Measuring and Managing Carbon Risk in Investment Portfolios*

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Abstract

This article studies the impact of carbon risk on stock pricing. To address this, we consider the seminal approach of Görgen *et al.* (2019), who proposed estimating the carbon financial risk of equities by their carbon beta. To achieve this, the primary task is to develop a brown-minus-green (or BMG) risk factor, similar to Fama and French (1992). Secondly, we must estimate the carbon beta using a multi-factor model. While Görgen *et al.* (2019) considered that the carbon beta is constant, we propose a time-varying estimation model to assess the dynamics of the carbon risk. Moreover, we test several specifications of the BMG factor to understand which climate change-related dimensions are priced in by the stock market. In the second part of the article, we focus on the carbon risk management of investment portfolios. First, we analyze how carbon risk impacts the construction of a minimum variance portfolio. As the goal of this portfolio is to reduce unrewarded financial risks of an investment, incorporating the carbon risk into this approach fulfils this objective. Second, we propose a new framework for building enhanced index portfolios with a lower exposure to carbon risk than capitalization-weighted stock indices. Finally, we explore how carbon sensitivities can improve the robustness of factor investing portfolios.

Keywords: Carbon, climate change, risk factor, Kalman filter, minimum variance portfolio, enhanced index portfolio, factor investing.

JEL classification: C61, G11.

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1 Introduction

The general approach to managing the carbon risk of an investment portfolio is to reduce or control the portfolio's carbon footprint, for instance by considering CO_2 emissions. This approach supposes that the carbon risk will materialize and having a portfolio with a lower exposure to CO_2 emissions will help to avoid some future losses. The main assumption of this approach is then to postulate that firms currently with high carbon footprints will be penalized in the future in comparison with firms currently with low carbon footprints.

In this article, we use an alternative approach. We define carbon risk from a financial point of view, and we consider that the carbon risk of equities corresponds to the market risk priced in by the stock market. This carbon financial risk can be decomposed into a common (or systematic) risk factor and a specific (or idiosyncratic) risk factor. Since identifying the specific risk is impossible, we focus on the common risk factor that drives the carbon risk. The objective is then to build a market-based carbon risk measure to manage this market risk in investment portfolios. This is exactly the framework proposed by Görgen *et al.* (2019) in their seminal paper.

Görgen *et al.* (2019) proposed extending the Fama-French-Carhart model by including a brown-minus-green (or BMG) risk factor. Using the sorted portfolios technique popularized by Fama and French (1992), they build a factor-mimicking portfolio based on a scoring model and more than fifty carbon risk variables. They then defined the carbon financial risk of a stock using its price sensitivity to the BMG factor or its carbon beta. In this paper, we explore the original approach of Görgen *et al.* (2019) and estimate a time-varying model in order to analyze the dynamics of the carbon risk. Moreover, we make the distinction between relative and absolute carbon risk. Relative carbon risk may be viewed as an extension or forward-looking measure of the carbon footprint, where the objective is to be more exposed to green firms than to brown firms. In this case, this is equivalent to promoting stocks with a negative carbon beta over stocks with a positive carbon beta. Absolute carbon risk considers that both large positive and negative carbon beta values incur a financial risk that must be reduced. This is an agnostic or neutral method, contrary to the first method which is more related to investors' moral values. In this paper, an important issue concerns the climate change-related dimensions that are priced in by the financial market. According to Delmas *et al.* (2013), the concept of environmental performance encompasses several dimensions but there is no consensus on universally accepted environmental performance indicators. To address this issue, we compare pricing models with different criteria: current carbon footprint, carbon management, climate change score and environmental risk.

The carbon beta of a stock can be interpreted as its carbon-related systematic risk. Therefore, it contains financial information that is extremely useful from a trading point of view. In particular, it can be used to improve the construction of a minimum variance portfolio, the main goal of which is to avoid unrewarded risks. It can also be used in the investment scope of enhanced indexing or factor investing. For these different illustrations, we develop an analytical framework to better understand the impact of carbon betas.

This paper is organized as follows. Section Two presents the seminal approach of Görgen *et al.* (2019), and reviews the pricing impact of the carbon risk factor. Besides the static analysis, we also consider a dynamic approach where the carbon beta is estimated using the Kalman filter. Then we test the contribution of the different climate-change related dimensions. Section Three is dedicated to investment portfolio management considering the information deduced from the carbon beta values. First, we focus on the minimum variance portfolio, before extending the analytical results to enhanced index portfolios and explaining how carbon betas can also be used in a factor investing framework. Finally, Section Four offers some concluding remarks.

2 Measuring carbon risk

To manage a portfolio's carbon risk, it is important to measure carbon risk at the company level. There are different ways to measure this risk, including the fundamental and market approaches. In this paper, we will favor the second approach because it provides a better assessment of the impact of climate-related transition risks on each company's stock price. Moreover, the market-based approach allows us to mitigate the issue of a lack of climate change-relevant information. In what follows, we present this latest approach by using the mimicking portfolio for carbon risk developed by Görgen *et al.* (2019). We also discuss the different climate change-relevant dimensions to determine which dimensions are priced in by the market.

2.1 The Carima approach

The goal of the carbon risk management (Carima) project, developed by Görgen *et al.* (2019), is to develop '*a quantitative tool in order to assess the opportunities of profits and the risks of losses that occur from the transition process*'. The Carima approach combines a market-based approach and a fundamental approach. Indeed, the carbon risk of a firm or a portfolio is measured by considering the dynamics of stock prices which are partly determined by climate policies and transition processes towards a green economy. Nevertheless, a prior fundamental approach is important to quantify carbon risk. In a practical manner, the fundamental approach consists in defining a carbon risk score for each stock of a universe using a set of objective measures, whereas the market approach consists in building a brown minus green or BMG carbon risk factor, and computing the risk sensitivity of stock prices with respect to this BMG factor. Therefore, the carbon factor is derived from climate change-relevant information from numerous firms.

2.1.1 Construction of the BMG factor

The development of the BMG factor is based on a large amount of climate-relevant information provided by different databases. In the following, we report the methodology used by the Carima project to construct the BMG factor and thereby obtain a deeper understanding of the results[1]. Two steps are required to develop this new common risk factor: (1) the development of a scoring system to determine if a firm is green, neutral or brown and (2) the construction of a mimicking factor portfolio for carbon risk which has a long exposure to brown firms and a short exposure to green firms.

The first step consists in defining a brown green score (BGS) with a fundamental approach to assess the carbon risk of different firms. This scoring system uses four ESG databases over the period from 2010 to 2016: Thomson Reuters ESG, MSCI ESG Ratings, Sustainalytics ESG ratings and the Carbon Disclosure Project (CDP) climate change questionnaire. Overall, 55 carbon risk proxy variables are retained[2]. Then, Görgen *et al.* (2019) classified the variables into three different dimensions that may affect the stock value of a firm in the event of unexpected shifts towards a low carbon economy:

1. Value chain (impact of a climate policy or a cap and trade system on the different activities of a firm: inbound logistics and supplier chain, manufacturing production, sales, etc.);

[1] A more exhaustive presentation is available in the Carima manual, which can be downloaded at the following address: `https://carima-project.de/downloads`.

[2] The governance and social variables of a traditional ESG analysis or even certain environmental variables such as waste recycling, water consumption or toxic emissions have been deleted.

2. Public perception (external environmental image of a firm: ratings, controversies, disclosure of environmental information, etc.);

3. Adaptability (capacity of the firm to shift towards a low carbon strategy without substantial efforts and losses).

The value chain dimension mainly deals with current emissions while the adaptability dimension reflects potential future emissions determined in particular by emission reduction targets and environmental R&D spending. The Carima project considers that the higher the variable, the browner the firm. Hence each variable (except the dummies) is transformed into a dummy derived with respect to the median, meaning that 1 corresponds to a brown value and 0 corresponds to a green value. Then, three scores are created and correspond to the average of all variables contained in each dimension: the *value chain* VC, the *public perception* PP and the *non-adaptability* NA. It follows that each score has a range between 0 and 1. Görgen *et al.* (2019) proposed defining the brown green score (BGS) by the following equation:

$$\mathrm{BGS}_i(t) = \frac{2}{3}\left(0.7 \cdot \mathrm{VC}_i(t) + 0.3 \cdot \mathrm{PP}_i(t)\right) + \frac{\mathrm{NA}_i(t)}{3}\left(0.7 \cdot \mathrm{VC}_i(t) + 0.3 \cdot \mathrm{PP}_i(t)\right) \tag{1}$$

The higher the BGS value, the browner the firm. The value chain and public perception axes directly influence stock prices in the case of unexpected changes in the transition process. However, Görgen *et al.* (2019) considered that the impact of the value chain score is more important than the impact of the public perception score. The adaptability axis influences the equity value in a different way. Indeed, it mitigates the upward or downward impacts of the two other axes. The less adaptable a firm is, the greater the impact of an unexpected acceleration in the transition process. In total, almost 1 650 firms are retained thanks to sufficient data covered.

The second step consists in constructing a BMG carbon risk factor. Here the Carima project considers an average BGS for each stock that corresponds to the mean value of the BGS over the period in question, from 2010 to 2016. The construction of the BMG factor follows the methodology of Fama and French (1992, 1993), which consists in splitting the stocks into six portfolios:

	Green	Neutral	Brown
Small	SG	SN	SB
Big	BG	BN	BB

where the classification is based on the terciles of the aggregating BGS and the median market capitalization. Then, the return of the BMG factor is defined as follows:

$$R_{\mathrm{bmg}}(t) = \frac{1}{2}\left(R_{\mathrm{SB}}(t) + R_{\mathrm{BB}}(t)\right) - \frac{1}{2}\left(R_{\mathrm{SG}}(t) + R_{\mathrm{BG}}(t)\right) \tag{2}$$

where the returns of each portfolio is value-weighted by market capitalization. The BMG factor can then be integrated as a new common risk factor into a multi-factor model. Some statistical details are reported in Table 9 on page 51, whereas the historical cumulative performance of the BMG factor is showed in Figure 19 on page 52. According to the factor developed for the Carima project, brown firms slightly outperform green firms from 2010 to the end of 2012. During the next three years, the cumulative return fell by almost 35% because of the unexpected path in the transition process towards a low carbon economy. From 2016 to the end of the study period, brown firms created a slight excess performance. Overall, the best-in-class green stocks outperform the worst-in-class green stocks over the study period with an annual return of 2.52%.

2.1.2 Advantages and limits

Many advantages can be attributed to the BMG factor. Some biases in the construction of ESG databases are offset since the BGS scores are derived from several databases. Moreover, the tests performed by Görgen *et al.* (2019) showed that there are no significant country-specific or sector-specific effects[3]. Even though the BMG factor has many benefits, it can be subject to some disadvantages, starting with the treatment of variables. The transformation of continuous and discrete variables into a dummy variable with respect to the median value fixes the problem of extreme values, but does not differentiate between values based on their distance from the median. Besides, the most important problem is that no rebalancing takes place. Some tests performed by Görgen *et al.* (2019) showed that less than five percent of firms shifted between the green, neutral and brown portfolios during the study period but such a decision presents some consistency problems in the long-run. For instance, the results obtained by the average BGS score for the 2010-2016 period have been generalized for the following two years.

Another limit involves the size-specific effects in the BMG factor. Table 1 reports the correlation matrix of common risk factors during the sample period. While the value (HML) and momentum (WML) factors are not significantly correlated to the size (SMB) factor, the BMG factor is influenced by size characteristics. Mitigating this problem can be difficult since the carbon risk factor has been derived from the methodology of Fama and French (1992). The most plausible explanation of this correlation is that among the studied firms, the green firms have the largest market capitalizations as we can see in Figure 28 on page 56. In this case, when big firms outperform small firms, both the SMB and BMG factor returns decrease. Furthermore, preventing the BMG factor from capturing size-specific effect is an important but difficult matter to solve.

Table 1: Correlation matrix of factor returns (in %)

Factor	MKT	SMB	HML	WML	BMG
MKT	100.00^{***}				
SMB	1.41	100.00^{***}			
HML	11.51	-8.93	100.00^{***}		
WML	-14.59	3.87	-41.43^{***}	100.00^{***}	
BMG	5.33	20.33^{**}	27.41^{***}	-21.28^{**}	100.00^{***}

Source: Görgen *et al.* (2019).

Selecting numerous variables allows us to avoid some important dependencies on a variable and incorporate a lot of climate change-relevant information. Nevertheless, we have double counting problems. For instance, the carbon emissions score and the climate change theme score in the MSCI ESG Ratings are both taken into account when developing the BGS score but the carbon emissions score is integrated into the climate change theme score. Moreover, some variables in the public perception dimension are not exclusive to the carbon risk dimension, such as the ESG score developed by Sustainalytics ESG ratings or the Industry-adjusted Overall score developed by MSCI ESG Ratings.

2.1.3 Static analysis

The first carbon risk objective is to assess the relevance of the BMG factor during the study period. To do this, we follow the analysis of Görgen *et al.* (2019), but our analysis is

[3] In the following, we will find some sector-specific effects for short periods.

slightly different because of the investment universe. Indeed, Görgen *et al.* (2019) considered a universe of 39 500 stocks, whereas we only consider the stocks that were present in the MSCI World index during the 2010-2018 period. As a result, our investment universe has less than 2 000 stocks, but we think that a restricted universe makes more sense than a very large universe. Indeed, the computation of a market beta is already difficult for some small and micro stocks because of OTC pricing and low trading activity. Therefore, calculating a carbon beta is even more difficult for such equities.

It may be worthwhile to compare different common factor models to measure the information gain related to the carbon risk factor. The first studied model is the CAPM model introduced by Sharpe (1964) which is defined by:

$$R_i(t) = \alpha_i + \beta_{\mathrm{mkt},i} R_{\mathrm{mkt}}(t) + \varepsilon_i(t) \tag{3}$$

where $R_i(t)$ is the return of asset i, α_i is the alpha of the asset i, $R_{\mathrm{mkt}}(t)$ is the return of the market factor, $\beta_{\mathrm{mkt},i}$ is the systematic risk (or the market beta) of stock i and $\varepsilon_i(t)$ is the idiosyncratic risk. We may also consider that the risk is multi-dimensional with the model developed by Fama and French (1992):

$$R_i(t) = \alpha_i + \beta_{\mathrm{mkt},i} R_{\mathrm{mkt}}(t) + \beta_{\mathrm{smb},i} R_{\mathrm{smb}}(t) + \beta_{\mathrm{hml},i} R_{\mathrm{hml}}(t) + \varepsilon_i(t) \tag{4}$$

where $R_{\mathrm{smb}}(t)$ is the return of the size (or small minus big) factor, $\beta_{\mathrm{smb},i}$ is the SMB sensitivity (or the size beta) of stock i, $R_{\mathrm{hml}}(t)$ is the return of the value (or high minus low) factor and $\beta_{\mathrm{hml},i}$ is the HML sensitivity (or the value beta) of stock i. Nevertheless, these two models do not include the carbon risk. Furthermore, we also consider the MKT+BMG model:

$$R_i(t) = \alpha_i + \beta_{\mathrm{mkt},i} R_{\mathrm{mkt}}(t) + \beta_{\mathrm{bmg},i} R_{\mathrm{bmg}}(t) + \varepsilon_i(t) \tag{5}$$

and the extended Fama-French (FF+BMG) model:

$$R_i(t) = \alpha_i + \beta_{\mathrm{mkt},i} R_{\mathrm{mkt}}(t) + \beta_{\mathrm{smb},i} R_{\mathrm{smb}}(t) + \beta_{\mathrm{hml},i} R_{\mathrm{hml}}(t) + \beta_{\mathrm{bmg},i} R_{\mathrm{bmg}}(t) + \varepsilon_i(t) \tag{6}$$

where $R_{\mathrm{bmg}}(t)$ is the return of the carbon risk factor and $\beta_{\mathrm{bmg},i}$ is the BMG sensitivity of stock i. Another well-known model is the four-factor model (4F) developed by Carhart (1997). This model corresponds to the following equation:

$$R_i(t) = \alpha_i + \beta_{\mathrm{mkt},i} R_{\mathrm{mkt}}(t) + \beta_{\mathrm{smb},i} R_{\mathrm{smb}}(t) + \beta_{\mathrm{hml},i} R_{\mathrm{hml}}(t) + \beta_{\mathrm{wml},i} R_{\mathrm{wml}}(t) + \varepsilon_i(t) \tag{7}$$

where $R_{\mathrm{wml}}(t)$ is the return of the momentum (or winners minus losers) factor and $\beta_{\mathrm{wml},i}$ is the WML sensitivity of stock i. Again, we may include the carbon risk factor to obtain a five-factor (4F+BMG) model:

$$\begin{aligned} R_i(t) = {} & \alpha_i + \beta_{\mathrm{mkt},i} R_{\mathrm{mkt}}(t) + \beta_{\mathrm{smb},i} R_{\mathrm{smb}}(t) + \beta_{\mathrm{hml},i} R_{\mathrm{hml}}(t) + \\ & \beta_{\mathrm{wml},i} R_{\mathrm{wml}}(t) + \beta_{\mathrm{bmg},i} R_{\mathrm{bmg}}(t) + \varepsilon_i(t) \end{aligned} \tag{8}$$

In minimum variance or enhanced index portfolios, we assume that the factor returns are uncorrelated.

Risk factor model estimates were performed on single stocks during the 2010-2018 period[4]. In Table 2, we have reported a comparison between the common factor models and their nested models by computing the average difference of the adjusted $\mathfrak{R}^2$ and the proportion of stocks for which the Fisher test is significant at 10%, 5% and 1%. According to

[4] In this article, we only consider the stocks that were in the MSCI World index for at least three years during the 2010-2018 period. Moreover, we do not consider the returns for the period during which the stock is outside the index.

Table 2: Comparison of cross-section regressions (in %)

	Adjusted $\mathfrak{R}^2$ difference	F-test 10%	5%	1%
CAPM vs FF	1.74	34.6	25.5	13.5
CAPM vs MKT+BMG	1.74	21.2	15.6	9.2
FF vs FF+BMG	1.73	22.5	17.5	9.7
FF vs FF+WML	0.22	6.6	3.0	0.8
4F vs 4F+BMG	1.76	23.6	18.6	10.0

the first two tests, we remark that the Fama-French and MKT+BMG models significantly increase the explanatory power compared to the CAPM model. This increase is significant for almost 26% and 16% of the stocks respectively for Models (4) and (5) at the threshold of 5%. Nevertheless, the difference between the two models for the Fisher test declines when the threshold significance is 1% and the effect on the explanatory power is at the same level for the SMB and HML factors together and the BMG factor alone. The following two tests study the relevance of the carbon factor against the momentum factor when they are added to the Fama and French model. We remark that the sensitivity of the stock returns to the carbon factor is higher than the sensitivity of the stocks returns to the momentum factor. The last test confirms the relevance of the BMG factor when we add the BMG factor to the Carhart model. Overall, we confirm the original results[5] obtained by Görgen *et al.* (2019), meaning that the carbon factor plays a key role in the variation of stock returns.

Figure 1 reports the sector[6] analysis of the carbon beta $\hat{\beta}_{\text{bmg},i}$ estimated with Model (5). The box plots provide the median, the quartiles and the 5% and 95% quantiles of the carbon beta. The energy, materials, real estate and, to a lesser extent, industrial sectors are negatively impacted by an unexpected acceleration in the transition process towards a green economy, certainly because these four sectors are responsible for a large part of greenhouse gas emissions (GHG). Indeed, the energy and the materials sectors have a large scope 1 mainly because of oil and gas drilling and refining for the former and the extraction and processing of raw materials for the latter[7]. Overall, the energy sector is the most sensitive to an unexpected acceleration in the transition process but the carbon beta range is widest for the materials sector, which indicates a high heterogenous risk for this sector. The latter is mostly influenced by the growth in material demand per capita. In terms of the industrial sector, construction and transport are responsible for much of global final energy consumption, which leads to a high carbon risk for this sector. In the real estate sector, the firms have a large scope 2 and energy efficiency can be improved in many cases. If we consider a long-run investment, a transition process that reduces climate change can protect households from physical risks like climate hazards. Nevertheless, a short-run vision supposes that a climate policy negatively impacts households that over-consume. Therefore, real estate investment trusts are highly sensitive to climate-related policies. One surprising result involves utility firms which do not have a substantial positive carbon beta whereas their scope 1 is on average the largest of any sector. This overall neutral carbon sensitivity for utilities is explained by their carbon emissions management and efforts to reduce carbon exposure. Indeed, Le Guenedal *et al.* (2020) have shown that utilities – power generation according to the Sectoral Decarbonization Approach (SDA) – have been aggressive in their inflexion of carbon intensity trajectories.

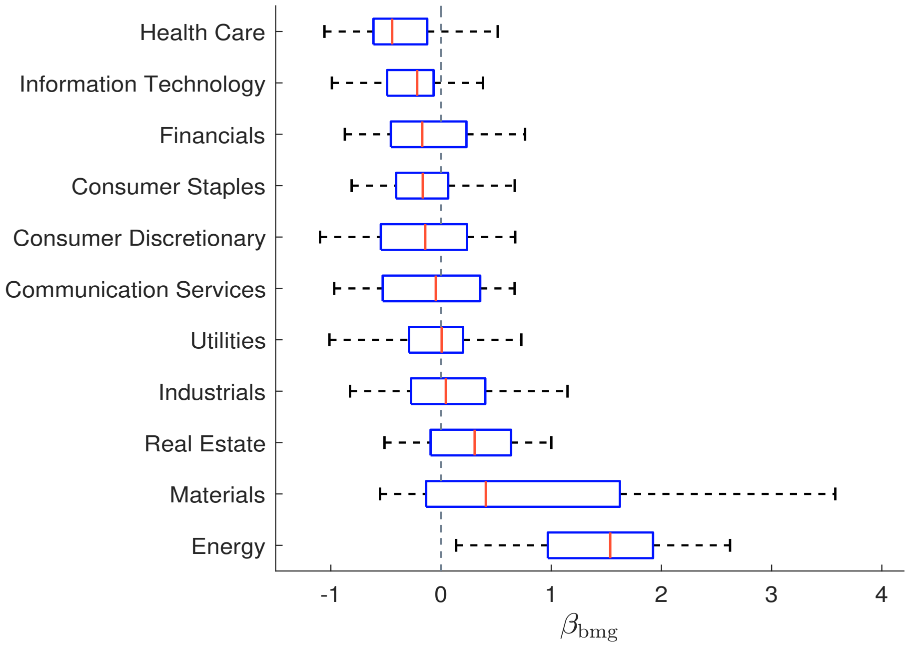
If we consider the sectors positively impacted by an unexpected shift towards a green economy, these primarily include health care, information technology and consumer sta-

[5] See Table IA.2 in Görgen *et al.* (2019).

[6] Sector taxonomy is based on the Global Industry Classification Standard (GICS).

[7] Moreover, scope 3 of the basic materials sector is very large.

Figure 1: Box plots of the carbon sensitivities



ples because of their low GHG emissions. Financials are also part of this group, but the interpretation of the carbon risk differs. Indeed, the carbon risk of financial institutions is less connected to their GHG emissions than their investments and financing programs. The greener a financial institution's investment, the lower its carbon beta. The low value of the median beta implies that financial firms integrate carbon risk into their investment strategies or that financials are not significantly disadvantaged by the relative carbon risk.

Remark 1. *These results are coherent, but slightly different from those obtained by Görgen et al. (2019). Certainly, this difference comes from the investment universe, which is more liquid in our case. For instance, we obtain less high median carbon betas – except for the energy and materials sectors – since our universe of stocks includes only the world's biggest firms.*

We have also reported in Figure 20 on page 52 the box plots for four other investment universes: Eurozone, Europe ex EMU, North America and Japan. The energy sector remains the most negatively impacted by an unexpected acceleration in the transition process regardless of the region under review. The integration of carbon risk in the Eurozone is substantial, especially in the financial sector where green investments are widely taken into account. The inclusion of this carbon risk is also significant in Europe ex EMU and Japan, whereas it is very mixed in North America. Concerning the real estate sector, BMG risk is highly integrated in Europe ex EMU[8] and slightly in North America because of their large exposure to climate risks[9], while the integration of carbon risk is different in Japan because real estate investment trusts are more short-sighted despite the vulnerability of this sector[10].

[8] Almost all the companies with a negative carbon beta are English, while the one remaining with a positive carbon beta is Swiss.

[9] Real estate in United States is especially exposed to rising sea levels and hurricanes.

[10] Real estate in Japan is exposed to the physical risk of typhoons.

2.1.4 Dynamic analysis

In this section, we suppose that the risks are time-varying. For instance, carbon risk may evolve with the introduction of a climate-related policy, a firm's environmental controversy, a change in the firm's environmental strategy, a greater integration of carbon risk into portfolio strategies, etc. Therefore, we use the following dynamic common factor model[11]:

$$R_i(t) = R(t)^\top \beta_i(t) + \varepsilon_i(t) \tag{9}$$

where $R(t) = \left(1, R_{\mathrm{mkt}}(t), R_{\mathrm{bmg}}(t)\right)$ is the vector of factor returns, $\beta_i(t)$ is the vector of factor betas[12]:

$$\beta_i(t) = \begin{pmatrix} \alpha_i(t) \\ \beta_{\mathrm{mkt},i}(t) \\ \beta_{\mathrm{bmg},i}(t) \end{pmatrix} \tag{10}$$

and $\varepsilon_i(t)$ is a white noise. We assume that the state vector $\beta_i(t)$ follows a random walk process:

$$\beta_i(t) = \beta_i(t-1) + \eta_i(t) \tag{11}$$

where $\eta_i(t) \sim \mathcal{N}\left(\mathbf{0}_3, \Sigma_{\beta,i}\right)$ is the white noise vector and $\Sigma_{\beta,i}$ is the covariance matrix of the white noise. Several specifications of $\Sigma_{\beta,i}$ may be used[13], but we assume that $\Sigma_{\beta,i}$ is a diagonal matrix in the following. As previously, the time-varying risk factor model is used on single stocks during the 2010-2018 period[14]. Below, we provide the average of two forecast error criteria between the OLS model and the SSM model:

Model	OLS	SSM
MAE	4.95%	4.63%
RMSE	6.45%	6.01%

We notice that the time-varying risk factor model reduces the forecast error. On average, the monthly return error is equal to 4.95% in the OLS model while it is equal to 4.63% in the SSM model. Overall, the SSM model reduces the mean absolute error value of the last observation date by 12.17% with respect to the OLS model.

