Course 2023-2024 in Sustainable Finance
Lecture 13. Climate Portfolio Construction

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[1] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Agenda

Quadratic programming

Definition

We have:

$$
\begin{array}{rcl}
x^{\star} & = & \arg\min \frac{1}{2} x^{\top} Q x - x^{\top} R \\
 & \text{s.t.} & \left\{ \begin{array}{l} Ax = B \\ Cx \leq D \\ x^{-} \leq x \leq x^{+} \end{array} \right.
\end{array}
$$

where x is a $n \times 1$ vector, Q is a $n \times n$ matrix, R is a $n \times 1$ vector, A is a $n_A \times n$ matrix, B is a $n_A \times 1$ vector, C is a $n_C \times n$ matrix, D is a $n_C \times 1$ vector, and x^{-} and x^{+} are two $n \times 1$ vectors

Quadratic form

A quadratic form is a polynomial with terms all of degree two

$$\mathcal{QF}(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} x_i x_j = x^\top A x$$

Canonical form

$$\mathcal{QF}(x_1, \ldots, x_n) = \frac{1}{2}\left(x^\top A x + x^\top A^\top x\right) = \frac{1}{2} x^\top \left(A + A^\top\right) x = \frac{1}{2} x^\top Q x$$

Generalized quadratic form

$$\mathcal{QF}(x; Q, R, c) = \frac{1}{2} x^\top Q x - x^\top R + c$$

Quadratic form

Main properties

1. $\varphi \cdot \mathcal{QF}(w; Q, R, c) = \mathcal{QF}(w; \varphi Q, \varphi R, \varphi c)$
2. $\mathcal{QF}(x; Q_1, R_1, c_1) + \mathcal{QF}(x; Q_2, R_2, c_2) = \mathcal{QF}(x; Q_1 + Q_2, R_1 + R_2, c_1 + c_2)$
3. $\mathcal{QF}(x - y; Q, R, c) = \mathcal{QF}\left(x; Q, R + Qy, \frac{1}{2}y^\top Qy + y^\top R + c\right)$
4. $\mathcal{QF}(x - y; Q, R, c) = \mathcal{QF}\left(y; Q, Qx - R, \frac{1}{2}x^\top Qx - x^\top R + c\right)$
5. $\dfrac{1}{2}\sum_{i=1}^n q_i x_i^2 = \mathcal{QF}(x; \mathcal{D}(q), \mathbf{0}_n, 0)$ where $q = (q_1, \ldots, q_n)$ is a $n \times 1$ vector and $\mathcal{D}(q) = \operatorname{diag}(q)$
6. $\dfrac{1}{2}\sum_{i=1}^n q_i (x_i - y_i)^2 = \mathcal{QF}\left(x; \mathcal{D}(q), \mathcal{D}(q)\, y, \frac{1}{2}y^\top \mathcal{D}(q)\, y\right)$
7. $\dfrac{1}{2}\left(\sum_{i=1}^n q_i x_i\right)^2 = \mathcal{QF}(x; \mathcal{T}(q), \mathbf{0}_n, 0)$ where $\mathcal{T}(q) = qq^\top$
8. $\dfrac{1}{2}\left(\sum_{i=1}^n q_i (x_i - y_i)\right)^2 = \mathcal{QF}\left(x; \mathcal{T}(q), \mathcal{T}(q)\, y, \frac{1}{2}y^\top \mathcal{T}(q)\, y\right)$

Quadratic form

Main properties

We note $\omega = (\omega_1, \ldots, \omega_n)$ where $\omega_i = \mathbb{1}\{i \in \Omega\}$

1. $\frac{1}{2}\sum_{i\in\Omega} q_i x_i^2 = \mathcal{QF}\left(x; \mathcal{D}(\omega \circ q), \mathbf{0}_n, 0\right)$
2. $\frac{1}{2}\sum_{i\in\Omega} q_i (x_i - y_i)^2 =$
$$\mathcal{QF}\left(x; \mathcal{D}(\omega \circ q), \mathcal{D}(\omega \circ q) y, \frac{1}{2} y^\top \mathcal{D}(\omega \circ q) y\right)$$
3. $\frac{1}{2}\left(\sum_{i\in\Omega} q_i x_i\right)^2 = \mathcal{QF}\left(x; \mathcal{T}(\omega \circ q), \mathbf{0}_n, 0\right)$
4. $\frac{1}{2}\left(\sum_{i\in\Omega} q_i (x_i - y_i)\right)^2 =$
$$\mathcal{QF}\left(x; \mathcal{T}(\omega \circ q), \mathcal{T}(\omega \circ q) y, \frac{1}{2} y^\top \mathcal{T}(\omega \circ q) y\right)$$
5. $\mathcal{D}(\omega \circ q) = \operatorname{diag}(\omega \circ q) = \mathcal{D}(\omega)\mathcal{D}(q)$
6. $\mathcal{T}(\omega \circ q) = (\omega \circ q)(\omega \circ q)^\top = \left(\omega\omega^\top\right) \circ qq^\top = \mathcal{T}(\omega) \circ \mathcal{T}(q)$

Equity portfolio

Basic optimization problems

Mean-variance optimization

The long-only mean-variance optimization problem is given by:

$$\begin{aligned} w^{\star} &= \arg\min \frac{1}{2} w^{\top} \Sigma w - \gamma w^{\top} \mu \\ &\text{s.t.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^{\top} w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

where:

- γ is the risk-tolerance coefficient
- the equality constraint is the budget constraint ($\sum_{i=1}^{n} w_i = 1$)
- the bounds correspond to the no short-selling restriction ($w_i \geq 0$)

QP form

$Q = \Sigma$, $R = \gamma \mu$, $A = \mathbf{1}_n^{\top}$, $B = 1$, $w^{-} = \mathbf{0}_n$ and $w^{+} = \mathbf{1}$

Equity portfolio

Basic optimization problems

Tracking error optimization

The tracking error optimization problem is defined as:

$$\begin{aligned} w^{\star} &= \arg\min \frac{1}{2} w^{\top} \Sigma w - w^{\top} (\gamma \mu + \Sigma b) \\ &\text{s.t.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^{\top} w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

QP form

$Q = \Sigma$, $\boxed{R = \gamma \mu + \Sigma b}$, $A = \mathbf{1}_n^{\top}$, $B = 1$, $w^{-} = \mathbf{0}_n$ and $w^{+} = \mathbf{1}$

$\Rightarrow$ Portfolio replication: $\boxed{R = \Sigma b}$

Specification of the constraints

Sector weight constraint

- We have

$$s_j^- \leq \sum_{i \in \mathcal{S}ector_j} w_i \leq s_j^+$$

- $\boldsymbol{s}_j$ is the $n \times 1$ sector-mapping vector: $\boldsymbol{s}_{i,j} = \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\}$
- We notice that:

$$\sum_{i \in \mathcal{S}ector_j} w_i = \boldsymbol{s}_j^\top w$$

- We deduce that:

$$s_j^- \leq \sum_{i \in \mathcal{S}ector_j} w_i \leq s_j^+ \Leftrightarrow \left\{ \begin{array}{l} s_j^- \leq \boldsymbol{s}_j^\top w \\ \boldsymbol{s}_j^\top w \leq s_j^+ \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} -\boldsymbol{s}_j^\top w \leq -s_j^- \\ \boldsymbol{s}_j^\top w \leq s_j^+ \end{array} \right.$$

QP form

$$\underbrace{\left(\begin{array}{c} -\boldsymbol{s}_j^\top \\ \boldsymbol{s}_j^\top \end{array}\right)}_{C} w \leq \underbrace{\left(\begin{array}{c} -s_j^- \\ s_j^+ \end{array}\right)}_{D}$$

Specification of the constraints

Score constraint

- General constraint:

$$\sum_{i=1}^{n} w_i \mathcal{S}_i \geq \mathcal{S}^\star \Leftrightarrow -\mathcal{S}^\top w \leq -\mathcal{S}^\star$$

QP form

- $C = -\mathcal{S}^\top$
- $D = -\mathcal{S}^\star$

Specification of the constraints

Score constraint

- Sector-specific constraint:

$$
\begin{aligned}
\sum_{i \in \mathcal{S}ector_j} w_i \mathcal{S}_i \geq \mathcal{S}_j^\star & \Leftrightarrow \sum_{i=1}^{n} \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\} \cdot w_i \mathcal{S}_i \geq \mathcal{S}_j^\star \\
& \Leftrightarrow \sum_{i=1}^{n} \boldsymbol{s}_{i,j} w_i \mathcal{S}_i \geq \mathcal{S}_j^\star \\
& \Leftrightarrow \sum_{i=1}^{n} w_i \cdot \left(\boldsymbol{s}_{i,j} \mathcal{S}_i\right) \geq \mathcal{S}_j^\star \\
& \Leftrightarrow \left(\boldsymbol{s}_j \circ \mathcal{S}\right)^\top w \geq \mathcal{S}_j^\star
\end{aligned}
$$

QP form

- $C = -\left(\boldsymbol{s}_j \circ \mathcal{S}\right)^\top$
- $D = -\mathcal{S}_j^\star$

Equity portfolios

Example #1

- The capitalization-weighted equity index is composed of 8 stocks
- The weights are equal to 23%, 19%, 17%, 13%, 9%, 8%, 6% and 5%
- The ESG score, carbon intensity and sector of the eight stocks are the following:

Stock	#1	#2	#3	#4	#5	#6	#7	#8
$\mathcal{S}$	−1.20	0.80	2.75	1.60	−2.75	−1.30	0.90	−1.70
$\mathcal{CI}$	125	75	254	822	109	17	341	741
$\mathcal{S}ector$	1	1	2	2	1	2	1	2

Equity portfolios

Example #1 (Cont'd)

- The stock volatilities are equal to 22%, 20%, 25%, 18%, 35%, 23%, 13% and 29%
- The correlation matrix is given by:

$$\mathbb{C} = \begin{pmatrix} 100\% & & & & & & & \\ 80\% & 100\% & & & & & & \\ 70\% & 75\% & 100\% & & & & & \\ 60\% & 65\% & 80\% & 100\% & & & & \\ 70\% & 50\% & 70\% & 85\% & 100\% & & & \\ 50\% & 60\% & 70\% & 80\% & 60\% & 100\% & & \\ 70\% & 50\% & 70\% & 75\% & 80\% & 50\% & 100\% & \\ 60\% & 65\% & 70\% & 75\% & 65\% & 70\% & 80\% & 100\% \end{pmatrix}$$

Equity portfolios

QP problem

- We have:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2} w^{\top} Q w - w^{\top} R \\
\text{s.t.} & \quad \left\{
\begin{array}{l}
Aw = B \\
Cw \leq D \\
w^{-} \leq w \leq w^{+}
\end{array}
\right.
\end{aligned}
$$

Equity portfolios

Objective function

- Using $\Sigma_{i,j} = \mathbb{C}_{i,j}\sigma_i\sigma_j$, we obtain:

$$Q = \Sigma = 10^{-4} \times$$

$$\begin{pmatrix} 484.00 & 352.00 & 385.00 & 237.60 & 539.00 & 253.00 & 200.20 & 382.80 \\ 352.00 & 400.00 & 375.00 & 234.00 & 350.00 & 276.00 & 130.00 & 377.00 \\ 385.00 & 375.00 & 625.00 & 360.00 & 612.50 & 402.50 & 227.50 & 507.50 \\ 237.60 & 234.00 & 360.00 & 324.00 & 535.50 & 331.20 & 175.50 & 391.50 \\ 539.00 & 350.00 & 612.50 & 535.50 & 1225.00 & 483.00 & 364.00 & 659.75 \\ 253.00 & 276.00 & 402.50 & 331.20 & 483.00 & 529.00 & 149.50 & 466.90 \\ 200.20 & 130.00 & 227.50 & 175.50 & 364.00 & 149.50 & 169.00 & 301.60 \\ 382.80 & 377.00 & 507.50 & 391.50 & 659.75 & 466.90 & 301.60 & 841.00 \end{pmatrix}$$

Equity portfolios
Objective function

- We have:

$$R = \Sigma b = \begin{pmatrix} 3.74 \\ 3.31 \\ 4.39 \\ 3.07 \\ 5.68 \\ 3.40 \\ 2.02 \\ 4.54 \end{pmatrix} \times 10^{-2}$$

Equity portfolios

Constraint specification (bounds)

- The portfolio is long-only

QP form

- $w^- = \mathbf{0}_8$
- $w^+ = \mathbf{1}_8$

Equity portfolios

Constraint specification (equality)

- The budget constraint $\sum_{i=1}^{8} w_i = 1 \Rightarrow$ a first linear equation $A_0 w = B_0$

QP form

- $A_0 = \mathbf{1}_8^\top$
- $B_0 = 1$

Equity portfolios

Constraint specification (equality)

- We can impose the sector neutrality of the portfolio meaning that:

$$\sum_{i \in \mathcal{S}ector_j} w_i = \sum_{i \in \mathcal{S}ector_j} b_i$$

The sector neutrality constraint can be written as:

$$\left(\begin{array}{c} A_1 \\ A_2 \end{array} \right) w = \left(\begin{array}{c} B_1 \\ B_2 \end{array} \right)$$

QP form

- $A_1 = \boldsymbol{s}_1^\top = \left(\begin{array}{cccccccc} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{array} \right)$
- $A_2 = \boldsymbol{s}_2^\top = \left(\begin{array}{cccccccc} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{array} \right)$
- $B_1 = \boldsymbol{s}_1^\top b = \sum_{i \in \mathcal{S}ector_1} b_i$
- $B_2 = \boldsymbol{s}_2^\top b = \sum_{i \in \mathcal{S}ector_2} b_i$

Equity portfolios

Constraint specification (inequality)

- We can impose a relative reduction of the benchmark carbon intensity:

$$\mathcal{CI}(w) \leq (1-\mathcal{R})\,\mathcal{CI}(b) \Leftrightarrow C_1 w \leq D_1$$

QP form

- $C_1 = \mathcal{CI}^\top$ (because $\mathcal{CI}(w) = \mathcal{CI}^\top w$)
- $D_1 = (1-\mathcal{R})\,\mathcal{CI}(b)$

- We can impose an absolute increase of the benchmark ESG score:

$$\mathcal{S}(w) \geq \mathcal{S}(b) + \Delta\mathcal{S}^\star$$

Since $\mathcal{S}(w) = \mathcal{S}^\top w$, we deduce that $C_2 w \leq D_2$

QP form

- $C_2 = -\mathcal{S}^\top$
- $D_2 = -\left(\mathcal{S}(b) + \Delta\mathcal{S}^\star\right)$

Equity portfolios

Combination of constraints

Set of constraints	Carbon intensity	ESG score	Sector neutrality	A	B	C	D
#1	✓			A_0	B_0	C_1	D_1
#2		✓		A_0	B_0	C_2	D_2
#3	✓	✓		A_0	B_0	$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$	$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$
#4	✓	✓	✓	$\begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix}$	$\begin{bmatrix} B_0 \\ B_1 \\ B_2 \end{bmatrix}$	$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$	$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$

Equity portfolios

Results

Table 1: $\mathcal{R} = 30\%$ and $\Delta\mathcal{S}^\star = 0.50$ (Example #1)

