Managing Risk Exposures using the Risk Budgeting Approach[1]

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[1] The opinions expressed in this presentation are those of the author and are not meant to represent the opinions or official positions of Lyxor Asset Management.

Outline

Some issues on the asset management industry
(after the 2008 financial crisis)

Transition in the investment industry

- Concentration of assets under management
- Risk aversion of large institutional investors becomes higher
- Funding ratios are smaller (weakness of retirement systems)
- Pressure for more transparency and robustness
- Risk management as important as performance management

Transition in the passive indexation

- Robustness of market-cap indexation?
- Equity indexes = trend-following strategy
- Lack of portfolio construction rules ⇒ Risk concentration
- Alternative-weighted indexation = passive indexes where the weights are not based on market capitalization

Some issues on the asset management industry
(after the 2008 financial crisis)

Emergence of heuristic solutions

- Strong criticism of Markowitz optimization
- 2011: success of minimum-variance, erc, mdp/msr and risk parity strategies ⇒ These portfolio constructions only depend on risks, not on expected returns
- Special cases of the risk budgeting approach

⇒ Risk budgeting allocation = widely used by market practitioners (multi-asset classes, strategic asset allocation, equity portfolios)

Weight budgeting versus risk budgeting

Let $w = (w_1, \ldots, w_n)$ be the weights of n assets in the portfolio. Let $\mathscr{R}(w_1, \ldots, w_n)$ be a coherent and convex risk measure. We have:

$$\begin{aligned} \mathscr{R}(w_1, \ldots, w_n) &= \sum_{i=1}^{n} w_i \cdot \frac{\partial \mathscr{R}(w_1, \ldots, w_n)}{\partial w_i} \\ &= \sum_{i=1}^{n} \mathrm{RC}_i(w_1, \ldots, w_n) \end{aligned}$$

Let $b = (b_1, \ldots, b_n)$ be a vector of budgets such that $b_i \geq 0$ and $\sum_{i=1}^{n} b_i = 1$. We consider two allocation schemes:

1. Weight budgeting (WB)

$$w_i = b_i$$

2. Risk budgeting[2] (RB)

$$\mathrm{RC}_i = b_i \cdot \mathscr{R}(w_1, \ldots, w_n)$$

[2] The ERC portfolio is a special case when $b_i = 1/n$.

Application to the volatility risk measure

Let Σ be the covariance matrix of the assets returns. We assume that the risk measure $\mathscr{R}(w)$ is the volatility of the portfolio $\sigma(w) = \sqrt{w^\top \Sigma w}$. We have:

$$\begin{aligned}
\frac{\partial\, \mathscr{R}(w)}{\partial\, w} &= \frac{\Sigma w}{\sqrt{w^\top \Sigma w}} \\
\mathrm{RC}_i(w_1, \ldots, w_n) &= w_i \cdot \frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}} \\
\sum_{i=1}^n \mathrm{RC}_i(w_1, \ldots, w_n) &= \sum_{i=1}^n w_i \cdot \frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}} = w^\top \frac{\Sigma w}{\sqrt{w^\top \Sigma w}} = \sigma(w)
\end{aligned}$$

The risk budgeting portfolio is defined by this system of equations:

$$\left\{ \begin{array}{l}
w_i \cdot (\Sigma w)_i / b_i = w_j \cdot (\Sigma w)_j / b_j \\
w_i \geq 0 \\
\sum_{i=1}^n w_i = 1
\end{array} \right.$$

An example

Illustration

- 3 assets
- Volatilities are respectively 20%, 30% and 15%
- Correlations are set to 60% between the 1st asset and the 2nd asset and 10% between the first two assets and the 3rd asset

The marginal risk for the first asset is:

$$\lim_{\varepsilon \to 0} \frac{\sigma(w_1+\varepsilon, w_2, w_3) - \sigma(w_1, w_2, w_3)}{(w_1+\varepsilon) - w_1}$$

If $\varepsilon = 1\%$, we have:

$$\frac{\partial \mathscr{R}(w)}{\partial w_1} \simeq \frac{16.72\% - 16.54\%}{1\%} = 18.01\%$$

Weight budgeting (or traditional) approach

Asset	Weight	Marginal Risk	Risk Contribution	
			Absolute	Relative
1	**50.00%**	17.99%	9.00%	54.40%
2	**25.00%**	25.17%	6.29%	38.06%
3	**25.00%**	4.99%	1.25%	7.54%
Volatility			16.54%	

Risk budgeting approach

Asset	Weight	Marginal Risk	Risk Contribution	
			Absolute	Relative
1	41.62%	16.84%	7.01%	**50.00%**
2	15.79%	22.19%	3.51%	**25.00%**
3	42.58%	8.23%	3.51%	**25.00%**
Volatility			14.02%	

ERC approach

Asset	Weight	Marginal Risk	Risk Contribution	
			Absolute	Relative
1	30.41%	15.15%	4.61%	**33.33%**
2	20.28%	22.73%	4.61%	**33.33%**
3	49.31%	9.35%	4.61%	**33.33%**
Volatility			13.82%	

The ERC portfolio of Maillard et al. (2010)

Definition

The Euler decomposition gives us:

$$\sigma(w) = \sum_{i=1}^{n} w_i \times \frac{\partial\, \sigma(w)}{\partial\, w_i} = \sum_{i=1}^{n} \mathrm{RC}_i$$

The ERC portfolio is the risk-balanced portfolio such that the risk contribution is the same for all assets of the portfolio:

$$\mathrm{RC}_i = \mathrm{RC}_j \quad \text{for all } i,j$$

The ERC portfolio is then the solution of the following non-linear system:

$$\left\{ \begin{array}{c} w_2 \times (\Sigma w)_2 = w_1 \times (\Sigma w)_1 \\ w_3 \times (\Sigma w)_3 = w_1 \times (\Sigma w)_1 \\ \vdots \\ w_n \times (\Sigma w)_n = w_1 \times (\Sigma w)_1 \\ w_1 + w_2 + \cdots + w_n = 1 \\ w_1 > 0, w_2 > 0, \ldots, w_n > 0 \end{array} \right.$$

The ERC portfolio of Maillard et al. (2010)

Properties

Consider the following optimization problem:

$$\begin{aligned} w^{\star}(c) &= \arg\min \sqrt{w^{\top}\Sigma w} \\ &\text{u.c.} \quad \begin{cases} \sum_{i=1}^{n} \ln w_i \geq c \\ \mathbf{1}^{\top} w = 1 \\ \mathbf{0} \leq w \leq \mathbf{1} \end{cases} \end{aligned}$$

We have $w^{\star}(-\infty) = w_{\mathrm{mv}}$ and $w^{\star}(-n\ln n) = w_{1/\mathrm{n}}$. The ERC portfolio corresponds to a particular value of c such that $-\infty \leq c \leq -n\ln n$.

1. The solution of the ERC problem exists and is **unique**.
2. We also obtain the following inequality:

$$\sigma_{\mathrm{mv}} \leq \sigma_{\mathrm{erc}} \leq \sigma_{1/\mathrm{n}}$$

   because if $c_1 \leq c_2$, we have $\sigma(w^{\star}(c_1)) \leq \sigma(w^{\star}(c_2))$. The ERC portfolio may be viewed as a portfolio "between" the 1/n portfolio and the minimum-variance portfolio.
3. The ERC portfolio is a form of variance-minimizing portfolio subject to a constraint of sufficient diversification in terms of weights.

- **If the correlations are the same**, the solution is:

$$w_i = \frac{\sigma_i^{-1}}{\sum_{j=1}^{n} \sigma_j^{-1}}$$

The weight allocated to each component i is given by the ratio of the inverse of its volatility with the harmonic average of the volatilities.

- **If the volatilities are the same**, we have:

$$w_i \propto \left(\sum_{k=1}^{n} w_k \rho_{ik} \right)^{-1}$$

The weight of the asset i is proportional to the inverse of the weighted average of correlations of component i with other components.

- **In the general case**, we obtain:

$$w_i \propto \beta_i^{-1}$$

The weight of the asset i is proportional to the inverse of its beta.

- The ERC portfolio is the tangency portfolio if all the assets have the same Sharpe ratio and if the correlation is uniform (one-factor model).
- Let us consider the minimum variance portfolio with a constant correlation matrix $C_n(\rho)$. The solution is:

$$w_i = \frac{-((n-1)\rho+1)\sigma_i^{-2} + \rho\sum_{j=1}^n(\sigma_i\sigma_j)^{-1}}{\sum_{k=1}^n\left(-((n-1)\rho+1)\sigma_k^{-2} + \rho\sum_{j=1}^n(\sigma_k\sigma_j)^{-1}\right)}$$

  The lower bound of $C_n(\rho)$ is achieved for $\rho = -(n-1)^{-1}$ and we have:

$$w_i = \frac{\sum_{j=1}^n(\sigma_i\sigma_j)^{-1}}{\sum_{k=1}^n\sum_{j=1}^n(\sigma_k\sigma_j)^{-1}} = \frac{\sigma_i^{-1}}{\sum_{k=1}^n\sigma_k^{-1}} \rightarrow \mathrm{erc}$$

- The ERC portfolio minimizes the Gini and Herfindal indexes applied to the risk measure.

Some analytical solutions

- The case of uniform correlation[3] $\rho_{i,j} = \rho$
  - ERC portfolio ($b_i = 1/n$)

$$w_i(\rho) = \frac{\sigma_i^{-1}}{\sum_{j=1}^n \sigma_j^{-1}}$$

  - RB portfolio

$$w_i\left(-\frac{1}{n-1}\right) = \frac{\sigma_i^{-1}}{\sum_{j=1}^n \sigma_j^{-1}}, \; w_i(0) = \frac{\sqrt{b_i}\sigma_i^{-1}}{\sum_{j=1}^n \sqrt{b_j}\sigma_j^{-1}}, \; w_i(1) = \frac{b_i\sigma_i^{-1}}{\sum_{j=1}^n b_j\sigma_j^{-1}}$$

- The general case

$$w_i = \frac{b_i\beta_i^{-1}}{\sum_{j=1}^n b_j\beta_j^{-1}}$$

where β_i is the beta of the asset i with respect to the RB portfolio.

