Alternative Risk Premia: What Do We Know?*

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Abstract

The concept of alternative risk premia is an extension of the factor investing approach. Factor investing consists in building long-only equity portfolios, which are directly exposed to common risk factors like size, value or momentum. Alternative risk premia designate non-traditional risk premia other than a long exposure to equities and bonds. They may involve equities, rates, credit, currencies or commodities and correspond to long/short portfolios. However, contrary to traditional risk premia, it is more difficult to define alternative risk premia and which risk premia really matter. In fact, the term "*alternative risk premia*" encompasses two different types of systematic risk factor: skewness risk premia and market anomalies. For example, the most frequent alternative risk premia are carry and momentum, which are respectively a skewness risk premium and a market anomaly. Because the returns of alternative risk premia exhibit heterogeneous patterns in terms of statistical properties, option profile and drawdown, asset allocation is more complex than with traditional risk premia. In this context, risk diversification cannot be reduced to volatility diversification and skewness risk becomes a key component of portfolio optimization. Understanding these different concepts and how they interconnect is essential for improving multi-asset allocation.

Keywords: Alternative risk premium, factor investing, skewness risk, market anomalies, systematic risk factor, diversification, carry, momentum, value, low beta, short volatility, payoff function, alternative beta, hedge funds, multi-asset allocation.

JEL classification: C50, C60, G11.

1 Introduction

After the emergence of risk-based investing, factor investing has been the new hot topic in the asset management industry since the 2008 Global Financial Crisis. The two concepts are related to the notion of diversification, but take different standpoints. The goal of risk-based investing is to build a better diversified portfolio than a mean-variance optimized portfolio. The idea is that mathematical optimization and volatility minimization do not always lead to financial diversification. The aim of factor investing is to extend the universe of assets for building a diversified allocation by capturing systematic risk factors. For instance, in the

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equity space, the capital asset pricing model has been supplemented by a five-factor model, which is based on size, value, momentum, low beta and quality risk factors.

The concept of alternative risk premia (ARP) is an extension of factor investing, which is a term generally reserved for long-only equity risk factors. Indeed, alternative risk premia concern all the asset classes, not only equities, but also rates, credit, currencies and commodities. Moreover, they may be implemented using long/short portfolios. To be more precise, a risk premium is compensation for taking a risk that cannot be hedged or diversified. Traditionally, we consider that there are two main risk premia, which correspond to a long exposure to equities and bonds. However, since the eighties, academics have shown that there are other sources of risk premia. For instance, cat bonds must incorporate a risk premium, because the investor takes a large risk that cannot be diversified. Therefore, alternative risk premia designate all the risk premia other than a long exposure to equities and bonds.

Contrary to traditional risk premia, whose risk/return profile is relatively easy to understand, the behavior of alternative risk premia is more heterogeneous. In fact, they cover two main categories of strategies: skewness risk premia and market anomalies. Skewness risk premia are 'pure' risk premia, meaning that they reward systematic risks in bad times. Conversely, market anomalies are strategies that have performed well in the past, but this performance cannot be explained by the existence of a risk premium. For example, momentum and trend-following strategies are market anomalies, whereas carry strategies are generally considered as skewness risk premia. As a result, statistical properties and option profiles are different from one risk premium to another. In particular, skewness risk premia may exhibit a high skewness risk. Whereas portfolio allocation between traditional risk premia is usually based on expected returns and the covariance matrix, portfolio management cannot ignore the third statistical moment. This issue is particularly important, because some investors see portfolios of alternatives risk premia as all-weather strategies. However, this is not the case in reality.

Diversification is the primary objective when investing in alternative risk premia. The second motivation is the search for higher returns, especially in a low-rate environment. In this context, alternative risk premia are performance assets, and not only diversification assets. It is therefore natural that the development of alternative risk premia impacts the hedge fund industry. First, it offers a new framework for analyzing the risk/return profile of hedge fund strategies and institutional portfolios invested in alternative assets. Second, it provides new investment products that replicate the alternative beta of hedge funds. But the most significant influence of alternative risk premia certainly involves multi-asset management, which cannot be reduced to an allocation between stocks and bonds. Indeed, alternative risk premia constitute the other building blocks of multi-asset portfolios. This is why they participate in the convergence of traditional and alternative investments.

This chapter is organized as follows. In section two, we present the rationale of alternative risk premia, in particular the difference between systematic, arbitrage and specific risk factors. The study of factor investing in the equity market also helps in understanding the motivations behind the emergence of this new framework. In section three, we define more precisely the concept of alternative risk premia and make the distinction between skewness risk premia and market anomalies. We can then review the different generic strategies. In particular, carry and momentum are the two most relevant alternative risk premia across the different asset classes. Section four deals with the issue of diversification and portfolio management in the presence of skewness risk. Finally, section five offers some concluding remarks.

2 The rationale of alternative risk premia

In order to understand the relationship between alternative risk premia and the concept of diversification, we have to go back to the works of Markowitz (1952) on this topic. In a first step, we show that diversification depends on the allocation model, but also on the definition of the common risk factors. In a second step, using the results on the equity asset class, we show that common risk factors are the only bets that are compatible with diversification.

2.1 Difference between common risk factors and arbitrage factors

We consider a universe of n assets. Let μ and Σ be the vector of expected returns and the covariance matrix of asset returns. We denote by $x = (x_1, \ldots, x_n)$ the vector of weights in the portfolio. For Markowitz (1952), the financial problem of the investor consists in maximizing the expected excess return of his portfolio subject to a constraint on the portfolio's volatility:

$$\begin{aligned} x^\star &= \arg\max x^\top (\mu - r\mathbf{1}) \\ \text{u.c.} & \quad \sqrt{x^\top \Sigma x} \leq \sigma^\star \end{aligned} \tag{1}$$

where r is the return of the risk-free asset. Markowitz (1956) showed that this non-linear optimization problem is equivalent to a quadratic optimization problem:

$$x^\star = \arg\min \frac{1}{2} x^\top \Sigma x - \gamma x^\top (\mu - r\mathbf{1}) \tag{2}$$

where γ is a parameter that controls the risk aversion of the investor. Without any constraints, the mean-variance optimized (MVO) portfolio $x^\star$ is equal to $\gamma \Sigma^{-1} (\mu - r\mathbf{1})$. More generally, in the presence of linear equality and inequality constraints, MVO portfolios are of the following form:

$$x^\star \propto f\left(\mu, \Sigma^{-1}\right) \tag{3}$$

where f is a complicated function that depends on the constraints. More precisely, the solution of the Markowitz optimization problem depends on the inverse of the covariance matrix and not the covariance matrix itself. Therefore, the important quantity in portfolio optimization is the information matrix $\mathcal{I} = \Sigma^{-1}$.

In order to better understand the notion of information matrix, we consider the eigendecomposition of the covariance matrix Σ:

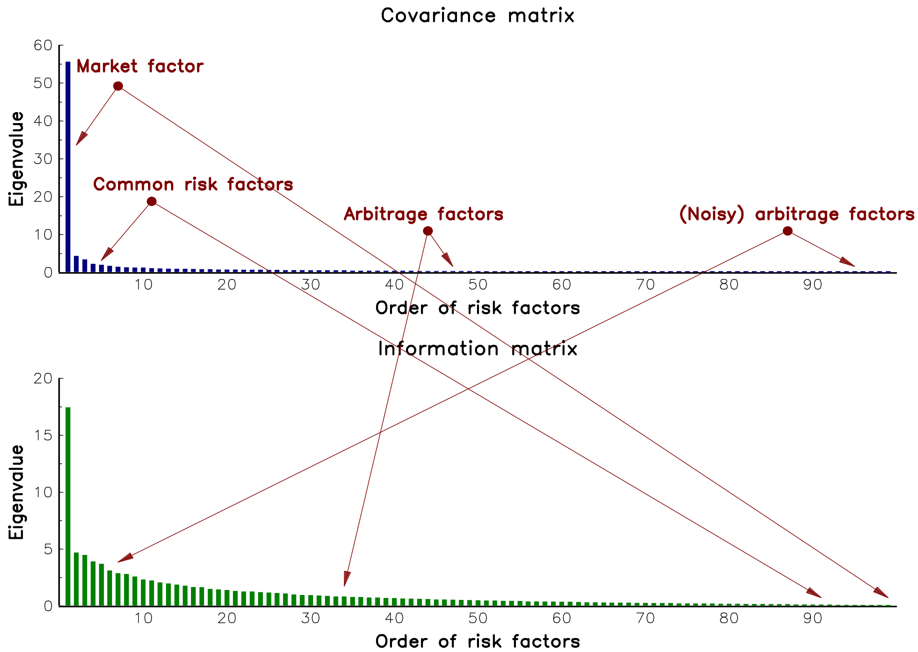
$$\Sigma = V \Lambda V^\top \tag{4}$$

where V is the matrix of eigenvectors of Σ and Λ is the diagonal matrix, whose elements are the eigenvalues of Σ. We have:

$$\begin{aligned} \Sigma^{-1} &= \left(V \Lambda V^\top\right)^{-1} \\ &= \left(V^\top\right)^{-1} \Lambda^{-1} V^{-1} \\ &= V \Lambda^{-1} V^\top \end{aligned} \tag{5}$$

because V is an orthogonal matrix. It follows that the eigenvectors of the information matrix $\mathcal{I}$ are the same as those of the covariance matrix. This is not the case of eigenvalues. Indeed, the eigenvalues of $\mathcal{I}$ are the inverse of the eigenvalues of Σ.

Figure 1: Eigenvalues of covariance and information matrices of stock returns (FTSE 100 index, June 2012)



In Figure 1, we consider the one-year empirical covariance matrix of stock returns that made up the FTSE 100 index in June 2012. In the top panel, we have reported the breakdown of the corresponding eigenvalues. In the case of an equity investment universe, the first risk factor of the covariance matrix is generally interpreted as the market risk factor. The next eigenvectors correspond to the common risk factors, whereas the last eigenvectors are the arbitrage factors[1]. The breakdown of the eigenvalues of the information matrix is given in the bottom panel. In this case, the most important eigenvectors are the arbitrage factors (Scherer, 2007). This implies that MVO portfolios are mainly exposed to the less significant risk factors of the covariance matrix[2]. We face an issue here, because the Markowitz framework is generally presented as a diversification approach. In fact, in the case of Markowitz optimization, the common risk factors are not very interesting, because they are not arbitrage factors. For instance, Markowitz optimization is not sensitive to the market risk factor. We face a paradox here, because when we speak about Markowitz diversification, this does not mean diversification of common risk factors. Markowitz diversification means

[1] An arbitrage factor is a long/short portfolio between a stock and the corresponding hedging portfolio of the stock. In this case, an arbitrage opportunity is detected if the long/short portfolio exhibits a significant excess return compared to the tracking error volatility of the hedging portfolio.