In Table 3, we have reported the proportion of firms for which the t-student test of the estimation of the covariance matrix $\Sigma_{\beta,i}$ is significant at 10%, 5% and 1% confidence levels. We notice that the coefficients of the covariance matrix are significant for a substantial number of firms implying that between 10% and 15% of stocks present time-varying market and carbon risks.

In Figure 2, we have reported the variation of the average carbon beta by region[15]. Whatever the study period, the carbon beta $\beta_{\mathrm{bmg},\mathcal{R}}(t)$ is positive in North America, which implies that American stocks are negatively influenced by an acceleration in the transition

[11] The beta estimates are based on the state space model (SSM) and the Kalman filter algorithm described in Appendix A.1 on page 41.

[12] In this model, we only consider the dynamics of market and carbon risks. We have also performed the same analysis with the 4F+BMG model, but the results are noisier.

[13] For instance, we can assume that $\left(\Sigma_{\beta,i}\right)_{1,1}$ is equal to zero, implying that the alpha coefficient $\alpha_i(t)$ is constant.

[14] As previously, we only consider the stocks that were in the MSCI World index for at least three years during the 2010-2018 period and we take into account only the returns for the period during which the stock is in the MSCI World index.

[15] The average carbon beta $\beta_{\mathrm{bmg},\mathcal{R}}(t)$ for the region $\mathcal{R}$ at time t is calculated as follows:

$$\beta_{\mathrm{bmg},\mathcal{R}}(t) = \frac{\sum_{i\in\mathcal{R}} \beta_{\mathrm{bmg},i}(t)}{\operatorname{card}\mathcal{R}}$$

Table 3: Significance test frequency for the white noise covariance matrix (in %)

Factor	10%	5%	1%
α	7.97	4.10	0.84
β_{mkt}	15.95	10.22	3.93
β_{bmg}	10.00	5.90	1.85

process towards a green economy. The average carbon beta is always negative in the Eurozone[16]. Overall, the Eurozone has always a lower average carbon beta than the world as a whole, whereas the opposite is true for North America. Nevertheless, the sensitivity of European equity returns to carbon risk dramatically increases and the BMG betas are getting closer for North America and the Eurozone. In Europe ex EMU, the BMG beta is higher than in the Eurozone but their trends are very similar. Regarding the Japanese firms, the trend has tracked the world as a whole since 2013 but with a lower carbon beta. We notice that the carbon risk is not driven by climate agreements in the short run. For instance, the 2030 climate and energy framework, which includes EU-wide targets and policy objectives for the period from 2021 to 2030 does not influence the average European carbon beta in 2014 certainly because of the lack of binding commitments. Another example is the 2015 Paris Climate Agreement, which does not include fiscal pressure mechanisms. Because of the differences between expectations and constraints, the Paris Climate Agreement has not been followed by a significant increase in the carbon beta and has been concomitant with the outperformance of brown stocks one quarter later[17]. In February 2016, the global increase of the carbon beta is related to a sector-specific effect. Indeed, the materials sector has largely outperformed because of a substantial increase in gold, silver and zinc prices whereas the market index has decreased. Furthermore, some firms in the materials sector were not driven by the market for a short period, implying a sharp increase in the carbon beta. This explains that the carbon beta returns to its long-term trend some months later.

Remark 2. *In terms of the results obtained, one issue concerns the increase in the average carbon beta in Europe, and we wonder if this is due to a geographical area effect or a carbon sensitivity specific effect – the firms with the most negative carbon betas may be less influenced by the BMG factor over time. To answer this question, we use the method of sorting portfolios, which has been popularized by Fama and French (1992). Every month, we rank the stocks with respect to their carbon beta, and form five quintile portfolios. Portfolio Q_1 corresponds to the 20% lowest carbon beta stocks while Portfolio Q_5 corresponds to the 20% highest carbon beta stocks. The stocks are equally weighted in each portfolio and the portfolios are rebalanced every month. In Figure 21 on page 53, we have reported the average carbon beta of the five sorted portfolios for each region. Since European stocks are the most positively impacted by an unexpected change in the transition process towards a green economy, we have to compare similar portfolios between the regions. For instance, Portfolio Q_3 in the Eurozone and Portfolio Q_1 in North America started with an average carbon beta around minus one. In the Eurozone, the increase in the carbon beta for Portfolio Q_3 is much higher than the increase of Portfolio Q_1 in North America. The Europe ex EMU region is another example. Portfolio Q_4 in this region is comparable with Portfolio Q_3 in North America since their average carbon betas started at a similar level. Nevertheless, the average carbon beta of Portfolio Q_4 in Europe ex EMU increases while it decreases for Portfolio Q_3 in North America. We can deduce that the increase of the carbon beta in Europe is not due to a sensitivity effect for the stocks that are the most negatively sensitive to the BMG factor, but to a geographical effect.*

[16] In Japan, it is also negative most of the time.

[17] We recall that the performance of the brown minus green portfolio is given in Figure 19 on page 52.

Figure 2: Dynamics of the average relative carbon risk $\beta_{\mathrm{bmg},\mathcal{R}}(t)$ by region

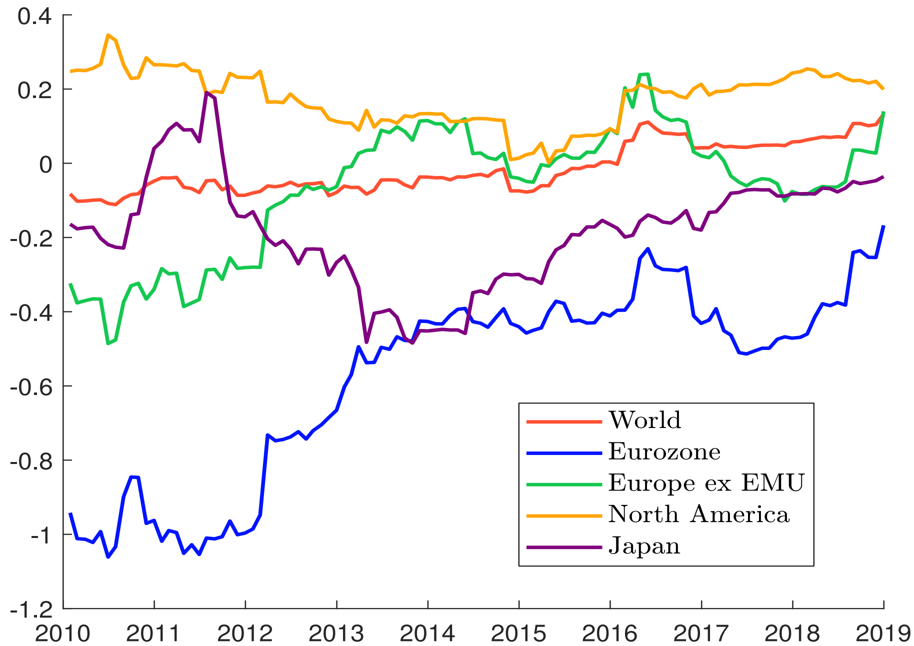


Figure 3: Dynamics of the average absolute carbon risk $|\beta|_{\mathrm{bmg},\mathcal{R}}(t)$ by region

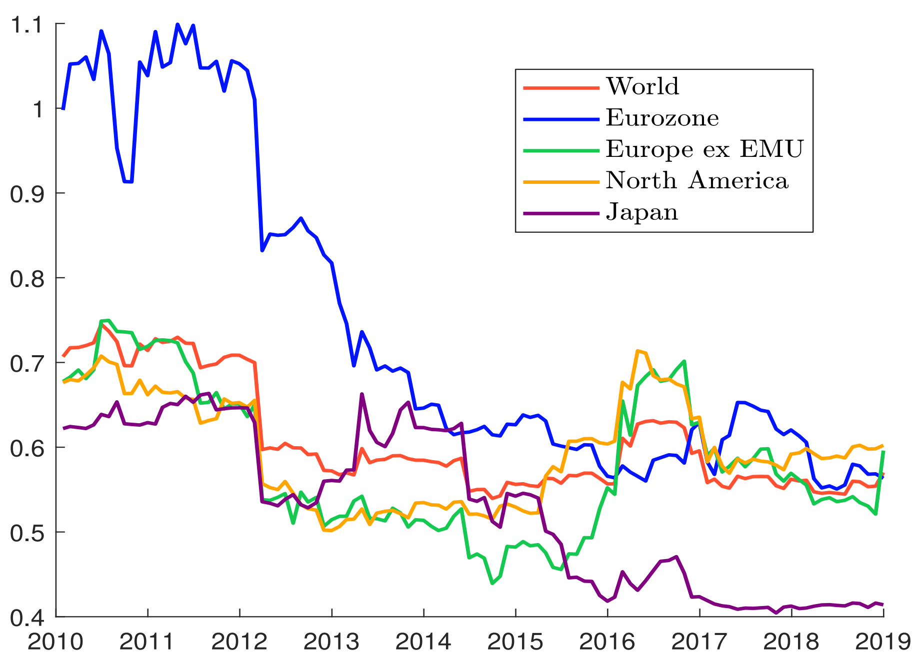


Figure 3 provides the dynamics of the average absolute carbon beta[18] $|\beta|_{\mathrm{bmg},\mathcal{R}}(t)$ for each region $\mathcal{R}$[19]. The higher the value of $|\beta|_{\mathrm{bmg},\mathcal{R}}(t)$, the greater the impact (positive or negative) of carbon risk on stock returns. Curiously, we notice that the integration of carbon risk in the financial market decreases over time. In particular, there is a substantial decrease in 2012 and then a stabilization of the global average absolute carbon beta[20]. The Eurozone was the region with the highest sensitivity to carbon risk but this decreased sharply by almost 44% between 2010 and 2018. However, we observe all regions converging except Japan[21]. The convergence of absolute sensitivities between large geographical regions indicates that investors see carbon risk as a global issue.

We may also be interested in carbon risk trends by sector. In the static analysis, we recall that the energy, materials and real estate sectors were the most negatively impacted by an unexpected acceleration in the transition process, whereas the opposite is true for the health care, information technology and consumer staples sectors. Figures 4 and 5 provide the trends in the median carbon beta $\beta_{\mathrm{bmg},\mathcal{S}}(t)$ for the sector $\mathcal{S}$ at time t, which is defined as follows:

$$\beta_{\mathrm{bmg},\mathcal{S}}(t) = \operatorname*{median}_{i\in\mathcal{S}} \beta_{\mathrm{bmg},i}(t)$$

We distinguish four categories. The first one concerns high positively sensitive sectors to the carbon factor. This includes only the energy sector. The stock price of the latter is increasingly negatively influenced by the movements of the carbon factor. The second category includes the materials and real estate sectors for which the positive sensitivity of stock price to carbon factor is much more moderate. The third category includes the sectors with a neutral or a low negative sensitivity to the carbon factor. This category is made up of the industrials, utilities, communication services, consumer discretionary, consumer staples, financials and information technology sectors. The last category, including only the health care sector, concerns a moderate negative sensitivity to the carbon factor. However, this sector is getting closer and closer to the carbon risk-neutral category over time, even though we continue to observe a gap.

Among the second category, we observe that the materials and real estate sectors started with a similar median carbon beta. However, the spread has been increasing between the two sectors since 2016, because the median carbon risk is stable in the case of the real estate sector whereas it is increasing for the materials sector. This gap may be persistent in the long run, implying that the materials sector may be increasingly affected by carbon risk. Concerning the third category, we observe that the industrials sector was mostly negatively influenced by the BMG factor, but it has become a carbon risk-neutral sector[22]. Overall, sector differentiation is more important than geographical breakdown for investors since market-based carbon risks converge both in absolute and relative values at the geographical level.

[18] It is calculated as follows:

$$|\beta|_{\mathrm{bmg},\mathcal{R}}(t) = \frac{\sum_{i\in\mathcal{R}} |\beta_{\mathrm{bmg},i}(t)|}{\operatorname{card}\mathcal{R}}$$

[19] As we have seen in Table 9 on page 51, the volatility of the BMG factor is lower than the volatility of the MKT factor. A variation of the carbon beta can not be interpreted in an ordinary scale.

[20] The decrease of $|\beta|_{\mathrm{bmg},\mathcal{R}}(t)$ in March 2012 is not due to a climate-related policy but to green stocks far outperforming as we can see in Figure 19 on page 52. At the same time, the European market declined while the American market increased. Therefore, the carbon beta considerably increased for green European stocks, which was driven mostly by the European market's return rather than their carbon return. In a similar way, the carbon beta decreased for brown American stocks, which was driven mostly by the American market's return rather than their carbon return.

[21] Carbon risk pricing in Japan is around 25% lower than globally.

[22] This increase in the median carbon beta may push the industrial sector to the second category in the future.

Figure 4: Dynamics of the median carbon risk $\beta_{\mathrm{bmg},\mathcal{S}}(t)$ by sector

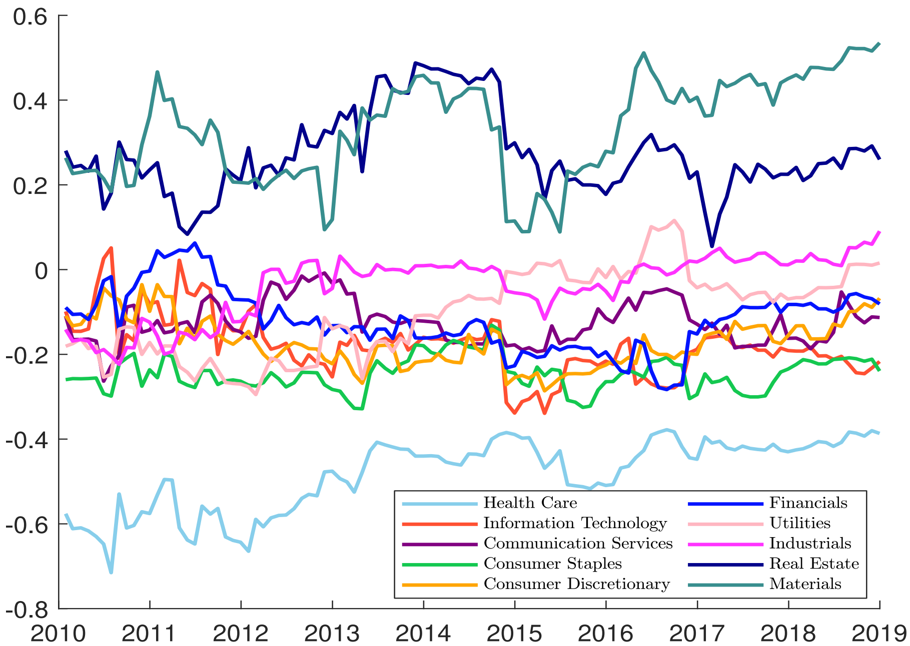
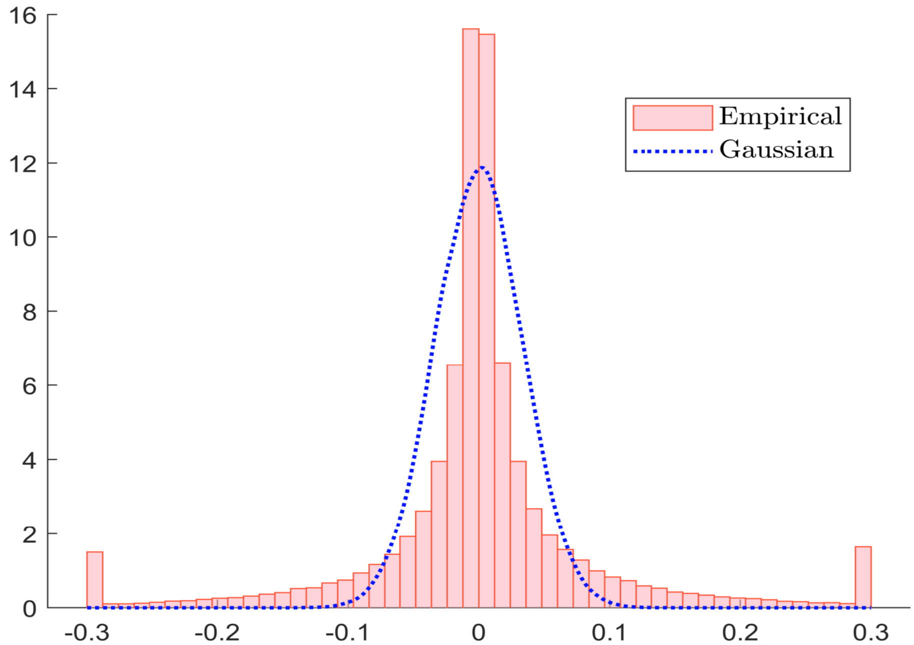


Figure 5: Dynamics of the median carbon risk $\beta_{\mathrm{bmg},\mathcal{S}}(t)$ for the energy sector

Remark 3. *We may also consider the absolute average carbon beta for each sector. Results are reported in Figure 22 on page 53. In this case, we distinguish two main categories. The first one corresponds to high carbon pricing. This category includes the energy and materials sectors. The second one includes the sectors with a low (either upward or downward) carbon sensitivity to stock prices.*

Figure 6: Density of the carbon risk first difference



The advantage of this dynamic analysis is to assume that common risks are time-varying. In Figure 6, we have reported the density of the monthly variations $\beta_{\mathrm{bmg},i}(t) - \beta_{\mathrm{bmg},i}(t-1)$. We observe that it is far from being a Gaussian distribution since we observe fat tails with a significant number of extreme variations. Hence, we deduce that the time-varying model allows us to take into account some extreme changes in carbon risk. However, a microeconomic analysis shows that, in the event of environmental controversies, the model is not able to substantially change the carbon risk in the short run[23]. The extreme changes are more explained by regional or sector-related effects. This confirms that $\beta_{\mathrm{bmg},i}(t)$ is more a low-frequency systematic measure than a high-frequency idiosyncratic measure of the carbon risk.

2.2 Alternative measures of the BMG factor

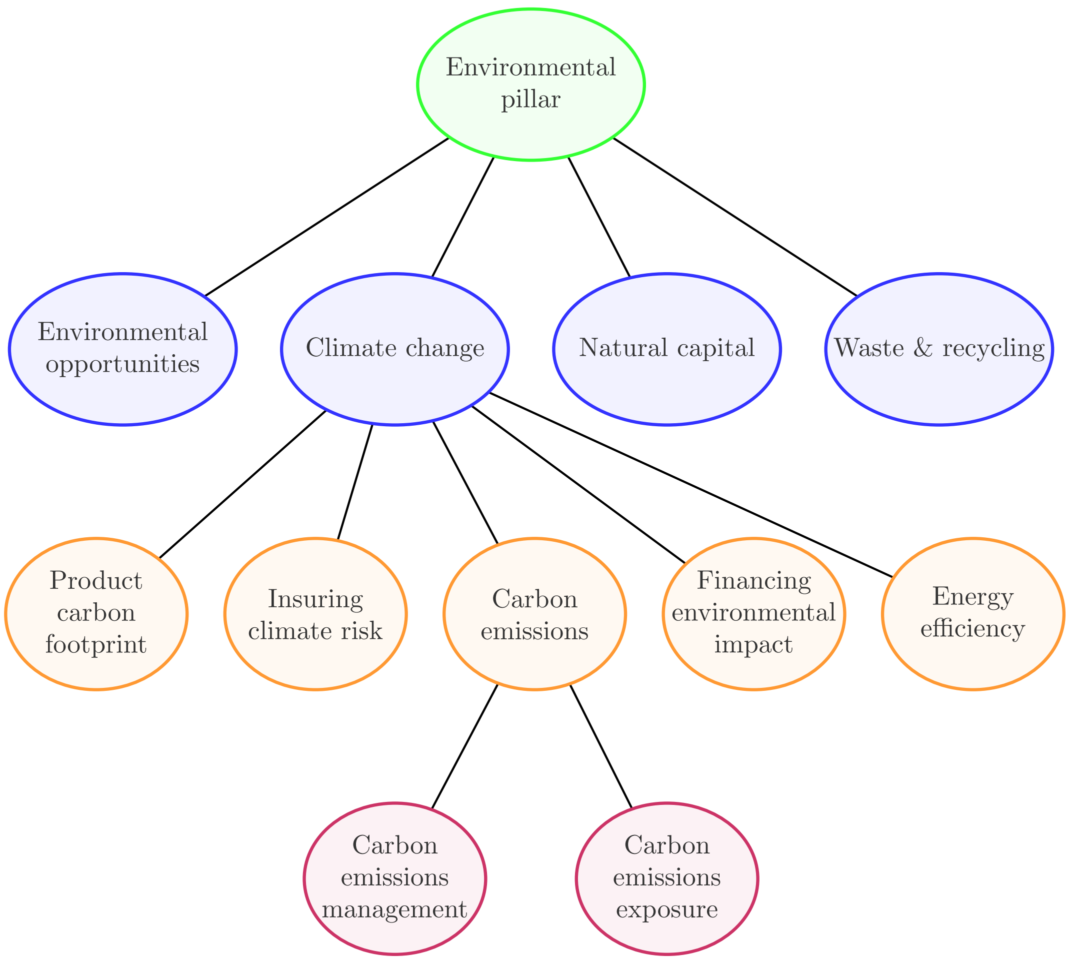
The brown minus green or BMG factor developed by Görgen *et al.* (2019) is an aggregation of numerous variables and we wonder what climate change-related dimensions[24] are most

[23] This is for example the case of some famous controversial events, e.g. Volkswagen, Bayer, etc. Whatever the variation of the carbon factor, the firm's stock return decreases in the case of an environmental controversy.

[24] In this section, the dimensions do not correspond to the dimensions previously introduced (value chain, public perception and adaptability). When we refer to climate-related dimensions, it concerns any variables involved in climate change.

priced in by the financial market. Indeed, some variables taken into account by the Carima project may create some noise without necessarily being relevant. With a set of more than 50 proxy variables, it is likely that some variables can include information that is not priced in by the market or that is not specifically related to carbon risk. For the value chain and adaptability dimensions, the variables provide climate-relevant information. Nevertheless, the public perception dimension mainly contains ESG or E pillar scores while these scores do not incorporate just climate-related information.

Figure 7: Dimension hierarchy in the environmental pillar (MSCI methodology)



Source: MSCI (2020).

In what follows, we present some risk factors built on different climate-related dimensions (Figure 7). These factors have long exposure to worst-in-class green stocks and short exposure to best-in-class green stocks. To obtain results that are comparable with the carbon risk factor developed for the Carima project, we use the methodology of Fama and French (1992, 1993). Nevertheless, the returns of the four portfolios (SG, BG, SB and BB) are equally weighted[25], the portfolio weights are rebalanced every month and the stock universe is the

[25] We have also derived the risk factors with a capitalization-weighted scheme. The results do not change

MSCI World index. Another difference from the Carima approach is that we integrate the financial firms into our factors even though their carbon risk is determined differently from the other sectors. According to Görgen *et al.* (2019), their carbon risk is more influenced by their investments than by their carbon emissions. We retain this sector because excluding financial firms would result in lower statistical significance of the factors. Moreover, we think that financial firms are exposed to some climate-related dimensions. Unless otherwise specified, all the variables come from the MSCI ESG Ratings dataset.

2.2.1 Exposure to carbon costs

The first comparison concerns the factors built on the exposure to carbon pricing and regulatory caps: (1) the carbon intensity[26] derived on the three scopes[27] and (2) the carbon emissions exposure score based on the carbon-intensive business activities and the current or potential future carbon regulations (MSCI, 2020). In Figure 8, we have reported the cumulative performance of these two factors and the carbon factor developed by Görgen *et al.* (2019). We observe that the three factors are very similar. For instance, we have a strong linear correlation greater than 90% between the carbon intensity and carbon emissions exposure factors. Because of the greater similarity with the Carima factor, we wonder if the carbon intensity risk measure is the only carbon dimension priced in by the market[28]. Table 10 on page 51 provides the correlation between the carbon risk factors and the reference factors. We have a close correlation between the two current carbon factors and the Carima factor, but not so much (58% and 64%). In Table 11 on page 51, we have reported a comparison between multi-factor models and their nested models. There is a slightly larger factor exposure for the two current carbon risk factors with respect to the Carima risk factor. Hence, the carbon exposure is a dimension widely taken into account by the market to determine changes in stock prices, but we cannot deduce that this dimension is the only dimension priced in because our methodology and our universe of stocks to build factors are different from the Carima project.

