Portfolio Allocation of Hedge Funds*

Benjamin Bruder
Research & Development
Lyxor Asset Management, Paris
benjamin.bruder@lyxor.com

Serge Darolles
Research & Development
Lyxor Asset Management, Paris
serge.darolles@lyxor.com

Abdul Koudiraty
Research & Development
Lyxor Asset Management, Paris
abdul.koudiraty@lyxor.com

Thierry Roncalli
Research & Development
Lyxor Asset Management, Paris
thierry.roncalli@lyxor.com

January 2010

Abstract

Research in hedge fund investing proposes different solutions to build optimal hedge fund portfolios. However, these solutions are direct extensions of the usual mean-variance framework, and still suffer from *model risks*. More complex approaches start to be used but are related to numerous *estimation risk*s. We compare in this paper the out-sample properties of different allocation models through a dynamic investment exercise using hedge fund indices. We show that the best out-of-sample properties are obtained by allocation models that take into account the specific statistical properties of hedge fund returns.

Keywords: Hedge funds, portfolio allocation, higher-order moments, regime-switching models.

JEL classification: G11, G24, C53.

1 Introduction

Hedge fund returns differ substantially from the returns of standard asset classes, making hedge funds of interest to investors seeking to diversify balanced portfolios. Research into hedge fund investing has therefore naturally focused on finding the optimal proportion in which to invest in hedge funds[1], measuring hedge fund performance[2], identifying hedge fund risk factors[3], and finally constructing optimal hedge fund portfolios[4]. However, despite the

*We are grateful to L. Erdely, J.M. Stenger and G. Comissiong for their helpful comments.

[1]see e.g. Terhaar *et al.* (2003), Cvitanic *et al.* (2003), Popova *et al.* (2003), Amin and Kat (2003).

[2]see e.g. Eling and Schumacher (2007), Darolles *et al.* (2009), Darolles and Gourieroux (2010).

[3]see e.g. Fung and Hsieh (1997), Ackermann *et al.* (1999), Brown *et al.* (1999), Liew (2003), Agarwal and Naik (2004), Agarwal *et al.* (2009), Buraschi *et al.* (2010), Darolles and Mero (2010).

[4]see e.g. Amenc and Martellini (2002), McFall Lamm (2003), Kat (2004), Agarwal and Naik (2004), Alexander and Dimitriu (2004), Morton *et al.* (2006), Giomouridis and Vrontos (2007), Adam *et al.* (2008).

nature of hedge fund investments (*dynamic trading strategies, use of derivatives and leverage)* and their observable consequences on hedge fund return characteristics (*time-varying covariance parameters, high kurtosis of returns distributions*), this research is mainly conducted in the usual framework of normal independent and identically distributed (*i.i.d.*) returns. For example, Giomouridis and Vrontos (2007) are the first to use dynamic specification for the covariance parameters and to evaluate the consequences at the portfolio allocation level. More generally, we can think that the standard methods used for portfolio allocation are inadequate since non null skewness and high kurtosis, relative to traditional asset classes, are observed in the monthly returns of hedge funds[5].

The mean-variance approach (Markovitz, 1952) is a standard in the asset management industry. This approach is founded on the assumption of normal *i.i.d.* returns and is naturally subject to criticisms, when applied to build portfolios involving hedge funds. These criticisms stress that mean-variance is in this case only appropriate for investors having quadratic preferences, thus making it not applicable in all situations. But this kind of analysis is widely used in practice as well as in the literature. Among others, Amenc and Martellini (2002), Terhaar *et al.* (2003) and Alexander and Dimitriu (2004) consider hedge fund allocation in this framework. However, alternative approaches have already been proposed. Bares *et al.* (2002) and Cvitanic *et al.* (2003) compute optimal portfolios in an expected utility framework. Amin and Kat (2003) discuss the problems arising in mean-variance allocation when the asset returns are not symmetric. Popova *et al.* (2003), Hagelin and Pramborg (2003), Jurczenko *et al.* (2005) and Davies *et al.* (2009) introduce higher moments analysis. Krokhmal *et al.* (2002), Favre and Galeano (2002), Agarwal and Naik (2004) and Adam *et al.* (2008) construct optimal portfolios using alternative risk measures. Given the increasing interest for risk management, there is indeed a multiplication of measures capturing different types of risks and several tentatives to unify these approaches. Rockafellar *et al.* (2006) for example propose generalized measures as substitutes for volatility.

In this paper, we consider a general set of asset allocation models to measure the differences in allocation obtained when using alternative specifications. We first consider the mean-variance model and then discuss basic extensions that do not involve expected return estimation and are presumed to be more robust against estimation errors. Such extensions include the minimum-variance (Clarke *et al.*, 2006), constant-Sharpe (Martellini, 2008), equally-weighted risk contribution (Maillard *et al.*, 2010) and most diversified (Choueifaty and Coignard, 2008) portfolios. We also consider optimal portfolios as determined by maximizing different utility functions and the Omega ratio risk measure. Each of these approaches is given a set of solutions that can be easily compared when applied to hedge fund returns. We then examine four extensions that seem particularly promising in addressing the appropriate allocation to hedge funds. First, we include higher-order moments in the objective function to maximize. This extension allows investors to recognize and use information embedded in the unusual distribution of hedge fund returns (Martellini and Ziemann, 2010). Second, we address the issue of time-varying variance covariance parameters and their potential impact on hedge fund portfolio allocation. We use the regime-switching dynamic correlation model[6] to include these stylized facts in the covariance matrix specification. Third, we include exogenous stress scenarios to reflect the possibility of extreme drawdowns attributable to certain rare events. Fourth, we consider the impact of inclusion of managers' views of portfolio performance in a Black-Litterman framework.

[5]see e.g. Fung and Hsieh (1997, 2001), Brooks and Kat (2002), Popova *et al.* (2003), Agarwal and Naik (2004), Martellini and Ziemann (2010).

[6]see, e.g. Pelletier (2006), Giomouridis and Vrontos (2007).

To evaluate the models ability to work with non-normal distribution and time-varying parameters, we illustrate our analysis by actively managing a fund of hedge funds invested in hedge fund indices from the CSFB/Tremont and HFR database. We use index data instead of single hedge fund data for essentially two reasons. First, the direct optimization of a hedge funds portfolio is impossible with regards to the variety of investment constraints related to each individual hedge fund. This explains why portfolio construction is in practice done in two steps. The first step corresponds to strategic and tactical allocation and determines the portfolio profile in terms of strategies. The second step is the implementation of the decision allocation, with the choice of specific hedge funds allowing the portfolio manager to implement the strategic/tactical allocation bets. As this second step necessarily involves deep qualitative analysis of each candidate hedge fund, quantitative models are more relevant during the first step, i.e. when allocating between hedge fund strategies. Second, Darolles and Vaissie (2010) find that the added value of funds of hedge funds at the tactical level is poor. It is then of great interest to evaluate quantitative allocation tools to actively manage the tactical allocation of hedge fund portfolios and enhance the total added value proposed by funds of hedge funds. The paper is organized as follow. Section 2 describes the classic allocation models used as benchmarks for our analysis. Section 3 presents two allocation models developed in the particular context of hedge funds portfolio construction. Section 4 applies benchmark and specific models to hedge fund data. Finally, Section 5 concludes.

2 Models of portfolio optimization

In what follows, we describe the different models that can be used to determinate the portfolio allocation of hedge funds. We consider a universe of n hedge fund strategies. We denote by $R_{i,t}$ the return of the i^{th} strategy at time t. μ and Σ are the expected return and the covariance matrix of the returns $(R_{1,t}, \ldots, R_{n,t})$. Let w be the weights of the portfolio. We suppose that the portfolio construction faces some constraints and we note $w \in \mathcal{A}$. For example, we may assume that the weights are positive and sum to one. In this case, $\mathcal{A}$ is the simplex set:

$$\mathcal{A} = \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i = 1 \right\}$$

In order to be more legible, we do not report the constraints $w \in \mathcal{A}$ in the optimization programs, but they are taken into account implicitly.

2.1 Mean-variance framework

2.1.1 Optimization program

The mean-variance framework was introduced by Markowitz (1952) and is now widely used. The main idea of that seminal work is to find an efficient equilibrium between risks and returns. More precisely, the market risks are entirely monitored by the variance of the portfolio while the returns are characterized through their expected value. Its popularity comes from the fact that this program produces an optimal allocation with respect to any risk-averse criterion based on probabilities, as long as the asset return distributions are supposed to be gaussian. Indeed, in that case, any portfolio has a gaussian behavior, which is therefore completely described by its return and its variance. On the other hand, many studies (see e.g. Cont, 2001) show that asset returns are not normally distributed, especially when considering hedge funds (Fung and Hsieh, 1999, Malkiel and Saha, 2005). Hence, considering only the variance as a measure of risks may be insufficient.

For any risk-averse investor, the expected return of the portfolio should be maximized, while risk should be minimized. Thus, it is natural to maximize expected returns while keeping variance under a given acceptable level:

$$\max w^\top \mu \quad \text{u.c.} \quad w^\top \Sigma w \leq \sigma^2_{\max} \tag{1}$$

where $w^\top \mu$ is the expected return of the portfolio and $w^\top \Sigma w$ denotes the variance of the portfolio. Another equivalent procedure is to minimize the variance while keeping the expected returns above a minimal level:

$$\min w^\top \Sigma w \quad \text{u.c.} \quad w^\top \mu \geq \mu_{\min} \tag{2}$$

Portfolio allocations given by those programs define the mean/variance efficient frontier, that is also the maximal attainable expected return for a given level of risk[7].

Another equivalent point of view is to set the following optimization problem:

$$\max w^\top \mu - \frac{\lambda}{2} w^\top \Sigma w \tag{3}$$

where λ is called the risk aversion parameter. It is clear that the solution of this problem will maximize expected returns while minimizing portfolio variance. Each parameter $\lambda > 0$ involves an optimal portfolio corresponding to a given value of $\mu_{\min}$ in problem (2). Unfortunately, the correspondence between the risk aversion parameter λ and the resulting average return $\mu_{\min}$ can only be found by solving problem (3). Nevertheless, with this formulation, this method could be seen as a special case of utility maximization involving a quadratic utility function[8].

2.1.2 Parameter uncertainty problem

The major issue of the mean-variance framework is the estimation of the asset returns covariance matrix Σ and of the expected returns μ. These parameters can be estimated with their empirical counterparts. However, we should emphasize that optimizing the mean-variance criterion through backtesting or with the empirical moments lead exactly to the same results.

The empirical estimator $\hat{\Sigma}$ is traditionally used for the covariance matrix, due to its appealing properties. Indeed, it corresponds to the maximum likelihood estimator. However, its convergence is very slow especially when the number of assets is large. In this case, $\hat{\Sigma}$ introduces some extreme errors, and any optimization procedure will focus on those errors, thus placing big bets on the mostly unreliable coefficients (Michaud, 1989). Shrinkage methods is a popular solution to obtain a biased but robust covariance matrix.

Let $\hat{\Phi}$ be a biased estimate of Σ. We assume that it converges faster than $\hat{\Sigma}$. In this case, we define $\hat{\Sigma}_\alpha$ as a convex combination of these two estimators:

$$\hat{\Sigma}_\alpha = \alpha \hat{\Phi} + (1 - \alpha)\, \hat{\Sigma}$$

The main problem is to compute the optimal value of α. Ledoit and Wolf (2003) consider the quadratic loss function $L(\alpha)$ defined as follows:

$$L(\alpha) = \left\| \alpha \hat{\Phi} + (1 - \alpha)\, \hat{\Sigma} - \Sigma \right\|^2$$

[7]This frontier can be obtained by computing the minimal attainable variance $w^\top \Sigma w$ for all the expected return level constraints $\mu_{\min}$.

[8]See Section 2.2 for details.

They define the optimal value as the solution of the problem $\alpha^\star = \arg\min \mathbb{E}\left[L\left(\alpha\right)\right]$. Ledoit and Wolf (2003) give the analytical solution of $\alpha^\star$ in the case of a one-factor model, whereas the optimal estimate $\alpha^\star$ in the case of a constant correlation matrix may be found in Ledoit and Wolf (2004).

2.1.3 Risk based portfolios

One of the issues of parameter uncertainty is to estimate the expected returns μ. Merton (1980) points that historical estimates fail to account for the effects of changes in the level of market risk and expected returns may be estimated using equilibrium models. Nevertheless, this approach makes sense only for traditional asset classes. In the case of hedge funds, an equilibrium model to estimate the risk premium of one strategy does not exist. That is why we may prefer to use some allocation methods which are presumed to be more robust as they are not dependent on expected returns.

The minimum-variance portfolio is the only portfolio on the mean-variance frontier which does not depend on expected returns. More formally, it is defined by the following optimization program:

$$w^\star = \arg\min w^\top \Sigma w$$

This approach is widely used by practitioners (Clarke *et al.*, 2006). However, it may suffer some drawbacks concerning the concentration.

In order to obtain a more diversified portfolio, Maillard *et al.* (2010) propose to use the equally-weighted risk contribution portfolio (known as the ERC portfolio). Let $\sigma\left(w\right) = \sqrt{w^\top \Sigma w}$ be the volatility of the portfolio. We have:

$$\sigma\left(w\right) = \sum_{i=1}^{n} \mathrm{RC}_i = \sum_{i=1}^{n} w_i \frac{\partial\, \sigma\left(w\right)}{\partial\, w_i}$$

$\mathrm{RC}_i = w_i \times \partial_{w_i} \sigma\left(w\right)$ is the risk contribution of the i^{th} strategy to the portfolio volatility. The ERC portfolio corresponds to the portfolio where all the risk contributions are the same:

$$\mathrm{RC}_i = \mathrm{RC}_j$$

As shown by Maillard *et al.* (2010), the ERC portfolio may be viewed as a minimum-variance portfolio subject to a constraint of sufficient diversification in terms of weights.