		Benchmark	Set #1	Set #2	Set #3	Set #4
Weights (in %)	$w_1^\star$	23.00	18.17	25.03	8.64	12.04
	$w_2^\star$	19.00	24.25	14.25	29.27	23.76
	$w_3^\star$	17.00	16.92	21.95	26.80	30.55
	$w_4^\star$	13.00	2.70	27.30	1.48	2.25
	$w_5^\star$	9.00	12.31	3.72	10.63	8.51
	$w_6^\star$	8.00	11.23	1.34	6.30	10.20
	$w_7^\star$	6.00	11.28	1.68	16.87	12.69
	$w_8^\star$	5.00	3.15	4.74	0.00	0.00
Statistics	$\sigma(w^\star \mid b)$ (in %)	0.00	0.50	1.18	1.90	2.12
	$\mathcal{CI}(w^\star)$	261.72	183.20	367.25	183.20	183.20
	$\mathcal{R}(w^\star \mid b)$ (in %)		30.00	-40.32	30.00	30.00
	$\mathcal{S}(w^\star)$	0.17	0.05	0.67	0.67	0.67
	$\mathcal{S}(w^\star) - \mathcal{S}(b)$		-0.12	0.50	0.50	0.50
	$w^\star(\mathcal{S}ector_1)$ (in %)	57.00	66.00	44.67	65.41	57.00
	$w^\star(\mathcal{S}ector_2)$ (in %)	43.00	34.00	55.33	34.59	43.00

Equity portfolios

Dealing with constraints on relative weights

- The carbon intensity of the j^{th} sector within the portfolio w is:

$$\mathcal{CI}\left(w;\mathcal{S}ector_j\right) = \sum_{i \in \mathcal{S}ector_j} \tilde{w}_i \mathcal{CI}_i$$

where $\tilde{w}_i$ is the normalized weight in the sector bucket:

$$\tilde{w}_i = \frac{w_i}{\sum_{k \in \mathcal{S}ector_j} w_k}$$

- Another expression of $\mathcal{CI}\left(w;\mathcal{S}ector_j\right)$ is:

$$\mathcal{CI}\left(w;\mathcal{S}ector_j\right) = \frac{\sum_{i \in \mathcal{S}ector_j} w_i \mathcal{CI}_i}{\sum_{i \in \mathcal{S}ector_j} w_i} = \frac{\left(\boldsymbol{s}_j \circ \mathcal{CI}\right)^\top w}{\boldsymbol{s}_j^\top w}$$

Equity portfolios

Dealing with constraints on relative weights

- If we consider the constraint $\mathcal{CI}\left(w; \mathcal{S}ector_j\right) \leq \mathcal{CI}_j^\star$, we obtain:

$$
\begin{aligned}
(*) \quad & \Leftrightarrow \quad \mathcal{CI}\left(w; \mathcal{S}ector_j\right) \leq \mathcal{CI}_j^\star \\
& \Leftrightarrow \quad \left(\boldsymbol{s}_j \circ \mathcal{CI}\right)^\top w \leq \mathcal{CI}_j^\star \left(\boldsymbol{s}_j^\top w\right) \\
& \Leftrightarrow \quad \left(\left(\boldsymbol{s}_j \circ \mathcal{CI}\right) - \mathcal{CI}_j^\star \boldsymbol{s}_j\right)^\top w \leq 0 \\
& \Leftrightarrow \quad \left(\boldsymbol{s}_j \circ \left(\mathcal{CI} - \mathcal{CI}_j^\star\right)\right)^\top w \leq 0
\end{aligned}
$$

QP form

- $C = \left(\boldsymbol{s}_j \circ \left(\mathcal{CI} - \mathcal{CI}_j^\star\right)\right)^\top$
- $D = 0$

Equity portfolios

Dealing with constraints on relative weights

Example #2

- Example #1
- We would like to reduce the carbon footprint of the benchmark by 30%
- We impose the sector neutrality

Equity portfolios

Dealing with constraints on relative weights

QP form

- $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$
- $B = \begin{pmatrix} 100\% \\ 57\% \\ 43\% \end{pmatrix}$
- $C = \begin{pmatrix} 125 & 75 & 254 & 822 & 109 & 17 & 341 & 741 \end{pmatrix}$
- $D = 183.2040$

Equity portfolios

Dealing with constraints on relative weights

- The optimal solution is:

$$w^{\star} = (21.54\%, 18.50\%, 21.15\%, 3.31\%, 10.02\%, 15.26\%, 6.94\%, 3.27\%)$$

- $\sigma(w^{\star} \mid b) = 112$ bps
- $\mathcal{CI}(w^{\star}) = 183.20$ vs. $\mathcal{CI}(b) = 261.72$

BUT

$$\left\{ \begin{array}{l} \mathcal{CI}(w^{\star}; \mathcal{S}ector_1) = 132.25 \\ \mathcal{CI}(w^{\star}; \mathcal{S}ector_2) = 250.74 \end{array} \right. \quad \text{versus} \quad \left\{ \begin{array}{l} \mathcal{CI}(b; \mathcal{S}ector_1) = 128.54 \\ \mathcal{CI}(b; \mathcal{S}ector_2) = 438.26 \end{array} \right.$$

The global reduction of 30% is explained by:

- an increase of 2.89% of the carbon footprint for the first sector
- a decrease of 42.79% of the carbon footprint for the second sector

Equity portfolios

Dealing with constraints on relative weights

- We impose $\mathcal{R}_1 = 20\%$

QP form

- $C = \begin{pmatrix} \mathcal{CI}^\top \\ \left(\boldsymbol{s}_1 \circ \left(\mathcal{CI} - (1 - \mathcal{R}_1)\, \mathcal{CI}\left(b; \mathcal{S}ector_1\right)\right)\right)^\top \end{pmatrix} =$
$\begin{pmatrix} 125 & 75 & 254 & 822 & 109 & 17 & 341 & 741 \\ 22.1649 & -27.8351 & 0 & 0 & 6.1649 & 0 & 238.1649 & 0 \end{pmatrix}$
- $D = \begin{pmatrix} 183.2040 \\ 0 \end{pmatrix}$

Equity portfolios

Dealing with constraints on relative weights

- Solving the new QP problem gives the following optimal portfolio:

$$w^{\star} = (22.70\%, 22.67\%, 19.23\%, 5.67\%, 11.39\%, 14.50\%, 0.24\%, 3.61\%)$$

- $\sigma(w^{\star} \mid b) = 144$ bps
- $\mathcal{CI}(w^{\star}) = 183.20$
  - $\mathcal{CI}(w^{\star}; \mathcal{S}ector_1) = 102.84$ (reduction of 20%)
  - $\mathcal{CI}(w^{\star}; \mathcal{S}ector_2) = 289.74$ (reduction of 33.89%)

Risk measure of a bond portfolio

- We consider a zero-coupon bond, whose price and maturity date are $B(t,T)$ and T:

$$B_t(t,T) = e^{-(r(t)+s(t))(T-t)+L(t)}$$

where $r(t)$, $s(t)$ and $L(t)$ are the interest rate, the credit spread and the liquidity premium

- We deduce that:

$$\begin{aligned} \mathrm{d}\ln B(t,T) &= -(T-t)\,\mathrm{d}r(t) - (T-t)\,\mathrm{d}s(t) + \mathrm{d}L(t) \\ &= -D\,\mathrm{d}r(t) - (D\,s(t))\frac{\mathrm{d}s(t)}{s(t)} + \mathrm{d}L(t) \\ &= -D\,\mathrm{d}r(t) - \mathrm{DTS}(t)\frac{\mathrm{d}s(t)}{s(t)} + \mathrm{d}L(t) \end{aligned}$$

where:

  - $D = T - t$ is the remaining maturity (or duration)
  - $\mathrm{DTS}(t)$ is the duration-times-spread factor

Risk measure of a bond portfolio

- If we assume that $r(t)$, $s(t)$ and $L(t)$ are independent, the risk of the defaultable bond is equal to:

$$\sigma^2 \left(\mathrm{d} \ln B(t,T)\right) = D^2 \sigma^2 \left(\mathrm{d} r(t)\right) + \mathrm{DTS}(t)^2 \sigma^2 \left(\frac{\mathrm{d} s(t)}{s(t)}\right) + \sigma^2 \left(\mathrm{d} L(t)\right)$$

- Three risk components

$$\sigma^2 \left(\mathrm{d} \ln B(t,T)\right) = D^2 \sigma_r^2 + \mathrm{DTS}(t)^2 \sigma_s^2 + \sigma_L^2$$

$\Longrightarrow$ **The historical volatility of a bond price is not a relevant risk measure**

Bond portfolio optimization

Without a benchmark

- Duration risk:

$$\mathrm{MD}(w) = \sum_{i=1}^{n} w_i \, \mathrm{MD}_i$$

- DTS risk:

$$\mathrm{DTS}(w) = \sum_{i=1}^{n} w_i \, \mathrm{DTS}_i$$

- Clustering approach = generalization of the sector approach, e.g. (EUR, Financials, AAA to A−, 1Y-3Y)
- We have:

$$\mathrm{MD}_j(w) = \sum_{i \in \mathcal{S}ector_j} w_i \, \mathrm{MD}_i$$

and:

$$\mathrm{DTS}_j(w) = \sum_{i \in \mathcal{S}ector_j} w_i \, \mathrm{DTS}_i$$

Bond portfolio optimization

Without a benchmark

Objective function without a benchmark

We have:

$$\begin{aligned} w^{\star} &= \arg\min \frac{\varphi_{\mathrm{MD}}}{2} \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\mathrm{MD}_j(w) - \mathrm{MD}_j^{\star}\right)^2 + \\ & \frac{\varphi_{\mathrm{DTS}}}{2} \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\mathrm{DTS}_j(w) - \mathrm{DTS}_j^{\star}\right)^2 - \gamma \sum_{i=1}^{n} w_i \mathcal{C}_i \end{aligned}$$

where:

- $\varphi_{\mathrm{MD}} \geq 0$ and $\varphi_{\mathrm{DTS}} \geq 0$ indicate the relative weight of each risk component
- $\mathcal{C}_i$ is the expected carry of bond i and γ is the risk-tolerance coefficient

Bond portfolio optimization

Without a benchmark

QP form

$$
\begin{aligned}
w^{\star} & = & \arg\min \mathcal{QF}\left(w; Q, R, c\right) \\
& \text{s.t.} & \left\{ \begin{array}{l} \mathbf{1}_n^\top w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array} \right.
\end{aligned}
$$

where $\mathcal{QF}\left(w; Q, R, c\right)$ is the quadratic form of the objective function

Bond portfolio optimization

Without a benchmark

We have:

$$\begin{aligned}
\frac{1}{2}\left(\mathrm{MD}_j(w)-\mathrm{MD}_j^{\star}\right)^2 &= \frac{1}{2}\left(\sum\nolimits_{i\in\mathcal{S}ector_j} w_i\,\mathrm{MD}_i-\mathrm{MD}_j^{\star}\right)^2 \\
&= \frac{1}{2}\left(\sum_{i=1}^{n} \boldsymbol{s}_{i,j} w_i\,\mathrm{MD}_i-\mathrm{MD}_j^{\star}\right)^2 \\
&= \frac{1}{2}\left(\sum_{i=1}^{n} \boldsymbol{s}_{i,j}\,\mathrm{MD}_i\,w_i\right)^2 - w^\top\left(\boldsymbol{s}_j\circ\mathrm{MD}\right)\mathrm{MD}_j^{\star}+\frac{1}{2}\,\mathrm{MD}_j^{\star 2} \\
&= \mathcal{QF}\left(w;\mathcal{T}\left(\boldsymbol{s}_j\circ\mathrm{MD}\right),\left(\boldsymbol{s}_j\circ\mathrm{MD}\right)\mathrm{MD}_j^{\star},\frac{1}{2}\,\mathrm{MD}_j^{\star 2}\right)
\end{aligned}$$

where $\mathrm{MD}=\left(\mathrm{MD}_1,\ldots,\mathrm{MD}_n\right)$ is the vector of modified durations and $\mathcal{T}\left(u\right)=uu^\top$

Bond portfolio optimization
Without a benchmark

We deduce that:

$$\frac{1}{2}\sum_{j=1}^{n_{\mathcal{S}ector}} \left(\mathrm{MD}_j(w) - \mathrm{MD}_j^\star\right)^2 = \mathcal{QF}\left(w; Q_{\mathrm{MD}}, R_{\mathrm{MD}}, c_{\mathrm{MD}}\right)$$

where:

$$\begin{cases} Q_{\mathrm{MD}} = \sum_{j=1}^{n_{\mathcal{S}ector}} \mathcal{T}\left(\boldsymbol{s}_j \circ \mathrm{MD}\right) \\ R_{\mathrm{MD}} = \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\boldsymbol{s}_j \circ \mathrm{MD}\right)\mathrm{MD}_j^\star \\ c_{\mathrm{MD}} = \frac{1}{2}\sum_{j=1}^{n_{\mathcal{S}ector}} \mathrm{MD}_j^{\star 2} \end{cases}$$

Bond portfolio optimization

Without a benchmark

In a similar way, we have:

$$\frac{1}{2}\sum_{j=1}^{n_{\mathcal{S}ector}}\left(\mathrm{DTS}_j\left(w\right)-\mathrm{DTS}_j^{\star}\right)^2=\mathcal{QF}\left(w;Q_{\mathrm{DTS}},R_{\mathrm{DTS}},c_{\mathrm{DTS}}\right)$$

where:

$$\begin{cases}Q_{\mathrm{DTS}}=\sum_{j=1}^{n_{\mathcal{S}ector}}\mathcal{T}\left(\boldsymbol{s}_j\circ\mathrm{DTS}\right)\\R_{\mathrm{MD}}=\sum_{j=1}^{n_{\mathcal{S}ector}}\left(\boldsymbol{s}_j\circ\mathrm{DTS}\right)\mathrm{DTS}_j^{\star}\\c_{\mathrm{DTS}}=\frac{1}{2}\sum_{j=1}^{n_{\mathcal{S}ector}}\mathrm{DTS}_j^{\star 2}\end{cases}$$

Bond portfolio optimization

Without a benchmark

We have:

$$-\gamma \sum_{i=1}^{n} w_i \mathcal{C}_i = \gamma \mathcal{QF}\left(w; \mathbf{0}_{n,n}, \mathcal{C}, 0\right) = \mathcal{QF}\left(w; \mathbf{0}_{n,n}, \gamma \mathcal{C}, 0\right)$$

where $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_n)$ is the vector of expected carry values

Bond portfolio optimization

Without a benchmark

Quadratic form of the objective function

The function to optimize is:

$$\begin{aligned}\mathcal{QF}(w; Q, R, c) &= \varphi_{\mathrm{MD}} \mathcal{QF}(w; Q_{\mathrm{MD}}, R_{\mathrm{MD}}, c_{\mathrm{MD}}) + \\ &\quad \varphi_{\mathrm{DTS}} \mathcal{QF}(w; Q_{\mathrm{DTS}}, R_{\mathrm{DTS}}, c_{\mathrm{DTS}}) + \\ &\quad \mathcal{QF}(w; \mathbf{0}_{n,n}, \gamma \mathcal{C}, 0)\end{aligned}$$

where:

$$\left\{\begin{array}{l} Q = \varphi_{\mathrm{MD}} Q_{\mathrm{MD}} + \varphi_{\mathrm{DTS}} Q_{\mathrm{DTS}} \\ R = \gamma \mathcal{C} + \varphi_{\mathrm{MD}} R_{\mathrm{MD}} + \varphi_{\mathrm{DTS}} R_{\mathrm{DTS}} \\ c = \varphi_{\mathrm{MD}} c_{\mathrm{MD}} + \varphi_{\mathrm{DTS}} c_{\mathrm{DTS}} \end{array}\right.$$

Bond portfolio optimization

With a benchmark

- The MD- and DTS-based tracking error variances are equal to:

$$\mathcal{R}_{\mathrm{MD}}(w \mid b) = \sigma^2_{\mathrm{MD}}(w \mid b) = \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\sum\nolimits_{i \in \mathcal{S}ector_j} (w_i - b_i) \, \mathrm{MD}_i \right)^2$$

  and:

$$\mathcal{R}_{\mathrm{DTS}}(w \mid b) = \sigma^2_{\mathrm{DTS}}(w \mid b) = \sum_{j=1}^{n_{\mathcal{S}ector}} \left(\sum\nolimits_{i \in \mathcal{S}ector_j} (w_i - b_i) \, \mathrm{DTS}_i \right)^2$$