[3] The solution is noted $w_i(\rho)$.

The RB portfolio is a minimum variance (MV) portfolio subject to a constraint of weight diversification

Let us consider the following minimum variance optimization problem:

$$
\begin{aligned}
w^{\star}(c) &= \arg\min \sqrt{w^{\top}\Sigma w} \\
\text{u.c.} &\quad \begin{cases} \sum_{i=1}^{n} b_i \ln w_i \geq c \\ \mathbf{1}^{\top} w = 1 \\ w \geq \mathbf{0} \end{cases}
\end{aligned}
$$

- if $c = c^{-} = -\infty$, $w^{\star}(c^{-}) = w_{\mathrm{mv}}$ (no weight diversification)
- if $c = c^{+} = \sum_{i=1}^{n} b_i \ln b_i$, $w^{\star}(c^{+}) = w_{\mathrm{wb}}$ (no variance minimization)
- $\exists c^{0} : w^{\star}(c^{0}) = w_{\mathrm{rb}}$ (variance minimization and weight diversification)

$\Longrightarrow$ if $b_i = 1/n$, $w_{\mathrm{rb}} = w_{\mathrm{erc}}$ (variance minimization, weight diversification and perfect risk diversification[4])

[4] The Gini coefficient of the risk measure is then equal to 0.

Existence and uniqueness

- If $b_i > 0$, the solution exists and is unique.
- If $b_i \geq 0$, there may be several solutions.
- If $\rho_{i,j} \geq 0$, the solution is unique.

An example with 3 assets: $\sigma_1 = 20\%$, $\sigma_2 = 10\%$, $\sigma_3 = 5\%$ and $\rho_{1,2} = 50\%$.

$\rho_{1,3} = \rho_{2,3}$	Solution		1	2	3	$\sigma(x)$
-25%	$\mathcal{S}_1$	w_i	20.00%	40.00%	40.00%	6.63%
		$\partial_{w_i} \sigma(w)$	16.58%	8.29%	0.00%	
		RC_i	50.00%	50.00%	0.00%	
	$\mathcal{S}_2$	w_i	33.33%	66.67%	0.00%	11.55%
		$\partial_{w_i} \sigma(w)$	17.32%	8.66%	-1.44%	
		RC_i	50.00%	50.00%	0.00%	
	$\mathcal{S}'_1$	w_i	19.23%	38.46%	42.31%	6.38%
		$\partial_{w_i} \sigma(w)$	16.42%	8.21%	0.15%	
		RC_i	49.50%	49.50%	1.00%	
25%	$\mathcal{S}_1$	w_i	33.33%	66.67%	0.00%	11.55%
		$\partial_{w_i} \sigma(w)$	17.32%	8.66%	1.44%	
		RC_i	50.00%	50.00%	0.00%	

Bounds for the risk of the risk budgeting portfolio

We have:

$$\sigma_{\mathrm{mv}} \leq \sigma_{\mathrm{rb}} \leq \sigma_{\mathrm{wb}}$$

Given a portfolio w^0, we could easily find a new portfolio w^1 with lower volatility by setting:

$$b_i = w_i^0$$

In this case, the minimum variance portfolio appears to be a limit portfolio.

The result of Maillard *et al.* (2010) for the ERC portfolio is a special case of this theorem with:

$$\mathrm{wb} = 1/n$$

Optimality of risk budgeting portfolios

If the RB portfolio is optimal (in the Markowitz sense), the ex-ante performance contributions are equal to the risk budgets:

Black-Litterman Approach

Budgeting the risk = budgeting the performance
(in an ex-ante point of view)

Let $\tilde{\mu}_i$ be the market price of the expected return. We have:

$$w_i \cdot \tilde{\mu}_i \propto w_i \cdot \frac{\partial\, \sigma(w)}{\partial\, w}$$

Optimality of risk budgeting portfolios

An example

$$\sigma = \begin{pmatrix} 10\% \\ 20\% \\ 30\% \\ 40\% \end{pmatrix} \quad \text{and} \quad \rho = \begin{pmatrix} 1.0 & & & \\ 0.8 & 1.0 & & \\ 0.2 & 0.2 & 1.0 & \\ 0.2 & 0.2 & 0.5 & 1.0 \end{pmatrix}$$

Example 1

b_i	w_i	$\partial_{w_i}\sigma(w)$	RC_i	$\tilde{\mu}_i$	PC_i
20.0	40.9	7.1	20.0	5.2	20.0
25.0	25.1	14.5	25.0	10.5	25.0
40.0	25.3	23.0	40.0	16.7	40.0
15.0	8.7	25.0	15.0	18.2	15.0

Example 2

b_i	w_i	$\partial_{w_i}\sigma(w)$	RC_i	$\tilde{\mu}_i$	PC_i
10.0	35.9	5.3	10.0	5.0	10.0
10.0	17.9	10.5	10.0	9.9	10.0
10.0	10.2	18.6	10.0	17.5	10.0
70.0	36.0	36.7	70.0	34.7	70.0

Generalization to other convex risk measures

If the risk measure is convex and satisfy the Euler principle, the following properties are verified:

1. Existence and uniqueness
2. Location between the minimum risk portfolio and the weight budgeting portfolio
3. Optimality

Some heuristic portfolios as RB portfolios

The EW, MV, MDP and ERC portfolios could be interpreted as RB portfolios.

	EW	MV	MDP	ERC
b_i				
PC_i	β_i	w_i	$w_i \sigma_i$	$1/n$

MV and MDP portfolios are two limit portfolios (explaining that the weights of some assets could be equal to zero).

What is the problem with optimized portfolios?

"The indifference of many investment practitioners to mean-variance optimization technology, despite its theoretical appeal, is understandable in many cases. The major problem with MV optimization is its tendency to maximize the effects of errors in the input assumptions. Unconstrained MV optimization can yield results that are inferior to those of simple equal-weighting schemes" (Michaud, 1989).

Optimal solutions are of the following form:

$$w^\star \propto \Sigma^{-1}\mu$$

The important quantity is then the information matrix:

$$\mathscr{I} = \Sigma^{-1}$$

The eigendecomposition of $\mathscr{I}$ is:

$$V_i(\mathscr{I}) = V_{n-i}(\Sigma) \quad \text{and} \quad \lambda_i(\mathscr{I}) = \frac{1}{\lambda_{n-i}(\Sigma)}$$

An illustration

$\mu_1 = \mu_2 = 8\%$, $\mu_3 = 5\%$, $\sigma_1 = 20\%$, $\sigma_2 = 21\%$, $\sigma_3 = 10\%$ and $\rho_{i,j} = 80\%$.

The eigendecomposition of the covariance and information matrices is:

Asset / Factor	Covariance matrix Σ 1	2	3	Information matrix $\mathcal{I}$ 1	2	3
1	65.35%	−72.29%	−22.43%	−22.43%	−72.29%	65.35%
2	69.38%	69.06%	−20.43%	−20.43%	69.06%	69.38%
3	30.26%	−2.21%	95.29%	95.29%	−2.21%	30.26%
Eigenvalue	8.31%	0.84%	0.26%	379.97	119.18	12.04
% cumulated	88.29%	97.20%	100.00%	74.33%	97.65%	100.00%

With a volatility of 15%, the optimal portfolio is $(38.3\%, 20.2\%, 41.5\%)$. The corresponding risk contributions are 49.0%, 25.8% and 25.2%.

What is the sensitivity to the input parameters?

	ρ / σ_2 / μ_1						
	ρ		70%	90%		90%	
	σ_2				18%	18%	
	μ_1						9%
MVO	x_1	38.3%	38.3%	44.6%	13.7%	0.0%	56.4%
	x_2	20.2%	25.9%	8.9%	56.1%	65.8%	0.0%
	x_3	41.5%	35.8%	46.5%	30.2%	34.2%	43.6%
RB	x_1	38.3%	37.7%	38.9%	37.1%	37.7%	38.3%
	x_2	20.2%	20.4%	20.0%	22.8%	22.6%	20.2%
	x_3	41.5%	41.9%	41.1%	40.1%	39.7%	41.5%

How to save portfolio optimization?

Because the optimal solution depends principally on the last factors of the covariance matrix, we have to introduce some regularization techniques:

- regularization of the objective function by using resampling techniques
- regularization of the covariance matrix:
  - Factor analysis
  - Shrinkage methods
  - Random matrix theory
  - etc.
- regularization of the program specification by introducing some constraints

In the 90's, there were some hope to save portfolio optimisation.

Today, this hope is tiny and portfolio optimization is perhaps dead.

Justification of diversified funds

Investor Profiles

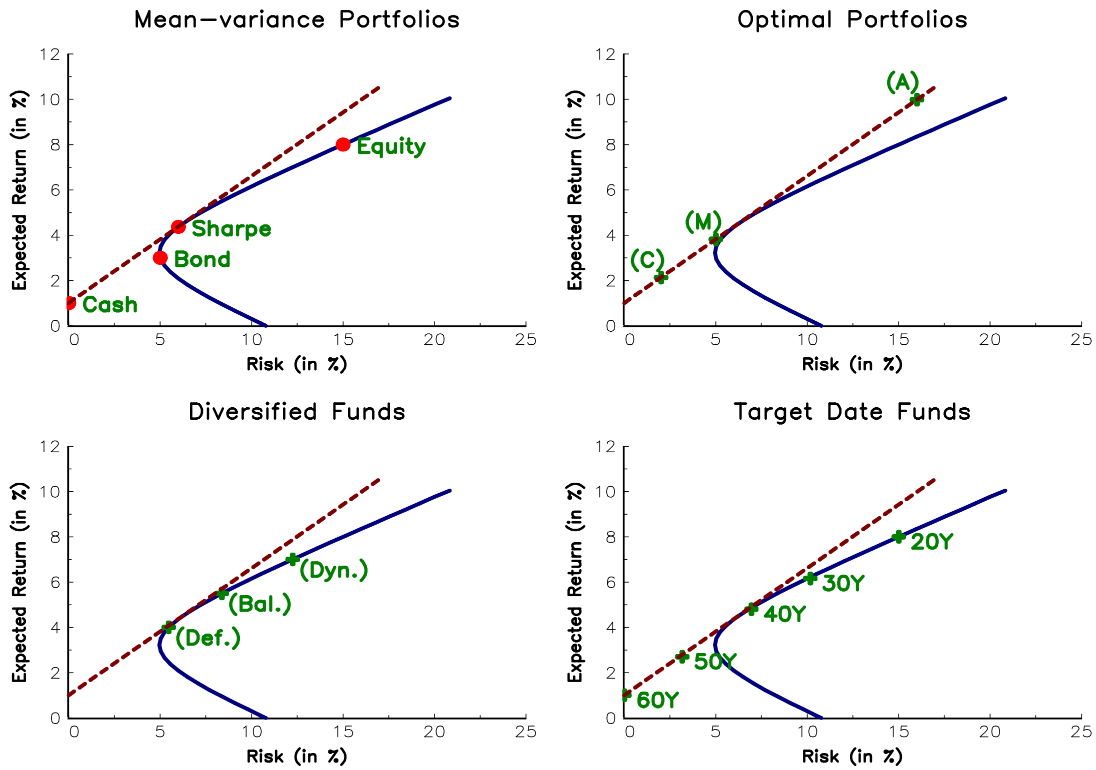
1. Moderate (medium risk tolerance)
2. Conservative (low risk tolerance)
3. Aggressive (high risk tolerance)