[2] Without any constraints, the MVO portfolio is defined by:

$$x_i^\star = \gamma \sum_{j=1}^{n} \frac{v_{i,j}}{\lambda_j} \tilde{\mu}_j \tag{6}$$

where λ_j is the j^{th} eigenvalue, $v_{i,j}$ is the i^{th} element of the j^{th} eigenvector and:

$$\tilde{\mu}_j = \sum_{k=1}^{n} v_{k,j} \mu_k \tag{7}$$

concentration on the most important arbitrage factors. Therefore, the Markowitz model is one of the most aggressive approaches in active management.

It is well-known that portfolio optimization based on common risk factors requires non-linear constraints or shrinkage. This is the case of risk budgeting portfolios. Let $\mathcal{R}(x)$ be a risk measure applied to the portfolio x. We consider a vector of risk budgets $(b_1, \ldots, b_n)$. In a risk budgeting (RB) approach, the portfolio manager chooses weights such that the risk contributions are proportional to the risk budgets:

$$\mathcal{RC}_i = x_i \cdot \frac{\partial\, \mathcal{R}(x)}{\partial\, x_i} = b_i \mathcal{R}(x) \tag{8}$$

If the risk budgets are the same for all the assets, the risk budgeting portfolio is called the equal risk contribution (ERC) portfolio. In the case where the risk measure is the volatility, we can show that an RB portfolio is a highly-regularized MVO portfolio (Roncalli, 2013). The two main differences between an RB portfolio and a traditional MVO portfolio are then the following:

1. the RB portfolio is always a long-only portfolio whereas an MVO portfolio can be a long/short portfolio;
2. the RB portfolio is sensitive to the covariance matrix, implying it is mainly exposed to the common risk factors.

The reference to the RB portfolio is important, because it is generally accepted that the risk budgeting approach is a robust way to build diversified portfolio. Whereas active management (and mean-variance optimization) is associated with arbitrage factors, diversification management (and risk budgeting optimization) is related to common risk factors. A next step is then to understand what these common risk factors are. The case of factor investing in equities helps us to better assess them.

2.2 Factor investing in the equity market

Since the seminal research of Fama and French (1992, 1992), it is accepted that the market factor defined by Sharpe (1964) is not the only common risk factor that explains the cross-section variance of expected returns. Among these factors, we find the low beta factor (Black, 1972), the value factor (Basu, 1977), the size factor (Banz, 1981), the momentum factor (Jegadeesh and Titman, 1993) or the quality factor (Piotroski, 2000). The concept of factor investing has been popularized by Ang (2014). It consists in building (long-only) equity portfolios, which are directly exposed to these common risk factors. Therefore, factor investing is a subset of smart beta or an extension of risk-based indexation. It may be curious that factor investing has become very popular since the 2008 Global Financial Crisis, but we are going to see that this development is related to the search for diversification by long-term institutional investors (Ilmanen and Kizer, 2012).

In the arbitrage pricing theory (APT) of Ross (1976), the return on asset i is driven by a standard linear factor model:

$$R_i = \alpha_i + \sum_{j=1}^{n_{\mathcal{F}}} \beta_i^j \mathcal{F}_j + \varepsilon_i \tag{9}$$

where α_i is the intercept, β_i^j is the sensitivity of asset i to factor j and $\mathcal{F}_j$ is the (random) value of factor j. ε_i is the idiosyncratic risk of asset i, implying that $\mathbb{E}[\varepsilon_i] = 0$, $\operatorname{cov}(\varepsilon_i, \varepsilon_k) = 0$

for $i \neq k$ and $\operatorname{cov}(\varepsilon_i, \mathcal{F}_j) = 0$. The $\mathfrak{R}$-squared coefficient associated with Model (9) is equal to:

$$\mathfrak{R}_i^2 = 1 - \frac{\sigma^2(\varepsilon_i)}{\sigma^2(R_i)} \tag{10}$$

It measures the part of the variance of asset returns explained by common factors. It follows that the part due to the idiosyncratic risk is equal to $1 - \mathfrak{R}_i^2$. In Table 1, we have reported the variance decomposition of daily returns of 6 stocks between common risk factors and the idiosyncratic risk factor. For that, we use the 4-factor model of Carhart (1997), which is based on market, size, value and momentum risk factors. For instance, if we consider the Google stock, 47% of the variance is explained by the four common risk factors and 53% by the idiosyncratic risk. If we consider the Netflix stock, 76% of the return variance corresponds to an idiosyncratic risk. At the level of individual stocks, there is a lot of alpha[3] and this alpha dominates common risk factors.

Table 1: Variance decomposition of stock returns

Stock	Common risk factors	Idiosyncratic risk factor
Google	47%	53%
Netflix	24%	76%
Mastercard	50%	50%
Nokia	32%	68%
Total	89%	11%
Airbus	56%	44%

Carhart's model with 4 factors, 2010-2014.

We may wonder what does this result become if we consider diversified portfolios in place of individual stocks? In 1968, Jensen defined the alpha of a portfolio as the intercept of the linear model (9). In the case of the CAPM, the alpha is then the performance of the portfolio minus the beta of the portfolio times the return of the market portfolio[4]. By applying this concept to 115 mutual funds, Jensen (1968) rejected the assumption that the alpha is positive. This implies that the active management does not produce alpha on average. Nevertheless, Hendricks *et al.* (1993) noticed that this alpha is positively autocorrelated. This implies that a fund manager that has outperformed in the past has a higher probability of outperforming than underperforming in the future. This result suggested then that there is a persistence of the performance of active management and this persistence is due to the persistence of the alpha.

In 1995, Grinblatt and his co-authors analyzed the quarterly portfolio holdings of 155 equity mutual funds between 1974 and 1984. They found that "77% *of these mutual funds*

[3]Here, the concept of alpha corresponds to the part of the return variance that is not explained by common risk factors. More generally, the alpha component is the random variable represented by the residual risk factor. In this case, the alpha can be measured as the expected return of this component (Jensen's alpha) or its variance.

[4]The alpha at time t is then equal to:

$$\alpha_t = (R_t - r) - \hat{\beta}\left(R_t^{\mathrm{MKT}} - r\right) \tag{11}$$

where R_t is the portfolio's return, R_t^{MKT} is the return of the market portfolio, r is the return of the risk-free asset and $\hat{\beta}$ is the estimated OLS coefficient.

were momentum investors". Two years later, Mark Carhart proposed a four-factor model for explaining the persistence of equity mutual funds. These four factors are the market (or the traditional beta), size, value and momentum. Carhart (1997) found that the alpha calculated with the four-factor model is not auto-correlated. Carhart concluded that the persistence of the performance of active management is not due to the persistence of the alpha, but it is due to the persistence of the performance of common risk factors.

Another important result concerns the relationship between diversification and risk factors. We can wonder what the optimal number of holdings of a stock picking portfolio is. It is commonly accepted that a well-diversified portfolio reduces the impact of alpha, because the beta dominates the alpha if the portfolio is not concentrated in a small number of bets. This idea is shared by Warren Buffett[5], David Swensen[6] and other successful investors. In Figure 2, we have reported the proportion of Carhart's alpha (or relative active risk) with respect to the number of stocks[7]. For individual stocks, the alpha represents about 60% of return variance. In the case of a well-diversified portfolio, the alpha is less than 10% on average.

We verify this rule with the Morningstar database. We consider the 880 mutual funds invested in European equities from 2010 to 2014. In Figure 3, we have reported the part of the performance explained by Carhart's model with respect to the logarithm of assets under management. Each symbol corresponds to one mutual fund, whereas the red dashed line corresponds to the median regression. It follows that the alpha is equal to 20% on average for small funds, whereas it is equal to 5% for large funds.

It follows from the previous results that idiosyncratic risks and specific bets disappear in large and diversified portfolios. Therefore, alpha is not scalable. Common risk factors are the only bets that are compatible with diversification. This explains why long-term investors including sovereign wealth funds and pension funds are so interested in factor investing. This conclusion has been reiterated by the report on the Norwegian Government Pension Fund. Ang *et al.* (2009) found that "*the active management activities of the Fund account for less than one percent of the overall variance*" from January 1998 to September 2009, and "*a significant part of even the very small component of the total Fund return represented by active return is linked to a number of well-recognized systematic factors*".

3 Defining alternative risk premia

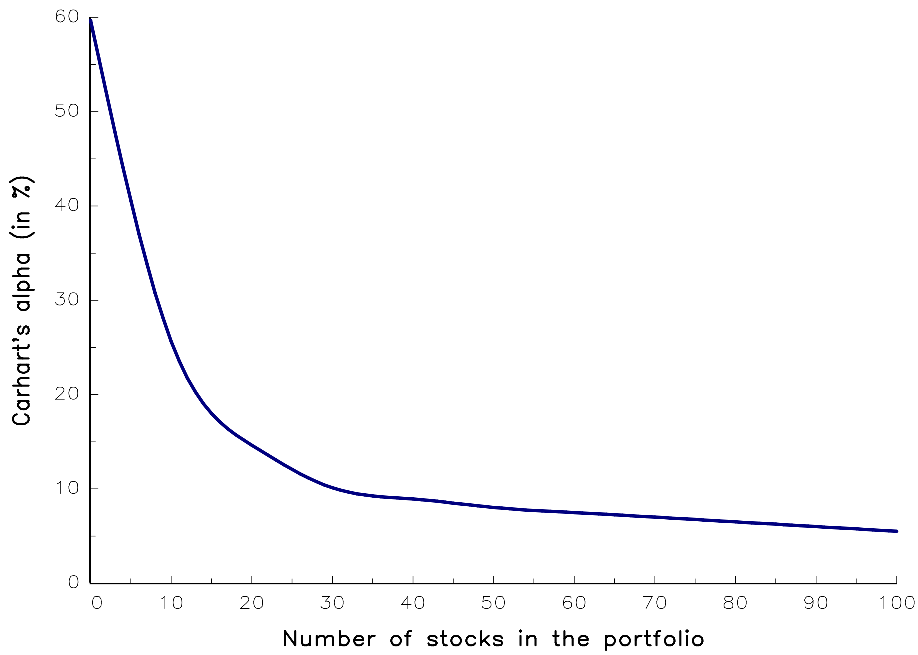
In the case of the equity asset class, we generally consider that the common risk factors are the beta, size, value, low beta, momentum and quality risk factors. The term "*factor investing*" is mostly used for designing long-only portfolios based on these risk factors. The concept of alternative risk premia is an extension of the concept of factor investing in the case of long/short portfolios for all asset classes, including rates, credit, currencies

[5] "*If you can identify six wonderful businesses, that is all the diversification you need. And you will make a lot of money. And I can guarantee that going into the seventh one instead of putting more money into your first one is going to be terrible mistake. Very few people have gotten rich on their seventh best idea.*" (Warren Buffett, University of Florida, 1998).

[6] "*Concentration is another important factor in generating high levels of incremental returns. We have managers in Yale's portfolio that will hold three or four or five stocks, or maybe eight or 10 stocks*" (David Swensen, WSJ, 2005).