Remark 4. *We have also built the factors with the capitalization-weighted scheme and excluding financial firms. In this case, the carbon intensity factor has a smaller factor exposure than the Carima factor because of a strong correlation with the market, value and momentum factors. However, this decrease in explanatory power is mainly due to the exclusion of financials. Indeed, this sector has little exposure to the potential risk of increased costs linked to carbon pricing or cap and trade systems. We have seen previously in Figure 1 on page 8 that the financial sector has a negative relative carbon risk. Even though carbon risk is constructed differently for this sector, excluding it means losing information. Concerning the CW carbon emissions exposure factor, the results are halfway between the two EW factors and the Carima factor. Nevertheless, the better results of explanatory power for this CW factor in comparison with the Carima factor are too low to conclude that the carbon exposure is the only carbon dimension priced in by the market.*

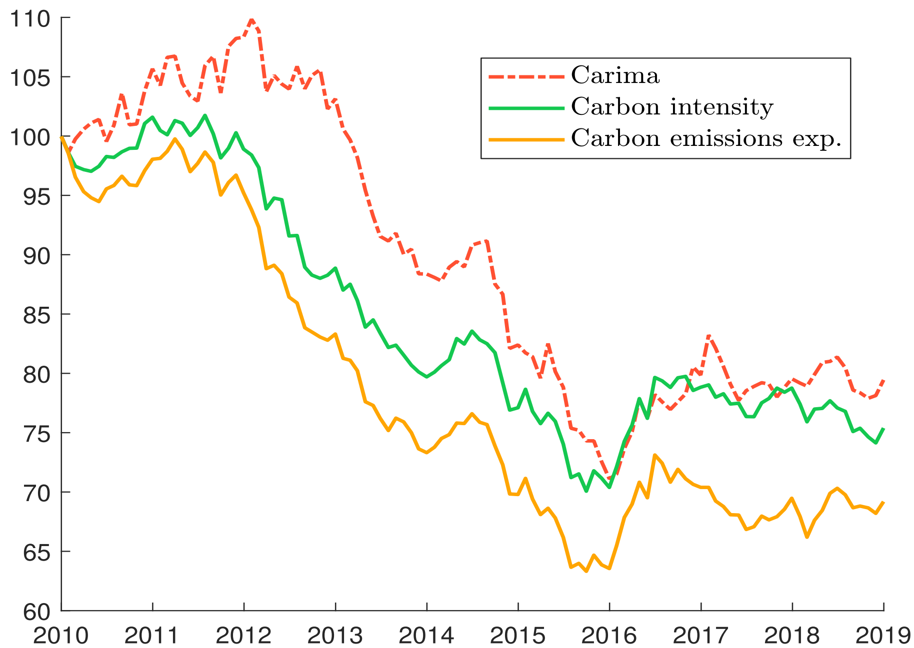
much. We have a correlation of around 95% between EW and CW factors. Nevertheless, we obtain a better explanatory power for the multi-factor regression models when we consider an equally-weighted scheme.

[26]This factor has already been proposed by In *et al.* (2017) on a universe of American stocks.

[27]The three scopes are available in the Trucost dataset. Assets are selected every month with a reporting lag of one year.

[28]It is obvious that numerous other carbon variables explain the fluctuations in stock prices but some (or all) of them increase the explanatory power of a multi-factor model because they are correlated with the carbon intensity.

Figure 8: Cumulative performance of the factors based on the exposure to carbon costs



2.2.2 Carbon exposure and management pricing

While prior literature (Semenova and Hassel, 2015) distinguishes between environmental performance (or management) and environmental risk (or exposure), we focus on the same distinction but on the carbon dimension. In what follows, we will provide an answer to whether carbon emissions exposure or carbon management contributes more to variations in stock prices. The carbon emissions exposure mainly involves current carbon emissions and factors inherent to the firm's business whereas the carbon emissions management is about future potential carbon emissions and measures the efforts to reduce this exposure. If we consider the carbon axes used by the Carima project, the first one corresponds to the value chain axis and the second one to the adaptability axis. In Table 11 on page 51, we remark that both carbon exposure and carbon management significantly increase the explanatory power of the common factor models but this variation is greater with the first factor. With respect to the other carbon risk factors, the carbon management factor has a lower correlation with the Carima factor as we can see in Table 10 on page 51. Moreover, the close correlation of this factor with the Carhart risk factors is a problem in the case of minimum variance and enhanced index portfolios where the risk factors are supposed to be uncorrelated. Furthermore, this carbon factor is not an alternative option to the reference carbon factor.

Despite the issues with the carbon management factor, abandoning this dimension would be unfortunate. The carbon emissions factor built on the carbon emissions score, which is derived from both the carbon emissions management and exposure scores, allows us to overcome this problem. The higher the exposure score and the lower the management score, the lower the carbon emissions score. The latter assesses the capacity of a firm to handle increasing carbon costs. As we can see in Table 11 on page 51, aggregating the two dimensions leads to an increase in explanatory power, which is significant for almost 19% of the stocks at the threshold of 5%. Moreover, as seen in Table 10 on page 51, the carbon emissions dimension is the most closely correlated to the Carima risk factor.

Remark 5. *The median sector carbon beta coefficients have almost the same ranks for the carbon exposure and the aggregated carbon factors. Nonetheless, we have very different results with the carbon management factor. In the case of carbon management, the carbon betas are very high for the health care sector because of the lack of environmental performances. For the utilities and materials sector, the carbon beta has significantly decreased – the utilities sector has the lower average carbon beta. These results are not surprising since the more environmentally responsible firms face greater environmental challenges (Delmas and Blass, 2010; Rahman and Post, 2012). We confirm again that the exposure factor is better than the management factor since the aggregated carbon factor is associated to a higher adjusted $\Re^2$ coefficient while it is very related to the carbon exposure dimension.*

2.2.3 Environmental, climate and carbon dimensions

Let us consider now the three main climate-related dimensions: environment, climate change and carbon emissions. The carbon dimension is nested into the climate dimension which is itself nested into the environmental dimension (see Figure 7 on page 15). The last two factors are based respectively on the environmental pillar score and the climate change theme score available in the MSCI ESG Ratings dataset. The environmental pillar includes the climate change dimension but also environmental opportunities, waste and recycling, and natural capital. The climate change scope includes carbon emissions, environmental risk financing, climate change vulnerability of insurance companies and the product carbon footprint (MSCI, 2020). The three new factors are very similar with a correlation between them around 75% and 80%. In Table 11 on page 51, we notice that the mimicking factor portfolio for environmental risk is more closely correlated with the factor built by Görgen *et al.* (2019) than the climate change factor, certainly because the Carima risk factor doesn't just incorporate carbon emissions variables. One issue with the environmental factor is that it is significantly negatively correlated with the market factor[29].

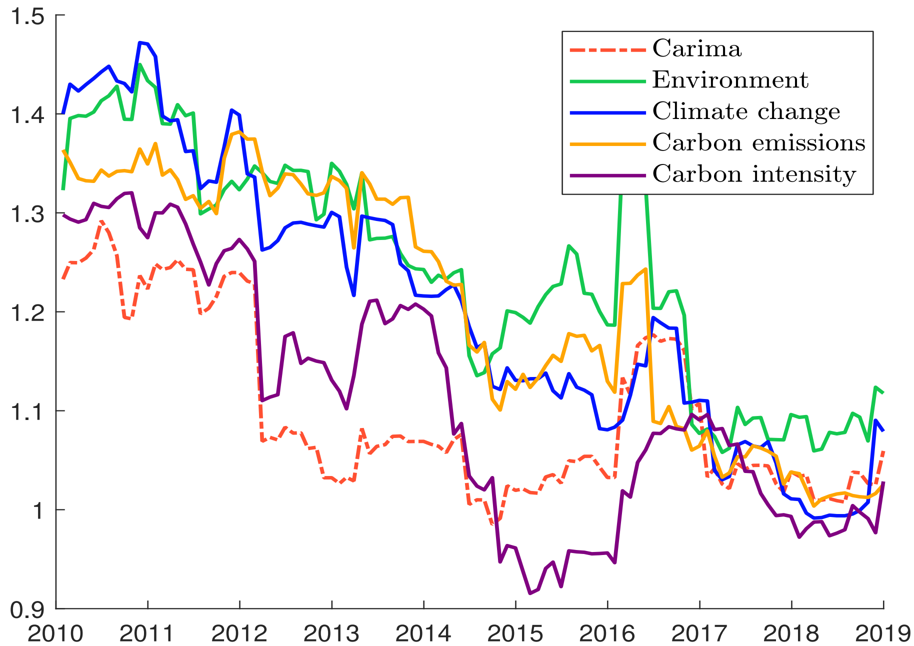
In Figure 24 on page 54, we have reported the sector analysis of the carbon beta $\hat{\beta}_{\mathrm{bmg},i}$ estimated with the MKT+BMG model. We notice that the carbon emissions and environmental factors are very similar while there are some differences with the climate change factor. For this latter factor, the median and quantiles are associated with small relative carbon risk $\beta_{\mathrm{bmg},i}$ for most sectors. These small carbon risks are mainly offset by the financial sector's higher carbon risk. While the financial sector has the lower median carbon beta with the carbon emissions factor, it ranks eighth with the climate change factor because it takes into account the vulnerability of insurance companies to insured individuals' physical risks and the integration of the environmental component into banks' or asset managers' business models. We also notice that the carbon beta of the consumer staples and discretionary sectors have overall increased because the climate change factor takes into account the product carbon footprint.

2.2.4 Overview of the factors

Numerous climate-related dimensions can be used to measure carbon risk. Among the studied factors, some of them are more appropriate for assessing stock price fluctuations. Figure 9 provides the dynamics of the average absolute carbon risk $|\beta|_{\mathrm{bmg},i}(t)$ for some climate change-related dimensions and the Carima factor. In order to have comparable

[29] In a minimum variance portfolio where the average relative carbon risk is negative, a bear market may imply a higher loss since the best-in-class green stocks underperform the other stocks.

Figure 9: Dynamics of the average absolute carbon risk $|\beta|_{\mathrm{bmg},i}(t)$



carbon betas, each carbon factor BMG_j has been standardized so that its monthly volatility is equal to 1%:

$$\tilde{R}_{\mathrm{bmg},j}(t) = \frac{R_{\mathrm{bmg},j}(t)}{100 \times \sigma_{\mathrm{bmg},j}(t)}$$

where $\tilde{R}_{\mathrm{bmg},j}(t)$ is the return of the standardized carbon factor and $\sigma_{\mathrm{bmg},j}(t)$ is the conditional volatility of the factor BMG_j estimated with a GARCH(1,1) model. In this case, the factors' volatilities are the same at any time t. Overall, the trends of absolute carbon risk in the environment, climate change and carbon emissions factors are very similar. Therefore, we think that the carbon emissions dimension is highly priced in, whereas adding other environmental variables does not significantly increase the average absolute carbon beta[30]. We can observe that the Carima factor is less priced in than the three main climate change-related factors. One reason might be the different methodologies between the factors. Nevertheless, our results do not change much with a capitalization-weighted scheme or by excluding financial firms. Therefore, we may wonder whether including a large range of environmental variables is more informative. The carbon intensity dimension is less priced in by the stock market at the beginning of the study period in comparison with the three main climate change-related dimensions, but this gap has been eliminated since 2017.

According to Table 11 on page 51, carbon emissions is the better climate change-related factor to explain stock price fluctuations, followed by the carbon emissions exposure and carbon intensity factors. By splitting the period into two equal subperiods[31], we obtain the

[30]Our results do not mean that the other environmental dimensions are barely priced in. For a wide range of environmental variables, the associated average absolute betas are often high but not so much as the carbon emissions dimension. By taking into account several variables into one factor like the climate change and environmental factors do, there is some collinearity between variables and it is difficult, from a statistical point of view, to determine which dimension is predominant in stock price variations.

[31]The first period starts at the beginning of 2010 and ends in mid-2014 and the second period starts in mid-2014 and ends at the end of 2018.

following adjusted $\mathfrak{R}^2$ difference for the CAPM model against the MKT+BMG model:

	1^{st} period 2010-2014	2^{nd} period 2014-2018
Carima	1.16	2.21
Carbon intensity	1.43	2.53
Carbon emissions	2.18	2.39
Climate change	1.98	1.83
Environment	1.35	2.17

The use of the BMG factors to explain stock price fluctuations is not consistent over time. According to these results, the carbon intensity dimension is currently the most important climate change-related axis in the two-factor model. Concerning the environment and climate change dimensions, the results and their correlations with the market factor lead to these factors being abandoned when it comes to managing carbon risk in investment portfolios. Overall, the carbon intensity and carbon emissions dimensions are the more interesting alternative factors to the Carima factor.

Remark 6. *Even though the climate change-related dimensions are less priced in by the stock market over time, their integration – except for the climate change dimension – in a CAPM model significantly increases the explanatory power during the second period. This is because the CAPM model has an average adjusted* $\mathfrak{R}^2$ *equal to* 34.31% *during the 1st period and* 24.15% *during the 2nd period. Moreover, if we compute a single carbon factor model, the adjusted* $\mathfrak{R}^2$ *is lower during the 2nd period. Therefore, our results are coherent with the observation that the carbon risk is less priced in today than before.*

3 Managing carbon risk

In what follows, we consider how to manage carbon risk in an investment portfolio. Three methods are used: the minimum variance strategy, the enhanced index portfolio and factor investing. For each method, we use an easily understood example and then we apply the method to the MSCI World index. In this last case, we take into account the dynamic betas of the MKT+BMG model estimated by the Kalman filter, implying that we use the beta coefficients and the weights of the MSCI World index at the end of December 2018.

In this section, it is important to keep in mind the distinction between absolute and relative carbon risks. In the first case, the underlying idea is to have a neutral exposure to the BMG factor. In other words, we search the closest carbon exposure to zero. In the second case, the objective is to have a negative exposure to carbon risk. These two approaches lead us to consider different objective functions or constraints of the portfolio optimization program.

3.1 Application to the minimum variance portfolio

We consider the global minimum variance (GMV) portfolio, which corresponds to this optimization program:

$$\begin{aligned} x^\star &= \arg\min \frac{1}{2} x^\top \Sigma x \\ &\text{s.t.} \quad \mathbf{1}_n^\top x = 1 \end{aligned} \tag{12}$$

where x is the vector of portfolio weights and Σ is the covariance matrix of stock returns. The solution is given by the well-known formula:

$$x^{\star} = \frac{\Sigma^{-1}\mathbf{1}_n}{\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n} \tag{13}$$

Problem (12) can be extended by considering other constraints:

$$\begin{aligned} x^{\star} &= \arg\min \frac{1}{2}x^{\top}\Sigma x \\ & \text{s.t.} \quad \left\{ \begin{array}{l} \mathbf{1}^{\top}x = 1 \\ x \in \Omega \end{array} \right. \end{aligned} \tag{14}$$

For instance, the most famous formulation is the long-only optimization problem where $\Omega = [0,1]^n$ (Jagannathan and Ma, 2003).

3.1.1 The CAPM risk factor model

In the capital asset pricing model, we recall that:

$$R_i(t) = \alpha_i + \beta_{\text{mkt},i}R_{\text{mkt}}(t) + \varepsilon_i(t) \tag{15}$$

where $R_i(t)$ is the return of asset i, $R_{\text{mkt}}(t)$ is the return of the market factor and $\varepsilon_i(t)$ is the idiosyncratic risk. It follows that the covariance matrix Σ can be decomposed as:

$$\Sigma = \beta_{\text{mkt}}\beta_{\text{mkt}}^{\top}\sigma_{\text{mkt}}^2 + D$$

where $\beta_{\text{mkt}} = \left(\beta_{\text{mkt},1},\ldots,\beta_{\text{mkt},n}\right)$ is the vector of betas, σ_{mkt}^2 is the variance of the market portfolio and $D = \operatorname{diag}\left(\tilde{\sigma}_1^2,\ldots,\tilde{\sigma}_n^2\right)$ is the diagonal matrix of specific variances. Using the Sherman-Morrison-Woodbury formula[32], we deduce that the inverse of the covariance matrix is:

$$\Sigma^{-1} = D^{-1} - \frac{\sigma_{\text{mkt}}^2}{1+\sigma_{\text{mkt}}^2\varphi\left(\tilde{\beta}_{\text{mkt}},\beta_{\text{mkt}}\right)}\tilde{\beta}_{\text{mkt}}\tilde{\beta}_{\text{mkt}}^{\top}$$

where $\tilde{\beta}_{\text{mkt},i} = \beta_{\text{mkt},i}/\tilde{\sigma}_i^2$ and $\varphi\left(\tilde{\beta}_{\text{mkt}},\beta_{\text{mkt}}\right) = \tilde{\beta}_{\text{mkt}}^{\top}\beta_{\text{mkt}}$. Solution (13) becomes:

$$x^{\star} = \sigma^2\left(x^{\star}\right)\left(D^{-1}\mathbf{1}_n - \frac{\sigma_{\text{mkt}}^2}{1+\sigma_{\text{mkt}}^2\varphi\left(\tilde{\beta}_{\text{mkt}},\beta_{\text{mkt}}\right)}\tilde{\beta}_{\text{mkt}}\tilde{\beta}_{\text{mkt}}^{\top}\mathbf{1}_n\right)$$

Using this new expression, Scherer (2011) showed that:

$$x_i^{\star} = \frac{\sigma^2\left(x^{\star}\right)}{\tilde{\sigma}_i^2}\left(1 - \frac{\beta_{\text{mkt},i}}{\beta_{\text{mkt}}^{\star}}\right) \tag{16}$$

where:

$$\beta_{\text{mkt}}^{\star} = \frac{1+\sigma_{\text{mkt}}^2\varphi\left(\tilde{\beta}_{\text{mkt}},\beta_{\text{mkt}}\right)}{\sigma_{\text{mkt}}^2\tilde{\beta}_{\text{mkt}}^{\top}\mathbf{1}_n} \tag{17}$$

[32] This is provided in Appendix A.2 on page 42. The expression of Σ^{-1} is obtained with $A = D$ and $u = v = \sigma_{\text{mkt}}\beta_{\text{mkt}}$.

If we consider this formula, we note that the minimum variance portfolio is exposed to stocks with low volatility and low beta. More precisely, if asset i has a beta $\beta_{\mathrm{mkt},i}$ smaller than $\beta^\star_{\mathrm{mkt}}$, the weight of this asset is positive ($x^\star_i > 0$). If $\beta_{\mathrm{mkt},i} > \beta^\star_{\mathrm{mkt}}$, then $x^\star_i < 0$. Clarke *et al.* (2011) extended Formula (16) to the long-only case with the threshold $\beta^\star_{\mathrm{mkt}}$ defined as follows:

$$\beta^\star_{\mathrm{mkt}} = \frac{1 + \sigma^2_{\mathrm{mkt}} \sum_{\beta_{\mathrm{mkt},i} < \beta^\star_{\mathrm{mkt}}} \tilde{\beta}_{\mathrm{mkt},i} \beta_{\mathrm{mkt},i}}{\sigma^2_{\mathrm{mkt}} \sum_{\beta_{\mathrm{mkt},i} < \beta^\star_{\mathrm{mkt}}} \tilde{\beta}_{\mathrm{mkt},i}} \tag{18}$$

In this case, if $\beta_{\mathrm{mkt},i} > \beta^\star_{\mathrm{mkt}}$, $x^\star_i = 0$.

3.1.2 Including the absolute carbon risk

We consider an extension of the CAPM by including the BMG risk factor:

$$R_i(t) = \alpha_i + \beta_{\mathrm{mkt},i} R_{\mathrm{mkt}}(t) + \beta_{\mathrm{bmg},i} R_{\mathrm{bmg}}(t) + \varepsilon_i(t) \tag{19}$$

where $R_{\mathrm{bmg}}(t)$ is the return of the BMG factor and $\beta_{\mathrm{bmg},i}$ is the BMG sensitivity (or the carbon beta) of stock i. Moreover, we assume that $R_{\mathrm{mkt}}(t)$ and $R_{\mathrm{bmg}}(t)$ are uncorrelated. It follows that the expression of the covariance matrix becomes:

$$\Sigma = \beta_{\mathrm{mkt}} \beta^\top_{\mathrm{mkt}} \sigma^2_{\mathrm{mkt}} + \beta_{\mathrm{bmg}} \beta^\top_{\mathrm{bmg}} \sigma^2_{\mathrm{bmg}} + D$$

In Appendix A.3 on page 43, we show that the GMV portfolio is defined as:

$$x^\star_i = \frac{\sigma^2(x^\star)}{\tilde{\sigma}^2_i} \left(1 - \frac{\beta_{\mathrm{mkt},i}}{\beta^\star_{\mathrm{mkt}}} - \frac{\beta_{\mathrm{bmg},i}}{\beta^\star_{\mathrm{bmg}}} \right) \tag{20}$$

where $\beta^\star_{\mathrm{mkt}}$ and $\beta^\star_{\mathrm{bmg}}$ are two threshold values given by Equations (43) and (44) on page 45. In the case of long-only portfolios, we obtain a similar formula:

$$x^\star_i = \begin{cases} \dfrac{\sigma^2(x^\star)}{\tilde{\sigma}^2_i} \left(1 - \dfrac{\beta_{\mathrm{mkt},i}}{\beta^\star_{\mathrm{mkt}}} - \dfrac{\beta_{\mathrm{bmg},i}}{\beta^\star_{\mathrm{bmg}}} \right) & \text{if } \dfrac{\beta_{\mathrm{mkt},i}}{\beta^\star_{\mathrm{mkt}}} + \dfrac{\beta_{\mathrm{bmg},i}}{\beta^\star_{\mathrm{bmg}}} \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

but the expressions of the thresholds[33] $\beta^\star_{\mathrm{mkt}}$ and $\beta^\star_{\mathrm{bmg}}$ are different from those obtained in the GMV case.

Contrary to the single-factor model, the impact of sensitivities is more complex in the two-factor model. Indeed, we know that $\bar{\beta}_{\mathrm{mkt}} \approx 1$ and $\bar{\beta}_{\mathrm{bmg}} \approx 0$. It follows that $\beta^\star_{\mathrm{mkt}}$ is positive, but $\beta^\star_{\mathrm{bmg}}$ may be positive or negative. We deduce that the ratio $\dfrac{\beta_{\mathrm{mkt},i}}{\beta^\star_{\mathrm{mkt}}}$ is an increasing function of $\beta_{\mathrm{mkt},i}$. Therefore, the MV portfolio selects assets that present a low MKT beta value. For the BMG factor, the impact of $\beta_{\mathrm{bmg},i}$ is more complex. Let us first compute the volatility of the asset i. We have:

$$\sigma^2_i = \beta^2_{\mathrm{mkt},i} \sigma^2_{\mathrm{mkt}} + \beta^2_{\mathrm{bmg},i} \sigma^2_{\mathrm{bmg}} + \tilde{\sigma}^2_i$$

Selecting low volatility assets is then equivalent to considering assets with a low absolute value $\left|\beta_{\mathrm{bmg},i}\right|$. If we calculate the correlation between assets i and j, we obtain:

$$\rho_{i,j} = \frac{\beta_{\mathrm{mkt},i} \beta_{\mathrm{mkt},j} \sigma^2_{\mathrm{mkt}} + \beta_{\mathrm{bmg},i} \beta_{\mathrm{bmg},j} \sigma^2_{\mathrm{bmg}}}{\sigma_i \sigma_j}$$

[33] They are given by Equations (47) and (48) on page 47.

In practical cases, the cross product $\beta_{\mathrm{mkt},i}\beta_{\mathrm{mkt},j}$ is generally positive, whereas the cross product $\beta_{\mathrm{bmg},i}\beta_{\mathrm{bmg},j}$ is positive or negative. In this context, diversifying a portfolio consists in selecting assets with low values of $\beta_{\mathrm{mkt},i}\beta_{\mathrm{mkt},j}$. We then observe consistency between low volatility and low correlated assets if we consider market beta contributions. In terms of BMG sensitivities, diversifying a portfolio consists in selecting assets with high negative values of $\beta_{\mathrm{bmg},i}\beta_{\mathrm{bmg},j}$. Therefore, we have to choose assets, with high absolute values $\left|\beta_{\mathrm{bmg},i}\beta_{\mathrm{bmg},j}\right|$, and opposite signs of $\beta_{\mathrm{bmg},i}$ and $\beta_{\mathrm{bmg},j}$. We do not have consistency between low volatility and low correlated assets if we consider BMG contributions. This explains that the ratio $\dfrac{\beta_{\mathrm{bmg},i}}{\beta^{\star}_{\mathrm{bmg}}}$ may be an increasing or decreasing function of $\beta_{\mathrm{bmg},i}$. The MV portfolio will then overweight assets with a negative value of $\beta_{\mathrm{bmg},i}$ only if $\beta^{\star}_{\mathrm{bmg}}$ is positive. Otherwise, the MV portfolio may prefer assets with a positive sensitivity to the BMG factor.

Remark 7. *Let us denote by $x^{\star}\left(\beta_{\mathrm{mkt}},\beta_{\mathrm{bmg}}\right)$ the minimum variance portfolio that depends on the parameters β_{mkt} and β_{bmg}. We have the following properties:*

$$\left\{\begin{array}{l} x^{\star}\left(\beta_{\mathrm{mkt}},-\beta_{\mathrm{bmg}}\right)=x^{\star}\left(\beta_{\mathrm{mkt}},\beta_{\mathrm{bmg}}\right)\\ \beta^{\star}_{\mathrm{mkt}}\left(\beta_{\mathrm{mkt}},-\beta_{\mathrm{bmg}}\right)=\beta^{\star}_{\mathrm{mkt}}\left(\beta_{\mathrm{mkt}},\beta_{\mathrm{bmg}}\right)\\ \beta^{\star}_{\mathrm{bmg}}\left(\beta_{\mathrm{mkt}},-\beta_{\mathrm{bmg}}\right)=-\beta^{\star}_{\mathrm{bmg}}\left(\beta_{\mathrm{mkt}},\beta_{\mathrm{bmg}}\right)\end{array}\right.$$

Changing the BMG sensitivities by their opposite values does not change the solution[34].

3.1.3 Some examples

We consider an example given in Roncalli (2013, Example 24, page 168). The investment universe is made up of five assets. Their market beta is respectively equal to 0.9, 0.8, 1.2, 0.7 and 1.3 whereas their specific volatility is 4%, 12%, 5%, 8% and 5%. The market portfolio volatility is equal to 25%. Using these figures, we have computed the composition of the minimum variance portfolio. The results are reported in Table 4. The fourth and fifth columns contain the weights (in %) of the unconstrained and long-only minimum variance portfolios. In the case of the unconstrained portfolio (GMV), we have $\beta^{\star}_{\mathrm{mkt}}=1.0972$. We deduce then that long exposures concern the first, second and fourth assets whereas the short exposures concern the third and fifth assets. For the long-only portfolio (MV), we obtain $\beta^{\star}_{\mathrm{mkt}}=0.8307$. This implies that only the second and fourth assets are represented in the long-only minimum variance portfolio.