Fund Profiles

1. Defensive (80% bonds and 20% equities)
2. Balanced (50% bonds and 50% equities)
3. Dynamic (20% bonds and 80% equities)

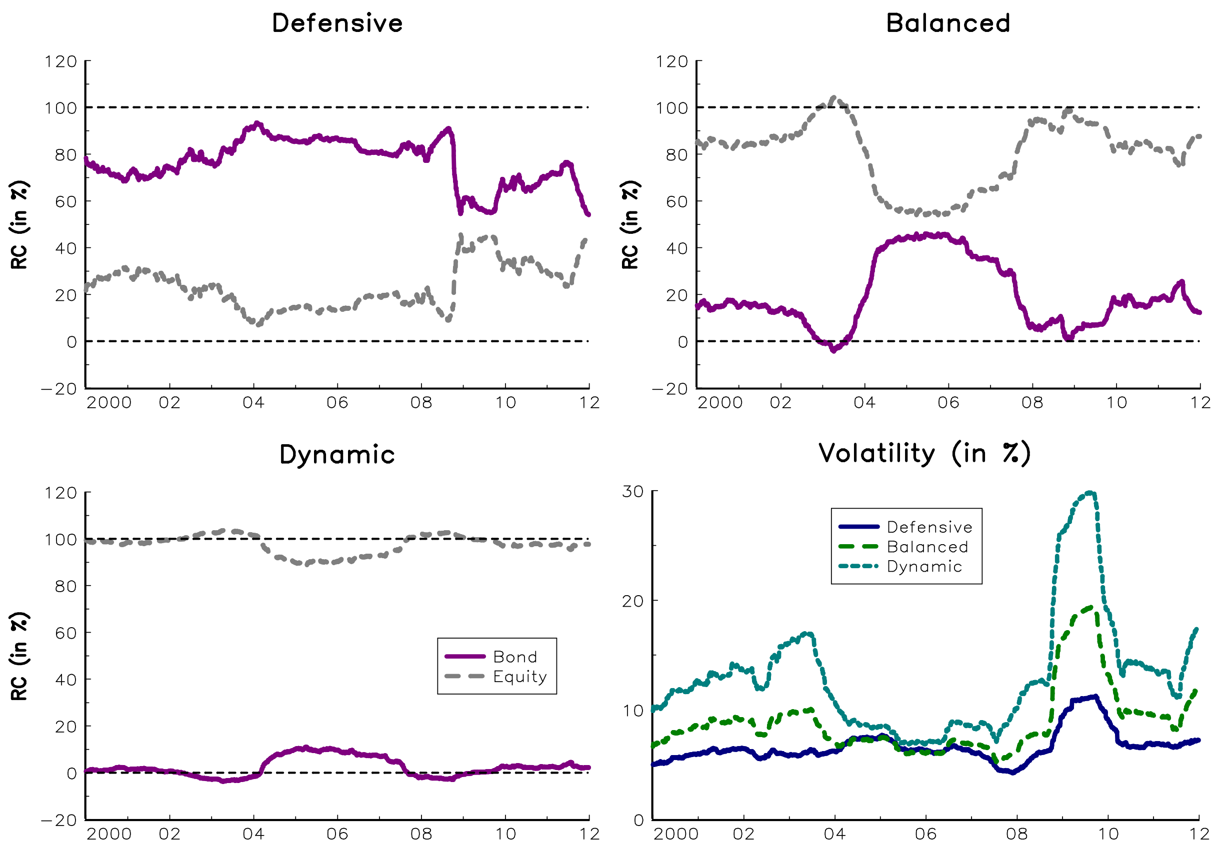
Relationship with portfolio theory?

Figure: The asset allocation puzzle



What type of diversification offer diversified funds?

Figure: Risk contribution of diversified funds[a]



[a] Backtest with CG WGBI Index and MSCI World

Diversified funds
=
Marketing idea?

- Deleverage of an equity exposure
- Diversification in weights $\neq$ Risk diversification
- No mapping between fund profiles and volatility profiles
- No mapping between fund profiles and investor profiles

Risk parity funds

Definition

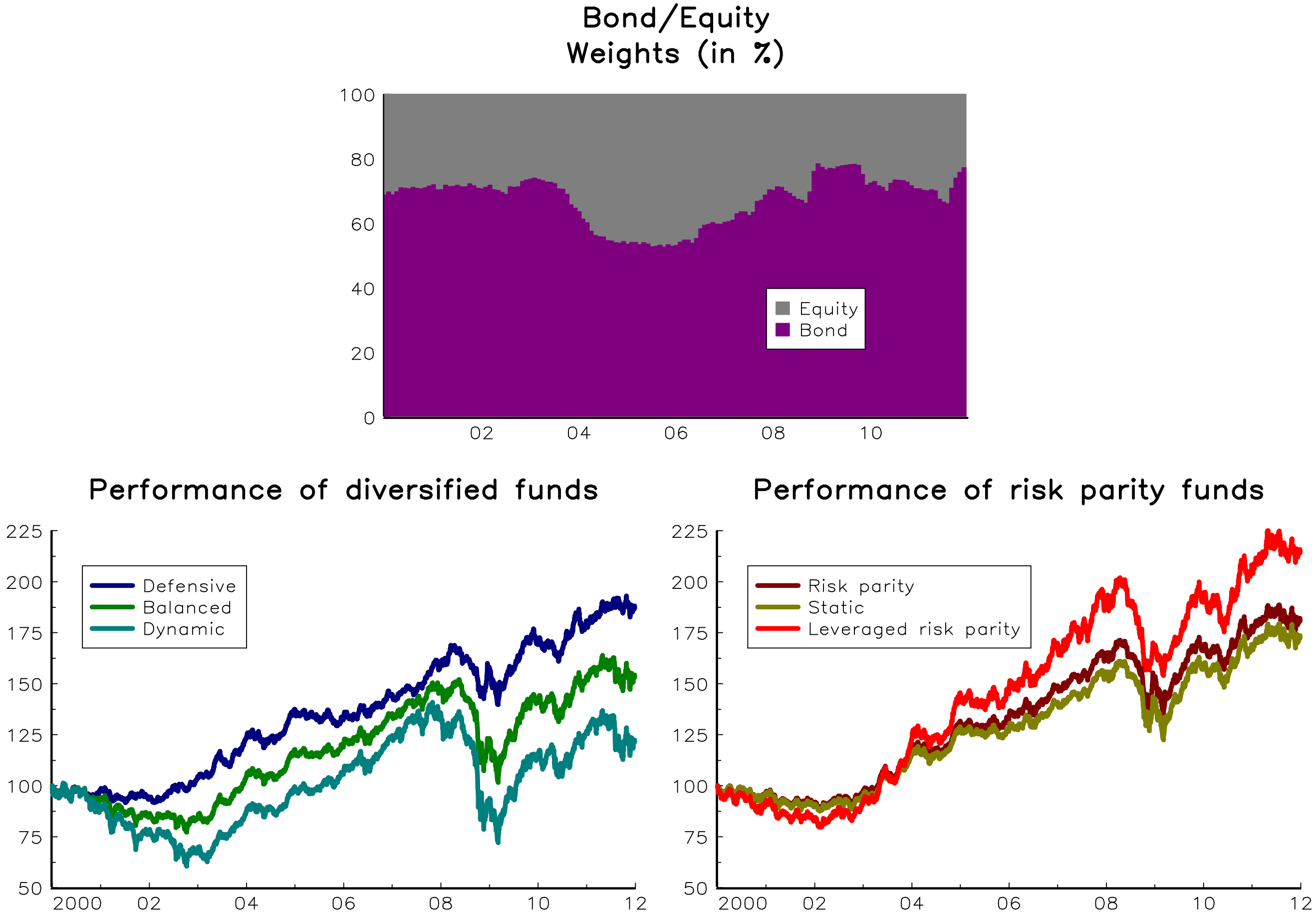
A risk parity fund is an ERC startegy on multi-assets classes.

Some examples

- AQR Capital Management
- Bridgewater
- Invesco
- Lyxor Asset Management
- PanAgora Asset Management
- Wegelin Asset Management

ERC diversified funds

Figure: Comparison of diversified and risk parity funds[5]



[5] Backtest with CG WGBI Index and MSCI World Index

Time to rethink the bond portfolios management

Two main problems:

1. Benchmarks = debt-weighted indexation (the weights are based on the notional amount of the debt)
2. Fund management driven by the search of yield with little consideration for risk (carry position $\neq$ arbitrage position)

$\Rightarrow$ Time to rethink bond indexes? (Toloui, 2010)

For the application, we consider the euro government bond portfolios. The benchmark is the Citigroup EGBI index[6].

[6] This index is very close to the EuroMTS index.

Bond indexation schemes

Debt weighting

It is defined by[a]:

$$w_i = \frac{\mathrm{DEBT}_i}{\sum_{i=1}^{n} \mathrm{DEBT}_i}$$

[a] Two forms of debt-weighting are considered : DEBT (with the 11 countries) and DEBT* (without Greece after July 2010). This last one corresponds to the weighting scheme of the EGBI index.

