \mathcal{C}_B}{\partial \Sigma} \cdot \frac{\partial \Sigma_B(T, K)}{\partial F_0}$$

and:

$$\boldsymbol{\upsilon} = \frac{\partial \mathcal{C}_B}{\partial \Sigma} \cdot \frac{\partial \Sigma_B(T, K)}{\partial \alpha}$$

- To obtain these formulas, we apply the chain rule on the Black formula by assuming that the volatility Σ is not constant and depends on F_0 and α

SABR model

We notice that the vega is defined with respect to the parameter α. This approach is little used in practice, because it is difficult to hedge this model parameter. This is why traders prefer to compute the vega with respect to the ATM volatility:

$$\boldsymbol{\upsilon} = \frac{\partial \boldsymbol{\mathcal{C}}_B}{\partial \Sigma} \cdot \frac{\partial \Sigma_B(T, K)}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial \Sigma_{\mathrm{ATM}}}$$

where $\Sigma_{\mathrm{ATM}} = \Sigma_B(T, F_0)$

SABR model

Bartlett (2006) proposes a refinement for computing the delta. Indeed, a shift in F_0 produces a shift in α, because the two processes $F(t)$ and $\alpha(t)$ are correlated. Since we have:

$$\begin{aligned} \mathrm{d}\alpha(t) &= \nu\alpha(t)\,\mathrm{d}W_2^{Q}(t) \\ &= \nu\alpha(t)\left(\rho\,\mathrm{d}W_1^{Q}(t)+\sqrt{1-\rho^2}\,\mathrm{d}W(t)\right) \end{aligned}$$

and:

$$\mathrm{d}W_1^{Q}(t)=\frac{\mathrm{d}F(t)}{\alpha(t)\,F(t)^{\beta}}$$

we deduce that:

$$\mathrm{d}\alpha(t)=\frac{\nu\rho}{F(t)^{\beta}}\,\mathrm{d}F(t)+\nu\alpha(t)\sqrt{1-\rho^2}\,\mathrm{d}W(t)$$

SABR model

The new delta is then:

$$\begin{aligned}
\boldsymbol{\Delta}^{\star} &= \frac{\partial\, \mathcal{C}_B}{\partial\, F_0} + \frac{\partial\, \mathcal{C}_B}{\partial\, \Sigma}\left(\frac{\partial\, \Sigma_B\,(T,K)}{\partial\, F_0} + \frac{\partial\, \Sigma_B\,(T,K)}{\partial\, \alpha}\cdot\frac{\partial\, \alpha}{\partial\, F_0}\right) \\
&= \frac{\partial\, \mathcal{C}_B}{\partial\, F_0} + \frac{\partial\, \mathcal{C}_B}{\partial\, \Sigma}\left(\frac{\partial\, \Sigma_B\,(T,K)}{\partial\, F_0} + \frac{\nu\rho}{F\,(t)^{\beta}}\frac{\partial\, \Sigma_B\,(T,K)}{\partial\, \alpha}\right) \\
&= \boldsymbol{\Delta} + \frac{\nu\rho}{F\,(t)^{\beta}}\boldsymbol{\upsilon}
\end{aligned}$$

$\Rightarrow$ The new delta incorporates a part of the vega risk

Factor models

- Factor models: Vasicek, CIR, HJM, etc.
- Interest rates are linked to some factors $X(t)$, which can be observable or not observable
- The factor is directly the instantaneous interest rate $r(t)$ in Vasicek or CIR models
- Multi-factor models by considering explicit factors (level, slope, convexity, etc.)
- Professional practice based on non-explicit factors

Linear and quadratic Gaussian models

- Let us assume that the instantaneous interest rate $r(t)$ is linked to the factors $X(t)$ under the risk-neutral probability $\mathbb{Q}$ as follows:

$$r(t) = \alpha(t) + \beta(t)^\top X(t) + X(t)^\top \Gamma(t) X(t)$$

where $\alpha(t)$ is a scalar, $\beta(t)$ is a $n \times 1$ vector and $\Gamma(t)$ is a $n \times n$ matrix

- The factors follow an Ornstein-Uhlenbeck process:

$$\mathrm{d}X(t) = (a(t) + B(t) X(t))\,\mathrm{d}t + \Sigma(t)\,\mathrm{d}W^{\mathbb{Q}}(t)$$

where $a(t)$ is a $n \times 1$ vector, $B(t)$ is a $n \times n$ matrix, $\Sigma(t)$ is a $n \times n$ matrix and $W^{\mathbb{Q}}(t)$ is a standard n-dimensional Brownian motion

Linear and quadratic Gaussian models

There exists a family of $\hat{\alpha}(t,T)$, $\hat{\beta}(t,T)$ and $\hat{\Gamma}(t,T)$ such that the price of the zero-coupon bond $B(t,T)$ is given by:

$$B(t,T) = \exp\left(-\hat{\alpha}(t,T) - \hat{\beta}(t,T)^\top X(t) - X(t)^\top \hat{\Gamma}(t,T) X(t)\right)$$

where $\hat{\alpha}(t,T)$, $\hat{\beta}(t,T)$ and $\hat{\Gamma}(t,T)$ solve a system of Riccati equations. If we assume that the matrix $\hat{\Gamma}(t,T)$ is symmetric, we obtain:

$$\begin{aligned}
\partial_t \hat{\alpha}(t,T) &= -\operatorname{tr}\left(\Sigma(t)\Sigma(t)^\top \hat{\Gamma}(t,T)\right) - \hat{\beta}(t,T)^\top a(t) + \\
&\quad \frac{1}{2}\hat{\beta}(t,T)^\top \Sigma(t)\Sigma(t)^\top \hat{\beta}(t,T) - \alpha(t) \\
\partial_t \hat{\beta}(t,T) &= -B(t)^\top \hat{\beta}(t,T) + 2\hat{\Gamma}(t,T)\Sigma(t)\Sigma(t)^\top \hat{\beta}(t,T) - \\
&\quad 2\hat{\Gamma}(t,T) a(t) - \beta(t) \\
\partial_t \hat{\Gamma}(t,T) &= 2\hat{\Gamma}(t,T)\Sigma(t)\Sigma(t)^\top \hat{\Gamma}(t,T) - \\
&\quad 2\hat{\Gamma}(t,T) B(t) - \Gamma(t)
\end{aligned}$$

with the boundary conditions $\hat{\alpha}(T,T) = \hat{\beta}(T,T) = \hat{\Gamma}(T,T) = \mathbf{0}$