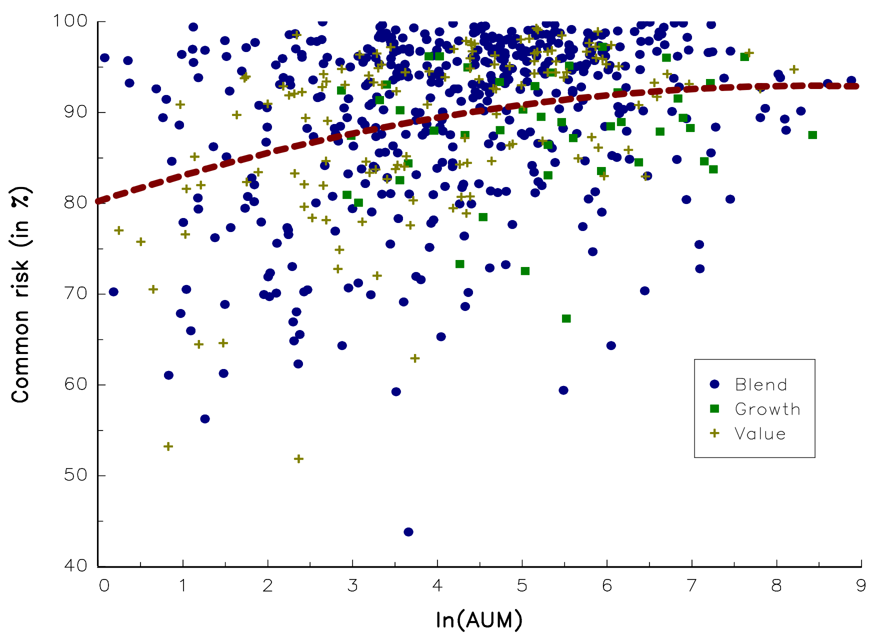
[7] The asset universe corresponds to stocks that belong to the S&P 500 index. The stocks are selected randomly and the allocation is equally-weighted. For a given number of stocks, we run 500 randomly portfolios and we calculate Carhart's alpha as one minus the mean of the $\mathfrak{R}$-squared coefficient obtained with the four-factor model.

Figure 2: Proportion of Carhart's active risk with respect to the number of holdings



Four-factor Carhart's model, equally-weighted portfolio of US stocks, 2010-2014.

Figure 3: Proportion of return variance explained by the four-factor Carhart's model



Morningstar database, 880 mutual funds, European equities, 2010-2014.

and commodities. It is interesting to notice that some asset classes, like currencies and commodities, may exhibit alternative risk premia, but no traditional risk premia.

Alternative risk premia refer to non-traditional risk premia other than long-only exposures on equities and bonds. However, alternative risk premia also refer to alternative investments and hedge fund strategies (Blin *et al.*, 2017). Whereas factor investing affects the industry of equity active management, alternative risk premia is clearly a new analysis and investment framework for multi-asset allocation and portfolios of hedge funds.

3.1 Skewness risk premia and market anomalies

Strictly speaking, a risk premium rewards an exposure to a non-diversifiable or systematic risk. For instance, the equity risk premium is defined as the reward that investors expect for being exposed to the equity risk. In fact, equity and bond risk premia are the two traditional risk premia. But there are other risk premia. A famous example is the premium embedded in cat bonds, which are insurance-linked securities that transfer catastrophe risks like hurricanes to investors. In this specific case, it is obvious that the risk taken by investors is non-diversifiable and non-hedgeable and must be rewarded. However, the existence of a risk premium is not always easy to justify for many strategies. Nevertheless, the consumption-based model of Lucas (1978) helps to better characterize the concept of risk premia. According to Cochrane (2001), the risk premium associated with an asset is equal to[8]:

$$\underbrace{\mathbb{E}_t\left[R_{t+1}-R_{f,t}\right]}_{\text{Risk premium}} \propto -\underbrace{\rho\left(u'\left(C_{t+1}\right),R_{t+1}\right)}_{\text{Correlation term}} \times \underbrace{\sigma\left(u'\left(C_{t+1}\right)\right)}_{\text{Smoothing term}} \times \underbrace{\sigma\left(R_{t+1}\right)}_{\text{Volatility term}} \tag{12}$$

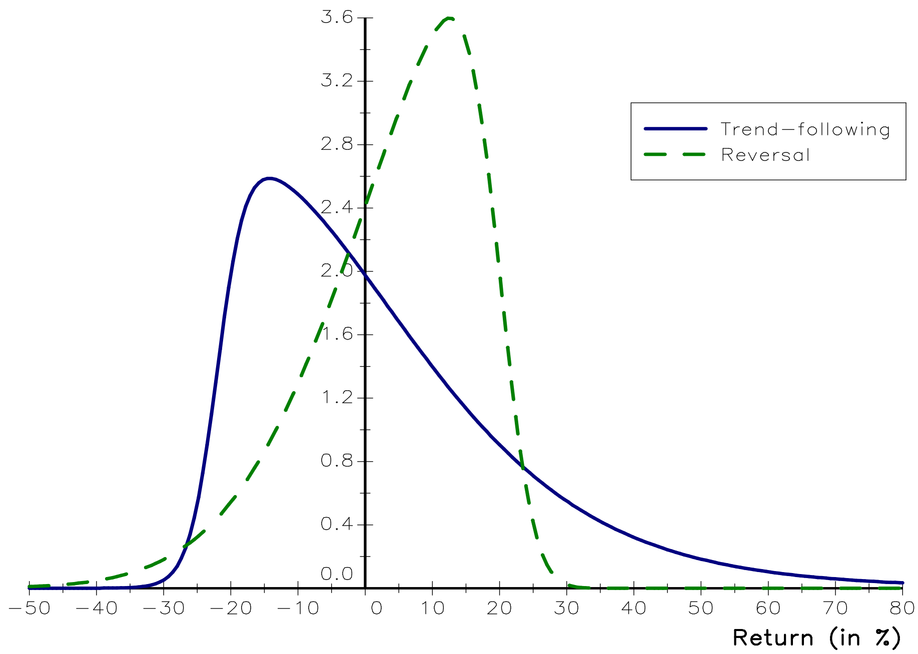
where R_{t+1} is the one-period return of the asset, $R_{f,t}$ is the risk-free rate, C_{t+1} is the future consumption and $u(C)$ is the utility function. In bad times, investors decrease their consumption and the marginal utility is high. Therefore, investors agree to pay a high price for an asset that helps to smooth their consumption. To hedge bad times, investors can use assets with a low or negative risk premium. They will invest in assets that are positively correlated with these bad times only if their risk premium is high. This is why investors require a high risk premium in order to buy assets that are negatively correlated with the marginal utility and are highly volatile. Therefore, in the consumption-based model, the risk premium is compensation for accepting risk in bad times (Ang, 2014).

The study of mean-reverting and trend-following strategies is of particular interest for understanding whether they exhibit a risk premium. In Appendix A.1 and A.2, we present a simple analytical framework in order to obtain the main properties of these two canonical strategies. We show that their probability distribution is very different (see Figure 4). The trend-following strategy has a positive skewness, a bounded loss and a significant probability of infinite gain (Potters and Bouchaud, 2006). On the contrary, the contrarian strategy has a negative skewness, a bounded gain and a significant probability of infinite loss. The contrarian strategy can then have a risk premium, but not the trend-following strategy. Moreover, the loss of the contrarian strategy generally occurs at bad times (or when the performance of traditional risk premia is very bad).

Let us come back to the equity factor investing framework. While size and value factors are two mean-reverting strategies, they can exhibit a risk premium (Hamdan *et al*, 2016).

[8] See also Martellini and Milhau (2015) or Hamdan *et al.* (2016).

Figure 4: Probability density function of contrarian and trend-following strategies



This is not the case of the momentum risk factor. Concerning low beta and quality factors, there is no evidence that they reward a non-diversifiable risk during bad times. Here we have precisely the opposite situation. During a stock market crisis, these two strategies are generally more resilient and outperform a buy-and-hold strategy in a cap-weighted index. Therefore, the good past performance of momentum, low beta and quality risk factors is not due to a risk premium, but is explained by the theory of behavioral finance[9]. When a strategy has performed well in the past and it is not due to the existence of a risk premium, it is called a market anomaly (Hou *et al.*, 2015).

In practice, investors and portfolio managers consider that alternative risk premia cover two types of strategies:

1. The pure risk premia that are also called skewness risk premia.
   They correspond to the previous definition (Lempérière *et al.*, 2014). For example, the size and value risk factors are two skewness risk premia.

2. The market anomalies.
   They correspond to trading strategies that have delivered good performance in the past, but their performance cannot be explained by the existence of a systematic risk at bad times. Their performance can only be explained by behavioral theories. For example, the momentum, low beta and quality risk factors are three market anomalies.

[9]For instance, the momentum pattern may be explained by either an under-reaction to earnings announcements and news, a delayed reaction, excessive optimism or pessimism, etc. (Barberis and Thaler, 2003). The strong performance of low beta and low volatility assets may be explained by investors' leverage aversion (Frazzini and Pedersen, 2014). The quality strategy is another good example of strong and consistent abnormal returns not related to risk (Asness *et al.*, 2014).

Figure 5: Which option profile may exhibit a risk premium?