Alternative weighting

1. Fundamental indexation
   The GDP-weighting is defined by:

$$w_i = \frac{\mathrm{GDP}_i}{\sum_{i=1}^{n} \mathrm{GDP}_i}$$

2. Risk-based indexation
   The DEBT-RB and GDP-RB weightings are defined by:

$$b_i = \frac{\mathrm{DEBT}_i}{\sum_{i=1}^{n} \mathrm{DEBT}_i}$$

$$b_i = \frac{\mathrm{GDP}_i}{\sum_{i=1}^{n} \mathrm{GDP}_i}$$

Choosing the right measure of credit risk

- Volatility of price returns $\neq$ a good measure of credit risk
- Correlation of price returns $\neq$ a good measure of contagion
- A better measure is the yield spread, but its computation can be difficult as it needs to first define a reference risk-free rate.

$\Rightarrow$ One of the best measure is the CDS spread (it does not depend on the currency, the yield curve or the duration).

The SABR CDS model

Let $S_i(t)$ be the spread of the i^{th} issuer. We have:

$$\mathrm{d}S_i(t) = \sigma_i^S \cdot S_i(t)^{\beta_i} \cdot \mathrm{d}W_i(t)$$

Moreover, we assume that the correlation between the brownian motions $W_i(t)$ and $W_j(t)$ is $\Gamma_{i,j}$.

Calibration of the β_i parameter

We assume that we observe spreads at some given known dates $t_0, \ldots, t_n$. Let $S_{i,j}$ be the observed spread for the i^{th} country at date t_j. The log-likelihood function for the i^{th} country is:

$$\begin{aligned}\ell &= -\frac{n}{2}\ln 2\pi - n\ln\sigma_i^S - \frac{1}{2}\sum_{j=1}^{n}\ln\left(t_j - t_{j-1}\right) - \\ &\quad \beta_i\sum_{j=1}^{n}\ln S_{i,j-1} - \frac{1}{2}\sum_{j=1}^{n}\frac{\left(S_{i,j} - S_{i,j-1}\right)^2}{\left(\sigma_i^S S_{i,j-1}^{\beta_i}\right)^2\left(t_j - t_{j-1}\right)}\end{aligned}$$

Figure: Results for the period January 2008-August 2011

Country	AT	BE	FI	FR	DE	GR	IE	IT	NL	PT	ES	Average
estimate	0.996	1.017	0.816	0.786	0.899	1.070	0.836	1.157	0.793	1.013	1.148	0.957
std-dev.	1.10%	2.00%	1.60%	1.60%	2.00%	1.10%	0.70%	1.70%	0.90%	1.10%	2.10%	1.45%

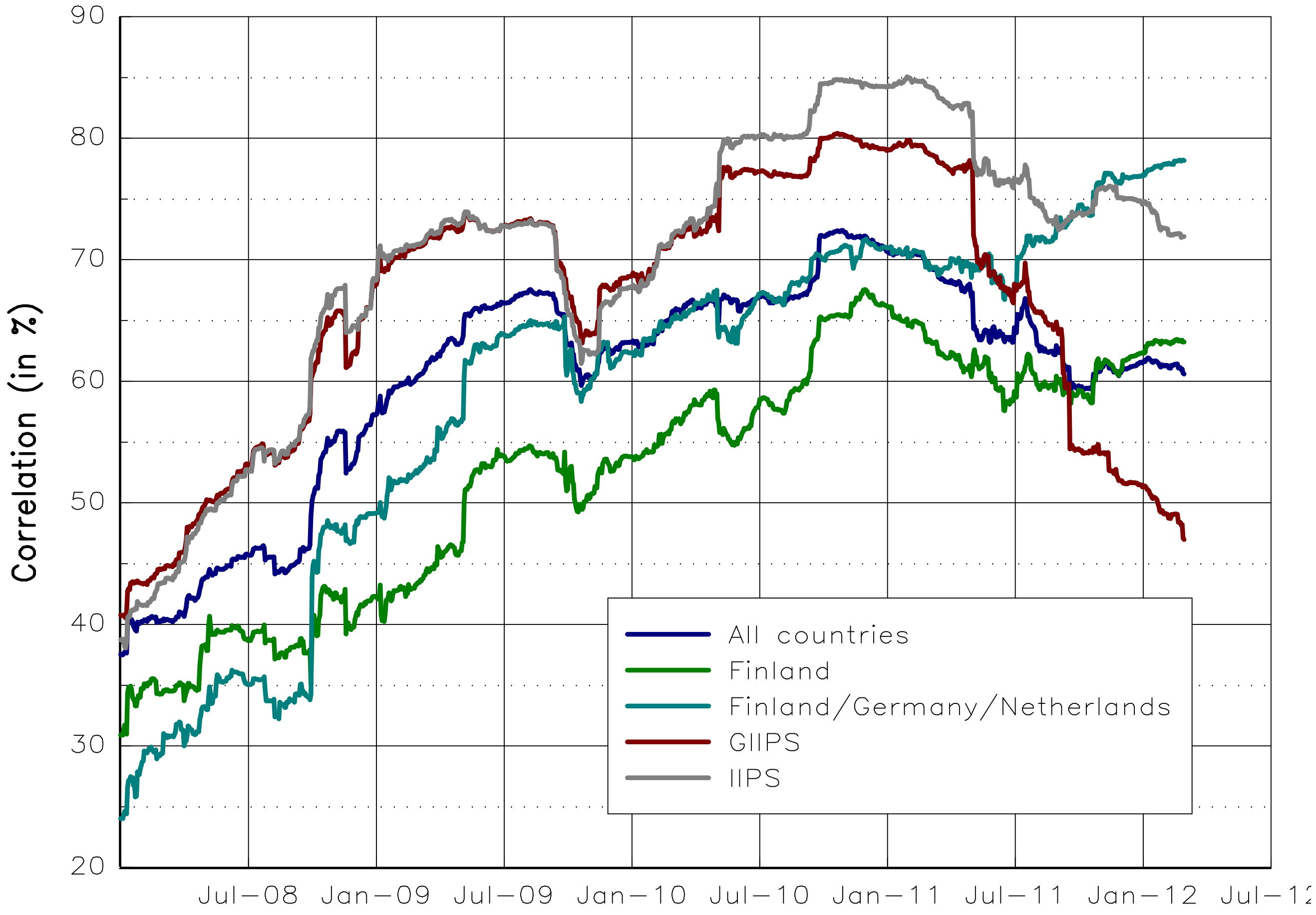
$\Rightarrow$ We assume that $\beta_i = 1$ (ML estimation is then easy to compute).

Statistics as of March 1st, 2012

Table: CDS levels $S_i(t)$, volatilities σ_i^S and correlations $\Gamma_{i,j}$

Country	Spread	Volatility	AT	BE	FI	FR	DE	GR	IE	IT	NL	PT	ES
Austria	158	73.5%	100%										
Belgium	223	73.1%	80%	100%									
Finland	64	68.8%	75%	75%	100%								
France	166	70.9%	87%	85%	78%	100%							
Germany	76	66.0%	82%	78%	73%	86%	100%						
Greece	8,871	163.4%	9%	12%	9%	6%	6%	100%					
Ireland	581	51.9%	62%	72%	57%	67%	66%	16%	100%				
Italy	356	74.2%	74%	86%	72%	80%	73%	11%	71%	100%			
Netherlands	94	67.7%	79%	79%	78%	85%	83%	6%	64%	74%	100%		
Portugal	1,175	56.1%	55%	66%	50%	60%	57%	15%	79%	67%	54%	100%	
Spain	356	72.5%	74%	80%	66%	75%	69%	9%	69%	81%	66%	64%	100%

Evolution of the correlation matrix



Computing the credit risk measure of a bond portfolio

Let $w = (w_1, \ldots, w_n)$ be the weights of bonds in the portfolio. The risk measure is[7]:

$$\mathscr{R}(x) = \sqrt{w^\top \Sigma w} = \sqrt{\sum_{i=1}^n \sum_{j=1}^n w_i w_j \Sigma_{i,j}}$$

with $\Sigma_{i,j} = \Gamma_{i,j} \cdot \sigma_i^B \cdot \sigma_j^B$ and $\sigma_i^B = D_i \cdot \sigma_i^S \cdot S_i(t)$, where D_i is the duration of the bond i, σ_i^S is the CDS volatility of the corresponding issuer, $S_i(t)$ is the CDS level and $\Gamma_{i,j}$ is the correlation between the CDS relative variations of issuers corresponding to the bonds i and j.

$\mathscr{R}(w)$ is the volatility of the CDS basket which would perfectly hedge the credit risk of the bond portfolio.

$\mathscr{R}(w)$ depends on 3 "CDS" parameters $S_i(t)$, σ_i^S and $\Gamma_{i,j}$ and two "portfolio" parameters w_i and D_i.

[7] We have $\mathrm{d}\ln B_t(D_i) = -D_i \cdot \mathrm{d}R(t) - D_i \cdot \mathrm{d}S_i(t)$ with $B_t(D_i)$ the zero-coupon of maturity D_i and $R(t)$ the "risk-free" interest rate. It comes that $\sigma_i^B = D_i \cdot \sigma_i^S \cdot S_i(t)$.

Defining the risk contribution

Our credit risk measure $\mathscr{R}(w) = \sqrt{w^\top \Sigma w}$ is a convex risk measure. It means that:

$$\begin{aligned}\mathscr{R}(w_1,\ldots,w_n) &= \sum_{i=1}^{n} w_i \cdot \frac{\partial\,\mathscr{R}(w_1,\ldots,w_m)}{\partial\, w_i} \\ &= \sum_{i=1}^{n} RC_i\end{aligned}$$

We can then break the risk measure down into n individual sources of risk.

The risk contribution RC_i is an increasing function of the parameters D_i, $S_i(t)$ and σ_i^S.

