Advanced Asset Management
Final Examination

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Remark 1 ***The final examination is composed of 2 exercises. Please write entirely your answers[1]. Precise the different concepts and the different statistics you use. Define the optimization program associated to each portfolio. Provide also one excel file or one program by exercise.***

- ***Concerning risk decomposition[2], present the results as follows:***

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1				
2				
$\vdots$				
n				
$\mathcal{R}(x)$			✓	

- ***The report is a zipped file, whose filename is <u>yourname.zip</u> if you do the project alone or <u>yourname1-yourname2.zip</u> if you do the project in groups of two.***
- ***The zipped file is composed of three files:***
  1. ***the pdf document that contains the answers to the two exercises and a cover sheet with your names;***
  2. ***the program of each exercise with an explicit filename, e.g. exercise1.xls (if you use excel), exercise1.m (if you use matlab), exercise1.py (if you use python), exercise1.r (if you use R), etc.***

[1] Read carefully the questions and answer to all the elements of the questions. For instance, when I say "Find the portfolio x and compute its volatility $\sigma(x)$", you have to give the numeric values of x and $\sigma(x)$. If you only give the numeric value of $\sigma(x)$, the answer is false because I don't know what the weights of the portfolio are.

[2] x_i is the weight (or the exposure) of the i^{th} asset in the portfolio, $\mathcal{MR}_i$ is the marginal risk, $\mathcal{RC}_i$ is the nominal risk contribution, $\mathcal{RC}_i^\star$ is the relative risk contribution and $\mathcal{R}(x)$ is the risk measure of the portfolio.

1 Portfolio optimization and risk budgeting

We consider the CAPM model:

$$R_i - r = \beta_i \cdot (R_m - r) + \varepsilon_i$$

where R_i is the return of asset i, R_m is the return of the market portfolio, r is the risk free asset, β_i is the beta of asset i with respect to the market portfolio and ε_i is the idiosyncratic risk. We assume that $R_m \perp \varepsilon_i$ and $\varepsilon_i \perp \varepsilon_j$. We note σ_m the volatility of the market portfolio and $\tilde{\sigma}_i$ the idiosyncratic volatility. We consider a universe of 5 assets with the following parameter values:

Asset i	1	2	3	4	5	6	7	8
β_i	-0.60	-0.40	0.00	0.25	0.90	1.00	2.00	3.00
$\tilde{\sigma}_i$	15%	16%	15%	13%	12%	10%	6%	5%

and $\sigma_m = 15\%$. The risk free return is set to 1% and we assume that the expected return of the market portfolio is equal to $\mu_m = 6\%$.

1. We assume that the CAPM is valid.
   (a) Calculate the vector μ of expected returns.
   (b) Compute the covariance matrix Σ.
   (c) Deduce the volatility σ_i of the assets and find the correlation matrix $C = (\rho_{i,j})$ between asset returns.

2. We consider **long-short** MVO portfolios such that:
$$\sum_{i=1}^{n} x_1 = 1$$
   (a) Give the QP formulation of the mean-variance optimization problem.
   (b) Using the γ-problem, find the optimal solution[3] $x^\star(\gamma)$ when the coefficient γ is equal to 0, 0.10, 0.20, 0.50 and 1.00. For each optimized portfolio, compute its expected return $\mu(x^\star(\gamma))$, its volatility $\sigma(x^\star(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^\star(\gamma) \mid r)$.
   (c) Draw the efficient frontier by considering granular values[4] of γ.
   (d) Using a bisection algorithm, calculate the MVO portfolio if we target an ex-ante volatility of 6% and 12%. Give the corresponding values of γ of the QP problem. For each optimized portfolio, compute its expected return $\mu(x^\star(\gamma))$, its volatility $\sigma(x^\star(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^\star(\gamma) \mid r)$.

3. We consider **long-only** portfolios such that:
$$x_i \geq 0 \qquad \text{and} \qquad \sum_{i=1}^{n} x_1 = 1$$
   (a) Draw the efficient frontier by considering granular values of γ.
   (b) Using a bisection algorithm, calculate the MVO portfolio if we target an ex-ante volatility of 6% and 12%. Give the corresponding values of γ of the QP problem. For each optimized portfolio, compute its expected return $\mu(x^\star(\gamma))$, its volatility $\sigma(x^\star(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^\star(\gamma) \mid r)$. Comment on these results.
   (c) Compute the tangency portfolio $x^\star_{\mathrm{MSR}}$. What is the value of its Sharpe ratio?

[3] You have to give the composition of each optimized portfolios.

[4] For instance, you can consider that $\gamma = -0.5, -0.4, \ldots, -0.1, 0, 0.05, 0.10, \ldots, 0.95, 1, 2, \ldots, 10$.

(d) Give the risk decomposition of the tangency portfolio $x^{\star}_{\mathrm{MSR}}$.

(e) Give the risk decomposition of the long-only minimum variance (MV) portfolio.

(f) Give the risk decomposition of the long-only most diversified portfolio (MDP).

(g) Compute the beta $\beta(x \mid b)$ of the portfolios MV and MDP with respect to the tangency portfolio $x^{\star}_{\mathrm{MSR}}$. Comment on these results.

4. We consider the following optimization program:

$$x^{\star} = \arg\min \frac{1}{2} x^{\top} \Sigma x - \sum_{i=1}^{n} \ln x_i$$

where Σ is the covariance matrix and x is the vector of portfolio weights.

(a) Write the first-order condition with respect to the vector x and show that the solution $x^{\star}$ corresponds to a scaling ERC portfolio.

(b) Write the first-order condition with respect to the coordinate x_i. Find the optimal value $x_i^{\star}$ when we consider the other coordinates $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ as fixed.

