From Portfolio Optimization to Risk Parity[1]

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[1] The opinions expressed in this presentation are those of the author and are not meant to represent the opinions or official positions of Lyxor Asset Management.

Outline

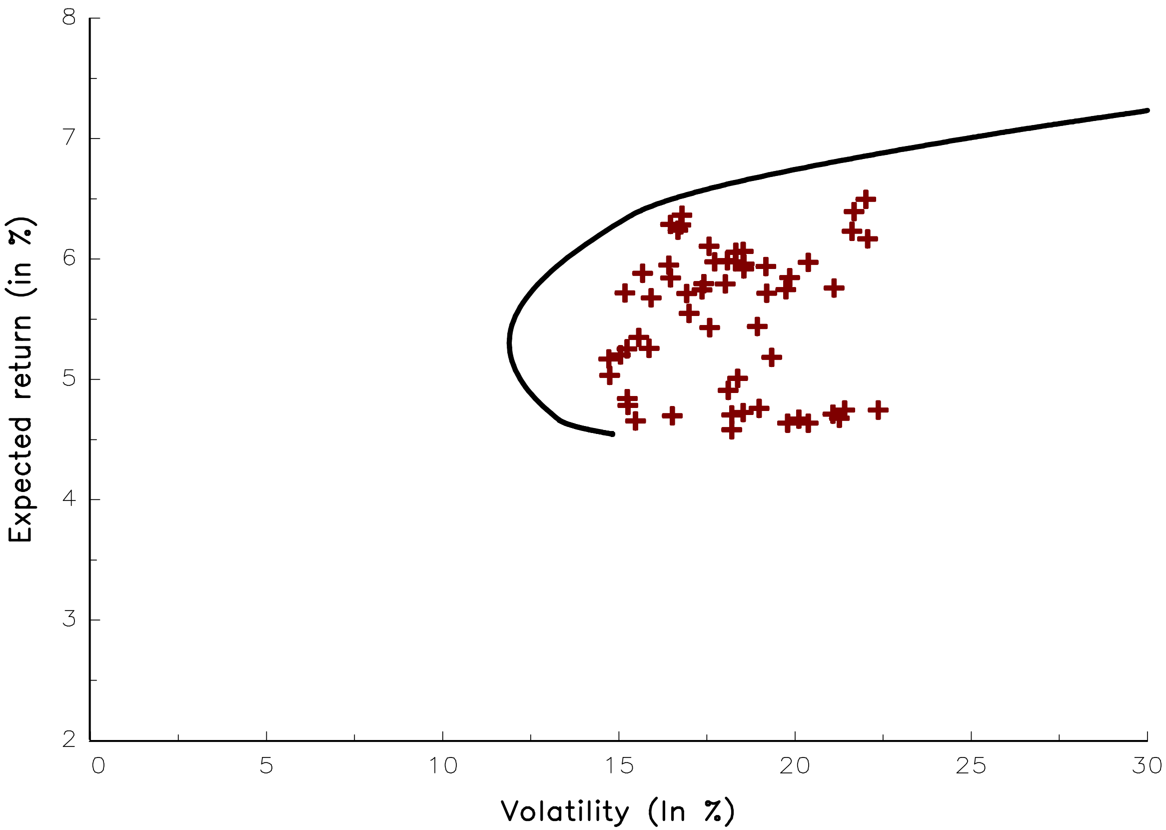
Executive summary

Over the last fifty years, mean-variance optimization has been widely used to manage asset portfolios and to build strategic asset allocations. However, it faces some stability issues because of its tendency to maximize the effects of estimation errors. At the end of the eighties, researchers began to develop some regularization methods to avoid these stability issues. For example, Michaud used resampling techniques of the objective function whereas Ledoit and Wolf introduced some new shrinkage estimators of the covariance matrix. More recently, results on ridge and lasso regressions have been considered to improve Markowitz portfolios. But, even if all these appealing methods give some answers to the regularization problem, portfolio managers prefer to use a less sophisticated method by constraining directly the weights of the portfolio. As shown by Jagannathan and Ma (2003), this approach could be viewed as a Black-Litterman approach or a shrinkage method. For some years now, another route has been explored by considering some heuristic methods like the minimum variance, equal risk contribution, or equally-weighted portfolios. These portfolios are special cases of a more general allocation approach based on risk budgeting methods (called also risk parity). This approach has opened a door to develop new equity and bond benchmarks (risk-based indexation) and to propose new multi-assets allocation styles (risk-balanced allocation).

Some issues on Markowitz portfolios

- The market portfolio theory
- Portfolio optimization and active management
- Stability issues

The market portfolio theory
The efficient frontier of Markowitz



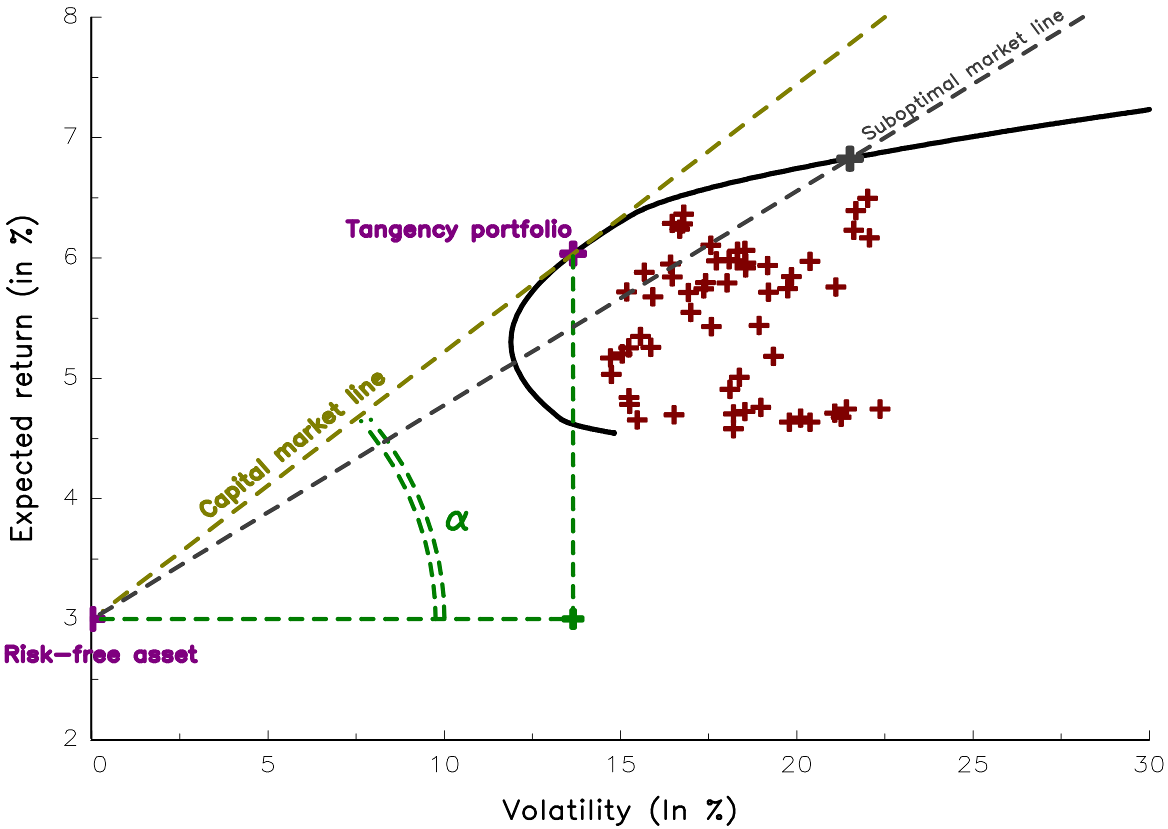
- "*the investor does (or should) consider expected return a desirable thing and variance of return an undesirable thing*"(Markowitz, 1952).
- We consider a universe of n assets. Let μ and Σ be the vector of expected returns and the covariance matrix of returns. We have:

$$\begin{aligned} & \max \mu(x) = \mu^\top x \\ \text{u.c.} \quad & \sigma(x) = \sqrt{x^\top \Sigma x} = \sigma^\star \end{aligned}$$

There isn't one optimal portfolio, but a set of optimal portfolios!

The market portfolio theory

Does one optimized portfolio dominate all the other portfolios?



- Tobin (1958) introduces the risk-free rate and shows that the efficient frontier is a straight line.
- Optimal portfolios are a combination of the tangency portfolio and the risk-free asset.
- *Separation theorem* (Lintner, 1965).

There is one optimal (risky) portfolio!

The market portfolio theory

How to compute the tangency portfolio?

- Sharpe (1964) develops the CAPM theory.
- If the market is at the equilibrium, the prices of assets are such that the tangency portfolio is the market portfolio (or the market-cap portfolio).
- Avoids assumptions on expected returns, volatilities and correlations!
- It is the beginning of passive management:
  - Jensen (1969): no alpha in mutual equity funds
  - John McQuown (Wells Fargo Bank, 1971)
  - Rex Sinquefield (American National Bank, 1973)

Portfolio optimization and active management

For active management, portfolio optimization continues to make sense.

However...

> *"The indifference of many investment practitioners to mean-variance optimization technology, despite its theoretical appeal, is understandable in many cases. The major problem with MV optimization is its tendency to maximize the effects of errors in the input assumptions. Unconstrained MV optimization can yield results that are inferior to those of simple equal-weighting schemes" (Michaud, 1989).*

Are optimized portfolios optimal?

Stability issues

An illustration

- We consider a universe of 3 assets.
- The parameters are: $\mu_1 = \mu_2 = 8\%$, $\mu_3 = 5\%$, $\sigma_1 = 20\%$, $\sigma_2 = 21\%$, $\sigma_3 = 10\%$ and $\rho_{i,j} = 80\%$.
- The objective is to maximize the expected return for a 15% volatility target.
- The optimal portfolio is $(38.3\%, 20.2\%, 41.5\%)$.

What is the sensitivity to the input parameters?

ρ		70%	90%		90%	
σ_2				18%	18%	
μ_1						9%
x_1	38.3%	38.3%	44.6%	13.7%	0.0%	56.4%
x_2	20.2%	25.9%	8.9%	56.1%	65.8%	0.0%
x_3	41.5%	35.8%	46.5%	30.2%	34.2%	43.6%

Stability issues
Solutions

In order to stabilize the optimal portfolio, we have to introduce some regularization techniques:

- regularization of the objective function by using resampling techniques
- regularization of the covariance matrix:
  - Factor analysis
  - Shrinkage methods
  - Random matrix theory
  - etc.
- regularization of the program specification by introducing some weight constraints

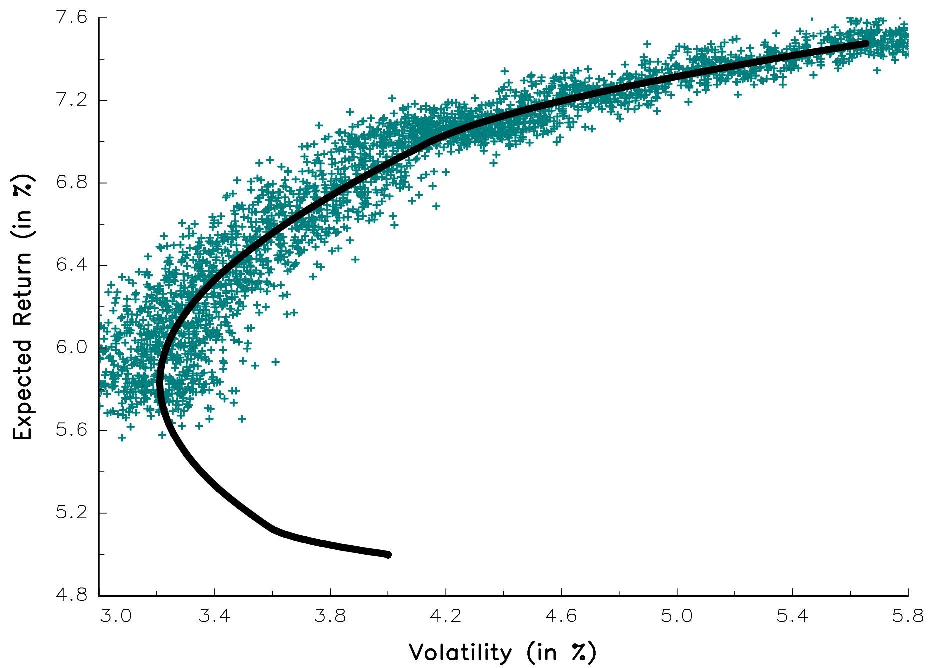
Regularization using resampling and shrinkage methods

- Resampling methods
- Factor analysis
- Shrinkage methods
- Why these regularization techniques are not enough?

Resampling methods

- Jacknife
- Cross validation
  - Hold-out
  - K-fold
- Bootstrap
  - Resubstitution
  - Out of the bag
  - .632

Figure: An example of resampled efficient frontier



Factor analysis

How to denoise the covariance matrix?

1. Factor analysis by imposing a correlation structure (MSCI Barra).
2. Factor analysis by filtering the correlation structure (APT).
3. Principal component analysis.
4. Random matrix theory[2] (Bouchaud *et al.*, 1999).

[2] In a random matrix of dimension $T \times n$, the maximal eigenvalue satisfies:

$$\lambda_{\max} \approx \sigma^2 \left(1 + n/T + 2\sqrt{n/T}\right)$$

Shrinkage methods

The Ledoit and Wolf approach

- Let $\hat{\Sigma}$ be the empirical covariance matrix. This estimator is without bias but converges slowly.
- Let $\hat{\Phi}$ be another estimator which is biased but converges faster.

Ledoit and Wolf (2003) propose to combine $\hat{\Sigma}$ and $\hat{\Phi}$:

$$\hat{\Sigma}_\alpha = \alpha \hat{\Phi} + (1 - \alpha) \hat{\Sigma}$$

The value of α is estimated by minimizing a quadratic loss:

$$\alpha^\star = \arg\min \mathbb{E}\left[\left\| \alpha \hat{\Phi} + (1 - \alpha) \hat{\Sigma} - \Sigma \right\|^2\right]$$

They find analytical expression of $\alpha^\star$ when:

- $\hat{\Phi}$ has a constant correlation structure;
- $\hat{\Phi}$ corresponds to a factor model or is deduced from PCA.

Shrinkage methods
Linear regression and characteristic portfolios

We consider the quadratic utility function

$$\mathcal{U}(x) = x^\top \mu - \frac{1}{2}\phi x^\top \Sigma x$$

The solution is:

$$x^\star = \left(X^\top X\right)^{-1} X^\top Y = \frac{1}{\phi}\hat{\Sigma}^{-1}\hat{\mu}$$

where X is the matrix of asset returns, $\hat{\Sigma} = n^{-1}\left(X^\top X\right)$ is the sample covariance matrix and $Y = \phi^{-1}\mathbf{1}$.