Some results for the EGBI index

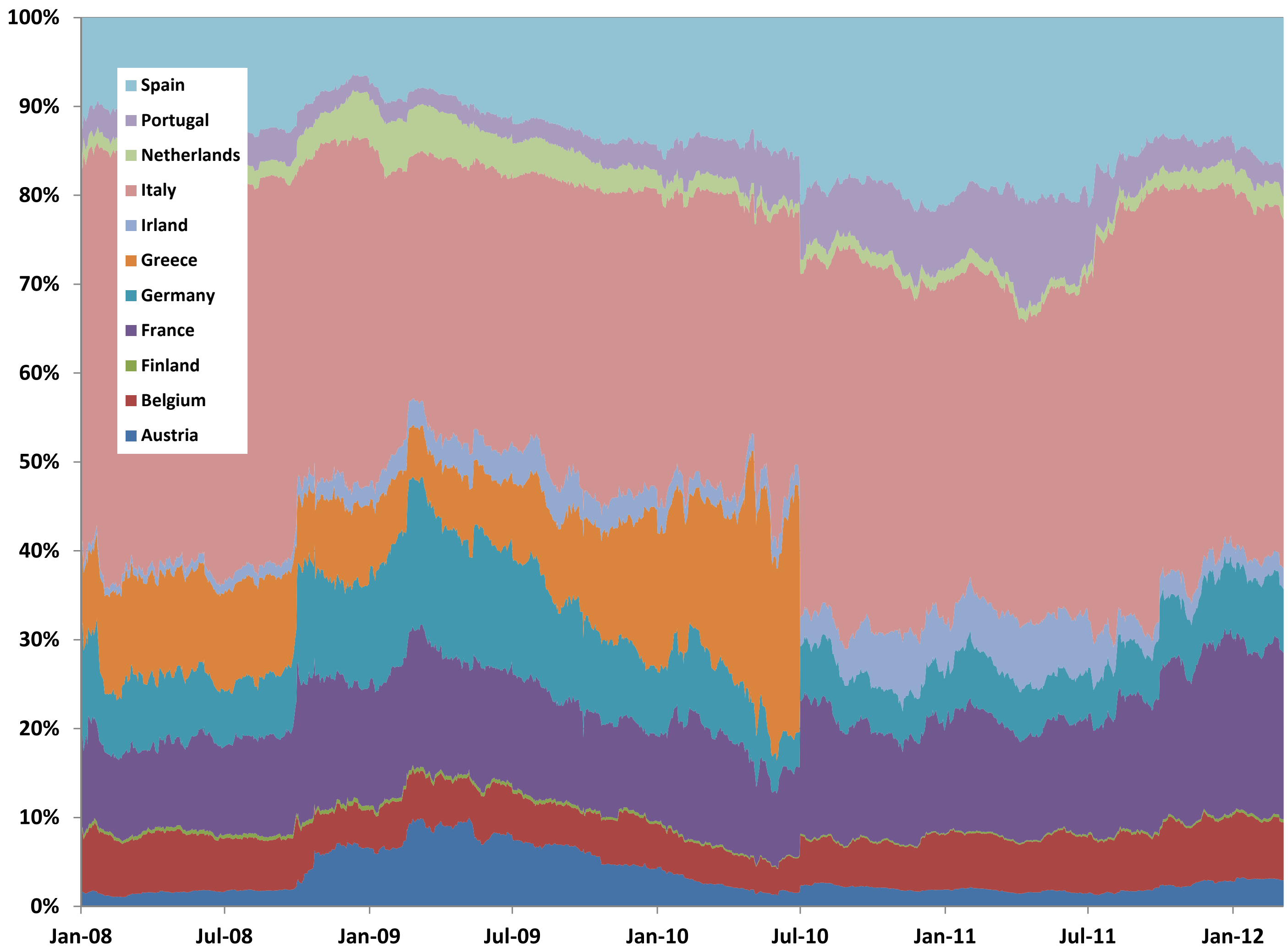
Figure: EGBI weights and risk contributions

Country	July-08		July-09		July-10		July-11		March-12	
	Weights	RC	Weights	RC	Weights	RC	Weights	RC	Weights	RC
Austria	4.1%	1.7%	3.6%	7.7%	4.1%	2.3%	4.3%	1.5%	4.2%	3.0%
Belgium	6.2%	6.1%	6.5%	5.1%	6.3%	5.7%	6.4%	6.5%	6.3%	6.6%
Finland	1.2%	0.4%	1.3%	0.5%	1.3%	0.2%	1.6%	0.2%	1.6%	0.3%
France	20.5%	9.8%	20.4%	13.2%	22.2%	15.1%	23.1%	13.3%	23.2%	19.0%
Germany	24.4%	6.1%	22.3%	13.0%	22.9%	6.0%	22.1%	5.3%	22.4%	7.3%
Greece	4.9%	11.4%	5.4%	8.5%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Ireland	1.0%	1.3%	1.5%	4.3%	2.1%	3.3%	1.4%	5.4%	1.7%	2.3%
Italy	22.1%	45.2%	22.4%	29.5%	23.4%	38.7%	23.1%	38.5%	22.1%	39.7%
Netherlands	5.3%	1.7%	5.3%	4.1%	6.1%	1.6%	6.2%	1.2%	6.2%	2.6%
Portugal	2.4%	3.9%	2.3%	2.3%	2.1%	6.3%	1.6%	6.6%	1.4%	3.0%
Spain	7.8%	12.4%	9.1%	11.8%	9.6%	20.9%	10.3%	21.3%	10.8%	16.2%
Sovereign Risk Measure	0.70%		2.59%		6.12%		4.02%		8.62%	

⇒ Small changes in weights but large changes in risk contributions.
⇒ The sovereign credit risk measure has highly increased (the largest value 12.5% is obtained in November 25$^{\text{th}}$ 2011).
⇒ If we think that the EGBI portfolio is optimal, we expect that 60% of the performance will come from Italy and France.

Some results for the EGBI index

Evolution of the risk contributions



GDP indexation

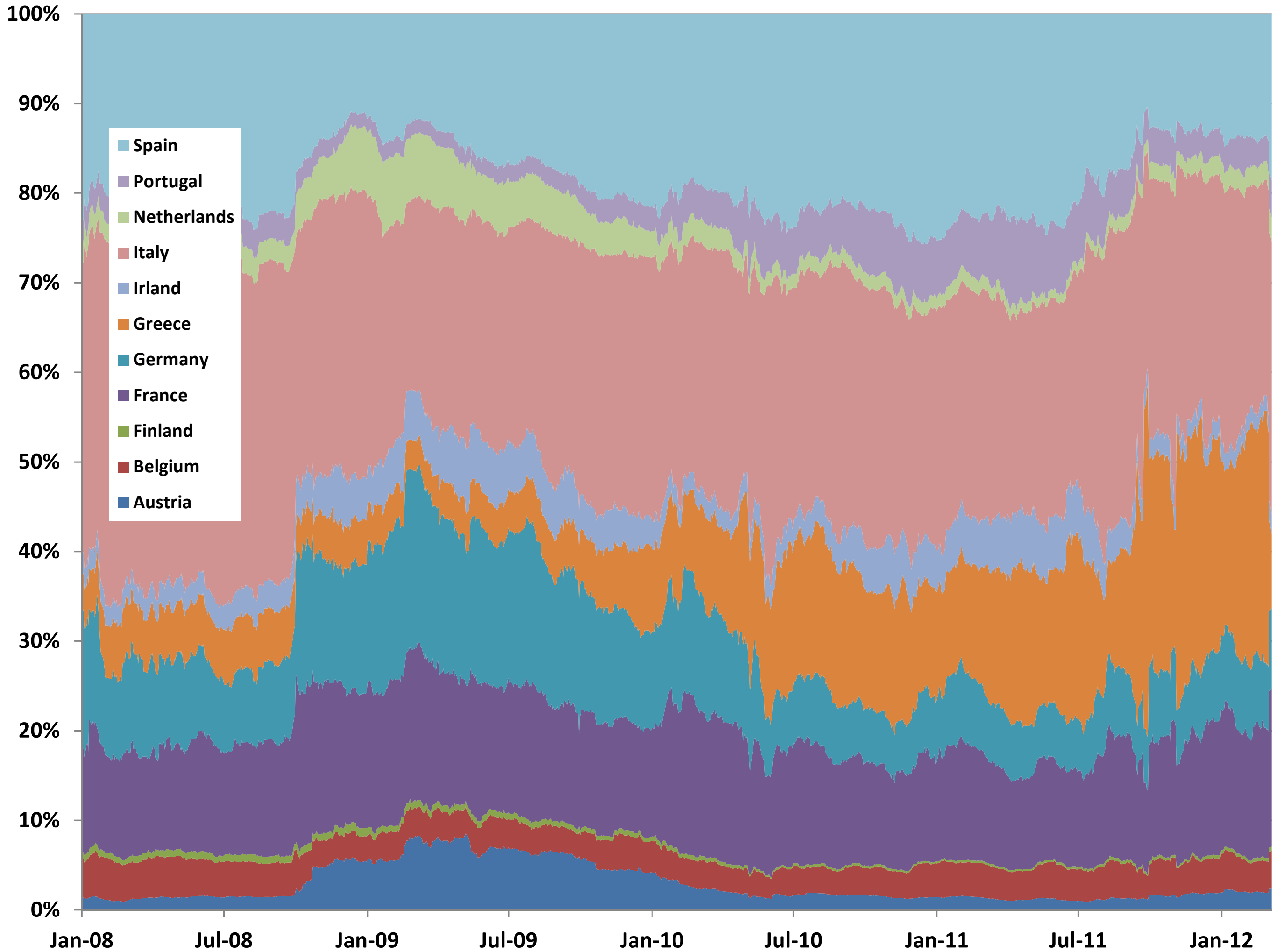
Figure: Weights and risk contributions of the GDP indexation

Country	July-08		July-09		July-10		July-11		March-12	
	GDP	RC	GDP	RC	GDP	RC	GDP	RC	GDP	RC
Austria	3.1%	1.4%	3.1%	7.0%	3.1%	1.7%	3.2%	1.0%	3.4%	2.0%
Belgium	3.8%	4.0%	3.8%	3.2%	3.9%	3.3%	4.0%	3.5%	4.0%	3.4%
Finland	2.0%	0.8%	1.9%	0.7%	2.0%	0.3%	2.1%	0.3%	2.1%	0.4%
France	21.2%	11.2%	21.5%	14.9%	21.4%	13.4%	21.5%	10.6%	21.7%	14.0%
Germany	27.4%	7.6%	27.2%	17.0%	27.7%	6.7%	27.9%	5.8%	27.8%	7.2%
Greece	2.6%	6.2%	2.7%	4.4%	2.6%	15.7%	2.4%	19.8%	2.4%	26.9%
Ireland	2.0%	3.0%	1.9%	5.6%	1.8%	2.6%	1.7%	5.9%	1.7%	1.9%
Italy	17.4%	37.5%	17.3%	23.5%	17.2%	25.8%	17.0%	23.9%	17.1%	24.6%
Netherlands	6.5%	2.5%	6.5%	5.3%	6.5%	1.6%	6.6%	1.2%	6.5%	2.1%
Portugal	1.9%	3.3%	1.9%	2.0%	1.9%	5.3%	1.9%	6.7%	1.9%	3.4%
Spain	12.0%	22.6%	12.0%	16.5%	11.8%	23.7%	11.8%	21.4%	11.6%	14.1%
Sovereign Risk Measure	0.64%		2.47%		6.59%		4.56%		9.41%	

⇒ RC of Debt and GDP indexations are different, but sovereign credit risk measures are similar.