Table 4: Composition of the MV portfolio (parameter set #1)

Asset	$\beta_{\mathrm{mkt},i}$	$\beta_{\mathrm{bmg},i}$	CAPM		MKT+BMG	
			GMV	MV	GMV	MV
1	0.90	−0.50	147.33	0.00	166.55	33.54
2	0.80	0.70	24.67	9.45	21.37	1.46
3	1.20	0.20	−49.19	0.00	−58.80	0.00
4	0.70	0.90	74.20	90.55	65.06	64.99
5	1.30	−0.30	−97.01	0.00	−94.18	0.00

We now consider the impact of the BMG factor. We assume that the BMG sensitivities are respectively equal to −0.5, 0.7, 0.2, 0.9 and −0.3, whereas the volatility of the BMG factor is set to 10%. In the case of the GMV, the thresholds are equal to $\beta^{\star}_{\mathrm{mkt}}=1.0906$ and

[34]This result holds for both the GMV portfolio and the long-only MV portfolio.

$\beta^{\star}_{\text{bmg}} = 19.7724$. In this case, we obtain the same long and short exposures with different magnitudes than previously. In the case of the long-only portfolio, the thresholds become $\beta^{\star}_{\text{mkt}} = 0.8667$ and $\beta^{\star}_{\text{bmg}} = 9.7394$. Compared to the CAPM solution, we observe that the weights in the second and fourth assets are reduced, because they have a positive BMG sensitivity. At the same time, the portfolio is exposed to the first asset. The reason lies in the negative value of BMG sensitivity. Indeed, we have:

$$\frac{\beta_{\text{mkt},1}}{\beta^{\star}_{\text{mkt}}} = \frac{0.90}{0.8307} > 1$$

but:

$$\frac{\beta_{\text{mkt},1}}{\beta^{\star}_{\text{mkt}}} + \frac{\beta_{\text{bmg},1}}{\beta^{\star}_{\text{bmg}}} = \frac{0.90}{0.8667} + \frac{-0.50}{9.7394} < 1$$

Therefore, thanks to the BMG factor, the first asset passes the eligibility test.

Table 5: Composition of the MV portfolio (parameter sets #2 and #3)

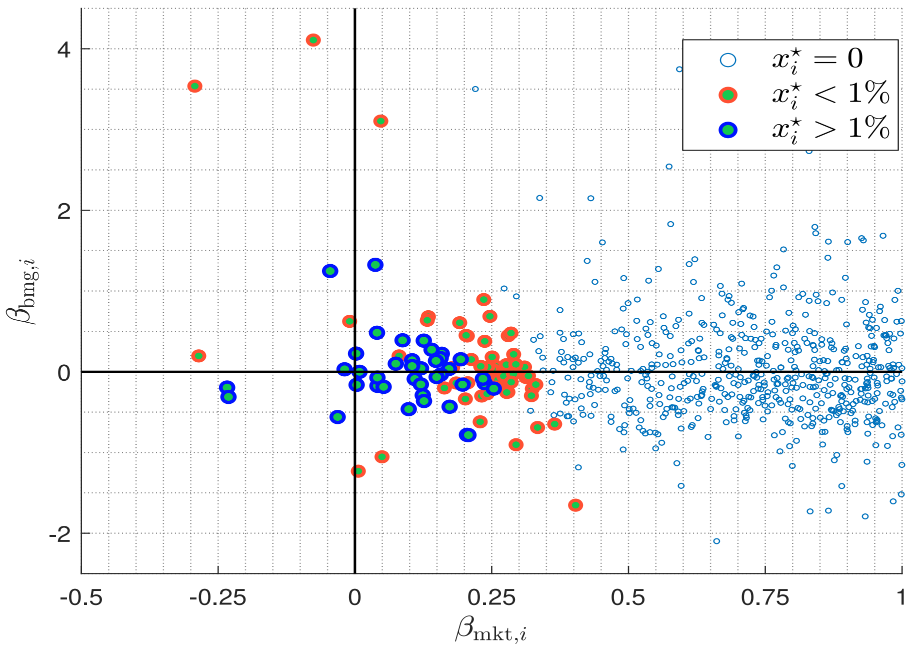
Asset	$\beta_{\text{mkt},i}$	Parameter set #2			Parameter set #3		
		$\beta_{\text{bmg},i}$	GMV	MV	$\beta_{\text{bmg},i}$	GMV	MV
1	0.90	−1.50	105.46	0.00	1.50	105.46	0.00
2	0.80	−0.50	27.88	19.48	0.50	27.88	19.48
3	1.20	3.00	40.19	13.61	−3.00	40.19	13.61
4	0.70	−1.20	76.77	66.91	1.20	76.77	66.91
5	1.30	−0.90	−150.30	0.00	0.90	−150.30	0.00

We now consider a variant of the previous example. We use the same parameter values, but different BMG sensitivity values. In the case of the parameter set #2, they are equal to −1.5, −0.5, 3.0, −1.2 and −0.9. For the long/short MV portfolio, we obtain $\beta^{\star}_{\text{mkt}} = 1.0982$ and $\beta^{\star}_{\text{bmg}} = -19.4470$. For the long-only MV portfolio, the thresholds become $\beta^{\star}_{\text{mkt}} = 0.9070$ and $\beta^{\star}_{\text{bmg}} = -9.0718$. We notice that the BMG threshold is negative, whereas it was positive in the case of the parameter set #1. Moreover, we observe a positive exposure on the third asset even though it has a high positive market beta. The reason is the high magnitude of the BMG sensitivity and the negative correlation with the BMG sensitivities of the other assets. In the case of the parameter set #3, we have only changed the sign of the BMG sensitivities. We obtain the same composition of the MV portfolio and the same value of $\beta^{\star}_{\text{mkt}}$, but the value of $\beta^{\star}_{\text{bmg}}$ is different. Indeed, we obtain $\beta^{\star}_{\text{bmg}} = +19.4470$ for the long/short portfolio and $\beta^{\star}_{\text{bmg}} = +9.0718$ for the long-only portfolio.

We apply the previous framework to the MSCI World index at December 2018. We have already estimated the MKT+BMG model[35] in Section 2.1.4 on page 9. By computing the long-only MV portfolio, we obtain $\beta^{\star}_{\text{mkt}} = 0.3465$ and $\beta^{\star}_{\text{bmg}} = 5.3278$. In Figure 10, we indicate the assets that make up the MV portfolio with respect to their beta values $\beta_{\text{mkt},i}$ and $\beta_{\text{bmg},i}$. We verify that the most important axis is the MKT beta. Indeed, the market risk of a stock determines whether the stock is included in the MV portfolio or not whereas the carbon risk adjusts the weights of the asset. As we can see, the portfolio overweights assets whose MKT and BMG sensitivities are both close to zero. This solution is satisfactory if the original motivation is to reduce the portfolio's absolute carbon risk, but it is not satisfactory if the objective is to manage the portfolio's relative carbon risk.

[35] We have reported the scatter plot of MKT and BMG sensitivities in Figure 23 on page 54. We observe a low positive correlation between $\beta_{\text{mkt},i}$ and $\beta_{\text{bmg},i}$.

Figure 10: Weights of the MV portfolio



3.1.4 New formulation of the minimum variance portfolio with relative carbon risk

In order to circumvent the previous drawback, we can directly add a BMG constraint in the optimization program:

$$x^\star = \arg\min \frac{1}{2} x^\top \Sigma x \quad (22)$$
$$\text{s.t.} \quad \begin{cases} \mathbf{1}_n^\top x = 1 \\ \beta_{\mathrm{bmg}}^\top x \leq \beta_{\mathrm{bmg}}^+ \\ x \geq \mathbf{0}_n \end{cases}$$

where β_{bmg}^+ is the maximum tolerance of the investor with respect to the relative BMG risk. In this case, the values of $\beta_{\mathrm{bmg},i}$ influence both the covariance matrix and the optimization problem. In this case, it is not possible to obtain an analytical solution. Nevertheless, we can always analyze the solution using the framework developed by Jagannathan and Ma (2003). Introducing the BMG constraint is equivalent to applying a shrinkage of the covariance matrix[36]:

$$\tilde{\Sigma} = \Sigma + \lambda_{\mathrm{bmg}} \left(\beta_{\mathrm{bmg}} \mathbf{1}_n^\top + \mathbf{1}_n \beta_{\mathrm{bmg}}^\top \right) \quad (23)$$

where $\lambda_{\mathrm{bmg}} \geq 0$ is the Lagrange coefficient associated to the inequality constraint $\beta_{\mathrm{bmg}}^\top x \leq \beta_{\mathrm{bmg}}^+$. We deduce that the shrinkage matrix is equal to:

$$\tilde{\Sigma} = \sigma_{\mathrm{mkt}}^2 \left(\beta_{\mathrm{mkt}} \beta_{\mathrm{mkt}}^\top - \left(\frac{\lambda_{\mathrm{bmg}}}{\sigma_{\mathrm{mkt}} \sigma_{\mathrm{bmg}}} \right)^2 \right) + \sigma_{\mathrm{bmg}}^2 \left(\dot{\beta}_{\mathrm{bmg}} \dot{\beta}_{\mathrm{bmg}}^\top \right) + D$$

[36]The analysis of Jagannathan and Ma (2003) only includes bound constraints. The extension to other linear constraints can be found in Roncalli (2013).

where $\dot{\beta}_{\text{bmg}} = \beta_{\text{bmg}} + \frac{\lambda_{\text{bmg}}}{\sigma^2_{\text{bmg}}}\mathbf{1}_n$. By imposing that the MV portfolio has a carbon beta lower than β^+_{bmg}, we implicitly introduce two effects:

1. first, we shift the BMG sensitivities by a positive scalar $\frac{\lambda_{\text{bmg}}}{\sigma^2_{\text{bmg}}}$;
2. second, we reduce the MKT covariance matrix by a uniform parallel shift, because of the term $\frac{\lambda_{\text{bmg}}}{\sigma_{\text{mkt}}\sigma_{\text{bmg}}}$.

Therefore, the BMG constraint $\beta^\top_{\text{bmg}} x \leq \beta^+_{\text{bmg}}$ can be interpreted as an active view.

In order to illustrate the BMG constraint, we consider the previous examples and impose that the BMG sensitivity of the MV portfolio cannot be positive. Results are reported in Table 6. We notice that the invariance of the BMG sign is broken. For instance, we do not obtain the same solution between parameter sets #2 and #3. Moreover, we verify that the constrained MV portfolio promotes negative BMG sensitivities[37].

Table 6: Composition of the constrained MV portfolio ($\beta^+_{\text{bmg}} = 0$)

Asset	$\beta_{\text{mkt},i}$	Parameter set #1		Parameter set #2		Parameter set #3	
		$\beta_{\text{bmg},i}$	MV	$\beta_{\text{bmg},i}$	MV	$\beta_{\text{bmg},i}$	MV
1	0.90	−0.50	64.29	−1.50	0.00	1.50	0.00
2	0.80	0.70	0.00	−0.50	19.48	0.50	16.11
3	1.20	0.20	0.00	3.00	13.61	−3.00	25.89
4	0.70	0.90	35.71	−1.20	66.91	1.20	58.00
5	1.30	−0.30	0.00	−0.90	0.00	0.90	0.00
λ_{bmg}		65 bps		0		56 bps	

We consider again the MSCI World index universe at December 2018. Since asset managers are mostly interested in long-only portfolios, we only report the long-only MV portfolio. If we would like to impose that the BMG sensitivity is lower than −0.5, we obtain results given in Figure 11. The comparison with the previous results (Figure 10) shows that the MV portfolio tends to select assets with both a low MKT beta and a negative BMG sensitivity. Moreover, large weights are associated with large negative values of $\beta_{\text{bmg},i}$ on average. These results can be explained because the Lagrange coefficient λ_{bmg} is equal to 29 bps. Of course, the magnitude of the shrinkage depends on the value of β^+_{bmg}. The lower the BMG constraint, the higher the Lagrange coefficient. For instance, we report the relationship between β^+_{bmg} and λ_{bmg} in Figure 12. This trade-off is not free since it will also impact the volatility of the MV portfolio.

For some time now, an important preoccupation of asset managers and asset owners is about the GHG emissions associated with their investment portfolios. There are many portfolio carbon footprint metrics but most of them are not consistent over time because of the equity ownership approach. To overcome this issue, we use the weighted average carbon intensity (or WACI) recommended by the Task Force on Climate-related Financial Disclosures (TCFD, 2017):

$$\text{WACI}(x) = \sum_{i=1}^{n} x_i \cdot \mathcal{CI}_i$$

[37] We recall that this is not the case of the unconstrained MV portfolio, which better promotes weak BMG sensitivities.

Figure 11: Weights of the constrained MV portfolio ($\beta^+_{\text{bmg}} = -0.50$)

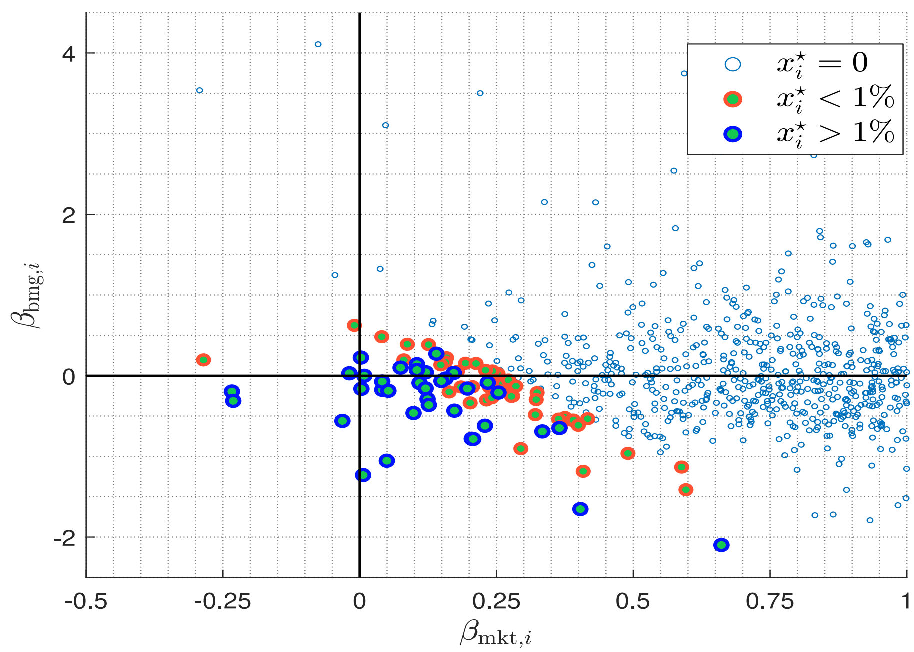


Figure 12: Relationship between β^+_{bmg} and λ_{bmg}

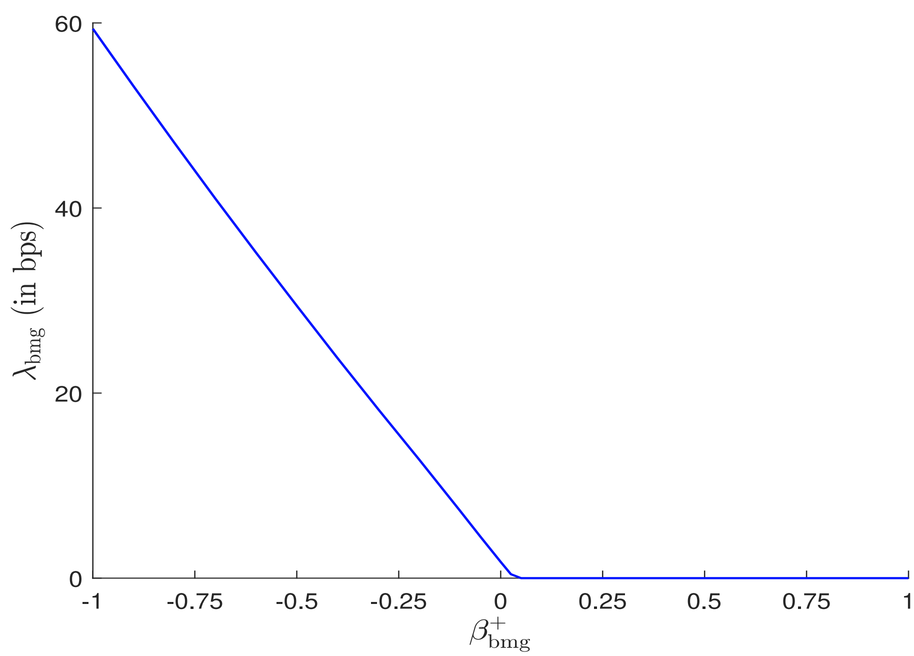
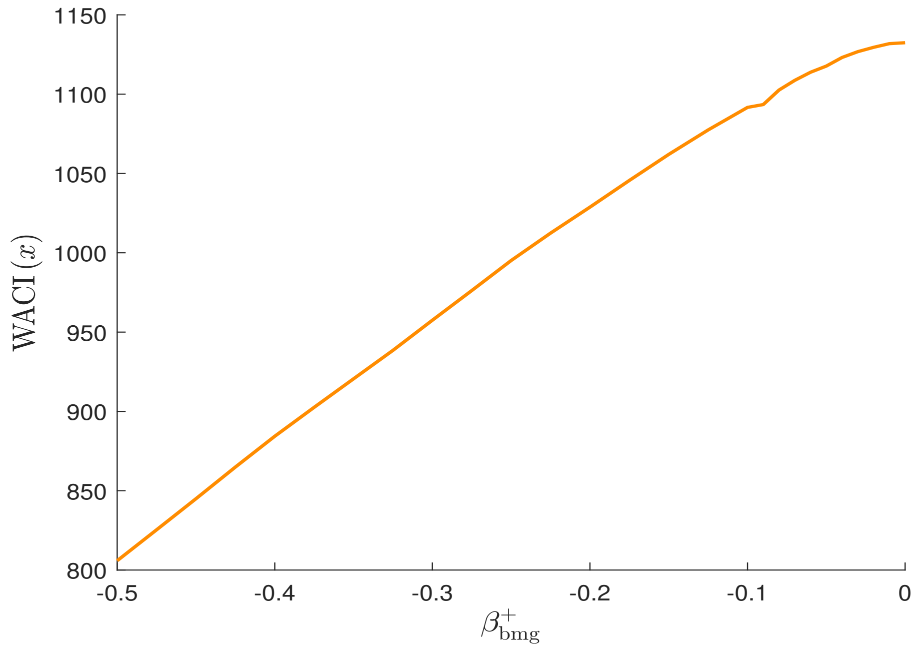


Figure 13: WACI of the constrained MV portfolio



where $\mathcal{CI}_i$ is the carbon intensity of issuer i. We define the carbon intensity $\mathcal{CI}$ as the issuer's direct and first-tier indirect GHG emissions[38] divided by the revenue. This measure is expressed in tons CO_2e per million dollars in revenue. In Figure 13, we have reported the relationship between WACI and β^{+}_{bmg} of the MV portfolio. We remark that the lower the relative carbon risk threshold, the lower the portfolio exposure to carbon-intensive companies. Nevertheless, the carbon intensities related to these different portfolios are very high in comparison with the equally-weighted portfolio whose WACI is approximatively equal to 315. The reason is that carbon footprint metrics cannot be necessarily interpreted as risk metrics. Some firms in the energy sector whose stock prices follow a singular pathway have both non-significant market and carbon betas, whereas they have a high carbon intensity. These unusual firms, whose carbon risk is captured in the idiosyncratic risk, contribute to dramatically increasing the WACI of the MV portfolio. In order to circumvent this issue, we can add a constraint to the MV problem:

$$
\begin{aligned}
x^{\star} &= \arg\min \frac{1}{2} x^{\top} \Sigma x \qquad (24)\\
&\text{s.t.} \quad \begin{cases} \mathbf{1}_n^{\top} x = 1 & \\ \beta_{\text{bmg}}^{\top} x \leq \beta^{+}_{\text{bmg}} & \\ x_i = 0 & \text{if } \mathcal{CI}_i > \mathcal{CI}^{+} \\ x \geq \mathbf{0}_n & \end{cases}
\end{aligned}
$$

where $\mathcal{CI}^{+}$ is a maximum carbon intensity threshold. We have reported in Figure 25 on page 55 the relationship between WACI and β^{+}_{bmg} of the MV portfolio when the carbon intensity threshold $\mathcal{CI}^{+}$ is equal to the WACI of the EW portfolio. In this case, we considerably

[38] The direct emissions correspond to the scope 1 emissions and the first-tier indirect emissions correspond to the GHG emissions of the firm's direct suppliers (scope 2 emissions + some upstream scope 3 emissions).

reduce the WACI regardless of the value of the portfolio carbon risk threshold β_{bmg}^{+}. Even if we impose $\beta_{\mathrm{bmg}}^{+} = 0$ and $\mathcal{CI}^{+} = 2\,000$, we obtain a WACI around 225, which is very low in comparison with the MV portfolio without a carbon intensity threshold. Figure 26 on page 55 provides the tradeoff between the portfolio volatility $\sigma(x)$, the carbon risk threshold β_{bmg}^{+} and the carbon intensity threshold $\mathcal{CI}^{+}$. This latter constraint has no substantial impact on portfolio volatility. Therefore, it is possible to reduce the weighted average carbon intensity without substantially increasing volatility.

Remark 8. *In order to highlight the difference between a market measure of carbon risk and a fundamental measure of carbon risk, we have reported in Figure 27 on page 56 the relationship between $\mathcal{CI}_i$ and $\beta_{\mathrm{bmg},i}$. On average, the linear correlation is equal to 17.5% for the Carima factor. It slightly increases if we consider the carbon intensity factor, but it remains lower than 30%.*

3.2 Enhanced index portfolio

3.2.1 Analysis of the optimization problem

Enhanced index portfolios may be obtained by considering the portfolio optimization method in the presence of a benchmark (Roncalli, 2013). For that, we define $b = (b_1, \ldots, b_n)$ and $x = (x_1, \ldots, x_n)$ as the asset weights in the benchmark and the portfolio. The tracking error between the active portfolio x and its benchmark b is the difference between the portfolio's return and the benchmark's return:

$$\begin{aligned} R\left(x \mid b\right) &= R(x) - R(b) \\ &= (x - b)^{\top} R \end{aligned}$$

where $R = (R_1, \ldots, R_n)$ is the vector of asset returns. The volatility of the tracking error $R\left(x \mid b\right)$ corresponds to the standard deviation of $R(x) - R(b)$:

$$\sigma\left(x \mid b\right) = \sqrt{(x - b)^{\top} \Sigma (x - b)}$$

The optimization problem of enhanced index portfolios consists in replacing the portfolio's volatility with the portfolio's tracking error volatility in a minimum variance framework and imposing long-only weights:

$$\begin{aligned} x^{\star} &= \arg\min \frac{1}{2} (x - b)^{\top} \Sigma (x - b) \\ & \text{s.t.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^{\top} x = 1 \\ x \geq \mathbf{0}_n \\ x \in \Omega \end{array} \right. \end{aligned} \tag{25}$$

If no other constraint is added ($\Omega = \mathbb{R}^n$), the optimal solution $x^{\star}$ is the benchmark b. But this framework only makes sense if we impose a second objective using the restriction $x \in \Omega$. For instance, we can impose that the optimal portfolio has a carbon beta less than a threshold as in the case of the minimum variance problem:

$$\Omega = \left\{ x \in \mathbb{R}^n : \beta_{\mathrm{bmg}}^{\top} x \leq \beta_{\mathrm{bmg}}^{+} \right\} \tag{26}$$

Another approach consists in excluding the first m stocks that present the largest carbon risk beta:

$$\Omega = \left\{ x \in \mathbb{R}^n : x_i = 0 \text{ if } \beta_{\mathrm{bmg},i} \geq \beta_{\mathrm{bmg}}^{(m,n)} \right\} \tag{27}$$

where $\beta_{\mathrm{bmg}}^{(m,n)} = \beta_{\mathrm{bmg},n-m+1:n}$ is the $(n-m+1)$-*th* order statistic of $\left(\beta_{\mathrm{bmg},1}, \ldots, \beta_{\mathrm{bmg},n}\right)$.

These two approaches are similar to the ones proposed by Andersson *et al.* (2016). The difference comes from the fact that we use a market measure to estimate the carbon risk, whereas Andersson *et al.* (2016) measured the carbon risk directly using the carbon intensity. The two methods have their own advantages and drawbacks. It is obvious that the method of Andersson *et al.* (2016) is more objective than our method, because it directly uses the carbon footprint of the issuer. However, it is difficult to know if the stock price is sensitive to this carbon footprint measure, especially since there are several carbon intensity measures[39]. We understand that the carbon footprint is an ecological environment risk for planet Earth. But it is less obvious that it corresponds to the carbon market risk which is priced in by the stock market at the security level. If this were the case, it would mean that two corporate firms with the same carbon footprint present the same carbon beta, regardless of the firms' other characteristics. Our method is less objective since the carbon risk is estimated through the dynamics of stock prices, and also depends on the methodology to build the carbon risk factor. Nevertheless, it is more relevant from a financial point of view, because we consider the carbon risk directly priced in by the stock market. In a sense, the first method has a longer-term horizon, whereas the second method is short-term by construction.