In the third approach, we assume that the risk premium is proportional to the volatility. It means that all components of the portfolio have the same Sharpe ratio s. The Sharpe ratio of the portfolio is defined by:

$$\mathrm{sh}\left(w\right) = \frac{w^\top\left(\mu - r\right)}{\sqrt{w^\top \Sigma w}} = s\, \frac{w^\top \sigma}{\sqrt{w^\top \Sigma w}}$$

Maximizing the Sharpe ratio is then equivalent to maximizing the dispersion ratio. This portfolio is known as the most diversified portfolio (Choueifaty and Coignard, 2008).

2.2 Utility theory

2.2.1 General framework

The utility function framework was introduced by Von Neumann and Morgenstern (1953) and has been widely used since by academics and practitioners to handle portfolio management problems. This theory states that, under some assumptions, the investor's preference in terms of returns distributions can be ordered with the expected value of a utility function. That is, an allocation is preferred to another if the expected utility of the resulting wealth at some fixed time T is higher. In mathematical terms, a portfolio w_1 resulting in some random wealth $W_T^{(w_1)}$ at time T is preferred to w_2 leading to $W_T^{(w_2)}$ if:

$$\mathbb{E}\left[U\left(W_T^{(w_1)}\right)\right] > \mathbb{E}\left[U\left(W_T^{(w_2)}\right)\right]$$

We can interpret $U(x)$ as the "satisfaction" of the investor to possess the wealth level x. Therefore, we must find the portfolio that achieves a maximum expectation of the utility function.

As described in Appendix A.2, this criterion features both appetite for gains and risk aversion. Of course, even if we suppose that the probability distribution of assets returns is perfectly known, we must specify the utility function for our problem. This is the major weakness of this kind of framework. Indeed, two difficulties appear. First of all, this utility function should be different for every investor, thus introducing difficulties in fund management. Second, the utility function of an investor is very difficult to formalize, and there is no clear method to build it. Nevertheless, in Appendix A.2 we come up with some intuitions on the behavior of utility functions and provide some fundamental examples.

2.2.2 Application to the empirical distribution

Let us consider the utility maximization problem. Once the utility function has been fixed, the main difficulty is to estimate the joint probability distribution of the asset returns. If we consider a gaussian distribution, the utility problem is strictly equivalent to the mean-variance problem. Indeed, with such assumptions, the returns of any portfolio is gaussian and is entirely characterized by its mean and variance parameters. Therefore, as the utility function introduces a risk adverse behavior together with an appetite for gains, the program consist in maximizing the average return while minimizing the volatility of the portfolio. This is equivalent to solving the optimization program (3). A non-gaussian probability distribution leads to a different optimal portfolio. Therefore, this choice is crucial. This problem is avoided by using the empirical distribution of the asset returns, i.e. maximizing the expected utility function through backtesting (Sharpe, 2007).

All allocations w are first backtested, leading to the historical monthly performance r_t^w of all corresponding portfolios. These backtests are then used to compute the historical utility of the corresponding portfolio as follows:

$$\hat{U}(w) = \frac{1}{n}\sum_{t=1}^{n} u\left(1 + r_t^w\right)$$

In the last step, we maximize the historical average utility function to find out the portfolio weights $w^\star$ such that:

$$w^\star = \arg\max \hat{U}(w)$$

This method can be interpreted as a "*let the data talk*" approach. No explicit statistical model assumptions are needed, and model risk is irrelevant. We only introduce assumption about the preferences of the investor (i.e. the utility function).

2.3 Omega ratio

2.3.1 Definition

The Omega ratio performance measure was introduced in Shadwick and Keating (2002). Let $\mathbf{F}$ be the cumulative distribution function of the portfolio returns. For a given threshold H, the Omega ratio is defined as follows:

$$\Omega(H) = \frac{\int_H^{+\infty} (1 - \mathbf{F}(x))\, \mathrm{d}x}{\int_{-\infty}^{H} \mathbf{F}(x)(x)\, \mathrm{d}x}$$

It can be also written as[9]:

$$\Omega(H) = \frac{\Pr\{R > H\} \times \mathbb{E}[R - H \mid R > H]}{\Pr\{R < H\} \times \mathbb{E}[R - H \mid R < H]}$$

Therefore, it can be interpreted as the ratio between the average returns above H and the average returns below H (or call-put ratio). If $H = 0$, it is the ratio between average gains and losses, weighted by the probabilities of those events. Therefore, H can be interpreted as the threshold above which a return is considered as a gain and below which the returns are considered as losses. An order of magnitude of Ω can be derived by remarking that setting the threshold H equal to the mean return $\mathbb{E}[R]$ of the portfolio leads to an Omega ratio equal to 1. In other words, $\Omega(\mathbb{E}[R]) = 1$ for any portfolio.

Let $\Omega^\star(H)$ be the optimal portfolio Omega ratio. Maximizing the Omega ratio for a given value H is equivalent to maximizing the following utility function:

$$\begin{aligned} U(x) &= (x - H)_+ - \Omega^\star(H)(H - x)_+ \\ &= x - H + (1 - \Omega^\star(H))(H - x)_+ \end{aligned}$$

When the optimal Omega ratio $\Omega^\star(H)$ is below 1, the utility function is convex, while an optimal ratio $\Omega^\star(H)$ above 1 corresponds to a concave utility function. The behavior of the investor is risk adverse when $\Omega^\star(H) > 1$, while it is risk-seeking when $\Omega^\star(H) < 1$. Lastly, an Omega ratio equal to 1 induces a risk-neutral behavior. When $\Omega^\star(H) < 1$, the risk seeking behavior is not conventional as an investor will try to maximize his risk for a given expected return level. Therefore, allocation based on the Omega ratio is misleading and inaccurate when $\Omega^\star(H) < 1$. This can be considered together with the property that the Omega ratio is equal to 1 when the target H is equal to the mean return of the portfolio. To conclude, the target H should be set carefully, ensuring that this performance level H is attainable on average by some portfolios based on available assets. For example, this is always the case if H is the return of the riskless asset.

[9] Let f be the density function of $\mathbf{F}$. We have:

$$\Omega(H) = \frac{\int_H^{+\infty}\int_x^{+\infty} f(y)\, \mathrm{d}y\, \mathrm{d}x}{\int_{-\infty}^{H}\int_{-\infty}^{x} f(y)\, \mathrm{d}y\, \mathrm{d}x} = \frac{\int_H^{+\infty} (x - H) f(x)\, \mathrm{d}x}{\int_{-\infty}^{H} (H - x) f(x)\, \mathrm{d}x} = \frac{\mathbb{E}\left[(X - H)_+\right]}{\mathbb{E}\left[(H - X)_+\right]}$$

2.3.2 Regularization of the Omega ratio using the Cornish-Fisher expansion

As in the case of utility maximization, the empirical distribution can be used to optimize the Omega ratio. However, this is a difficult task, as standard maximization procedures tend to perform poorly. Furthermore, as we use a limited set of returns (typically 24 monthly returns), there often exists an asset that never performs below the threshold H, thus leading to an infinite Omega ratio. This problem gets even worse when several assets exhibit this behavior, as any combination of these assets lead to an infinite Omega ratio, and the optimal allocation cannot be uniquely defined. Thus, we need a smoother criterion that takes into account the possibility of losses in any case. To this end, we introduce the Cornish-Fisher approximation of the empirical distribution of the portfolio. We compute the four first moments of the returns of every allocation we consider. These moments characterize the distribution of returns. The Cornish Fisher approximation takes those moments as an input to build a distribution, which we use to calculate the past Omega ratio of the considered portfolio. We do not consider the moments and distribution of each asset separately, but only the moments of the historical performance obtained by each portfolio constructed with those assets. Therefore, we do not need to consider explicitly the assets covariances, or higher order cross-moments, which may be poorly estimated. Details are given in Appendix A.3.

3 Hedge Funds portfolio specific models

There is a trade-off between model risk and estimation risk in the choice of an allocation model (see, e.g. Amenc and Martellini, 2002, in the case of hedge funds investing). If the model is simple, then the risk to miss some important characteristics of the return distribution is high. But this kind of model is likely to be more robust and associated with small estimation errors. On the contrary, the use of a more sophisticated model in general decreases the model risk, but also increases dramatically the estimation risk and generates robustness issues. The literature of hedge funds investing addresses this issue and proposes two extensions of the usual mean-variance framework that take into account the estimation risk dimension. The first extension starts from the expected utility maximization approach, but use finite dimension approximation of the utility function to first reduce the problem dimension, and then impose some structure on the covariance and higher-order comoments (Martellini and Ziemann, 2010). The second extension involves more complex specifications of the covariance matrix that is the central parameter driving portfolio allocation. A parsimonious dynamic specification of these covariance parameters are proposed in Giomouridis and Vrontos (2007).

3.1 Taking into account higher-order moments

A way to reduce model risk is to consider higher comoments, i.e. coskewness and cokurtosis, in addition to covariance. However, this approach increases further the estimation risk which is already high in the classic mean-variance framework when the number of assets is high. Therefore, one could wonder whether the framework using higher-order moments could be implemented in a realistic way. Martellini and Ziemann (2010) apply some statistical techniques already used in the mean-variance framework to higher-order moments estimation. These extensions are dimension reduction techniques based on constant correlation (Elton and Gruber, 1973) and single-factor based (Sharpe, 1963) estimators.

3.1.1 Finite dimensional expansion

To account for the impact of higher-order moments of asset returns on the portfolio allocation, Martellini and Ziemann (2010) consider the finite dimensional expansion of a standard expected utility maximization framework introduced in Section 2.2. Assuming infinite differentiability, the utility function U can be approximated as follows:

$$U(W) = \sum_{k=0}^{\infty} \frac{U^k(\mathbb{E}[W])}{k!}(W - \mathbb{E}[W])^k$$

where W is a random variable representing investor's terminal wealth. One can typically assume that the fourth-order development is sufficient to get a good approximation. We cut the sum after the fourth term and apply the expectation operator to both sides of the previous equation. We then get the approximated expected utility:

$$\mathbb{E}[U(W)] \simeq U(\mathbb{E}[W]) + \frac{1}{2}U^{(2)}(\mathbb{E}[W])\,\mu^{(2)} + \frac{1}{6}U^{(3)}(\mathbb{E}[W])\,\mu^{(3)} + \frac{1}{24}U^{(4)}(\mathbb{E}[W])\,\mu^{(4)}$$

where $\mu^{(n)} = \mathbb{E}[W - \mathbb{E}[W]]^n$ denotes the n^{th}-order centered moment of W. Thus, maximizing this approximated expected utility allows investors to account for first, second, but also third (skewness) and fourth (kurtosis) moments and comoments of the underlying assets in the portfolio. Portfolio choice is no longer a trade-off between expected return and volatility as in the mean-variance case. It involves high moments considerations that can be written as direct functions of portfolio weights.

To get an explicit form of the higher-order moments and comoments, Martellini and Ziemann (2010) introduce higher-order moment tensors M_2, M_3 and M_4 (see Jondeau and Rockinger, 2006 and Appendix A.4 for explicit expressions of these tensors). As in the general expected utility maximization framework, we choose a given U to completely specify the objective function. We use in the following a CARA utility function. More generally, this approach can be seen as a midpoint between the mean-variance and the expected utility maximization framework. Indeed, the two-order approximation corresponds to the mean-variance case, as the limiting case is the expected utility maximization case. This suggests applying covariance cleaning techniques developed in the mean-variance case to more complex allocation models.

3.1.2 Taking into account estimation risk

Even in the mean-variance case, the number of parameters to estimate increases dramatically with the number of assets. This problem is of course even more important when we consider higher-order moments. For an illustration, we develop the example of $n = 10$ assets. The number parameters involved in M_2 (respectively M_3 and M_4) is 55 (respectively 220 and 715). Martellini and Ziemann (2010) then address the dimension reduction issue when estimating the M_2, M_3 and M_4 tensors. Two improved estimators for the higher-order moment tensor matrices are proposed. The constant correlation estimator (Elton and Gruber, 1973) is a classic solution to overcome the covariance matrix estimator for portfolios containing a large number of assets. A constant correlation across the underlying assets improves the out-of-sample mean-variance portfolio performance despite the specification error. Martellini and Ziemann (2010) extend this approach to M_3 and M_4 estimators.

An unbiased estimator for the constant correlation parameter is given by the average of all sample correlation parameters:

$$\hat{\rho} = \frac{2}{N(N-1)} \sum_{i<j} \frac{\hat{\Sigma}_{i,j}}{\sqrt{\hat{\Sigma}_{i,i}\hat{\Sigma}_{j,j}}}$$

where $\hat{\Sigma}_{i,j}$ denotes the sample covariance between assets i and j. Thus, the covariance coefficient can be estimated as follows:

$$\hat{\sigma}_{i,j} = \hat{\rho}\sqrt{\hat{\Sigma}_{i,i}\hat{\Sigma}_{j,j}}$$

Martellini and Ziemann (2010) extend the idea of constant correlation to the context of higher-order moment tensors by including appropriate combinations of higher-order comoments according to the definition of the matrices M_3 and M_4.

The single-factor based estimator (Sharpe, 1963) assumes that a single factor explain the n individual asset returns:

$$R_{i,t} = c + \beta_i F_t + \varepsilon_{i,t}$$

where F_t is a well-diversified market index at time t and $\varepsilon_{i,t}$ is the idiosyncratic error term of asset i. The regression residual are assumed to be homoscedastic and cross-sectionally uncorrelated with covariance matrix Ψ. Based on this specification, the elements of the covariance matrix and the higher-order moment tensors have the following forms:

$$\begin{aligned} M_2 &= \beta\beta^\top \mu_0^{(2)} + \Psi \\ M_3 &= (\beta\beta^\top \otimes \beta^\top)\mu_0^{(3)} + \Phi \\ M_4 &= (\beta\beta^\top \otimes \beta^\top \otimes \beta^\top)\mu_0^{(4)} + \Gamma \end{aligned}$$

where $\mu_0^{(2)}$, $\mu_0^{(3)}$ and $\mu_0^{(4)}$ are respectively the second, third and fourth centered moments of the single factor and $\otimes$ designates the tensor product between two matrices. The detailed formulas of Φ, Ψ and Γ are provided in Martellini and Ziemann (2010).