Optimized (characteristic) portfolios $\Leftrightarrow$ Linear regression

Regularization of linear regression:

- the ridge approach (L^2 norm or $\sum \beta_i^2$)
- the lasso approach (L^1 norm or $\sum |\beta_i|$)

Shrinkage methods

The ridge approach

We consider the ridge regression:

$$x^{\star}(\lambda) = \arg\min \frac{1}{2}\phi x^{\top}\Sigma x - x^{\top}\mu + \frac{\lambda}{2}x^{\top}Ax$$

The solution is (with $\lambda' = \lambda/\phi$):

$$x^{\star}(\lambda) = \left(\mathbf{I} + \lambda'\hat{\Sigma}^{-1}A\right)^{-1} x^{\star}$$

- If $A = \mathbf{I}$, we obtain:

$$x^{\star}(\lambda) = \left(\mathbf{I} + \lambda'\hat{\Sigma}^{-\mathbf{1}}\right)^{-\mathbf{1}} x^{\star}$$

- If $A = \mathcal{V}$ with $\mathcal{V}_{\boldsymbol{i},\boldsymbol{i}} = \sigma_{\boldsymbol{i}}^{\mathbf{2}}$ and $\mathcal{V}_{\boldsymbol{i},\boldsymbol{j}} = 0$, we obtain:

$$\begin{aligned} x^{\star}(\lambda) &= \frac{1}{\phi}\left(\frac{1}{1+\lambda'}\hat{\Sigma} + \left(1 - \frac{1}{1+\lambda'}\right)\mathcal{V}\right)^{-\mathbf{1}}\hat{\mu} \\ &= \left(\frac{1}{1+\lambda'}\mathbf{I} + \left(1 - \frac{1}{1+\lambda'}\right)\hat{\mathbf{C}}^{-\mathbf{1}}\right)^{-\mathbf{1}} x^{\star} \end{aligned}$$

$x^{\star}(\lambda)$ is a combination of a portfolio with correlations and a portfolio without correlations.

Shrinkage methods

Extension to lasso regression and dynamic allocation

- Extension to lasso regression:

$$x^{\star}(\lambda) = \arg\min \frac{1}{2}\phi x^{\top}\Sigma x - x^{\top}\mu + \frac{\lambda}{2}\mathbf{a}^{\top}|x|$$

$\Rightarrow$ Deleveraged portfolios & asset selection.

- Extension to dynamic allocation:
  - Lasso approach

$$x^{\star}(\lambda) = \arg\min \frac{1}{2}\phi x^{\top}\Sigma x - x^{\top}\mu + \frac{\lambda}{2}\mathbf{a}^{\top}|x - x_0|$$

  $\Rightarrow$ Interpretation in terms of turnover and trading costs (Scherer, 2007).

  - Ridge approach

$$x^{\star}(\lambda) = \arg\min \frac{1}{2}\phi x^{\top}\Sigma x - x^{\top}\mu + \frac{\lambda}{2}(x - x_0)^{\top}A(x - x_0)$$

Why these regularization techniques are not enough?

On the importance of the information matrix

Optimized portfolios are solutions of the following quadratic program:

$$x^{\star} = \arg\max x^{\top}\mu - \frac{1}{2}\phi x^{\top}\Sigma x$$
$$\text{u.c.} \quad \begin{cases} \mathbf{1}^{\top}x = 1 \\ x \in \mathbb{R}^{n} \end{cases}$$

Let $\mathscr{C} = \mathbb{R}^{n}$ (no constraints). We have:

$$x^{\star} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^{\top}\Sigma^{-1}\mathbf{1}} - \frac{1}{\phi}\cdot\frac{\left(\left(\mathbf{1}^{\top}\Sigma^{-1}\mu\right)\Sigma^{-1}\mathbf{1} - \left(\mathbf{1}^{\top}\Sigma^{-1}\mathbf{1}\right)\Sigma^{-1}\mu\right)}{\mathbf{1}^{\top}\Sigma^{-1}\mathbf{1}}$$

Optimal solutions are of the following form: $x^{\star} \propto f\left(\Sigma^{-1}\right)$.
The important quantity is then the information matrix $\mathscr{I} = \Sigma^{-1}$ and the eigendecomposition of $\mathscr{I}$ is:

$$V_i\left(\mathscr{I}\right) = V_{n-i}\left(\Sigma\right) \quad \text{and} \quad \lambda_i\left(\mathscr{I}\right) = \frac{1}{\lambda_{n-i}\left(\Sigma\right)}$$

Why these regularization techniques are not enough?

An illustration

We consider the example of Slide 9:
$\mu_1 = \mu_2 = 8\%$, $\mu_3 = 5\%$, $\sigma_1 = 20\%$, $\sigma_2 = 21\%$, $\sigma_3 = 10\%$ and $\rho_{i,j} = 80\%$.

The eigendecomposition of the covariance and information matrices is:

Asset / Factor	Covariance matrix Σ 1	2	3	Information matrix $\mathcal{I}$ 1	2	3
1	65.35%	−72.29%	−22.43%	−22.43%	−72.29%	65.35%
2	69.38%	69.06%	−20.43%	−20.43%	69.06%	69.38%
3	30.26%	−2.21%	95.29%	95.29%	−2.21%	30.26%
Eigenvalue	8.31%	0.84%	0.26%	379.97	119.18	12.04
% cumulated	88.29%	97.20%	100.00%	74.33%	97.65%	100.00%

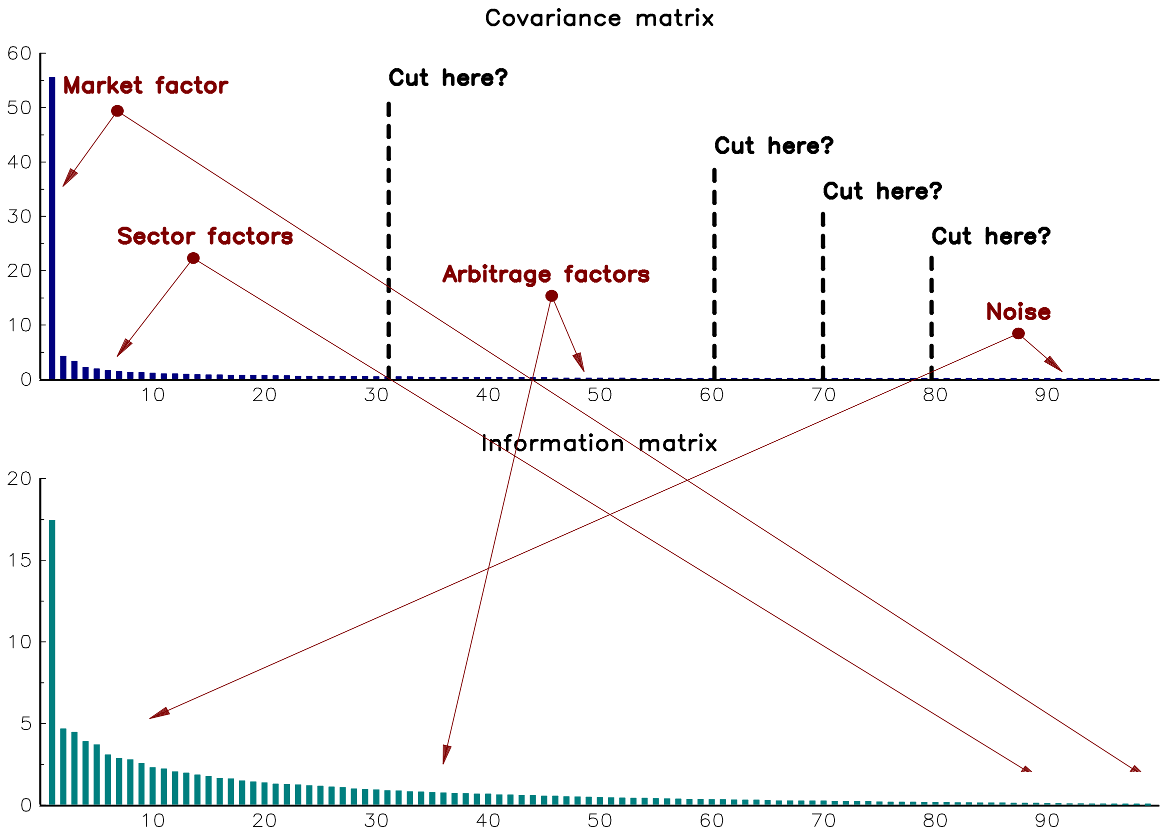
⇒ It means that the first factor of the information matrix corresponds to the last factor of the covariance matrix and that the last factor of the information matrix corresponds to the first factor.

⇒ Optimization on arbitrage risk factors, idiosyncratic risk factors and (certainly) noise factors!

Why these regularization techniques are not enough?

Working with a large universe of assets

Figure: Eigendecomposition of the FTSE 100 covariance matrix



$\Rightarrow$ Shrinkage is then necessary to eliminate the noise factors, but is not sufficient because it is extremely difficult to filter the arbitrage factors!

The impact of the weight constraints

- Shrinkage interpretation of weight constraints
- Some examples
- Myopic behavior of portfolio managers?

Shrinkage interpretation of weight constraints

The framework

We consider a universe of n assets. We denote by μ the vector of their expected returns and by Σ the corresponding covariance matrix. We specify the optimization problem as follows:

$$\min \frac{1}{2} x^\top \Sigma x$$
$$\text{u.c.} \left\{ \begin{array}{l} \mathbf{1}^\top x = 1 \\ \mu^\top x \geq \mu^\star \\ x \in \mathbb{R}^n \bigcap \mathscr{C} \end{array} \right.$$

where x is the vector of weights in the portfolio and $\mathscr{C}$ is the set of weights constraints. We define:

- the unconstrained portfolio $x^\star$ or $x^\star(\mu, \Sigma)$:

$$\mathscr{C} = \mathbb{R}^n$$

- the constrained portfolio $\tilde{x}$:

$$\mathscr{C}\left(x^-, x^+\right) = \left\{x \in \mathbb{R}^n : x_i^- \leq x_i \leq x_i^+\right\}$$

Shrinkage interpretation of weights constraints

Main result

Theorem

Jagannathan and Ma (2003) show that the constrained portfolio is the solution of the unconstrained problem:

$$\tilde{x} = x^{\star}\left(\tilde{\mu}, \tilde{\Sigma}\right)$$

with:

$$\left\{ \begin{array}{l} \tilde{\mu} = \mu \\ \tilde{\Sigma} = \Sigma + \left(\lambda^{+} - \lambda^{-}\right)\mathbf{1}^{\top} + \mathbf{1}\left(\lambda^{+} - \lambda^{-}\right)^{\top} \end{array} \right.$$

where λ^{-} and λ^{+} are the Lagrange coefficients vectors associated to the lower and upper bounds.

$\Rightarrow$ Introducing weights constraints is equivalent to introduce a shrinkage method or to introduce some relative views (similar to the **Black-Litterman** approach).

Shrinkage interpretation of weights constraints

Proof for the global minimum variance portfolio

We define the Lagrange function as $f(x;\lambda_0) = \frac{1}{2}x^\top \Sigma x - \lambda_0 \left(\mathbf{1}^\top x - 1\right)$ with $\lambda_0 \geq 0$. The first order conditions are $\Sigma x - \lambda_0 \mathbf{1} = 0$ and $\mathbf{1}^\top x - 1 = 0$. We deduce that the optimal solution is:

$$x^\star = \lambda_0^\star \Sigma^{-1}\mathbf{1} = \frac{1}{\mathbf{1}^\top \Sigma \mathbf{1}} \Sigma^{-1}\mathbf{1}$$

With weights constraints $\mathscr{C}(x^-, x^+)$, we have:

$$f\left(x;\lambda_0,\lambda^-,\lambda^+\right) = \frac{1}{2}x^\top \Sigma x - \lambda_0 \left(\mathbf{1}^\top x - 1\right) - \lambda^{-\top}\left(x - x^-\right) - \lambda^{+\top}\left(x^+ - x\right)$$

with $\lambda_0 \geq 0$, $\lambda_i^- \geq 0$ and $\lambda_i^+ \geq 0$. In this case, the first-order conditions becomes $\Sigma x - \lambda_0 \mathbf{1} - \lambda^- + \lambda^+ = 0$ and $\mathbf{1}^\top x - 1 = 0$. We have:

$$\tilde{\Sigma}\tilde{x} = \left(\Sigma + \left(\lambda^+ - \lambda^-\right)\mathbf{1}^\top + \mathbf{1}\left(\lambda^+ - \lambda^-\right)^\top\right)\tilde{x} = \left(2\tilde{\lambda}_0 - \tilde{x}^\top \Sigma \tilde{x}\right)\mathbf{1}$$

Because $\tilde{\Sigma}\tilde{x}$ is a constant vector, it proves that $\tilde{x}$ is the solution of the unconstrained optimisation problem with $\lambda_0^\star = \left(2\tilde{\lambda}_0 - \tilde{x}^\top \Sigma \tilde{x}\right)$.

Some examples

The minimum variance portfolio

Table: Specification of the covariance matrix Σ (in %)

σ_i	$\rho_{i,j}$			
15.00	100.00			
20.00	10.00	100.00		
25.00	40.00	70.00	100.00	
30.00	50.00	40.00	80.00	100.00

Given these parameters, the global minimum variance portfolio is equal to:

$$x^{\star} = \begin{pmatrix} 72.742\% \\ 49.464\% \\ -20.454\% \\ -1.753\% \end{pmatrix}$$

Some examples

The minimum variance portfolio

Table: Global minimum variance portfolio when $x_i \geq 10\%$

$\tilde{x}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
56.195	0.000	0.000	15.000	100.000			
23.805	0.000	0.000	20.000	10.000	100.000		
10.000	1.190	0.000	19.671	10.496	58.709	100.000	
10.000	1.625	0.000	23.980	17.378	16.161	67.518	100.000

Table: Global minimum variance portfolio when $0\% \leq x_i \leq 50\%$

$\tilde{x}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
50.000	0.000	1.050	20.857	100.000			
50.000	0.000	0.175	20.857	35.057	100.000		
0.000	0.175	0.000	24.290	46.881	69.087	100.000	
0.000	0.000	0.000	30.000	52.741	41.154	79.937	100.000

Myopic behavior of portfolio managers?

Weight constraints → **Shrinkage methods**

By using weight constraints, the portfolio manager changes (implicitly):

1. the values of the volatilities;
2. the ordering of the volatilities;
3. the values of the correlations;
4. the ordering of the correlations;
5. the sign of the correlations.

The question is then the following:

> Is the portfolio manager aware and agreed upon these changes?

The risk budgeting (or risk parity) approach

- Definition
- Main properties (From Bruder and Roncalli, 2012)
- Some popular RB portfolios
- RB portfolios vs optimized portfolios

Remark

What is risk parity?