Remark 9. *We can replace the absolute threshold β_{bmg}^{+} with a relative threshold. Indeed, imposing a reduction of the carbon risk with respect to the benchmark, e.g. $\beta_{\mathrm{bmg}}^{\top}(x-b) \leq -\Delta_{\mathrm{bmg}}$, is equivalent to using an absolute threshold, e.g. $\beta_{\mathrm{bmg}}^{+} = \beta_{\mathrm{bmg}}^{\top} b - \Delta_{\mathrm{bmg}}$ where Δ_{bmg} corresponds to the relative or absolute difference between the benchmark's carbon risk and the threshold value of relative carbon risk.*

The mathematical analysis of the optimization problem (25) with the constraint (26) is given in Appendix A.4 on page 47. We show that $\Delta_i = x_i^\star - b_i$ is a decreasing function of the scaled BMG sensitivity $\breve{\beta}_{\mathrm{bmg},i}$, which is equal to $\left(\Sigma^{-1}\beta_{\mathrm{bmg}}\right)_i$. To illustrate this property, we consider the example used on page 23 (parameter set #1) and assume that the benchmark is the equally-weighted (EW) portfolio. Moreover, we impose that the relative carbon risk of the optimized portfolio is less than zero – $\beta_{\mathrm{bmg}}^{+} = 0$. Results are reported in Table 7. We verify that underweights and overweights depend on the sign of $\breve{\beta}_{\mathrm{bmg},i}$. If $\breve{\beta}_{\mathrm{bmg},i}$ is negative, Δ_i is positive and the asset is overweighted with respect to the benchmark. Otherwise, the asset is underweighted if $\breve{\beta}_{\mathrm{bmg},i}$ is positive.

Table 7: Enhanced index portfolio

Asset	b_i	$x_i^\star$	Δ_i	$\beta_{\mathrm{bmg},i}$	$\breve{\beta}_{\mathrm{bmg},i}$
1	20.00%	36.77	16.77%	−0.5	−56.38
2	20.00%	17.12	−2.88%	0.7	12.22
3	20.00%	11.61	−8.39%	0.2	29.46
4	20.00%	12.03	−7.97%	0.9	34.10
5	20.00%	22.48	2.48%	−0.3	−14.33

3.2.2 Application to the MSCI World index

The previous example gives the impression that underweights and overweights can also be predicted thanks to the BMG sensitivity $\beta_{\mathrm{bmg},i}$. In this example, $\beta_{\mathrm{bmg},i}$ and $\breve{\beta}_{\mathrm{bmg},i}$ have

[39] We generally distinguish scope 1, 2 and 3 carbon emissions. According to the Greenhouse Gas Protocol (2013), scope 1 corresponds to all direct emissions from the firm's activities, scope 2 includes indirect emissions from electricity purchased and used by the firm, whereas scope 3 measures all other indirect emissions from the firm's activities.

the same sign. In order to illustrate that the statistic $\beta_{\text{bmg},i}$ is less relevant than $\breve{\beta}_{\text{bmg},i}$, we apply the previous framework to the MSCI World index universe. We consider that the benchmark is the EW portfolio, and we impose that the BMG sensitivity is less than zero – $\beta^{+}_{\text{bmg}} = 0$. In Figures 14 and 15, we have reported the relationships between $\beta_{\text{bmg},i}$, $\breve{\beta}_{\text{bmg},i}$ and $\Delta_i = x^\star_i - b_i$. We verify that $\breve{\beta}_{\text{bmg},i}$ is a better statistic than $\beta_{\text{bmg},i}$ for estimating the weighting direction. Indeed, the relationship between $\beta_{\text{bmg},i}$ and $\Delta_i = x^\star_i - b_i$ is noisier than the relationship between $\breve{\beta}_{\text{bmg},i}$ and $\Delta_i = x^\star_i - b_i$. Moreover, in this last case, the relationship is almost linear.

In what follows, we always consider that the benchmark is the capitalization-weighted (CW) portfolio. We have again reported the relationships between $\beta_{\text{bmg},i}$, $\breve{\beta}_{\text{bmg},i}$ and $\Delta_i = x^\star_i - b_i$ in Figures 29 and 30 on page 57. In this case, we impose a relative carbon risk of the portfolio[40] less than -0.3 – $\beta^{+}_{\text{bmg}} = -0.3$. We can notice that some assets are not in line with the linear relationship between $\breve{\beta}_{\text{bmg},i}$ and $\Delta_i = x^\star_i - b_i$. Since we consider a long-only portfolio, these assets are excluded in the optimized portfolio. The slope of the relationship between $\breve{\beta}_{\text{bmg},i}$ and Δ_i is steeper in the current case than in the case where the benchmark is the EW portfolio for two reasons. The first one is that we have set a higher Δ_{bmg} and the second one is that a steeper slope allows us to offset the assets whose weights $x^\star_i$ have already reached the value of zero.

Remark 10. *Since the relationship between $\beta_{\text{bmg},i}$ and $\Delta_i = x^\star_i - b_i$ is not a monotonically decreasing function, the order-statistic optimization problem defined by the restriction (27) is not equivalent to the max-threshold optimization problem defined by the inequality (26).*

Table 8: Regional composition of the portfolio (in %)

Region	$b_{\mathcal{R}}$	$x^\star_{\mathcal{R}}$	$\Delta_{\mathcal{R}}$
Eurozone	10.89	12.77	1.88
Europe ex EMU	10.83	10.73	−0.09
North America	65.16	64.97	−0.19
Japan	8.70	8.90	0.19
Others	4.41	2.63	−1.79

Table 8 reports some results about the portfolio's regional exposure. Since the exposures of the capitalization-weighted and optimized portfolios to a region $\mathcal{R}$ are respectively equal to $b_{\mathcal{R}} = \sum_{i\in\mathcal{R}} b_i$ and $x^\star_{\mathcal{R}} = \sum_{i\in\mathcal{R}} x^\star_i$, the long/short regional exposure $\Delta_{\mathcal{R}}$ of the optimized portfolio with respect to the benchmark is equal to $\Delta_{\mathcal{R}} = x^\star_{\mathcal{R}} - b_{\mathcal{R}}$. The long/short exposure for the Eurozone and Japan is positive while it is negative for Europe ex EMU and North America. We notice that these results are consistent with the regional trend analysis provided by Figure 2 on page 11. Indeed, the higher a region's relative carbon risk, the lower the long/short exposure to a region. Long/short exposure to the Eurozone is high in absolute and relative values. Indeed, the optimized portfolio has a long exposure to the Eurozone of almost 17.2% higher than the benchmark. Nevertheless, the exposure to European ex EMU and North American stocks has not changed significantly with respect to the benchmark. This result is quite surprising for North America since the average relative carbon risk $\beta_{\text{bmg},\mathcal{R}}(t)$ was high in December 2018. For the rest of the world, long/short

[40] We impose a smaller threshold value of carbon risk when we consider the CW portfolio as the benchmark because the relative carbon risk is equal to 0.1491 for the EW portfolio whereas it is equal to -0.0851 for the CW portfolio. Indeed, Figure 28 on page 56 shows that the higher the market capitalisation of a firm, the lower its relative carbon risk. If we set $\beta^{+}_{\text{bmg}} = 0$ in the case where the benchmark is the CW portfolio, the optimized portfolio $x^\star$ is then equal to the benchmark b.

Figure 14: Relationship between $\beta_{\text{bmg},i}$ and $\Delta_i = x_i^\star - b_i$ for the EW benchmark

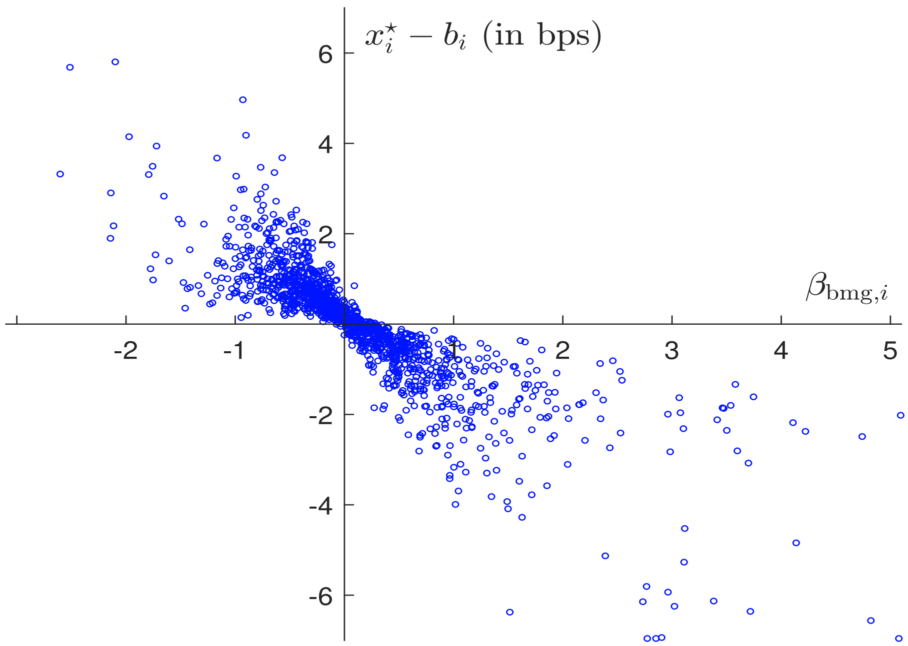
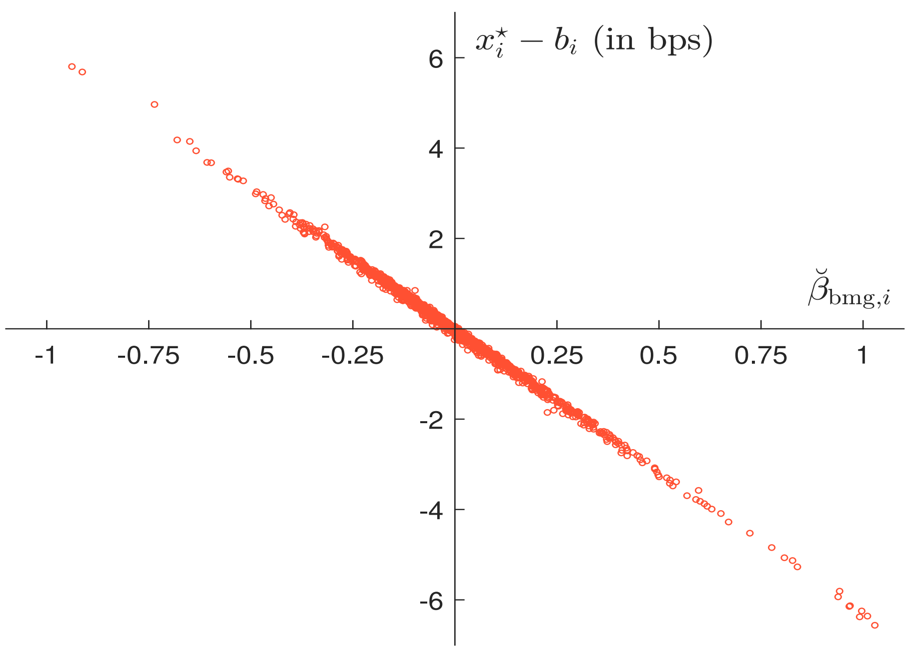


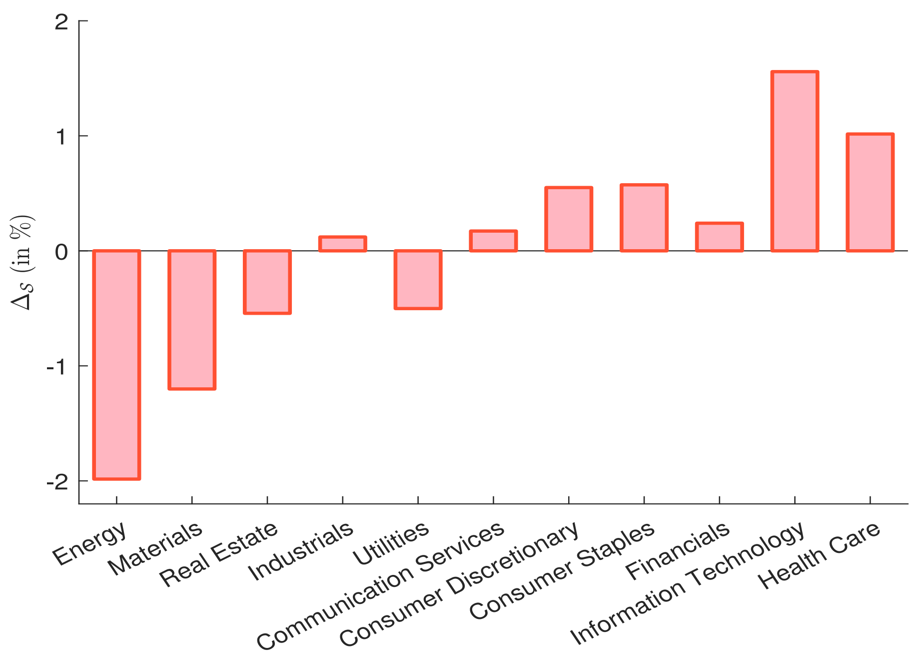
Figure 15: Relationship between $\breve{\beta}_{\text{bmg},i}$ and $\Delta_i = x_i^\star - b_i$ for the EW benchmark



exposure has significantly decreased by almost 40.5% with respect to the benchmark. We obtain such a result because of the very high average relative carbon risk $\beta_{\text{bmg},\mathcal{R}}(t)$ of the other regions which is around 0.75 at the end of December 2018.

We may also be interested in the sector composition of the optimized portfolio[41]. Figure 16 provides the portfolio's long/short exposure $\Delta_{\mathcal{S}}$ with respect to the benchmark, which is defined as $\Delta_{\mathcal{S}} = \sum_{i \in \mathcal{S}} \left(x_i^{\star} - b_i\right)$ where $\mathcal{S}$ is the sector. In this case, the difference (both in absolute and relative terms) between the CW and optimized portfolios is high. The energy sector is the most impacted. Indeed, the weight of the energy sector is 6.00% in the benchmark while it is equal to 4.02% in the optimized portfolio which represents a decrease of 33%. The materials sector closely follows the energy sector with a decrease of its exposure from 4.58% to 3.38%. The real estate and utilities sectors also see decreases in their weights, of 17.42% and 14.80%, respectively. In contrast, the sectors whose exposure has increased are mainly information technology and health care. Indeed, the weight increases from 14.95% to 16.51% for the information technology sector and 13.32% to 14.34% for the health care sector. The weight for the other sectors has changed slightly. Overall, the results are consistent with the results obtained in Figures 1, 4 and 5. In comparison with the benchmark, the optimized portfolio is more exposed to the sectors which are positively impacted by an unexpected acceleration in the transition process towards a green economy.

Figure 16: Long/short sector exposure $\Delta_{\mathcal{R}}$ of the portfolio (in %)



In what follows, we again consider that the benchmark is the CW portfolio. Figure 17 provides the relationships between the difference between the benchmark's carbon risk and the portfolio's carbon risk Δ_{bmg}, the tracking error $\sigma\left(x \mid b\right)$, the active share $\text{AS}\left(x \mid b\right)$, the number of excluding stocks $\mathcal{N}_0\left(x \mid b\right)$ and the weighted average carbon intensity WACI for the max-threshold optimization problem. We notice that the relationship between Δ_{bmg}

[41] We recall that $\beta_{\text{bmg}}^{+} = -0.3$.

and $\sigma\left(x \mid b\right)$ is linear. Indeed, we can demonstrate that[42]:

$$\sigma\left(x^{\star} \mid b\right) \approx c\Delta_{\text{bmg}}$$

By decreasing the relative carbon risk of the portfolio by 0.1, the tracking error increases by almost 65 bps whatever the initial value of the portfolio's carbon risk. In the current optimization problem, the active share remains relatively low for any value of Δ_{bmg}. Moreover, we verify that the higher the Δ_{bmg}, the lower the WACI.

Figure 17: Solution of the max-threshold optimization problem

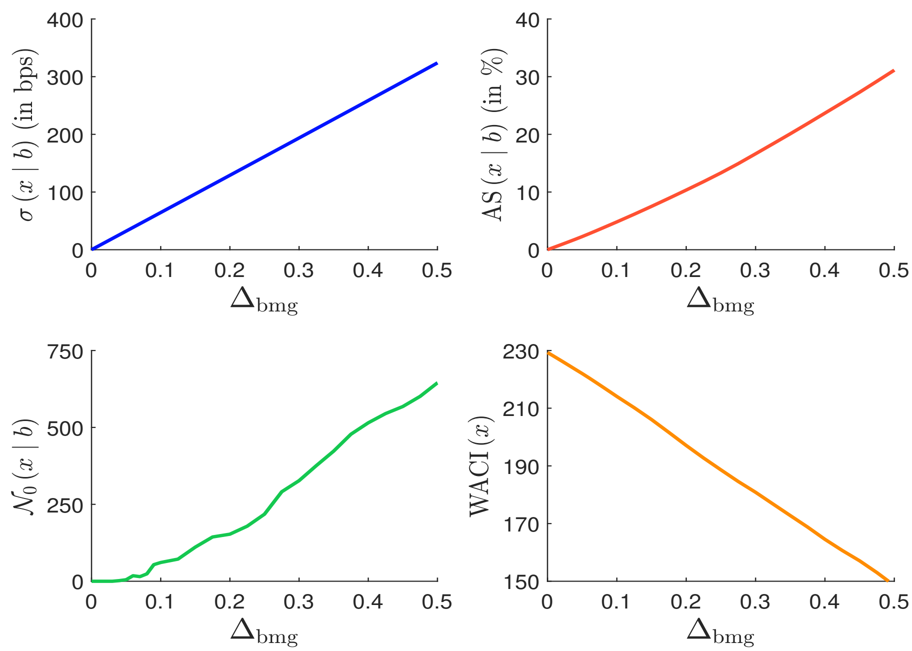


Figure 31 on page 58 provides the solution to the order-statistic optimization problem defined by the constraint (27). In this case, we solve a standard tracking error problem by having zero exposure to the first m stocks with higher carbon betas. We can notice that for a same level of relative carbon risk tolerance, the tracking error and the active share are always larger in the case of the order-statistic problem in comparison with the max-threshold problem. By excluding the assets with the higher relative carbon risk, the optimization problem excludes itself the assets with a low relative carbon risk in order to have a carbon exposure similar to the benchmark. Hence, we must exclude a large number of stocks to move from $\Delta_{\text{bmg}} = 0$ to $\Delta_{\text{bmg}} = 0.1$. This implies both high active share and tracking error. For instance, the tracking error is almost twice that of the max-threshold optimization problem when Δ_{bmg} is equal to 0.1.

Some variants can be used to define the enhanced index portfolio problem. For instance, we can use the following constraint:

$$\Omega = \left\{ x \in \mathbb{R}^n : x_i = 0 \text{ if } b_i \beta_{\text{bmg},i} \geq \left(b \odot \beta_{\text{bmg}}\right)^{(m,n)} \right\} \tag{28}$$

where $\left(b \odot \beta_{\text{bmg}}\right)^{(m,n)} = \left(b \odot \beta_{\text{bmg}}\right)_{n-m+1:n}$ is the $(n-m+1)$-*th* order statistic of the vector $\left(b_1\beta_{\text{bmg},1}, \ldots, b_n\beta_{\text{bmg},n}\right)$. In this case, we exclude the assets with both high weight

[42] The semi-formal proof is given in Appendix A.4.3 on page 50.

and high carbon beta in order to considerably reduce the portfolio's relative carbon risk. Figure 32 on page 58 provides the solution to this optimization problem. Surprising, the outcomes are slightly less successful than those of the standard order-statistic problem. Therefore, we can deduce that excluding the assets with both high weight and high relative carbon risk implies that the tracking error is more impacted.

3.2.3 New formulation of the optimization problem with absolute carbon risk

For the minimum variance portfolio, we have seen that managing the relative or absolute carbon risk leads to two different portfolio optimization programs. In the previous section, we considered the case where the fund manager would like to reduce exposure to the brownest stocks. In order to decrease the BMG sensitivity, the optimized portfolio increases its exposure to the greenest stocks. This is normal if the investor's moral values are to fight climate risk. However, this implies taking a bet particularly in the short run. Indeed, if the investor chooses a negative value of β^{+}_{bmg} (e.g. $\beta^{+}_{\mathrm{bmg}} = -30\%$), his optimized investment portfolio may substantially underperform the benchmark if the BMG factor posts a positive performance. Therefore, by choosing a negative BMG sensitivity, the investor expects that the performance of the BMG factor will be negative in the future.

Another enhanced index management approach is to limit the exposure to absolute carbon risk. In this case, we obtain the same optimization problem (25), but with another system of constraints:

$$\Omega = \left\{ x \in \mathbb{R}^n : \left| \beta_{\mathrm{bmg}}^{\top} x \right| \leq |\beta|^{+}_{\mathrm{bmg}} \right\}$$

where $|\beta|^{+}_{\mathrm{bmg}}$ is the maximum sensitivity to absolute carbon risk. This is equivalent to imposing the following inequality constraints[43]:

$$\begin{pmatrix} \beta_{\mathrm{bmg}} & -\beta_{\mathrm{bmg}} \end{pmatrix}^{\top} x \leq \begin{pmatrix} |\beta|^{+}_{\mathrm{bmg}} \\ |\beta|^{+}_{\mathrm{bmg}} \end{pmatrix}$$

Again, we obtain a QP problem, which is easy to solve. The special case $|\beta|^{+}_{\mathrm{bmg}} = 0$ corresponds to the neutral exposure to the absolute carbon risk[44]. From the viewpoint of passive management, imposing that $\beta_{\mathrm{bmg}}^{\top} x = 0$ can be justified because the objective of passive management is to implement no active bets.

Remark 11. *While we have two specific optimization programs in the case of the minimum variance portfolios, the boundary between relative and absolute carbon risk is not obvious when we consider tracking error optimization problems. Indeed, imposing a neutral absolute carbon risk is equivalent to using an inequality constraint on the relative carbon risk*[45].

3.3 A factor investing perspective

Bennani *et al.* (2018) and Drei *et al.* (2019) discussed the relationships between ESG investing and factor investing. Among the different results, they showed that ESG may be

[43] Indeed, we have:

$$\left| \beta_{\mathrm{bmg}}^{\top} x \right| \leq |\beta|^{+}_{\mathrm{bmg}} \Leftrightarrow -|\beta|^{+}_{\mathrm{bmg}} \leq \beta_{\mathrm{bmg}}^{\top} x \leq +|\beta|^{+}_{\mathrm{bmg}} \Leftrightarrow \left\{ \begin{array}{l} \beta_{\mathrm{bmg}}^{\top} x \leq |\beta|^{+}_{\mathrm{bmg}} \\ |\beta|^{+}_{\mathrm{bmg}} \geq -\beta_{\mathrm{bmg}}^{\top} x \end{array} \right.$$

[44] In this case, the inequality constraints are replaced by the equality constraint $\beta_{\mathrm{bmg}}^{\top} x = 0$, and the optimization program remains a QP problem.

[45] If $\beta_{\mathrm{bmg}}^{\top} b > 0$ (resp. $\beta_{\mathrm{bmg}}^{\top} b < 0$), a neutral absolute carbon risk is achieved using the constraint $\beta_{\mathrm{bmg}}^{\top} x \leq 0$ (resp. $\beta_{\mathrm{bmg}}^{\top} x \geq 0$).

considered as a beta strategy in the Eurozone while it continues to remain an alpha strategy in North America (Roncalli, 2020b). We may wonder if the carbon risk can play a similar role from a factor investing point of view. Indeed, the results obtained in Section Two show that it helps to explain the cross-section of stock returns.

We recall that the objective of factor investing is to build a well-diversified portfolio of risk factors to better capture the equity risk premium. The underlying idea is that there is not only one risk premium, but several risk premia, implying that the equity risk premium cannot be reduced to the CAPM risk factor. The traditional approach is then to capture the equity risk premia with five risk factors: size, value, momentum, low-volatility and quality.

Based on the previous analysis, it is tempting to include a carbon risk factor as a sixth risk premium. However, we think that carbon risk is too specific and cannot be considered as a risk premium. First, it is obviously not a skewness risk premium (Roncalli, 2017). Second, it may be a market anomaly, but we must be cautious with the '*factor zoo*' (Cochrane, 2011). Indeed, we think that the ESG risk factor does make more sense than the carbon risk factor in a factor investing framework, because it represents a broader investment type. Moreover, ESG investing is today a big investment topic of institutional investors, whereas carbon risk is embedded in the Environment pillar of ESG. Carbon risk is more a risk management subject than an investing approach. This is why we speak about ESG investing, but not about carbon investing.

Nevertheless, as shown previously, carbon risk is a financial risk and is interesting to manage. Since equity factor investing is benchmarked against capitalization-weighted indices, we can use the framework developed in Section 3.2 on page 29. Instead of using a bottom-up approach to define a carbon risk factor, it is easier to implement an overlay or a top-down approach that controls the relative or absolute carbon risk of the factor investing portfolio with respect to its benchmark.

4 Conclusion

This paper studies the methodology proposed by Görgen *et al.* (2019) for measuring carbon risk in investment portfolios. We confirm the results of these authors, that showed that carbon risk is priced in at the stock level and is relevant in a cross-sectional multi-factor analysis. By considering a dynamic framework, we highlight several stylized facts. First, we notice that carbon risk was priced in more at the beginning of the 2010s than it is today. Nevertheless, this is mainly due to the Eurozone. Second, we observe a convergence of absolute carbon risk pricing among the different regions, except in Japan. If we focus on relative carbon risk, we confirm another transatlantic divide that we generally observe in ESG investing (Bennani *et al.*, 2018; Drei *et al.*, 2019). On average, European stocks have a negative carbon beta, whereas it is positive for North America. We also observe some differences between sectors. For instance, there is clearly a difference in the dynamics of the carbon beta between the materials and energy sectors, and the other sectors.

Because the brown-minus-green (BMG) factor developed by Görgen *et al.* (2019) is based on more than 50 proxy variables, investors may have some difficulty understanding which risk dimension is priced in by the market. For each stock, these authors calculate a brown green score, which is based on three dimensions: value chain, public perception and adaptability. Each dimension is the result of mixing several sub-dimensions. In this article, we have preferred to consider very basic dimensions based on the Trucost and MSCI databases. We have focused our analysis on carbon intensity data provided by Trucost, and data on carbon emissions exposure, carbon management, the climate change score and the environment pillar provided by MSCI. All these dimensions have an explanatory power that is in line with the Carima factor of Görgen *et al.* (2019). However, at first order, we find that carbon intensity and carbon emissions exposure are the best alternative approaches to the Carima factor.