GDP indexation

Evolution of the risk contributions



GDP-RB indexation

Figure: Weights and risk contributions of the GDP-RB indexation

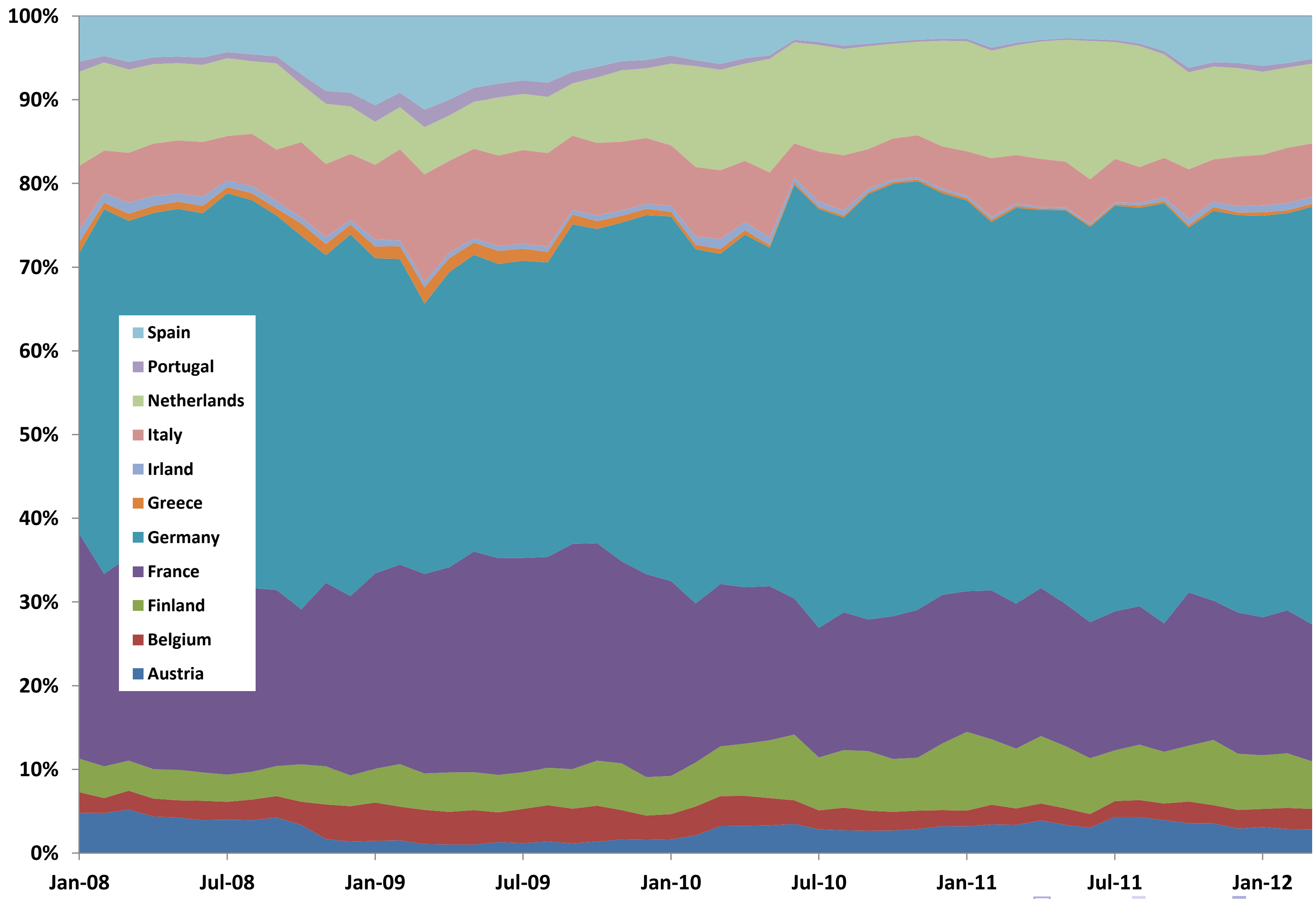
Country	July-08		July-09		July-10		July-11		March-12	
	RC	Weights	RC	Weights	RC	Weights	RC	Weights	RC	Weights
Austria	3.1%	3.9%	3.1%	1.2%	3.1%	2.9%	3.2%	4.2%	3.4%	2.8%
Belgium	3.8%	2.1%	3.8%	4.1%	3.9%	2.2%	4.0%	1.9%	4.0%	2.4%
Finland	2.0%	3.2%	1.9%	4.4%	2.0%	6.3%	2.1%	6.0%	2.1%	5.7%
France	21.2%	22.0%	21.5%	25.6%	21.4%	15.5%	21.5%	16.5%	21.7%	16.4%
Germany	27.4%	47.8%	27.2%	35.5%	27.7%	50.0%	27.9%	48.7%	27.8%	49.9%
Greece	2.6%	0.7%	2.7%	1.4%	2.6%	0.2%	2.4%	0.2%	2.4%	0.3%
Ireland	2.0%	0.8%	1.9%	0.6%	1.8%	0.6%	1.7%	0.2%	1.7%	0.8%
Italy	17.4%	5.3%	17.3%	11.2%	17.2%	6.0%	17.0%	5.2%	17.1%	6.4%
Netherlands	6.5%	9.2%	6.5%	6.7%	6.5%	12.8%	6.6%	14.0%	6.5%	9.5%
Portugal	1.9%	0.7%	1.9%	1.6%	1.9%	0.4%	1.9%	0.2%	1.9%	0.6%
Spain	12.0%	4.2%	12.0%	7.7%	11.8%	3.1%	11.8%	2.9%	11.6%	5.1%
Sovereign Risk Measure	0.39%		2.10%		3.25%		1.91%		5.43%	

$\Rightarrow$ RB indexation is very different from WB indexation, in terms of weights, RC and credit risk measures.

$\Rightarrow$ The dynamics of the GDP-RB is relatively smooth (monthly turnover $\simeq 7\%$, max $= 20\%$, min $= 1.8\%$).