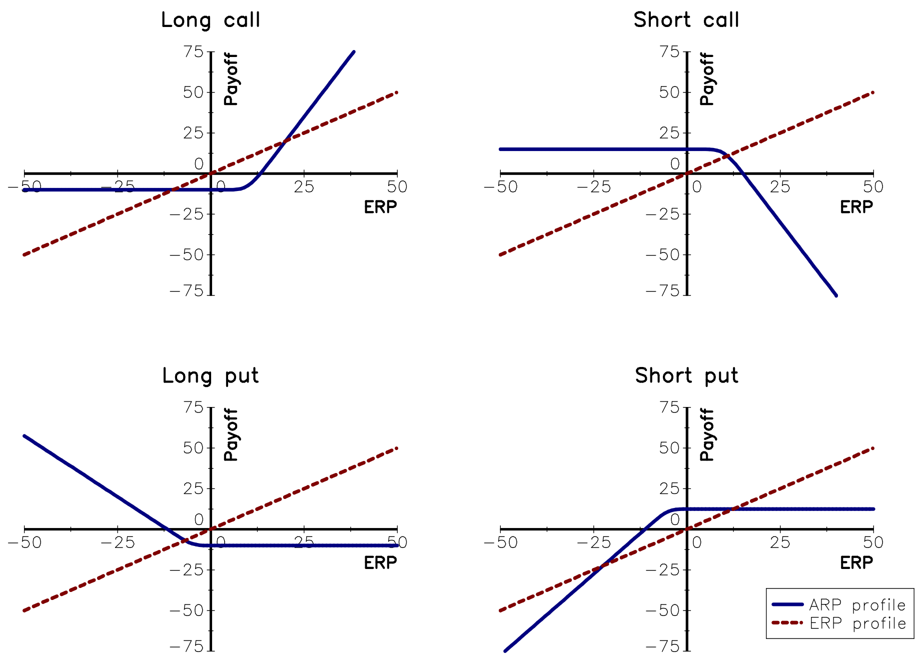
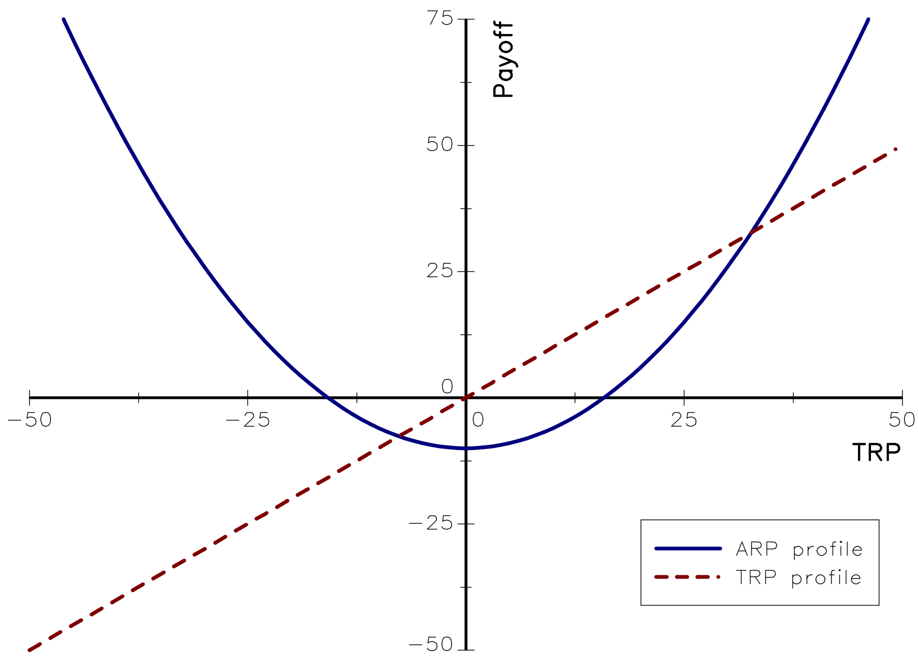


Figure 6: The case of a long straddle profile



In order to better understand the difference between a skewness risk premium and a market anomaly, we report generic payoffs of trading strategies with respect to the equity risk premium (ERP) in Figure 5. In this case, bad times correspond to the drawdown of the stock market. If the payoff function of the trading strategy is a long call, it cannot be a risk premium, because the investor is not exposed to a skewness risk. Indeed, the loss of the trading strategy is limited and small. If the payoff function of the trading strategy is a long put, again it cannot be a risk premium, because the investor is rewarded in a bear market and this strategy hedges bad times. Therefore, this is an insurance premium and not a risk premium. The case of the short call profile is interesting, because it exhibits a drawdown when the market is up. This means that the drawdown occurs in good times[10] and not in bad times. If this trading strategy has a positive expected return, it can only be a market anomaly, not a skewness risk premium. However, if the payoff function of the trading strategy is a short put, the investor takes a risk at bad times, when the performance of the equity market is negative. In this case, this type of strategy is a skewness risk premium[11]. It is interesting to relate this analysis to the trend-following strategy on multi-asset classes[12]. Fung and Hsieh (2001) showed that this strategy has a long straddle option profile[13] (Figure 6). Based on our analysis, it is obvious that this strategy is a market anomaly, because its drawdown is not correlated to bad times.

3.2 Identification of alternative risk premia

Identifying alternative risk premia is not an easy task, because there is no consensus. For instance, Harvey *et al.* (2016) found more than 300 academic publications that have exhibited new risk factors and tried to explain the cross-section of expected returns. They finally concluded that "*most claimed research findings in financial economics are likely false*". Therefore, identifying alternative risk premia cannot be reduced to backtesting a strategy and performing a statistical analysis of past performance (Cochrane, 2011). In fact, the existence of an alternative risk premium must be backed by the existence of investment products, whose goal is indeed to harvest and replicate this risk premium. Otherwise, this means that the asset management industry does not believe in this risk premium. This underlying idea is the starting point of the empirical study of Hamdan *et al.* (2016), who have compiled a database of 1 120 existing indices, which are sponsored and calculated by asset managers, banks and index providers. They have classified these products according to the mapping shown in Table 2.

The different categories of risk premia are the following: carry, event, growth, liquidity, low beta, momentum, quality, reversal, size, value, volatility. This list is certainly non-exhaustive according to academic research. However, the asset management industry has either not developed, or developed to a lesser extent, products based on the other categories, meaning that they are marginal from an investment point of view. Moreover, we notice that some categories of risk premia are not present in all asset classes. For instance, the event, growth, low beta, quality and size categories only concern the equity market. We also notice that some risk premia can be implemented in several ways and correspond to different strategies. For instance, the equity carry risk premium corresponds to the high dividend

[10] When the stock market posts a very good performance.

[11] This strategy has a negative skewness. However, a strategy that exhibits a short call option payoff may also have a negative skewness. So, the value of the skewness can not be the only criterion. Indeed, the important point is when the skewness events occur. In some sense, the concept of skewness risk premia can be related to the concept of the conditional co-skewness (Ilmanen, 2012).

[12] In the hedge fund industry, this strategy is known as the CTA strategy.

[13] This strategy performs well when the market presents a significant (positive or negative) trend and posts negative returns in rangy or reversal markets.

Table 2: Mapping of alternative risk premia

Strategy	Equities	Rates	Credit	Currencies	Commodities
Carry	Dividend futures High dividend yield	Forward rate bias Term structure slope Cross-term-structure	Forward rate bias	Forward rate bias	Forward rate bias Term structure slope Cross-term-structure
Event	Buyback Merger arbitrage				
Growth	Growth				
Liquidity	Amihud liquidity	Turn-of-the-month	Turn-of-the-month		Turn-of-the-month
Low beta	Low beta Low volatility				
Momentum	Cross-section Time-series	Cross-section Time-series	Time-series	Cross-section Time-series	Cross-section Time-series
Quality	Quality				
Reversal	Time-series Variance	Time-series		Time-series	Time-series
Size	Size				
Value	Value	Value	Value	PPP Economic model	Value
Volatility	Carry Term structure	Carry		Carry	Carry

yield strategy and the dividend futures strategy. The first strategy consists in building a portfolio that is long on stocks with high dividend yields and short on stocks with low dividend yields. The aim of the second strategy consists in capturing the difference between implied and realized dividends.

Let us briefly define the different categories[14]. The underlying idea of a carry strategy is to capture a spread or a return by betting that the underlying risk will not occur or that market conditions will stay the same (Koijen *et al.*, 2017; Baltas, 2017). One famous example of such a strategy is the currency carry trade. It consists in being long on currencies with high interest rates and short on currencies with low interest rates. If exchange rates do not change, this portfolio generates a positive return. In the case of bonds and commodities, we generally distinguish between several forms of carry strategies, depending on whether the carry is calculated using one maturity of the term structure (forward rate bias), two maturities of the same term structure (term structure slope) or one maturity of two different term structures (cross-yield curve).

The event category covers several idiosyncratic risk strategies, like merger arbitrage, convertible arbitrage and buyback strategy. The growth strategy consists in selecting stocks of companies that are growing substantially faster than others. Contrary to popular belief, this is not the same as the anti-value strategy.

In the liquidity category, we find strategies whose goal is to capture the illiquidity premium of some assets (Pàstor and Stambaugh, 2003). In the equity asset class, the most popular illiquidity measure is the Amihud ratio (Amihud, 2002). In the other asset classes, liquidity strategies consist in market timing strategies and generally exploit the turn-of-the-month effect. Indeed, some (passive) investors have to roll futures contracts at some pre-defined periods, resulting in liquidity pressures around these rolling periods. The low beta anomaly consists in building a portfolio with exposure to low volatility stocks.

Two strategies define the momentum risk premium: cross-section momentum (Jegadeesh and Titman, 1993) and time-series momentum (Moskowitz *et al.*, 2012). The two strategies assume that the past trend is a predictor of the future trend. The cross-section momentum strategy consists in building a portfolio that is long on assets that have outperformed and short on assets that have underperformed. In the case of the time-series momentum strategy, the portfolio is long on assets with a positive past trend and short on assets with a negative past trend[15].

The quality factor is a market anomaly that cannot be explained by a risk premium. It has been exhibited by Piotroski (2000) and the strategy corresponds to a portfolio long on quality stocks and short on junk stocks without any reference to market prices (Asness *et al.*, 2014). Typical quality measures include equity-to-debt, return-on-equity or income-to-sales financial ratios.

The reversal strategy is also known as the contrarian or the mean-reverting strategy. In some sense, it is the opposite of the trend-following strategy. For an asset class, the two strategies can coexist because they do not involve the same time frequency. For instance, in the case of equities, it is widely recognized that the market is contrarian in the short term

[14] See Hamdan *et al.* (2016) for a detailed explanation of each category of risk premia and the related strategies.

[15] Whereas cross-section momentum is related to relative returns, time-series momentum considers absolute returns.

(less than one month), trend-following in the medium term (between one month and two years) and mean-reverting in the long run (greater than two years). When we speak about the reversal premium, we generally consider the short-term contrarian strategy, whereas the long-term mean reverting strategy is classified with the value risk premium. Like many alternative risk premia, there are several ways to implement such a strategy. For example, it can use short-term trends (time-series reversal) or variance differences of returns between two time horizons (variance reversal).

The underlying idea of the size factor is that small stocks have a natural excess return with respect to large stocks. This excess return may be explained by a liquidity premium or because this market is less efficient than a market of large caps. In the asset management industry, this factor is only implemented in the equity asset class.

The value equity factor was popularized by Fama and French (1993, 1998). This strategy goes long on under-valuated stocks and short on over-valuated stocks. Whereas Fama and French use the price-to-book value ratio as the value measure, asset managers generally combine different financial ratios (earnings yield, dividend yield, etc.). Choosing the approach to implement the value factor is crucial, because it impacts the nature of the captured risk premium. Some products focus on the short-term value premium, whereas the majority of products try to capture the long-term value premium or the fundamental component of the value premium. In the other asset classes, the value strategy corresponds more to a long-term contrarian strategy. There are generally two main approaches for defining the long-run fundamental price. The first approach uses economic models whereas the second approach consists in estimating the long-run equilibrium price using statistical methods.

The last risk premium concerns the volatility asset class. The volatility carry risk premium corresponds to a portfolio that captures the spread between implied volatility and realized volatility. It is also known as the short volatility strategy. Another strategy concerns the term structure of VIX futures contracts, and aims to capture the roll-down effect of the slope of the term structure.