ESG rating agencies and other NGOs have developed many fundamental measures and scores to assess a firm's carbon risk. Carbon beta is not a new measure. However, the approach of Görgen *et al.* (2019) is definitively original since it is a market-based measure and not a fundamental-based measure. The carbon beta of a stock corresponds to the carbon risk of the stock priced in by the financial market. Therefore, we may observe wide discrepancies between the market perception of the carbon risk, and for instance the direct value of the carbon intensity. Our analysis shows that they are weakly correlated (less than 30%). We can draw a parallel with the value risk factor. Indeed, the value beta of a stock may be related to its book-to-price, but they are two different measures. And the financial market may consider that a stock with a high book-to-price is not necessarily a value stock, but a growth stock.

The carbon beta therefore constitutes useful information for managing carbon risk in investment portfolios. In this article, we have mainly studied two investment strategies: the minimum variance portfolio and the enhanced index portfolio. Our results highlight the difference between managing absolute carbon risk and relative carbon risk. Managing absolute carbon risk implies having zero exposure to the BMG factor, whereas managing relative carbon risk implies having negative exposure to the BMG factor. In the first case, the objective is to propose an immunization-hedging investment strategy against carbon risk. In the second case, we explicitly take an active management bet by overweighting green stocks and underweighting brown stocks. This second approach is certainly the most frequently observed, even in passive management, because of investors' moral values. We show that the two approaches led us to consider different objective functions or constraints of the portfolio optimization program, implying that we obtain very different solutions. Another finding is that managing market-based carbon risk and fundamental-based carbon risk[46] does not give the same solution, even though we observe similar properties between the optimized portfolios. This is particularly true in the case of enhanced index portfolios. Finally, we discuss whether the BMG factor can be considered as a new factor or not, alongside traditional factors (size, value, momentum, etc.). Our conviction is that carbon risk is a risk management subject, and not an investment style such as ESG investing. This is why we consider that carbon risk is more appropriate for better defining a minimum variance portfolio than improving the diversification of a factor investing portfolio. Nevertheless, all our results confirm the initial findings of Görgen *et al.* (2019). Investors must be aware that carbon risk is priced in by the stock market. This is why they must measure and manage this risk, especially when it is too high or when it is incompatible with the fiduciary duties of their investment portfolios.

[46] By considering for instance a direct measure of carbon risk such as carbon intensity.

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Appendix

A Mathematical results

A.1 Time-varying estimation with Kalman filter

The time-varying risk factor model can be written as a state space model:

$$\left\{ \begin{array}{l} y(t) = x(t)^\top \beta(t) + \varepsilon(t) \\ \beta(t) = \beta(t-1) + \eta(t) \end{array} \right. \tag{29}$$

where $\varepsilon(t) \sim \mathcal{N}(0, \sigma_\varepsilon^2)$, $\eta(t) \sim \mathcal{N}(\mathbf{0}_{K+1}, \Sigma_\beta)$ and K is the number of risk factors. For instance, in the case of the MKT+BMG model, $y(t)$ corresponds to the asset return $R_i(t)$, $x(t)$ is a 3×1 vector, whose elements are 1, $R_{\text{mkt}}(t)$ and $R_{\text{bmg}}(t)$ and:

$$\beta(t) = \left(\begin{array}{c} \alpha_i(t) \\ \beta_{\text{mkt},i}(t) \\ \beta_{\text{bmg},i}(t) \end{array} \right) \tag{30}$$

It follows that the variable y_t is observable, but this is not the case for the state vector $\beta(t)$. The Kalman filter is a statistical tool to estimate the distribution function of $\beta(t)$. Let $\beta(0) \sim \mathcal{N}(\beta_0, P_0)$ be the initial position of the state vector. We note $\hat{\beta}(t \mid t-1) = \mathbb{E}[\beta(t) \mid \mathcal{F}(t-1)]$ and $\hat{\beta}(t \mid t) = \mathbb{E}[\beta(t) \mid \mathcal{F}(t)]$ as the optimal estimators of $\beta(t)$ given the available information until time $t-1$ and t. $P(t \mid t-1)$ and $P(t \mid t)$ are the covariance matrices associated with $\hat{\beta}(t \mid t-1)$ and $\hat{\beta}(t \mid t)$. Since the estimate of $y(t)$ is equal to $\hat{y}(t \mid t-1) = x(t)^\top \hat{\beta}(t \mid t-1)$, we can compute the variance $F(t)$ of the innovation process $v(t) = y(t) - \hat{y}(t \mid t-1)$. These different quantities can be calculated thanks to the Kalman filter, which consists in the following recursive algorithm[47] (Roncalli, 2020a, page 654):

$$\left\{ \begin{array}{l} \hat{\beta}(t \mid t-1) = \hat{\beta}(t-1 \mid t-1) \\ P(t \mid t-1) = P(t-1 \mid t-1) + \Sigma_\beta \\ v(t) = y(t) - x(t)^\top \hat{\beta}(t \mid t-1) \\ F(t) = x(t)^\top P(t \mid t-1)\, x(t) + \sigma_\varepsilon^2 \\ \hat{\beta}(t \mid t) = \hat{\beta}(t \mid t-1) + \left(\dfrac{P(t \mid t-1)}{F(t)} \right) x(t)\, v(t) \\ P(t \mid t) = \left(I_{K+1} - \left(\dfrac{P(t \mid t-1)}{F(t)} \right) x(t)\, x(t)^\top \right) P(t \mid t-1) \end{array} \right. \tag{31}$$

In this model, the parameters σ_u^2 and Σ_β are unknown and can be estimated by the method of maximum likelihood. Since $v(t) \sim \mathcal{N}(0, F(t))$, the log-likelihood function is equal to:

$$\boldsymbol{\ell}(\theta) = -\frac{T}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^{T} \left(\ln F(t) + \frac{v^2(t)}{F(t)} \right) \tag{32}$$

where $\theta = (\sigma^2, \Sigma)$. Maximizing the log-likelihood function requires specifying the initial conditions β_0 and P_0, which are not necessarily known. In this case, we use the linear regression $y(t) = x(t)^\top \beta + \varepsilon(t)$, and the OLS estimates $\hat{\beta}_{\text{ols}}$ and $\hat{\sigma}_\varepsilon^2 \left(X^\top X\right)^{-1}$ to initialize β_0 and P_0.

[47] The algorithm is initialized with values $\hat{\beta}(0 \mid 0) = \beta_0$ and $P(0 \mid 0) = P_0$.

A.2 Sherman-Morrison-Woodbury formula

Suppose u and v are two $n \times 1$ vectors and A is an invertible $n \times n$ matrix. We can show that (Golub and Van Loan, 2013):

$$\left(A + uv^\top\right)^{-1} = A^{-1} - \frac{1}{1 + v^\top A^{-1} u} A^{-1} uv^\top A^{-1} \tag{33}$$

Batista (2008) and Batista and Karawia (2009) extended the SMW formula when the outer product is a sum:

$$\left(A + \sum_{k=1}^{m} u_k v_k^\top\right)^{-1} = A^{-1} - A^{-1} U S^{-1} V^\top A^{-1} \tag{34}$$

where $U = \begin{pmatrix} u_1 & \cdots & u_m \end{pmatrix}$ and $V = \begin{pmatrix} v_1 & \cdots & v_m \end{pmatrix}$ are two $n \times m$ matrices, and $S = I + T$ and $T = \left(T_{i,j}\right)$ are two $m \times m$ matrices where $T_{i,j} = v_i^\top A^{-1} u_j$.

In the case $m = 2$, the SMW formula becomes:

$$\left(A + u_1 v_1^\top + u_2 v_2^\top\right)^{-1} = A^{-1} - A^{-1} U S^{-1} V^\top A^{-1} \tag{35}$$

where:

$$S = \begin{pmatrix} 1 + v_1^\top A^{-1} u_1 & v_1^\top A^{-1} u_2 \\ v_2^\top A^{-1} u_1 & 1 + v_2^\top A^{-1} u_2 \end{pmatrix}$$

Since we have:

$$S^{-1} = \frac{1}{|S|} \begin{pmatrix} 1 + v_2^\top A^{-1} u_2 & -v_1^\top A^{-1} u_2 \\ -v_2^\top A^{-1} u_1 & 1 + v_1^\top A^{-1} u_1 \end{pmatrix}$$

where

$$\begin{aligned} |S| &= \left(1 + v_1^\top A^{-1} u_1\right)\left(1 + v_2^\top A^{-1} u_2\right) - v_2^\top A^{-1} u_1 v_1^\top A^{-1} u_2 \\ &= 1 + v_1^\top A^{-1} u_1 + v_2^\top A^{-1} u_2 + v_1^\top A^{-1} u_1 v_2^\top A^{-1} u_2 - v_2^\top A^{-1} u_1 v_1^\top A^{-1} u_2 \end{aligned}$$

We deduce that:

$$\begin{aligned} |S| \cdot U S^{-1} V^\top &= \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} 1 + v_2^\top A^{-1} u_2 & -v_1^\top A^{-1} u_2 \\ -v_2^\top A^{-1} u_1 & 1 + v_1^\top A^{-1} u_1 \end{pmatrix} \begin{pmatrix} v_1^\top \\ v_2^\top \end{pmatrix} \\ &= u_1 v_1^\top + u_1 v_2^\top A^{-1} u_2 v_1^\top - u_2 v_2^\top A^{-1} u_1 v_1^\top - \\ &\quad u_1 v_1^\top A^{-1} u_2 v_2^\top + u_2 v_2^\top + u_2 v_1^\top A^{-1} u_1 v_2^\top \end{aligned}$$

If A is a diagonal matrix, we can simplify the previous SMW formula (35). Indeed, we have:

$$\begin{aligned} |S| &= 1 + \sum_{s=1}^{n} \frac{u_{1,s} v_{1,s} + u_{2,s} v_{2,s}}{a_{s,s}} + \sum_{s=1}^{n} \frac{u_{1,s} v_{1,s}}{a_{s,s}} \sum_{s=1}^{n} \frac{u_{2,s} v_{2,s}}{a_{s,s}} - \\ &\quad \sum_{s=1}^{n} \frac{u_{1,s} v_{2,s}}{a_{s,s}} \sum_{s=1}^{n} \frac{u_{2,s} v_{1,s}}{a_{s,s}} \end{aligned} \tag{36}$$

and[48]:

$$\begin{aligned}|S|\cdot US^{-1}V^{\top} &= \left(1+\sum_{s=1}^{n}\frac{u_{2,s}v_{2,s}}{a_{s,s}}\right)u_1v_1^{\top}+\left(1+\sum_{s=1}^{n}\frac{u_{1,s}v_{1,s}}{a_{s,s}}\right)u_2v_2^{\top}- \\ &\quad \left(\sum_{s=1}^{n}\frac{u_{1,s}v_{2,s}}{a_{s,s}}\right)u_2v_1^{\top}-\left(\sum_{s=1}^{n}\frac{u_{2,s}v_{1,s}}{a_{s,s}}\right)u_1v_2^{\top} \qquad (37)\end{aligned}$$

If we assume that u_1 is uncorrelated to v_2, u_2 is uncorrelated to v_1, u_2 and v_2 are centered around 0, we obtain:

$$|S|\approx 1+\sum_{s=1}^{n}\frac{u_{1,s}v_{1,s}+u_{2,s}v_{2,s}}{a_{s,s}}+\sum_{s=1}^{n}\frac{u_{1,s}v_{1,s}}{a_{s,s}}\sum_{s=1}^{n}\frac{u_{2,s}v_{2,s}}{a_{s,s}} \qquad (38)$$

and:

$$|S|\cdot US^{-1}V^{\top}\approx\left(1+\sum_{s=1}^{n}\frac{u_{2,s}v_{2,s}}{a_{s,s}}\right)u_1v_1^{\top}+\left(1+\sum_{s=1}^{n}\frac{u_{1,s}v_{1,s}}{a_{s,s}}\right)u_2v_2^{\top} \qquad (39)$$

A.3 Minimum variance portfolio in the MKT+BMG model

Remark 12. *In what follows, we use the notations β_i and γ_i instead of $\beta_{\mathrm{mkt},i}$ and $\beta_{\mathrm{bmg},i}$ to simplify the notations.*

We have:

$$R_i(t)=\alpha_i+\beta_iR_{\mathrm{mkt}}(t)+\gamma_iR_{\mathrm{bmg}}(t)+\varepsilon_i(t)$$

It follows that the covariance matrix is:

$$\Sigma=\beta\beta^{\top}\sigma_{\mathrm{mkt}}^2+\gamma\gamma^{\top}\sigma_{\mathrm{bmg}}^2+D$$

where $\beta=(\beta_1,\ldots,\beta_n)$ is the vector of MKT betas, $\gamma=(\gamma_1,\ldots,\gamma_n)$ is the vector of BMG betas, σ_{mkt}^2 is the variance of the market portfolio, σ_{bmg}^2 is the variance of the BMG factor and $D=\operatorname{diag}\left(\tilde\sigma_1^2,\ldots,\tilde\sigma_n^2\right)$ is the diagonal matrix of specific variances. We recall that the GMV portfolio is equal to:

$$\begin{aligned}x^{\star} &= \frac{\Sigma^{-1}\mathbf{1}_n}{\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n} \\ &= \sigma^2(x^{\star})\cdot\Sigma^{-1}\mathbf{1}_n \qquad (40)\end{aligned}$$

because we have:

$$\begin{aligned}\sigma^2(x^{\star}) &= {x^{\star}}^{\top}\Sigma x^{\star} \\ &= \frac{\mathbf{1}_n^{\top}\Sigma^{-1}}{\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n}\Sigma\frac{\Sigma^{-1}\mathbf{1}_n}{\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n} \\ &= \frac{1}{\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n}\end{aligned}$$

[48] Let $A=(a_{i,j})$ and $C=(c_{i,j})$ be two $n\times n$ matrices and b a vector of dimension n. We recall that:

$$\left(A\operatorname{diag}(b)\,C\right)_{i,j}=\sum_{s=1}^{n}a_{i,s}b_sc_{s,j}$$

We deduce that:

$$\left(a_1a_2^{\top}\operatorname{diag}(b)\,c_1c_2^{\top}\right)_{i,j}=a_{1,i}c_{2,j}\sum_{s=1}^{n}a_{2,s}b_sc_{1,s}$$

where a_1, a_2, $c_1$and a_2 are $n\times 1$ vectors.

A.3.1 General formula

We use the generalized Sherman-Morrison-Woodbury with $A = D$, $u_1 = v_1 = \sigma_{\mathrm{mkt}}\beta$ and $u_2 = v_2 = \sigma_{\mathrm{bmg}}\gamma$. It follows that the inverse of the covariance matrix is equal to:

$$\Sigma^{-1} = D^{-1} - D^{-1}US^{-1}V^{\top}D^{-1}$$

where $U = V = \begin{pmatrix} \sigma_{\mathrm{mkt}}\beta & \sigma_{\mathrm{bmg}}\gamma \end{pmatrix}$ and:

$$S = \begin{pmatrix} 1 + \sigma_{\mathrm{mkt}}^2 \beta^{\top} D^{-1}\beta & \sigma_{\mathrm{mkt}}\sigma_{\mathrm{bmg}}\beta^{\top}D^{-1}\gamma \\ \sigma_{\mathrm{mkt}}\sigma_{\mathrm{bmg}}\beta^{\top}D^{-1}\gamma & 1 + \sigma_{\mathrm{bmg}}^2 \gamma^{\top}D^{-1}\gamma \end{pmatrix}$$

We notice that:

$$\begin{aligned} \sum_{s=1}^{n} \frac{u_{1,s}v_{1,s}}{a_{s,s}} &= \sigma_{\mathrm{mkt}}^2 \sum_{s=1}^{n} \frac{\beta_s^2}{\tilde{\sigma}_s^2} = \sigma_{\mathrm{mkt}}^2 \varphi\left(\tilde{\beta}, \beta\right) \\ \sum_{s=1}^{n} \frac{u_{2,s}v_{2,s}}{a_{s,s}} &= \sigma_{\mathrm{bmg}}^2 \sum_{s=1}^{n} \frac{\gamma_s^2}{\tilde{\sigma}_s^2} = \sigma_{\mathrm{bmg}}^2 \varphi\left(\tilde{\gamma}, \gamma\right) \\ \sum_{s=1}^{n} \frac{u_{1,s}v_{2,s}}{a_{s,s}} &= \sum_{s=1}^{n} \frac{u_{2,s}v_{1,s}}{a_{s,s}} = \sigma_{\mathrm{mkt}}\sigma_{\mathrm{bmg}} \sum_{s=1}^{n} \frac{\beta_s\gamma_s}{\tilde{\sigma}_s^2} = \sigma_{\mathrm{mkt}}\sigma_{\mathrm{bmg}}\varphi\left(\tilde{\beta}, \gamma\right) \end{aligned}$$

where $\tilde{\beta}$ and $\tilde{\gamma}$ are the standardized vectors of β and γ by the idiosyncratic variances and $\varphi(x, y) = x \circ y$ is the outer product. Using Equation (36), we obtain:

$$|S| = 1 + \sigma_{\mathrm{mkt}}^2 \varphi\left(\tilde{\beta}, \beta\right) + \sigma_{\mathrm{bmg}}^2 \varphi\left(\tilde{\gamma}, \gamma\right) + \sigma_{\mathrm{mkt}}^2\sigma_{\mathrm{bmg}}^2 \left(\varphi\left(\tilde{\beta}, \beta\right)\varphi\left(\tilde{\gamma}, \gamma\right) - \varphi^2\left(\tilde{\beta}, \gamma\right)\right)$$

Equation (37) becomes:

$$\begin{aligned} |S| \cdot US^{-1}V^{\top} &= \sigma_{\mathrm{mkt}}^2 \left(1 + \sigma_{\mathrm{bmg}}^2 \varphi\left(\tilde{\gamma}, \gamma\right)\right)\beta\beta^{\top} + \sigma_{\mathrm{bmg}}^2 \left(1 + \sigma_{\mathrm{mkt}}^2 \varphi\left(\tilde{\beta}, \beta\right)\right)\gamma\gamma^{\top} - \\ &\quad \sigma_{\mathrm{mkt}}^2\sigma_{\mathrm{bmg}}^2 \varphi\left(\tilde{\beta}, \gamma\right)\left(\gamma\beta^{\top} + \beta\gamma^{\top}\right) \end{aligned}$$

Finally, the inverse of the covariance matrix has the following expression:

$$\Sigma^{-1} = D^{-1} - M^{-1}$$

where:

$$M^{-1} = \omega_1 \tilde{\beta}\tilde{\beta}^{\top} + \omega_2 \tilde{\gamma}\tilde{\gamma}^{\top} - \omega_3\left(\tilde{\gamma}\tilde{\beta}^{\top} + \tilde{\beta}\tilde{\gamma}^{\top}\right)$$

and:

$$\begin{aligned} \omega_0 &= |S| \\ \omega_1 &= \omega_0^{-1} \cdot \sigma_{\mathrm{mkt}}^2 \left(1 + \sigma_{\mathrm{bmg}}^2 \varphi\left(\tilde{\gamma}, \gamma\right)\right) \\ \omega_2 &= \omega_0^{-1} \cdot \sigma_{\mathrm{bmg}}^2 \left(1 + \sigma_{\mathrm{mkt}}^2 \varphi\left(\tilde{\beta}, \beta\right)\right) \\ \omega_3 &= \omega_0^{-1} \cdot \sigma_{\mathrm{mkt}}^2\sigma_{\mathrm{bmg}}^2 \varphi\left(\tilde{\beta}, \gamma\right) \end{aligned}$$

Therefore, the solution (40) becomes:

$$x^{\star} = \sigma^2\left(x^{\star}\right)\left(D^{-1}\mathbf{1}_n - M^{-1}\mathbf{1}_n\right)$$

It follows that:

$$x^{\star} = \sigma^{2}\left(x^{\star}\right)\left(\xi - \tilde{\omega}_{1}\tilde{\beta} - \tilde{\omega}_{2}\tilde{\gamma}\right) \tag{41}$$

where $\xi = \left(\tilde{\sigma}_{1}^{-2}, \ldots, \tilde{\sigma}_{n}^{-2}\right)$, $\tilde{\omega}_{1} = \omega_{1}\tilde{\beta}^{\top}\mathbf{1}_{n} - \omega_{3}\tilde{\gamma}^{\top}\mathbf{1}_{n}$ and $\tilde{\omega}_{2} = \omega_{2}\tilde{\gamma}^{\top}\mathbf{1}_{n} - \omega_{3}\tilde{\beta}^{\top}\mathbf{1}_{n}$. We conclude that:

$$x_{i}^{\star} = \frac{\sigma^{2}\left(x^{\star}\right)}{\tilde{\sigma}_{i}^{2}}\left(1 - \frac{\beta_{i}}{\beta^{\star}} - \frac{\gamma_{i}}{\gamma^{\star}}\right) \tag{42}$$

where $\beta^{\star} = \tilde{\omega}_{1}^{-1}$ and $\gamma^{\star} = \tilde{\omega}_{2}^{-1}$.

Remark 13. *If we develop the expression of $\beta^{\star}$ and $\gamma^{\star}$, we obtain:*

$$\beta^{\star} = \frac{1 + \sigma_{\mathrm{mkt}}^{2}\varphi\left(\tilde{\beta}, \beta\right) + \sigma_{\mathrm{bmg}}^{2}\varphi\left(\tilde{\gamma}, \gamma\right) + \sigma_{\mathrm{mkt}}^{2}\sigma_{\mathrm{bmg}}^{2}\left(\varphi\left(\tilde{\beta}, \beta\right)\varphi\left(\tilde{\gamma}, \gamma\right) - \varphi^{2}\left(\tilde{\beta}, \gamma\right)\right)}{\sigma_{\mathrm{mkt}}^{2}\left(\tilde{\beta}^{\top}\mathbf{1}_{n} + \sigma_{\mathrm{bmg}}^{2}\left(\varphi\left(\tilde{\gamma}, \gamma\right)\tilde{\beta}^{\top}\mathbf{1}_{n} - \varphi\left(\tilde{\beta}, \gamma\right)\tilde{\gamma}^{\top}\mathbf{1}_{n}\right)\right)} \tag{43}$$

and:

$$\gamma^{\star} = \frac{1 + \sigma_{\mathrm{mkt}}^{2}\varphi\left(\tilde{\beta}, \beta\right) + \sigma_{\mathrm{bmg}}^{2}\varphi\left(\tilde{\gamma}, \gamma\right) + \sigma_{\mathrm{mkt}}^{2}\sigma_{\mathrm{bmg}}^{2}\left(\varphi\left(\tilde{\beta}, \beta\right)\varphi\left(\tilde{\gamma}, \gamma\right) - \varphi^{2}\left(\tilde{\beta}, \gamma\right)\right)}{\sigma_{\mathrm{bmg}}^{2}\left(\tilde{\gamma}^{\top}\mathbf{1}_{n} + \sigma_{\mathrm{mkt}}^{2}\left(\varphi\left(\tilde{\beta}, \beta\right)\tilde{\gamma}^{\top}\mathbf{1}_{n} - \varphi\left(\tilde{\beta}, \gamma\right)\tilde{\beta}^{\top}\mathbf{1}_{n}\right)\right)} \tag{44}$$

A.3.2 Special cases

Let us assume that stock returns are not sensitive to the BMG risk factor, i.e. $\gamma_{i} = 0$. It follows that:

$$x_{i}^{\star} = \frac{\sigma^{2}\left(x^{\star}\right)}{\tilde{\sigma}_{i}^{2}}\left(1 - \frac{\beta_{i}}{\beta^{\star}}\right)$$

and:

$$\beta^{\star} = \frac{1 + \sigma_{\mathrm{mkt}}^{2}\varphi\left(\tilde{\beta}, \beta\right)}{\sigma_{\mathrm{mkt}}^{2}\tilde{\beta}^{\top}\mathbf{1}_{n}}$$

We retrieve Equations (16) and (17) that have been formulated by Scherer (2011).