Linear and quadratic Gaussian models

The forward interest rate $F(t, T_1, T_2)$ is given by:

$$
\begin{aligned}
F(t, T_1, T_2) &= -\frac{1}{T_2 - T_1} \ln \frac{B(t, T_2)}{B(t, T_1)} \\
&= \frac{\hat{\alpha}(t, T_2) - \hat{\alpha}(t, T_1) + \left(\hat{\beta}(t, T_2) - \hat{\beta}(t, T_2)\right)^\top X(t)}{T_2 - T_1} + \\
&\quad \frac{X(t)^\top \left(\hat{\Gamma}(t, T_2) - \hat{\Gamma}(t, T_1)\right) X(t)}{T_2 - T_1}
\end{aligned}
$$

We deduce that the instantaneous forward rate is equal to:

$$
f(t, T) = \alpha(t, T) + \beta(t, T)^\top X(t) + X(t)^\top \Gamma(t, T) X(t)
$$

where $\alpha(t, T) = \partial_T \hat{\alpha}(t, T)$, $\beta(t, T) = \partial_T \hat{\beta}(t, T)$ and $\Gamma(t, T) = \partial_T \hat{\Gamma}(t, T)$

It follows that $\alpha(t) = \alpha(t, t) = \partial_t \hat{\alpha}(t, t)$, $\beta(t) = \beta(t, t) = \partial_t \hat{\beta}(t, t)$ and $\Gamma(t) = \Gamma(t, t) = \partial_t \hat{\Gamma}(t, t)$

Linear and quadratic Gaussian models

Let $V(t, X)$ be the price of the option, whose payoff is $f(x)$. It satisfies the following PDE:

$$\begin{aligned}\frac{1}{2}\operatorname{trace}\left(\Sigma(t)\,\partial_X^2 V(t,X)\,\Sigma(t)^\top\right) + (a(t) + B(t)X)\,\partial_X V(t,X) + \\ \partial_t V(t,X) - \left(\alpha(t) + \beta(t)^\top X + X^\top \Gamma(t) X\right) V(t,X) &= 0\end{aligned}$$

Once we have specified the functions $\alpha(t)$, $\beta(t)$, $\Gamma(t)$, $a(t)$, $B(t)$ and $\Sigma(t)$, we can then price the option by solving numerically the previous multidimensional PDE with the terminal condition $V(T, X) = f(X)$
Most of the time, the payoff is not specified with respect to the state variables X, but depends on the interest rate $r(t)$. In this case, we use the following transformation:

$$f(r) = f\left(\alpha(T) + \beta(T)^\top X + X^\top \Gamma(T) X\right)$$

Dynamics of risk factors under the forward probability measure

We have:

$$\frac{\mathrm{d}B(t,T)}{B(t,T)} = r(t)\,\mathrm{d}t - \left(2\hat{\Gamma}(t,T)\,X(t) + \hat{\beta}(t,T)\right)^{\top}\Sigma(t)\,\mathrm{d}W^{\mathbb{Q}}(t)$$

We deduce that:

$$W^{\mathbb{Q}^{\star}(T)}(t) = W^{\mathbb{Q}}(t) + \int_0^t \Sigma(s)^{\top}\left(2\hat{\Gamma}(s,T)\,X(s) + \hat{\beta}(s,T)\right)\mathrm{d}s$$

defines a Brownian motion under $\mathbb{Q}^{\star}(T)$

Dynamics of risk factors under the forward probability measure

It follows that:

$$\mathrm{d}X(t) = \left(\tilde{a}(t) + \tilde{B}(t) X(t)\right) \mathrm{d}t + \Sigma(t)\, \mathrm{d}W^{\mathbb{Q}^{\star}(T)}(t)$$

where:

$$\tilde{a}(t) = a(t) - \Sigma(t) \Sigma(t)^{\top} \hat{\beta}(t, T)$$

and:

$$\tilde{B}(t) = B(t) - 2\Sigma(t) \Sigma(t)^{\top} \hat{\Gamma}(t, T)$$

We conclude that $X(t)$ is Gaussian under any forward probability measure $\mathbb{Q}^{\star}(T)$:

$$X(t) \sim \mathcal{N}\left(m(0, t), V(0, t)\right)$$

Dynamics of risk factors under the forward probability measure

El Karoui *et al.* (1992) showed that the conditional mean and variance satisfies the following forward differential equations:

$$\partial_T m(t,T) = a(T) + B(T)\, m(t,T) - 2V(t,T)\,\Gamma(T)\, m(t,T) - V(t,T)\,\beta(T)$$

and:

$$\begin{aligned}\partial_T V(t,T) &= V(t,T)\, B(T)^\top + B(T)\, V(t,T) - 2V(t,T)\,\Gamma(T)\, V(t,T) + \\ &\quad \Sigma(T)\,\Sigma(T)^\top\end{aligned}$$

- If t is equal to zero, the initial conditions are $m(0,0) = X(0) = \mathbf{0}$ and $V(0,0) = \mathbf{0}$
- If $t \neq 0$, we proceed in two steps: first, we calculate numerically the solutions $m(0,t)$ and $V(0,t)$, and second, we initialize the system with $m(t,t) = m(0,t)$ and $V(t,t) = V(0,t)$

Dynamics of risk factors under the forward probability measure

In fact, the previous forward differential equations are not obtained under the traditional forward probability measure $\mathbb{Q}^{\star}(T)$, but under the probability measure $\mathbb{Q}^{\star}(t,T)$ defined by the following Radon-Nykodin derivative:

$$\frac{\mathrm{d}\mathbb{Q}^{\star}(t,T)}{\mathrm{d}\mathbb{P}} = e^{-\int_0^T r(s)\,\mathrm{d}s} e^{\int_t^T f(t,s)\,\mathrm{d}s}$$