  This means that $\mathrm{MD}_j^\star = \sum_{i \in \mathcal{S}ector_j} b_i \, \mathrm{MD}_i$ and $\mathrm{DTS}_j^\star = \sum_{i \in \mathcal{S}ector_j} b_i \, \mathrm{DTS}_i$.
- The active share risk is defined as:

$$\mathcal{R}_{\mathrm{AS}}(w \mid b) = \sigma^2_{\mathrm{AS}}(w \mid b) = \sum_{i=1}^{n} (w_i - b_i)^2$$

Bond portfolio optimization

With a benchmark

Objective function with a benchmark

The optimization problem becomes:

$$\begin{aligned} w^{\star} &= \arg\min \frac{1}{2}\mathcal{R}\left(w \mid b\right) - \gamma \sum_{i=1}^{n} \left(w_i - b_i\right) \mathcal{C}_i \\ & \text{s.t.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^\top w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

where the synthetic risk measure is equal to:

$$\mathcal{R}\left(w \mid b\right) = \varphi_{\mathrm{AS}} \mathcal{R}_{\mathrm{AS}}\left(w \mid b\right) + \varphi_{\mathrm{MD}} \mathcal{R}_{\mathrm{MD}}\left(w \mid b\right) + \varphi_{\mathrm{DTS}} \mathcal{R}_{\mathrm{DTS}}\left(w \mid b\right)$$

Bond portfolio optimization

With a benchmark

We can show that

$$\begin{aligned} w^{\star} &= \arg\min \mathcal{QF}\left(w; Q\left(b\right), R\left(b\right), c\left(b\right)\right) \\ &\text{s.t.} \left\{ \begin{array}{l} \mathbf{1}_n^\top w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

where:

$$\left\{ \begin{array}{l} Q\left(b\right) = \varphi_{\mathrm{AS}} Q_{\mathrm{AS}}\left(b\right) + \varphi_{\mathrm{MD}} Q_{\mathrm{MD}}\left(b\right) + \varphi_{\mathrm{DTS}} Q_{\mathrm{DTS}}\left(b\right) \\ R\left(b\right) = \gamma \mathcal{C} + \varphi_{\mathrm{AS}} R_{\mathrm{AS}}\left(b\right) + \varphi_{\mathrm{MD}} R_{\mathrm{MD}}\left(b\right) + \varphi_{\mathrm{DTS}} R_{\mathrm{DTS}}\left(b\right) \\ c\left(b\right) = \gamma b^\top \mathcal{C} + \varphi_{\mathrm{AS}} c_{\mathrm{AS}}\left(b\right) + \varphi_{\mathrm{MD}} c_{\mathrm{MD}}\left(b\right) + \varphi_{\mathrm{DTS}} c_{\mathrm{DTS}}\left(b\right) \end{array} \right.$$

$Q_{\mathrm{AS}}\left(b\right) = I_n$, $R_{\mathrm{AS}}\left(b\right) = b$, $c_{\mathrm{AS}}\left(b\right) = \frac{1}{2} b^\top b$, $Q_{\mathrm{MD}}\left(b\right) = Q_{\mathrm{MD}}$, $R_{\mathrm{MD}}\left(b\right) = Q_{\mathrm{MD}} b = R_{\mathrm{MD}}$, $c_{\mathrm{MD}}\left(b\right) = \frac{1}{2} b^\top Q_{\mathrm{MD}} b = c_{\mathrm{MD}}$, $Q_{\mathrm{DTS}}\left(b\right) = Q_{\mathrm{DTS}}$, $R_{\mathrm{DTS}}\left(b\right) = Q_{\mathrm{DTS}} b = R_{\mathrm{DTS}}$, and $c_{\mathrm{DTS}}\left(b\right) = \frac{1}{2} b^\top Q_{\mathrm{DTS}} b = c_{\mathrm{DTS}}$

Bond portfolio optimization

With a benchmark

Example #3

We consider an investment universe of 9 corporate bonds with the following characteristics[a]:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8	#9
b_i	21	19	16	12	11	8	6	4	3
$\mathcal{CI}_i$	111	52	369	157	18	415	17	253	900
MD_i	3.16	6.48	3.54	9.23	6.40	2.30	8.12	7.96	5.48
DTS_i	107	255	75	996	289	45	620	285	125
$\mathcal{S}ector$	1	1	1	2	2	2	3	3	3

We impose that $0.25 \times b_i \leq w_i \leq 4 \times b_i$. We have $\varphi_{\mathrm{AS}} = 100$, $\varphi_{\mathrm{MD}} = 25$ and $\varphi_{\mathrm{DTS}} = 0.001$.

[a] The units are: b_i in %, $\mathcal{CI}_i$ in $\mathrm{tCO_2e/\$\ mn}$, MD_i in years and DTS_i in bps

Bond portfolio optimization

With a benchmark

The optimization problem is defined as:

$$
\begin{aligned}
w^{\star}(\mathcal{R}) &= \arg\min \frac{1}{2} w^{\top} Q(b) w - w^{\top} R(b) \\
\text{s.t.} & \left\{
\begin{array}{l}
\mathbf{1}_9^{\top} w = 1 \\
\mathcal{CI}^{\top} w \leq (1 - \mathcal{R}) \mathcal{CI}(b) \\
\dfrac{b}{4} \leq w \leq 4b
\end{array}
\right.
\end{aligned}
$$

where $\mathcal{R}$ is the reduction rate

Bond portfolio optimization

With a benchmark

Since the bonds are ordering by sectors, $Q(b)$ is a block diagonal matrix:

$$Q(b) = \begin{pmatrix} Q_1 & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & Q_2 & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & Q_3 \end{pmatrix} \times 10^3$$

where:

$$Q_1 = \begin{pmatrix} 0.3611 & 0.5392 & 0.2877 \\ 0.5392 & 1.2148 & 0.5926 \\ 0.2877 & 0.5926 & 0.4189 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 3.2218 & 1.7646 & 0.5755 \\ 1.7646 & 1.2075 & 0.3810 \\ 0.5755 & 0.3810 & 0.2343 \end{pmatrix}$$

and:

$$Q_3 = \begin{pmatrix} 2.1328 & 1.7926 & 1.1899 \\ 1.7926 & 1.7653 & 1.1261 \\ 1.1899 & 1.1261 & 0.8664 \end{pmatrix}$$

$R(b) = (2.243, 4.389, 2.400, 6.268, 3.751, 1.297, 2.354, 2.120, 1.424) \times 10^2$

Bond portfolio optimization

With a benchmark

Table 2: Weights in % of optimized bond portfolios (Example #3)

Portfolio	#1	#2	#3	#4	#5	#6	#7	#8	#9
b	21.00	19.00	16.00	12.00	11.00	8.00	6.00	4.00	3.00
$w^\star$ (10%)	21.92	19.01	15.53	11.72	11.68	7.82	6.68	4.71	0.94
$w^\star$ (30%)	26.29	20.24	10.90	10.24	16.13	3.74	9.21	2.50	0.75
$w^\star$ (50%)	27.48	23.97	4.00	6.94	22.70	2.00	11.15	1.00	0.75

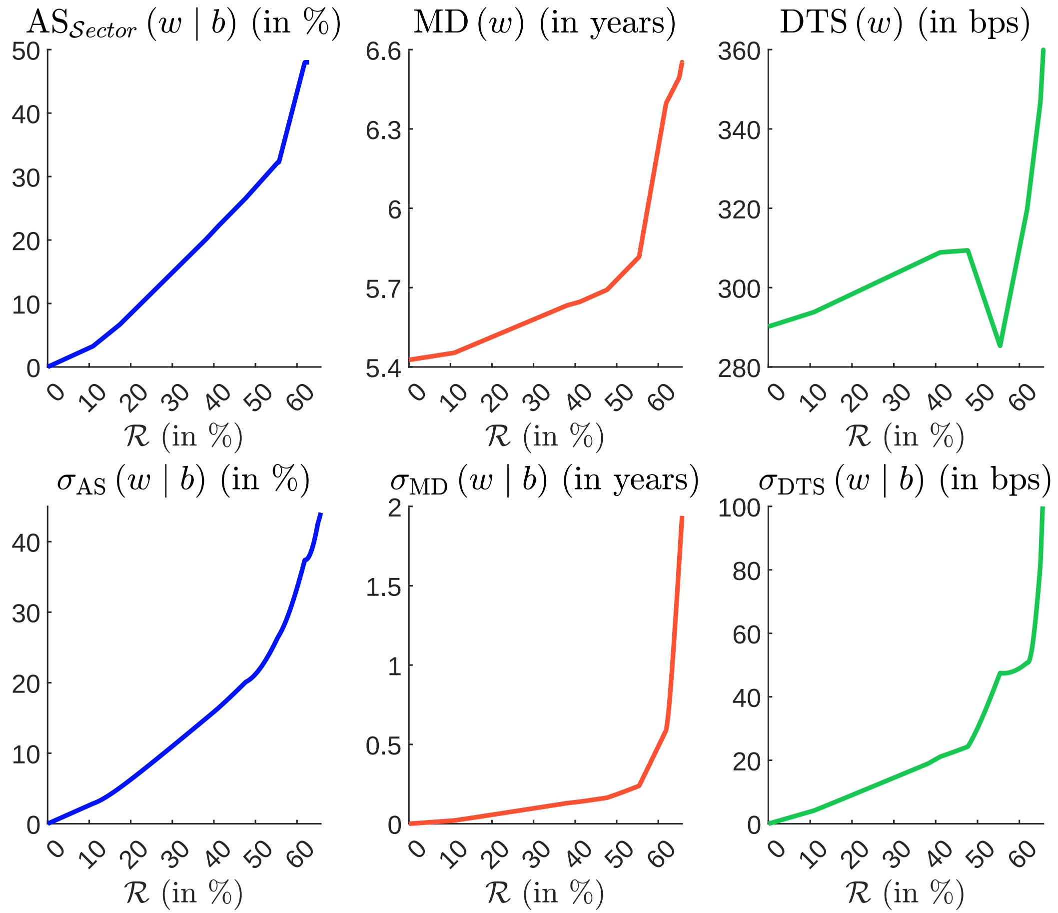
Table 3: Risk statistics of optimized bond portfolios (Example #3)

Portfolio	$\mathrm{AS}_{\mathcal{S}ector}$ (in %)	$\mathrm{MD}(w)$ (in years)	$\mathrm{DTS}(w)$ (in bps)	$\sigma_{\mathrm{AS}}(w \mid b)$ (in %)	$\sigma_{\mathrm{MD}}(w \mid b)$ (in years)	$\sigma_{\mathrm{DTS}}(w \mid b)$ (in bps)	$\mathcal{CI}(w)$ $\mathrm{gCO_2e}/\$$
b	0.00	5.43	290.18	0.00	0.00	0.00	184.39
$w^\star$ (10%)	3.00	5.45	293.53	2.62	0.02	3.80	165.95
$w^\star$ (30%)	14.87	5.58	303.36	10.98	0.10	14.49	129.07
$w^\star$ (50%)	28.31	5.73	302.14	21.21	0.19	30.11	92.19

Bond portfolio optimization

With a benchmark

Figure 1: Relationship between the reduction rate and the tracking risk (Example #3)



Advanced optimization problems
Large bond universe

- QP: $\mathbf{n} \leq \mathbf{5\,000}$ (the dimension of Q is $n \times n$)
- LP: $\mathbf{n} \gg \mathbf{10^6}$

- Some figures as of 31/01/2023
  - MSCI World Index (DM): $n = 1\,508$ stocks
  - MSCI World IMI (DM): $n = 5\,942$ stocks
  - MSCI World AC (DM + EM): $n = 2\,882$ stocks
  - MSCI World AC IMI (DM + EM): $n = 7\,928$ stocks

  - Bloomberg Global Aggregate Total Return Index: $n = 28\,799$ securities
  - ICE BOFA Global Broad Market Index: $n = 33\,575$ securities

- Trick: $\mathcal{L}_2$-norm risk measures $\Rightarrow$ $\mathcal{L}_1$-norm risk measures

Advanced optimization problems

Large bond universe

We replace the synthetic risk measure by:

$$\mathcal{D}\left(w \mid b\right) = \varphi'_{\mathrm{AS}}\mathcal{D}_{\mathrm{AS}}\left(w \mid b\right) + \varphi'_{\mathrm{MD}}\mathcal{D}_{\mathrm{MD}}\left(w \mid b\right) + \varphi'_{\mathrm{DTS}}\mathcal{D}_{\mathrm{DTS}}\left(w \mid b\right)$$

where:

$$\begin{aligned}
\mathcal{D}_{\mathrm{AS}}\left(w \mid b\right) &= \frac{1}{2}\sum_{i=1}^{n}\left|w_i - b_i\right| \\
\mathcal{D}_{\mathrm{MD}}\left(w \mid b\right) &= \sum_{j=1}^{n_{\mathcal{S}ector}}\left|\sum\nolimits_{i\in\mathcal{S}ector_j}\left(w_i - b_i\right)\mathrm{MD}_i\right| \\
\mathcal{D}_{\mathrm{DTS}}\left(w \mid b\right) &= \sum_{j=1}^{n_{\mathcal{S}ector}}\left|\sum\nolimits_{i\in\mathcal{S}ector_j}\left(w_i - b_i\right)\mathrm{DTS}_i\right|
\end{aligned}$$

Advanced optimization problems
Large bond universe

The optimization problem becomes:

$$\begin{aligned} w^{\star} & = \arg\min \mathcal{D}\left(w \mid b\right) - \gamma \sum_{i=1}^{n} \left(w_i - b_i\right) \mathcal{C}_i \\ & \text{s.t.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^{\top} w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

Advanced optimization problems

Large bond universe

Absolute value trick

If $c_i \geq 0$, then:

$$\min \sum_{i=1}^{n} c_i \left| f_i\left(x\right)\right| + g\left(x\right) \Leftrightarrow \left\{ \begin{array}{ll} \min & \displaystyle\sum_{i=1}^{n} c_i \tau_i + g\left(x\right) \\ \text{s.t.} & \left\{ \begin{array}{l} \left| f_i\left(x\right)\right| \leq \tau_i \\ \tau_i \geq 0 \end{array} \right. \end{array} \right.$$

The problem becomes linear:

$$\left| f_i\left(x\right)\right| \leq \tau_i \Leftrightarrow -\tau_i \leq f_i\left(x\right) \wedge f_i\left(x\right) \leq \tau_i$$

Advanced optimization problems

Large bond universe

Linear programming

The standard formulation of a linear programming problem is:

$$\begin{aligned} x^{\star} & = & \arg\min c^{\top} x \\ & \text{s.t.} & \left\{ \begin{array}{l} Ax = b \\ Cx \leq D \\ x^{-} \leq x \leq x^{+} \end{array} \right. \end{aligned}$$

where x is a $n \times 1$ vector, c is a $n \times 1$ vector, A is a $n_A \times n$ matrix, B is a $n_A \times 1$ vector, C is a $n_C \times n$ matrix, D is a $n_C \times 1$ vector, and x^{-} and x^{+} are two $n \times 1$ vectors.