In Table 2, we notice that some risk premia are not present in all asset classes, because they are not implemented in the industry of financial indices[16]. This mapping was valid at the end of December 2015. It does not mean that it will continue to be valid in the coming years. For example, there have been some recent attempts by asset managers to apply the quality factor to the fixed-income universe. Another identification issue is the robustness of a given category. If a category contains very few products, we can consider that the risk premium is anecdotal. For example, Hamdan *et al.* (2006) only found three momentum risk premium indices on the US credit asset class. In this case, we may wonder if this risk premium really exists.

For a risk premium to be robust, there must be a sufficient number of products but they also must be sufficiently homogenous in order to represent the same common risk factor. Let us consider the case of the traditional equity risk premium in the US market. The investor has the choice between different indices to harvest this risk premium. Selecting the index is a minor problem, because the correlation between the different indices is very high[17]. This

[16] Of course, they can be implemented in other forms by the asset management industry. For example, the event factor on fixed-income instruments is implemented by some hedge funds. The fact that there is no index means that it is more a 'discretionary' strategy than a risk premium. In this case, the skill of the fund manager is essential to deliver good performance.

[17] For example, the cross-correlation between the daily returns of the S&P 500, FTSE USA, MSCI USA, Russell 1000 and Russell 3000 indices was greater than 99.5% between 2000 and 2015.

is not the case with alternative risk premia. Suppose that we have a category with five indices and that the cross-correlation between them is lower than 50%. In this case, we can believe that this category is more representative of a strategy than a risk premium. Indeed, the performance will be explained more by the portfolio construction than the intrinsic return of the common risk factor. In order to obtain a homogeneous category, Hamdan *et al.* (2016) proposed a selection procedure in order to estimate the generic performance of the risk premium. They found that some categories are so heterogeneous that it is not possible to obtain a subset of indices that present the same patterns. This is the case with the following strategies: the carry risk premium based on dividend futures, the liquidity premium in equities, rates and currencies, the value risk premium in rates and commodities, the reversal risk premium based on the variance approach and risk premia in the credit market.

3.3 Carry and momentum everywhere

According to Hamdan *et al.* (2016), the most important[18] risk premia in equities are the value risk factor, followed by the carry based on the high dividend yield approach, the low volatility, the short volatility and the momentum risk factor. In the case of currencies and commodities, the two important risk premia are the carry and momentum strategies. For the fixed-income asset class, these same risk premia are important, in addition to the short volatility strategy.

We notice that carry and momentum are the most relevant alternative risk premia. We find them in the four asset classes, even if they are differently implemented. This is particularly true for the carry risk premium. It corresponds to strategies on the term structure for rates and commodities, and income strategies for equities and currencies. It also encompasses the famous short volatility strategy. For the momentum risk premium, both cross-section and time-series strategies are appropriate.

The title of this section refers to the article of Asness *et al.* (2013) entitled 'Value and Momentum Everywhere', that found "*significant return premia to value and momentum in every asset class*". The difference comes from the fact that the approach of Asness *et al.* (2013) is based on backtesting whereas the approach of Hamdan *et al.* (2016) is based on the existence of current investment indices. It is interesting to notice that the asset management industry believes more in carry than in value, except for the equity asset class. This result may change in the future. For example, some recent research also exhibits a value pattern in the universe of corporate bonds (Bektic *et al.*, 2017; Houweling and van Zundert, 2017; Israel *et al.*, 2016). However, it is unlikely that the value risk premium enjoys the same status as carry and momentum in the case of commodities and currencies. The issue comes from the mean-reverting frequency of the value strategy. When the frequency is very low (e.g. five years), it is extremely difficult for the asset management industry to propose investment vehicles with such a long time horizon, but investors can always implement such a strategy at their own level. In the case of equities, two value strategies exist with two different mean-reverting frequencies[19]. The success of the value strategy in the equity space comes from the mixing of these two time horizons, which are shorter than the value frequency observed in the other asset classes.

[18]The importance is measured in terms of the number of homogenous indices within the category.

[19]A short-term strategy with a one-month frequency and a strategy, whose frequency is more than two years. For instance, Bourguignon and de Jong (2006) broke down the performance of the value strategy into a transitory time component and a structural time component. They showed that a large part of the performance is explained by the short-term component.

It is especially interesting to analyze all the assets with respect to these three dimensions: carry, momentum and value (see Figure 7). As seen previously, the three dimensions can be reduced to two dimensions when we consider currencies and commodities. In the case of stocks, three dimensions are not sufficient and we have to include quality, size and volatility. The case of bonds is less obvious. If we consider the results of Hamdan *et al.* (2016), they only have two dimensions. However, as explained before, new results reopen the debate, especially with the emergence of factor investing in the fixed-income asset class.

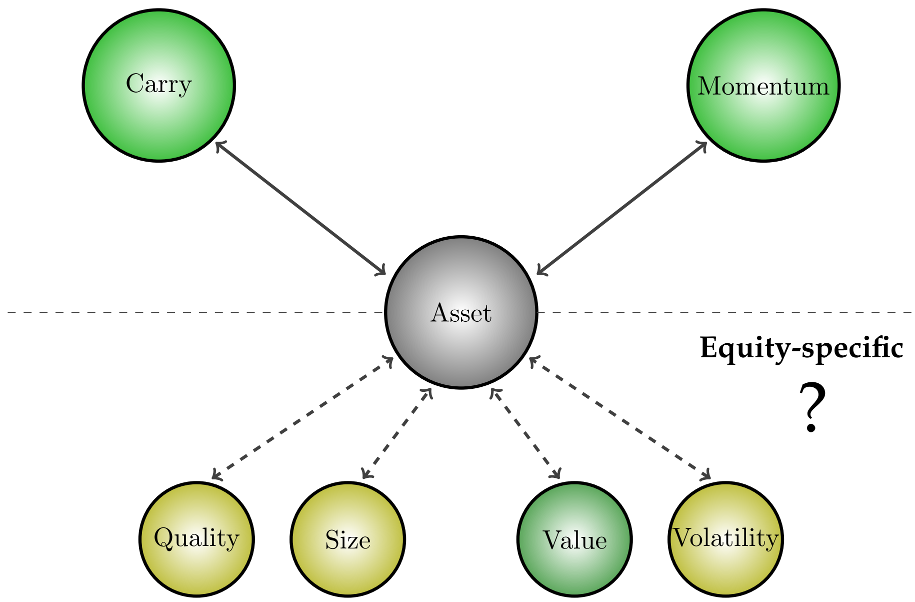


Figure 7: Risk premium analysis of an asset

Whereas equity factor investing had a big impact on the active management, alternative risk premia questions the place of hedge funds in a strategic asset allocation. Investing in hedge funds has been generally motivated by their diversification properties and ability to generate alpha with respect to a stock-bond allocation. The goal of alternative risk premia is the same. They are the primary assets of the diversification and they claim to be the new sources of performance. In fact, hedge funds and alternative risk premia are two sides of the same coin. It is no coincidence that most alternative risk premia are also hedge fund strategies. Moreover, an analysis of hedge funds shows that a part of their performance is explained by alternative risk premia (Maeso and Martellini, 2016). The results of Hamdan *et al.* (2016) exhibit that equity beta, carry and momentum are the three main factors of hedge fund returns. The carry factor takes different forms: it can be a long credit position (traditional carry), carry risk premia in rates, currencies and commodities, but also a short volatility exposure. Carry is also particularly present in relative value and event-driven hedge fund strategies. The momentum factor is the other important pillar of hedge fund strategies, particularly for CTA and managed futures strategies. In this context, alternative risk premia will have a significant impact on the hedge fund sector. But the impact will certainly be more significant on the multi-asset management industry and the design of diversified portfolios.

4 Portfolio allocation with alternative risk premia

Using a universe of alternative risk premia makes the asset allocation policy more difficult than using traditional risk premia. First, alternative risk premia are generally long-short strategies. It may be difficult to understand the behavior of some ARP with respect to a traditional long exposure on equities or bonds. Second, the skewness risk cannot be ignored and must be managed.

4.1 Volatility diversification

Let X_1 and X_2 be two random variables. The volatility of the sum is less than the sum of individual volatilities:

$$\sigma(X_1 + X_2) \leq \sigma(X_1) + \sigma(X_2) \tag{13}$$

We deduce that volatility is a convex risk measure, implying that the volatility risk can be diversified. This is one of the main objectives of stock-bond asset mix policies. However, when considering a universe of equity and bond capitalization-weighted (CW) indices for different regions, we observe a limitation to the volatility diversification. Indeed, the marginal diversification becomes very quickly close to zero. The problem comes from the fact that the asset correlation is very high within the set of equity CW indices or the set of bond CW indices. In Figure 8, we report the breakdown of eigenvalues of a covariance matrix calculated with 17 traditional risk premia[20]. We notice that the two principal components explain about 75% of the total variance of the investment universe. If we now consider the universe of alternative risk premia, we observe that there is more volatility diversification. Indeed, the two principal components explains about 50% of the total variance of the investment universe. Five principal components are sufficient to explain more than 90% of the total variance of the TRP universe. In the case of the ARP universe, we need more than 20 principal components.

The reason for this impressive volatility diversification comes from the fact that the average correlation between alternative risk premia is very low and close to 10%. For traditional risk premia, the average correlation is higher and about 50%. This difference in correlation has a big impact on diversified portfolios. Whereas the volatility of a diversified equity-bond portfolio is between 6% and 9%, the volatility of a well-diversified ARP portfolio may easily be below[21] 2%. However, even if the volatility risk of an ARP portfolio is low, it does not mean that the drawdown risk is low.

4.2 Skewness aggregation

The skewness of a random variable X is defined as:

$$\gamma_1(X) = \frac{\mu_3(X)}{\mu_2(X)^{3/2}} \tag{14}$$

where $\mu_n(X)$ is the n^{th} central moment of X. Contrary to the volatility, the skewness is not a convex risk measure, meaning that[22]:

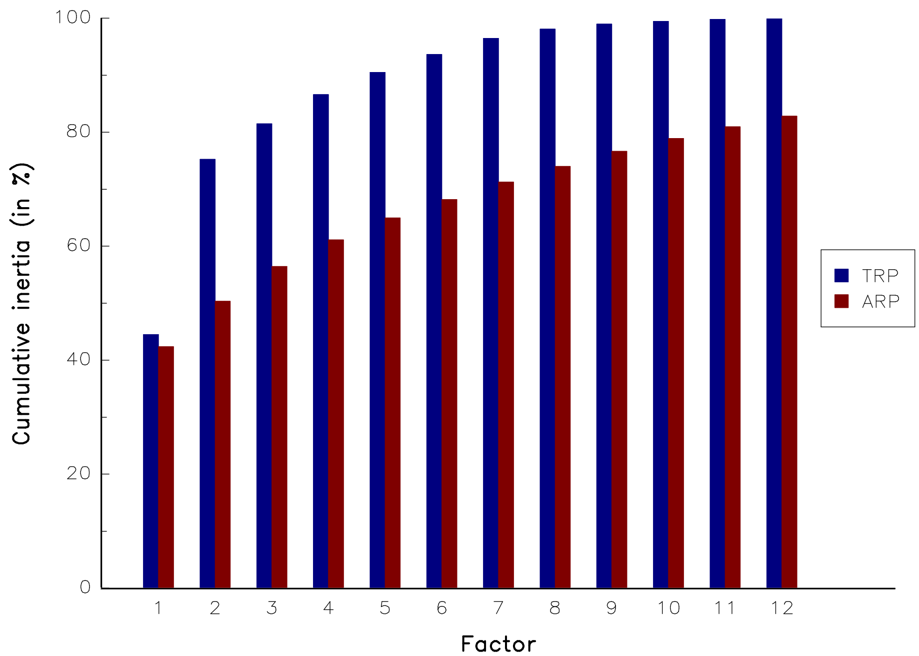
$$|\gamma_1(X_1 + X_2)| \gtreqless |\gamma_1(X_1) + \gamma_1(X_2)| \tag{15}$$

[20] The set is composed of 8 equity indices, 7 bond indices, 2 currency indices and 1 commodity index.