The reason is that we would like to price at time t any caplet with maturity T. Therefore, this is the maturity T and not the filtration $\mathcal{F}_t$ that moves

Pricing caplets and swaptions

The formula of the Libor rate $L(t, T_{i-1}, T_i)$ at time t between the dates T_{i-1} and T_i is:

$$L(t, T_{i-1}, T_i) = \frac{1}{T_i - T_{i-1}} \left(\frac{B(t, T_{i-1})}{B(t, T_i)} - 1 \right)$$

It follows that the price of the caplet is given by:

$$\mathrm{Caplet} = B(0, t)\, \mathbb{E}^{\mathbb{Q}^\star(t)} \left[\left(B(t, T_{i-1}) - (1 + (T_i - T_{i-1}) K) B(t, T_i) \right)^+ \right]$$

where $\mathbb{Q}^\star(t)$ is the forward probability measure. We can then calculate the price using two approaches:

1. we can solve the partial differential equation
2. we can calculate the mathematical expectation using numerical integration

Pricing caplets and swaptions

In the first approach, we consider the PDE with the following payoff:

$$f(X) = \max(0, g(X))$$

where:

$$\begin{aligned} g(X) &= \exp\left(-\hat{\alpha}(t, T_{i-1}) - \hat{\beta}(t, T_{i-1})^\top X - X^\top \hat{\Gamma}(t, T_{i-1}) X\right) - \\ & (1 + \delta_{i-1} K) \exp\left(-\hat{\alpha}(t, T_i) - \hat{\beta}(t, T_i)^\top X - X^\top \hat{\Gamma}(t, T_i) X\right) \end{aligned}$$

In the second approach, we have $X(t) \sim \mathcal{N}(m(0,t), V(0,t))$ under the forward probability $\mathbb{Q}^\star(t)$. We deduce that:

$$\text{Caplet}(t, T_{i-1}, T_i) = B(0,t) \int f(x)\, \phi_n(x; m(0,t), V(0,t))\, \mathrm{d}x$$

This integral can be computed numerically using Gauss-Legendre quadrature methods

Calibration and practice of factor models

- The calibration of the model consists in fitting the functions $\alpha(t)$, $\beta(t)$, $\Gamma(t)$, $a(t)$, $B(t)$ and $\Sigma(t)$
- Generally, professionals assume that $a(t) = 0$ and $B(t) = \mathbf{0}$

Calibration and practice of factor models

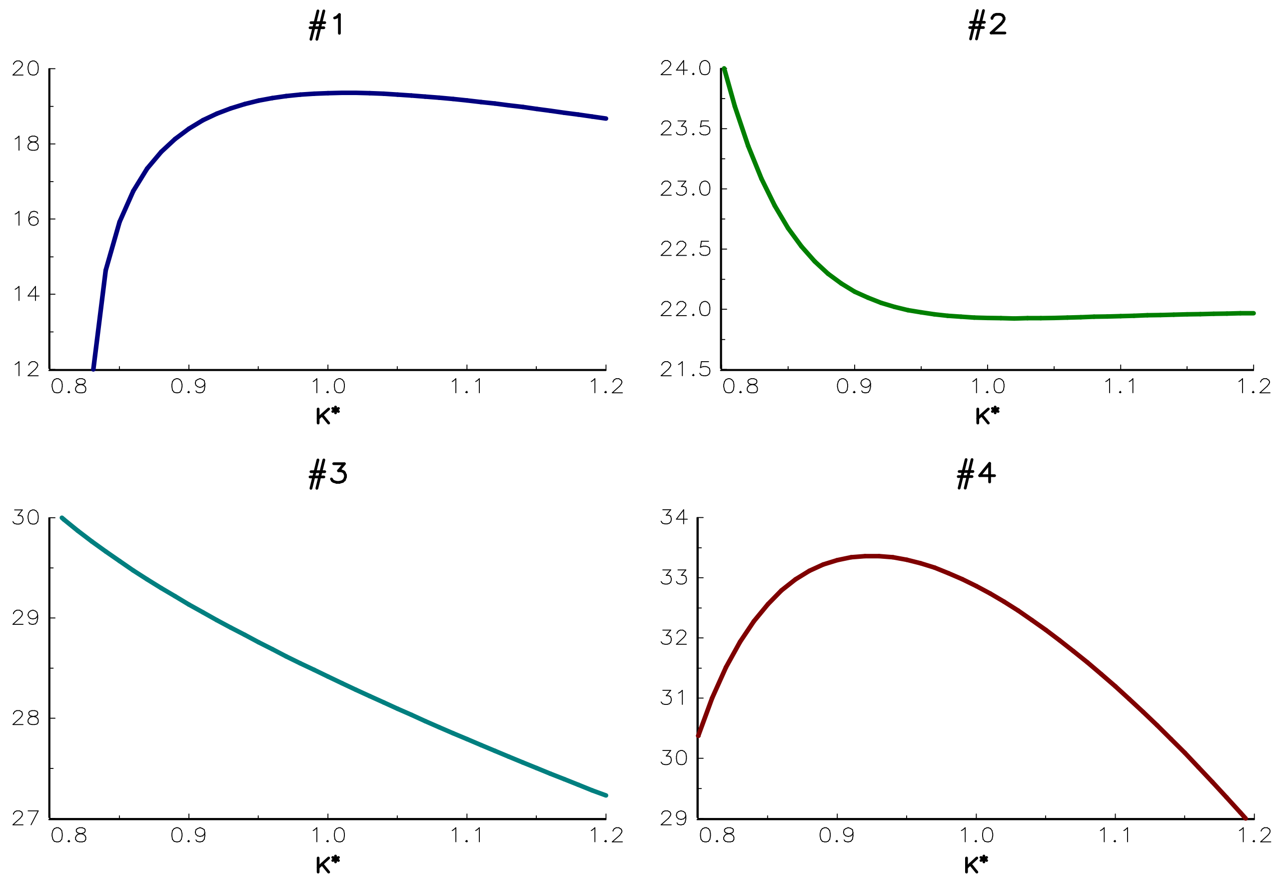


Figure: Volatility smiles generated by the quadratic Gaussian model

Impact of dividends on option prices

- Let us consider that the underlying asset pays a continuous dividend yield d during the life of the option
- The risk-neutral dynamics become:

$$\mathrm{d}S(t) = (r - d) S(t) \, \mathrm{d}t + \sigma S(t) \, \mathrm{d}W(t)$$

- We deduce that the Black-Scholes formula is equal to:

$$\mathcal{C}_0 = S_0 e^{-dT} \Phi(d_1) - K e^{-rT} \Phi(d_2)$$

where:

