Financial Risk Management Examination

Thierry Roncalli

January 7^{th} 2015

Please write entirely your answers.

1 The BCBS regulation

1. What are the main differences between the first Basle Accord and the second Basle Accord?
2. What are the main differences between the second Basle Accord and the third Basle Accord?
3. Define precisely the second pillar of the Basle Accord. Give an example in the case of credit risk and an example in the case of operational risk.

2 Market risk

1. What is the difference between the banking book and the trading book? Define the perimeter of assets for capital requirement.
2. How is computed the capital requirement with the internal model-based approach?
3. Why do you need to compute two VaR measures for the internal model-based approach?

3 Credit risk

1. What is the definition of the default in Basle II?
2. Describe the standard approach (SA) to compute the capital requirement.
3. Define the different parameters of the internal ratings-based (IRB) approach. What are the differences between FIRB and AIRB?

4 Counterparty credit risk

1. Define the concept of counterparty credit risk. Give two examples.
2. How is computed the capital requirement for this type of risk? Explain why the exposure-at-default (EAD) has to be estimated.
3. Define the three methods to estimate the EAD parameter.

5 Operational risk

1. What is the definition of operational risk? Give two examples.
2. Describe the standardized approach (TSA) to compute the capital charge.
3. Describe the loss distribution approach (LDA) to compute the capital charge.

6 Value-at-risk of a long-short portfolio

We consider a long-short portfolio composed of a long (buying) position in the asset A and a short (selling) position in the asset B. The long exposure is 2 millions of euros whereas the short exposure is 1 million of euros. Using the historical prices of the last 250 trading days of assets A and B, we estimate that the asset volatilities σ_A and σ_B are both equal to 20% per year and that the correlation $\rho_{A,B}$ between asset returns is equal to 50%. In what follows, we ignore the mean effect.

1. Calculate the Gaussian VaR of the long-short portfolio for a time horizon of 1 day and a confidence level of 99%.

2. How do you calculate the historical VaR? Using the historical returns of the last 250 trading days, the five worst scenarios of the 250 simulated daily PnLs of the portfolio are $-58\,700$ €, $-56\,850$ €, $-54\,270$ €, $-52\,170$ € and $-49\,231$ €. Calculate the historical VaR for a time horizon of 1 day and a confidence level of 99%.

3. Show that the Gaussian VaR is multiplied by a factor equal to $\sqrt{7/3}$ if the correlation $\rho_{A,B}$ is equal to -50%. How do you explain this result?

4. The portfolio manager sells a call option on the stock A. The delta of the option is equal to 50%. What does the Gaussian value-at-risk of the long-short portfolio become if the nominal of the option is equal to 2 millions of euros? Same question if the nominal of the option is equal to 4 millions of euros. How do you explain this result?

5. The portfolio manager replaces the short position on the stock B by selling a call option on the stock B. The delta of the option is equal to 50%. Show that the Gaussian value-at-risk is minimum when the nominal (expressed in millions of euros) is equal to four times the correlation $\rho_{A,B}$. Deduce then an expression of the lowest Gaussian value-at-risk. Comment on these results.

7 Estimating frequency and severity distributions for operational risk

We consider a sample of individual losses $\{L_1, \ldots, L_n\}$. We assume that they can be described by different probability distributions:

(i) L_i follows a log-normal distribution: $L_i \sim \mathcal{LN}(\mu, \sigma)$;

(ii) L_i follows a Pareto distribution $\mathcal{P}(\theta; x^-)$ defined by:

$$\Pr\{L \leq x\} = 1 - \left(\frac{x}{x_-}\right)^{-\theta}$$

with $x \geq x_-$ and $\theta > 1$;

(iii) L_i follows a Gamma distribution $\Gamma(\alpha, \beta)$ defined by[1]:

$$\Pr\{L \leq x\} = \int_0^x \frac{\beta^\alpha t^{\alpha-1} e^{-\beta t}}{\Gamma(\alpha)} \,\mathrm{d}t$$

with $x \geq 0$, $\alpha > 0$ and $\beta > 0$.

(iv) The natural logarithm of the loss L_i follows a Gamma distribution: $\ln L_i \sim \Gamma(\alpha; \beta)$.

1. We define different estimators for each distribution.

   (a) We consider the case *(i)*. Calculate the first two moments of L_i. Write the generalized method of moments (GMM) associated to the sample of individual losses $\{L_1, \ldots, L_n\}$.

   (b) We consider the case *(i)*. Find the maximum likelihood (ML) estimators $\hat{\mu}$ and $\hat{\sigma}$.

[1]Here $\Gamma(\alpha)$ is the gamma function evaluated at α.

(c) We consider the case *(ii)*. Calculate the first two moments of L_i. Deduce the generalized method of moments for estimating θ.

(d) We consider the case *(ii)*. Find the maximum likelihood estimator $\hat{\theta}$.

(e) We consider the case *(iii)*. Write the log-likelihood function associated to the sample of individual losses $\{L_1, \ldots, L_n\}$. Deduce the first-order condition of the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$.