3.2 Regime-switching dynamic correlation model

As our focus on hedge funds is portfolio allocation, it is natural to discuss the specification of the covariance matrix, the essential input of the mean-variance approach. Empirical evidence observed on hedge fund returns exclude static specifications and clearly support the use of dynamic models. There are numerous dynamic specifications but, following Giomouridis and Vrontos (2007), we consider regime-switching models for two reasons. First, they show that this approach is the most relevant to model hedge fund data. Second, we can control the increase in the number of parameters by limiting the number of states.

The notion of regime-switching models for correlation was introduced by Ang and Chen (2002) for explaining asymmetry in the correlation of an equity portfolio. This kind of stylized facts is also observed on hedge fund returns during the last subprime crisis, where correlations where submitted to brutal changes. Therefore, a model in which the correlation could switch between states of nature should be able to provide a better explanation of the asset's joint behavior. Intuitively, this switching approach can be seen as a midpoint between the constant correlation coefficient (Bollerslev, 1990) and the dynamic correlation coefficient (Engle 2002) models. The first model assumes constant correlation over time. The second allows correlations to change at every period. In the switching approach, correlation can only take a finite number of values. We use the dynamic specification proposed by Pelletier (2006) and applied to hedge fund allocation problem by Giomouridis and Vrontos (2007). More generally, regime-switching models are used in the hedge funds literature in a number of contexts including measuring the systemic risk (Chan *et al.*, 2006, Billio *et al.*, 2010), studying serial correlations (Getmansky *et al.*, 2004) and detecting switching strategies (Alexander and Dimitiu, 2004).

In this section we provide a short presentation of the regime-switching dynamic correlation (RSDC hereafter). The hedge fund returns are defined by:

$$R_t = \mu + \varepsilon_t$$

where $\varepsilon_t | \mathcal{I}_{t-1} \sim \mathcal{N}(0, V_t)$, μ is a $n \times 1$ vector of constants, ε_t is a $n \times 1$ innovation vector and $\mathcal{I}_t$ is the information available up to time t. The $n \times n$ covariance matrix V_t is decomposed into:

$$V_t = \Sigma_t C_t \Sigma_t$$

where Σ_t is a diagonal matrix composed of the standard deviations $\sigma_{i,t}$ $(i = 1, \ldots, n)$ and C_t is the correlation matrix. Both matrices are time-varying. In particular, the conditional variances $\sigma_{i,t}$ are modeled using a GARCH(1,1) specification of the form:

$$\sigma_{i,t}^2 = \alpha_i + \beta_i \varepsilon_{i,t}^2 + \gamma_i \sigma_{i,t-1}^2$$

while the correlation matrix C_t is modeled in a dynamic framework by using:

$$C_t = \sum_{k=1}^{K} \mathbf{1}\left\{S_t = k\right\} \cdot C_k$$

where $\mathbf{1}$ is the indicator function, S_t is an unobserved Markov chain process independent of ε_t which can take K possible values $(S_t = 1, 2, ..., K)$ and C_k are correlation matrices with $C_k \neq C_{k'}$ for $k \neq k'$. Regime switches in the state variable S_t are assumed to be governed by the transition probability matrix $\Pi = (\pi_{i,j})$. The transition probabilities between states follow a first order Markov chain:

$$\Pr\left\{ S_t = j \middle| S_{t-1} = i, S_{t-2} = k, \ldots \right\} = \Pr\left\{ S_t = j \middle| S_{t-1} = i \right\} = \pi_{i,j}$$

As in Pelletier (2006) and Giomouridis and Vrontos (2007), we assume that $K = 2$. The estimation of the RSDC model can be achieved by using a two-steps procedure (Engle, 2002). In the first step, we estimate the univariate GARCH model parameters. In the second step, we estimate the parameters in the correlation matrix and the transition probabilities $\pi_{i,j}$ conditional on the first step estimates. Details of the estimation procedure can be found in Pelletier (2006).

3.3 Taking into account stress scenarios

Hedge fund returns exhibit negative skewness and positive excess kurtosis. These mathematical properties can be explained through the exposure of these funds to extreme events. Indeed, hedge fund returns show stable positive returns with little volatility most of the time, but may experience severe drawdowns on rare occasions, with a magnitude exceeding several times their usual standard deviation. Furthermore, these rare events might not be observed on historical data. Finding a parametric law that matches this behavior is difficult, as the scope of such probability distributions is huge, and involve a precise knowledge of events that typically never happen.

The stress testing methodology precisely addresses this question (Berkowitz, 2000). It introduces changes in the asset simulation method, such as simulating shocks that occur more likely than historical observation suggests, or introducing shocks that never happened, to reflect a potential structural change or a major crisis. Therefore, we consider a generic and intuitive method, with the introduction of stress scenarios into the previously used probability distribution. This framework can be applied to any parametric or empirical distribution, as well as in the mean-variance framework. We add to the probability distribution of asset returns a set of scenarios that may happen with a given probability (see Appendix A.5 for details).

3.4 How to incorporate the manager's views?

The classical method to incorporate the manager's view is to use the Black-Litterman (BL) method described in Appendix A.6. Given a reference portfolio, the idea of this model is to modify the allocation in order to take into account the views of the manager. It may be viewed as a model of tactical asset allocation.

The BL portfolio is a weighted average of the reference portfolio w_0 and the portfolio $w^\star$ reflecting the views of the manager:

$$w = \alpha w_0 + (1 - \alpha)\, w^\star$$

One difficulty is to define the weight α. Generally, one considers a tracking error constraint to compute α. The input of the model is the value and the uncertainty of the views. We have:

$$P\mu = Q + \varepsilon$$

with $\varepsilon \sim \mathcal{N}(0, \Omega)$. Suppose for example that the manager thinks that the second strategy will outperform the first strategy by 3% in mean and that the expected return of the third strategy will be 10%. We then have:

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} 3\% \\ 10\% \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$$

Now, he has to define the uncertainty of his views. This is done by scenario analysis. For example, if he think that $\Pr\{\mu_2 > \mu_1 + x\%\} = y\%$, we deduce that the volatility of ε_1 is $(x\% - 3\%)/\,\Phi^{-1}(1 - y\%)$. For example, if the manager thinks that the probability that the second strategy will outperform the first strategy by 5% is 20%, the uncertainty or the volatility of ε_1 is 2.54%.

4 Empirical Results

4.1 Data

We illustrate our analysis by constructing an actively managed fund of funds investing in hedge fund indices from the CSFB/Tremont database. CSFB/Tremont computes monthly return data for the 10 strategy indices: convertible arbitrage (CA), dedicated short bias (SB), emerging markets (EM), equity market neutral (EMN), event driven (ED), fixed income arbitrage (FI), global macro (GM), long/short equity (LS), managed futures (MF) and multi-strategy (MS). CSFB/Tremont also publish a global index (CST Index) that corresponds to an asset-weighted average of all strategy index performances, with some additional concentration risk limit. The returns cover the time period from January 1994 through May 2010 for a total of 197 monthly returns. It includes a number of crises that occurred in the 1990s, i.e. the Mexican, Asian, Russian, LTCM crises as well as the IT bubble in 2000 and the subprime crisis in 2008. This last crisis is particularly interesting since hedge fund performances during this crisis are disappointing and surprisingly correlated. Figure 1 plots the historical evolution of the 10 strategy indices compared to the global index and shows that this last crisis has a huge impact on most hedge fund strategies.

Figure 1: CSFB/Tremont strategy indices

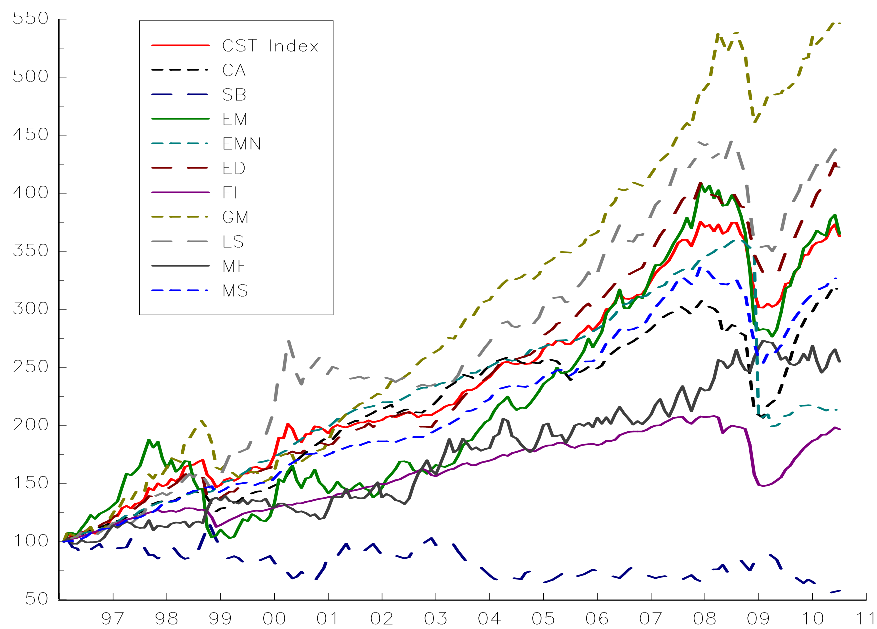


Table 1 reports summary statistics for the hedge fund indices returns over the period. It presents the annualized return, annualized volatility, Sharpe ratio, skewness, kurtosis, historical drawdown period (MaxDD), with additional information of the first and last month corresponding to this period (Start MDD and End MDD, respectively). The returns of the 10 hedge fund strategies are very heterogeneous in terms of risk profile. Some strategies have relatively high annualized volatilities, such as dedicated short bias, emerging markets, long/short equity and managed futures and can be considered as equity type investments. On the contrary, convertible arbitrage, event driven, fixed income arbitrage and multi-strategy exhibit low volatility levels and could be used in a portfolio to substitute some percentage of the fixed income holdings. Differences in the higher-order moments are also important. The kurtosis of the 10 indices returns ranges from 0.07 (managed futures) to 156.44 (equity market neutral), indicating fat-tailness in the return distributions of some strategies, in particular the ones related to arbitrage trading. Concerning the skewness, 5 strategies have null or slightly positive asymmetry (dedicated short bias, emerging markets, global macro, long/short equity and managed futures, i.e. directional strategies). The 5 remaining strategies exhibit significant negative asymmetry (convertible arbitrage, equity market neutral, even driven, fixed income arbitrage and multi-strategy, i.e. essentially arbitrage trading strategies). Finally, we observe that the strategies do not produce their historical drawdown during the same period. If the subprime crisis is the most difficult period for 6 strategies (convertible arbitrage, equity market neutral, even driven, fixed income arbitrage, long/short equity and multistrategy), global macro and emerging markets hit their drawdown in 1998 and managed futures in 1995. Interestingly, the last crisis allows the dedicated short bias strategy to end its drawdown period.

Table 1: CSFB/Tremont single strategy indices descriptive statistics

Hedge Funds strategy	Ann. Ret	Ann. Vol	Skew	Kurtosis	Max DD	Start MDD	End MDD
Convertible arbitrage	7.65%	7.18%	-2.72	15.70	-32.86%	Oct-07	Dec-08
Dedicated short bias	-2.92%	16.92%	0.75	1.62	-53.54%	Aug-98	Apr-10
Emerging markets	7.76%	15.43%	-0.76	4.85	-45.15%	Jul-97	Jan-99
Equity market neutral	5.10%	10.75%	-11.86	156.44	-45.11%	Jun-08	Feb-09
Event driven	10.20%	6.09%	-2.55	13.86	-19.15%	Oct-07	Feb-09
Fixed income arbitrage	4.98%	6.02%	-4.25	28.06	-29.03%	Jan-08	Dec-08
Global macro	12.32%	10.18%	-0.02	3.40	-26.78 %	Jul-98	Sep-99
Long/short equity	9.95%	10.02%	0.00	3.53	-21.97 %	Oct-07	Feb-09
Managed futures	6.12%	11.79%	0.02	0.07	-17.74 %	Mar-95	Nov-95
Multi-strategy	7.89%	5.45%	-1.78	6.29	-24.75 %	Oct-07	Dec-08

Table 2 reports correlation coefficients computed for the CSFB/Tremont strategy index returns. Index returns exhibit low to medium absolute pairwise correlation. Indeed, correlations range from a minimum of 6% between managed futures and long short equity, to a maximum of 78% for fixed income and convertible arbitrage. The average pairwise correlation is 22%. These low correlations indicate that the potential for risk diversification in hedge fund investment portfolios is high. However, high kurtosis observed in hedge fund returns suggests the analysis of covariance dynamics. Figure 2 plots the $24-$months rolling pairwise correlation analysis and clearly reveals the time-variation behavior of correlations. We observe that pairwise correlations vary over time suggesting that modeling time-varying correlations may improve optimal portfolio construction.