- *sensu strictissimo: an ERC portfolio on bonds and equities*
- *sensu stricto: all the assets have the same risk contribution*
- *sensu lato: a risk budgeting portfolio*

Three methods to build a portfolio

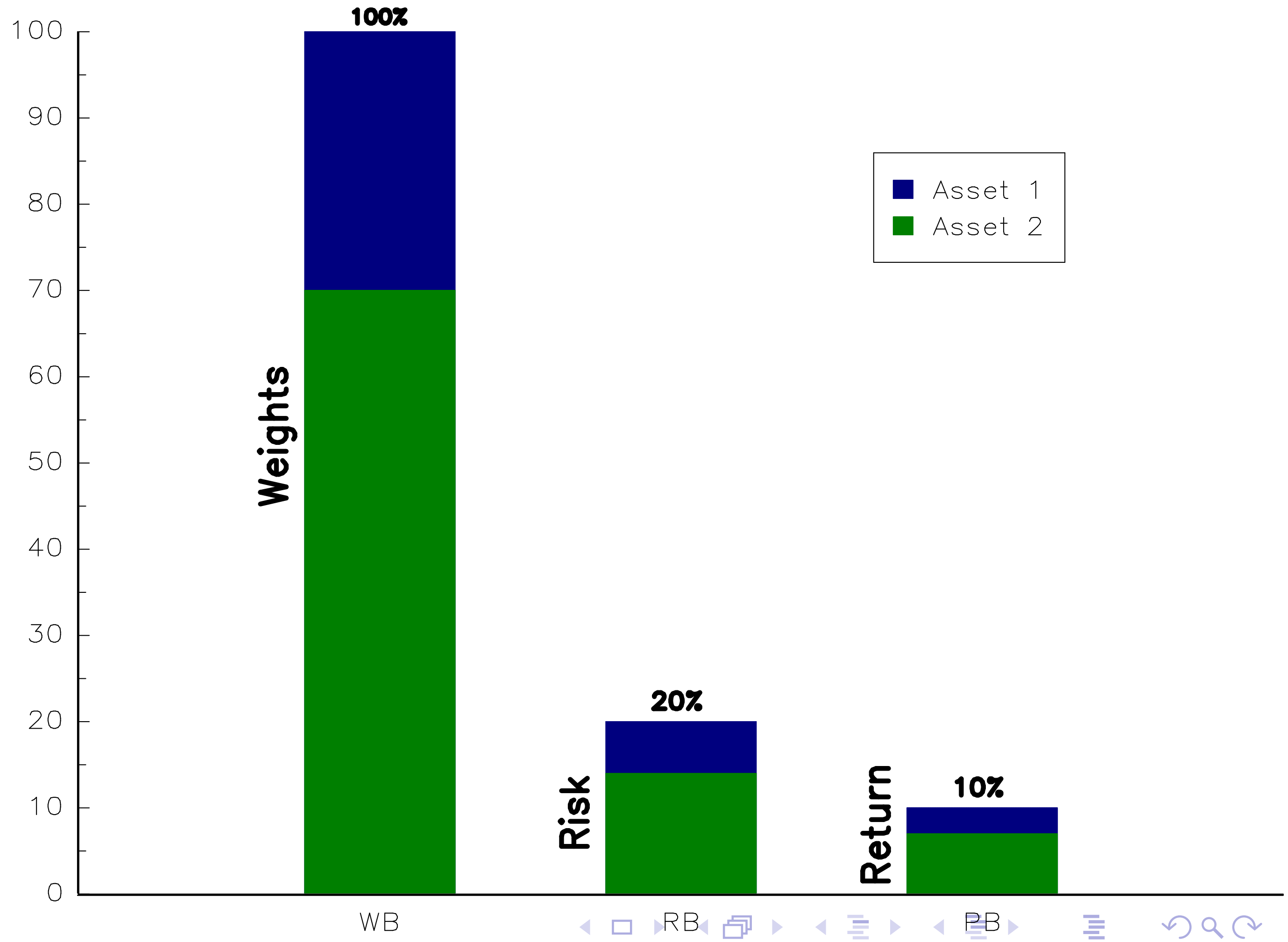
1. Weight budgeting (WB)
2. Risk budgeting (RB)
3. Performance budgeting (PB)

Ex-ante analysis
$\neq$
Ex-post analysis

Important result

RB = PB

Figure: The 30/70 rule



Weight budgeting versus risk budgeting

Let $x = (x_1, \ldots, x_n)$ be the weights of n assets in the portfolio. Let $\mathscr{R}(x_1, \ldots, x_n)$ be a coherent and convex risk measure. We have:

$$\begin{aligned} \mathscr{R}(x_1, \ldots, x_n) &= \sum_{i=1}^{n} x_i \cdot \frac{\partial \mathscr{R}(x_1, \ldots, x_n)}{\partial x_i} \\ &= \sum_{i=1}^{n} \mathrm{RC}_i(x_1, \ldots, x_n) \end{aligned}$$

Let $b = (b_1, \ldots, b_n)$ be a vector of budgets such that $b_i \geq 0$ and $\sum_{i=1}^{n} b_i = 1$. We consider two allocation schemes:

1. Weight budgeting (WB)

$$x_i = b_i$$

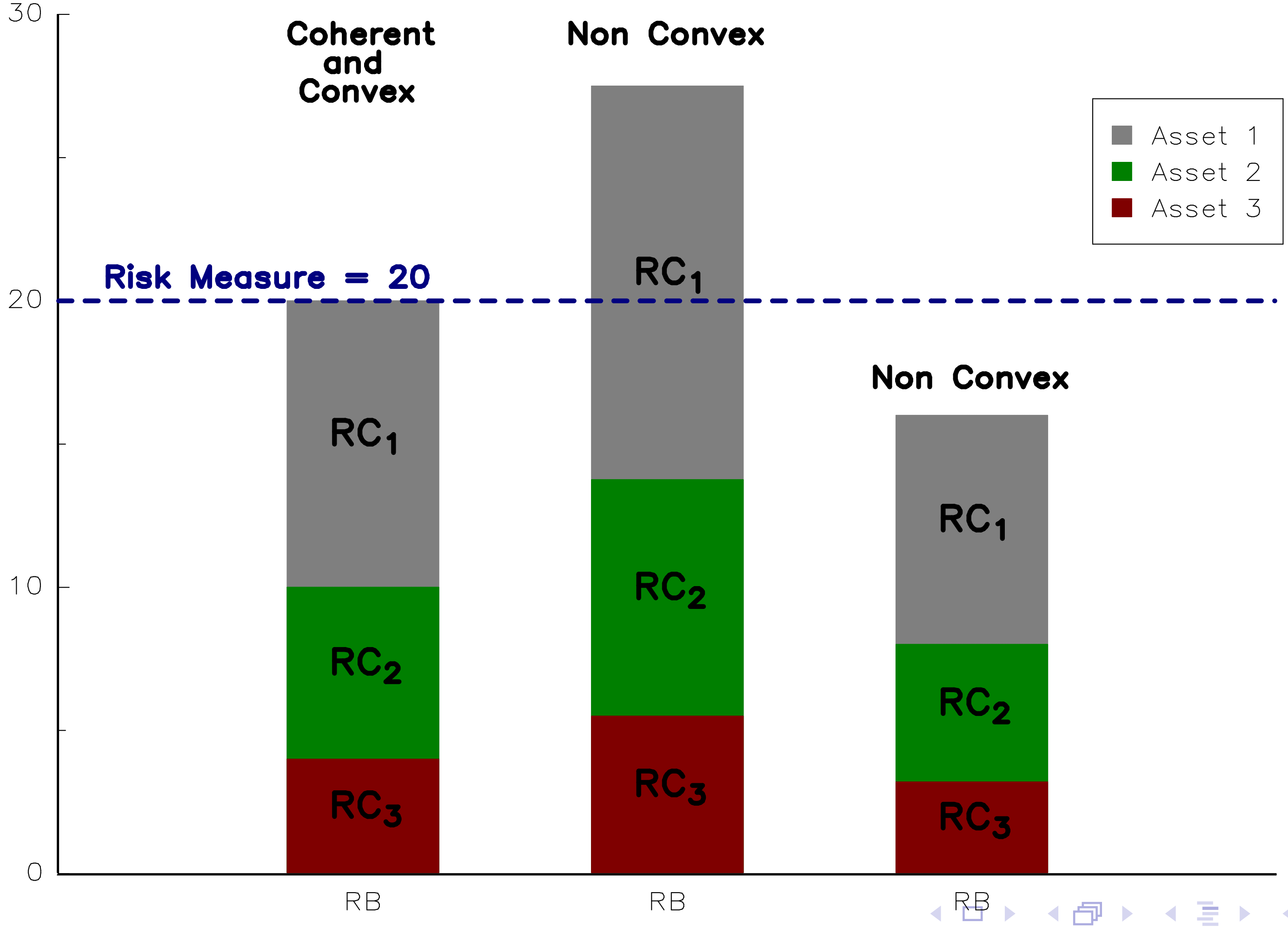
2. Risk budgeting[3] (RB)

$$\mathrm{RC}_i = b_i \cdot \mathscr{R}(x_1, \ldots, x_n)$$

[3] The ERC portfolio is a special case when $b_i = 1/n$.

Importance of the coherency and convexity properties

Figure: Risk Measure = 20 with a 50/30/20 budget rule



Application to the volatility risk measure

Let Σ be the covariance matrix of the assets returns. We assume that the risk measure $\mathscr{R}(x)$ is the volatility of the portfolio $\sigma(x) = \sqrt{x^\top \Sigma x}$. We have:

$$\begin{aligned}
\frac{\partial \mathscr{R}(x)}{\partial x} &= \frac{\Sigma x}{\sqrt{x^\top \Sigma x}} \\
\mathrm{RC}_i(x_1,\ldots,x_n) &= x_i \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \\
\sum_{i=1}^n \mathrm{RC}_i(x_1,\ldots,x_n) &= \sum_{i=1}^n x_i \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} = x^\top \frac{\Sigma x}{\sqrt{x^\top \Sigma x}} = \sigma(x)
\end{aligned}$$

The risk budgeting portfolio is defined by this system of equations:

$$\begin{cases}
x_i \cdot (\Sigma x)_i = b_i \cdot \left(x^\top \Sigma x\right) \\
x_i \geq 0 \\
\sum_{i=1}^n x_i = 1
\end{cases}$$

An example

Illustration

- 3 assets
- Volatilities are respectively 30%, 20% and 15%
- Correlations are set to 80% between the 1st asset and the 2nd asset, 50% between the 1st asset and the 3rd asset and 30% between the 2nd asset and the 3rd asset
- Budgets are set to 50%, 20% and 30%
- For the ERC (Equal Risk Contribution) portfolio, all the assets have the same risk budget

Weight budgeting (or traditional) approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	**50.00%**	29.40%	14.70%	70.43%
2	**20.00%**	16.63%	3.33%	15.93%
3	**30.00%**	9.49%	2.85%	13.64%
Volatility			20.87%	

Risk budgeting approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	31.15%	28.08%	8.74%	**50.00%**
2	21.90%	15.97%	3.50%	**20.00%**
3	46.96%	11.17%	5.25%	**30.00%**
Volatility			17.49%	

ERC approach

Asset	Weight	Marginal Risk	Risk Contribution Absolute	Risk Contribution Relative
1	19.69%	27.31%	5.38%	**33.33%**
2	32.44%	16.57%	5.38%	**33.33%**
3	47.87%	11.23%	5.38%	**33.33%**
Volatility			16.13%	

RB is (a little) more complex than ERC

Let us consider the two-asset case. Let ρ be the correlation and $x = (w, 1-w)$ be the vector of weights. The ERC portfolio is:

$$w^* = \frac{1}{\sigma_1} \Big/ \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} \right)$$

The RB portfolio with $(b, 1-b)$ as the vector of risk budgets is:

$$w^\star = \frac{(b - 1/2)\,\rho\sigma_1\sigma_2 - b\sigma_2^2 + \sigma_1\sigma_2\sqrt{(b-1/2)^2\rho^2 + b(1-b)}}{(1-b)\,\sigma_1^2 - b\sigma_2^2 + 2\,(b - 1/2)\,\rho\sigma_1\sigma_2}$$

It introduces some convexity with respect to b and ρ.

Table: Weights $w^\star$ with respect to some values of b and ρ

		$\sigma_2 = \sigma_1$				$\sigma_2 = 3 \times \sigma_1$			
	b	20%	50%	70%	90%	20%	50%	70%	90%
ρ	−99.9%	50.0%	50.0%	50.0%	50.0%	75.0%	75.0%	75.0%	75.0%
	−50%	41.9%	50.0%	55.2%	61.6%	68.4%	75.0%	78.7%	82.8%
	0%	33.3%	50.0%	60.4%	75.0%	60.0%	75.0%	82.1%	90.0%
	25%	29.3%	50.0%	63.0%	80.6%	55.5%	75.0%	83.6%	92.6%
	50%	25.7%	50.0%	65.5%	84.9%	51.0%	75.0%	85.1%	94.4%
	75%	22.6%	50.0%	67.8%	87.9%	46.7%	75.0%	86.3%	95.6%
	90%	21.0%	50.0%	69.1%	89.2%	44.4%	75.0%	87.1%	96.1%

Some analytical solutions

- The case of uniform correlation[4] $\rho_{i,j} = \rho$
  - ERC portfolio ($b_i = 1/n$)

$$x_i(\rho) = \frac{\sigma_i^{-1}}{\sum_{j=1}^n \sigma_j^{-1}}$$

  - RB portfolio

$$x_i\left(-\frac{1}{n-1}\right) = \frac{\sigma_i^{-1}}{\sum_{j=1}^n \sigma_j^{-1}}, \; x_i(0) = \frac{\sqrt{b_i}\sigma_i^{-1}}{\sum_{j=1}^n \sqrt{b_j}\sigma_j^{-1}}, \; x_i(1) = \frac{b_i \sigma_i^{-1}}{\sum_{j=1}^n b_j \sigma_j^{-1}}$$

- The general case

$$x_i = \frac{b_i \beta_i^{-1}}{\sum_{j=1}^n b_j \beta_j^{-1}}$$

where β_i is the beta of the asset i with respect to the RB portfolio.

[4] The solution is noted $x_i(\rho)$.

The RB portfolio is a minimum variance (MV) portfolio subject to a constraint of weight diversification

Let us consider the following minimum variance optimization problem:

$$\begin{aligned} x^{\star}(c) &= \arg\min \sqrt{x^{\top}\Sigma x} \\ &\text{u.c.} \quad \begin{cases} \sum_{i=1}^{n} b_i \ln x_i \geq c \\ \mathbf{1}^{\top} x = 1 \\ x \geq \mathbf{0} \end{cases} \end{aligned}$$

- if $c = c^{-} = -\infty$, $x^{\star}(c^{-}) = x_{\mathrm{MV}}$ (no weight diversification)
- if $c = c^{+} = \sum_{i=1}^{n} b_i \ln b_i$, $x^{\star}(c^{+}) = x_{\mathrm{WB}}$ (no variance minimization)
- $\exists\, c^{0} : x^{\star}(c^{0}) = x_{\mathrm{RB}}$ (variance minimization and weight diversification)

$\Longrightarrow$ if $b_i = 1/n$, $x_{\mathrm{RB}} = x_{\mathrm{ERC}}$ (variance minimization, weight diversification and perfect risk diversification[5])

[5] The Gini coefficient of the risk measure is then equal to 0.