(f) We consider the case *(iv)*. Show that the probability density function of L_i is:

$$f(x) = \frac{\beta^\alpha (\ln x)^{\alpha-1}}{\Gamma(\alpha) x^{\beta+1}}$$

What is the support of this probability density function? Write the log-likelihood function associated to the sample of individual losses $\{L_1, \ldots, L_n\}$.

2. We now assume that the losses $\{L_1, \ldots, L_n\}$ have been collected beyond a threshold H meaning that $L_i \geq H$.

   (a) Calculate the maximum likelihood estimator $\hat{\theta}$ in the case of distribution *(ii)*.

   (b) Write the log-likelihood function in the case of distribution *(iii)*.

3. The sample of individual losses $\{L_1, \ldots, L_n\}$ corresponds to a sample of T annual loss numbers $\{N_{Y_1}, \ldots, N_{Y_T}\}$. This implies that:

$$\sum_{t=1}^{T} N_{Y_t} = n$$

If we measure the number of losses per quarter $\{N_{Q_1}, \ldots, N_{Q_{4T}}\}$, we have:

$$\sum_{t=1}^{4T} N_{Q_t} = n$$

   (a) We assume that the annual number of losses follows a Poisson distribution $\mathcal{P}(\lambda_Y)$. Calculate the maximum likelihood estimator $\hat{\lambda}_Y$ associated to the sample $\{N_{Y_1}, \ldots, N_{Y_T}\}$.

   (b) We assume that the quarterly number of losses follows a Poisson distribution $\mathcal{P}(\lambda_Q)$. Calculate the maximum likelihood estimator $\hat{\lambda}_Q$ associated to the sample $\{N_{Q_1}, \ldots, N_{Q_{4T}}\}$.

   (c) What is the impact of considering a quarterly or annual basis on the computation of the capital charge?

   (d) What does this result become if you consider a method of moments based on the first moment?

   (e) Same question if you consider a method of moments based on the second moment.

8 Counterparty credit risk

1. Define the notion of counterparty credit risk (CCR). Give an example.

2. We define the loss related to a counterparty credit risk as follows:

$$L = \mathrm{EaD}(\tau) \times \mathrm{LGD} \times \mathbb{1}\{\tau < T\}$$

Why is the exposure-at-default random? Why does it correspond to the positive part of the mark-to-market of the OTC transaction when the counterparty defaults?

3. In the table below, we report the actual mark-to-market of 7 OTC contracts between Bank A and bank B:

	Equity			Fixed income		Currency	
	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$
A	10	-5	6	17	-5	-5	1
B	-11	6	-3	-12	9	5	1

We read figures in the following way: Bank A has a mark-to-market equal to 10 for the contract $\mathcal{C}_1$ whereas Bank B has a mark-to-market equal to -11 for the same contract.

(a) Explain why there are differences between the mark-to-market values of a same contract.

(b) Calculate the exposure-at-default of bank A.

(c) Same question if there is a global netting agreement.

(d) Same question if the netting agreement only concerns equity products.

4. We note $e(t)$ the random exposure-at-default of the OTC contract with a one-year maturity. The actual date t_0 is equal to 0. Let $\mathbf{F}_{[t_0,t]}$ be the cumulative distribution function of $e(t)$ for a future date t. We recall the following definitions. The *potential future exposure* is equal to the quantile $\mathbf{F}_{[t_0,t]}^{-1}(\alpha)$:

$$\mathrm{PFE}_\alpha(t_0;t) = \mathbf{F}_{[t_0,t]}^{-1}(\alpha)$$

The *peak exposure* (PE) is the maximum potential future exposure that occurs before the maturity date:

$$\mathrm{PE}_\alpha(t_0) = \sup_t \mathrm{PFE}_\alpha(t_0;t)$$

The *expected exposure* (EE) is the average of the distribution of exposures at any particular future date t:

$$\mathrm{EE}(t_0;t) = \mathbb{E}[e(t)] = \int e \, \mathrm{d}\mathbf{F}_{[t_0,t]}(e)$$

The *effective expected exposure* (EEE) at a specific date t is the maximum expected exposure that occurs at that date or any prior date:

$$\mathrm{EEE}(t_0;t) = \sup_{\theta \le t} \mathrm{EE}(t_0;\theta)$$

The *expected positive exposure* (EPE) is the average over time of expected exposures:

$$\mathrm{EPE}(t_0;h) = \frac{1}{h}\int_{t_0}^{t_0+h} \mathrm{EE}(t_0;t)\,\mathrm{d}t$$

The effective expected positive exposure (EEPE) is the average over time of effective expected exposures:

$$\mathrm{EEPE}(t_0;h) = \frac{1}{h}\int_{t_0}^{t_0+h} \mathrm{EEE}(t_0;t)\,\mathrm{d}t$$

(a) Comment on the different notions of exposure amount. Define the (non-random) exposure-at-default when using an internal model. What is the rationale of this choice?