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}} \left(\ln \frac{S_0}{K} + (r - d) T \right) + \frac{1}{2} \sigma \sqrt{T} \\ d_2 &= d_1 - \sigma \sqrt{T} \end{aligned}$$

Impact of dividends on option prices

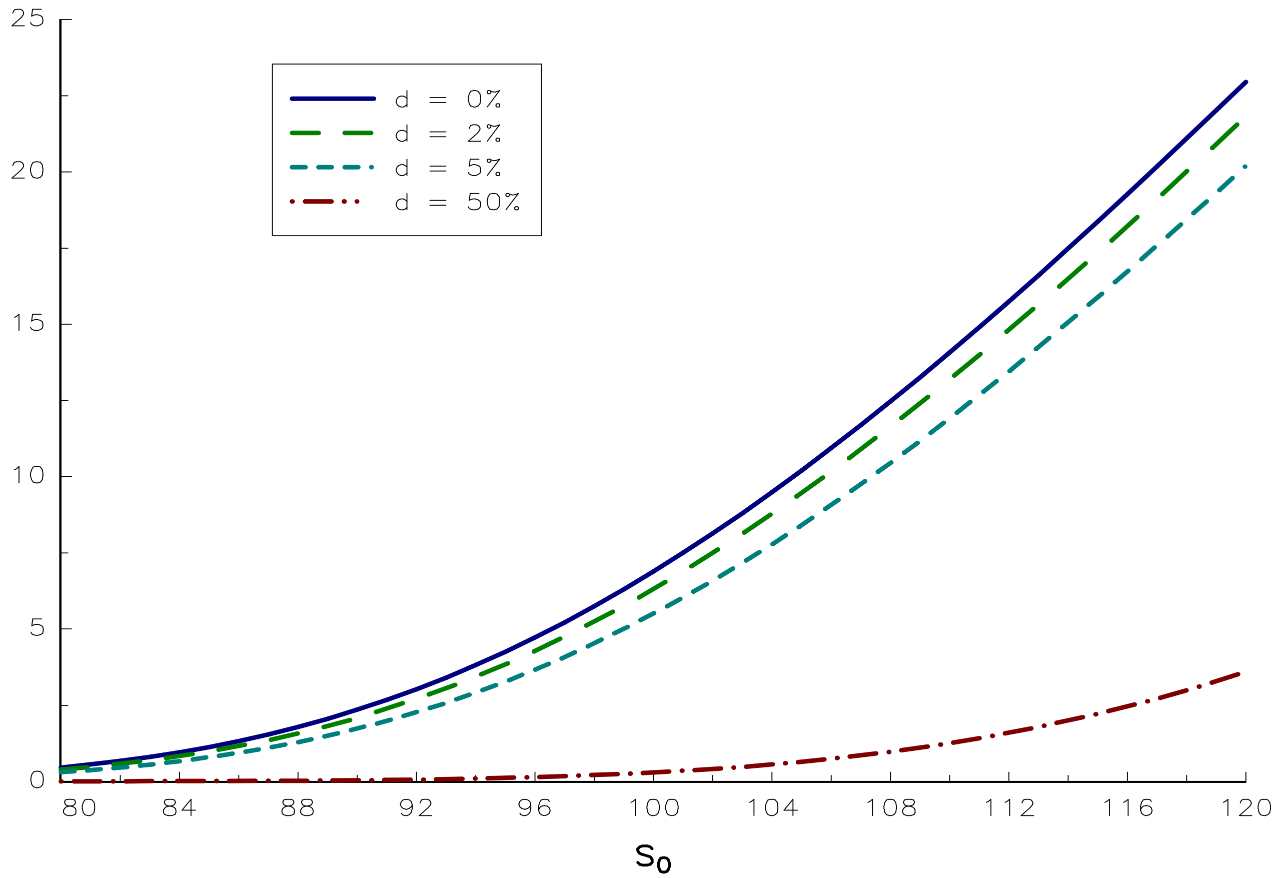


Figure: Impact of dividends on the call option price

Models of discrete dividends

We denote by $S(t)$ the market price and $Y(t)$ an additional process that is assumed to be a geometric Brownian motion:

$$\mathrm{d}Y(t) = rY(t)\,\mathrm{d}t + \sigma Y(t)\,\mathrm{d}W^{\mathbb{Q}}(t)$$

$\Rightarrow$ Three main approaches to take into account discrete dividends

Models of discrete dividends

1. $Y(t)$ is the capital price process excluding the dividends and the market price $S(t)$ is equal to the sum of the capital price and the discounted value of future dividends:

$$S(t) = Y(t) + \sum_{t_k \in [t, T]} D(t_k) e^{-r(t_k - t)}$$

To price European options, we then replace the price S_0 by the adjusted price $Y_0 = S_0 - \sum_{t_k \leq T} D(t_k) e^{-rt_k}$

Models of discrete dividends

2. We define $D(t)$ as the sum of capitalized dividends paid until time t:

$$D(t) = \sum \mathbb{1}\{t_k < t\} \cdot D(t_k) e^{r(t-t_k)}$$

The market price $S(t)$ is equal to the difference between the cum-dividend price $Y(t)$ and the capitalized dividends :

$$S(t) = Y(t) - D(t)$$

We deduce that:

$$\begin{aligned} (S(T) - K)^{+} &= (Y(T) - D(T) - K)^{+} \\ &= (Y(T) - (K + D(T)))^{+} \\ &= (Y(T) - K')^{+} \end{aligned}$$