The RB portfolio is located between the MV portfolio and the WB portfolio

- The RB portfolio is a combination of the MV and WB portfolios:

$$\partial_{x_i}\,\sigma(x) = \partial_{x_j}\,\sigma(x) \qquad \text{(MV)}$$
$$x_i/b_i = x_j/b_j \qquad \text{(WB)}$$

$$x_i\partial_{x_i}\,\sigma(x)/b_i = x_j\partial_{x_j}\,\sigma(x)/b_j \qquad \text{(RB)}$$

- The volatility of the RB portfolio is between the volatility of the MV portfolio and the volatility of the WB portfolio:

$$\sigma_{\mathrm{MV}} \leq \sigma_{\mathrm{RB}} \leq \sigma_{\mathrm{WB}}$$

With risk budgeting, we always diminish the volatility compared to the weight budgeting

⇒ For the ERC portfolio, we retrieve the famous relationship:

$$\sigma_{\mathrm{MV}} \leq \sigma_{\mathrm{ERC}} \leq \sigma_{1/n}$$

Existence and uniqueness

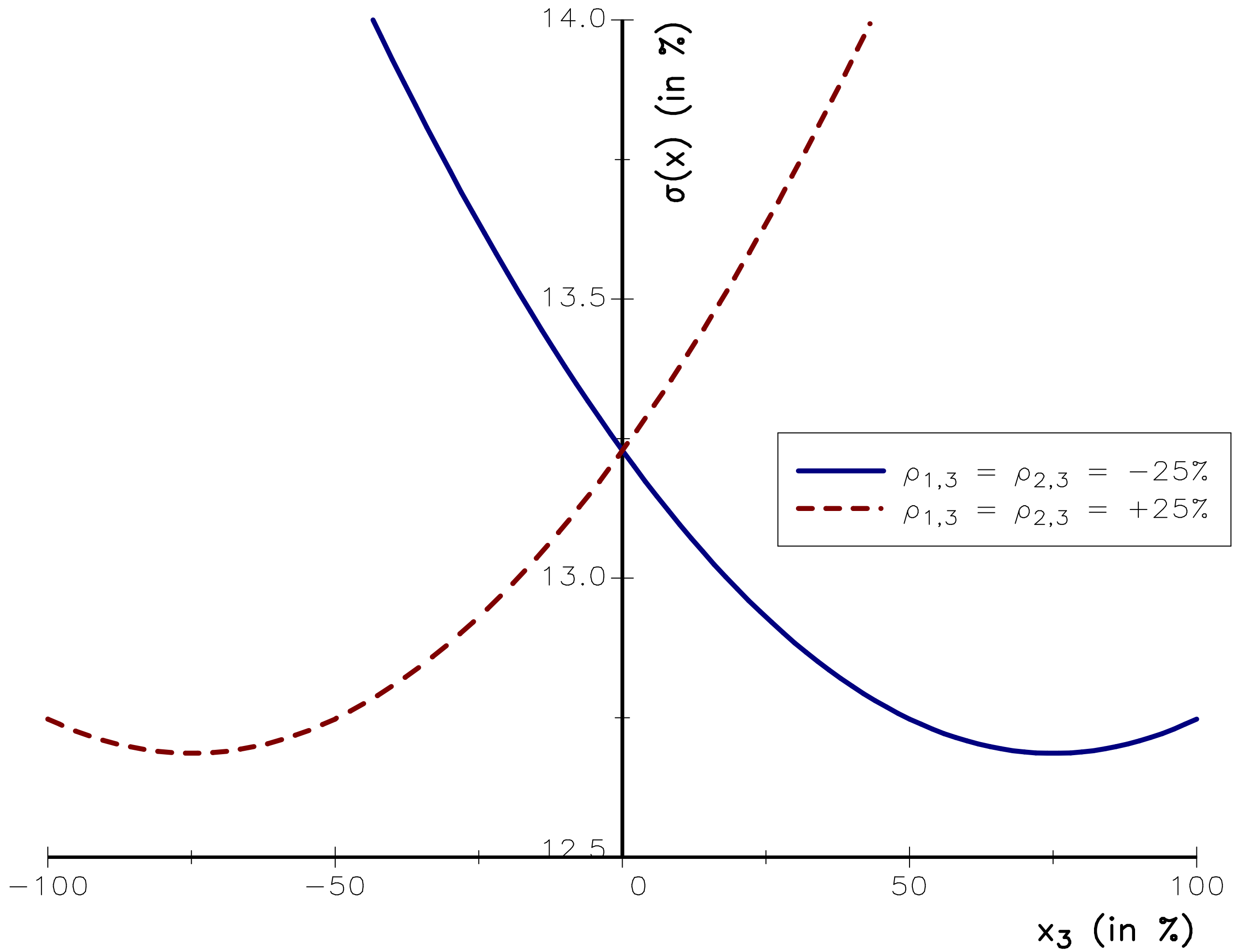
- If $b_i > 0$, the solution exists and is unique.
- If $b_i \geq 0$, there may be several solutions.
- If $\rho_{i,j} \geq 0$, the solution is unique.

An example with 3 assets: $\sigma_1 = 20\%$, $\sigma_2 = 10\%$, $\sigma_3 = 5\%$ and $\rho_{1,2} = 50\%$.

$\rho_{1,3} = \rho_{2,3}$	Solution		1	2	3	$\sigma(x)$
-25%	$\mathscr{S}_1$	x_i	20.00%	40.00%	40.00%	
		$\partial_{x_i}\sigma(x)$	16.58%	8.29%	0.00%	6.63%
		RC_i	50.00%	50.00%	0.00%	
	$\mathscr{S}_2$	x_i	33.33%	66.67%	0.00%	
		$\partial_{x_i}\sigma(x)$	17.32%	8.66%	-1.44%	11.55%
		RC_i	50.00%	50.00%	0.00%	
	$\mathscr{S}'_1$	x_i	19.23%	38.46%	42.31%	
		$\partial_{x_i}\sigma(x)$	16.42%	8.21%	0.15%	6.38%
		RC_i	49.50%	49.50%	1.00%	
25%	$\mathscr{S}_1$	x_i	33.33%	66.67%	0.00%	
		$\partial_{x_i}\sigma(x)$	17.32%	8.66%	1.44%	11.55%
		RC_i	50.00%	50.00%	0.00%	

Existence and uniqueness

Figure: Evolution of the volatility with respect to the weights $(50\%, 50\%, x_3)$



Existence and uniqueness

Characterization of the solutions

Let $\mathcal{N}$ be the set of assets such that $b_i = 0$. The solution $\mathcal{S}_1$ satisfies the following relationships:

$$\begin{cases} \mathrm{RC}_i = x_i \cdot \partial_{x_i} \sigma(x) = b_i & \text{if} \quad i \notin \mathcal{N} \\ \begin{cases} x_i = 0 \text{ and } \partial_{x_i} \sigma(x) > 0 \quad (i) \\ \text{or} \\ x_i > 0 \text{ and } \partial_{x_i} \sigma(x) = 0 \quad (ii) \end{cases} & \text{if} \quad i \in \mathcal{N} \end{cases}$$

The conditions (i) and (ii) are mutually exclusive for one asset $i \in \mathcal{N}$, but not necessarily for all the assets $i \in \mathcal{N}$.

Let $\mathcal{N} = \mathcal{N}_1 \bigsqcup \mathcal{N}_2$ where $\mathcal{N}_1$ is the set of assets verifying the condition (i) and $\mathcal{N}_2$ is the set of assets verifying the condition (ii). The number of solutions is equal to 2^m where $m = |\mathcal{N}_1|$ is the cardinality of $\mathcal{N}_1$.

Optimality of risk budgeting portfolios

If the RB portfolio is optimal (in the Markowitz sense), the ex-ante performance contributions are equal to the risk budgets:

Black-Litterman Approach

Budgeting the risk = budgeting the performance
(in an ex-ante point of view)

Let $\tilde{\mu}_i$ be the market price of the expected return. We have:

$$x_i \cdot \tilde{\mu}_i \propto x_i \cdot \frac{\partial\, \sigma(x)}{\partial\, x}$$

In the ERC portfolio, the (ex-ante) performance contributions are equal. The ERC portfolio is then the less concentrated portfolio in terms of risk contributions, but also in terms of performance contributions.

Optimality of risk budgeting portfolios

Proof

We consider the quadratic utility function $\mathcal{U}(x) = x^\top \mu - \frac{1}{2}\phi x^\top \Sigma x$ of Markowitz. The portfolio x is optimal if the vector of expected returns satisfies this relationship:

$$\partial_x \mathcal{U}(x) = 0 \Leftrightarrow \tilde{\mu} = \frac{1}{\phi}\Sigma x$$

If the RB portfolio is optimal, the performance contribution PC_i of the asset i is then proportional to its risk contribution (or risk budget):

$$\begin{array}{rcl} \mathrm{PC}_i & = & x_i \tilde{\mu}_i \\ & = & \frac{1}{\phi} x_i (\Sigma x)_i \\ & = & \frac{\sqrt{x^\top \Sigma x}}{\phi} \cdot \frac{x_i (\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \\ & \propto & \mathrm{RC} \\ & \propto & b_i \end{array}$$

Optimality of risk budgeting portfolios

An example

$$\sigma = \begin{pmatrix} 10\% \\ 20\% \\ 30\% \\ 40\% \end{pmatrix} \quad \text{and} \quad \rho = \begin{pmatrix} 1.0 & & & \\ 0.8 & 1.0 & & \\ 0.2 & 0.2 & 1.0 & \\ 0.2 & 0.2 & 0.5 & 1.0 \end{pmatrix}$$

Example 1

b_i	x_i	$\partial_{x_i}\sigma(x)$	RC_i	$\tilde{\mu}_i$	PC_i
20.0	40.9	7.1	20.0	5.2	20.0
25.0	25.1	14.5	25.0	10.5	25.0
40.0	25.3	23.0	40.0	16.7	40.0
15.0	8.7	25.0	15.0	18.2	15.0

Example 2

b_i	x_i	$\partial_{x_i}\sigma(x)$	RC_i	$\tilde{\mu}_i$	PC_i
10.0	35.9	5.3	10.0	5.0	10.0
10.0	17.9	10.5	10.0	9.9	10.0
10.0	10.2	18.6	10.0	17.5	10.0
70.0	36.0	36.7	70.0	34.7	70.0

Generalization to other convex risk measures

If the risk measure is coherent and satisfies the Euler principle (convexity property), the following properties are verified:

1. Existence and uniqueness
2. Location between the minimum risk portfolio and the weight budgeting portfolio
3. Optimality

Some heuristic portfolios as RB portfolios

The EW, MV, MDP and ERC portfolios could be interpreted as (endogenous) RB portfolios.

	EW	MV	MDP	ERC
b_i / PC_i	β_i	x_i	$x_i \sigma_i$	$\frac{1}{n}$

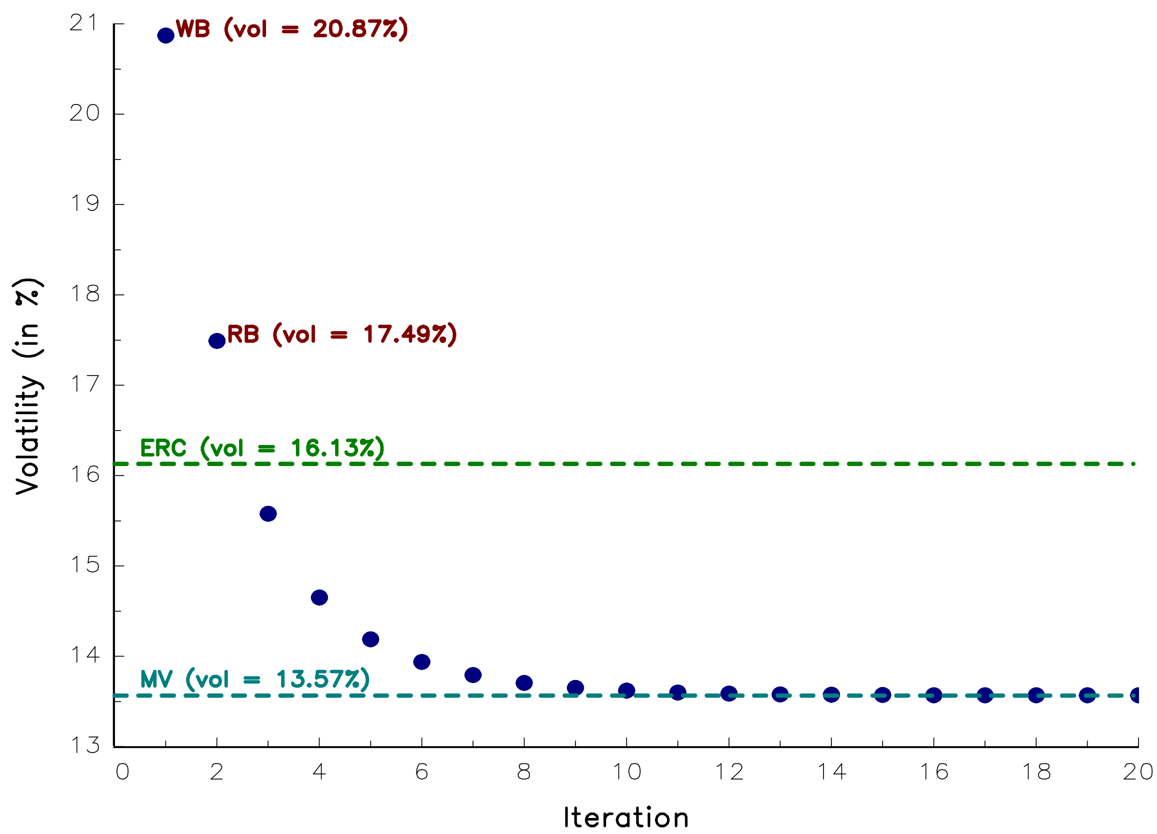
MV and MDP portfolios are two limit portfolios (explaining that the weights of some assets could be equal to zero).

Some heuristic portfolios as RB portfolios

MV portfolio as a limit portfolio

Let us consider an iterated portfolio $\left(x_1^{(t)}, \ldots, x_n^{(t)}\right)$ where t represents the iteration. The portfolio is defined such that the risk budget $b_i^{(t)}$ of the asset i at iteration t corresponds to the weight $x_i^{(t-1)}$ at iteration $t-1$. If the portfolio $\left(x_1^{(t)}, \ldots, x_n^{(t)}\right)$ admits a limit when $t \longrightarrow \infty$, it is equal to the minimum variance portfolio.

Figure: Illustration with the example of Slide 33



RB portfolios vs optimized portfolios

An illustration

With the example of Slide 9, the optimal portfolio is (38.3%, 20.2%, 41.5%) for a volatility of 15%. The corresponding risk contributions are 49.0%, 25.8% and 25.2%.

1. MVO: the objective is to target a volatility of 15%.
2. RB: the objective is to target the budgets (49.0%, 25.8%, 25.2%).

What is the sensitivity to the input parameters?