Advanced optimization problems

Large bond universe

We have:

$$w^{\star} = \arg\min \frac{1}{2}\varphi'_{\mathrm{AS}}\sum_{i=1}^{n}\tau_{i,w} + \varphi'_{\mathrm{MD}}\sum_{j=1}^{n_{\mathcal{S}ector}}\tau_{j,\mathrm{MD}} + \varphi'_{\mathrm{DTS}}\sum_{j=1}^{n_{\mathcal{S}ector}}\tau_{j,\mathrm{DTS}} - \gamma\sum_{i=1}^{n}(w_i - b_i)\,\mathcal{C}_i$$

$$\text{s.t.} \left\{\begin{array}{l} \mathbf{1}_n^\top w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \\ \left|w_i - b_i\right| \leq \tau_{i,w} \\ \left|\sum_{i\in\mathcal{S}ector_j}(w_i - b_i)\,\mathrm{MD}_i\right| \leq \tau_{j,\mathrm{MD}} \\ \left|\sum_{i\in\mathcal{S}ector_j}(w_i - b_i)\,\mathrm{DTS}_i\right| \leq \tau_{j,\mathrm{DTS}} \\ \tau_{i,w} \geq 0, \tau_{j,\mathrm{MD}} \geq 0, \tau_{j,\mathrm{DTS}} \geq 0 \end{array}\right.$$

Advanced optimization problems

Large bond universe

$$\left| w_i - b_i \right| \leq \tau_{i,w} \Leftrightarrow \left\{ \begin{array}{l} w_i - \tau_{i,w} \leq b_i \\ -w_i - \tau_{i,w} \leq -b_i \end{array} \right.$$

Advanced optimization problems

Large bond universe

$$
\begin{aligned}
(*) \quad & \Leftrightarrow \quad \left| \sum_{i \in \mathcal{S}ector_j} (w_i - b_i) \, \mathrm{MD}_i \right| \leq \tau_{j,\mathrm{MD}} \\
& \Leftrightarrow \quad -\tau_{j,\mathrm{MD}} \leq \sum_{i \in \mathcal{S}ector_j} (w_i - b_i) \, \mathrm{MD}_i \leq \tau_{j,\mathrm{MD}} \\
& \Leftrightarrow \quad -\tau_{j,\mathrm{MD}} + \sum_{i \in \mathcal{S}ector_j} b_i \, \mathrm{MD}_i \leq \sum_{i \in \mathcal{S}ector_j} w_i \, \mathrm{MD}_i \leq \tau_{j,\mathrm{MD}} + \\
& \qquad \sum_{i \in \mathcal{S}ector_j} b_i \, \mathrm{MD}_i \\
& \Leftrightarrow \quad -\tau_{j,\mathrm{MD}} + \mathrm{MD}_j^\star \leq \left(\boldsymbol{s}_j \circ \mathrm{MD} \right)^\top w \leq \tau_{j,\mathrm{MD}} + \mathrm{MD}_j^\star \\
& \Leftrightarrow \quad \left\{ \begin{array}{l} \left(\boldsymbol{s}_j \circ \mathrm{MD} \right)^\top w - \tau_{j,\mathrm{MD}} \leq \mathrm{MD}_j^\star \\ -\left(\boldsymbol{s}_j \circ \mathrm{MD} \right)^\top w - \tau_{j,\mathrm{MD}} \leq -\mathrm{MD}_j^\star \end{array} \right.
\end{aligned}
$$

Advanced optimization problems

Large bond universe

$$\left|\sum_{i\in\mathcal{S}ector_j}(w_i - b_i)\,\mathrm{DTS}_i\right| \leq \tau_{j,\mathrm{DTS}} \Leftrightarrow \left\{\begin{array}{l}(\boldsymbol{s}_j \circ \mathrm{DTS})^\top w - \tau_{j,\mathrm{DTS}} \leq \mathrm{DTS}_j^\star \\ -(\boldsymbol{s}_j \circ \mathrm{DTS})^\top w - \tau_{j,\mathrm{DTS}} \leq -\mathrm{DTS}_j^\star\end{array}\right.$$

Advanced optimization problems
LP formulation

- x is a vector of dimension $n_x = 2 \times (n + n_{\mathcal{S}ector})$:

$$x = \begin{pmatrix} w \\ \tau_w \\ \tau_{\mathrm{MD}} \\ \tau_{\mathrm{DTS}} \end{pmatrix}$$

Advanced optimization problems

LP formulation

- The vector c is equal to:

$$c = \begin{pmatrix} -\gamma \mathcal{C} \\ \frac{1}{2} \varphi'_{\mathrm{AS}} \mathbf{1}_n \\ \varphi'_{\mathrm{MD}} \mathbf{1}_{n_{\mathcal{S}ector}} \\ \varphi'_{\mathrm{DTS}} \mathbf{1}_{n_{\mathcal{S}ector}} \end{pmatrix}$$

Advanced optimization problems

LP formulation

- The linear equality constraint $Ax = B$ is defined by:

$$A = \begin{pmatrix} \mathbf{1}_n^\top & \mathbf{0}_n^\top & \mathbf{0}_{n_{\mathcal{S}ector}}^\top & \mathbf{0}_{n_{\mathcal{S}ector}}^\top \end{pmatrix}$$

and:

$$B = 1$$

Advanced optimization problems
LP formulation

- The linear inequality constraint $Cx \leq D$ is defined by:

$$C = \begin{pmatrix} I_n & -I_n & \mathbf{0}_{n,n_{\mathcal{S}ector}} & \mathbf{0}_{n,n_{\mathcal{S}ector}} \\ -I_n & -I_n & \mathbf{0}_{n,n_{\mathcal{S}ector}} & \mathbf{0}_{n,n_{\mathcal{S}ector}} \\ C_{\mathrm{MD}} & \mathbf{0}_{n_{\mathcal{S}ector},n} & -I_{n_{\mathcal{S}ector}} & \mathbf{0}_{n_{\mathcal{S}ector},n_{\mathcal{S}ector}} \\ -C_{\mathrm{MD}} & \mathbf{0}_{n_{\mathcal{S}ector},n} & -I_{n_{\mathcal{S}ector}} & \mathbf{0}_{n_{\mathcal{S}ector},n_{\mathcal{S}ector}} \\ C_{\mathrm{DTS}} & \mathbf{0}_{n_{\mathcal{S}ector},n} & \mathbf{0}_{n_{\mathcal{S}ector},n_{\mathcal{S}ector}} & -I_{n_{\mathcal{S}ector}} \\ -C_{\mathrm{DTS}} & \mathbf{0}_{n_{\mathcal{S}ector},n} & \mathbf{0}_{n_{\mathcal{S}ector},n_{\mathcal{S}ector}} & -I_{n_{\mathcal{S}ector}} \end{pmatrix}$$

end:

$$D = \begin{pmatrix} b \\ -b \\ \mathrm{MD}^{\star} \\ -\mathrm{MD}^{\star} \\ \mathrm{DTS}^{\star} \\ -\mathrm{DTS}^{\star} \end{pmatrix}$$

Advanced optimization problems
LP formulation

- C_{MD} and C_{DTS} are two $n_{\mathcal{S}ector} \times n$ matrices, whose elements are:

$$\left(C_{\mathrm{MD}}\right)_{j,i} = \boldsymbol{s}_{i,j} \, \mathrm{MD}_i$$

and:

$$\left(C_{\mathrm{DTS}}\right)_{j,i} = \boldsymbol{s}_{i,j} \, \mathrm{DTS}_i$$

- We have:

$$\mathrm{MD}^\star = \left(\mathrm{MD}_1^\star, \ldots, \mathrm{MD}_{n_{\mathcal{S}ector}}^\star\right)$$

and

$$\mathrm{DTS}^\star = \left(\mathrm{DTS}_1^\star, \ldots, \mathrm{DTS}_{n_{\mathcal{S}ector}}^\star\right)$$

Advanced optimization problems

LP formulation

- The bounds are:

$$x^{-} = \mathbf{0}_{n_x}$$

and:

$$x^{+} = \infty \cdot \mathbf{1}_{n_x}$$

Advanced optimization problems

LP formulation

- Additional constraints:

$$\left\{ \begin{array}{l} A'w = B' \\ C'w \leq D' \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \left(\begin{array}{cc} A' & \mathbf{0}_{n_A, n_x - n} \end{array} \right) x = B' \\ \left(\begin{array}{cc} C' & \mathbf{0}_{n_A, n_x - n} \end{array} \right) x \leq D' \end{array} \right.$$

Advanced optimization problems

Large bond universe

Toy example

We consider a toy example with four corporate bonds:

Issuer	#1	#2	#3	#4
b_i (in %)	35	15	20	30
$\mathcal{CI}_i$ (in $\text{tCO}_2\text{e}/\$\text{ mn}$)	117	284	162.5	359
MD_i (in years)	3.0	5.0	2.0	6.0
DTS_i (in bps)	100	150	200	250
$\mathcal{S}ector$	1	1	2	2

We would like to reduce the carbon footprint by 20%, and we set $\varphi'_{\text{AS}} = 100$, $\varphi'_{\text{MD}} = 25$ and $\varphi'_{\text{DTS}} = 1$

Advanced optimization problems

Large bond universe

We have $n = 4$, $n_{\mathcal{S}ector} = 2$ and:

$$x = (\underbrace{w_1, w_2, w_3, w_4}_{w}, \underbrace{\tau_{w_1}, \tau_{w_2}, \tau_{w_3}, \tau_{w_4}}_{\tau_w}, \underbrace{\tau_{\mathrm{MD}_1}, \tau_{\mathrm{MD}_2}}_{\tau_{\mathrm{MD}}}, \underbrace{\tau_{\mathrm{DTS}_1}, \tau_{\mathrm{DTS}_2}}_{\tau_{\mathrm{DTS}}})$$

Since the vector $\mathcal{C}$ is equal to $\mathbf{0}_4$, we obtain:

$$c = (0, 0, 0, 0, 50, 50, 50, 50, 25, 25, 1, 1)$$

Advanced optimization problems

Large bond universe

The equality system $Ax = B$ is defined by:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and:

$$B = 1$$

Advanced optimization problems

Large bond universe

The inequality system $Cx \leq D$ is given by:

$$C = \left(\begin{array}{cccc|c|cccc}
\multicolumn{4}{c|}{I_4} & -I_4 & \multicolumn{4}{c}{\mathbf{0}_{4,4}} \\
\hline
\multicolumn{4}{c|}{-I_4} & -I_4 & \multicolumn{4}{c}{\mathbf{0}_{4,4}} \\
\hline
3 & 5 & 0 & 0 & & -1 & 0 & 0 & 0 \\
0 & 0 & 2 & 6 & \mathbf{0}_{4,4} & 0 & -1 & 0 & 0 \\
-3 & -5 & 0 & 0 & & -1 & 0 & 0 & 0 \\
0 & 0 & -2 & -6 & & 0 & -1 & 0 & 0 \\
\hline
100 & 150 & 0 & 0 & & 0 & 0 & -1 & 0 \\
0 & 0 & 200 & 250 & \mathbf{0}_{4,4} & 0 & 0 & 0 & -1 \\
-100 & -150 & 0 & 0 & & 0 & 0 & -1 & 0 \\
0 & 0 & -200 & -250 & & 0 & 0 & 0 & -1 \\
\hline
117 & 284 & 162.5 & 359 & \mathbf{0}_{1,4} & 0 & 0 & 0 & 0
\end{array}\right)$$

and:

$$\begin{array}{rcl}
D & = & (0.35, 0.15, 0.2, 0.3, \quad -0.35, -0.15, -0.2, -0.3, \ldots \\
 & & 1.8, 2.2, -1.8, -2.2, \quad 57.5, 115, -57.5, -115, \quad 179)
\end{array}$$

Advanced optimization problems
Large bond universe

- The last row of $Cx \leq D$ corresponds to the carbon footprint constraint
- We have:

$$\mathcal{CI}(b) = 223.75 \text{ tCO}_2\text{e}/\$ \text{ mn}$$

and:

$$(1 - \mathcal{R})\,\mathcal{CI}(b) = 0.80 \times 223.75 = 179.00 \text{ tCO}_2\text{e}/\$ \text{ mn}$$

Advanced optimization problems

Large bond universe

We solve the LP program, and we obtain the following solution:

$$\begin{aligned} w^{\star} &= (47.34\%, 0\%, 33.3\%, 19.36\%) \\ \tau_{w}^{\star} &= (12.34\%, 15\%, 13.3\%, 10.64\%) \\ \tau_{\mathrm{MD}}^{\star} &= (0.3798, 0.3725) \\ \tau_{\mathrm{DTS}}^{\star} &= (10.1604, 0) \end{aligned}$$

Advanced optimization problems

Large bond universe

- Interpretation of $\tau_w^\star$:

$$w^\star \pm \tau_w^\star = \begin{pmatrix} 47.34\% \\ 0.00\% \\ 33.30\% \\ 19.36\% \end{pmatrix} \begin{pmatrix} - \\ + \\ - \\ + \end{pmatrix} \begin{pmatrix} 12.34\% \\ 15.00\% \\ 13.30\% \\ 10.64\% \end{pmatrix} = \begin{pmatrix} 35\% \\ 15\% \\ 20\% \\ 30\% \end{pmatrix} = b$$

- Interpretation of $\tau_{\mathrm{MD}}^\star$:

$$\begin{pmatrix} \sum_{i \in \mathcal{S}ector_1} w_i^\star \, \mathrm{MD}_i \\ \sum_{i \in \mathcal{S}ector_2} w_i^\star \, \mathrm{MD}_i \end{pmatrix} \pm \tau_{\mathrm{MD}}^\star = \begin{pmatrix} 1.42 \\ 1.83 \end{pmatrix} \begin{pmatrix} + \\ + \end{pmatrix} \begin{pmatrix} 0.38 \\ 0.37 \end{pmatrix} = \begin{pmatrix} 1.80 \\ 2.20 \end{pmatrix} = \begin{pmatrix} \mathrm{MD}_1^\star \\ \mathrm{MD}_2^\star \end{pmatrix}$$

- Interpretation of $\tau_{\mathrm{DTS}}^\star$:

$$\begin{pmatrix} \sum_{i \in \mathcal{S}ector_1} w_i^\star \, \mathrm{DTS}_i \\ \sum_{i \in \mathcal{S}ector_2} w_i^\star \, \mathrm{DTS}_i \end{pmatrix} \pm \tau_{\mathrm{DTS}}^\star = \begin{pmatrix} 47.34 \\ 115.00 \end{pmatrix} \begin{pmatrix} + \\ + \end{pmatrix} \begin{pmatrix} 10.16 \\ 0.00 \end{pmatrix} = \begin{pmatrix} 57.50 \\ 115.00 \end{pmatrix} = \begin{pmatrix} \mathrm{DTS}_1^\star \\ \mathrm{DTS}_2^\star \end{pmatrix}$$

Advanced optimization problems

Large bond universe

Example #4 (Example #3 again)

We consider an investment universe of 9 corporate bonds with the following characteristics[a]:

Issuer	#1	#2	#3	#4	#5	#6	#7	#8	#9
b_i	21	19	16	12	11	8	6	4	3
$\mathcal{CI}_i$	111	52	369	157	18	415	17	253	900
MD_i	3.16	6.48	3.54	9.23	6.40	2.30	8.12	7.96	5.48
DTS_i	107	255	75	996	289	45	620	285	125
$\mathcal{S}ector$	1	1	1	2	2	2	3	3	3

We impose that $0.25 \times b_i \leq w_i \leq 4 \times b_i$ and assume that $\varphi'_{\mathrm{AS}} = \varphi_{\mathrm{AS}} = 100$, $\varphi'_{\mathrm{MD}} = \varphi_{\mathrm{MD}} = 25$ and $\varphi'_{\mathrm{DTS}} = \varphi'_{\mathrm{DTS}} = 0.001$

[a] The units are: b_i in %, $\mathcal{CI}_i$ in $\mathrm{tCO_2e}/\$\ \mathrm{mn}$, MD_i in years and DTS_i in bps

Advanced optimization problems

Large bond universe

Table 4: Weights in % of optimized bond portfolios (Example #4)