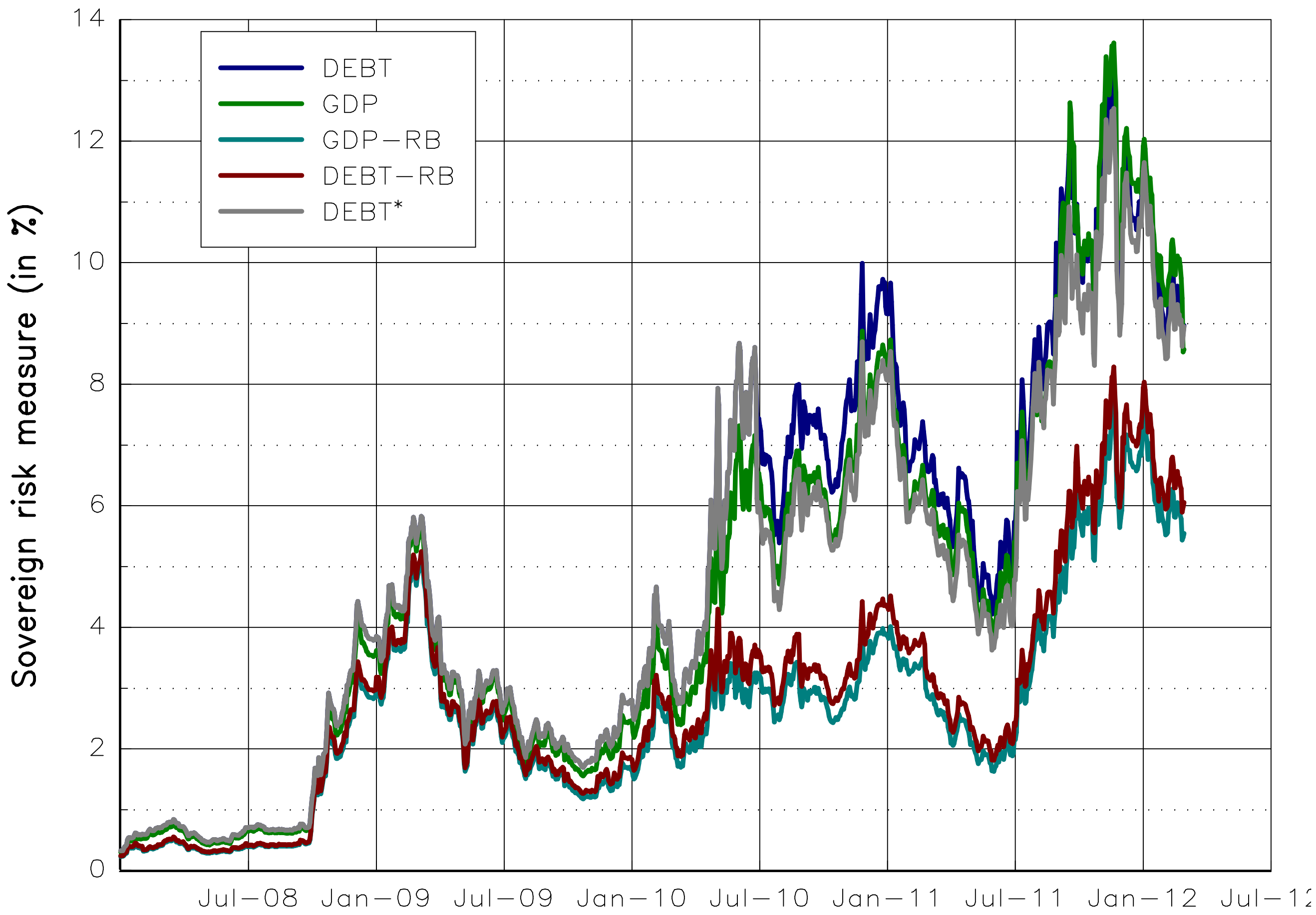
GDP-RB indexation

Evolution of weights



Comparison of the indexing schemes

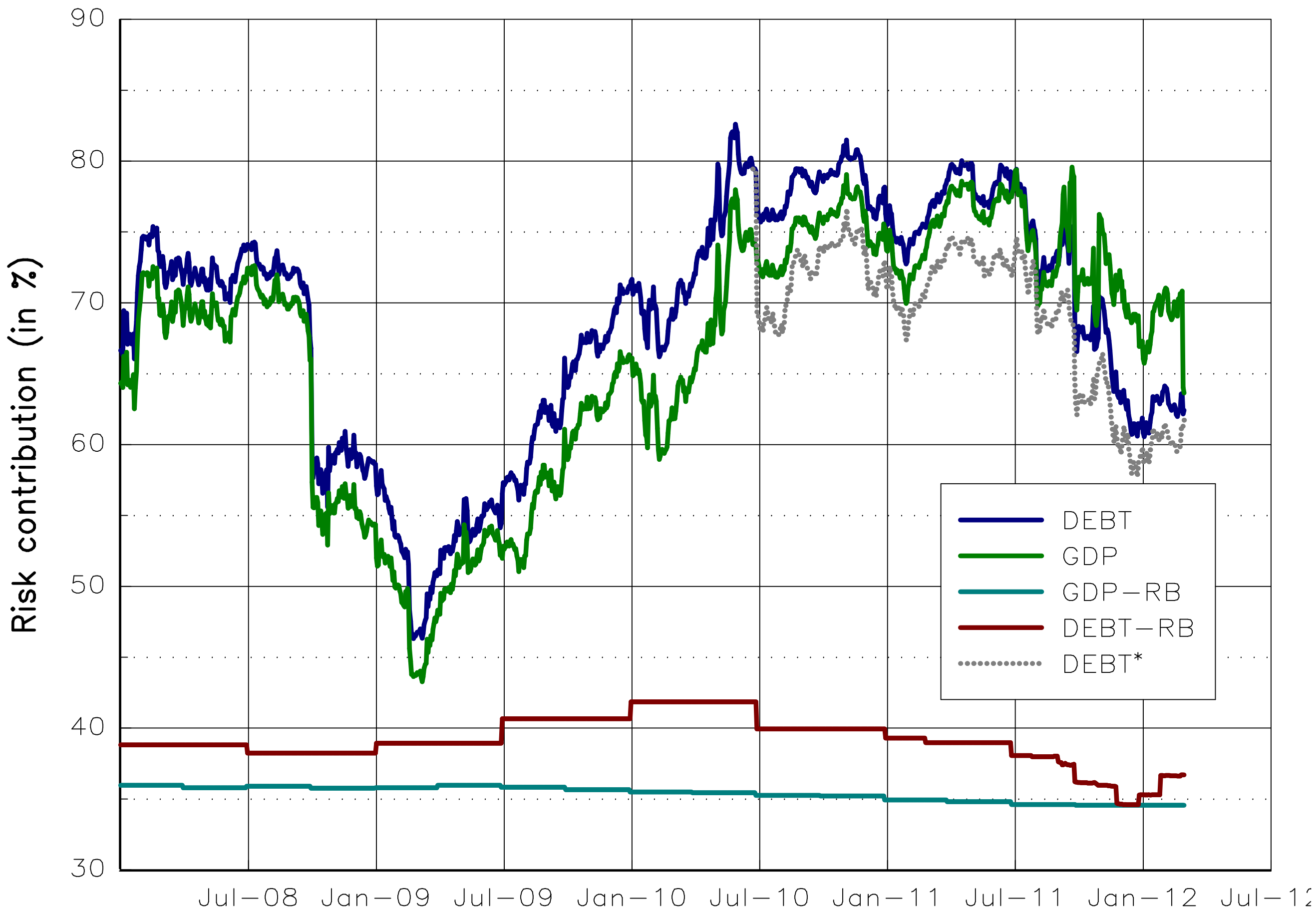
Evolution of the risk measure



⇒ We verify that the risk measure of the RB indexation is smaller than the one of the WB indexation.

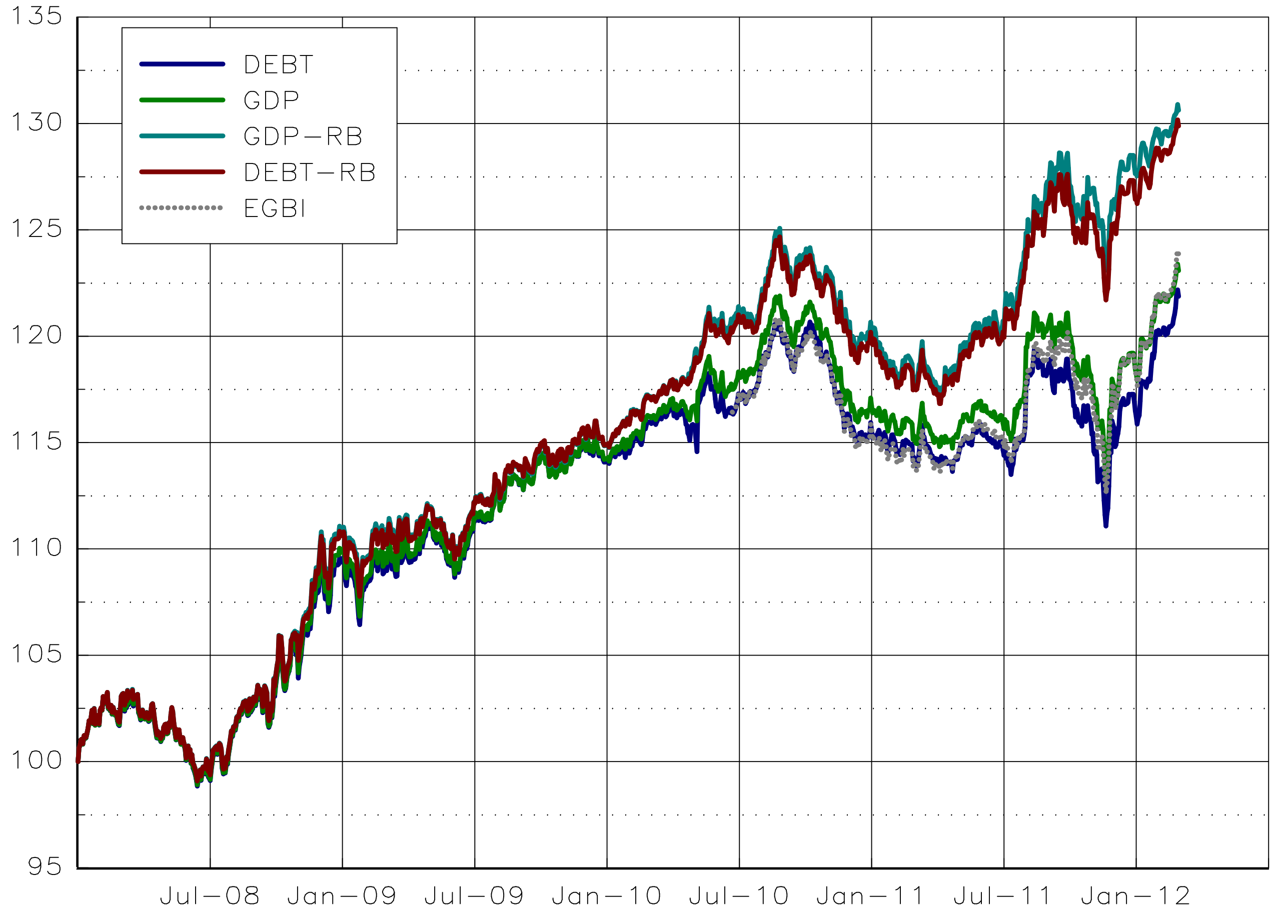
Comparison of the indexing schemes

Evolution of the GIIPS risk contribution



Comparison of the indexing schemes

Performance simulations



⇒ RB indexation / WB indexation = better performance, same volatility and smaller drawdowns.