In the case of European options, we replace the strike K by the adjusted strike $K' = K + \sum_{t_k \leq T} D(t_k) e^{r(T-t_k)}$

Models of discrete dividends

3. The last approach considers the market price process as a discontinuous process:

$$\begin{cases} \mathrm{d}S(t) = rS(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W^{\mathbb{Q}}(t) & \text{if } t_{k-1} < t < t_k \\ S(t) = S\left(t_k^{-}\right) - D(t_k) & \text{if } t = t_k \end{cases}$$

Models of discrete dividends

Example #11

We assume that $S_0 = 100$, $K = 100$, $\sigma = 30\%$, $T = 1$, $r = 5\%$ and $b = 5\%$. A dividend $D(t_1)$ will be paid at time $t_1 = 0.5$

Table: Impact of the dividend on the option price

	Call			Put		
$D(t_1)$	(#1)	(#2)	(#3)	(#1)	(#2)	(#3)
0	14.23	14.23	14.23	9.35	9.35	9.35
3	12.46	12.81	12.69	10.51	10.86	10.64
5	11.34	11.92	11.69	11.34	11.92	11.59
10	8.78	9.93	9.42	13.66	14.80	14.20

The two-asset case

- We consider the example of a basket option on two assets
- Let $S_i(t)$ be the price process of asset i at time t. According to the Black-Scholes model, we have:

$$\begin{cases} \mathrm{d}S_1(t) = b_1 S_1(t)\,\mathrm{d}t + \sigma_1 S_1(t)\,\mathrm{d}W_1^{\mathbb{Q}}(t) \\ \mathrm{d}S_2(t) = b_2 S_2(t)\,\mathrm{d}t + \sigma_2 S_2(t)\,\mathrm{d}W_2^{\mathbb{Q}}(t) \end{cases}$$

  where b_i and σ_i are the cost-of-carry and the volatility of asset i
- Under the risk-neutral probability measure $\mathbb{Q}$, $W_1^{\mathbb{Q}}(t)$ and $W_2^{\mathbb{Q}}(t)$ are two correlated Brownian motions:

$$\mathbb{E}\left[W_1^{\mathbb{Q}}(t)\,W_2^{\mathbb{Q}}(t)\right] = \rho\,t$$

The two-asset case

- The option price associated to the payoff $(\alpha_1 S_1(T) + \alpha_2 S_2(T) - K)^+$ is the solution of the two-dimensional PDE:

$$\begin{aligned}\frac{1}{2}\sigma_1^2 S_1^2 \partial_{S_1}^2 \mathcal{C} + \frac{1}{2}\sigma_2^2 S_2^2 \partial_{S_2}^2 \mathcal{C} + \rho\sigma_1\sigma_2 S_1 S_2 \partial_{S_1,S_2}^2 \mathcal{C} + & \\ b_1 S_1 \partial_{S_1} \mathcal{C} + b_2 S_2 \partial_{S_2} \mathcal{C} + \partial_t \mathcal{C} - r\mathcal{C} &= 0\end{aligned}$$

  with the terminal condition:

$$\mathcal{C}(T, S_1, S_2) = (\alpha_1 S_1 + \alpha_2 S_2 - K)^+$$

- Using the Feynman-Kac representation theorem, we have:

$$\mathcal{C}_0 = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r\,\mathrm{d}t}(\alpha_1 S_1(T) + \alpha_2 S_2(T) - K)^+\right]$$

The two-asset case

- The value $\mathcal{C}_0$ can be calculated using numerical integration
- In some cases, the two-dimensional problem can be reduced to one-dimensional integration. For instance, if $\alpha_1 < 0$, $\alpha_2 > 0$ and $K > 0$, we obtain:

$$\mathcal{C}_0 = \int_{\mathbb{R}} \mathrm{BS}\left(S^{\star}(x), K^{\star}(x), \sigma^{\star}, T, b^{\star}, r\right) \phi(x) \,\mathrm{d}x$$

where:

$$\begin{aligned}
S^{\star}(x) &= \alpha_2 S_2(0) e^{\rho\sigma_2\sqrt{T}x} \\
K^{\star}(x) &= K - \alpha_1 S_1(0) e^{\left(b_1 - \frac{1}{2}\sigma_1^2\right)T + \sigma_1\sqrt{T}x} \\
\sigma^{\star} &= \sigma_2\sqrt{1-\rho^2} \\
b^{\star} &= b_2 - \frac{1}{2}\rho^2\sigma_2^2
\end{aligned}$$

The two-asset case

Example #12

We assume that $S_1(0) = S_2(0) = 100$, $\sigma_1 = \sigma_2 = 20\%$, $b_1 = 10\%$, $b_2 = 0$ and $r = 5\%$. We calculate the price of a basket option, whose maturity T is equal to one year. For the other characteristics (α_1, α_2, K), we consider different set of parameters: $(1, -1, 1)$, $(1, -1, 5)$, $(0.5, 0.5, 100)$, $(0.5, 0.5, 110)$ and $(0.1, 0.1, -5)$

The two-asset case

Table: Impact of the correlation on the basket option price

α_1		1.0	1.0	0.5	0.5	0.1
α_2		−1.0	−1.0	0.5	0.5	0.1
K		1	5	100	110	−5
ρ	−0.90	20.41	18.23	5.39	0.66	24.78
	−0.75	19.81	17.62	6.06	1.35	24.78
	−0.50	18.76	16.55	6.97	2.31	24.78
	−0.25	17.61	15.37	7.73	3.12	24.78
	0.00	16.35	14.08	8.39	3.83	24.78
	0.25	14.94	12.61	8.99	4.46	24.78
	0.50	13.30	10.88	9.54	5.05	24.78
	0.75	11.29	8.66	10.05	5.59	24.78
	0.90	9.78	6.81	10.34	5.90	24.78

Cega sensitivity

Table: Relationship between the basket option price and the correlation parameter ρ