Portfolio	#1	#2	#3	#4	#5	#6	#7	#8	#9
b	21.00	19.00	16.00	12.00	11.00	8.00	6.00	4.00	3.00
$w^{\star}$ (10%)	21.70	19.00	16.00	12.00	11.00	8.00	7.46	4.00	0.84
$w^{\star}$ (30%)	34.44	19.00	4.00	11.65	11.98	6.65	7.52	4.00	0.75
$w^{\star}$ (50%)	33.69	19.37	4.00	3.91	24.82	2.00	10.46	1.00	0.75

Table 5: Risk statistics of optimized bond portfolios (Example #4)

Portfolio	$\mathrm{AS}_{\mathcal{S}ector}$ (in %)	$\mathrm{MD}(w)$ (in years)	$\mathrm{DTS}(w)$ (in bps)	$\sigma_{\mathrm{AS}}(w \mid b)$ (in %)	$\sigma_{\mathrm{MD}}(w \mid b)$ (in years)	$\sigma_{\mathrm{DTS}}(w \mid b)$ (in bps)	$\mathcal{CI}(w)$ $\mathrm{gCO_2e/\$}$
b	0.00	5.43	290.18	0.00	0.00	0.00	184.39
$w^{\star}$ (10%)	2.16	5.45	297.28	2.16	0.02	7.10	165.95
$w^{\star}$ (30%)	15.95	5.43	300.96	15.95	0.00	13.20	129.07
$w^{\star}$ (50%)	31.34	5.43	268.66	31.34	0.00	65.12	92.19

Equity portfolios

Threshold approach

The optimization problem is:

$$
\begin{aligned}
w^{\star} &= \arg\min \frac{1}{2}\left(w-b\right)^{\top}\Sigma\left(w-b\right) \\
\text{s.t.} &\quad \left\{
\begin{array}{l}
\mathbf{1}_n^{\top} w = 1 \\
w \in \Omega \\
\mathbf{0}_n \leq w \leq \mathbf{1}_n \\
\mathcal{CI}\left(w\right) \leq \left(1-\mathcal{R}\right)\mathcal{CI}\left(b\right)
\end{array}
\right.
\end{aligned}
$$

Equity portfolios

Order-statistic approach

- $\mathcal{CI}_{i:n}$ is the order statistics of $(\mathcal{CI}_1, \ldots, \mathcal{CI}_n)$:

$$\min \mathcal{CI}_i = \mathcal{CI}_{1:n} \leq \mathcal{CI}_{2:n} \leq \cdots \leq \mathcal{CI}_{i:n} \leq \cdots \leq \mathcal{CI}_{n:n} = \max \mathcal{CI}_i$$

- The carbon intensity bound $\mathcal{CI}^{(m,n)}$ is defined as:

$$\mathcal{CI}^{(m,n)} = \mathcal{CI}_{n-m+1:n}$$

  where $\mathcal{CI}_{n-m+1:n}$ is the $(n-m+1)$-th order statistic of $(\mathcal{CI}_1, \ldots, \mathcal{CI}_n)$
- Exclusion process:

$$\mathcal{CI}_i \geq \mathcal{CI}^{(m,n)} \Rightarrow w_i = 0$$

Equity portfolios

Order-statistic approach (Cont'd)

The optimization problem is:

$$\begin{aligned} w^{\star} &= \arg\min \frac{1}{2} (w - b)^{\top} \Sigma (w - b) \\ \text{s.t.} & \left\{ \begin{array}{l} \mathbf{1}_n^{\top} w = 1 \\ w \in \Omega \\ \mathbf{0}_n \leq w \leq \mathbb{1}\left\{ \mathcal{CI} < \mathcal{CI}^{(m,n)} \right\} \end{array} \right. \end{aligned}$$

Equity portfolios

Naive approach

We re-weight the remaining assets:

$$w_i^\star = \frac{\mathbb{1}\left\{\mathcal{CI}_i < \mathcal{CI}^{(m,n)}\right\} \cdot b_i}{\sum_{k=1}^n \mathbb{1}\left\{\mathcal{CI}_k < \mathcal{CI}^{(m,n)}\right\} \cdot b_k}$$

Equity portfolios

Example #5

We consider a capitalization-weighted equity index, which is composed of eight stocks. Their weights are equal to 20%, 19%, 17%, 13%, 12%, 8%, 6% and 5%. The carbon intensities (expressed in $\mathrm{tCO_2e}/\$ \mathrm{mn}$) are respectively equal to 100.5, 97.2, 250.4, 352.3, 27.1, 54.2, 78.6 and 426.7. To evaluate the risk of the portfolio, we use the market one-factor model: the beta β_i of each stock is equal to 0.30, 1.80, 0.85, 0.83, 1.47, 0.94, 1.67 and 1.08, the idiosyncratic volatilities $\tilde{\sigma}_i$ are respectively equal to 10%, 5%, 6%, 12%, 15%, 4%, 8% and 7%, and the estimated market volatility σ_m is 18%.

Equity portfolios

The covariance matrix is:

$$\Sigma = \beta\beta^\top \sigma_m^2 + D$$

where:

1. β is the vector of beta coefficients
2. σ_m^2 is the variance of the market portfolio
3. $D = diag\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2\right)$ is the diagonal matrix of idiosyncratic variances

Equity portfolios

Table 6: Optimal decarbonization portfolios (Example #5, threshold approach)

$\mathcal{R}$	0	10	20	30	40	50	$\mathcal{CI}_i$
$w_1^\star$	20.00	20.54	21.14	21.86	22.58	22.96	100.5
$w_2^\star$	19.00	19.33	19.29	18.70	18.11	17.23	97.2
$w_3^\star$	17.00	15.67	12.91	8.06	3.22	0.00	250.4
$w_4^\star$	13.00	12.28	10.95	8.74	6.53	3.36	352.3
$w_5^\star$	12.00	12.26	12.60	13.07	13.53	14.08	27.1
$w_6^\star$	8.00	11.71	16.42	22.57	28.73	34.77	54.2
$w_7^\star$	6.00	6.36	6.69	7.00	7.30	7.59	78.6
$w_8^\star$	5.00	1.86	0.00	0.00	0.00	0.00	426.7
$\sigma(w^\star \mid b)$	0.00	30.01	61.90	104.10	149.65	196.87	
$\mathcal{CI}(w)$	160.57	144.52	128.46	112.40	96.34	80.29	
$\mathcal{R}(w \mid b)$	0.00	10.00	20.00	30.00	40.00	50.00	

The reduction rate and the weights are expressed in % whereas the tracking error volatility is measured in bps

Equity portfolios

Table 7: Optimal decarbonization portfolios (Example #5, order-statistic approach)

m	0	1	2	3	4	5	6	7	$\mathcal{CI}_i$
$w_1^\star$	20.00	20.40	22.35	26.46	0.00	0.00	0.00	0.00	100.5
$w_2^\star$	19.00	19.90	20.07	20.83	7.57	0.00	0.00	0.00	97.2
$w_3^\star$	17.00	17.94	21.41	0.00	0.00	0.00	0.00	0.00	250.4
$w_4^\star$	13.00	13.24	0.00	0.00	0.00	0.00	0.00	0.00	352.3
$w_5^\star$	12.00	12.12	12.32	12.79	13.04	14.26	18.78	100.00	27.1
$w_6^\star$	8.00	10.04	17.14	32.38	74.66	75.12	81.22	0.00	54.2
$w_7^\star$	6.00	6.37	6.70	7.53	4.73	10.62	0.00	0.00	78.6
$w_8^\star$	5.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	426.7
$\sigma(w^\star \mid b)$	0.00	0.37	1.68	2.25	3.98	4.04	4.30	15.41	
$\mathcal{CI}(w)$	160.57	145.12	113.48	73.78	55.08	52.93	49.11	27.10	
$\mathcal{R}(w \mid b)$	0.00	9.62	29.33	54.05	65.70	67.04	69.42	83.12	

The reduction rate, the weights and the tracking error volatility are expressed in %

Equity portfolios

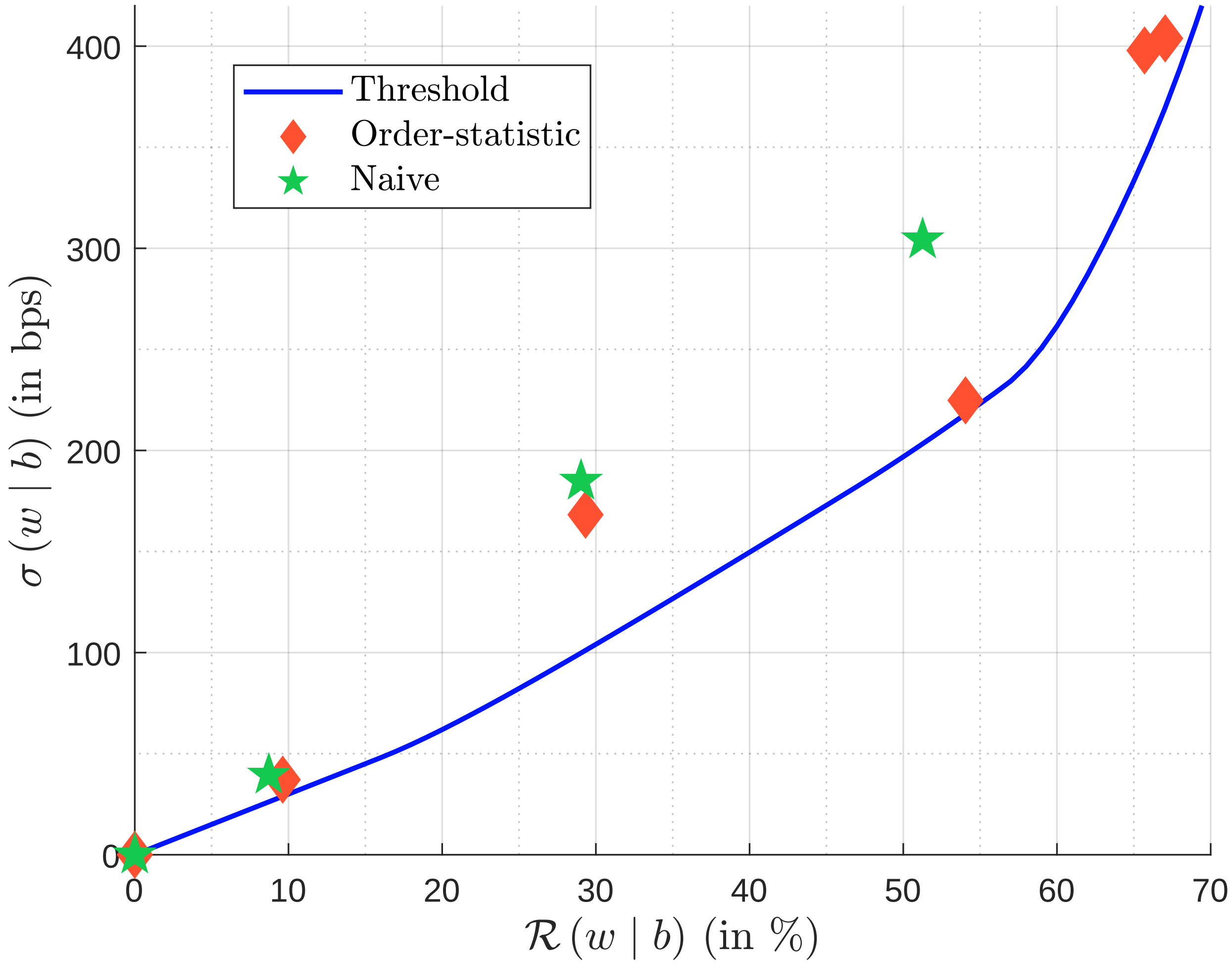
Table 8: Optimal decarbonization portfolios (Example #5, naive approach)

m	0	1	2	3	4	5	6	7	$\mathcal{CI}_i$
$w_1^\star$	20.00	21.05	24.39	30.77	0.00	0.00	0.00	0.00	100.5
$w_2^\star$	19.00	20.00	23.17	29.23	42.22	0.00	0.00	0.00	97.2
$w_3^\star$	17.00	17.89	20.73	0.00	0.00	0.00	0.00	0.00	250.4
$w_4^\star$	13.00	13.68	0.00	0.00	0.00	0.00	0.00	0.00	352.3
$w_5^\star$	12.00	12.63	14.63	18.46	26.67	46.15	60.00	100.00	27.1
$w_6^\star$	8.00	8.42	9.76	12.31	17.78	30.77	40.00	0.00	54.2
$w_7^\star$	6.00	6.32	7.32	9.23	13.33	23.08	0.00	0.00	78.6
$w_8^\star$	5.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	426.7
$\sigma(w^\star \mid b)$	0.00	0.39	1.85	3.04	9.46	8.08	8.65	15.41	
$\mathcal{CI}(w)$	160.57	146.57	113.95	78.26	68.38	47.32	37.94	27.10	
$\mathcal{R}(w \mid b)$	0.00	8.72	29.04	51.26	57.41	70.53	76.37	83.12	

The reduction rate, the weights and the tracking error volatility are expressed in %.

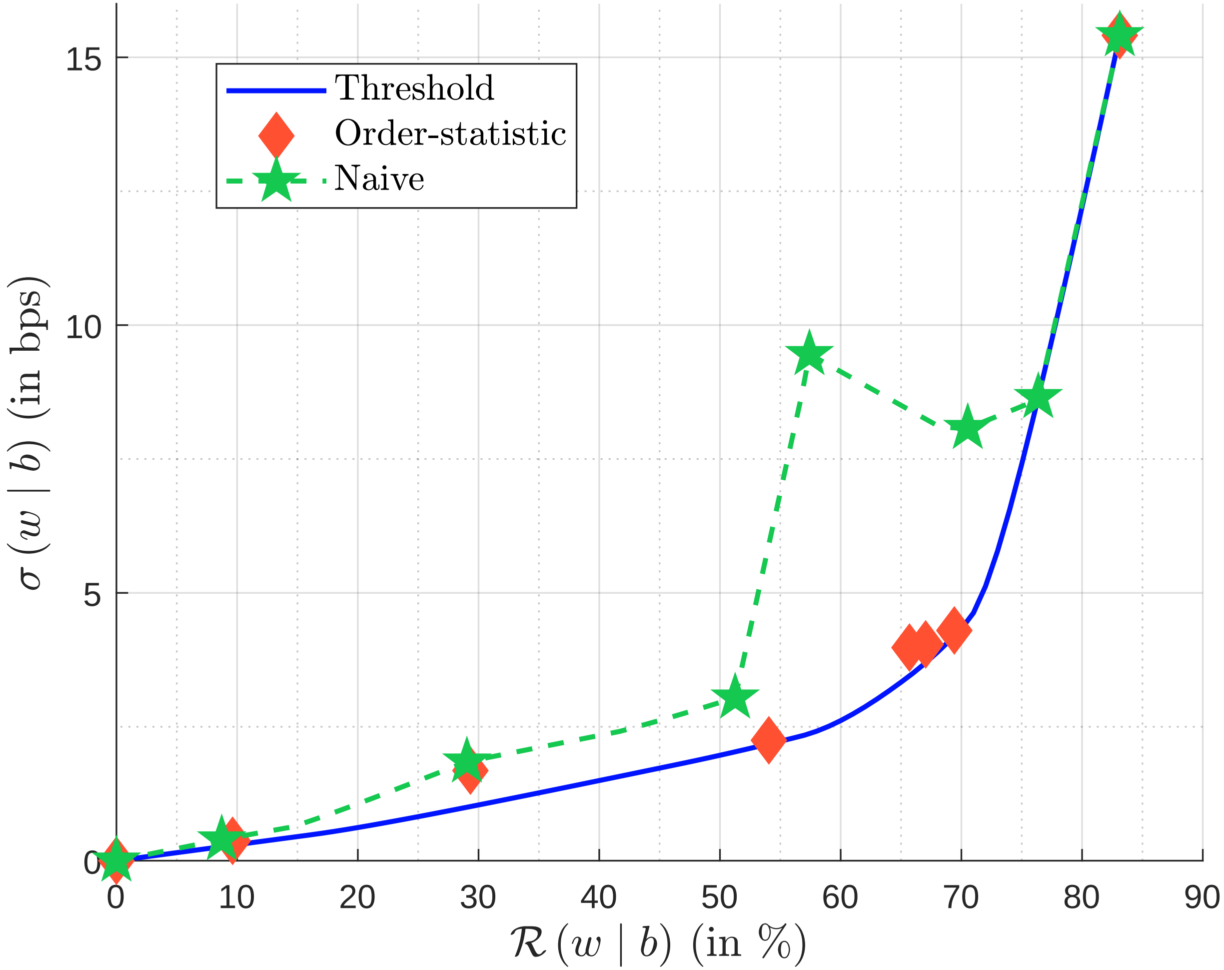
Equity portfolios

Figure 2: Efficient decarbonization frontier (Example #5)