Comparison with active management

- Database: Morningstar
- Category: Bond EURO Government
- 216 funds[8]

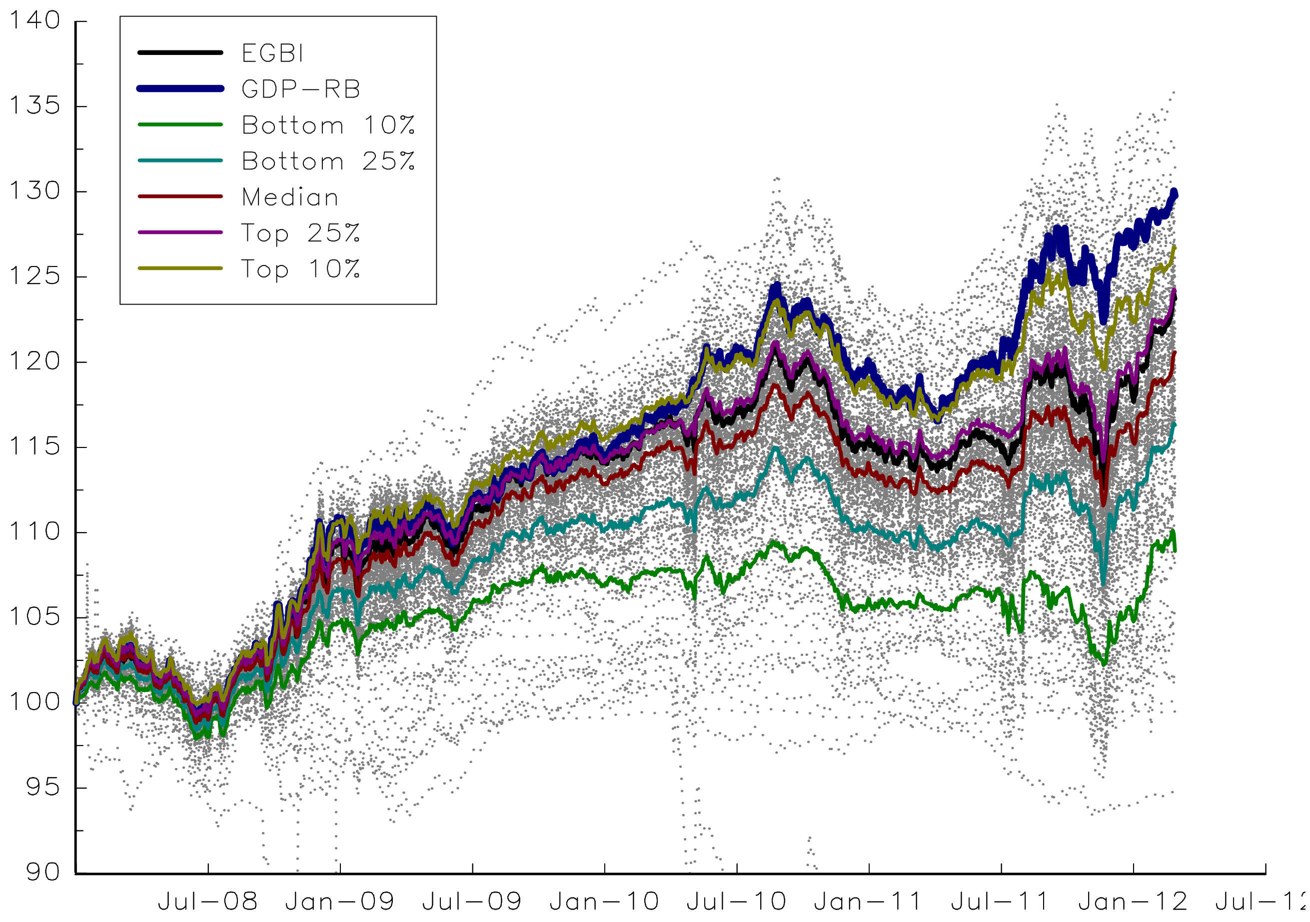
The Academic Rule[9]:

$$\text{Average Performance of Active Management} = \text{Performance of the Index} - \text{Management Fees}$$

$\Rightarrow$ Implied fees for Bond EURO Government: 61 bps / year[10]

[8] We don't take into account the survivorship bias.
[9] There is a large literature on this subject, see e.g. Blake *et al.* (1993).
[10] This figure was only 36 bps / year for the period 01/2008 - 08/2011.

Comparison with active management



#(funds > GDP-RB) = 6

Perf. of GDP-RB Index†
=
Perf. of Top 10%
+ 58 bps / year

† Transaction costs = 15 bps / year

Conclusion

- Risk-budgeting approach = a better approach than portfolio optimization
- Risk-budgeting approach = new theoretical results (Bruder and Roncalli, 2012)

- The risk-budgeting approach could be applied to:
  - Risk-balanced allocation
    - Risk parity funds
    - Strategic asset allocation
  - Risk-based indexation
    - Equity indexes (see for example Lyxor SmartIX ERC Index Series[11])
    - Bond indexes

[11] The web site is www.ftse.com/Indices.

For Further Reading

- B. Bruder, T. Roncalli.
  Managing Risk Exposures using the Risk Budgeting Approach.
  SSRN, www.ssrn.com/abstract=2009778, January 2012.
- B. Bruder, P. Hereil, T. Roncalli.
  Managing Sovereign Credit Risk in Bond Portfolios.
  SSRN, www.ssrn.com/abstract=1957050, October 2011.
- B. Bruder, P. Hereil, T. Roncalli.
  Managing Sovereign Credit Risk.
  Journal of Indexes Europe, 1(4), November 2011.
- S. Maillard, T. Roncalli, J. Teiletche.
  The Properties of Equally Weighted Risk Contribution Portfolios.
  Journal of Portfolio Management, 36(4), Summer 2010.

Weights constraints and Portfolio Theory

Main result

We consider a universe of n assets. We denote by μ the vector of their expected returns and by Σ the corresponding covariance matrix. We specify the optimization problem as follows:

$$\min \frac{1}{2} w^\top \Sigma w$$

$$\text{u.c.} \begin{cases} \mathbf{1}^\top w = 1 \\ \mu^\top w \geq \mu^\star \\ w \in \mathbb{R}^n \bigcap \mathscr{C} \end{cases}$$

where w is the vector of weights in the portfolio and $\mathscr{C}$ is the set of weights constraints. We define:

- the unconstrained portfolio $w^\star$ or $w^\star(\mu, \Sigma)$:

$$\mathscr{C} = \mathbb{R}^n$$

- the constrained portfolio $\tilde{w}$:

$$\mathscr{C}\left(w^-, w^+\right) = \left\{w \in \mathbb{R}^n : w_i^- \leq w_i \leq w_i^+\right\}$$

Weights constraints and Portfolio Theory

Main result

Theorem

Jagannathan and Ma (2003) show that the constrained portfolio is the solution of the unconstrained problem:

$$\tilde{w} = w^{\star}\left(\tilde{\mu}, \tilde{\Sigma}\right)$$

with:

$$\begin{cases} \tilde{\mu} = \mu \\ \tilde{\Sigma} = \Sigma + \left(\lambda^{+} - \lambda^{-}\right)\mathbf{1}^{\top} + \mathbf{1}\left(\lambda^{+} - \lambda^{-}\right)^{\top} \end{cases}$$

where λ^{-} and λ^{+} are the Lagrange coefficients vectors associated to the lower and upper bounds.

$\Rightarrow$ Introducing weights constraints is equivalent to introduce some relative views (similar to the **Black-Litterman** approach).

Proof for the global minimum-variance portfolio

We define the Lagrange function as $f(w;\lambda_0) = \frac{1}{2} w^\top \Sigma w - \lambda_0 \left(\mathbf{1}^\top w - 1\right)$ with $\lambda_0 \geq 0$. The first order conditions are $\Sigma w - \lambda_0 \mathbf{1} = 0$ and $\mathbf{1}^\top w - 1 = 0$. We deduce that the optimal solution is:

$$w^\star = \lambda_0^\star \Sigma^{-1} \mathbf{1} = \frac{1}{\mathbf{1}^\top \Sigma \mathbf{1}} \Sigma^{-1} \mathbf{1}$$

With weights constraints $\mathscr{C}(w^-, w^+)$, we have:

$$f\left(w;\lambda_0,\lambda^-,\lambda^+\right) = \frac{1}{2} w^\top \Sigma w - \lambda_0 \left(\mathbf{1}^\top w - 1\right) - \lambda^{-\top}\left(w - w^-\right) - \lambda^{+\top}\left(w^+ - w\right)$$

with $\lambda_0 \geq 0$, $\lambda_i^- \geq 0$ and $\lambda_i^+ \geq 0$. In this case, the first-order conditions becomes $\Sigma w - \lambda_0 \mathbf{1} - \lambda^- + \lambda^+ = 0$ and $\mathbf{1}^\top w - 1 = 0$. We have:

$$\tilde{\Sigma} \tilde{w} = \left(\Sigma + \left(\lambda^+ - \lambda^-\right)\mathbf{1}^\top + \mathbf{1}\left(\lambda^+ - \lambda^-\right)^\top\right)\tilde{w} = \left(2\tilde{\lambda}_0 - \tilde{w}^\top \Sigma \tilde{w}\right)\mathbf{1}$$

Because $\tilde{\Sigma}\tilde{w}$ is a constant vector, it proves that $\tilde{w}$ is the solution of the unconstrained optimisation problem with $\lambda_0^\star = \left(2\tilde{\lambda}_0 - \tilde{w}^\top \Sigma \tilde{w}\right)$.

Weights constraints and Portfolio Theory

Examples

Table: Specification of the covariance matrix Σ (in %)

σ_i	$\rho_{i,j}$			
15.00	100.00			
20.00	10.00	100.00		
25.00	40.00	70.00	100.00	
30.00	50.00	40.00	80.00	100.00

Given these parameters, the global minimum variance portfolio is equal to:

$$w^{\star} = \begin{pmatrix} 72.742\% \\ 49.464\% \\ -20.454\% \\ -1.753\% \end{pmatrix}$$

Weights constraints and Portfolio Theory

Examples

Table: Global minimum variance portfolio when $w_i \geq 10\%$

$\tilde{w}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
56.195	0.000	0.000	15.000	100.000			
23.805	0.000	0.000	20.000	10.000	100.000		
10.000	1.190	0.000	19.671	10.496	58.709	100.000	
10.000	1.625	0.000	23.980	17.378	16.161	67.518	100.000

Table: Global minimum variance portfolio when $0\% \leq w_i \leq 50\%$

$\tilde{w}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
50.000	0.000	1.050	20.857	100.000			
50.000	0.000	0.175	20.857	35.057	100.000		
0.000	0.175	0.000	24.290	46.881	69.087	100.000	
0.000	0.000	0.000	30.000	52.741	41.154	79.937	100.000

Weights constraints and Portfolio Theory

Examples

Table: MSR tangency portfolio when $0\% \leq w_i \leq 40\%$ and $\text{sh}^\star = 0.5$

$\tilde{w}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
40.000	0.000	0.810	19.672	100.000			
40.000	0.000	0.540	22.539	37.213	100.000		
0.000	0.000	0.000	25.000	46.970	71.698	100.000	
20.000	0.000	0.000	30.000	51.850	43.481	80.000	100.000

We obtain:

$$\widetilde{\text{sh}} = \begin{pmatrix} 0.381 \\ 0.444 \\ 0.5 \\ 0.5 \end{pmatrix}$$