ρ			70%	90%		90%	
σ_2					18%	18%	
μ_1							9%
MVO	x_1	38.3%	38.3%	44.6%	13.7%	0.0%	56.4%
	x_2	20.2%	25.9%	8.9%	56.1%	65.8%	0.0%
	x_3	41.5%	35.8%	46.5%	30.2%	34.2%	43.6%
RB	x_1	38.3%	37.7%	38.9%	37.1%	37.7%	38.3%
	x_2	20.2%	20.4%	20.0%	22.8%	22.6%	20.2%
	x_3	41.5%	41.9%	41.1%	40.1%	39.7%	41.5%

$\Rightarrow$ RB portfolios are less sensitive to specification errors than optimized portfolios.

Some applications

- Risk-based indexation
  - Equity indexation
  - Bond indexation
- Risk-balanced allocation
  - Strategic asset allocation
  - Risk parity funds

Equity indexation

Pros and cons of market-cap indexation

Pros of market-cap indexation

- A convenient and **recognized approach** to participate to broad equity markets.
- **Management simplicity**: low turnover & transaction costs.

Cons of market-cap indexation

- Trend-following strategy: momentum bias leads to bubble risk exposure as weight of best performers ever increases.
  ⇒ Mid 2007, financial stocks represent 40% of the Eurostoxx 50 index.
- Growth biais as high valuation multiples stocks weight more than low-multiple stocks with equivalent realised earnings.
  ⇒ Mid 2000, the 8 stocks of the technology/telecom sectors represent 35% of the Eurostoxx 50 index.
  ⇒ $2^1/_2$ years later after the dot.com bubble, these two sectors represent 12%.
- Concentrated portfolios.
  ⇒ The top 100 market caps of the S&P 500 account for around 70%.
- Lack of risk diversification and high drawdown risk: no portfolio construction rules leads to concentration issues (e.g. sectors, stocks).

Equity indexation

Alternative-weighted indexation

Alternative-weighted indexation aims at building passive indexes where the weights are not based on market capitalization.

Two sets of responses:

1. Fundamental indexation $\Rightarrow$ promising alpha
    1. Dividend yield indexation
    2. RAFI indexation
2. Risk-based indexation $\Rightarrow$ promising diversification
    1. Equally weighted (1/n)
    2. Minimum-variance portfolio
    3. ERC portfolio
    4. MDP/MSR portfolio

Equity indexation

Application to the Eurostoxx 50 index

Table: Composition in % (January 2010)

	CW	MV	ERC	MDP	1/n	MV 10%	MDP 10%	MV 5%	MDP 5%
TOTAL	6.1		2.1		2			5.0	
BANCO SANTANDER	5.8		1.3		2				
TELEFONICA SA	5.0	31.2	3.5		2	10.0		5.0	5.0
SANOFI-AVENTIS	3.6	12.1	4.5	15.5	2	10.0	10.0	5.0	5.0
E.ON AG	3.6		2.1		2				1.4
BNP PARIBAS	3.4		1.1		2				
SIEMENS AG	3.2		1.5		2				
BBVA(BILB-VIZ-ARG)	2.9		1.4		2				
BAYER AG	2.9		2.6	3.7	2	2.2	5.0	5.0	5.0
ENI	2.7		2.1		2				
GDF SUEZ	2.5		2.6	4.5	2		5.4	5.0	5.0
BASF SE	2.5		1.5		2				
ALLIANZ SE	2.4		1.4		2				
UNICREDIT SPA	2.3		1.1		2				
SOC GENERALE	2.2		1.2	3.9	2		3.7		5.0
UNILEVER NV	2.2	11.4	3.7	10.8	2	10.0	10.0	5.0	5.0
FRANCE TELECOM	2.1	14.9	4.1	10.2	2	10.0	10.0	5.0	5.0
NOKIA OYJ	2.1		1.8	4.5	2		4.8		5.0
DAIMLER AG	2.1		1.3		2				
DEUTSCHE BANK AG	1.9		1.0		2				
DEUTSCHE TELEKOM	1.9		3.2	2.6	2	5.7	3.7	5.0	5.0
INTESA SANPAOLO	1.9		1.3		2				
AXA	1.8		1.0		2				
ARCELORMITTAL	1.8		1.0		2				
SAP AG	1.8	21.0	3.4	11.2	2	10.0	10.0	5.0	5.0

	CW	MV	ERC	MDP	1/n	MV 10%	MDP 10%	MV 5%	MDP 5%
RWE AG (NEU)	1.7	2.7	2.7		2	7.0		5.0	
ING GROEP NV	1.6		0.8	0.4	2				
DANONE	1.6	1.9	3.4	1.8	2	8.7	3.3	5.0	5.0
IBERDROLA SA	1.6		2.5		2	5.1		5.0	1.2
ENEL	1.6		2.1		2			5.0	2.9
VIVENDI SA	1.6	2.8	3.1	4.5	2	10.0	5.9	5.0	5.0
ANHEUSER-BUSCH INB	1.6	0.2	2.7	10.9	2	2.1	10.0	5.0	5.0
ASSIC GENERALI SPA	1.6		1.8		2				
AIR LIQUIDE(L')	1.4		2.1		2			5.0	
MUENCHENER RUECKVE	1.3		2.1	2.1	2		3.1	5.0	5.0
SCHNEIDER ELECTRIC	1.3		1.5		2				
CARREFOUR	1.3	1.0	2.7	1.3	2	3.7	2.5	5.0	5.0
VINCI	1.3		1.6		2				
LVMH MOET HENNESSY	1.2		1.8		2				
PHILIPS ELEC(KON)	1.2		1.4		2				
L'OREAL	1.1	0.8	2.8		2	5.5		5.0	5.0
CIE DE ST-GOBAIN	1.0		1.1		2				
REPSOL YPF SA	0.9		2.0		2			5.0	
CRH	0.8		1.7	5.1	2		5.2		5.0
CREDIT AGRICOLE SA	0.8		1.1		2				
DEUTSCHE BOERSE AG	0.7		1.5		2				1.9
TELECOM ITALIA SPA	0.7		2.0		2				2.5
ALSTOM	0.6		1.5		2				
AEGON NV	0.4		0.7		2				
VOLKSWAGEN AG	0.2		1.8	7.1	2		7.4		5.0
Total of components	50	11	50	17	50	14	16	20	23

Equity indexation

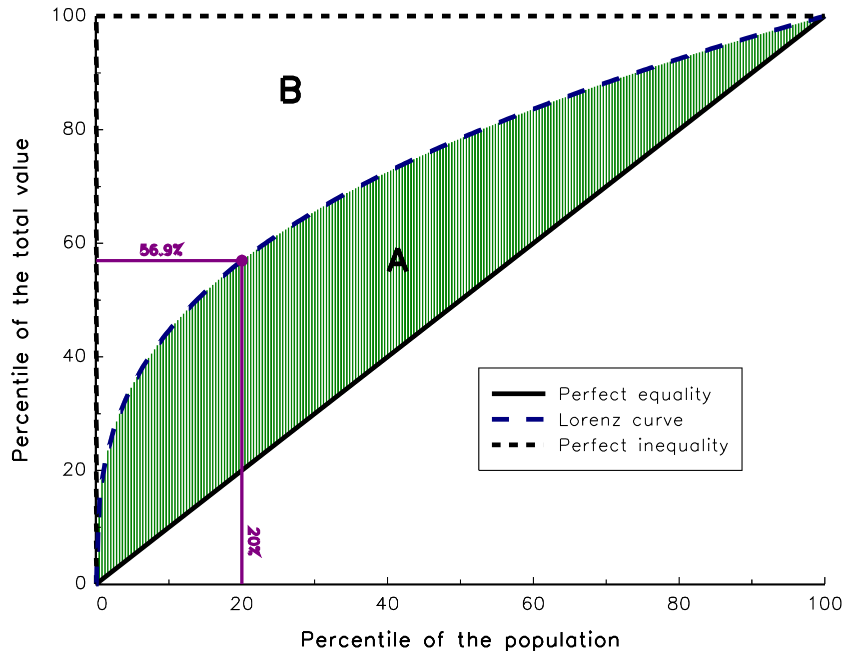
Measuring the concentration of an equity portfolio

- The Lorenz curve $\mathscr{L}(x)$
  It is a graphical representation of the concentration. It represents the cumulative weight of the first $x\%$ most representative stocks.

- The Gini coefficient
  It is a dispersion measure based on the Lorenz curve:

$$G = \frac{A}{A+B} = 2\int_0^1 \mathscr{L}(x)\,\mathrm{d}x - 1$$

  G takes the value 1 for a perfectly concentrated portfolio and 0 for the equally-weighted portfolio.

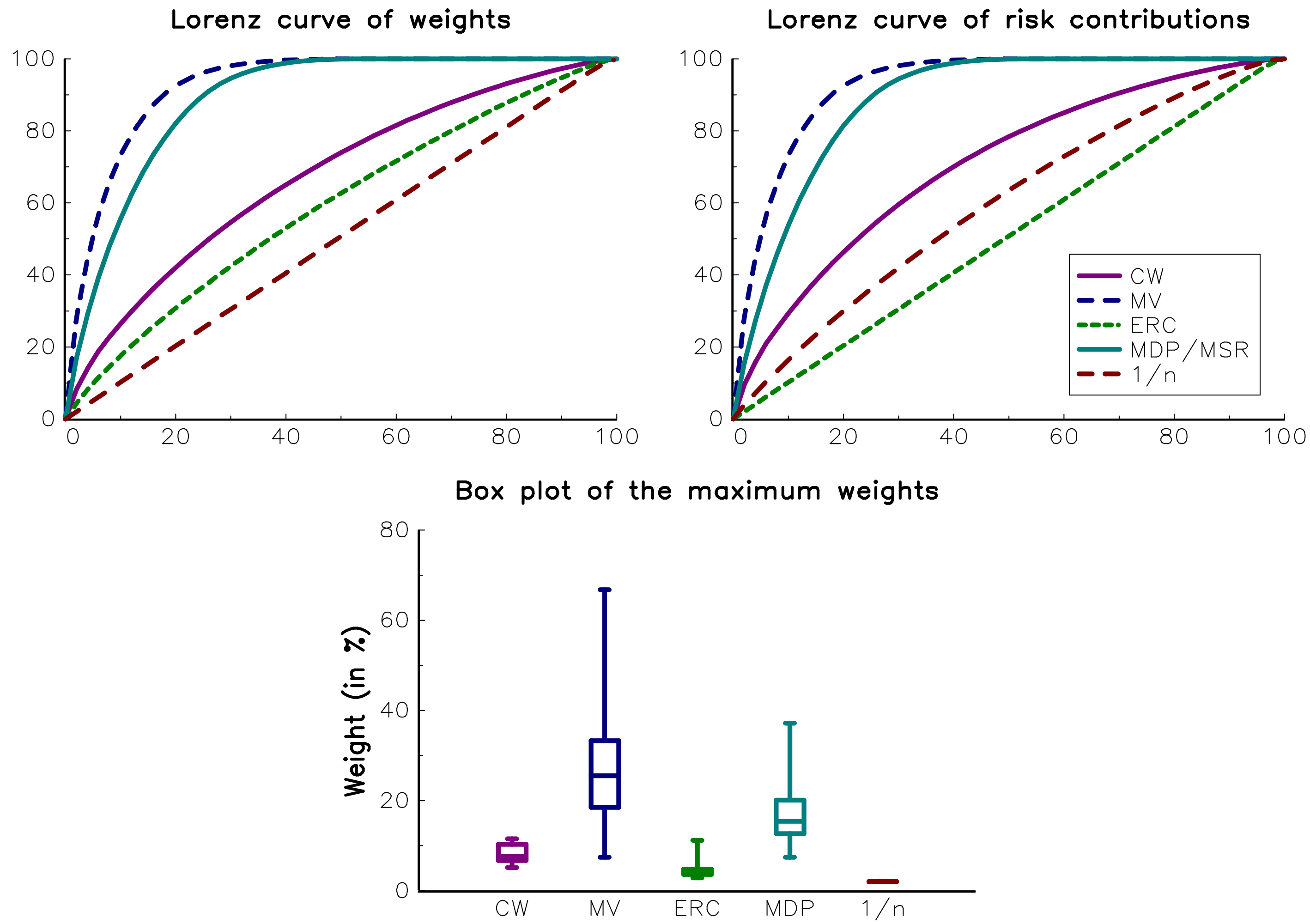


- The risk concentration of a portfolio is analyzed using Lorenz curve and Gini coefficient applied to risk contributions.

Equity indexation

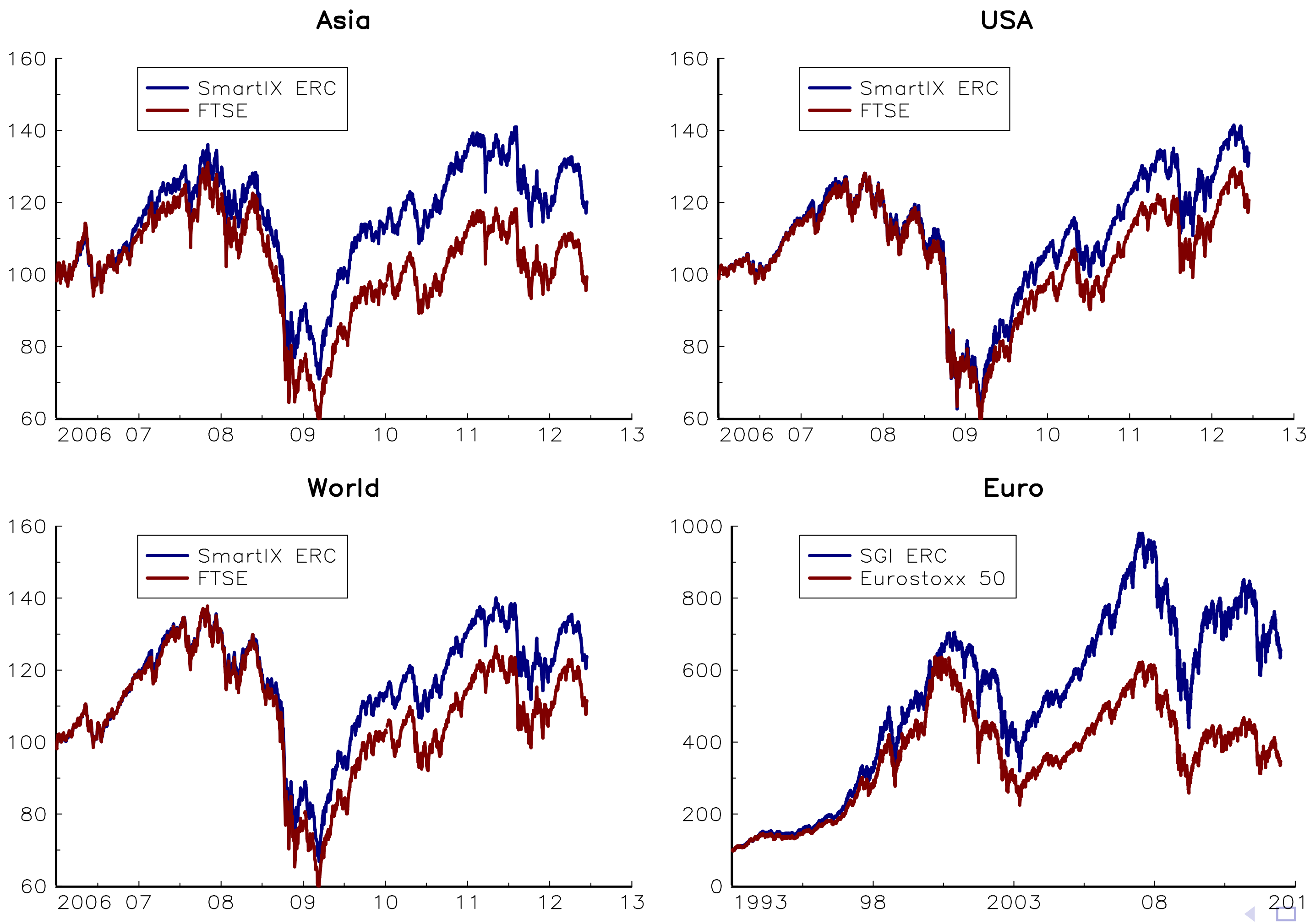
Concentration of the Eurostoxx 50 index

Figure: Weight and risk concentration (January 1993-December 2009)



Equity indexation
Examples

Figure: Performance of some ERC indexes



- Better performance
- Smaller volatility
- Smaller drawdown
- Controlled tracking error ($\simeq 5\%$)

Bond indexation

Time to rethink the bond portfolios management

Two main problems:

1. Benchmarks = debt-weighted indexation (the weights are based on the notional amount of the debt)
2. Fund management driven by the search of yield with little consideration for **credit risk** (carry position $\neq$ arbitrage position)

$\Rightarrow$ Time to rethink bond indexes? (Toloui, 2010)

We need to develop a framework to measure the credit risk of bond portfolios with two goals:

1. managing the credit risk of bond portfolios;
2. building alternative-weighted indexes.