Option type	Payoff	Increasing	Decreasing
Spread	$(S_2 - S_1 - K)^+$		✓
Basket	$(\alpha_1 S_1 + \alpha_2 S_2 - K)^+$	$\alpha_1\alpha_2 > 0$	$\alpha_1\alpha_2 < 0$
Max	$(\max(S_1, S_2) - K)^+$		✓
Min	$(\min(S_1, S_2) - K)^+$	✓	
Best-of call/call	$\max\left((S_1 - K_1)^+, (S_2 - K_2)^+\right)$		✓
Best-of put/put	$\max\left((K_1 - S_1)^+, (K_2 - S_2)^+\right)$		✓
Worst-of call/call	$\min\left((S_1 - K_1)^+, (S_2 - K_2)^+\right)$	✓	
Option!Worst-of Worst-of put/put	$\min\left((K_1 - S_1)^+, (K_2 - S_2)^+\right)$	✓	

Cega sensitivity

The sensitivity of the option price with respect to the correlation parameter ρ is called the cega:

$$\boldsymbol{c} = \frac{\partial \, \mathcal{C}_0}{\partial \rho}$$

The previous analysis leads us to define the lower and upper bounds of the option price when the cega is either positive or negative:

$$\mathcal{C}_0 \in \left\{ \begin{array}{ll} \left[\mathcal{C}_0\left(\rho^-\right), \mathcal{C}_0\left(\rho^+\right)\right] & \text{if } \boldsymbol{c} \geq 0 \\ \left[\mathcal{C}_0\left(\rho^+\right), \mathcal{C}_0\left(\rho^-\right)\right] & \text{if } \boldsymbol{c} \leq 0 \end{array} \right.$$

We can define the conservative price by taking the maximum between $\mathcal{C}_0\left(\rho^-\right)$ and $\mathcal{C}_0\left(\rho^+\right)$

Cega sensitivity

In the case where $\rho^- = -1$ and $\rho^+ = 1$, the bounds satisfy the one-dimensional PDE:

$$\left\{ \begin{array}{l} \frac{1}{2}\sigma_1^2 S^2 \partial_S^2 \mathcal{C}(t,S) + b_1 S \partial_S \mathcal{C}(t,S) + \partial_t \mathcal{C}(t,S) - r\mathcal{C}(t,S) = 0 \\ \mathcal{C}(T,S) = f(S, g(S)) \end{array} \right.$$

where:

$$g(S) = S_2(0) \left(\frac{S}{S_1(0)} \right)^{\pm \sigma_2/\sigma_1} \exp\left(\left(b_2 - \frac{1}{2}\sigma_2^2 \pm \left(\frac{1}{2}\sigma_1\sigma_2 - \frac{\sigma_2}{\sigma_1} b_1 \right) \right) T \right)$$

The implied correlation

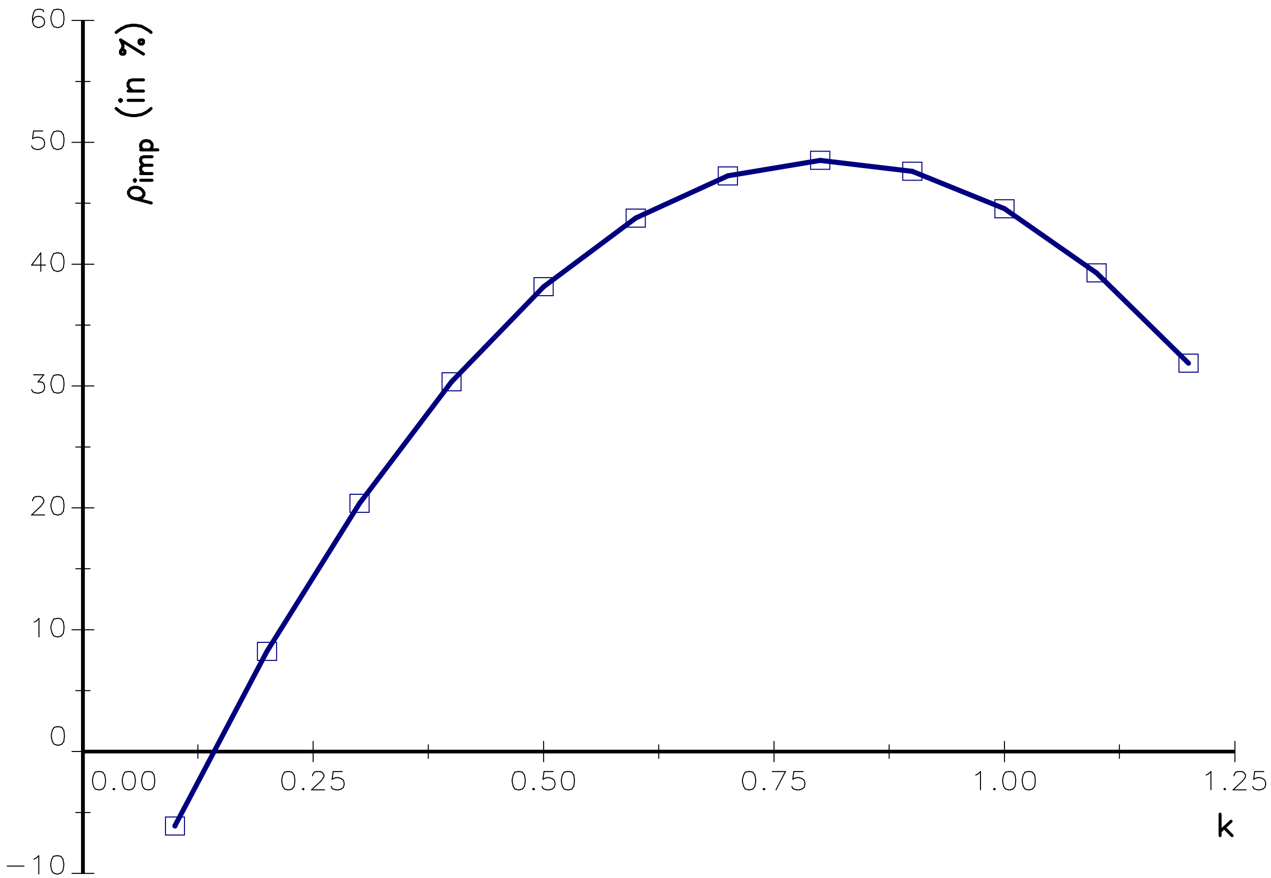


Figure: Correlation smile

Riding on the smiles

- In practice, the two volatilities are unknown and must be deduced from the volatility smiles $\Sigma_1(K_1, T)$ and $\Sigma_2(K_2, T)$ of the two assets
- The difficulty is then to find the corresponding strikes K_1 and K_2