[21] This explains that ARP investment products are generally leveraged in order to obtain a higher volatility.

[22] We use the absolute value because the skewness can be either positive or negative.

Figure 8: Principal component analysis of TRP and ARP investment universes



Source: Hamdan *et al.* (2016).

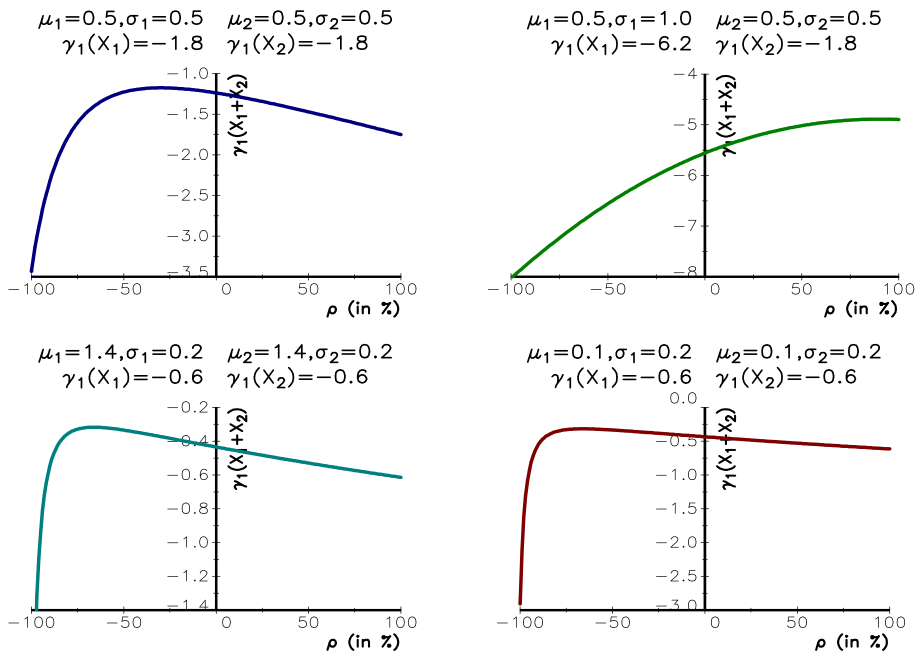
Therefore, the skewness of the sum may be lower or greater than the sum of individual skewness coefficients. We illustrate this property in Figure 9. For that, we assume that the opposite of the random vector $X = (X_1, X_2)$ follows a bivariate log-normal distribution:

$$-\left(\begin{array}{c} X_1 \\ X_2 \end{array}\right) \sim \mathcal{LN}\left(\left(\begin{array}{c} \mu_1 \\ \mu_2 \end{array}\right), \left(\begin{array}{cc} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{array}\right)\right) \tag{16}$$

For different sets of parameters, we report the relationship between the correlation parameter ρ of the log-normal distribution and the aggregated skewness coefficient $\gamma_1(X_1 + X_2)$. We notice that the highest skewness (in absolute value) is always reached when the parameter ρ is equal to -1 or when the aggregated volatility is minimum. This means that the diversification of the second moment is faster than the diversification of the third moment. In the case of the fourth panel in Figure 9, we notice that $\gamma_1(X_1 + X_2) \in [-2.91, -0.31]$ whereas the individual skewness is equal to -0.6. The skewness risk of a portfolio can therefore be larger than the skewness risk of the assets that belong to the portfolio.

These examples show that there is maximum diversification if we consider the skewness risk measure. The problem is twofold. First, volatility diversification is a limiting factor for skewness diversification. Indeed, by decreasing the volatility, we implicitly increase the skewness coefficient, all other things being equal. Second, the diversification of the third moment is an issue too, in that it is extremely difficult to hedge large losses. How can we explain this discrepancy between the behavior of the second moment and the behavior of the third moment? The answer lies in understanding the stochastic dependence between skewness risk premia. When a stochastic process exhibits high skewness, we generally break it down into a trend component, a Brownian component and a singular component. Unlike

Figure 9: Illustration of skewness aggregation with the bivariate log-normal distribution



regular and irregular variations that are easy to diversify, it is difficult to hedge discontinuous variations. In their simplest form, these singular variations are jumps. The worst-case scenario concerning skewness aggregation is thus to build a well-diversified portfolio by dramatically reducing the volatility of the portfolio. Indeed, it is extremely difficult to diversify the negative jump of an asset. For that, we need to find a second asset that jumps at the same time and has a positive jump. Moreover, bad times of skewness risk premia tend to occur at the same time. By accumulating alternative risk premia, we then increase the volatility diversification and reduce the absolute value of the drawdown, but the drawdown of the portfolio compared to its realized volatility appears to be very high. This explains that the Sharpe ratio is not the right measure for evaluating the risk/return ratio of an ARP portfolio.

4.3 Portfolio management

In order to establish clear rules about asset allocation, we have to understand the significance of the skewness risk[23]. In the top panel in Figure 10, we report the cumulative performance of US equities[24] and the US volatility carry premium[25]. If we consider weekly returns, it appears that the skewness of the US volatility carry premium is 13 times the skewness of US equities. This high skewness risk is explained by the magnitude of historical drawdowns with respect to the historical volatility. Indeed, we notice that the short volatility strategy experienced very low volatility most of times, implying that this risk premium seems to have a very low risk during long historical periods. However, in a period of stress, the short volatility strategy may suffer greatly, and its drawdowns appear very large compared to the

[23] See Jurczenko and Maillet (2006) for a review of the literature on portfolio management with skewness.
[24] It is approximated by the S&P 500 index
[25] We use the generic performance of the US short volatility strategy obtained by Hamdan *et al.* (2016).

observed volatility. Moreover, the drawdowns occur suddenly and correspond to negative jumps. In the case of equities, the drawdowns are also very large in absolute value, but they are relatively in line with the volatility of the stock market. Moreover, the drawdowns are generally accompanied by an increase in volatility, implying that generating said drawdowns is a more gradual process. Therefore, the skewness risk corresponds to a drawdown risk produced by a sudden jump. The short volatility strategy is emblematic of the skewness risk as it is certainly the most skewed alternative risk premium.

In Appendix A.4, we consider a classic jump-diffusion process for modeling asset returns. It follows that the associated density function can be approximated by a Gaussian mixture model with two regimes:

- a normal regime, whose returns are driven by a multivariate Gaussian distribution;
- a jump regime, whose returns are driven by another multivariate Gaussian distribution.

By construction, the occurrence probability of the jump regime is very low compared to the normal regime. This framework, which has been developed by Bruder *et al.* (2016), is very appealing because it reproduces many stylized facts concerning alternative risk premia. In Figure 11, we have reported the Sharpe ratio, the volatility and the skewness of a portfolio invested in n ARP strategies[26], whose density function of returns is given by Equation (34) in page 30. We notice that the Sharpe ratio increases dramatically with the number of ARP in the portfolio. This is due to the volatility diversification. However, we also notice that the skewness risk increases, even if the third moment decreases in absolute value. Therefore, a short-sighted investor feels that the risk decreases by accumulating skewness risk premia, because the volatility goes to zero. However, the relative drawdown[27] becomes higher. As this drawdown appears suddenly at a very low frequency, the short-sighted investor believes that its portfolio has low risk until the occurrence of the drawdown. It follows that the Sharpe ratio is not a good risk-return measure when considering alternative risk premia. In fact, the volatility risk is not a big concern. Investors are more focused on the absolute performance and the expected drawdown of such strategies.

By applying the Gaussian mixture model to weekly returns, we obtained the probability density functions given in Figure 10. Here, the frequency parameter λ is equal to 26%, meaning that we observe a jump every four years on average[28]. We notice that the density function of the normal regime in the Gaussian mixture normal is relatively close to the density function of the traditional Gaussian model in the case of US equities. The volatility is then an acceptable risk measure for such assets. In the case of the short volatility strategy, we obtain another story. The jump regime has a big impact on the behavior of this risk premia. In the normal regime, the volatility carry strategy has a Sharpe ratio of about 3. However, this strong risk-return ratio is offset by a high jump risk, which cannot be modeled by the normal regime. This is the story of most investors between 2003 and 2007, who underestimated the risk of such a strategy.

[26] We assume that the ARP strategies have the same characteristics: $\mu_i = 7\%$, $\sigma_i = 4\%$, $\tilde{\mu}_i = -3\%$ and $\tilde{\sigma}_i = 4\%$. The parameter λ is equal to 25% meaning that we observe a jump every four years on average. The correlation between two ARP strategies is uniform and we have $\rho_{i,j} = 10\%$ for the normal regime and $\rho_{i,j} = 50\%$ for the jump regime.

[27] The drawdown measured in absolute value decreases as shown by the behavior of the third moment. The relative drawdown is computed as the ratio between the absolute drawdown and the volatility. It is less than 2.5 for traditional risk premia. For some alternative risk premia, it may be equal to 5.

[28] In the case of the traditional Gaussian model, the estimated parameters are $\mu = 6.09\%$ and $\sigma = 18.38\%$ for US equities and $\mu = 6.00\%$ and $\sigma = 5.50\%$ for the US short volatility. In the case of the Gaussian mixture model, the estimated parameters are $\mu = 7.89\%$, $\sigma = 15.64\%$, $\tilde{\mu} = -1.20\%$ and $\tilde{\sigma} = 6.76\%$, for US equities and $\mu = 10.10\%$, $\sigma = 2.91\%$, $\tilde{\mu} = -2.23\%$ and $\tilde{\sigma} = 2.57\%$ for the US short volatility.