For the application, we consider the euro government bond portfolios. The benchmark is the Citigroup EGBI index.

Bond indexation

Defining the credit risk measure of a bond portfolio

- Volatility of price returns $\neq$ a good measure of credit risk
- Correlation of price returns $\neq$ a good measure of contagion
- A better measure is the asset swap spread, but it is an OTC data. That's why we use the CDS spread.

Our credit risk measure $\mathscr{R}(w)$ is the (integrated) volatility of the CDS basket which would perfectly hedge the credit risk of the bond portfolio[6].

Remark

$\mathscr{R}(w)$ depends on 3 "CDS" parameters $S_i(t)$ (the level of the CDS), σ_i^S (the volatility of the CDS) and $\Gamma_{i,j}$ (the cross-correlation between CDS) and two "portfolio" parameters w_i (the weight) and D_i (the duration).

[6] We use a SABR model for the dynamics of CDS spreads.

Bond indexation

Weighting schemes

Debt weighting

It is defined by[a]:

$$w_i = \frac{\mathrm{DEBT}_i}{\sum_{i=1}^{n} \mathrm{DEBT}_i}$$

[a] Two forms of debt-weighting are considered : DEBT (with the 11 countries) and DEBT* (without Greece after July 2010). This last one corresponds to the weighting scheme of the EGBI index.

Alternative weighting

1. Fundamental indexation
   The GDP-weighting is defined by:

$$w_i = \frac{\mathrm{GDP}_i}{\sum_{i=1}^{n} \mathrm{GDP}_i}$$

2. Risk-based indexation
   The DEBT-RB and GDP-RB weightings are defined by:

$$b_i = \frac{\mathrm{DEBT}_i}{\sum_{i=1}^{n} \mathrm{DEBT}_i}$$

$$b_i = \frac{\mathrm{GDP}_i}{\sum_{i=1}^{n} \mathrm{GDP}_i}$$

Bond indexation

Some results for the EGBI index

Figure: EGBI weights and risk contributions

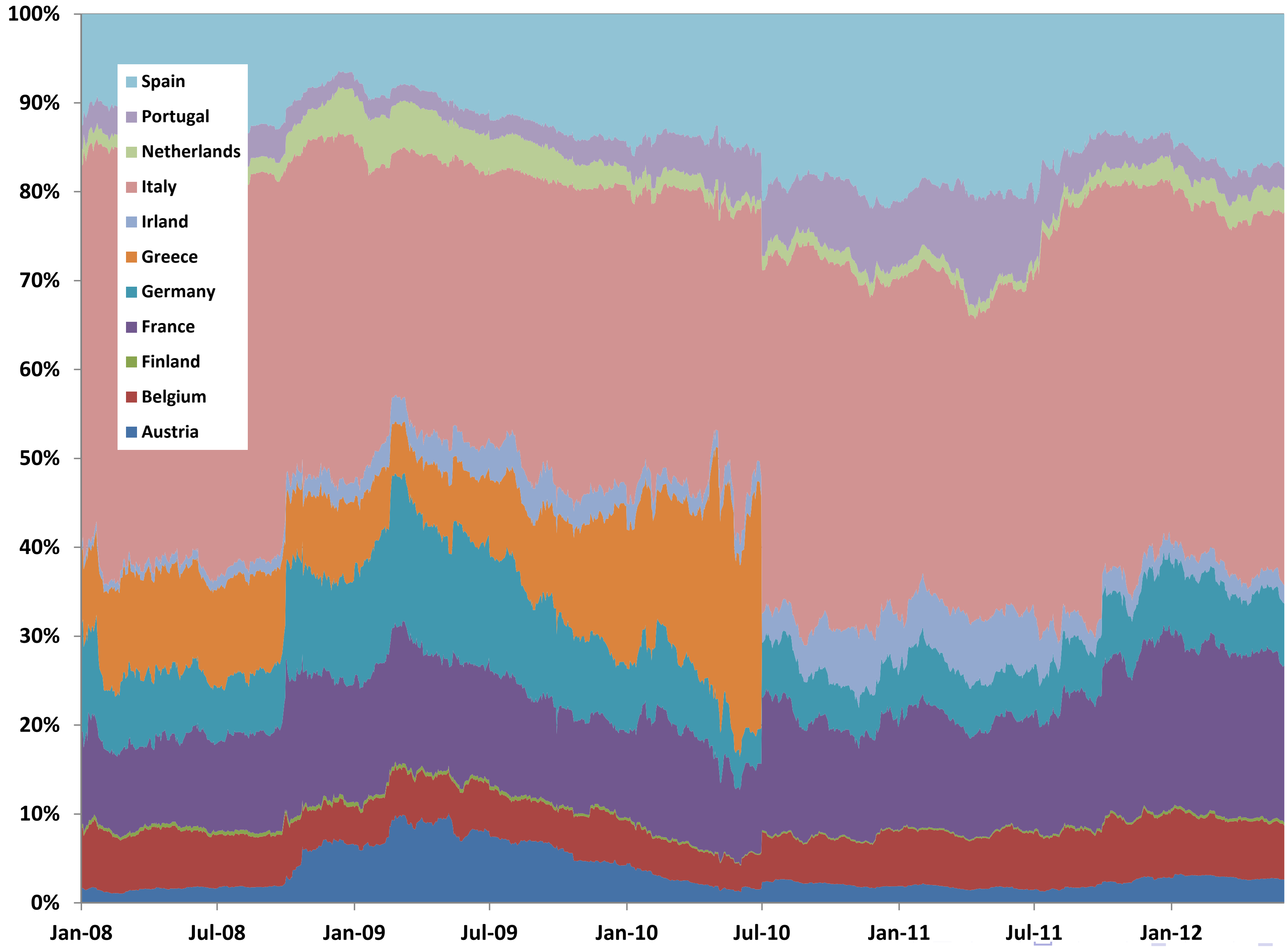
Country	July-08		July-09		July-10		July-11		March-12		June-12	
	Weights	RC	Weights	RC	Weights	RC	Weights	RC	Weights	RC	Weights	RC
Austria	4.1%	1.7%	3.6%	7.7%	4.1%	2.3%	4.3%	1.5%	4.2%	3.0%	4.3%	2.6%
Belgium	6.2%	6.1%	6.5%	5.1%	6.3%	5.7%	6.4%	6.5%	6.3%	6.6%	6.2%	6.1%
Finland	1.2%	0.4%	1.3%	0.5%	1.3%	0.2%	1.6%	0.2%	1.6%	0.3%	1.6%	0.3%
France	20.5%	9.8%	20.4%	13.2%	22.2%	15.1%	23.1%	13.3%	23.2%	19.0%	23.2%	17.6%
Germany	24.4%	6.1%	22.3%	13.0%	22.9%	6.0%	22.1%	5.3%	22.4%	7.3%	22.4%	7.0%
Greece	4.9%	11.4%	5.4%	8.5%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Ireland	1.0%	1.3%	1.5%	4.3%	2.1%	3.3%	1.4%	5.4%	1.7%	2.3%	1.7%	2.0%
Italy	22.1%	45.2%	22.4%	29.5%	23.4%	38.7%	23.1%	38.5%	22.1%	39.7%	21.8%	42.0%
Netherlands	5.3%	1.7%	5.3%	4.1%	6.1%	1.6%	6.2%	1.2%	6.2%	2.6%	6.5%	2.5%
Portugal	2.4%	3.9%	2.3%	2.3%	2.1%	6.3%	1.6%	6.6%	1.4%	3.0%	1.7%	2.6%
Spain	7.8%	12.4%	9.1%	11.8%	9.6%	20.9%	10.3%	21.3%	10.8%	16.2%	10.7%	17.2%
Sovereign Risk Measure	0.70%		2.59%		6.12%		4.02%		8.62%		12.16%	

$\Rightarrow$ Small changes in weights but large changes in risk contributions.

$\Rightarrow$ The sovereign credit risk measure has highly increased (the largest value 12.5% is obtained in November 25^{th} 2011).

Bond indexation

Evolution of the risk contributions in the EGBI index



Bond indexation

GDP-RB indexation

Figure: Weights and risk contributions of the GDP-RB indexation

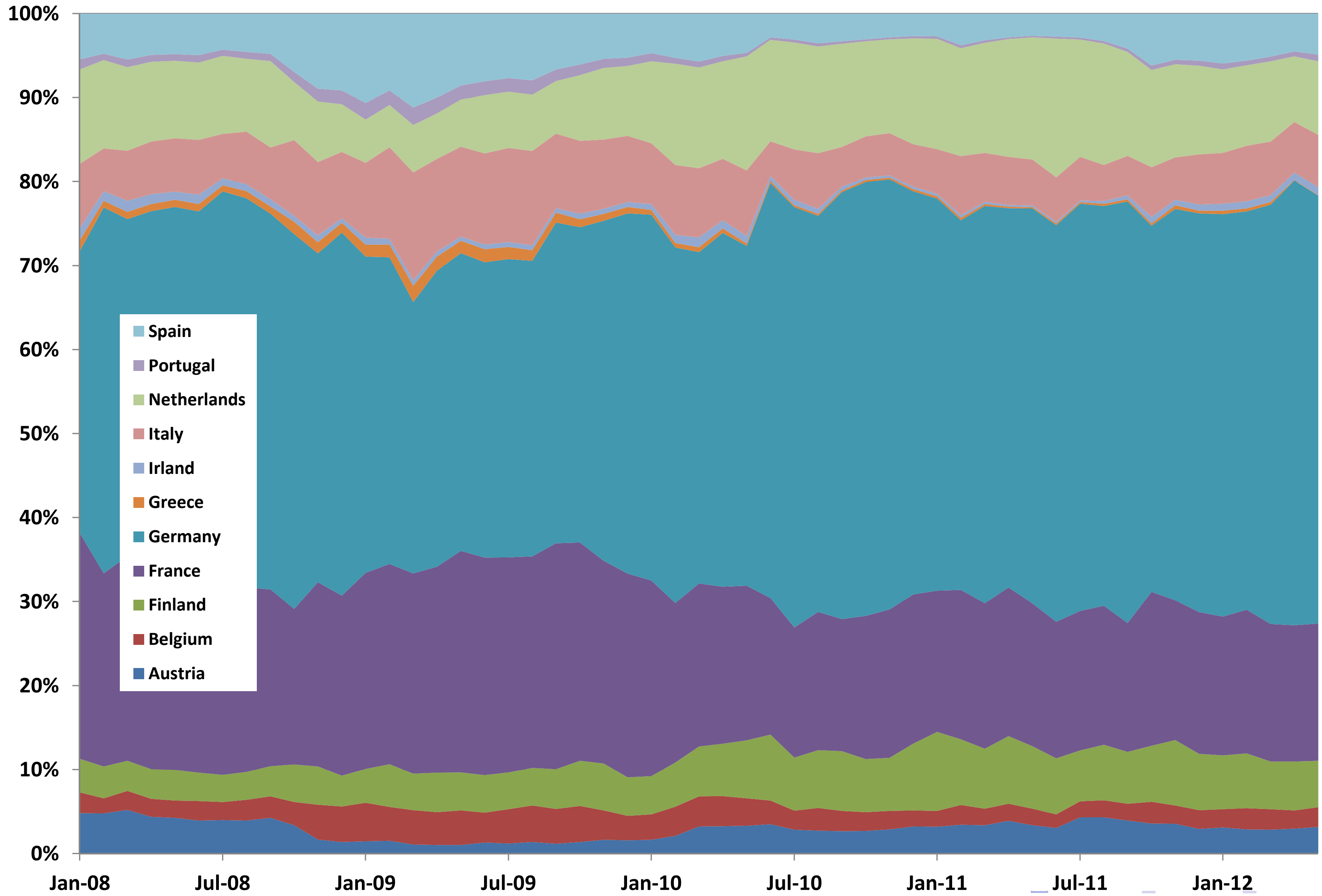
Country	July-08		July-09		July-10		July-11		March-12		June-12	
	RC	Weights	RC	Weights	RC	Weights	RC	Weights	RC	Weights	RC	Weights
Austria	3.1%	3.9%	3.1%	1.2%	3.1%	2.9%	3.2%	4.2%	3.4%	2.8%	3.4%	3.2%
Belgium	3.8%	2.1%	3.8%	4.1%	3.9%	2.2%	4.0%	1.9%	4.0%	2.4%	4.0%	2.4%
Finland	2.0%	3.2%	1.9%	4.4%	2.0%	6.3%	2.1%	6.0%	2.1%	5.7%	2.1%	5.5%
France	21.2%	22.0%	21.5%	25.6%	21.4%	15.5%	21.5%	16.5%	21.7%	16.4%	21.7%	16.3%
Germany	27.4%	47.8%	27.2%	35.5%	27.7%	50.0%	27.9%	48.7%	27.8%	49.9%	27.8%	51.0%
Greece	2.6%	0.7%	2.7%	1.4%	2.6%	0.2%	2.4%	0.2%	2.4%	0.3%	2.4%	0.1%
Ireland	2.0%	0.8%	1.9%	0.6%	1.8%	0.6%	1.7%	0.2%	1.7%	0.8%	1.7%	0.9%
Italy	17.4%	5.3%	17.3%	11.2%	17.2%	6.0%	17.0%	5.2%	17.1%	6.4%	17.1%	6.2%
Netherlands	6.5%	9.2%	6.5%	6.7%	6.5%	12.8%	6.6%	14.0%	6.5%	9.5%	6.5%	8.8%
Portugal	1.9%	0.7%	1.9%	1.6%	1.9%	0.4%	1.9%	0.2%	1.9%	0.6%	1.9%	0.8%
Spain	12.0%	4.2%	12.0%	7.7%	11.8%	3.1%	11.8%	2.9%	11.6%	5.1%	11.6%	4.9%
Sovereign Risk Measure	0.39%		2.10%		3.25%		1.91%		5.43%		7.43%	

$\Rightarrow$ RB indexation is very different from WB indexation, in terms of weights, RC and credit risk measures.