4.2 Traditional allocation approaches

In this section, we present the results of an investment exercise which compares the empirical out-of-sample performance of the 8 benchmark allocation models introduced in Section 2, i.e. mean-variance (MV), constant-Sharpe (CS), minimum-variance (MIN), equally-weighted risk contribution (ERC), most diversified portfolio (MDP), the two CARA and CRRA optimal portfolios (CARA & CRRA) and finally the portfolio obtained when maximizing the

Table 2: CSFB/Tremont single strategy indices correlations

		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Convertible arbitrage	(1)	1.00									
Dedicated short bias	(2)	-0.26	1.00								
Emerging markets	(3)	0.43	-0.54	1.00							
Equity market neutral	(4)	0.21	-0.13	0.14	1.00						
Event driven	(5)	0.66	-0.57	0.70	0.30	1.00					
Fixed income arbitrage	(6)	0.78	-0.20	0.41	0.32	0.55	1.00				
Global macro	(7)	0.34	-0.12	0.45	0.07	0.41	0.40	1.00			
Long/short equity	(8)	0.44	-0.68	0.65	0.19	0.72	0.38	0.46	1.00		
Managed futures	(9)	-0.09	0.09	-0.04	0.00	-0.06	-0.07	0.28	0.06	1.00	
Multi-strategy	(10)	0.70	-0.19	0.26	0.35	0.52	0.63	0.26	0.42	0.04	1.00

Figure 2: CSFB/Tremont strategy index rolling correlations

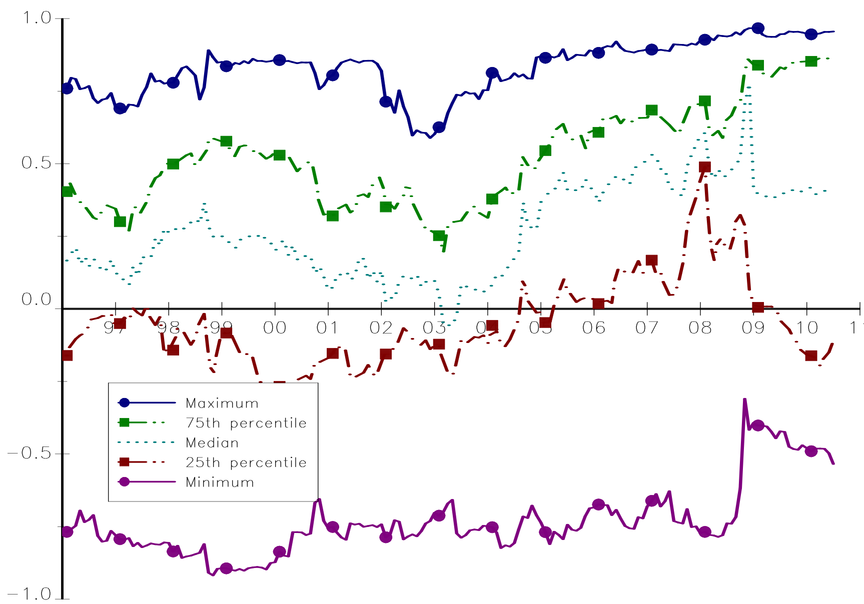
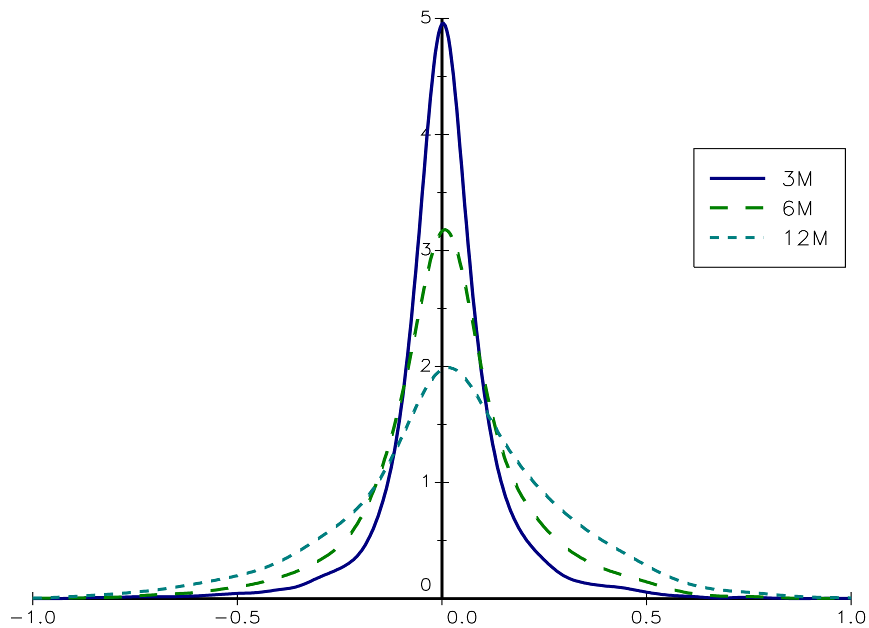


Figure 3: Density of the correlation variations



Omega ratio (OMEGA). For the mean-variance and constant-Sharpe problems, the volatility target is set to 7% by year. This value is also used to calibrate the parameters of the two utility functions CARA and CRRA. We also include in our analysis the portfolio invested in the global index and the equal-weighted portfolio. The setup of our experiments is the following. We use the first 24 months history of data to calibrate the empirical distribution or the model parameters. We then construct optimal hedge fund portfolios. Given the optimized weights, we calculate buy-and-hold returns on the portfolio for a holding period of 1 month, at the end of which the estimation and optimization procedures are repeated on the last 24 months until the end of the dataset. This exercise produces 173 out-of-sample observations that cover the period January 1996-May 2010.

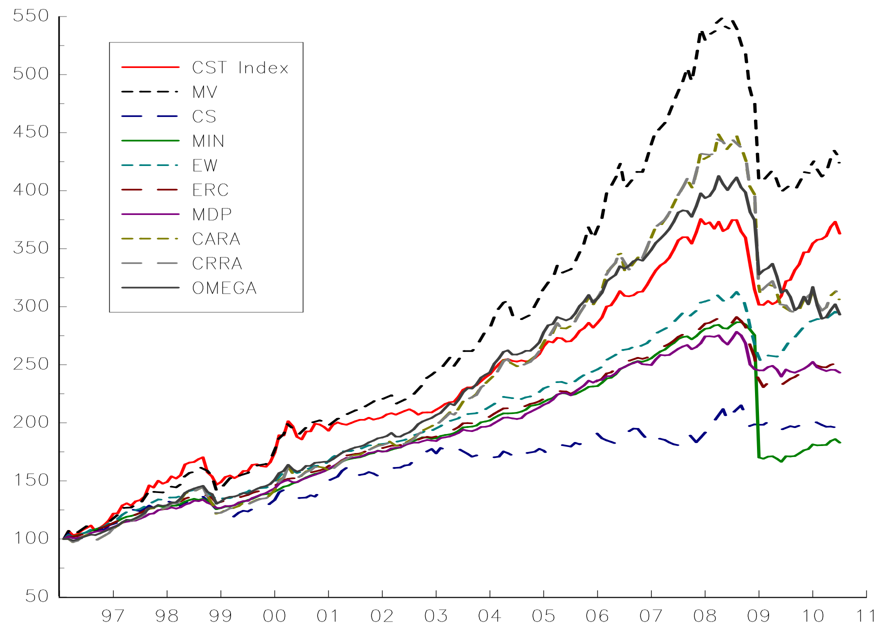
First, we examine the realized returns of the constructed portfolios. Given the fund weights $w_t = (w_{1,t}, w_{2,t}, ..., w_{10,t})$ at time t and the realized returns of the 10 indices at time $t+1$, the realized return $R_{p,t+1}$ of the portfolio at time $t+1$ is computed. Figure 4 report the backtests of these realized returns for the benchmark models. We analyze the portfolios performance both before and during the subprime crisis. The 10 backtested portfolios exhibit very heterogeneous performance and risk profiles. Mean-variance, CARA and CRRA are the most aggressive portfolios, with high positive returns during the first subperiod. However, these three portfolios suffer during the last period. On the contrary, minimum-variance, constant-Sharpe, ERC and MDP correspond to conservative portfolios, with low performance and risk during the first period when compared to the global index portfolio. However, if the drawdowns of the constant-Sharpe, ERC and MPD are limited during the last period, the recovery after this crisis is non significant versus the gains observed on the global index and the equal-weighted portfolios over the same period. The OMEGA portfolio has very disappointing returns during the post crisis period.

Table 3: Out-of-sample performance of efficient portfolios with 1-month rebalancing

Model	Ann. Ret	Ann. Vol	Skew	Kurtosis	Max DD	Start MDD	End MDD
Panel A: Raw statistics							
Global index	9.3%	7.8%	-0.2	5.7	-19.7%	Oct-07	Dec-08
Equal-weighted	7.7%	4.7%	-1.6	9.4	-18.8%	Jun-08	Dec-08
Mean-variance	10.5%	9.0%	-1.5	9.6	-28.2%	Feb-08	Apr-09
Constant-Sharpe	4.7%	7.0%	-0.2	3.2	-14.7%	May-98	Jan-99
Minimum-variance	4.3%	10.6%	-11.6	146.7	-41.9%	Jul-08	Apr-09
ERC	6.5%	4.5%	-4.1	29.7	-20.7%	Jun-08	Dec-08
MDP	6.4%	3.9%	-2.0	13.0	-13.8%	Jun-08	Apr-09
CARA	8.0%	9.7%	-3.2	24.6	-34.3%	Feb-08	Jul-09
CRRA	8.0%	9.7%	-3.2	24.4	-34.3%	Feb-08	Jul-09
Omega	7.7%	6.8%	-2.4	14.5	-29.8%	Feb-08	Jan-10

Model	SR	TE	IR	Alpha	DD 1M	DD 6M	DD 1Y
Panel B: Risk-adjusted statistics and drawdown analysis							
Global index	0.72				-7.5%	-19.5%	-19.1%
Equal-weighted	0.82	4.5%	-0.34	-1.5%	-6.4%	-18.8%	-17.4%
Mean-variance	0.74	5.5%	0.19	1.0%	-14.3%	-25.3%	-25.8%
Constant-Sharpe	0.13	9.2%	-0.46	-4.2%	-5.4%	-14.3%	-10.6%
Minimum-variance	0.05	11.0%	-0.42	-4.6%	-38.2%	-41.0%	-40.8%
ERC	0.61	6.1%	-0.43	-2.6%	-9.8%	-20.7%	-19.3%
MDP	0.66	6.3%	-0.43	-2.7%	-6.9%	-11.9%	-12.2%
CARA	0.44	7.0%	-0.17	-1.2%	-21.4%	-29.6%	-33.7%
CRRA	0.44	7.0%	-0.17	-1.2%	-21.4%	-29.6%	-33.8%
Omega	0.58	6.3%	-0.23	-1.5%	-12.5%	-19.4%	-26.3%

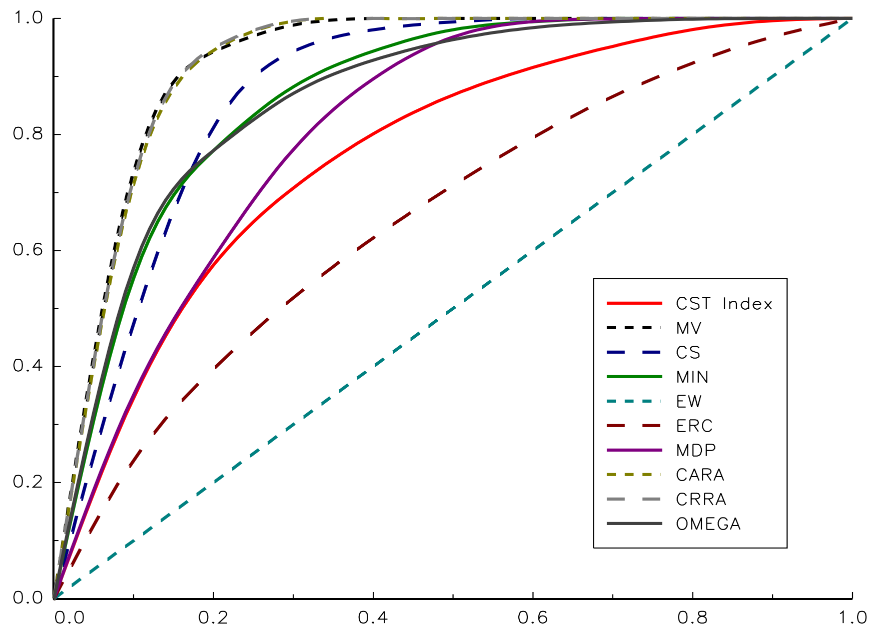
Figure 4: Backtest of traditional allocation approach with CSFB/Tremont strategy indices



Second, we compare the different allocation models in terms of risk-adjusted returns. Portfolio optimization generally produces heterogeneous volatility portfolios. As a result, realized returns are not directly comparable across models since they represent portfolios bearing different risks. Table 3 reports these results of the risk-adjusted analysis. Panel A presents the annualized return, annualized volatility, skewness, kurtosis and historical drawdown period information. All the portfolios, except the constant-Sharpe, hit their drawdown during the subprime crisis. The particular behavior of the mean-variance portfolio appears immediately on the higher-order moments of this portfolio's returns distribution. Panel B displays the risk-adjusted statistics (Sharpe ratio (SR), tracking error (TE), information ratio (IR) and Jensen alpha) and a detailed analysis of the drawdown over 1-month (DD 1M), 6-month (DD 6M) and 1-year (DD 1Y) periods. The best portfolios in terms of Sharpe ratio are the equal-weight (0.82) and the mean-variance (0.74). The other portfolios have lower Sharpe ratio than the global index (0.72). The range goes from 0.66 for the most diversified portfolio to 0.12 for the constant-Shape portfolio. These results confirm the poor performance of actively managed portfolios. Yearly returns and annually volatility may be found in Tables 3 and 4 in Appendix A. Figure 5 indicates the average Lorenz curve of the weights concentration.

Third, we discuss the model choice in terms of transaction costs. Transaction costs associated with hedge funds, however, are not generally easy to compute (Alexander and Dimitriu, 2004). Nevertheless, if the gain in the performance does not cover the extra transaction costs, less accurate, but less variable weighting strategies would be preferred. To study this issue we define portfolio turnover as in Greyserman *et al.* (2006), that is the sum of the absolute changes in the portfolio weights from the previous month to that month. This metric intuitively represents the fraction of the portfolio value that has to be liquidated/reallocated at the point of rebalancing.

Figure 5: Average Lorenz curve of the portfolio weights



4.3 Including higher-order moments

We reproduce in this section the same investment exercise, but with the allocation models introduced in Section 3.1. These portfolios are obtained by maximizing a finite dimensional approximation of a CARA utility function. They differ on two points: the approximation order (2 or 4), and the estimators of the higher-order moments (sample moments, constant correlation or single factor estimators). We finally get 6 different portfolios. The corresponding cumulative returns are plotted in Figure 6.