Riding on the smiles

- In the case of the general payoff $(\alpha_1 S_1(T) + \alpha_2 S_2(T) - K)^+$, we have:

$$\begin{cases} (\alpha_1 = 1, \alpha_2 = 0, K \geq 0) \Rightarrow K_1 = K \\ (\alpha_1 = -1, \alpha_2 = 0, K \leq 0) \Rightarrow K_1 = -K \end{cases}$$

and:

$$\begin{cases} (\alpha_1 = 0, \alpha_2 = 1, K \geq 0) \Rightarrow K_2 = K \\ (\alpha_1 = 0, \alpha_2 = -1, K \leq 0) \Rightarrow K_2 = -K \end{cases}$$

- The payoff of the spread option can be written as follows:

$$\begin{aligned} (S_1(T) - S_2(T) - K)^+ &= ((S_1(T) - K_1) + (K_2 - S_2(T)))^+ \\ &\leq \underbrace{(S_1(T) - K_1)^+}_{\text{Call}} + \underbrace{(K_2 - S_2(T))^+}_{\text{Put}} \end{aligned}$$

where $K_1 = K_2 + K$

- Therefore, the price of the spread option can be bounded above by a call price on S_1 plus a put price on S_2
- However, the implicit strikes can take different values

Riding on the smiles

Let us assume that $S_1(0) = S_2(0) = 100$ and $K = 4$. Below, we give five pairs (K_1, K_2) and the associated implied volatilities $(\Sigma_1(K_1, T), \Sigma_2(K_2, T))$:

Pair	#1	#2	#3	#4	#5
K_1	104	103	102	101	100
K_2	100	99	98	97	96
$\Sigma_1(K_1, T)$	16%	17%	18%	19%	20%
$\Sigma_2(K_2, T)$	20%	22%	24%	26%	28%
$\mathcal{C}_0$	10.77	11.37	11.99	12.61	13.24

The multi-asset case

How to define a conservative price?

- In the multivariate case, the PDE becomes:

$$\frac{1}{2}\sum_{i=1}^{n}\sigma_i^2 S_i^2 \partial^2_{S_i}\mathcal{C} + \sum_{i<j}^{n}\rho_{i,j}\sigma_i\sigma_j S_i S_j \partial^2_{S_i,S_j}\mathcal{C} + \sum_{i=1}^{n} b_i S_i \partial_i \mathcal{C} + \partial_t \mathcal{C} - r\mathcal{C} = 0$$

with the terminal value:

$$\mathcal{C}\left(T, S_1, \ldots, S_n\right) = f\left(S_1\left(T\right), \ldots, S_n\left(T\right)\right)$$

- Here, $\rho_{i,j}$ is the correlation between the Brownian motions of S_i and S_j
- Most of the time, the trader uses the same value ρ for all asset correlations $\rho_{i,j}$

The multi-asset case

How to define a conservative price?

- We can show that the price is increasing (resp. decreasing) with respect to ρ if $\sum_{i<j}^{n} \sigma_i \sigma_j \partial^2_{S_i,S_j} f$ is a positive (resp. negative) measure
- Let us consider the payoff function $f(S_1, S_2, S_3) = (S_1 + S_2 - S_3 - K)^+$, we have:

$$\sum_{i<j}^{n} \sigma_i \sigma_j \partial^2_{S_i,S_j} f = (\sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3) \cdot \mathbb{1}\{S_1 + S_2 - S_3 - K = 0\}$$

- If $\sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3 > 0$, the price increases with respect to ρ, and if $\sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3 < 0$, the price decreases with respect to ρ

The multi-asset case

Issues with constant correlation matrices

- We consider a basket of n stocks
- The basket volatility is given by:

$$\sigma_B = \sqrt{\sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2 \sum_{i>j}^{n} \rho_{i,j} w_i w_j \sigma_i \sigma_j}$$

where w_i is the weight of asset i in the basket, σ_i the volatility of asset i and $\rho_{i,j}$ the correlation between asset i and asset j

The multi-asset case

Issues with constant correlation matrices

- The implied correlation ρ_{imp} of the basket is defined as the root of the following equation:

$$\sigma_B^2 - \sum_{i=1}^{n} w_i^2 \sigma_i^2 - 2\rho_{\mathrm{imp}} \sum_{i>j}^{n} w_i w_j \sigma_i \sigma_j = 0$$

- We deduce that:

$$\rho_{\mathrm{imp}} = \frac{\sigma_B^2 - \sum_{i=1}^{n} w_i^2 \sigma_i^2}{2 \sum_{i>j}^{n} w_i w_j \sigma_i \sigma_j}$$

- Another expression of the implied correlation is:

$$\rho_{\mathrm{imp}} = \frac{\sigma_B^2 - \sum_{i=1}^{n} w_i^2 \sigma_i^2}{\left(\sum_{i=1}^{n} w_i \sigma_i\right)^2 - \sum_{i=1}^{n} w_i^2 \sigma_i^2}$$

The multi-asset case

Issues with constant correlation matrices

We consider the following payoff:

$$\left(S_1(T) - S_2(T) + S_3(T) - S_4(T) - K\right)_+ \cdot \mathbb{1}\left\{S_5(T) > L\right\}$$

We calculate the option price of maturity 3 months using the Black-Scholes model. We assume that $S_i(0) = 100$ and $\Sigma_i = 20\%$ for the five underlying assets, the strike K is equal to 5, the barrier L is equal to 105, and the interest rate r is set to 5%.