Equity portfolios

Figure 3: Efficient decarbonization frontier of the interpolated naive approach (Example #5)



Bond portfolios

Example #6

We consider a debt-weighted bond index, which is composed of eight bonds. Their weights are equal to 20%, 19%, 17%, 13%, 12%, 8%, 6% and 5%. The carbon intensities (expressed in $\mathrm{tCO_2e/\$\ mn}$) are respectively equal to 100.5, 97.2, 250.4, 352.3, 27.1, 54.2, 78.6 and 426.7. To evaluate the risk of the portfolio, we use the modified duration which is respectively equal to 3.1, 6.6, 7.2, 5, 4.7, 2.1, 8.1 and 2.6 years, and the duration-times-spread factor, which is respectively equal to 100, 155, 575, 436, 159, 145, 804 and 365 bps. There are two sectors. Bonds #1, #3, #4 and #8 belong to $\mathcal{S}ector_1$ while Bonds #2, #5, #6 and #7 belong to $\mathcal{S}ector_2$

Bond portfolios

Table 9: Optimal decarbonization portfolios (Example #6, threshold approach)

$\mathcal{R}$	0	10	20	30	40	50	$\mathcal{CI}_i$
$w_1^\star$	20.00	21.62	23.93	26.72	30.08	33.44	100.5
$w_2^\star$	19.00	18.18	16.98	14.18	7.88	1.58	97.2
$w_3^\star$	17.00	18.92	21.94	22.65	16.82	11.00	250.4
$w_4^\star$	13.00	11.34	5.35	0.00	0.00	0.00	352.3
$w_5^\star$	12.00	13.72	16.14	21.63	33.89	46.14	27.1
$w_6^\star$	8.00	9.60	10.47	10.06	7.21	4.36	54.2
$w_7^\star$	6.00	5.56	5.19	4.75	4.11	3.48	78.6
$w_8^\star$	5.00	1.05	0.00	0.00	0.00	0.00	426.7
$\mathrm{AS}_{\mathcal{S}ector}$	0.00	6.87	15.49	24.07	31.97	47.58	
$\mathrm{MD}(w)$	5.48	5.49	5.45	5.29	4.90	4.51	
$\mathrm{DTS}(w)$	301.05	292.34	282.28	266.12	236.45	206.78	
$\sigma_{\mathrm{AS}}(w \mid b)$	0.00	5.57	12.31	19.82	30.04	43.58	
$\sigma_{\mathrm{MD}}(w \mid b)$	0.00	0.01	0.04	0.17	0.49	0.81	
$\sigma_{\mathrm{DTS}}(w \mid b)$	0.00	8.99	19.29	35.74	65.88	96.01	
$\mathcal{CI}(w)$	160.57	144.52	128.46	112.40	96.34	80.29	
$\mathcal{R}(w \mid b)$	0.00	10.00	20.00	30.00	40.00	50.00	

Bond portfolios

Table 10: Optimal decarbonization portfolios (Example #6, order-statistic approach)

m	0	1	2	3	4	5	6	7	$\mathcal{CI}_i$
$w_1^\star$	20.00	20.83	24.62	64.64	0.00	0.00	0.00	0.00	100.5
$w_2^\star$	19.00	18.60	18.13	21.32	3.32	0.00	0.00	0.00	97.2
$w_3^\star$	17.00	17.79	26.30	0.00	0.00	0.00	0.00	0.00	250.4
$w_4^\star$	13.00	14.53	0.00	0.00	0.00	0.00	0.00	0.00	352.3
$w_5^\star$	12.00	12.89	13.96	6.00	36.57	41.27	41.27	100.00	27.1
$w_6^\star$	8.00	9.74	11.85	0.00	60.11	58.73	58.73	0.00	54.2
$w_7^\star$	6.00	5.62	5.15	8.03	0.00	0.00	0.00	0.00	78.6
$w_8^\star$	5.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	426.7
$\mathrm{AS}_{\mathcal{S}ector}$	0.00	5.78	19.72	49.00	76.68	80.00	80.00	88.00	
$\mathrm{MD}(w)$	5.48	5.52	5.54	4.77	3.27	3.17	3.17	4.70	
$\mathrm{DTS}(w)$	301.05	295.08	284.71	171.82	150.45	150.78	150.78	159.00	
$\sigma_{\mathrm{AS}}(w \mid b)$	0.00	5.73	17.94	50.85	66.96	68.63	68.63	95.33	
$\sigma_{\mathrm{MD}}(w \mid b)$	0.00	0.03	0.04	0.63	2.66	2.64	2.64	3.21	
$\sigma_{\mathrm{DTS}}(w \mid b)$	0.00	6.21	16.87	128.04	197.22	197.29	197.29	199.22	
$\mathcal{CI}(w)$	160.57	147.94	122.46	93.63	45.72	43.02	43.02	27.10	
$\mathcal{R}(w \mid b)$	0.00	7.87	23.74	41.69	71.53	73.21	73.21	83.12	

Sector-specific constraints

Sector scenario

- Decarbonization scenario per sector:

$$\mathcal{CI}\left(w;\mathcal{S}ector_j\right) \leq (1-\mathcal{R}_j)\,\mathcal{CI}\left(b;\mathcal{S}ector_j\right)$$

- We have:

$$\left(\boldsymbol{s}_j \circ \left(\boldsymbol{\mathcal{CI}} - \mathcal{CI}_j^{\star}\right)\right)^{\top} w \leq 0$$

where $\mathcal{CI}_j^{\star} = (1-\mathcal{R}_j)\,\mathcal{CI}\left(b;\mathcal{S}ector_j\right)$

Sector-specific constraints

Sector scenario

QP form

$$C = \begin{pmatrix} \left(\boldsymbol{s}_1 \circ \left(\mathcal{CI} - \mathcal{CI}_1^{\star}\right)\right)^\top \\ \vdots \\ \left(\boldsymbol{s}_j \circ \left(\mathcal{CI} - \mathcal{CI}_j^{\star}\right)\right)^\top \\ \vdots \\ \left(\boldsymbol{s}_{n_{\mathcal{S}ector}} \circ \left(\mathcal{CI} - \mathcal{CI}_{n_{\mathcal{S}ector}}^{\star}\right)\right)^\top \end{pmatrix}$$

$$D = \begin{pmatrix} \left(1 - \mathcal{R}_1\right) \mathcal{CI}\left(b; \mathcal{S}ector_1\right) \\ \vdots \\ \left(1 - \mathcal{R}_j\right) \mathcal{CI}\left(b; \mathcal{S}ector_j\right) \\ \vdots \\ \left(1 - \mathcal{R}_{n_{\mathcal{S}ector}}\right) \mathcal{CI}\left(b; \mathcal{S}ector_{n_{\mathcal{S}ector}}\right) \end{pmatrix}$$

Sector-specific constraints

Sector scenario

Table 11: Carbon intensity and threshold in $\mathrm{tCO_2e/\$ \ mn}$ per GICS sector (MSCI World, 2030)

Sector	$\mathcal{CI}(b; \mathcal{S}ector_j)$				$\mathcal{R}_j$	$\mathcal{CI}_j^\star$			
	$\mathcal{SC}_1$	$\mathcal{SC}_{1-2}$	$\mathcal{SC}_{1-3}^{\mathrm{up}}$	$\mathcal{SC}_{1-3}$	(in %)	$\mathcal{SC}_1$	$\mathcal{SC}_{1-2}$	$\mathcal{SC}_{1-3}^{\mathrm{up}}$	$\mathcal{SC}_{1-3}$
Communication Services	2	28	134	172	52.4	1	13	64	82
Consumer Discretionary	23	65	206	590	52.4	11	31	98	281
Consumer Staples	28	55	401	929	52.4	13	26	191	442
Energy	632	698	1 006	6 823	56.9	272	301	434	2 941
Financials	13	19	52	244	52.4	6	9	25	116
Health Care	10	22	120	146	52.4	5	10	57	70
Industrials	111	130	298	1 662	18.8	90	106	242	1 350
Information Technology	7	23	112	239	52.4	3	11	53	114
Materials	478	702	1 113	2 957	36.7	303	445	704	1 872
Real Estate	22	101	167	571	36.7	14	64	106	361
Utilities	1 744	1 794	2 053	2 840	56.9	752	773	885	1 224
MSCI World	130	163	310	992	36.6	82	103	196	629

Sector-specific constraints

Sector and weight deviation constraints (equity portfolio)

1. Asset weight deviation constraint:

$$\Omega := \mathcal{C}_1\left(m_w^-, m_w^+\right) = \left\{w : m_w^- b \leq w \leq m_w^+ b\right\}$$

2. Sector weight deviation constraint:

$$\Omega := \mathcal{C}_2\left(m_s^-, m_s^+\right) == \left\{\forall j : m_s^- \sum_{i \in \mathcal{S}ector_j} b_i \leq \sum_{i \in \mathcal{S}ector_j} w_i \leq m_s^+ \sum_{i \in \mathcal{S}ector_j} b_i\right\}$$

3. $\mathcal{C}_2\left(m_s\right) = \mathcal{C}_2\left(1/m_s, m_s\right)$
4. $\mathcal{C}_3\left(m_w^-, m_w^+, m_s\right) = \mathcal{C}_1\left(m_w^-, m_w^+\right) \cap \mathcal{C}_2\left(m_s\right)$

Sector-specific constraints

Sector and weight deviation constraints (bond portfolio)

1. Modified duration constraint:

$$\Omega := \mathcal{C}_1' = \{w : \mathrm{MD}(w) = \mathrm{MD}(b)\} = \left\{ w : \sum_{i=1}^{n} (x_i - b_i) \, \mathrm{MD}_i = 0 \right\}$$

2. DTS constraint

$$\Omega := \mathcal{C}_2' = \{w : \mathrm{DTS}(w) = \mathrm{DTS}(b)\} = \left\{ w : \sum_{i=1}^{n} (x_i - b_i) \, \mathrm{DTS}_i = 0 \right\}$$

3. Maturity/rating buckets:

$$\Omega := \left\{ w : \sum_{i \in \mathcal{B}ucket_j} (x_i - b_i) = 0 \right\}$$

   1. $\mathcal{C}_3'$: $\mathcal{B}ucket_j$ is the j^{th} maturity bucket, e.g., 0–1, 1–3, 3–5, 5–7, 7–10 and 10+
   2. $\mathcal{C}_4'$: $\mathcal{B}ucket_j$ is the j^{th} rating category, e.g., AAA–AA (AAA, AA+, AA and AA−), A (A+, A and A−) and BBB (BBB+, BBB, BBB−)

Sector-specific constraints

HCIS constraint

Two types of sectors:

1. High climate impact sectors (HCIS):
   "**sectors that are key to the low-carbon transition**" (TEG, 2019)
2. Low climate impact sectors (LCIS)

Let $\mathcal{HCIS}(w) = \sum_{i \in \text{HCIS}} w_i$ be the HCIS weight of portfolio w:

$$\mathcal{HCIS}(w) \geq \mathcal{HCIS}(b)$$

Sector-specific constraints

HCIS constraint

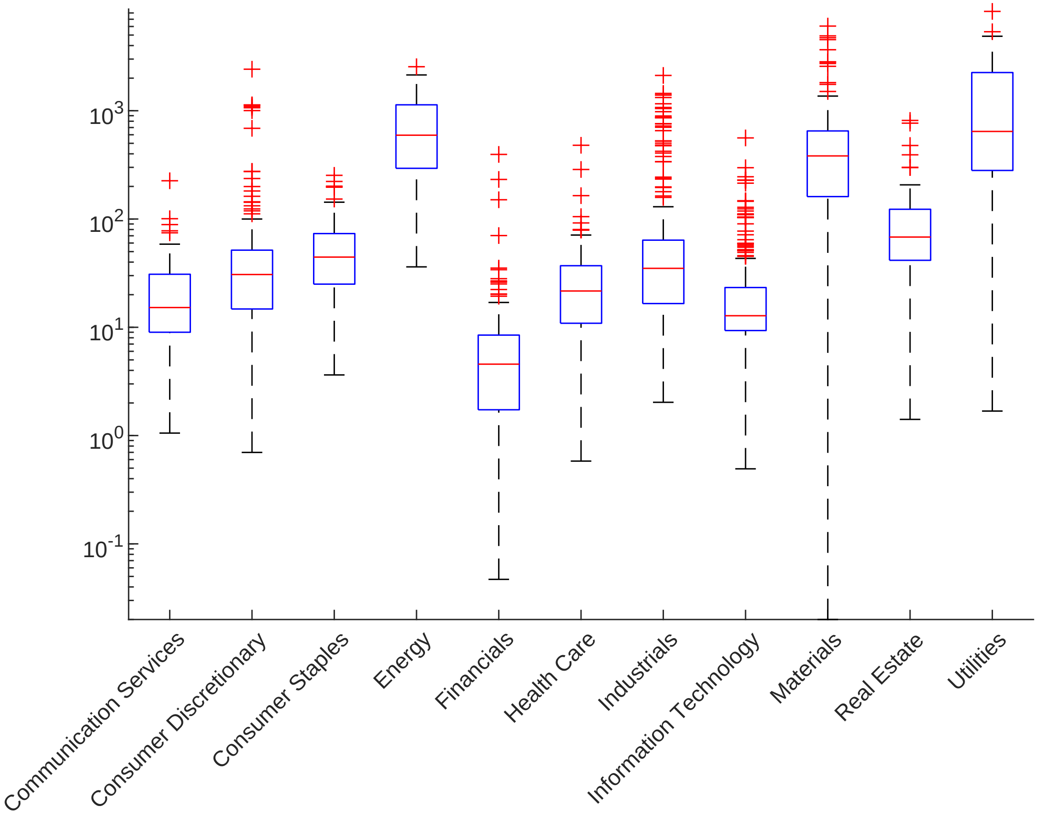
Table 12: Weight and carbon intensity when applying the HCIS filter (MSCI World, June 2022)