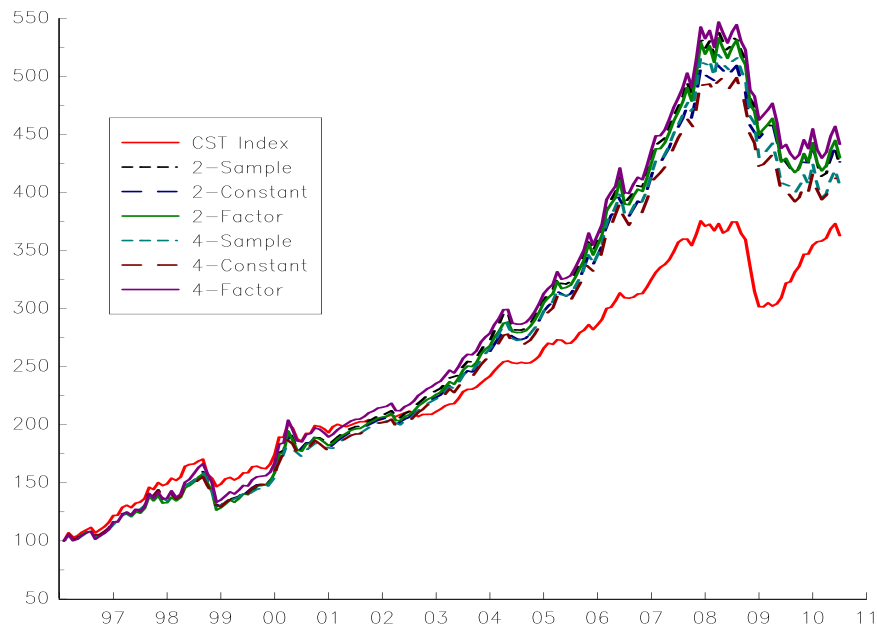
The risk-adjusted analysis of these six portfolios is presented in Table 4. Panel A reports the annualized statistics. We observe that all portfolios have very similar characteristics. However, two main comments can be done. First, we do not significantly increase the skewness and decrease the kurtosis of the out-of-sample portfolio when higher-order moments are included in the objective function. Similarly, the historical drawdown is of the same order and occurred during the same period. Second, the sample estimators are always given the less performing portfolios. In other words, it is always interesting to add structure to the estimation of covariance and higher-order moments. Martellini and Ziemann (2010) report similar results. Panel B confirms these findings. The Sharpe performance decreases if higher-order moments are introduced in the analysis without limiting the estimation risk. The only way to recover the initial global index performance is to use the constant correlation estimator together with the *4th* order expansion. These disappointing results can be explained by the use of index data. The non-normality on these indices, i.e. average returns, is not as severe as what we observe on single hedge fund returns. It would be interesting to apply these utility expansion approaches to less normal distributed return assets. The moment component analysis proposed by Jondeau *et al.* (2010) is also a promising solution to control the estimation risk related to the higher moments.

Table 4: Out-of-sample performance of efficient portfolios with 1-month rebalancing

Model	Estimation	Ann. Ret	Ann. Vol	Skew	Kurtosis	Max DD	Start MDD	End MDD
Panel A: Raw statistics								
Global index		9.3%	7.8%	-0.2	5.7	-19.7%	Oct-07	Dec-08
2nd order	Sample	10.5%	10.1%	-0.3	5.3	-23.3%	Feb-08	Jan-10
2nd order	Constant	10.6%	9.6%	-0.2	4.7	-19.0%	Feb-08	Jul-09
2nd order	Factor	10.6%	9.8%	-0.4	5.5	-21.7%	Feb-08	Jul-09
4th order	Sample	10.2%	10.0%	-0.5	5.0	-24.1%	Feb-08	Jan-10
4th order	Constant	10.1%	9.4%	-0.4	4.4	-22.1%	Feb-08	Jul-09
4th order	Factor	10.8%	10.1%	-0.4	5.2	-21.6%	Feb-08	Jul-09

Model	Estimation	SR	TE	IR	Alpha	DD 1M	DD 6M	DD 1Y
Panel B: Risk-adjusted statistics and drawdown analysis								
Global index		0.72				-7.5%	-19.5%	-19.1%
2nd order	Sample	0.67	5.7%	0.19	1.1%	-10.6%	-15.5%	-21.4%
2nd order	Constant	0.71	5.3%	0.21	1.1%	-9.1%	-14.4%	-17.4%
2nd order	Factor	0.70	5.4%	0.22	1.2%	-11.1%	-15.0%	-20.6%
4th order	Sample	0.64	5.6%	0.14	0.8%	-10.5%	-16.3%	-22.2%
4th order	Constant	0.67	5.2%	0.14	0.7%	-9.1%	-15.2%	-20.6%
4th order	Factor	0.70	5.6%	0.24	1.4%	-11.2%	-15.1%	-20.4%

Figure 6: Backtest of the allocation methods with higher-order moments



4.4 Including regime-switching dynamic correlation

We now examine the benefits of introducing dynamic structure for the covariance of hedge fund returns in hedge funds portfolio construction. We need to adapt the previous investment exercise. Indeed, the calibration of a regime-switching model demands a long history of data, and then out-of-sample analysis is impossible with monthly data. We consider then in this section a subset of HFRX strategy indices that display daily returns: convertible arbitrage, equity hedge, equity market neutral, event driven, macro and merger arbitrage. Our sample goes from 1 April 2003 to 16 August 2010, i.e. 1 848 daily returns. We compare performance of the HFRX global index, the constant-Sharpe portfolio with a static covariance specification, and the constant-Sharpe portfolio with a 2 states regime-switching covariance specification. We use the 300 first days of data to calibrate the model parameters. We then construct optimal portfolios and compute the buy and hold portfolios returns for 1 day. At the end of the day, we add the current return data to the calibration set and restart the optimization procedures. This exercise produces 1 548 out-of-sample observations.

First, we comment portfolio performance in terms of cumulative returns using Figure 7. Compared to the global index and the constant-Sharpe portfolio, the regime-switching portfolio gives the portfolio with the highest cumulative return. Table 5 reports the usual corresponding statistics. The regime-switching model gives the portfolio that realizes the highest out-of-sample performance. The most interesting feature is the portfolio returns during the crisis. The switch in the correlation coefficient limits the maximum drawdown of the actively managed portfolio (-10.0% instead of -26.3% for the passive portfolio). As a consequence, the annualized return of this portfolio is higher, the volatility lower and the Sharpe ratio multiplied by 4.

Figure 7: Backtest of the allocation methods with HFR Hedge Funds sub-indexes

Table 5: Out-of-sample performance of efficient portfolios with 1-day rebalancing

Model	Ann. Ret	Ann. Vol	Skew	Kurtosis	Max DD	Start MDD	End MDD
Panel A: Raw statistics							
Global index	0.7%	4.4%	-1.2	7.9	-26.3%	Jul-07	Dec-08
Equal-weighted	0.6%	3.7%	-1.7	11.2	-23.2%	Jul-07	Dec-08
Constant-Sharpe	-0.5%	3.3%	-1.9	16.1	-24.5%	Jul-07	Dec-08
regime-switching	2.9%	2.5%	-0.8	4.4	-10.0%	Jul-07	Nov-08
Model	**SR**	**TE**	**IR**	**Alpha**	**DD 1M**	**DD 6M**	**DD 1Y**
Panel B: Risk-adjusted statistics and drawdown analysis							
Global index					-12.3%	-22.9%	-23.4%
Equal-weighted		1.6%	-0.1	-0.61%	-12.1%	-20.9%	-20.1%
Constant-Sharpe		2.9%	-0.4	-2.13%	-13.2%	-21.3%	-21.8%
regime-switching	0.02	3.4%	0.6	0.80%	-6.7%	-5.6%	-7.9%

Finally, we briefly discuss the average structure of hedge fund portfolios constructed with the static and dynamic models. The weights of the assets in the regime-switching portfolio move faster than the ones obtained in the static case. We note that four strategies, merger arbitrage, macro, equity market neutral and convertible arbitrage have the highest weights in the average structure of the regime-switching portfolio. One of the main interesting characteristics of the model is to cut the convertible arbitrage exposure during the last crisis and increase it at the beginning of 2009.

In summary, we have found that modeling time-varying covariance of hedge fund returns improves our ability to optimize hedge fund portfolio risk. This is reflected in the reduced risk of the portfolios constructed with the dynamic covariance models relative to the risk of the portfolios constructed with the other models.

4.5 Including stress scenarios

Stress scenarios reflect the possibility of extreme drawdowns linked to some rare events. Table 6 gives a canonical example of stress scenarios definition.

Table 6: An example of stress scenarios

Intensity	CA	SB	EM	EMN	ED	FI	GM	LS	CTA	MS
10%	−15%	0	0	0	0	0	0	0	0	0
10%	0	−15%	0	0	0	0	0	0	0	0
10%	0	0	−15%	0	0	0	0	0	0	0
10%	0	0	0	−15%	0	0	0	0	0	0
10%	0	0	0	0	−15%	0	0	0	0	0
10%	0	0	0	0	0	−15%	0	0	0	0
10%	0	0	0	0	0	0	−15%	0	0	0
10%	0	0	0	0	0	0	0	−15%	0	0
10%	0	0	0	0	0	0	0	0	−15%	0
10%	0	0	0	0	0	0	0	0	0	−15%

The possibility of individual drawdowns introduces idiosyncratic risks to the probability measure, which favor portfolio diversification. As the stress scenarios are the same for all components of the index, the risk level of low volatility components is increased more proportionally. In other words, less faith is given to low volatility assets. As a result, the allocation procedure leads naturally to better diversified and more homogeneous portfolios, without imposing any diversification constraint. This is confirmed by the following historical simulations.

The stress scenarios given in Table 6 illustrate how they can favor diversification. We only consider in this example individual drawdowns of the same magnitude, with equal intensities. A deeper study of hedge fund strategies types could lead to more sophisticated features, such as different magnitudes, intensities, or joined stress scenarios involving some or all of the hedge fund strategies types. Such a study could be based on both quantitative and qualitative properties of those strategies, for example their risk profiles or their typical exposure to main asset classes. Furthermore, such fundamental studies on hedge funds or hedge fund classes could highlight hidden risks that a fund of funds manager would be reluctant to take. The stress scenario approach brings an adapted formalization of those views, introducing those expectations into the allocation procedure. Therefore, it can be a way to reduce exposure to hedge funds that are considered hazardous by the manager.

Figure 8: Backtest of the Stress Scenarios approach

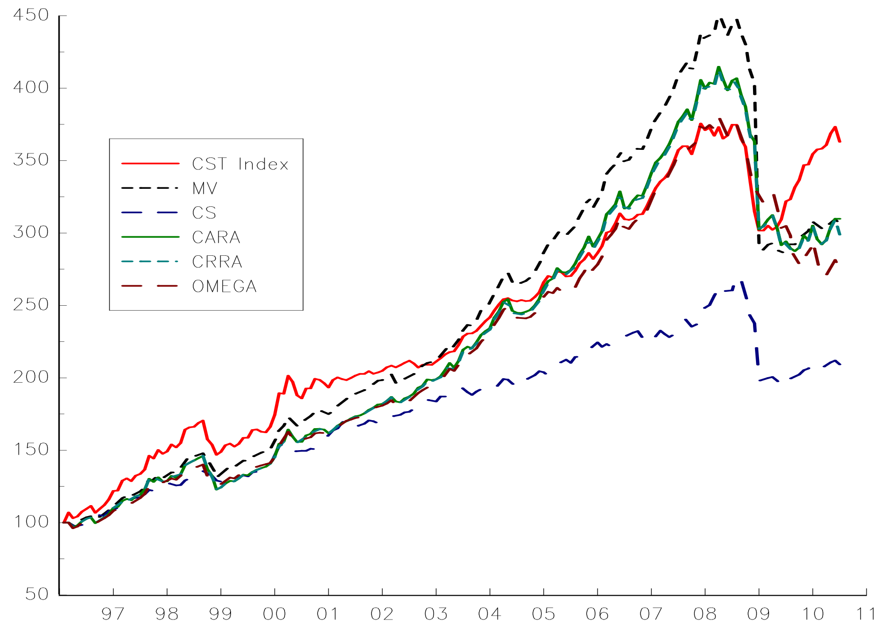
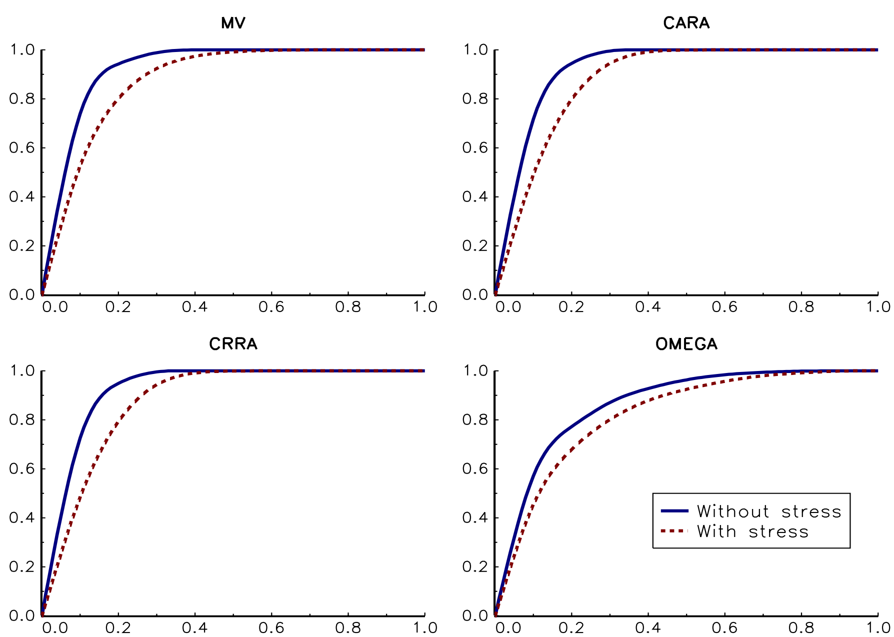