On average, we observe that the sensitivities β_{i} are distributed around 1.0, whereas the sensitivities γ_{i} are distributed around 0.0. Then, we can assume the hypothesis $\mathcal{H}_{0}$ that $\sum_{s=1}^{n} w_{s}\gamma_{s}$ is equal to zero when w is a vector that is not correlated to γ. Under $\mathcal{H}_{0}$, we deduce that:

$$x_{i}^{\star} = \frac{\sigma^{2}\left(x^{\star}\right)}{\tilde{\sigma}_{i}^{2}}\left(1 - \frac{\beta_{i}}{\beta^{\star}} - \frac{\gamma_{i}}{\gamma^{\star}}\right)$$

and:

$$\beta^{\star} = \frac{1 + \sigma_{\mathrm{mkt}}^{2}\varphi\left(\tilde{\beta}, \beta\right) + \sigma_{\mathrm{bmg}}^{2}\varphi\left(\tilde{\gamma}, \gamma\right) + \sigma_{\mathrm{mkt}}^{2}\sigma_{\mathrm{bmg}}^{2}\varphi\left(\tilde{\beta}, \beta\right)\varphi\left(\tilde{\gamma}, \gamma\right)}{\sigma_{\mathrm{mkt}}^{2}\left(1 + \sigma_{\mathrm{bmg}}^{2}\varphi\left(\tilde{\gamma}, \gamma\right)\right)\tilde{\beta}^{\top}\mathbf{1}_{n}}$$

and:

$$\gamma^{\star} = \frac{1 + \sigma_{\mathrm{mkt}}^{2}\varphi\left(\tilde{\beta}, \beta\right) + \sigma_{\mathrm{bmg}}^{2}\varphi\left(\tilde{\gamma}, \gamma\right) + \sigma_{\mathrm{mkt}}^{2}\sigma_{\mathrm{bmg}}^{2}\varphi\left(\tilde{\beta}, \beta\right)\varphi\left(\tilde{\gamma}, \gamma\right)}{\sigma_{\mathrm{bmg}}^{2}\left(1 + \sigma_{\mathrm{mkt}}^{2}\varphi\left(\tilde{\beta}, \beta\right)\right)\tilde{\gamma}^{\top}\mathbf{1}_{n}}$$

A.3.3 Extension to the long-only minimum variance portfolio

The extension to the long-only case follows the semi-formal proof formulated by Clarke *et al.* (2011). We note $\mathcal{I}_u = \{1, \ldots, n\}$ as the investment universe. We consider the long-only minimum variance (MV) portfolio, which corresponds to the optimization program:

$$\begin{aligned} x^\star &= \arg\min \frac{1}{2} x^\top \Sigma x \\ &\text{s.t.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^\top x = 1 \\ x \geq \mathbf{0}_n \end{array} \right. \end{aligned} \tag{45}$$

The associated Lagrange function is then equal to:

$$\mathcal{L}(x; \lambda_0, \lambda) = \frac{1}{2} x^\top \Sigma x - \lambda_0 \left(\mathbf{1}_n^\top x - 1 \right) - \lambda^\top (x - \mathbf{0}_n)$$

The first-order condition is:

$$\frac{\partial \, \mathcal{L}(x; \lambda_0, \lambda)}{\partial \, x} = \Sigma x - \lambda_0 \mathbf{1}_n - \lambda = \mathbf{0}_n$$

whereas the Kuhn-Tucker conditions are $\min(\lambda_i, x_i) = 0$ for $i = 1, \ldots, n$. The optimal solution is given by:

$$x^\star = \Sigma^{-1} (\lambda_0 \mathbf{1}_n + \lambda)$$

We deduce that[49]:

$$x_{mv} = x_{gmv} + \delta^+$$

where:

$$x_{gmv} = \frac{\Sigma^{-1} \mathbf{1}_n}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n}$$

and:

$$\delta^+ = \left(I_n - \frac{\Sigma^{-1} \mathbf{1}_n \mathbf{1}_n^\top}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n} \right) \Sigma^{-1} \lambda$$

Therefore, imposing the long-only constraint is equivalent to adding a correction term δ^+ to the GMV portfolio $x_{gmv}^\star$ in order to satisfy the constraint $x_i \geq 0$.

Let us assume that we know the assets that make up the long-only MV portfolio. We note $\mathcal{I}'_u \subset \mathcal{I}_u$ as this constrained investment universe. If we restrict the analysis to the set $\mathcal{I}'_u$, the GMV portfolio is equal to:

$$\tilde{x}_{gmv} = \frac{\tilde{\Sigma}^{-1} \mathbf{1}_{n'}}{\mathbf{1}_{n'}^\top \tilde{\Sigma}^{-1} \mathbf{1}_{n'}}$$

where $\tilde{\Sigma}$ is the submatrix of Σ corresponding to the restricted universe and $n' = \operatorname{card} \mathcal{I}'_u$ is the number of assets that belong to the long-only MV portfolio. By construction, we have the equivalence between long-only MV and restricted GMV portfolios:

$$x_{mv} \equiv \tilde{x}_{gmv}$$

[49] Because we have:

$$\begin{aligned} \mathbf{1}_n^\top x = 1 \quad &\Leftrightarrow \quad \lambda_0 \mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n + \mathbf{1}_n^\top \Sigma^{-1} \lambda = 1 \\ &\Leftrightarrow \quad \lambda_0 = \frac{1 - \mathbf{1}_n^\top \Sigma^{-1} \lambda}{\mathbf{1}_n^\top \Sigma^{-1} \mathbf{1}_n} \end{aligned}$$

It follows that:

$$x_i^\star = \begin{cases} \dfrac{\sigma^2(x^\star)}{\tilde{\sigma}_i^2}\left(1-\dfrac{\beta_i}{\beta^\star}-\dfrac{\gamma_i}{\gamma^\star}\right) & \text{if } \dfrac{\beta_i}{\beta^\star}+\dfrac{\gamma_i}{\gamma^\star}\leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{46}$$

However, contrary to the GMV case, the threshold values are endogenous and not exogenous:

$$\beta^\star = \frac{1+\sum_{i\in\mathcal{I}'_u}\left(\sigma^2_{\mathrm{mkt}}\tilde{\beta}_i\beta_i+\sigma^2_{\mathrm{bmg}}\tilde{\gamma}_i\gamma_i\right)+\sigma^2_{\mathrm{mkt}}\sigma^2_{\mathrm{bmg}}\left(\sum_{i\in\mathcal{I}'_u}\tilde{\beta}_i\beta_i\sum_{i\in\mathcal{I}'_u}\tilde{\gamma}_i\gamma_i-\left(\sum_{i\in\mathcal{I}'_u}\tilde{\beta}_i\gamma_i\right)^2\right)}{\sigma^2_{\mathrm{mkt}}\left(\sum_{i\in\mathcal{I}'_u}\tilde{\beta}_i+\sigma^2_{\mathrm{bmg}}\left(\sum_{i\in\mathcal{I}'_u}\tilde{\gamma}_i\gamma_i\sum_{i\in\mathcal{I}'_u}\tilde{\beta}_i-\sum_{i\in\mathcal{I}'_u}\tilde{\beta}_i\gamma_i\sum_{i\in\mathcal{I}'_u}\tilde{\gamma}_i\right)\right)} \tag{47}$$

and:

$$\gamma^\star = \frac{1+\sum_{i\in\mathcal{I}'_u}\left(\sigma^2_{\mathrm{mkt}}\tilde{\beta}_i\beta_i+\sigma^2_{\mathrm{bmg}}\tilde{\gamma}_i\gamma_i\right)+\sigma^2_{\mathrm{mkt}}\sigma^2_{\mathrm{bmg}}\left(\sum_{i\in\mathcal{I}'_u}\tilde{\beta}_i\beta_i\sum_{i\in\mathcal{I}'_u}\tilde{\gamma}_i\gamma_i-\left(\sum_{i\in\mathcal{I}'_u}\tilde{\beta}_i\gamma_i\right)^2\right)}{\sigma^2_{\mathrm{bmg}}\left(\sum_{i\in\mathcal{I}'_u}\tilde{\gamma}_i+\sigma^2_{\mathrm{mkt}}\left(\sum_{i\in\mathcal{I}'_u}\tilde{\beta}_i\beta_i\sum_{i\in\mathcal{I}'_u}\tilde{\gamma}_i-\sum_{i\in\mathcal{I}'_u}\tilde{\beta}_i\gamma_i\sum_{i\in\mathcal{I}'_u}\tilde{\beta}_i\right)\right)} \tag{48}$$

Indeed, we first need to determine $\mathcal{I}'_u$ in order to calculate $\beta^\star$ and $\gamma^\star$.

A.4 Analysis of the tracking error optimization problem

A.4.1 General formulation of the optimization program

We consider the following optimization program:

$$\begin{aligned} x^\star &= \arg\min \frac{1}{2}(x-b)^\top \Sigma (x-b) \\ \text{s.t.} &\quad \mathbf{1}_n^\top x = 1 \end{aligned} \tag{49}$$

The associated Lagrange function is then equal to:

$$\begin{aligned} \mathcal{L}(x;\lambda_0) &= \frac{1}{2}x^\top\Sigma x - x^\top\Sigma b - \lambda_0\left(\mathbf{1}_n^\top x - 1\right) \\ &= \frac{1}{2}x^\top\Sigma x - x^\top(\Sigma b + \lambda_0\mathbf{1}_n) + \lambda_0 \end{aligned}$$

We deduce that the first-order condition is:

$$\frac{\partial\,\mathcal{L}(x;\lambda_0)}{\partial\,x} = \Sigma x - (\Sigma b + \lambda_0\mathbf{1}_n) = \mathbf{0}_n$$

Since we have $x = \Sigma^{-1}(\Sigma b + \lambda_0\mathbf{1}_n)$ and $\mathbf{1}_n^\top x = 1$, it follows that:

$$\begin{aligned} \mathbf{1}_n^\top x = 1 &\Leftrightarrow \mathbf{1}_n^\top\Sigma^{-1}(\Sigma b + \lambda_0\mathbf{1}_n) = 1 \\ &\Leftrightarrow 1 + \lambda_0\left(\mathbf{1}_n^\top\Sigma^{-1}\mathbf{1}_n\right) = 1 \\ &\Leftrightarrow \lambda_0 = 0 \end{aligned}$$

and we obtain the trivial solution $x^\star = b$. We notice that this solution remains valid if we introduce long-only constraints $x \geq \mathbf{0}_n$ because the Kuhn-Tucker conditions $\min(\lambda_i, x_i) = 0$ are already satisfied.

We now consider the optimization program:

$$x^{\star} = \arg\min \frac{1}{2}(x-b)^{\top}\Sigma(x-b) \quad (50)$$
$$\text{s.t.} \begin{cases} \mathbf{1}_n^{\top}x = 1 \\ x \geq \mathbf{0}_n \\ \beta_{\text{bmg}}^{\top}x \leq \beta_{\text{bmg}}^{+} \end{cases}$$

The associated Lagrange function is then equal to:

$$\begin{aligned}\mathcal{L}(x;\lambda_0,\lambda,\lambda_{\text{bmg}}) &= \frac{1}{2}x^{\top}\Sigma x - x^{\top}\Sigma b - \lambda_0\left(\mathbf{1}_n^{\top}x - 1\right) - \lambda^{\top}x + \lambda_{\text{bmg}}\left(\beta_{\text{bmg}}^{\top}x - \beta_{\text{bmg}}^{+}\right) \\ &= \frac{1}{2}x^{\top}\Sigma x - x^{\top}\left(\Sigma b + \lambda_0\mathbf{1}_n + \lambda - \lambda_{\text{bmg}}\beta_{\text{bmg}}\right) + \lambda_0 - \lambda_{\text{bmg}}\beta_{\text{bmg}}^{+}\end{aligned}$$

where $\lambda_0 \geq 0$ is the Lagrange multiplier of the equality constraint $\mathbf{1}_n^{\top}x = 1$, $\lambda \geq \mathbf{0}_n$ is the vector of Lagrange multipliers of the bound constraints $x \geq \mathbf{0}_n$ and $\lambda_{\text{bmg}} \geq 0$ is the Lagrange multiplier of the inequality constraint $\beta_{\text{bmg}}^{\top}x \leq \beta_{\text{bmg}}^{+}$. We deduce that the first-order condition is:

$$\frac{\partial\,\mathcal{L}(x;\lambda_0,\lambda,\lambda_{\text{bmg}})}{\partial\,x} = \Sigma x - \left(\Sigma b + \lambda_0\mathbf{1}_n + \lambda - \lambda_{\text{bmg}}\beta_{\text{bmg}}\right) = \mathbf{0}_n$$

implying that:

$$\begin{aligned}x^{\star} &= \Sigma^{-1}\left(\Sigma b + \lambda_0\mathbf{1}_n + \lambda - \lambda_{\text{bmg}}\beta_{\text{bmg}}\right) \\ &= b + \lambda_0\Sigma^{-1}\mathbf{1}_n + \Sigma^{-1}\lambda - \lambda_{\text{bmg}}\Sigma^{-1}\beta_{\text{bmg}} \\ &= x^{\star}(b) + x^{\star}\left(\beta_{\text{bmg}}\right) \quad (51)\end{aligned}$$

where $x^{\star}(b) = b + \lambda_0\Sigma^{-1}\mathbf{1}_n$ and $x^{\star}\left(\beta_{\text{bmg}}\right) = \Sigma^{-1}\lambda - \lambda_{\text{bmg}}\Sigma^{-1}\beta_{\text{bmg}}$. We notice that Portfolio $x^{\star}(b)$ only depends on the benchmark b, the covariance matrix Σ and the Lagrange multiplier λ_0. It does not depend on the BMG sensitivities, and may be considered as a constant term. Moreover, we can set $\lambda_0 = 0$, because the constraint $\mathbf{1}_n^{\top}x = 1$ has no impact on the solution $x^{\star}$, but only scales the optimal portfolio such as the weights sum up to 100%. In this case, we have $x^{\star}(b) = b$ and $x^{\star}\left(\beta_{\text{bmg}}\right)$ is the long/short portfolio $x^{\star} - b$ that depends on several parameters:

$$\Delta = x^{\star} - b \approx \Sigma^{-1}\lambda - \lambda_{\text{bmg}}\Sigma^{-1}\beta_{\text{bmg}} \quad (52)$$

At first sight, these parameters are the covariance matrix Σ, the vector λ of Lagrange coefficients, the Lagrange coefficient λ_{bmg} and the vector β_{bmg} of carbon risk sensitivities. In fact, this interpretation of Equation (52) is misleading. The two important quantities are the scaled BMG beta vector $\breve{\beta}_{\text{bmg}} = \Sigma^{-1}\beta_{\text{bmg}}$ and the threshold value β_{bmg}^{+}, since λ and λ_{bmg} are endogenous[50].

A.4.2 Special cases

Let us assume that $\Sigma = \operatorname{diag}\left(\sigma_1^2,\ldots,\sigma_n^2\right)$ is a diagonal matrix. We have

$$\Delta_i = \frac{\lambda_i}{\sigma_i^2} - \lambda_{\text{bmg}}\frac{\beta_{\text{bmg},i}}{\sigma_i^2} \quad (53)$$

[50] The vector λ and the scalar λ_{bmg} are related to these two quantities. They do not add more degrees of freedom to the optimization problem.

We notice that Δ_i is a decreasing function of $\beta_{\mathrm{bmg},i}$. Moreover, the Kuhn-Tucker conditions implies the following property:

$$x_i > 0 \Rightarrow \lambda_i = 0 \Rightarrow \left\{ \begin{array}{ll} \Delta_i > 0 & \text{if } \beta_{\mathrm{bmg},i} < 0 \\ \Delta_i < 0 & \text{if } \beta_{\mathrm{bmg},i} > 0 \end{array} \right. \tag{54}$$

We now assume a constant correlation matrix. We have $\Sigma = \sigma\sigma^\top \odot R$ where $R = C_n(\rho)$. Maillard *et al.* (2010) showed that $\Sigma^{-1} = \Gamma \odot R^{-1}$ where $\Gamma_{i,j} = \frac{1}{\sigma_i \sigma_j}$ and:

$$R^{-1} = \frac{\rho \mathbf{1}_n \mathbf{1}_n^\top - \left((n-1)\rho + 1\right) I_n}{(n-1)\rho^2 - (n-2)\rho - 1}$$

It follows that:

$$\begin{array}{rcl} \Sigma^{-1} & = & \dfrac{\rho\Gamma \odot \left(\mathbf{1}_n \mathbf{1}_n^\top\right) - \left((n-1)\rho + 1\right)\Gamma \odot I_n}{(n-1)\rho^2 - (n-2)\rho - 1} \\ & = & \rho\xi\Gamma - \left((n-1)\rho + 1\right)\xi\tilde{\Gamma} \end{array}$$

where $\tilde{\Gamma} = \operatorname{diag}(\Gamma)$ is the diagonal matrix with $\tilde{\Gamma}_{i,i} = \Gamma_{i,i}$, and:

$$\xi = \frac{1}{(n-1)\rho^2 - (n-2)\rho - 1}$$

Let $u \in \mathbb{R}^n$ be a vector. We deduce that:

$$\begin{array}{rcl} v & = & \Sigma^{-1} u \\ & = & \rho\xi\Gamma u - \left((n-1)\rho + 1\right)\xi\tilde{u} \end{array}$$

where $\tilde{u}$ is a vector with $\tilde{u}_i = \sigma_i^{-2} u_i$. Finally, we obtain:

$$\begin{array}{rcl} v_i & = & \rho\xi \displaystyle\sum_{j=1}^n \Gamma_{i,j} u_j - \left((n-1)\rho + 1\right)\xi\tilde{u}_i \\ & = & \dfrac{n\rho\xi}{\sigma_i}\bar{s} - \dfrac{\left((n-1)\rho + 1\right)\xi}{\sigma_i} s_i \end{array} \tag{55}$$

where $s_i = \sigma_i^{-1} u_i$ and $\bar{s} = n^{-1}\sum_{j=1}^n s_j$. If n is very large and we assume that $\rho > 0$, we have:

$$\xi \approx \frac{1}{n\rho(\rho - 1)}$$

and Equation (55) reduces to:

$$v_i \approx \frac{1}{1-\rho}\left(\frac{s_i - \bar{s}}{\sigma_i}\right) \tag{56}$$

We only consider this case in order to simplify the expression of Δ_i, which is otherwise complex and does not help to interpret the impact of the BMG constraint. We obtain:

$$\begin{array}{rcl} \Delta_i & \approx & \dfrac{1}{(1-\rho)\sigma_i}\left(\dfrac{\lambda_i}{\sigma_i} - \overline{\dfrac{\lambda}{\sigma}}\right) + \dfrac{\lambda_{\mathrm{bmg}}}{(1-\rho)\sigma_i}\left(\overline{\dfrac{\beta_{\mathrm{bmg}}}{\sigma}} - \dfrac{\beta_{\mathrm{bmg},i}}{\sigma_i}\right) \\ & = & \Delta_i^1 + \Delta_i^2 \end{array} \tag{57}$$

We notice that the overweight or underweight of an asset will depend on the relative position of the statistic $\sigma_i^{-1}\beta_{\mathrm{bmg},i}$ with respect to its mean. If it is below the mean, the second term Δ_i^2 is positive. Generally, we would observe an overweight of asset i.

In the general case, we have:

$$\Delta_i = \left(\Sigma^{-1}\lambda\right)_i - \lambda_{\mathrm{bmg}}\breve{\beta}_{\mathrm{bmg},i} \tag{58}$$

If we omit the impact of the lower bound, Δ_i is positive if $\breve{\beta}_{\mathrm{bmg},i}$ is negative.

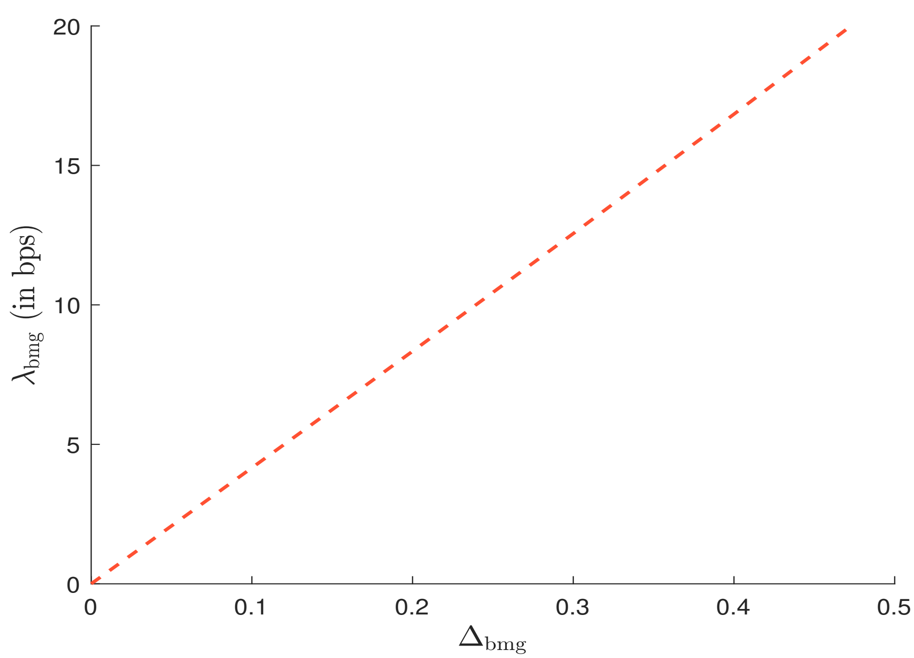
A.4.3 Approximation of the tracking error volatility

We recall that the first-order condition is[51] $\Sigma\left(x^\star - b\right) = \lambda_0 \mathbf{1}_n + \lambda - \lambda_{\mathrm{bmg}}\beta_{\mathrm{bmg}}$ where λ_0, λ and λ_{bmg} are the Lagrange coefficients associated with the constraints $\mathbf{1}_n^\top x = 1$, $x \geq \mathbf{0}_n$ and $\beta_{\mathrm{bmg}}^\top x \leq \beta_{\mathrm{bmg}}^+$. We deduce that:

$$\begin{aligned}
\sigma^2\left(x^\star \mid b\right) &= \left(x^\star - b\right)^\top \Sigma\left(x^\star - b\right) \\
&= \left(x^\star - b\right)^\top \left(\lambda_0 \mathbf{1}_n + \lambda - \lambda_{\mathrm{bmg}}\beta_{\mathrm{bmg}}\right) \\
&= \left(x^\star - b\right)^\top \lambda - \lambda_{\mathrm{bmg}}\left(x^\star - b\right)^\top \beta_{\mathrm{bmg}} \\
&= \left(x^\star - b\right)^\top \lambda + \lambda_{\mathrm{bmg}}\Delta_{\mathrm{bmg}} \\
&\approx c\Delta_{\mathrm{bmg}}^2
\end{aligned}$$

The last result is obtained because we have $\left(x^\star - b\right)^\top \lambda \approx 0$ and we also notice that the Lagrange coefficient λ_{bmg} is almost a linear function of Δ_{bmg} for reasonable values of Δ_{bmg}. For instance, we report in Figure 18 the relationship between Δ_{bmg} and λ_{bmg} when the benchmark corresponds to the CW index. This explains that the tracking error volatility is approximatively a linear function of Δ_{bmg}.

Figure 18: Relationship between Δ_{bmg} and λ_{bmg}



[51] See Equation (51) on page 48.

B Additional results

Table 9: Statistics of factor returns

Factor	μ	σ	γ_1	γ_2	$\mathcal{MDD}$	$\mathcal{BM}$	$\mathcal{WM}$
MKT	7.63%	13.23%	−0.34	3.63	−20.4%	10.0%	−9.5%
SMB	−0.10%	4.65%	0.14	2.82	−13.6%	3.9%	−3.1%
HML	−1.82%	5.92%	0.25	2.96	−20.0%	4.3%	−4.5%
WML	6.74%	8.24%	−0.14	2.92	−11.1%	6.3%	−6.8%
BMG	−2.52%	6.41%	−0.06	3.08	−35.3%	4.2%	−5.3%
Carbon intensity	−3.09%	5.13%	0.14	3.38	−31.1%	4.5%	−3.8%
Carbon emissions exp.	−4.01%	5.47%	0.36	3.58	−36.7%	5.2%	−3.8%
Carbon emissions mgmt.	4.15%	3.71%	0.20	2.93	−4.7%	3.1%	−2.2%
Carbon emissions	0.05%	4.68%	0.31	4.22	−15.8%	5.0%	−3.6%
Climate change	−0.78%	4.56%	0.25	2.69	−20.8%	3.7%	−2.7%
Environment	−0.26%	4.63%	0.48	3.24	−17.8%	4.1%	−3.1%

The statistics μ and σ are the annualized return and volatility, γ_1 and γ_2 are the skewness and kurtosis coefficients, $\mathcal{MDD}$ is the observed maximum drawdown, $\mathcal{BM}$ and $\mathcal{WM}$ are the best and worst monthly returns. BMG corresponds to the Carima factor developed by Görgen *et al.* (2019).

Table 10: Correlation matrix of factor returns (in %)

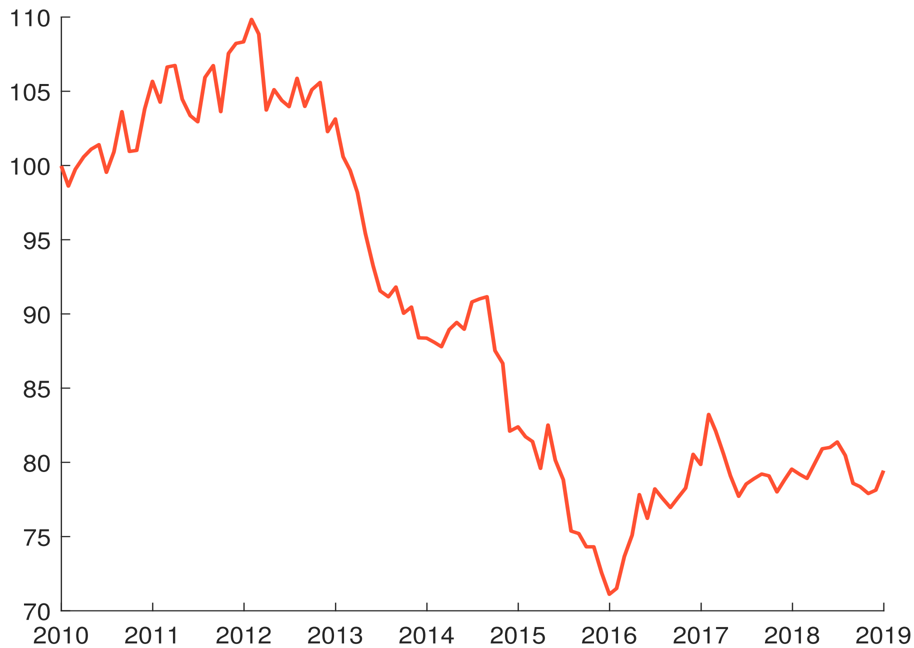
Factor	MKT	SMB	HML	WML	BMG
Carbon intensity	−6.46	13.71	8.71	−3.04	58.40***
Carbon emissions exp.	−6.71	14.95	4.03	−4.03	64.02***
Carbon emissions mgmt.	−17.93*	24.16**	−20.91**	20.93**	38.66***
Carbon emissions	1.22	25.85***	−0.23	5.15	72.36***
Climate change	−15.02	16.30*	11.43	2.07	61.11***
Environment	−28.20***	21.16**	−0.33	3.70	68.53***

BMG corresponds to the Carima factor developed by Görgen *et al.* (2019).