(c) We note $x_i^{(k)}$ the value of the i^{th} coordinate at the k^{th} iteration. Deduce the corresponding CCD algorithm.

(d) Starting from the initial value $x^{(0)} = (1/8, \ldots, 1/8)$, find the optimal coordinate $x_1^{(1)}$ for the first asset.

(e) Compute then the optimal coordinate $x_2^{(1)}$ for the second asset.

(f) Compute then the optimal coordinate $x_3^{(1)}$ for the third asset.

(g) Give the CCD solutions $x^{(1)}$, $x^{(2)}$, $x^{(5)}$ and $x^{(10)}$.

(h) Deduce the ERC portfolio.

(i) Give the risk decomposition of the ERC portfolio.

2 Portfolio decarbonization

We consider a capitalization-weighted equity index, which is composed of 8 assets. The asset weights are equal to 23%, 19%, 17%, 13%, 9%, 8%, 6% and 5%. We assume that their volatilities are equal to 22%, 20%, 25%, 18%, 35%, 23%, 13% and 29%. The correlation matrix between these assets is given by:

$$\rho = \begin{pmatrix} 100\% & & & & & & & \\ 80\% & 100\% & & & & & & \\ 70\% & 75\% & 100\% & & & & & \\ 60\% & 65\% & 80\% & 100\% & & & & \\ 70\% & 50\% & 70\% & 85\% & 100\% & & & \\ 50\% & 60\% & 70\% & 80\% & 60\% & 100\% & & \\ 70\% & 50\% & 70\% & 75\% & 80\% & 50\% & 100\% & \\ 60\% & 65\% & 70\% & 75\% & 65\% & 70\% & 80\% & 100\% \end{pmatrix}$$

We also report in the table below the carbon emissions $\mathcal{CE}_{i,j}$ of these companies and their revenue Y_i. Moreover, we indicate in the last row of the table if a company belongs to a high climate impact sector (HCIS = 1) or not (HICS = 0).

Stock	#1	#2	#3	#4	#5	#6	#7	#8
CW weight (in %)	23	19	17	13	9	8	6	5
Volatility (in %)	22	20	25	18	35	23	13	29
$\mathcal{CE}_{i,1}$ (in ktCO_2e)	75	5 000	720	50	2 500	25	30 000	5
$\mathcal{CE}_{i,2}$ (in ktCO_2e)	75	5 000	1 030	350	4 500	5	2000	64
$\mathcal{CE}_{i,3}$ (in ktCO_2e)	24 000	15 000	1 210	550	500	187	30 000	199
Y_i (in \$ bn)	300	320	125	100	200	102	100	25
HCIS	0	1	0	0	1	0	1	0

In what follows, we consider long-only portfolios, and the benchmark is the CW portfolio.

1. Compute the covariance matrix Σ.
2. Compute the carbon intensities $\mathcal{CI}_{i,j}$ of each company i for the scopes 1, 2 and 3.
3. Deduce the carbon intensities $\mathcal{CI}_{i,j}$ of each company i for the scopes $1+2$ and $1+2+3$.
4. The market capitalization of the index is equal to \$2 500 bn.
   (a) Deduce the market capitalization of each company (expressed in \$ bn).
   (b) Compute the carbon emissions (scopes 1, $1+2$ and $1+2+3$) of the index portfolio[5] if we invest \$1 bn.
   (c) Deduce the (exact) carbon intensity of the index portfolio.
5. Compute the weighted-average carbon intensity (scopes 1, $1+2$ and $1+2+3$) of the index portfolio. Comment on these results.
6. We would like to reduce the **weighted-average carbon intensity** of the index by the rate $\mathcal{R}$.
   (a) Write the optimization problem if the objective function is to minimize the tracking error risk.
   (b) Give the QP formulation of the optimization problem.
   (c) $\mathcal{R}$ is equal to 20%. Find the optimal portfolio if we target scope $1+2$. What is the value of the tracking error volatility?

[5] We assume that the float percentage is equal to 100% for all the 8 companies.

(d) Same question if $\mathcal{R}$ is equal to 30%, 40%, 50%, 60%, 70%, 80% and 90%.

(e) We target scope $1+2+3$. Find the optimal portfolio if $\mathcal{R}$ is equal to 20%, 30%, 40%, 50%, 60%, 70%, 80% and 90%. Give the value of the tracking error volatility for each optimized portfolio.

7. We would like to reduce the **weighted-average carbon intensity** of the index by the rate $\mathcal{R}$ and have an exposure to **high climate impact sectors** that is at least equal to the HCIS exposure of the benchmark.

   (a) What is the proportion of HCIS sectors in the benchmark?

   (b) Write the optimization problem if the objective function is to minimize the tracking error risk. Give the QP formulation of the optimization problem.

   (c) We target scope $1+2$. Find the optimal portfolio if $\mathcal{R}$ is equal to 20%, 30%, 40%, 50%, 60%, 70%, 80% and 90%. Give the value of the tracking error volatility for each optimized portfolio.

   (d) Same question if we target scope $1+2+3$.

   (e) Comment on these results.

8. We would like to reduce the **absolute carbon emissions** of the index by the rate $\mathcal{R}$ and have an exposure to **high climate impact sectors** that is at least equal to the HCIS exposure of the benchmark.

   (a) Write the carbon footprint constraint.

   (b) Write the optimization problem if the objective function is to minimize the tracking error risk. Give the QP formulation of the optimization problem.

   (c) We target scope $1+2$. Find the optimal portfolio if $\mathcal{R}$ is equal to 50%. Give the tracking error volatility.

   (d) Same question if we target scope $1+2+3$.

   (e) Compare the solutions given in Questions 6.c, 6.d, 7.c, 7.d, 8.c and 8.d.