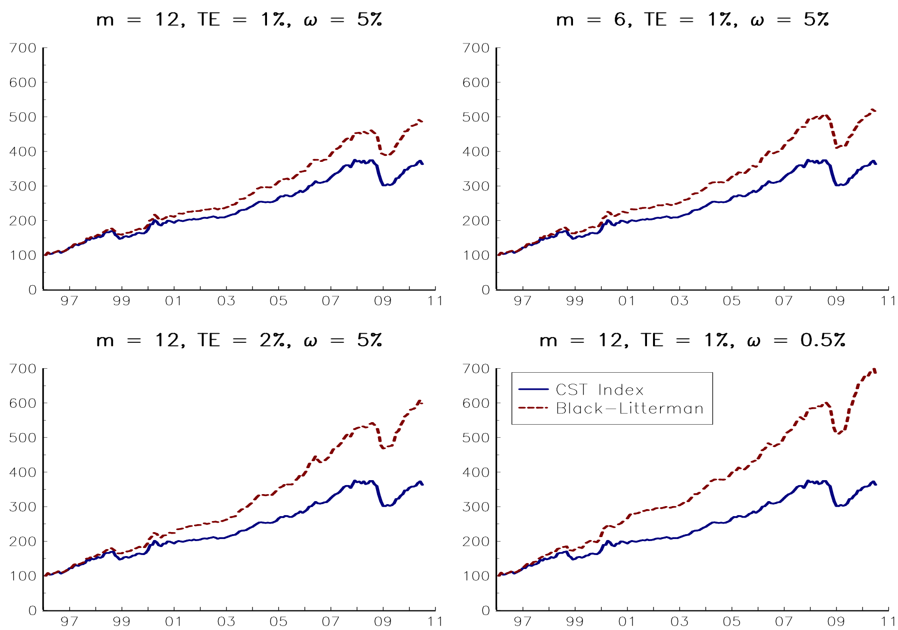
Figure 9: Comparison of average Lorenz curve with and without stress



4.6 Including views

We suppose that the manager knows the average returns for the next m months. We assume that the uncertainty matrix Ω is a diagonal matrix with entry ω. We report the results in Figure 10. As expected, the Black-Litterman approach produces better results than the previous ones. We notice than the performance depends on two main parameters. Active bets increase with the tracking-error (TE) level and decrease with the uncertainty parameter.

Figure 10: Backtest of the Black-Litterman approach



Of course, the previous example is not realistic, but it gives an idea about the calibration of views in a tactical allocation framework. In particular, the key point is not the tracking error target, but the joint statistics of tracking error and uncertainty. Indeed, in an allocation process, the portfolio manager has to manage the tracking error level with respect to the uncertainty of the views given by the hedge funds analysts.

5 Conclusions

To evaluate the impact of non-normal distribution and time-varying parameters, we illustrate our analysis using an actively managed fund of hedge funds invested in hedge fund indices from the CSFB/Tremont and HFR databases. Our main findings are summarized in the five following points. First, the application of mean-variance and extension models does not result in performing portfolios of hedge funds. In terms of risk-adjusted returns, all such portfolios underperformed the CSFB/Tremont global index portfolio. In other words, active management destroyed value as compared to a static investment in the same underlyings. Second, the inclusion of higher-order moments at the objective function level does not solve this problem. If the model takes into account more hedge fund returns characteristics, the loss in terms of estimation error is too high. This result persists even when some robustness techniques are used at the estimation level. Third, regime-switching dynamic correlation models are able to capture changes in correlation between hedge fund strategies and result in better performing hedge fund portfolios with better diversification. This gives some intuition about the failings of classic models, and implies that the correlation dynamics between hedge fund returns are the main feature allocation models must integrate. Fourth, considering stress scenarios in the allocation process also increases the diversification level of efficient portfolios. Fifth, views considered in a Black-Litterman framework result in optimal active portfolios that outperform a static allocation.

Appendix

A Mathematical aspects of allocation models

A.1 Shrinkage methods

We remind that the shrinkage estimator is:

$$\hat{\Sigma}_{\alpha^\star} = \alpha^\star \hat{\Phi} + (1 - \alpha^\star)\,\hat{\Sigma}$$

with:

$$\alpha^\star = \max\left(0, \min\left(\frac{1}{T}\frac{\pi - \varrho}{\gamma}, 1\right)\right)$$

If $\hat{\Phi}$ is the covariance matrix with a constant correlation[10] $\bar{\rho}$, we obtain (Ledoit and Wolf, 2004):

$$\begin{aligned}
\pi_{i,j} &= \frac{1}{T}\sum_{t=1}^{n}\left((x_{i,t} - \bar{x}_i)(x_{j,t} - \bar{x}_j) - \hat{\Sigma}_{i,j}\right)^2 \\
\vartheta_{i,j} &= \frac{1}{T}\sum_{t=1}^{n}\left((x_{i,t} - \bar{x}_i)^2 - \hat{\Sigma}_{i,j}\right)\left((x_{i,t} - \bar{x}_i)(x_{j,t} - \bar{x}_j) - \hat{\Sigma}_{i,j}\right) \\
\pi &= \sum_{i=1}^{n}\sum_{j=1}^{n}\pi_{i,j} \\
\varrho &= \sum_{i=1}^{n}\pi_{i,i} + \sum_{i=1}^{n}\sum_{j\neq i}\frac{\bar{\rho}}{2}\left(\sqrt{\frac{\hat{\Sigma}_{j,j}}{\hat{\Sigma}_{i,i}}}\vartheta_{i,j} + \sqrt{\frac{\hat{\Sigma}_{i,i}}{\hat{\Sigma}_{j,j}}}\vartheta_{j,i}\right) \\
\gamma &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\hat{\Phi}_{i,j} - \hat{\Sigma}_{i,j}\right)^2
\end{aligned}$$

In the more general case where $\hat{\Phi}$ is the covariance matrix of the one-factor model, the expressions of π and γ are the same and ϱ becomes:

$$\varrho = \sum_{i=1}^{n}\sum_{j=1}^{n}\varrho_{i,j}$$

with $\varrho_{i,i} = \pi_{i,j}$ and:

$$\begin{aligned}
\varrho_{i,j} &= \frac{1}{T}\sum_{t=1}^{n}\varrho_{i,j,t} \\
\varrho_{i,j,t} &= \left(\hat{\Sigma}_{j,0}\hat{\Sigma}_{0,0}(x_{i,t} - \bar{x}_i) + \hat{\Sigma}_{i,0}\hat{\Sigma}_{0,0}(x_{j,t} - \bar{x}_j) - \hat{\Sigma}_{i,0}\hat{\Sigma}_{j,0}\left(f_t - \bar{f}\right)\right) \times \\
&\quad (x_{i,t} - \bar{x}_i)(x_{j,t} - \bar{x}_j)\left(f_t - \bar{f}\right)\big/\,\hat{\Sigma}_{0,0}^2 - \hat{\Phi}_{i,j}\hat{\Sigma}_{i,j}
\end{aligned}$$

In the expression of $\varrho_{i,j,t}$, we use the augmented matrix $\hat{\Sigma}$ corresponding to the empirical covariance matrix of (f_t, X_t) with the convention that the position of the factor in the matrix is 0. Thus, $\hat{\Sigma}_{i,0}$ is the empirical covariance between f_t and $x_{i,t}$, and $\hat{\Sigma}_{0,0}$ is the empirical variance of the factor.

[10] We have $\hat{\Phi}_{i,i} = \hat{\Sigma}_{i,i}$ and $\hat{\Phi}_{i,j} = \bar{\rho}\sqrt{\hat{\Sigma}_{i,i}\hat{\Sigma}_{j,j}}$.

A.2 Utility function characteristics

The two main features of a utility function are the appetite for gain and the risk aversion. The appetite for gain is very natural, and can be translated as "more is better than less".

A.2.1 Some properties

The appetite for gain is captured by the increasing property of U, i.e. its derivative U' is positive. The other main feature is the investor's risk aversion, translating the fact that the investor would always prefer a fixed wealth to a random wealth with the same expectation. This feature corresponds to the concavity of the function U with respect to terminal wealth. Therefore, for a fixed level of risk, the investor would try to maximize his average return, and controversy, for a fixed level of return the investor would try to minimize his risks. On the contrary, a convex utility function (with $U'' > 0$) would imply that the investor is risk seeking and prefers to maximize his risk for a fixed average return.

The qualitative properties of utility functions are now stated, and we can study the quantitative properties of these functions. In particular, there is a clear dilemma between the appetite for high returns and the risk aversion. Indeed, if we consider only one risky asset and a riskless asset, and if this asset has a known positive expected return, then the money invested in the risky asset may range from 0 to $+\infty$ depending of the shape of the utility function. But fortunately, that behavior can be synthesized in a risk aversion coefficient A given by $-U''(x)/U'(x)$ (or $-U''(x)/(xU'(x))$ when relative wealth is considered. This coefficient may depend of the level of wealth. Two particular cases are interesting: the constant absolute risk aversion (CARA) or the constant relative risk aversion (CRRA). In the CARA case, with stable market conditions, the optimal investment strategy is to keep a fixed absolute amount of risk (i.e. a fixed amount of money invested in the risky asset). Meanwhile, with a CRRA utility, the optimal strategy keeps a fixed proportion of the investor's wealth invested in the risky asset. Therefore, those two fundamental examples differ on the absolute or relative reference to risks.

A.2.2 Fundamental examples

The CARA utility can be shown to be of the form:

$$U(x) = 1 - \exp\left(-\frac{x}{\gamma}\right)$$

Where the parameter γ is homogeneous to some wealth, and is therefore referred to as the wealth at risk of the utility function. The absolute risk aversion coefficient is equal to $1/\gamma$. Thus, the risk aversion parameter increases naturally as the target wealth at risk decreases. In the case of a single risky asset, it can be shown that the amount w invested in that risky asset is equal to:

$$w = \gamma\, \frac{\mu - r}{\sigma^2}$$

In other words, the volatility of the optimal investment strategy is equal to:

$$\sigma(w) = w \times \sigma = \gamma \times \mathrm{sh}$$

where sh is the Sharpe ratio of the risky asset. As the Sharpe ratio of a good risky asset can be in general estimated around 50%, the parameter γ actually deserves its name of wealth at risk, as it is of the same order of magnitude as the volatility of the optimal portfolio for this utility function.

Meanwhile, the CRRA utility can be written as:

$$U(x) = \frac{x^{\phi}}{\phi}$$

with $\phi < 1$ (in order to ensure a risk adverse behavior). Therefore the relative risk aversion coefficient is given by:

$$A = -\frac{U''(x)}{xU'(x)} = 1 - \phi$$

and the volatility of the optimal allocation is given by:

$$\sigma(w) = \frac{1}{1-\phi}\frac{\mu - r}{\sigma} = \frac{\text{sh}}{1-\phi}$$

Therefore, the volatility of the optimal allocation has the order of magnitude of the product of the Sharpe ratio of the risky asset and a factor $1/(1-\phi)$ which represents the risk tolerance. Hence, given the value of the Sharpe ratio of the allocation, the risk aversion coefficient $A = 1 - \phi$ may be calibrated to set the volatility of the optimal portfolio at a given value.

A.3 Computation of the Omega ratio with the Cornish-Fisher expansion

The Cornish Fisher approximation can be described as follows. To obtain a random variable Z with an average μ, a standard deviation σ, a skewness s and an excess Kurtosis κ, it is sufficient to consider a polynomial of a standard Gaussian random variable X, that is:

$$Z = P(X) = \mu + \sigma\left(X + \left(X^2 - 1\right)\frac{s}{6} + \left(X^3 - 3X\right)\frac{\kappa}{24} - \left(2X^3 - 5X\right)\frac{s^2}{36}\right)$$

as long as the following condition is satisfied:

$$\frac{s^2}{9} - 4\left(\frac{\kappa}{8} - \frac{s^2}{6}\right)\left(1 - \frac{\kappa}{8} + \frac{5s^2}{36}\right) \leq 0$$

Therefore, to calculate the Omega ratio:

$$\Omega(H) = \frac{\mathbb{E}\left[(Z-H)_+\right]}{\mathbb{E}\left[(H-Z)_+\right]} = \frac{\mathbb{E}\left[(Z-H)_+\right]}{H - \mu + \mathbb{E}\left[(Z-H)_+\right]}$$

we just have to calculate the expectation of a the positive part of a gaussian polynomial.