Table 11: Comparison of cross-section regressions (in %)

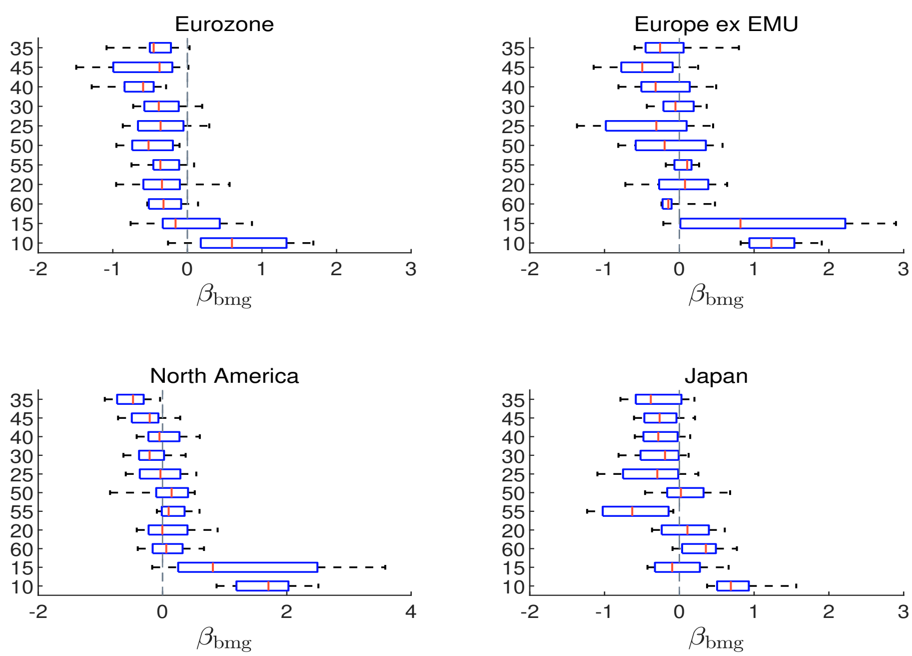
	Adjusted $\mathfrak{R}^2$ difference	F-test 10%	F-test 5%	F-test 1%	Adjusted $\mathfrak{R}^2$ difference	F-test 10%	F-test 5%	F-test 1%
	CAPM vs MKT+BMG				FF vs FF+BMG			
Carima	1.74	21.2	15.6	9.2	1.73	22.5	17.5	9.7
Carbon intensity	1.77	22.0	16.1	9.0	1.80	23.3	17.5	10.6
Carbon emissions exp.	1.79	21.2	16.3	9.8	1.85	23.0	18.2	10.5
Carbon emissions mgmt.	1.64	23.3	17.8	9.5	1.64	25.1	19.4	10.7
Carbon emissions	2.00	24.2	18.8	11.7	2.06	25.7	20.1	12.7
Climate change	1.58	20.9	15.9	8.6	1.48	21.2	16.5	9.4
Environment	1.63	21.9	17.1	10.5	1.60	23.0	18.1	10.2

Figure 19: Cumulative performance of the BMG factor



Source: Görgen *et al.* (2019).

Figure 20: Box plots of the carbon sensitivities by sector and region



The sorted GICS classification is: energy (10), materials (15), industrials (20), consumer discretionary (25), consumer staples (30), health care (35), financials (40), information technology (45), communication services (50), utilities (55) and real estate (60). The box plots provide the median, the quartiles and the 10% and 90% quantiles of the carbon beta.

Figure 21: Dynamics of the carbon beta of sorted-portfolios

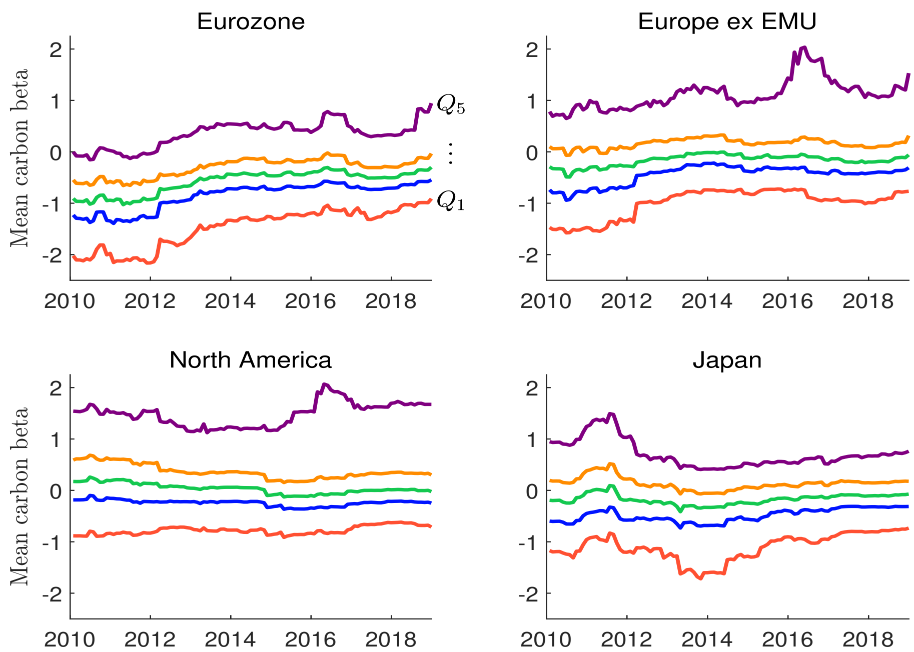


Figure 22: Dynamics of the average absolute carbon risk $|\beta|_{\mathrm{bmg},\mathcal{S}}(t)$ by sector

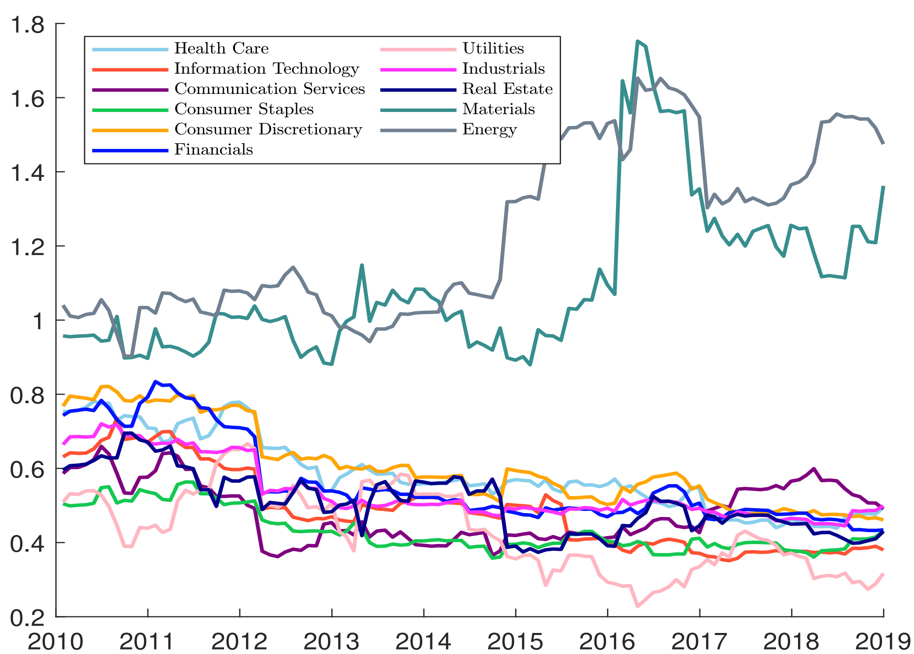


Figure 23: Scatter plot of MKT and BMG sensitivities

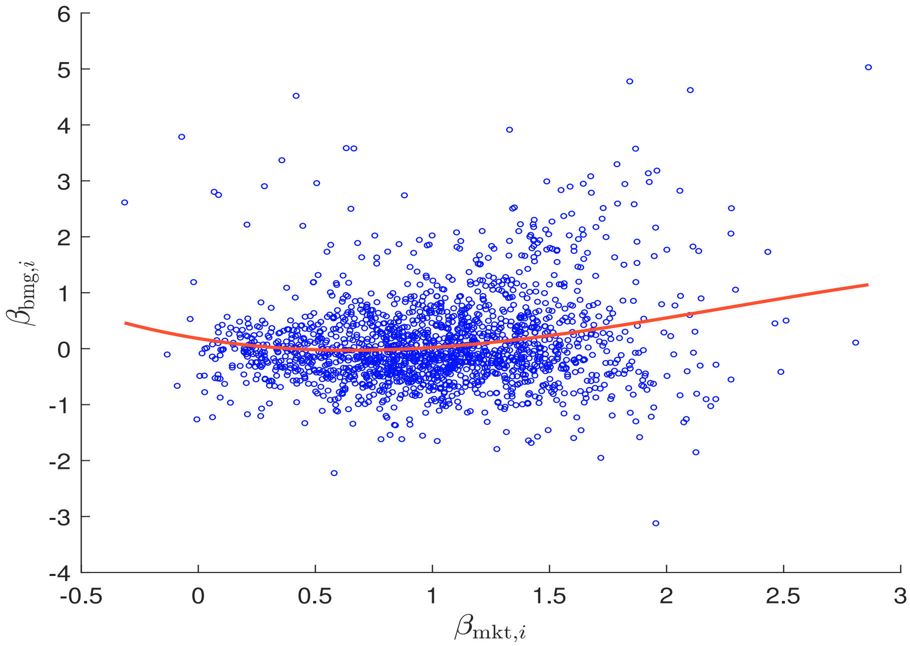
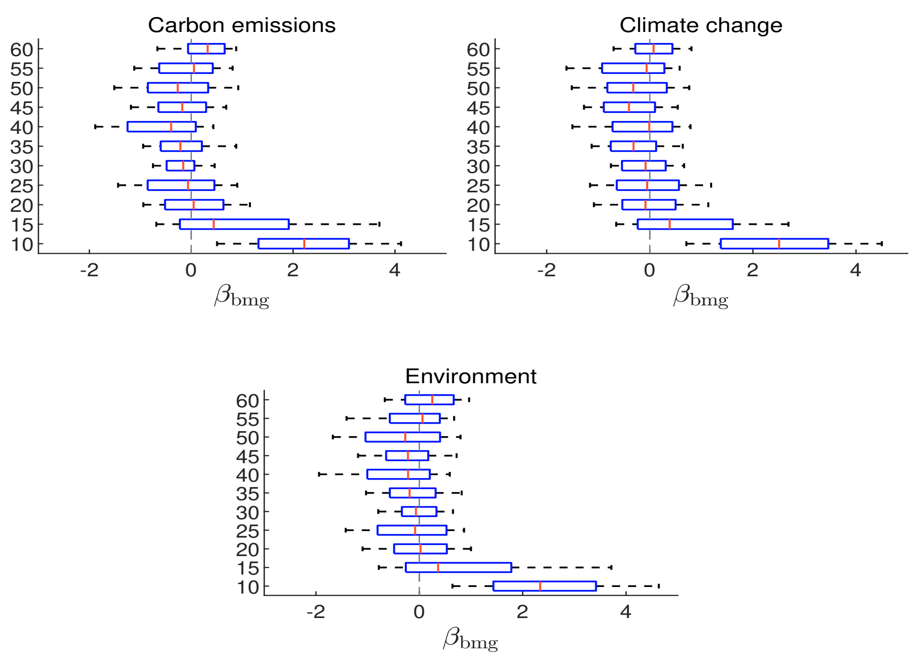


Figure 24: Box plots of the carbon sensitivities by sector and factor



The sorted GICS classification is: energy (10), materials (15), industrials (20), consumer discretionary (25), consumer staples (30), health care (35), financials (40), information technology (45), communication services (50), utilities (55) and real estate (60). The box plots provide the median, the quartiles and the 10% and 90% quantiles of the carbon beta.

Figure 25: WACI of the constrained MV portfolios ($\mathcal{CI}^+ = 315$)

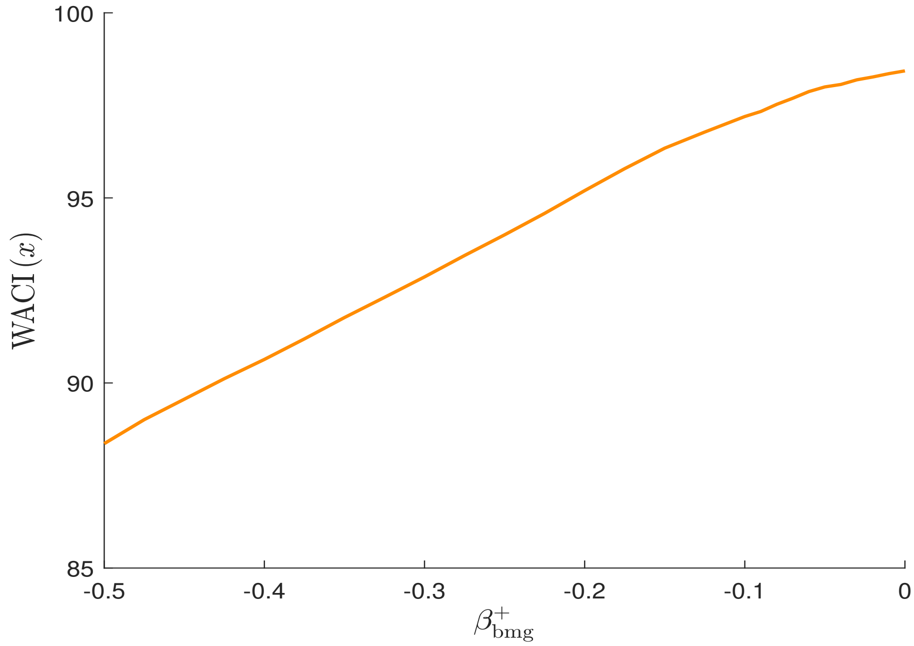
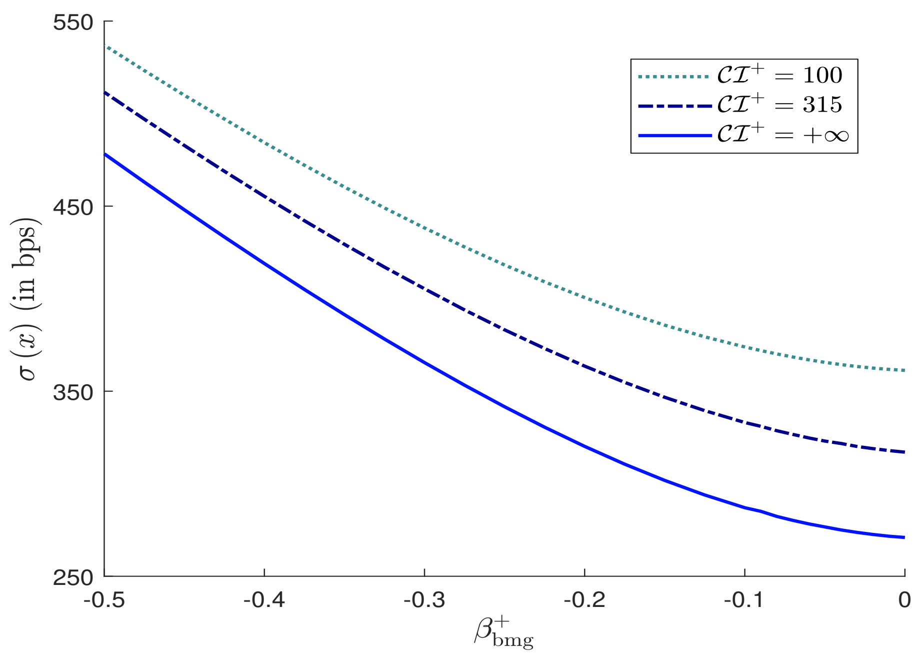


Figure 26: Volatility of the constrained MV portfolios



Based on the MKT+BMG model, the monthly volatility of the capitalisation-weighted MSCI World index is 13.25%.

Figure 27: Market and fundamental measures of carbon risk

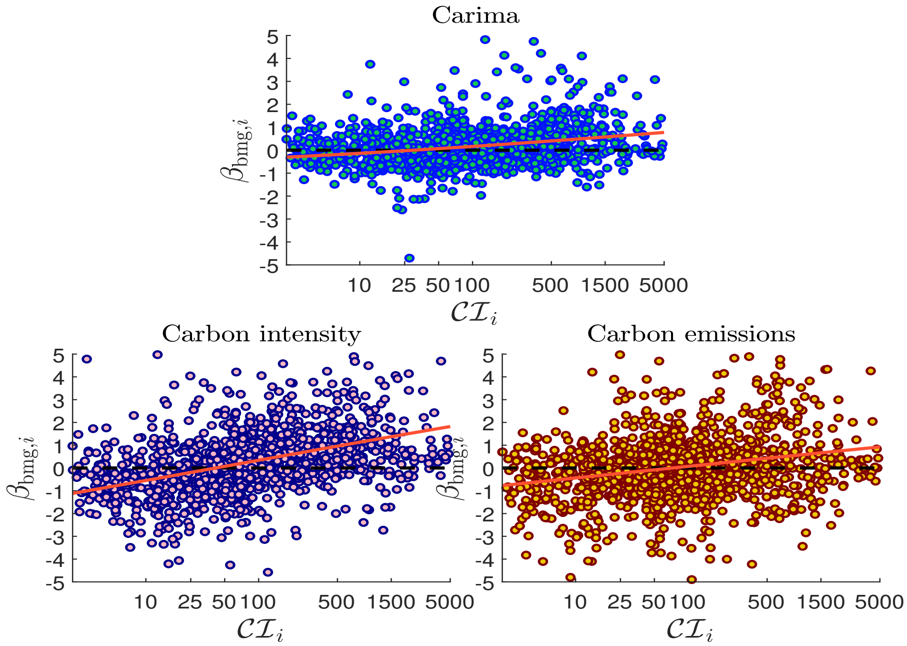


Figure 28: Scatter plot of CW weights and BMG sensitivities

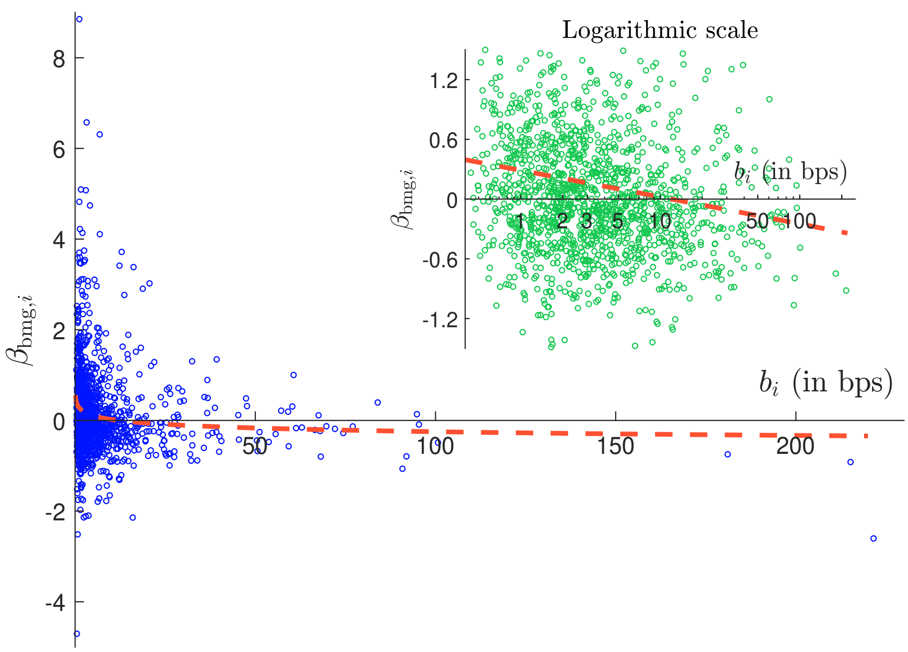


Figure 29: Relationship between $\beta_{\text{bmg},i}$ and $\Delta_i = x_i^\star - b_i$ for the CW benchmark

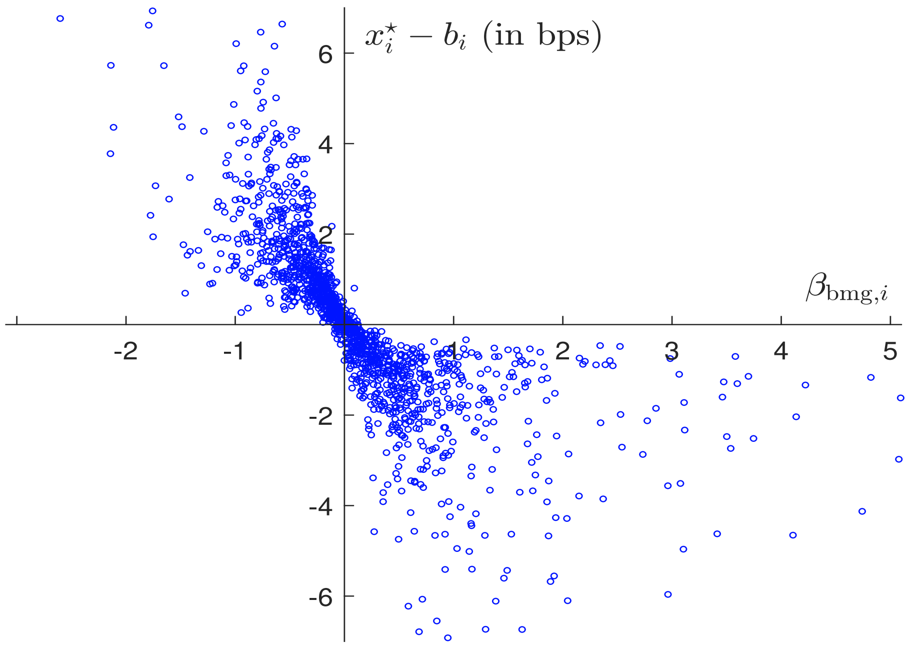


Figure 30: Relationship between $\breve{\beta}_{\text{bmg},i}$ and $\Delta_i = x_i^\star - b_i$ for the CW benchmark

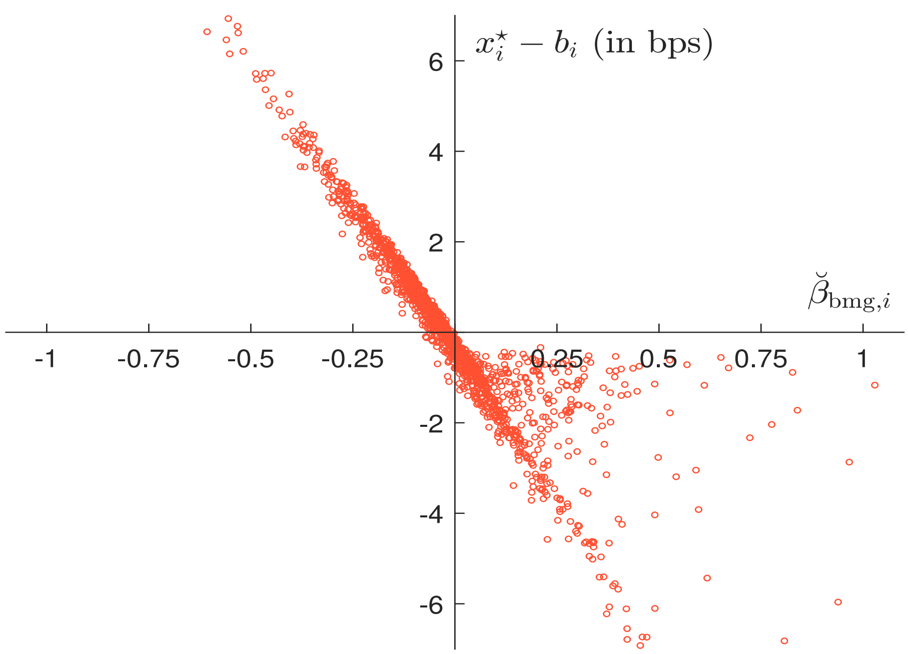


Figure 31: Solution of the order-statistic optimization problem

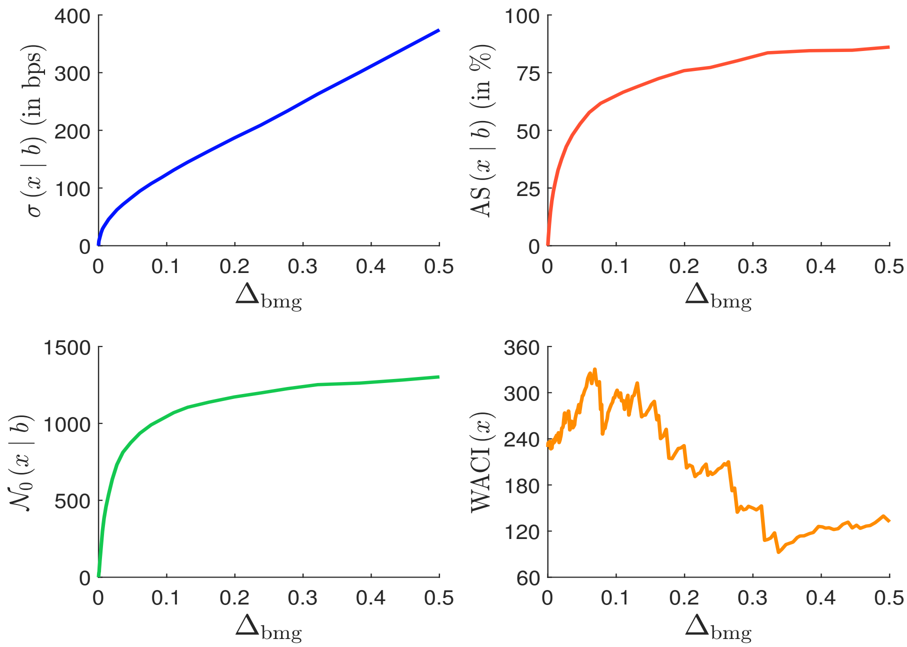


Figure 32: Solution of the alternative order-statistic optimization problem