Sector	Index b_j	HCIS b'_j	$\mathcal{SC}_1$ $\mathcal{CI}$	$\mathcal{SC}_1$ $\mathcal{CI}'$	$\mathcal{SC}_{1-2}$ $\mathcal{CI}$	$\mathcal{SC}_{1-2}$ $\mathcal{CI}'$	$\mathcal{SC}_{1-3}^{\text{up}}$ $\mathcal{CI}$	$\mathcal{SC}_{1-3}^{\text{up}}$ $\mathcal{CI}'$	$\mathcal{SC}_{1-3}$ $\mathcal{CI}$	$\mathcal{SC}_{1-3}$ $\mathcal{CI}'$
Communication Services	7.58	0.00	2		28		134		172	
Consumer Discretionary	10.56	8.01	23	14	65	31	206	189	590	462
Consumer Staples	7.80	7.80	28	28	55	55	401	401	929	929
Energy	4.99	4.99	632	632	698	698	1 006	1 006	6 823	6 823
Financials	13.56	0.00	13		19		52		244	
Health Care	14.15	9.98	10	13	22	26	120	141	146	177
Industrials	9.90	7.96	111	132	130	151	298	332	1 662	1 921
Information Technology	21.08	10.67	7	12	23	30	112	165	239	390
Materials	4.28	4.28	478	478	702	702	1 113	1 113	2 957	2 957
Real Estate	2.90	2.90	22	22	101	101	167	167	571	571
Utilities	3.21	3.21	1 744	1 744	1 794	1 794	2 053	2 053	2 840	2 840
MSCI World	100.00	59.79	130	210	163	252	310	458	992	1 498

Source: MSCI (2022), Trucost (2022) & Author's calculations

Empirical results (equity portfolios)

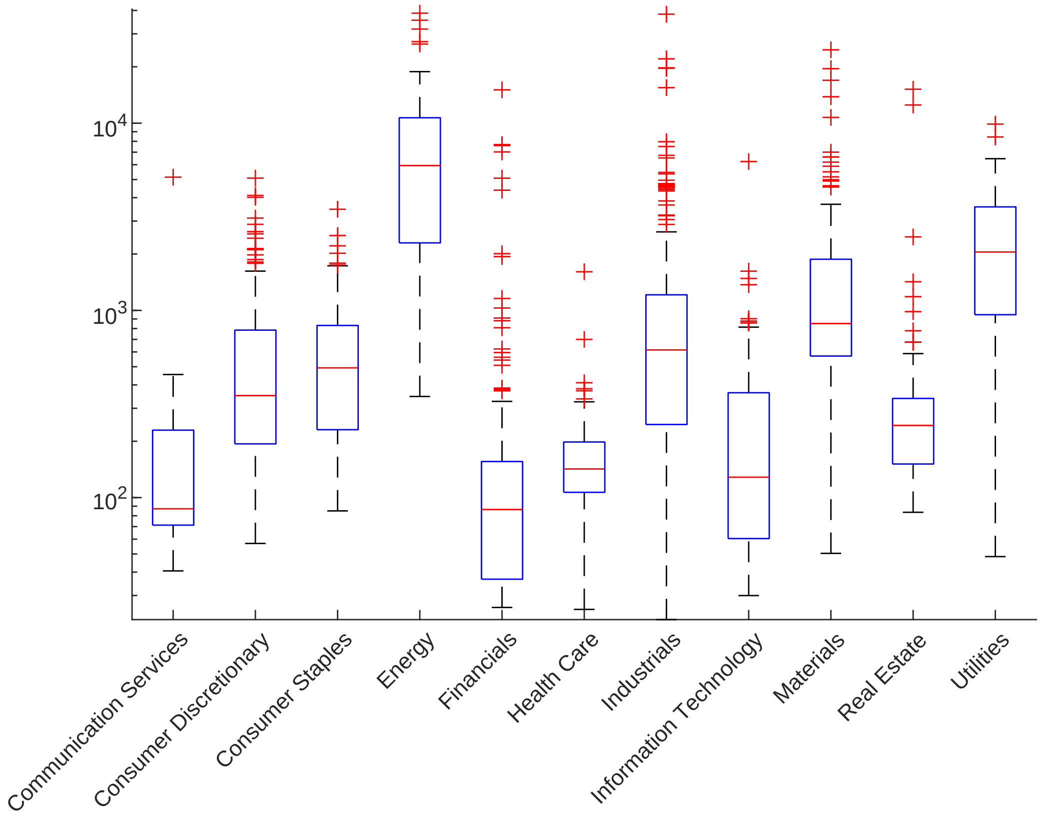
Figure 4: Boxplot of carbon intensity per sector (MSCI World, June 2022, scope $\mathcal{SC}_{1-2}$)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Empirical results (equity portfolios)

Figure 5: Boxplot of carbon intensity per sector (MSCI World, June 2022, scope $\mathcal{SC}_{1-3}$)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Equity portfolios

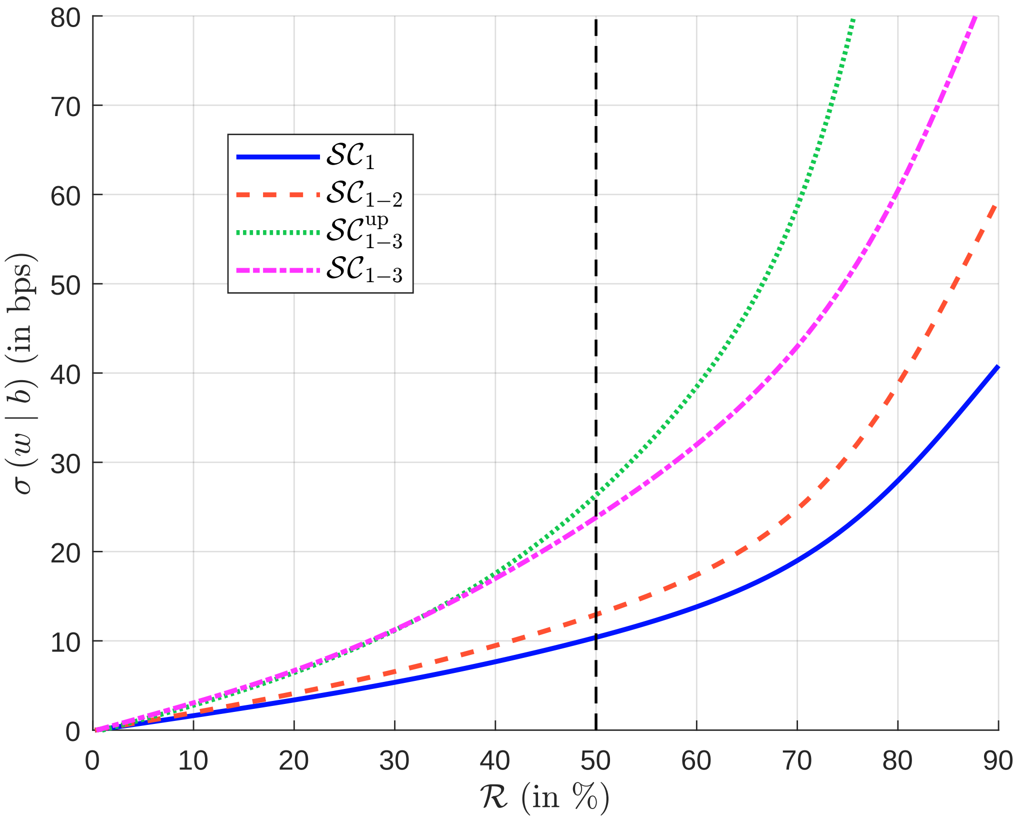
Barahhou *et al.* (2022) consider the basic optimization problem:

$$\begin{aligned} w^\star & = & \arg\min \frac{1}{2} (w - b)^\top \Sigma (w - b) \\ & \text{s.t.} & \left\{ \begin{array}{l} \mathcal{CI}(w) \leq (1 - \mathcal{R}) \, \mathcal{CI}(b) \\ w \in \Omega_0 \cap \Omega \end{array} \right. \end{aligned}$$

What is the impact of constraints $\Omega_0 \cap \Omega$?

Equity portfolios

Figure 6: Impact of the carbon scope on the tracking error volatility (MSCI World, June 2022, $\mathcal{C}_0$ constraint)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Equity portfolios

Table 13: Sector allocation in % (MSCI World, June 2022, scope $\mathcal{SC}_{1-3}$)

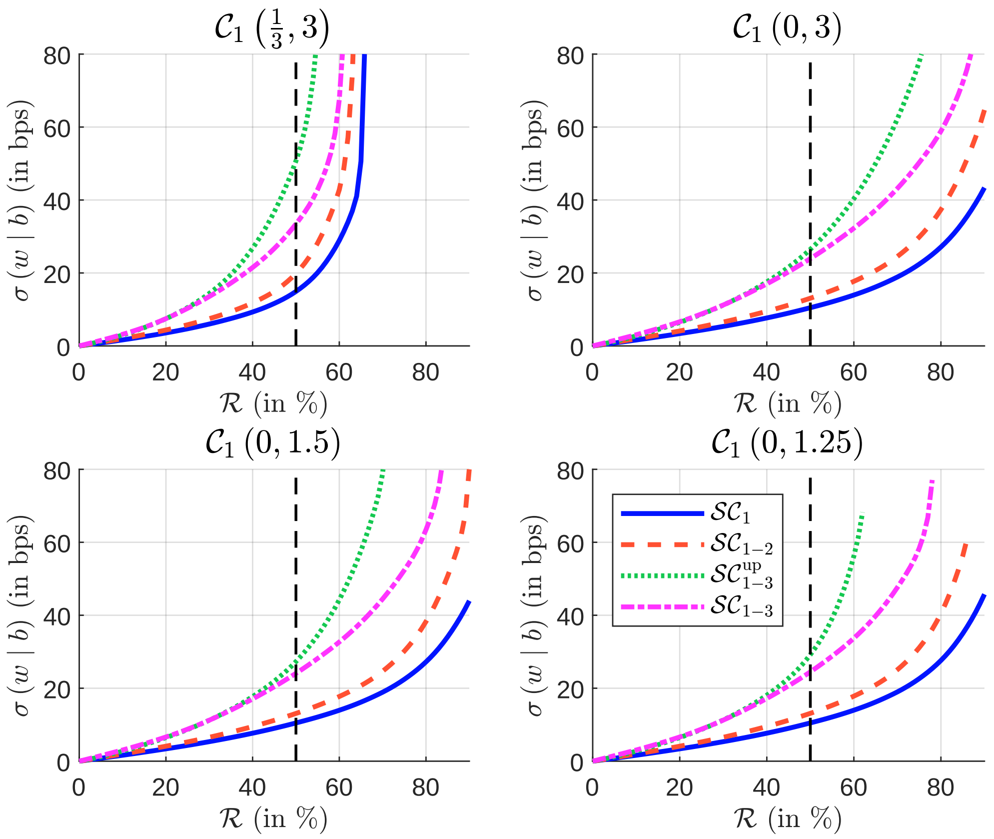
Sector	Index	Reduction rate $\mathcal{R}$ 30%	40%	50%	60%	70%	80%	90%
Communication Services	7.58	7.95	8.15	8.42	8.78	9.34	10.13	12.27
Consumer Discretionary	10.56	10.69	10.69	10.65	10.52	10.23	9.62	6.74
Consumer Staples	7.80	7.80	7.69	7.48	7.11	6.35	5.03	1.77
Energy	**4.99**	4.14	3.65	3.10	2.45	**1.50**	**0.49**	**0.00**
Financials	**13.56**	14.53	15.17	15.94	16.90	**18.39**	**20.55**	**28.62**
Health Care	14.15	14.74	15.09	15.50	16.00	16.78	17.77	17.69
Industrials	9.90	9.28	9.01	8.71	8.36	7.79	7.21	6.03
Information Technology	21.08	21.68	22.03	22.39	22.88	23.51	24.12	24.02
Materials	**4.28**	3.78	3.46	3.06	2.56	**1.85**	**1.14**	**0.24**
Real Estate	2.90	3.12	3.27	3.41	3.57	3.72	3.71	2.51
Utilities	**3.21**	2.28	1.79	1.36	0.90	**0.54**	**0.24**	**0.12**

Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Portfolio decarbonization = strategy **long on Financials** and **short on Energy, Materials and Utilities**

Equity portfolios

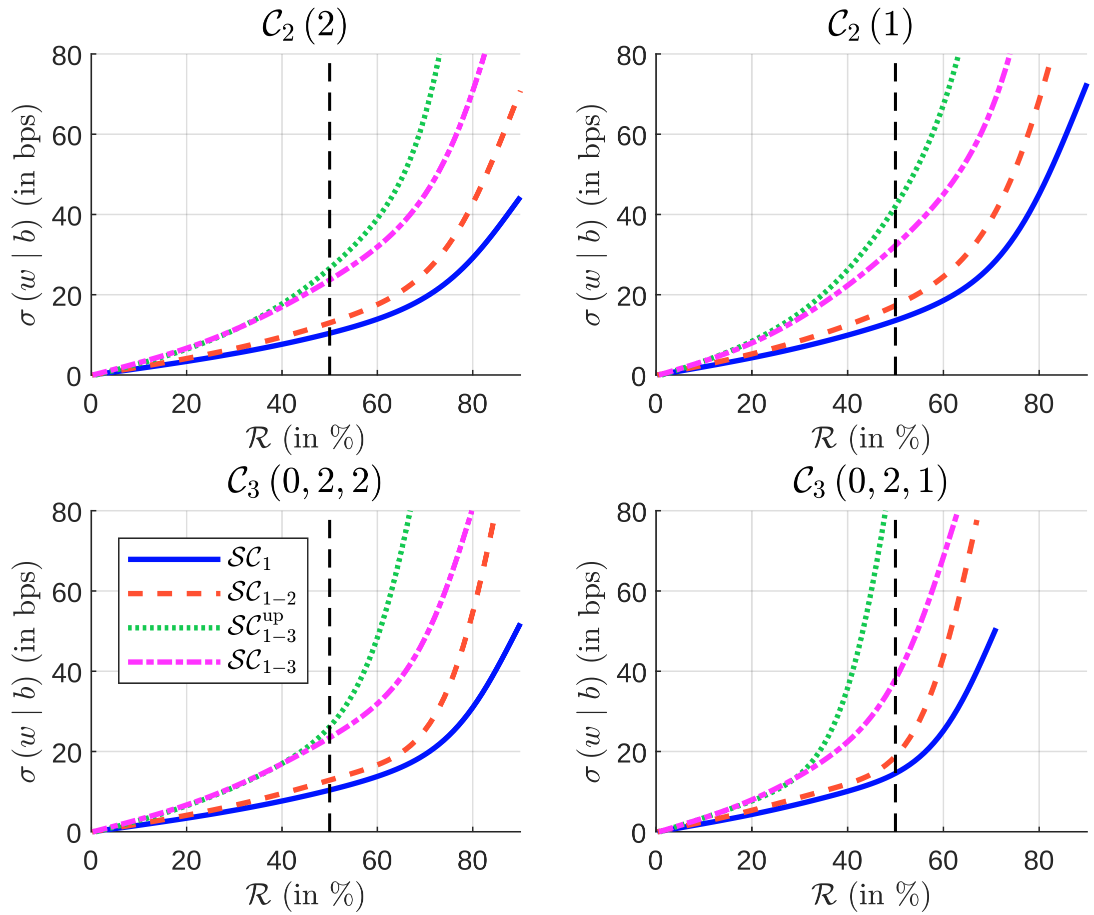
Figure 7: Impact of $\mathcal{C}_1$ constraint on the tracking error volatility (MSCI World, June 2022)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Equity portfolios