Figure 10: Skewness risk of US equities and US short volatility premium

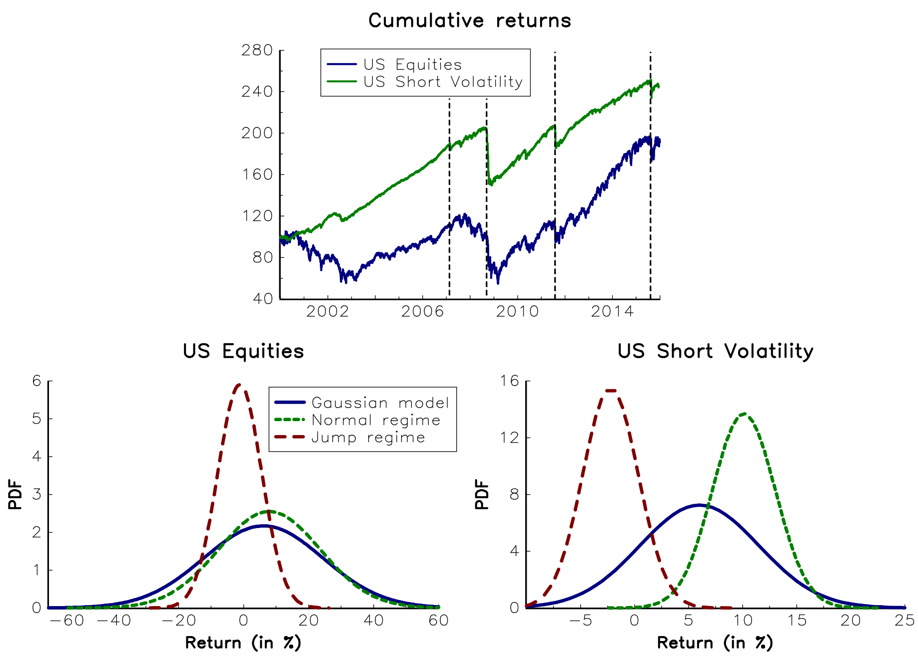
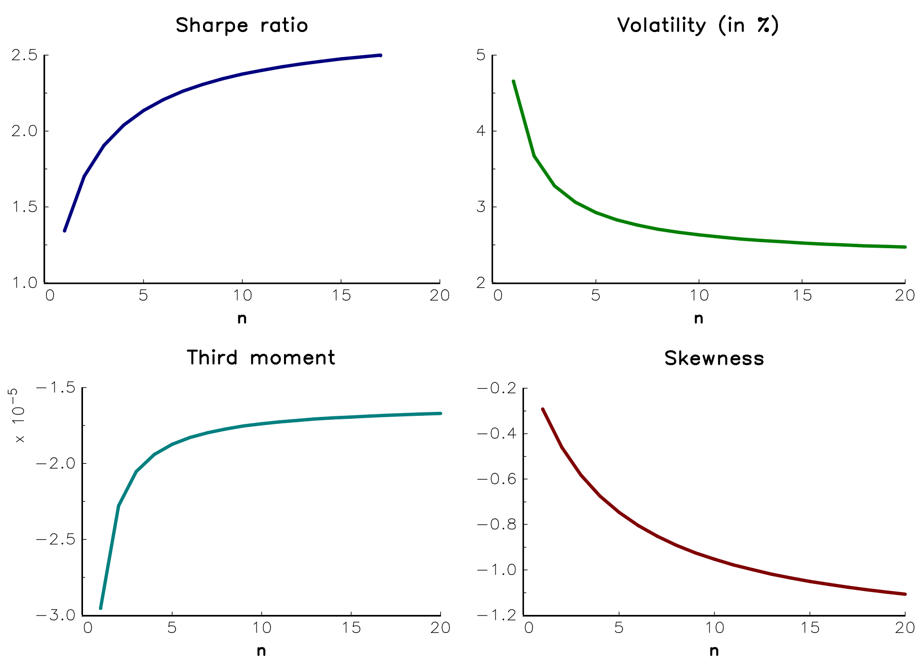


Figure 11: Risk of the portfolio with respect to the number of assets



We have seen previously that risk budgeting is the right approach for building a diversified portfolio, and alternative risk premia are the common risk factors for diversifying a strategic asset allocation. Therefore, professionals have naturally combined the two approaches in order to provide well-diversified multi-asset portfolios. Generally, the construction of the portfolio is a two-stage process. First, the manager selects the best alternative risk premia. Second, the portfolio is rebalanced at a fixed frequency by defining volatility risk budgets. However, we have seen that the volatility is certainly not relevant to assess the risk of skewness risk premia, because we cannot manage their bad times with a traditional risk parity method[29]. Moreover, the occurrence of a drawdown of a given skewness risk premium is followed by an increase of the realized volatility, implying that the risk parity portfolio reduces dramatically the allocation on this strategy. However, it is generally too late. If we consider again the short volatility strategy, we notice that the strategy rebounds sharply after a drawdown. Therefore, the optimal investment decision is not to reduce, but to maintain or increase the exposure.

Bruder *et al.* (2016) propose to replace the volatility risk measure of the risk budgeting method by the expected shortfall[30] based on the Gaussian mixture model. Their approach has the advantage of taking into account the skewness risk and eliminating the jumps in the allocation. This allocation is then more stable, because the risk measure integrates ex-ante the jump risk, meaning that the dynamic of the allocation is mainly driven by the true volatility and not by jumps. This point is very important, because we understand that the nature of the skewness risk is different than the nature of the volatility risk in terms of allocation dynamics. The skewness risk is a decision of strategic asset allocation, implying that the investor must allocate a skewness risk budget for each risk premium in the long-run and stick to this allocation even if a drawdown occurs. The volatility risk is a decision of tactical asset allocation, implying that the investor may dynamically change the allocation by considering the true volatility of risk premia. Therefore, the challenge is to separate volatility and skewness effects. For instance, the empirical volatility is a biased estimator of the true volatility, because it incorporates jumps. This is why we have to adopt filtering approaches for estimating the volatility of alternative risk premia.

The approach of Bruder *et al.* (2016) can be simplified as follows. Suppose that we would like to allocate the risk budgets $b_1, \ldots, b_n$ to a universe of n risk premia. The idea is to transform these risk budgets that incorporate the skewness risk into new risk budgets $b'_1, \ldots, b'_n$ that are only based on the volatility risk. We can then manage the portfolio by using a traditional risk budgeting approach and a filtered covariance matrix, which do not take into account skewness events. This simplified approach shows that skewness risk premia and market anomalies do not have the same status. For instance, if we wanted to allocate the same risk budget between a skewness risk premium and a market anomaly, this implies that the volatility budget will be higher for the market anomaly.

[29]By traditional risk party, we mean an equal risk contribution portfolio based on the volatility risk measure.

[30]See also Jurczenko and Teïletche (2015) and Roncalli (2015) for risk budgeting methods based on the expected shortfall.

5 Conclusion

Alternative risk premia cover two types of strategy: skewness risk premia and market anomalies. Skewness risk premia reward systematic risks taken by investors in bad times. An example is the short volatility strategy, and more generally carry strategies. Market anomalies correspond to trading strategies that have delivered good performance in the past, but their performance can be explained by behavioral theories, but not by a skewness risk. For instance, momentum is a market anomaly.

Diversification covers two main risks: volatility risk and skewness risk. It is very important to understand that volatility diversification is very different to skewness diversification. In particular, managing the skewness risk is a strategic asset allocation decision, whereas managing the volatility risk is a tactical asset allocation decision. Moreover, we notice that it is extremely difficult to hedge the skewness risk, because there is a floor to skewness diversification.

Alternative risk premia and diversification are highly related. Until recently, multi-asset allocation was reduced to stock-bond and country allocation. Alternative risk premia are now an extension to the traditional risk premia universe. Investors have then a large choice of building blocks or primary assets. Of course, this new approach challenges the place of hedge funds in a strategic asset allocation. Moreover, it also participates in the debate about alpha versus beta, but also in the debate about passive management versus active management. Every day, the importance of alpha is decreasing alarmingly, implying that the portfolio performance is mainly explained by systematic risk factors and not by specific risk factors. And the emergence of alternative risk premia renews risk/return and benchmarking analysis. However, it does not mean that active management does not play an important role in this context. Whereas it is more efficient to capture traditional betas using passive management, it is not straightforward that it is the same thing for alternative betas. Let us take the case of carry and momentum risk premia. Even if these two premia are theoretically well-defined, there are many ways for implementing them. We can harvest them using an index that encapsulates a fully detailed systematic strategy or using a portfolio manager that considers a more sophisticated quantitative model, which can be adapted to the investment and liquidity environment[31]. Certainly, these two approaches will co-exist, meaning that the shift of active management from alpha towards alternative risk premia has just begun.

This chapter is dedicated to alternative risk premia based on traditional financial assets (equities, rates, credit, currencies and commodities). Another important question concerns the place of risk premia on 'alternative' assets (real estate, private debt, private equity, infrastructure) in a strategic asset allocation. By construction, the asset allocation policy between these risk premia cannot be driven by volatility diversification. Therefore, skewness[32] diversification remains the main issue when managing a portfolio of real assets.

[31]Moreover, the allocation between alternative risk premia with respect to macro-economic factors remains an open question for active management.

[32]in a very broad sense including cross-skewness and time-skewness management.

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A Mathematical results

A.1 The contrarian (or reversal) strategy with a price target

Let S_t be the price of an asset. We assume that S_t follows a geometric Brownian motion:

$$\mathrm{d}S_t = \mu_t S_t \,\mathrm{d}t + \sigma_t S_t \,\mathrm{d}W_t \tag{17}$$

The reversal strategy is described by the number of assets $f(S_t)$ held at time t:

$$f(S_t) = m\frac{\left(\bar{S} - S_t\right)}{S_t} \tag{18}$$

where $\bar{S}$ is the price target of the asset and $m > 0$. If the current price is lower than the target level ($S_t \leq \bar{S}$), the nominal exposure $f(S_t) S_t$ is positive. On the contrary, we obtain a short exposure if the current price is higher than the target level. Hamdan *et al.* (2016) showed that:

$$X_T - X_0 = m\bar{S}\ln\frac{S_T}{S_0} - m\left(S_T - S_0\right) + \frac{m}{2}\bar{S}\int_0^T \sigma_t^2 \,\mathrm{d}t \tag{19}$$

We obtain a concave payoff with positive vega. Therefore, the strategy benefits from the volatility risk. Hamdan *et al.* (2016) also demonstrated that the skewness of this strategy is always negative.