$\Rightarrow$ The dynamics of the GDP-RB is relatively smooth.

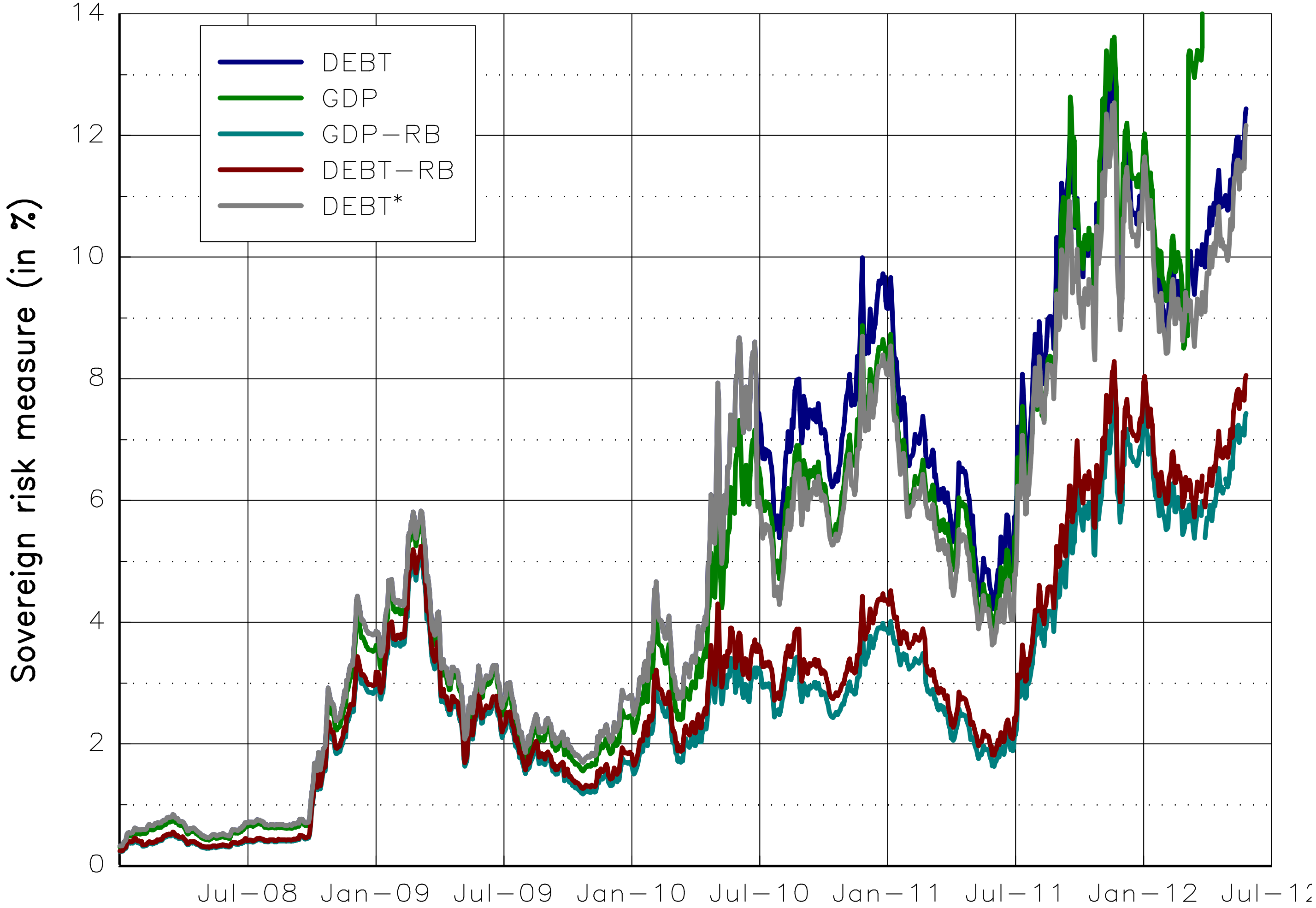
Bond indexation

Evolution of weights for the GDP-RB indexation



Bond indexation

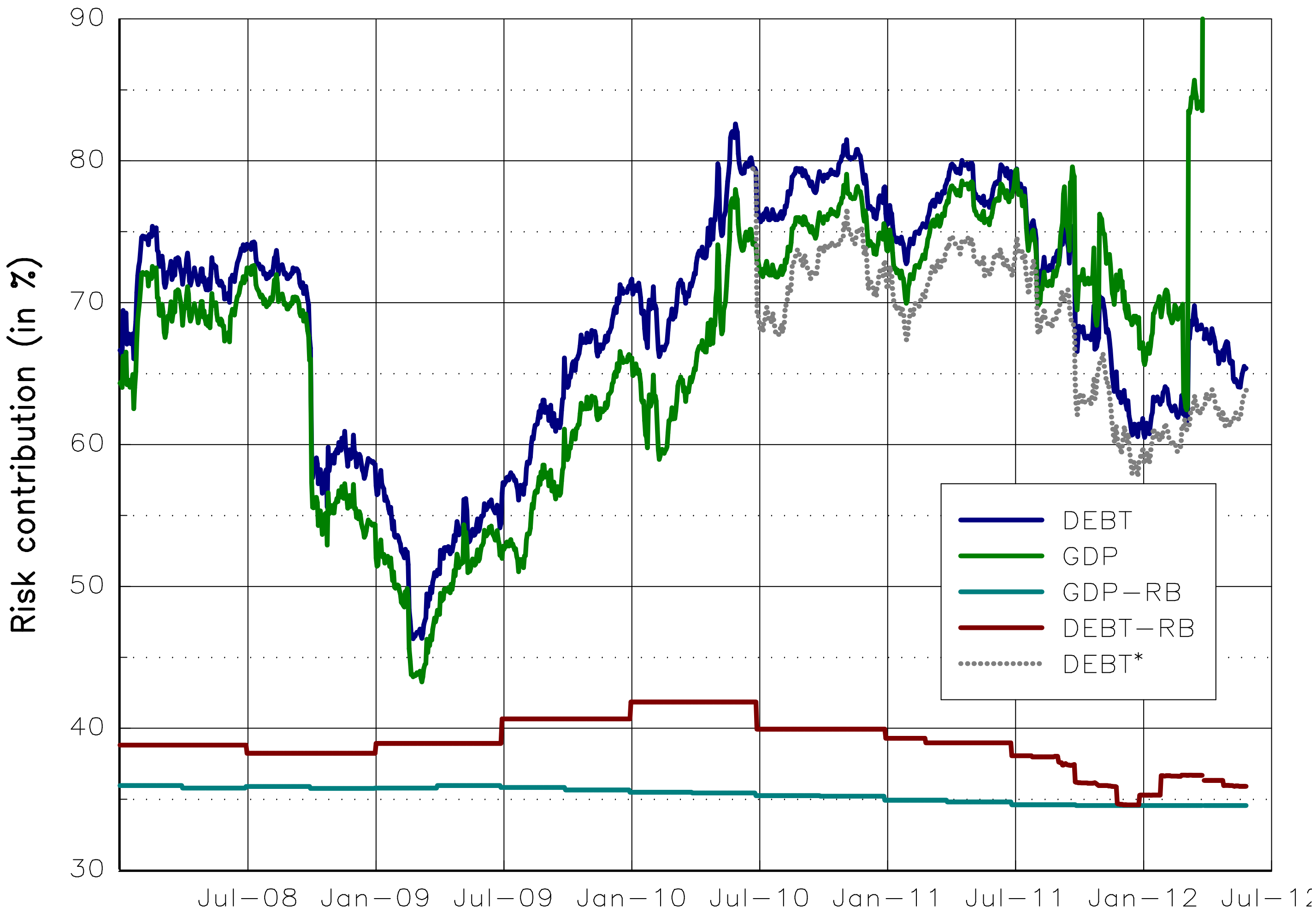
Evolution of the risk measure



⇒ We verify that the risk measure of the RB indexation is smaller than the one of the WB indexation.

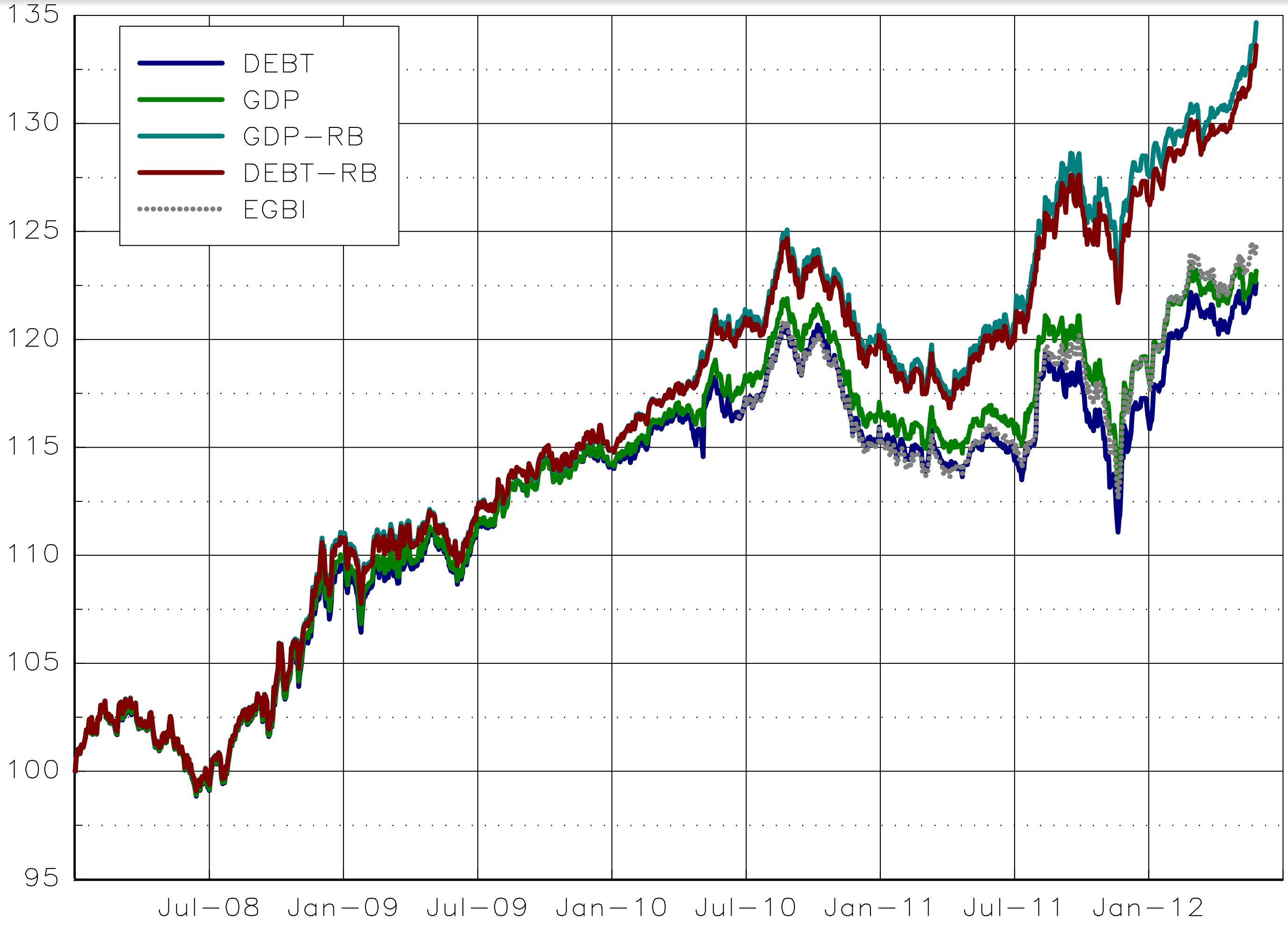
Bond indexation

Evolution of the GIIPS risk contribution



Bond indexation

Performance simulations



$\Rightarrow$ RB indexation / WB indexation = better performance, same volatility and smaller drawdowns.

Strategic asset allocation

Investment policy of long-term investors

Definition

Strategic asset allocation (SAA) is the choice of equities, bonds, and alternative assets that the investor wishes to hold for the long-run, usually from 10 to 50 years. Combined with tactical asset allocation (TAA) and constraints on liabilities, it defines the investment policy of pension funds.

Process of SAA:

- Universe definition of assets
- Expected returns, risks and correlations for the asset classes which compose the universe
- Portfolio optimization to target a given level of performance (subject to investor's constraints)

Strategic asset allocation

An example (input parameters)

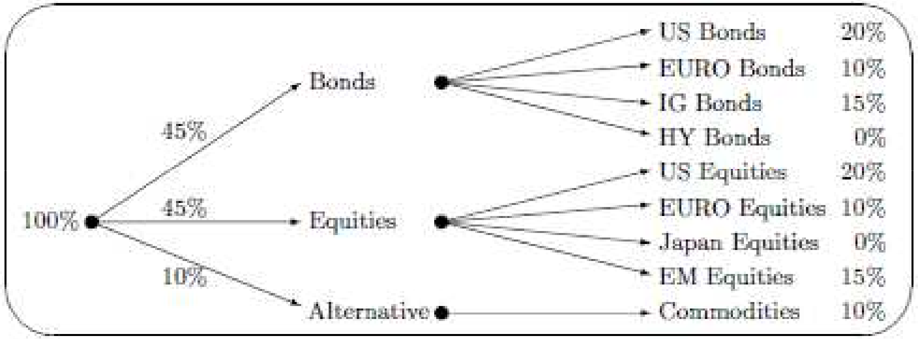
9 asset classes : US Bonds 10Y (1), EURO Bonds 10Y (2), Investment Grade Bonds (3), High Yield Bonds (4), US Equities (5), Euro Equities (6), Japan Equities (7), EM Equities (8) and Commodities (9).