Let $P(x)$ the function defined by:

$$P(x) = \mu + \sigma\left(x + \left(x^2 - 1\right)\frac{s}{6} + \left(x^3 - 3x\right)\frac{\kappa}{24} - \left(2x^3 - 5x\right)\frac{s^2}{36}\right)$$

A simple calculus gives:

$$\begin{aligned} P(x) &= \mu - \sigma\frac{s}{6} + \sigma\left(\left(1 - \frac{\kappa}{8} + \frac{5s^2}{36}\right)x + \frac{s}{6}x^2 + \left(\frac{\kappa}{24} - \frac{s^2}{18}\right)x^3\right) \\ &= Q(x) \end{aligned}$$

Using this result, the formula of the expected return above H when the distribution is given by the Cornish Fisher expansion is:

$$\begin{aligned}
\mathbb{E}\left((Z-H)_+\right) &= \int_{P^{-1}(H)}^{\infty} \frac{1}{\sqrt{2\pi}} P(x)\, e^{-\frac{x^2}{2}}\, \mathrm{d}x \\
&= \int_{P^{-1}(H)}^{\infty} \frac{1}{\sqrt{2\pi}} Q(x)\, e^{-\frac{x^2}{2}}\, \mathrm{d}x \\
&= \left(\mu - \sigma\frac{s}{6}\right) A_0(H) + \sigma\left(1 - \frac{\kappa}{8} + \frac{5s^2}{36}\right) A_1(H) + \\
&\quad \sigma\frac{s}{6} A_2(H) + \sigma\left(\frac{\kappa}{24} - \frac{s^2}{18}\right) A_3(H)
\end{aligned}$$

where the coefficient A_0, A_1, A_2 and A_3 are given by the following expressions :

$$\begin{aligned}
A_0(H) &= \int_{P^{-1}(H)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, \mathrm{d}x \\
&= 1 - \Phi\left(P^{-1}(H)\right) \\
A_1(H) &= \int_{P^{-1}(H)}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-\frac{x^2}{2}}\, \mathrm{d}x \\
&= \left[-\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right]_{P^{-1}(H)}^{\infty} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(P^{-1}(H)\right)^2}{2}} \\
A_2(H) &= \int_{P^{-1}(H)}^{\infty} \frac{1}{\sqrt{2\pi}} x^2 e^{-\frac{x^2}{2}}\, \mathrm{d}x \\
&= \left[-\frac{1}{\sqrt{2\pi}} x e^{-\frac{x^2}{2}}\right]_{P^{-1}(H)}^{\infty} + \int_{P^{-1}(H)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} P^{-1}(H)\, e^{-\frac{\left(P^{-1}(H)\right)^2}{2}} + 1 - \Phi\left(P^{-1}(H)\right)
\end{aligned}$$

$$\begin{aligned}
A_3(H) &= \int_{P^{-1}(H)}^{\infty} \frac{1}{\sqrt{2\pi}} x^3 e^{-\frac{x^2}{2}}\, \mathrm{d}x \\
&= \left[-\frac{1}{\sqrt{2\pi}} x^2 e^{-\frac{x^2}{2}}\right]_{P^{-1}(H)}^{\infty} + \int_{P^{-1}(H)}^{\infty} \frac{2x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \left(P^{-1}(H)\right)^2 e^{-\frac{\left(P^{-1}(H)\right)^2}{2}} + \frac{2}{\sqrt{2\pi}} e^{-\frac{\left(P^{-1}(H)\right)^2}{2}}
\end{aligned}$$

Anyway, we must make sure that the Cornish Fisher expansion is well defined. This random variable defined by this expansion must be an increasing function of the underlying gaussian variable. This is insured by the following condition:

$$\frac{s^2}{9} - 4\left(\frac{\kappa}{8} - \frac{s^2}{6}\right)\left(1 - \frac{\kappa}{8} + \frac{5s^2}{36}\right) \leq 0$$

By denoting $\tilde{\kappa} = \frac{\kappa}{8}$ and $\tilde{s} = \frac{s^2}{9}$, we obtain:

$$\tilde{s} - 4\left(\tilde{\kappa} - \frac{3\tilde{s}}{2}\right)\left(1 - \tilde{\kappa} + \frac{5\tilde{s}}{4}\right) \leq 0$$

As we only look for a sufficient condition, we first consider the case where $\tilde{s} = 0$. In that case, the condition reduces to $-4\tilde{\kappa}\left(1-\tilde{\kappa}\right) \leq 0$. Thus the condition is satisfied for $0 \leq \tilde{\kappa} \leq 1$. If we suppose that it is fulfilled, we have the condition:

$$\begin{aligned}
\tilde{s} - 4\left(\tilde{\kappa} - \frac{3\tilde{s}}{2}\right)\left(1 - \tilde{\kappa} + \frac{5\tilde{s}}{4}\right) &= \tilde{s} - 4\left(\tilde{\kappa} - \tilde{\kappa}^2 + \frac{5\tilde{s}\tilde{\kappa}}{4} - \frac{3\tilde{s}}{2} + \frac{3\tilde{s}\tilde{\kappa}}{2} - \frac{15\tilde{s}^2}{8}\right) \\
&= 4\left(\tilde{\kappa}^2 - \tilde{\kappa}\right) + \left(7 - 11\tilde{\kappa}\right)\tilde{s}^2 + \frac{15\tilde{s}^2}{2} \\
&\leq 0
\end{aligned}$$

Resolving this polynomial inequation, we finally find the sufficient condition:

$$\tilde{s} \leq \frac{11\tilde{\kappa} - 7 + \sqrt{\left(11\tilde{\kappa} - 7\right)^2 - 120\left(\tilde{\kappa}^2 - \tilde{\kappa}\right)}}{15}$$

A.4 Higher-order moments

The second moment tensor M_2 corresponds to the usual variance-covariance matrix:

$$M_2 = \mathbb{E}\left[\left(R - \mathbb{E}\left[R\right]\right)\left(R - \mathbb{E}\left[R\right]\right)^\top\right]$$

The representations of M_3 and M_4 use a column-wise approach that gives the following expression of the coskewness of assets i, j and k:

$$s_{i,j,k} = \mathbb{E}\left[\left(R_i - \mu_i\right)\left(R_j - \mu_j\right)\left(R_k - \mu_k\right)\right]$$

and the cokurtosis of assets i, j, k and l:

$$\kappa_{i,j,k,l} = \mathbb{E}\left[\left(R_i - \mu_i\right)\left(R_j - \mu_j\right)\left(R_k - \mu_k\right)\left(R_l - \mu_l\right)\right]$$

where R_i denotes the return of asset i and μ_i its expected return. To illustrate this column-wise representation, we give the higher-order moment tensors M_3 and M_4 for $n = 3$ assets:

$$\begin{aligned}
M_3 &= \left[\begin{array}{c|c|c} S_1 & S_2 & S_3 \end{array}\right] \\
M_4 &= \left[\begin{array}{ccc|ccc|ccc} K_{1,1} & K_{1,2} & K_{1,3} & K_{2,1} & K_{2,2} & K_{2,3} & K_{3,1} & K_{3,2} & K_{3,3} \end{array}\right]
\end{aligned}$$

where:

$$S_p = \begin{bmatrix} s_{p,1,1} & s_{p,1,2} & s_{p,1,3} \\ s_{p,2,1} & s_{p,2,2} & s_{p,2,3} \\ s_{p,3,1} & s_{p,3,2} & s_{p,3,3} \end{bmatrix}, \qquad K_{p,q} = \begin{bmatrix} \kappa_{p,q,1,1} & \kappa_{p,q,1,2} & \kappa_{p,q,1,3} \\ \kappa_{p,q,2,1} & \kappa_{p,q,2,2} & \kappa_{p,q,2,3} \\ \kappa_{p,q,3,1} & \kappa_{p,q,3,2} & \kappa_{p,q,3,3} \end{bmatrix}$$

are $n \times n$ matrices. Using the Kronecker product, the higher-order moment tensors can be represented as follows:

$$\begin{aligned}
M_3 &= \mathbb{E}\left[\left(R - \mathbb{E}\left[R\right]\right)\left(R - \mathbb{E}\left[R\right]\right)^\top \otimes \left(R - \mathbb{E}\left[R\right]\right)^\top\right] \\
M_4 &= \mathbb{E}\left[\left(R - \mathbb{E}\left[R\right]\right)\left(R - \mathbb{E}\left[R\right]\right)^\top \otimes \left(R - \mathbb{E}\left[R\right]\right)^\top \otimes \left(R - \mathbb{E}\left[R\right]\right)^\top\right]
\end{aligned}$$

Thus, the expressions of the portfolio centered moments are polynomial functions in the $n \times 1$ vector of the underlying asset weights w:

$$\begin{aligned}
\mu^{(2)} &= w^\top M_2 w \\
\mu^{(3)} &= w^\top M_3 \left(w \otimes w\right) \\
\mu^{(4)} &= w^\top M_4 \left(w \otimes w \otimes w\right)
\end{aligned}$$

Assuming that the wealth W is equal to the final portfolio value and with an initial wealth equal to one, we can write the expected utility as a function of the portfolio weights:

$$\begin{aligned}\mathbb{E}\left[U\left(W\right)\right] &\simeq U\left(w^\top\mu\right)+\frac{1}{2}U^{(2)}\left(w^\top\mu\right)w^\top M_2 w+\frac{1}{6}U^{(3)}\left(w^\top\mu\right)w^\top M_3\left(w\otimes w\right)+\\ &\quad \frac{1}{24}U^{(4)}\left(w^\top\mu\right)w^\top M_4\left(w\otimes w\otimes w\right)\end{aligned}$$

The investor's optimization problem is then to maximize this approximated expected utility with respect to the w.

A.5 Stress scenarios properties

A.5.1 Average time between two stress scenarios

The probability of occurrence of a stress scenario is defined through the concept of intensity, derived from the theory of Poisson processes. A given level of intensity λ means that the stress scenario has a probability $\lambda\Delta t$ of occurrence during any small time interval Δt. Thus, the parameter λ can be interpreted as the yearly probability of occurrence of the scenario. Therefore, a scenario is observed more frequently if its intensity λ is high. The frequency of stress scenario observation is therefore proportional to the intensity. From this definition, we can easily calculate the average time between two observations of the same scenario. If we denote the last scenario observation date as $t=0$, and the next occurrence date of the same scenario as τ, the probability that the same scenario occurs after a given time t is be denoted as:

$$P\left(t\right)=\mathbb{P}\left\{\tau>t\right\}$$

For small $\mathrm{d}t$, it satisfies the following equation:

$$\begin{aligned}\mathbb{P}\left\{\tau>t\right\}-\mathbb{P}\left\{\tau>t+\mathrm{d}t\right\} &= \mathbb{P}\left\{t+\mathrm{d}t>\tau>t\right\}\\ &= \mathbb{P}\left\{t+\mathrm{d}t>\tau\middle|\,\tau>t\right\}\cdot\mathbb{P}\left\{\tau>t\right\}\\ &= \lambda\mathbb{P}\left\{\tau>t\right\}\,\mathrm{d}t\end{aligned}$$

Therefore, we get the differential equation:

$$\mathrm{d}P\left(t\right)=-\lambda P\left(t\right)\,\mathrm{d}t$$

Using the fact that $P\left(0\right)=1$, we obtain:

$$P\left(t\right)=e^{-\lambda t}$$

Then, integrating the probability distribution of t gives us the average time between two stress scenarios:

$$\begin{aligned}\mathbb{E}\left[\tau\right] &= \int_0^{+\infty}-t\frac{\mathrm{d}P}{\mathrm{d}t}\,\mathrm{d}t\\ &= \int_0^{+\infty}\lambda t e^{-\lambda t}\,\mathrm{d}t\\ &= \frac{1}{\lambda}\end{aligned}$$

The average time between two stress scenarios is indeed the inverse of its intensity.

A.5.2 Stressing the probability distribution

Stress scenarios can be added when using either parametric or empirical probability distributions. Any criterion relying on an expectation of a function of the portfolio returns is based on a computation of:

$$\mathbb{E}^{\mathbb{P}}\left[f\left(r_t^w\right)\right]$$

where r_t^w is the random return of the portfolio w over some period t, characterized by its distribution $\mathbb{P}$. We introduce K stress scenarios, which would lead to a performance $r_{s_k}^w$ for strategy w and scenario s_k. The probability of each of those scenarios to occur is proportional to the length of the backtesting period. Typically, a stress scenario has a 10% probability to happen each year. We denote by λ_k the yearly probability of appearance of the k-th scenario, i.e. the stress intensity. The average number of occurrence of each scenario is then $\lambda_k t$, while the probabilities of other events sum to 1. Thus, the new formula to compute the stress expectation is given by:

$$\mathbb{E}\left[f\left(r_t^w\right)\middle|\, s_1,\ldots,s_K\right]=\frac{1}{1+\sum_{k=1}^K\lambda_k t}\left(\mathbb{E}^{\mathbb{P}}\left[f\left(r_t^w\right)\right]+\sum_{k=1}^{k}\lambda_k t f\left(r_{s_k}^w\right)\right)$$

A.5.3 Case of the empirical distributions

With the empirical distribution, we include some hypothetical scenarios into the set of historical returns. The difference between those fictitious stress scenarios and actual historical observations is their probabilities. In particular, without stress scenarios, backtesting over n months an allocation w leads to an historical monthly performance r_t^w, with t ranging from 1 to n. In that case, the standard empirical distribution is obtained by considering that each return occurs with an equal probability $\frac{1}{n}$. Thus, the expected utility given by this allocation is:

$$\hat{U}(w)=\frac{1}{\sum_e\alpha_e}\times\left(\sum_e\alpha_e\times U(e)\right)$$

where e is the event and α_e is the weight of the event. We obtain, with stress scenarios:

$$\begin{aligned}\hat{U}(w)&=\frac{1}{n+n\Delta_t\sum_{k=1}^K\lambda_k}\left(\sum_{t=1}^n U\left(1+r_t^w\right)+n\Delta_t\sum_{k=1}^K\lambda_k U\left(1+r_{s_k}^w\right)\right)\\&=\frac{1}{1+\Delta_t\sum_{k=1}^K\lambda_k}\left(n^{-1}\sum_{t=1}^n U\left(1+r_t^w\right)+\sum_{k=1}^K\Delta_t\lambda_k U\left(1+r_{s_k}^w\right)\right)\end{aligned}$$

In general, the expectation of any function of the assets returns can be written as:

$$\hat{\mathbb{E}}\left[f(r)\middle|\, s_1,\ldots,s_K\right]=\frac{1}{1+\Delta_t\sum_{k=1}^K\lambda_k}\left(n^{-1}\sum_{t=1}^n f\left(r_t\right)+\sum_{k=1}^K\Delta_t\lambda_k f\left(r_{s_k}\right)\right)$$

A.6 Black-Litterman approach

We assume that the vector μ of expected returns is unknown and we have:

$$\mu\sim\mathcal{N}(\pi,\Gamma)$$

The views of the fund manager are given by:

$$P\mu=Q+\varepsilon$$

where P is a $k \times n$ matrix Q is a $k \times 1$ vector and $\varepsilon \sim \mathcal{N}(0, \Omega)$ is a gaussian random vector. Let μ_{cond} the conditional expected return defined by the following optimization program:

$$\begin{aligned} \mu_{\mathrm{cond}} &= \arg\min \left(\mu - \pi\right)^\top \Gamma^{-1} \left(\mu - \pi\right) \\ &\text{u.c.} \quad P\mu = Q + \varepsilon \end{aligned}$$

We may show the solution is:

$$\begin{aligned} \mu_{\mathrm{cond}} &= \mathbb{E}\left[\mu \mid P\mu = Q + \varepsilon\right] \\ &= \left(\Gamma^{-1} + P^\top \Omega^{-1} P\right)^{-1} \left(\Gamma^{-1}\pi + P^\top \Omega^{-1} Q\right) \end{aligned}$$

After some computations, we finally obtain[11]:

$$\mu_{\mathrm{cond}} = \pi + \Gamma P^\top \left(P\Gamma P^\top + \Omega\right)^{-1} \left(Q - P\pi\right) \tag{4}$$

From a practical point of view, the following steps to implement the Black-Litterman model are:

1. We compute the empirical covariance matrix Σ. Given the portfolio allocation w_0, we deduce the expected returns π as the solution of the inverse mean-variance problem:

$$\pi = \frac{2}{\lambda} \Sigma^{-1} w$$

2. We define the views of the fund manager by specifying the matrices P, Q, $\Gamma$and Ω. We then compute the conditional expected returns μ_{cond} given by equation (4).