A.2 The trend-following strategy with an EWMA trend

We assume that S_t follows a geometric Brownian motion with constant volatility, but a time-varying unobservable trend:

$$\left\{\begin{array}{l} \mathrm{d}S_t = \mu_t S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t \\ \mathrm{d}\mu_t = \gamma \,\mathrm{d}W_t^{\star} \end{array}\right. \tag{20}$$

We estimate the trend using the exponentially moving average estimator defined as follows:

$$\hat{\mu}_t = \lambda \int_0^t e^{-\lambda(t-s)} \,\mathrm{d}y_s + e^{-\lambda t}\hat{\mu}_0 \tag{21}$$

where $y_t = \ln S_t$ and $\lambda = \gamma/\sigma$. The trend-following strategy is defined by the following nominal exposure:

$$\frac{\mathrm{d}X_t}{X_t} = m\hat{\mu}_t \frac{\mathrm{d}S_t}{S_t} \tag{22}$$

where m is the parameter of position sizing. The exposure is an increasing function of the estimated trend. In particular, we obtain a long portfolio if $\hat{\mu}_t > 0$ and a short portfolio otherwise. Hamdan *et al.* (2016) showed that the performance of the trend-following strategy is equal to:

$$\ln\frac{X_T}{X_0} = m\frac{\left(\hat{\mu}_T^2 - \hat{\mu}_0^2\right)}{2\lambda} + \frac{m}{2}\left(\int_0^T \hat{\mu}_t^2 \left(2 - m\sigma^2\right) \,\mathrm{d}t - \lambda\sigma^2 T\right) \tag{23}$$

We obtain a convex payoff with negative vega. Therefore, the strategy is penalized by the volatility risk. Hamdan *et al.* (2016) also demonstrated that the skewness of this strategy is always positive.

A.3 Skewness aggregation of two log-normal random variables

We assume that (X_1, X_2) follows a bivariate log-normal distribution. This implies that $\ln X_1 \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right)$ and $\ln X_2 \sim \mathcal{N}\left(\mu_2, \sigma_2^2\right)$. Moreover, we note ρ the correlation between $\ln X_1$ and $\ln X_2$. The skewness of X_1 is equal to:

$$\gamma_1\left(X_1\right) = \frac{e^{3\sigma_1^2} - 3e^{\sigma_1^2} + 2}{\left(e^{\sigma_1^2} - 1\right)^{3/2}} \tag{24}$$

whereas the skewness of $X_1 + X_2$ is equal to:

$$\gamma_1\left(X_1 + X_2\right) = \frac{\mu_3\left(X_1 + X_2\right)}{\mu_2^{3/2}\left(X_1 + X_2\right)} \tag{25}$$

where $\mu_n(X)$ is the n^{th} central moment of X. We can show that:

$$\mu_2\left(X_1 + X_2\right) = \mu_2\left(X_1\right) + \mu_2\left(X_2\right) + 2\operatorname{cov}\left(X_1, X_2\right) \tag{26}$$

where:

$$\mu_2\left(X_1\right) = e^{2\mu_1 + \sigma_1^2}\left(e^{\sigma_1^2} - 1\right) \tag{27}$$

and:

$$\operatorname{cov}\left(X_1, X_2\right) = \left(e^{\rho\sigma_1\sigma_2} - 1\right) e^{\mu_1 + \frac{1}{2}\sigma_1^2} e^{\mu_2 + \frac{1}{2}\sigma_2^2} \tag{28}$$

For the third moment of $X_1 + X_2$, we use the following formula:

$$\mu_3\left(X_1 + X_2\right) = \mu_3\left(X_1\right) + \mu_3\left(X_2\right) + 3\left(\operatorname{cov}\left(X_1, X_1, X_2\right) + \operatorname{cov}\left(X_1, X_2, X_2\right)\right) \tag{29}$$

where:

$$\mu_3\left(X_1\right) = e^{2\mu_3 + \frac{3}{2}\sigma_1^2}\left(e^{3\sigma_1^2} - 3e^{\sigma_1^2} + 2\right) \tag{30}$$

and:

$$\operatorname{cov}\left(X_1, X_1, X_2\right) = \left(e^{\rho\sigma_1\sigma_2} - 1\right) e^{2\mu_1 + \sigma_1^2 + \mu_2 + \frac{\sigma_2^2}{2}}\left(e^{\sigma_1^2 + \rho\sigma_1\sigma_2} + e^{\sigma_2^2} - 2\right) \tag{31}$$

A.4 A skewness model of asset returns

For modeling the skewness risk of a portfolio, Bruder *et al.* (2016) assume that the vector of asset prices $S_t = (S_{1,t}, \ldots, S_{n,t})$ follows a jump-diffusion process:

$$\left\{\begin{array}{l} \mathrm{d}S_t = \operatorname{diag}\left(S_t\right)\mathrm{d}L_t \\ \mathrm{d}L_t = \mu\,\mathrm{d}t + \Sigma^{1/2}\,\mathrm{d}W_t + \mathrm{d}Z_t \end{array}\right. \tag{32}$$

where μ and Σ are the vector of expected returns and the covariance matrix, W_t is a n-dimensional standard Brownian motion and Z_t is the irregular component independent from W_t. More precisely, $Z_t = \sum_{i=1}^{N_t} Z_i$ is a pure n-dimensional compound Poisson process with a finite number of jumps, where N_t is a scalar Poisson process with constant intensity parameter $\lambda > 0$, and $Z_1, \ldots, Z_{N_t}$ are vectors of *i.i.d.* random jump amplitudes with law $\nu(\mathrm{d}z)$. They also assume that $\nu(\mathrm{d}z) = \lambda f(z)\,\mathrm{d}z$ where $f(z)$ is the probability density function of the multivariate Gaussian distribution $\mathcal{N}\left(\tilde{\mu}, \tilde{\Sigma}\right)$, $\tilde{\mu}$ is the expected value of jump amplitudes and $\tilde{\Sigma}$ is the associated covariance matrix.

A.4.1 Probability distribution of asset returns

When λ is sufficiently small, we can show that asset returns[33] $R_t = (R_{1,t}, \ldots, R_{n,t})$ have the following multivariate density function:

$$\begin{aligned} f(y) &= \frac{1-\lambda\,\mathrm{d}t}{(2\pi)^{n/2}\left|\Sigma\,\mathrm{d}t\right|^{1/2}} e^{-\frac{1}{2}(y-\mu\,\mathrm{d}t)^\top(\Sigma\,\mathrm{d}t)^{-1}(y-\mu\,\mathrm{d}t)} + \\ &\quad \frac{\lambda\,\mathrm{d}t}{(2\pi)^{n/2}\left|\Sigma\,\mathrm{d}t+\tilde{\Sigma}\right|^{1/2}} e^{-\frac{1}{2}(y-(\mu\,\mathrm{d}t+\tilde{\mu}))^\top\left(\Sigma\,\mathrm{d}t+\tilde{\Sigma}\right)^{-1}(y-(\mu\,\mathrm{d}t+\tilde{\mu}))} \end{aligned} \tag{34}$$

It follows that it is equivalent to using a Gaussian mixture distribution for modeling asset returns. There are two regimes:

- The '*normal*' regime has the probability $(1-\lambda\,\mathrm{d}t)$ of occurring. In this case, asset returns are driven by the Gaussian distribution $\mathcal{N}(\mu\,\mathrm{d}t, \Sigma\,\mathrm{d}t)$.
- The 'jump' regime has the probability $\lambda\,\mathrm{d}t$ of occurring. In this case, asset returns jump simultaneously and the jump amplitudes are driven by the Gaussian distribution $\mathcal{N}\left(\tilde{\mu}, \tilde{\Sigma}\right)$.

We can show that the two first moments of asset returns are:

$$\mathbb{E}[R_t] = \mu\,\mathrm{d}t + \pi\tilde{\mu} \tag{35}$$

and:

$$\operatorname{cov}(R_t) = \left(\Sigma + \lambda\tilde{\Sigma}\right)\mathrm{d}t + \lambda(1-\lambda\,\mathrm{d}t)\,\tilde{\mu}\tilde{\mu}^\top\,\mathrm{d}t \tag{36}$$

For the skewness coefficient of Asset i, we obtain the following expression:

$$\gamma_1(R_{i,t}) = \frac{\lambda(1-\lambda\,\mathrm{d}t)\left((1-2\lambda\,\mathrm{d}t)\,\tilde{\mu}_i^3 + 3\tilde{\mu}_i\tilde{\sigma}_i^2\right)\mathrm{d}t}{\left(\left(\sigma_i^2+\lambda\tilde{\sigma}_i^2\right)\mathrm{d}t + \lambda(1-\lambda\,\mathrm{d}t)\,\tilde{\mu}_i^2\,\mathrm{d}t\right)^{3/2}} \tag{37}$$

A.4.2 Probability distribution of the portfolio's return

Let $x = (x_1, \ldots, x_n)$ be the vector of weights in the portfolio satisfying $\sum_{i=1}^n x_i = 1$. We note $R_t(x)$ the portfolio's return:

$$R_t(x) = \sum_{i=1}^n x_i R_{i,t} \tag{38}$$

Bruder *et al.* (2016) show that $R_t(x)$ has the following probability density function:

$$\begin{aligned} f(y) &= (1-\lambda\,\mathrm{d}t)\frac{1}{\sqrt{x^\top(\Sigma\,\mathrm{d}t)\,x}}\phi\left(\frac{y - x^\top(\mu\,\mathrm{d}t)}{\sqrt{x^\top(\Sigma\,\mathrm{d}t)\,x}}\right) + \\ &\quad (\lambda\,\mathrm{d}t)\frac{1}{\sqrt{x^\top\left(\Sigma\,\mathrm{d}t+\tilde{\Sigma}\right)x}}\phi\left(\frac{y - x^\top(\mu\,\mathrm{d}t+\tilde{\mu})}{\sqrt{x^\top\left(\Sigma\,\mathrm{d}t+\tilde{\Sigma}\right)x}}\right) \end{aligned} \tag{39}$$

[33] The return $R_{i,t}$ of asset i is defined for the holding period $[t-\mathrm{d}t, t]$:

$$R_{i,t} = \ln S_{i,t} - \ln S_{i,t-\mathrm{d}t} \tag{33}$$

We obtain a Gaussian mixture distribution. We can show that:

$$\mathbb{E}\left[R_t\left(x\right)\right] = x^\top\left(\mu + \lambda\tilde{\mu}\right)\,\mathrm{d}t \tag{40}$$

and:

$$\sigma\left(R_t\left(x\right)\right) = \sqrt{x^\top\left(\Sigma + \lambda\tilde{\Sigma}\right)x\,\mathrm{d}t + \lambda\left(1 - \lambda\,\mathrm{d}t\right)\left(x^\top\tilde{\mu}\right)^2\,\mathrm{d}t} \tag{41}$$

For the skewness coefficient, we obtain:

$$\gamma_1\left(R_t\left(x\right)\right) = \frac{\lambda\left(1 - \lambda\,\mathrm{d}t\right)\left(\left(1 - 2\lambda\,\mathrm{d}t\right)\left(x^\top\tilde{\mu}\right)^3 + 3\left(x^\top\tilde{\mu}\right)\left(x^\top\tilde{\Sigma}x\right)\right)\,\mathrm{d}t}{\left(x^\top\left(\Sigma + \lambda\tilde{\Sigma}\right)x\,\mathrm{d}t + \lambda\left(1 - \lambda\,\mathrm{d}t\right)\left(x^\top\tilde{\mu}\right)^2\,\mathrm{d}t\right)^{3/2}} \tag{42}$$