(b) Calculate $\mathrm{EEPE}(0;h)$ when the random exposure is:

$$e(t) = \sigma\sqrt{t}X$$

X is a random variable defined $[0,1]$ and its probability density function is:

$$f(x) = \frac{x^a}{a+1}$$

with $a > 0$.

(c) Why does this exposure-at-default look like an option profile more than a swap profile?

9 Risk contribution in the Basle II model

We note L the loss portfolio of n loans and x_i the exposure-at-default of the i^{th} loan. We have:

$$L = \mathbf{x}^\top \mathbf{e} = \sum_{i=1}^n x_i \times e_i$$

where e_i is the loss of the i^{th} loan. Let $\mathbf{F}$ be the cumulative distribution function of L.

1. We assume that $\mathbf{e} = (e_1, \ldots, e_n) \sim \mathcal{N}(\mathbf{0}, \Sigma)$. Compute the value-at-risk $\mathrm{VaR}(x;\alpha)$ of the portfolio x when the confidence level is equal to α.

2. Deduce the marginal value-at-risk of the i^{th} loan. Define then the risk contribution $\mathcal{RC}_i$ of the i^{th} loan.

3. Check that the marginal value-at-risk is equal to:

$$\frac{\partial\, \mathrm{VaR}(x;\alpha)}{\partial\, x_i} = \mathbb{E}\left[e_i \mid L = \mathbf{F}^{-1}(\alpha)\right]$$

Comment on this result.

4. We consider the Basle II model of credit risk. We have:

$$e_i = \mathrm{LGD}_i \times D_i$$

where LGD_i is the loss-given-default, $D_i = \mathbb{1}\{\tau_i < M_i\}$ is the default indicator and M_i is the maturity of the i^{th} loan. What are the necessary assumptions $(\mathcal{H})$ to obtain this result:

$$\mathbb{E}\left[e_i \mid L = \mathbf{F}^{-1}(\alpha)\right] = \mathbb{E}[\mathrm{LGD}_i] \times \mathbb{E}\left[D_i \mid L = \mathbf{F}^{-1}(\alpha)\right]$$

5. We assume that the loan i defaults before the maturity M_i if a latent variable Z_i goes below a barrier B_i:

$$\tau_i \leq M_i \Leftrightarrow Z_i \leq B_i$$

We consider that $Z_i = \sqrt{\rho}X + \sqrt{1-\rho}\,\varepsilon_i$ where Z_i, X and ε_i are three independent Gaussian variables $\mathcal{N}(0,1)$. X is the factor (or the systemic risk) and ε_i is the idiosyncratic risk. Let p_i be the unconditional default probability of the i^{th} loan for the maturity M_i. Compute the conditional default probability $\Pr\{\tau_i \leq M_i \mid X = x\}$.

6. Show that, under the previous assumptions $(\mathcal{H})$, you obtain:

$$\mathbb{E}\left[e_i \mid L = \mathbf{F}^{-1}(\alpha)\right] = \mathbb{E}[\mathrm{LGD}_i] \times \mathbb{E}\left[D_i \mid X = \Phi^{-1}(1-\alpha)\right]$$

7. Deduce that the risk contribution $\mathcal{RC}_i$ of the i^{th} loan in the Basle II model is:

$$\mathcal{RC}_i = x_i \times \mathbb{E}[\mathrm{LGD}_i] \times \Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right) \tag{1}$$

when the risk measure is the value-at-risk.

8. We now assume that the risk measure is the expected shortfall:

$$\mathrm{ES}(x;\alpha) = \mathbb{E}[L \mid L \geq \mathrm{VaR}(x;\alpha)]$$

   (a) Show that, in the case of the Basle II framework, you have:

$$\mathrm{ES}(x;\alpha) = \sum_{i=1}^{n} x_i \times \mathbb{E}[\mathrm{LGD}_i] \times \mathbb{E}\left[D_i \mid X \leq \Phi^{-1}(1-\alpha)\right]$$

   (b) By using the following result:

$$\int_{-\infty}^{c} \Phi(a+bx)\phi(x)\,\mathrm{d}x = \Phi_2\left(c, \frac{a}{\sqrt{1+b^2}}; \frac{-b}{\sqrt{1+b^2}}\right)$$

   where $\Phi_2(x,y;\rho)$ is the cdf of the bivariate Gaussian distribution with correlation ρ on the space $[-\infty, x] \times [-\infty, y]$, deduce that the risk contribution $\mathcal{RC}_i$ of the i^{th} loan in the Basle II model is:

$$\mathcal{RC}_i = x_i \times \mathbb{E}[\mathrm{LGD}_i] \times \frac{\mathbf{C}\left(1-\alpha, p_i; \sqrt{\rho}\right)}{1-\alpha} \tag{2}$$

   when the risk measure is the expected shortfall. Here $\mathbf{C}(u_1, u_2; \theta)$ is the Normal copula with parameter θ.

   (c) What do the results (1) and (2) become if the correlation ρ is equal to zero? Same question if $\rho = 1$.