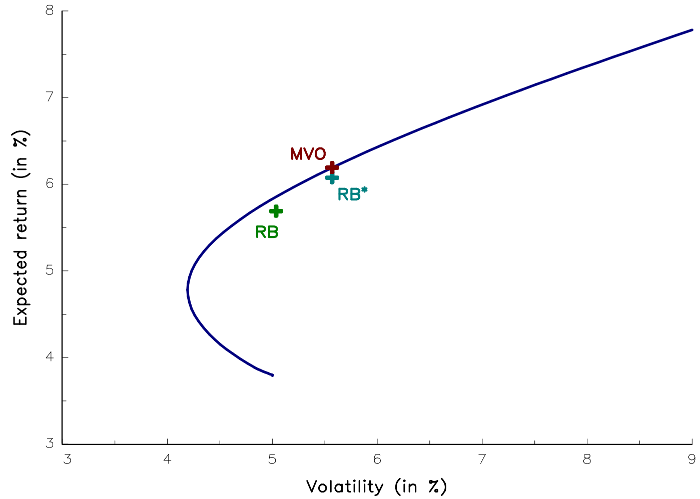
Table: Expected returns, risks and correlations (in %)

	μ_i	σ_i	$\rho_{i,j}$ (1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
(1)	4.2	5.0	100								
(2)	3.8	5.0	80	100							
(3)	5.3	7.0	60	40	100						
(4)	10.4	10.0	−20	−20	50	100					
(5)	9.2	15.0	−10	−20	30	60	100				
(6)	8.6	15.0	−20	−10	20	60	90	100			
(7)	5.3	15.0	−20	−20	20	50	70	60	100		
(8)	11.0	18.0	−20	−20	30	60	70	70	70	100	
(9)	8.8	30.0	0	0	10	20	20	20	30	30	100

Strategic asset allocation

Risk budgeting policy of the pension fund





Asset class	RB		RB*		MVO	
	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$	x_i	$\mathcal{RC}_i$
(1)	36.8%	20.0%	45.9%	18.1%	66.7%	25.5%
(2)	21.8%	10.0%	8.3%	2.4%	0.0%	0.0%
(3)	14.7%	15.0%	13.5%	11.8%	0.0%	0.0%
(5)	10.2%	20.0%	10.8%	21.4%	7.8%	15.1%
(6)	5.5%	10.0%	6.2%	11.1%	4.4%	7.6%
(8)	7.0%	15.0%	11.0%	24.9%	19.7%	49.2%
(9)	3.9%	10.0%	4.3%	10.3%	1.5%	2.7%

RB* = A BL portfolio with a tracking error of 1% wrt RB / MVO = Markowitz portfolio with the RB* volatility

Risk parity funds

Justification of diversified funds

Investor Profiles

1. Moderate (medium risk)
2. Conservative (low risk)
3. Aggressive (high risk)

Fund Profiles

1. Defensive (80% bonds and 20% equities)
2. Balanced (50% bonds and 50% equities)
3. Dynamic (20% bonds and 80% equities)

Relationship with portfolio theory?

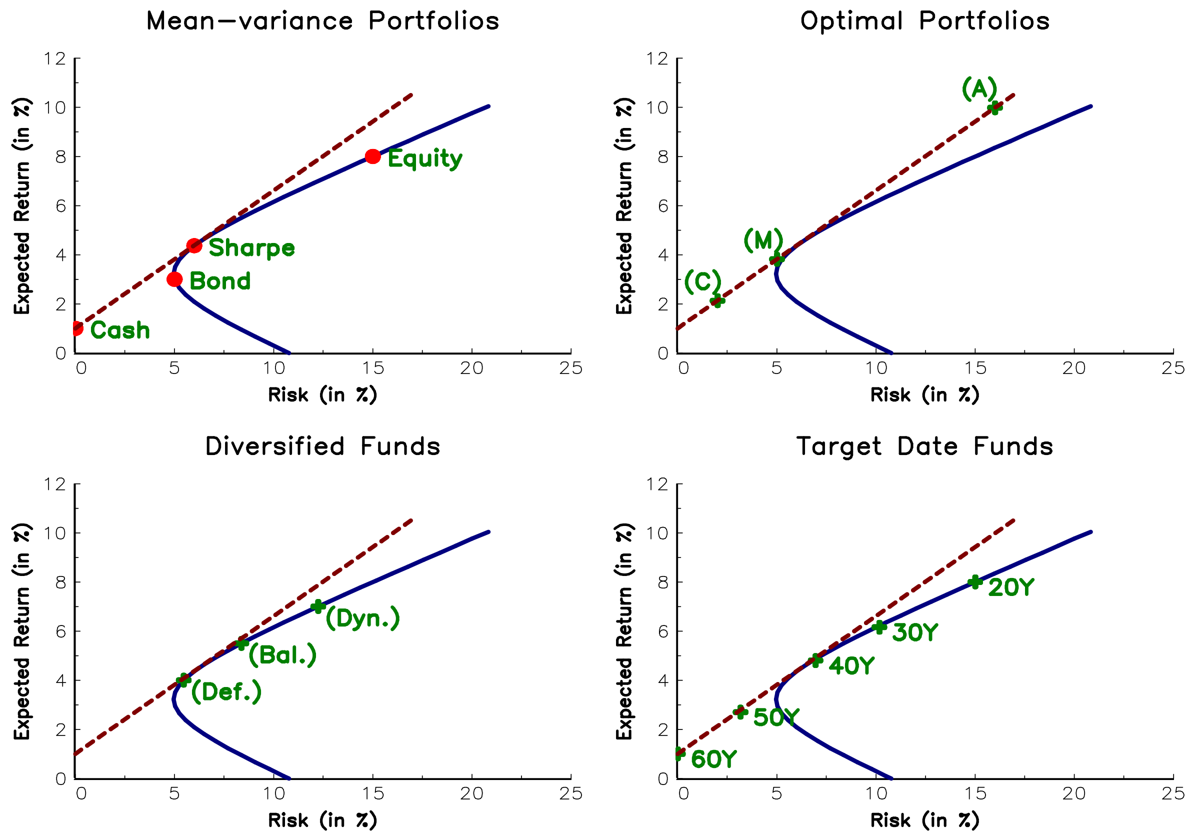
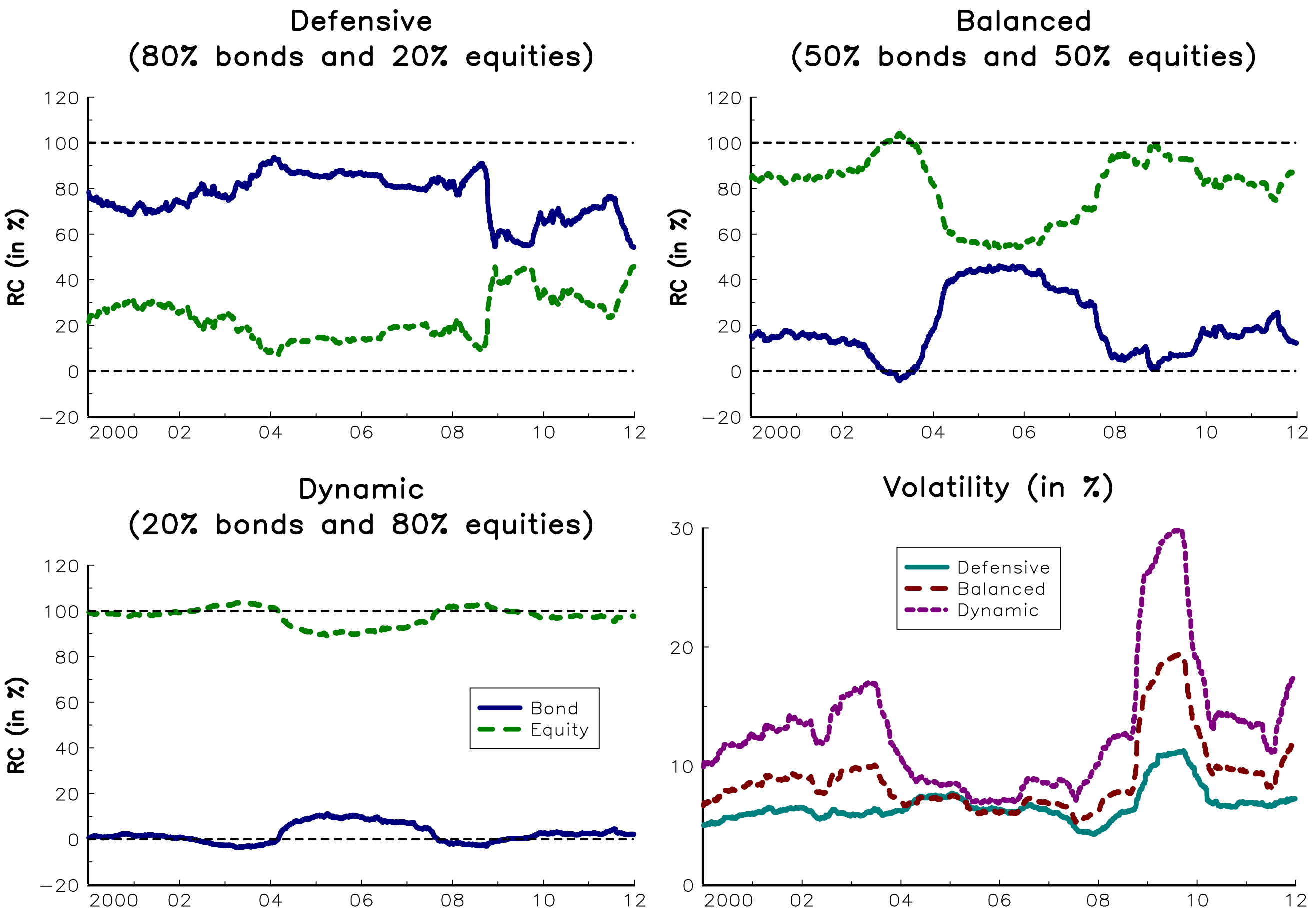


Figure: The asset allocation puzzle

Risk parity funds

What type of diversification is offered by diversified funds?

Figure: Risk contribution of diversified funds[a]



[a] Backtest with CG WGBI Index and MSCI World

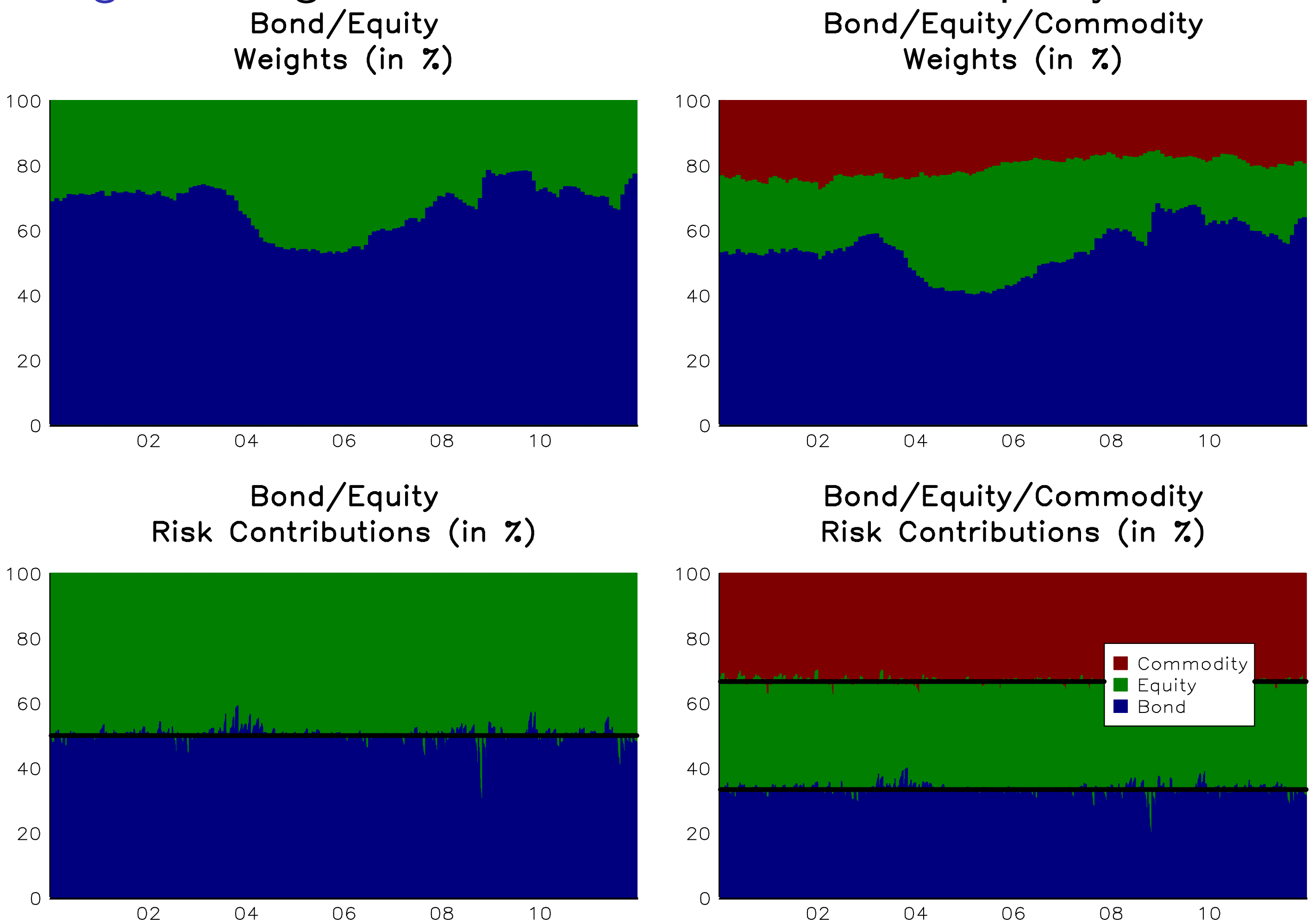
Diversified funds
=
Marketing idea?

- Deleverage of an equity exposure
- Diversification in weights $\neq$ Risk diversification
- No mapping between fund profiles and volatility profiles
- No mapping between fund profiles and investor profiles

Risk parity funds

Illustration of diversification

Figure: Weights and risk contributions of risk parity funds[7]



[7] Backtest with CG WGBI Index, MSCI World and DJ UBS Commodity Index

Risk parity funds

Some examples

Some examples

- AQR Capital Management (AQR Risk Parity)
- Aquila Capital
- Bridgewater (All Weather fund)
- First Quadrant
- Invesco (Invesco Balanced-Risk Allocation Fund)
- Lyxor Asset Management (ARMA fund)
- LODH
- PanAgora Asset Management
- Putnam Investments (Putnam Dynamic Risk Allocation)
- Wegelin Asset Management

Risk parity funds

Backtests (with equities and IG bonds)

Figure: With a developed countries universe

Conclusion

- Portfolio optimization leads to concentrated portfolios in terms of weights and risk.
- The use of weights constraints to diversify is equivalent to a discretionary shrinkage method.
- The risk parity approach is a better method to diversify portfolios.
- Risk parity strategies already implemented in:
  - Equity indexation (e.g. the SmartIX ERC indexes sponsored by Lyxor and calculated by FTSE)
  - Bond indexation (e.g. the RB EGBI index sponsored by Lyxor and calculated by Citigroup)
  - Commodity allocation (e.g. the Lyxor Commodity Active Fund)
  - Global asset allocation (e.g. the All Weather Strategy of Bridgewater or the IBRA fund of Invesco)

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