3. We then solve the traditional tracking error problem:

$$\max w^\top \mu_{\mathrm{cond}} \quad \text{u.c.} \quad \left(w - w_0\right)^\top \Sigma \left(w - w_0\right) \leq \sigma_{\mathrm{TE}}^2$$

with σ_{TE} the maximum level of the tracking error volatility.

[11] Using the following result on matrix inversion:

$$\left(A + XBX^\top\right)^{-1} = A^{-1} - A^{-1}X\left(B^{-1} + X^\top A^{-1} X\right)^{-1} X^\top A^{-1}$$

we obtain:

$$\left(\Gamma^{-1} + P^\top \Omega^{-1} P\right)^{-1} = \Gamma - \Gamma P^\top \left(\Omega + P\Gamma P^\top\right)^{-1} P\Gamma$$

We deduce that:

$$\begin{aligned} \mu_{\mathrm{cond}} &= \left(\Gamma - \Gamma P^\top \left(\Omega + P\Gamma P^\top\right)^{-1} P\Gamma\right)\left(\Gamma^{-1}\pi + P^\top \Omega^{-1} Q\right) \\ &= \pi - \Gamma P^\top \left(\Omega + P\Gamma P^\top\right)^{-1} \left(P\pi + P\Gamma P^\top \Omega^{-1} Q\right) + \Gamma P^\top \Omega^{-1} Q \\ &= \pi - \Gamma P^\top \left(\Omega + P\Gamma P^\top\right)^{-1} P\pi + \Gamma P^\top \left(\Omega^{-1} - \left(\Omega + P\Gamma P^\top\right)^{-1} P\Gamma P^\top \Omega^{-1}\right) Q \\ &= \pi - \Gamma P^\top \left(\Omega + P\Gamma P^\top\right)^{-1} P\pi + \Gamma P^\top \left(\Omega + P\Gamma P^\top\right)^{-1} Q \end{aligned}$$

References

[1] Ackermann C., McEnally R. and Ravenscraft D. (1999), The Performance of Hedge Funds: Risk, Return, and Incentives, *Journal of Finance*, 54, pp. 833-874.

[2] Adam A., Houkari M. and Laurent J.P. (2008), Spectral Risk Measures and Portfolio Selection, *Journal of Banking and Finance*, 32, pp. 1870-1882.

[3] Agarwal V., Bakshi G. and Huij J. (2009), Do Higher-Moment Equity Risks Explain Hedge Fund Returns? *Robert H. Smith School Research Paper,* 06-066.

[4] Agarwal V. and Naik N. (2004), Risks and Portfolio Decisions Involving Hedge Funds, *Review of Financial Studies*, 17, pp. 63-98.

[5] Alexander C. and Dimitriu A. (2004), The Art of Investing in Hedge Funds: Fund Selection and Optimal Allocation, *ICMA Centre Working Papers*.

[6] Amenc N. and Martellini L. (2002), Portfolio Optimization and Hedge Fund Style Allocation Decisions, *Journal of Alternative Investments*, 5, pp. 7-20.

[7] Amin G. and Kat H. (2003), Stocks, Bonds, and Hedge Funds, *Journal of Portfolio Management*, 29, pp. 113-120.

[8] Ang A. and Chen J. (2002), Asymmetric Correlations of Equity Portfolios, *Journal of Financial Economics*, 63, pp. 443-494.

[9] Bares P., Gibson R. and Gyger S. (2002), Hedge Fund Allocation with Survival Uncertainty and Investment Constraints, *Working Paper*.

[10] Berkowitz J. (2000), A Coherent Framework for Stress Testing, *Journal of Risk*, 2, pp. 1-11.

[11] Billio M., Getmanski M. and Pelizon S. (2010), Crises and Hedge Funds Risk, *Working Paper*.

[12] Bollerslev T. (1990), Modelling the Coherence in Short-run Nominal Exchange Rates: A Multivariate Generalized ARCH Model, *Review of Economics and Statistics*, 72, pp. 498-505.

[13] Brooks C. and Kat (2002), The Statistical Properties of Hedge Fund Index Returns and their Implications for Investors, *Journal of Alternative Investments*, 5, pp. 26-44.

[14] Brown S., Goetzmann W. and Ibbotson R. (1999), Offshore Hedge Funds: Survival and Performance 1989-1995, *Journal of Business*, 72, pp. 91-118.

[15] Buraschi A., Kosowski R. and Trojani F. (2010), When There Is No Place to Hide: Correlation Risk and the Cross-Section of Hedge Fund Returns, *Imperial College, Working Paper*.

[16] Chan N., Getmansky M., Haas S. and Lo A. (2006), Do Hedge Funds Increase Systemic Risk?, *Federal Reserve Bank of Atlanta Economic Review*, Q4, pp. 49-80.

[17] Choueifaty Y. and Coignard Y. (2008), Towards Maximum Diversification, *Journal of Portfolio Management*, 35, pp. 40-51.

[18] Clarke R., de Silva H. and Thorley S. (2006), Minimum-variance Portfolios in the U.S. Equity Market,*Journal of Portfolio Management*, 33, pp. 10-24.

[19] Cont R. (2001), Empirical Properties of Asset Returns: Stylized Facts and Statistical Issues, *Quantitative Finance*, 1, pp. 223-236.

[20] Cvitanic J., Lazrak A., Martellini L. and Zapatero F. (2003), Optimal Allocation to Hedge Funds: An Empirical Analysis, *Quantitative Finance*, 3, pp. 1-12.

[21] Darolles S. and Gourieroux C. (2010), Conditionally Fitted Sharpe Performance with an Application to Hedge Fund Rating, *Journal of Banking and Finance*, 34, pp. 578-593

[22] Darolles S., Gourieroux C. and Jasiak J. (2009), L-Performance with an Application to Hedge Funds, *Journal of Empirical Finance*, 16, pp. 671-685

[23] Darolles S. and Mero M. (2010), Hedge Fund Returns and Factor Models: The Factor Selection Puzzle, *Bankers Markets and Investors*, forthcoming.

[24] Darolles S. and Vaissie M. (2010), Do Funds of Hedge Funds Really Add Value?, *Working paper*.

[25] Davies R., Kat H. and Lu S. (2009), Fund of Hedge Funds Portfolio Selection: A Multiple-objective Approach, *Journal of Derivatives and Hedge Funds*, 15, pp. 91-115.

[26] Eling M. and Schumacher F. (2007), Does the Choice of Performance Measure Influence the Evaluation of Hedge Funds?, *Journal of Banking and Finance*, 31, pp. 2632-2647.

[27] Elton E.J. and Gruber M.J. (1973), Estimating the Dependence Structure of Share Prices — Implications for Portfolio Selection, *Journal of Finance*, 28, pp. 1203-1233.

[28] Engle R. (2002), Dynamic Conditional Correlation, *Journal of Business and Economic Statistics*, 20, pp. 339-350.

[29] Favre L. and Galeano J. (2002), Mean-Modified Value-at-Risk Optimization with Hedge Funds, *Journal of Alternative Investments*, 5, pp. 21-25.

[30] Fung W. and Hsieh D. (1997), Empirical Characteristics of Dynamic Trading Strategies: The Case of Hedge Funds, *Review of Financial Studies*, 10, pp. 275-302.

[31] Fung W. and Hsieh D. (1999), A Primer on Hedge Funds, *Journal of Empirical Finance*, 6, pp. 309-331.

[32] Fung W. and Hsieh D. (2001), The Risk in Hedge Fund Strategies: Theory and Evidence from Trend Followers, *Review of Financial Studies*, 14, pp. 313-341.

[33] Getmansky M, Lo A. and Makarov I. (2004), An Econometric Model of Serial Correlation and Illiquidity in Hedge Fund Returns, *Journal of Financial Economics*, 74, pp. 529-610.

[34] Giomouridis D. and Vrontos I. (2007), Hedge Fund Portfolio Construction: A Comparison of Static and Dynamic Approaches, *Journal of Banking and Finance*, 31, pp. 199-217.

[35] Greyserman A., Jones D. and Strawderman W. (2006), Portfolio Selection using Hierarchical Bayesian Analysis and MCMC Methods, *Journal of Banking and Finance*, 30, pp. 669-678.

[36] Hagelin N. and Pramborg B. (2003), Evaluating Gains from Diversifying into Hedge Funds Using Dynamic Investment Strategies, in Schachter, B. (eds), *Intelligent Hedge Fund Investing*, Risk Books, London.

[37] Jondeau E. and Rockinger M., (2006), Optimal Portfolio Allocation Under Higher Moments, *European Financial Management*, 12, pp. 29-55.

[38] Jondeau E., Jurczenko E. and Rockinger M., (2010), Moment Component Analysis: An Illustration with International Stock Markets, *Working paper*.

[39] Jurczenko E, Maillet B. and Merlin P. (2005), Hedge Funds Portfolio Selection with Higher-order Moments: A Non-parametric Mean-Variance-Skewness-Kurtosis Efficient Frontier, *Working paper*.

[40] Kat H. (2004), In Search of the Optimal Fund of Hedge Funds, *Journal of Wealth Management*, 6, pp. 43-51.

[41] Krokhmal P., Uryasev S. and Zrazhevsky G. (2002), Risk Management for Hedge Fund Portfolios: A Comparative Analysis of Linear Portfolio Rebalancing Strategies, *Journal of Alternative Investments*, 5, pp. 10-29.

[42] Ledoit O and Wolf M. (2003), Improved Estimation of the Covariance Matrix of Stock Returns With an Application to Portfolio Selection, *Journal of Empirical Finance*, 10, pp. 603-621.

[43] Ledoit O. and Wolf M. (2004), Honey, I Shrunk the Sample Covariance Matrix, *Journal of Portfolio Management*, 30, pp. 110-119.

[44] Liew J. (2003), Hedge Fund Index Investing Examined, *Journal of Portfolio Management*, 29, pp. 113-123.

[45] Maillard S., Roncalli T. and Teiletche J. (2010), The Properties of Equally Weighted Risk Contribution Portfolios, *Journal of Portfolio Management*, 36, pp. 1-11.

[46] Malkiel B.G. and Saha A. (2005), Hedge Funds: Risk and Return, *Financial Analysts Journal*, 61, pp. 80-88.

[47] Martellini L. (2008), Toward the Design of Better Equity Benchmarks, *Journal of Portfolio Management*, 34, pp. 1-8.

[48] Martellini L. and Ziemann V. (2010), Improved Estimates of Higher-Order Comoments and Implications for Portfolio Selection, *Review of Financial Studies*, 23, pp. 1467-1502.

[49] Markowitz H. (1952), Portfolio Selection, *Journal of Finance*, 7, pp. 77-91.

[50] McFall Lamm R. (2003), Asymmetric Returns and Optimal Hedge Fund Portfolios, *Journal of Alternative Investments*, 6, pp. 9-21.

[51] Michaud R. (1989), The Markowitz Optimization Enigma: Is Optimized Optimal?, *Financial Analysts Journal*, 45, pp. 31-42.

[52] Merton R. (1980), On estimating the Expected Return on the Market: An Exploratory Investigation, *Journal of Financial Economics*, 8, pp. 323-361.

[53] Morton D., Popova E. and Popova I. (2006), Efficient Fund of Hedge Funds Construction Under Downside Risk Measures, *Journal of Banking and Finance*, 30, pp. 503-518.

[54] Pelletier D. (2006), Regime Switching for Dynamic Correlations, *Journal of Econometrics*, 131, pp. 445-473.

[55] Popova I., Morton, D. and Popova E. (2003), Optimal Hedge Fund Allocation with Asymmetric Preferences and Distributions, *University of Texas, Working paper.*

[56] Rockafellar R., Uryasev S. and Zabarankin M. (2006), Master Funds in Portfolio Analysis with General Deviation Measures, *Journal of Banking and Finance*, 30, pp. 743-778.

[57] Shadwick W. and Keating C. (2002), A Universal Performance Measure, *Journal of Performance Measurement*, 6, pp. 59-84.

[58] Sharpe W. (1963), A Simplified Model for Portfolio Analysis, *Management Science*, 8, pp. 277-293.

[59] Sharpe W. (2007), Expected Utility Asset Allocation, *Financial Analysts Journal*, 63, pp. 18-30.

[60] Terhaar K., Staub R. and Singer B. (2003), Appropriate Policy Allocation for Alternative Investments, *Journal of Portfolio Management*, 29, pp. 101-111.

[61] Von Neumann J. and Morgenstern O. (1953), *Theory of Games and Economic Behavior*, Princeton University Press.