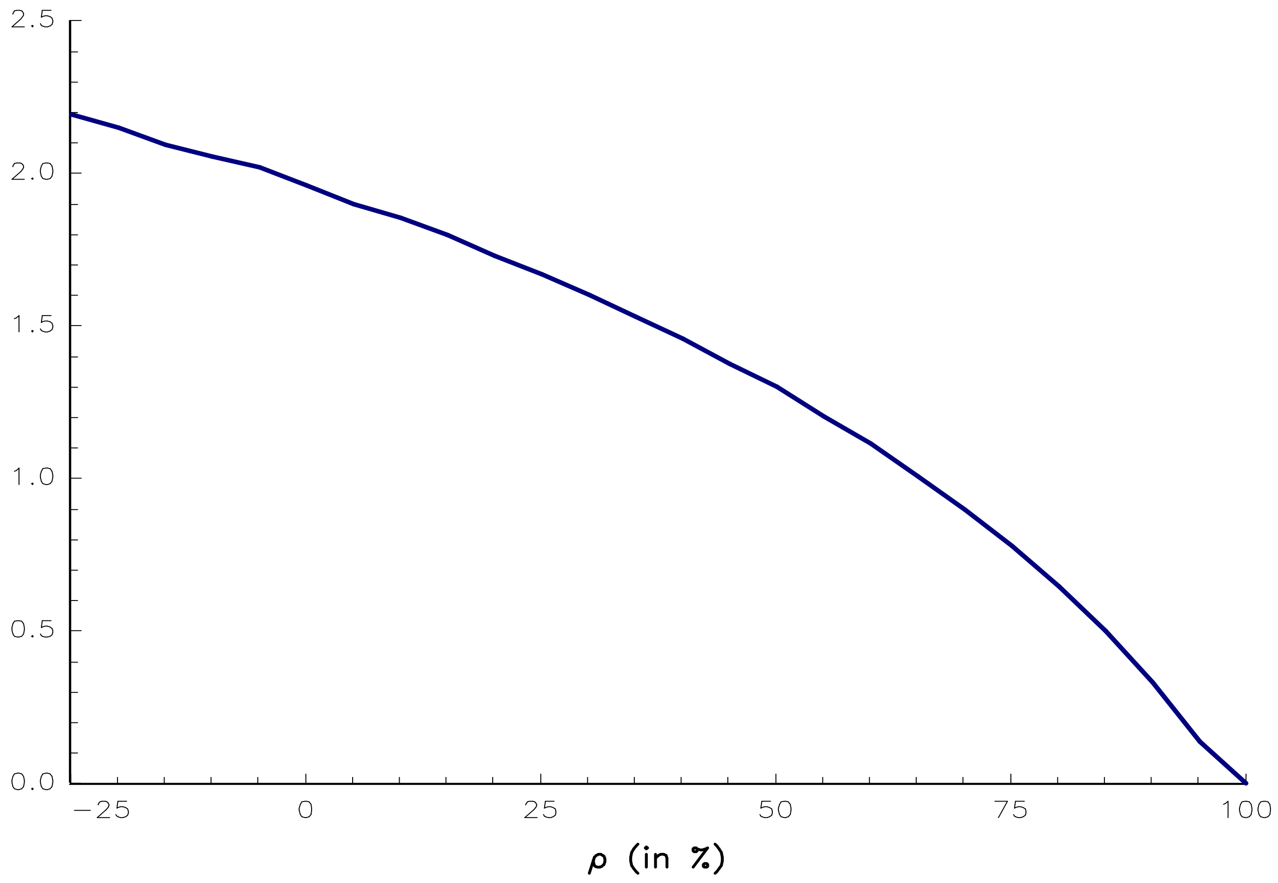
When the correlation matrix is $\mathbb{C}_5(\rho)$, the maximum price of 2.20 is not a conservative price. For instance, if we consider the correlation matrix below, we obtain an option price of 3.99:

$$\mathbb{C} = \begin{pmatrix} 1.0000 & 0.2397 & 0.7435 & -0.1207 & 0.0563 \\ 0.2397 & 1.0000 & -0.0476 & -0.0260 & -0.1958 \\ 0.7435 & -0.0476 & 1.0000 & 0.2597 & 0.1153 \\ -0.1207 & -0.0260 & 0.2597 & 1.0000 & -0.7568 \\ 0.0563 & -0.1958 & 0.1153 & -0.7568 & 1.0000 \end{pmatrix}$$

The multi-asset case

Issues with constant correlation matrices

Figure: Price of the basket option with respect to the constant correlation



Liquidity risk

Liquidity risk impacts trading costs of the hedging strategy

An example is the put option $\Rightarrow$ short strategy (can we be short on the underlying asset?)

Liquidity risk

Let us consider the replication of a call option. If the price of the underlying asset decreases sharply, the delta is reduced and the option trader has to sell asset shares. Because of their trend-following aspect, option traders generally buy assets when the market goes up and sell assets when the market goes down. However, we know that liquidity is asymmetric between these two market regimes. Therefore, it is more difficult to adjust the delta exposure when the market goes down, because of the lack of liquidity

Liquidity risk

Let us consider one of the most famous examples, which concerns call options on Sharpe ratio. Starting from 2004, some banks proposed to investors a payoff of the form $(\mathrm{SR}(0; T) - K)^{+}$ where $\mathrm{SR}(0; T)$ is the Sharpe ratio of the underlying asset during the option period. This payoff is relatively easy to replicate. However, most of call options on Sharpe ratio have been written on mutual funds and hedge funds. The difficulty comes from the liquidity of these underlying assets. For instance, the trader does not know exactly the price of the asset when he executes his order because of the notice period. This can be a big issue when the fund offers weekly or monthly liquidity. The second problem comes from the fact that the fund manager can impose lock-up period and gates. For instance, a gate limits the amount of withdrawals. During the 2008/2009 hedge fund crisis, many traders faced gate provisions and were unable to adjust their delta. This crisis marketed the end of call options on Sharpe ratio.

Exercises

- Option pricing models
  - Exercise 9.4.1 – Option pricing and martingale measure
  - Exercise 9.4.2 – The Vasicek model
  - Exercise 2.4.4 – Change of numéraire and Girsanov theorem

- Volatility
  - Exercise 9.4.8 – Dupire local volatility model
  - Exercise 2.4.9 – The stochastic normal model

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