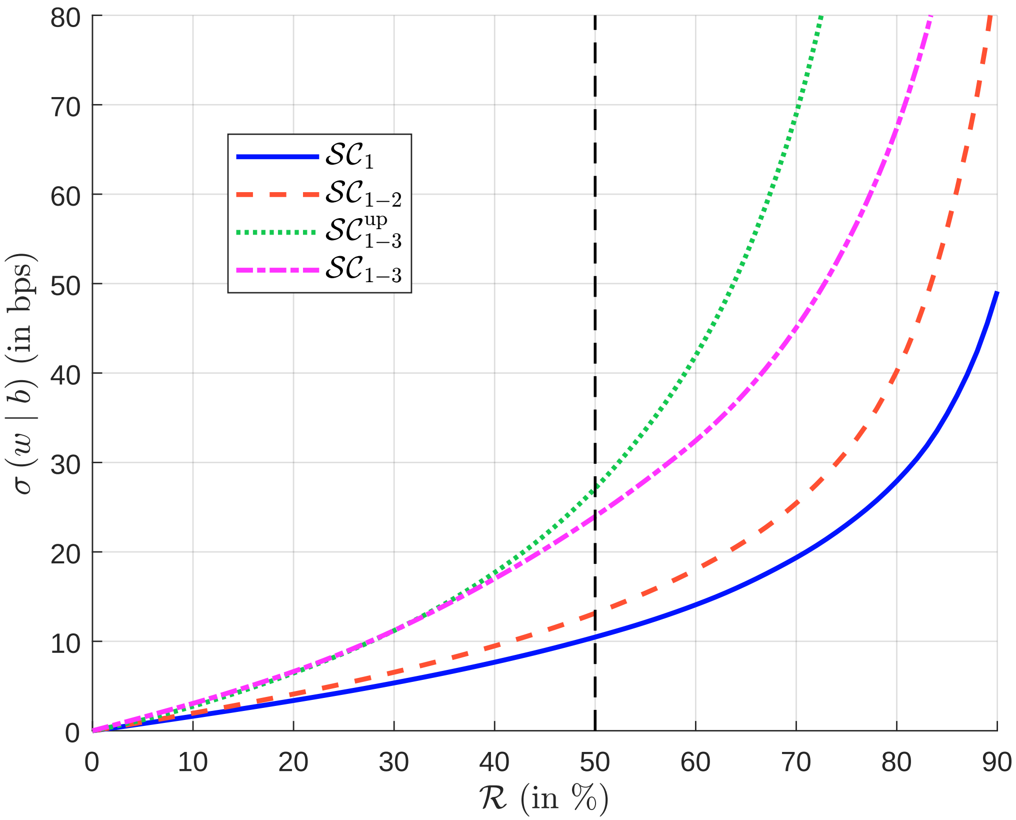
Figure 8: Impact of $\mathcal{C}_2$ and $\mathcal{C}_3$ constraints (MSCI World, June 2022)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Equity portfolios

Figure 9: Tracking error volatility with $\mathcal{C}_3\,(0, 10, 2)$ constraint (MSCI World, June 2022)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022)

Equity portfolios

First approach

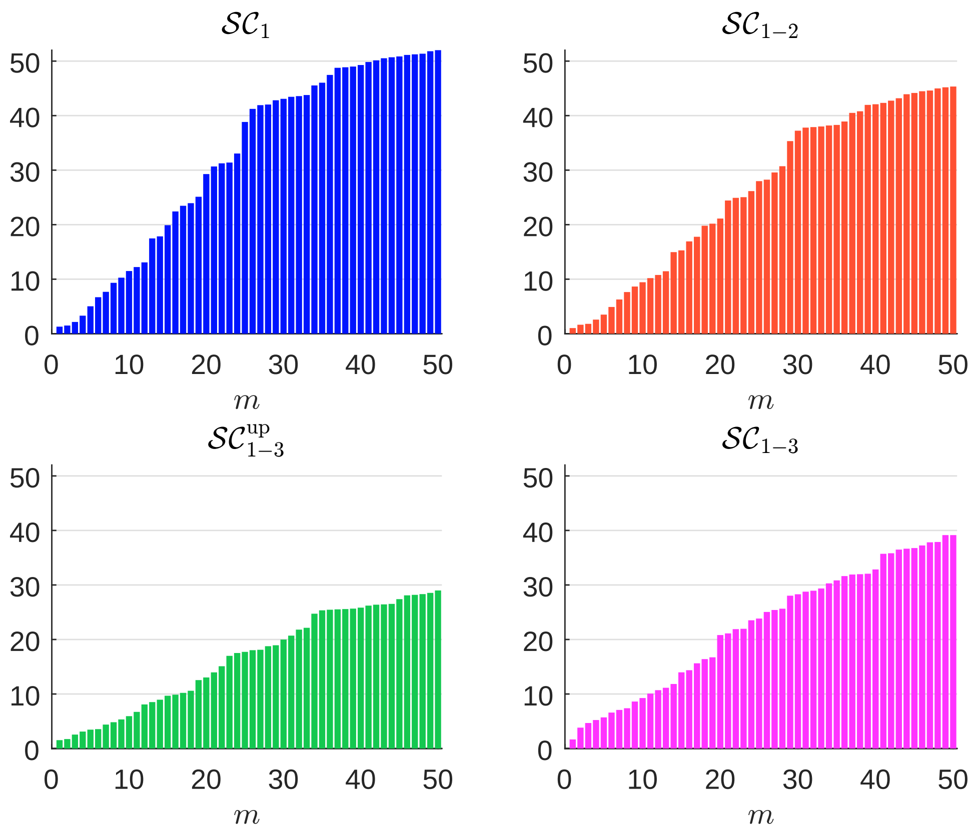
- The carbon footprint contribution of the m worst performing assets is:

$$\mathcal{CFC}^{(m,n)} = \frac{\sum_{i=1}^{n} \mathbb{1}\left\{\boldsymbol{\mathcal{CI}}_i \geq \boldsymbol{\mathcal{CI}}^{(m,n)}\right\} \cdot b_i \boldsymbol{\mathcal{CI}}_i}{CI\left(b\right)}$$

where $\boldsymbol{\mathcal{CI}}^{(m,n)} = \boldsymbol{\mathcal{CI}}_{n-m+1:n}$ is the $(n-m+1)$-th order statistic

Equity portfolios

Figure 10: Carbon footprint contribution $\mathcal{CFC}^{(m,n)}$ in % (MSCI World, June 2022, first approach)



Source: MSCI (2022), Trucost (2022) & Author's calculations

Equity portfolios

Second approach

- Another definition:

$$\mathcal{CFC}^{(m,n)} = \frac{\sum_{i=1}^{n} \mathbb{1}\left\{\mathcal{CIC}_i \geq \mathcal{CIC}^{(m,n)}\right\} \cdot b_i \boldsymbol{\mathcal{CI}}_i}{CI\left(b\right)}$$

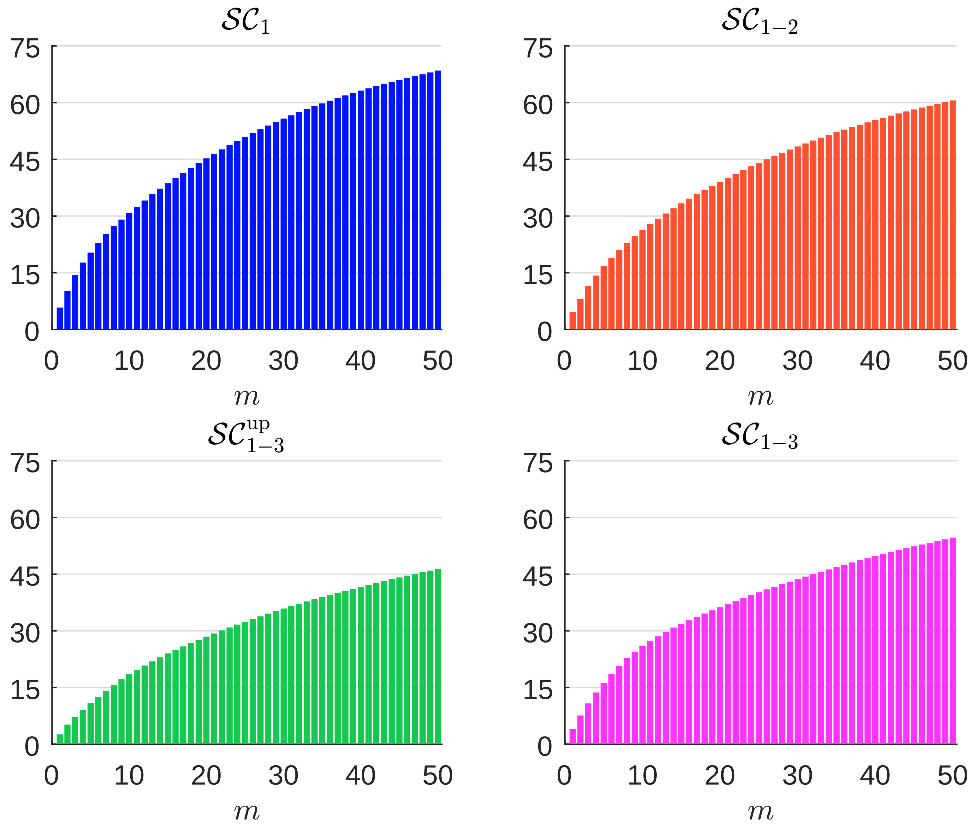
  where $\mathcal{CIC}_i = b_i \boldsymbol{\mathcal{CI}}_i$ and $\mathcal{CIC}^{(m,n)} = \mathcal{CIC}_{n-m+1:n}$

- Weight contribution:

$$\mathcal{WC}^{(m,n)} = \sum_{i=1}^{n} \mathbb{1}\left\{\mathcal{CIC}_i \geq \mathcal{CIC}^{(m,n)}\right\} \cdot b_i$$

Equity portfolios

Figure 11: Carbon footprint contribution $\mathcal{CFC}^{(m,n)}$ in % (MSCI World, June 2022, second approach)



Source: MSCI (2022), Trucost (2022) & Author's calculations

Equity portfolios

Table 14: Carbon footprint contribution $\mathcal{CFC}^{(m,n)}$ in % (MSCI World, June 2022, second approach, $\boldsymbol{\mathcal{SC}}_{1-3}$)

Sector	m: 1	5	10	25	50	75	100	200
Communication Services						0.44	0.44	0.73
Consumer Discretionary				0.78	1.37	2.44	2.93	4.28
Consumer Staples		2.46	2.46	2.46	3.75	4.44	4.92	5.62
Energy		9.61	17.35	23.78	29.56	31.78	33.02	33.89
Financials						0.72	1.53	1.88
Health Care							0.21	0.37
Industrials			2.16	5.59	7.13	8.70	9.48	13.05
Information Technology				0.98	1.58	1.94	2.15	3.30
Materials	4.08	4.08	4.08	5.81	7.31	8.81	9.59	10.75
Real Estate					0.77	0.77	0.77	0.85
Utilities				0.81	3.20	3.89	5.24	7.98
Total	4.08	16.15	26.06	40.21	54.66	63.94	70.29	82.70

Source: MSCI (2022), Trucost (2022) & Author's calculations

Equity portfolios

Table 15: Weight contribution $\mathcal{WC}^{(m,n)}$ in % (MSCI World, June 2022, second approach, $\boldsymbol{\mathcal{SC}}_{1-3}$)

Sector	b_j (in %)	m = 1	5	10	25	50	75	100	200
Communication Services	7.58						0.08	0.08	3.03
Consumer Discretionary	10.56				0.58	1.79	2.44	4.51	5.89
Consumer Staples	7.80		0.70	0.70	0.70	1.90	2.50	2.84	3.84
Energy	4.99		1.71	2.25	2.96	3.62	3.99	4.33	4.65
Financials	13.56						0.74	1.17	2.33
Health Care	14.15							0.95	1.34
Industrials	9.90			0.06	0.32	0.70	0.96	1.20	4.12
Information Technology	21.08				0.16	4.70	8.42	8.78	11.62
Materials	4.28	0.29	0.29	0.29	0.47	0.88	1.10	1.40	1.87
Real Estate	2.90					0.05	0.05	0.05	0.23
Utilities	3.21				0.31	0.86	1.04	1.31	2.33
Total		0.29	2.71	3.30	5.49	14.50	21.32	26.63	41.24

Source: MSCI (2022), Trucost (2022) & Author's calculations

Equity portfolios

- The order-statistic optimization problem is:

$$\begin{aligned} w^{\star} &= \arg\min \frac{1}{2} (w - b)^{\top} \Sigma (w - b) \\ \text{s.t.} & \left\{ \begin{array}{l} \mathbf{1}_n^{\top} w = 1 \\ \mathbf{0}_n \leq w \leq w^{(m,n)} \end{array} \right. \end{aligned}$$

where the upper bound $w^{(m,n)}$ is equal to $\mathbb{1}\left\{\boldsymbol{\mathcal{CI}} < \boldsymbol{\mathcal{CI}}^{(m,n)}\right\}$ for the first ordering approach and $\mathbb{1}\left\{\mathcal{CIC} < \mathcal{CIC}^{(m,n)}\right\}$ for the second ordering approach

Equity portfolios

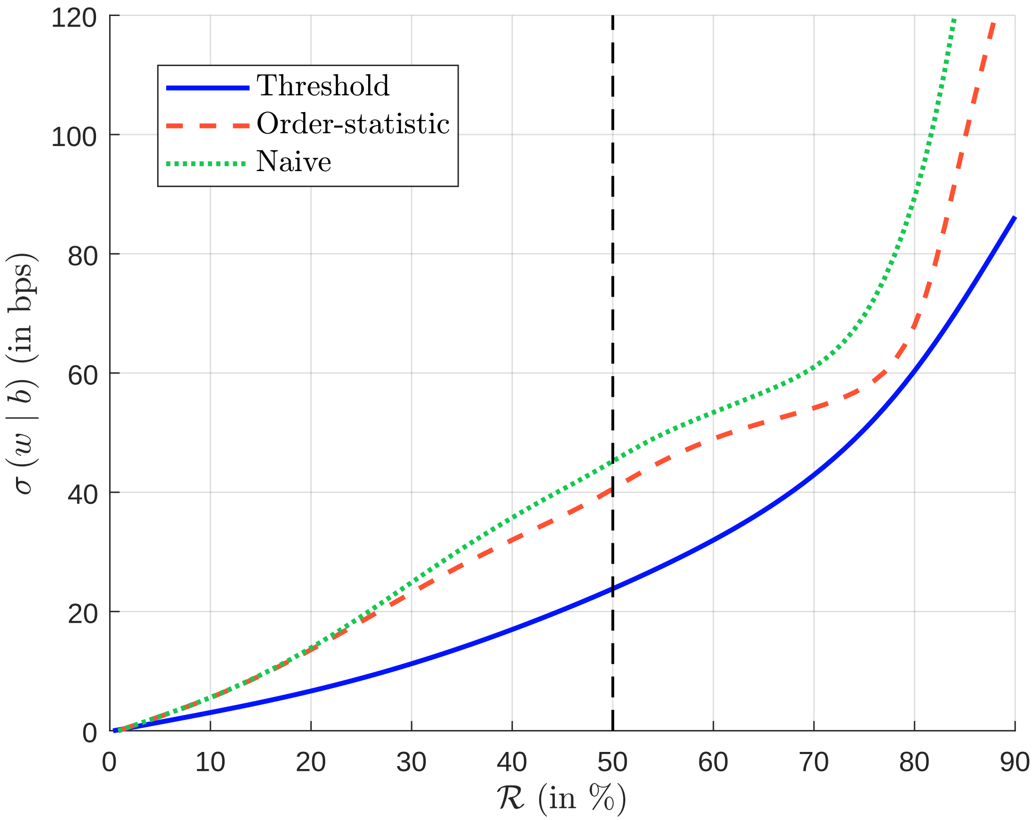
- The naive method is:

$$w_i^\star = \frac{e_i b_i}{\sum_{k=1}^n e_k b_k}$$

where e_i is defined as $\mathbb{1}\left\{\boldsymbol{\mathcal{CI}}_i < \boldsymbol{\mathcal{CI}}^{(m,n)}\right\}$ for the first ordering approach and $\mathbb{1}\left\{\mathcal{CIC}_i < \mathcal{CIC}^{(m,n)}\right\}$ for the second ordering approach

Equity portfolios

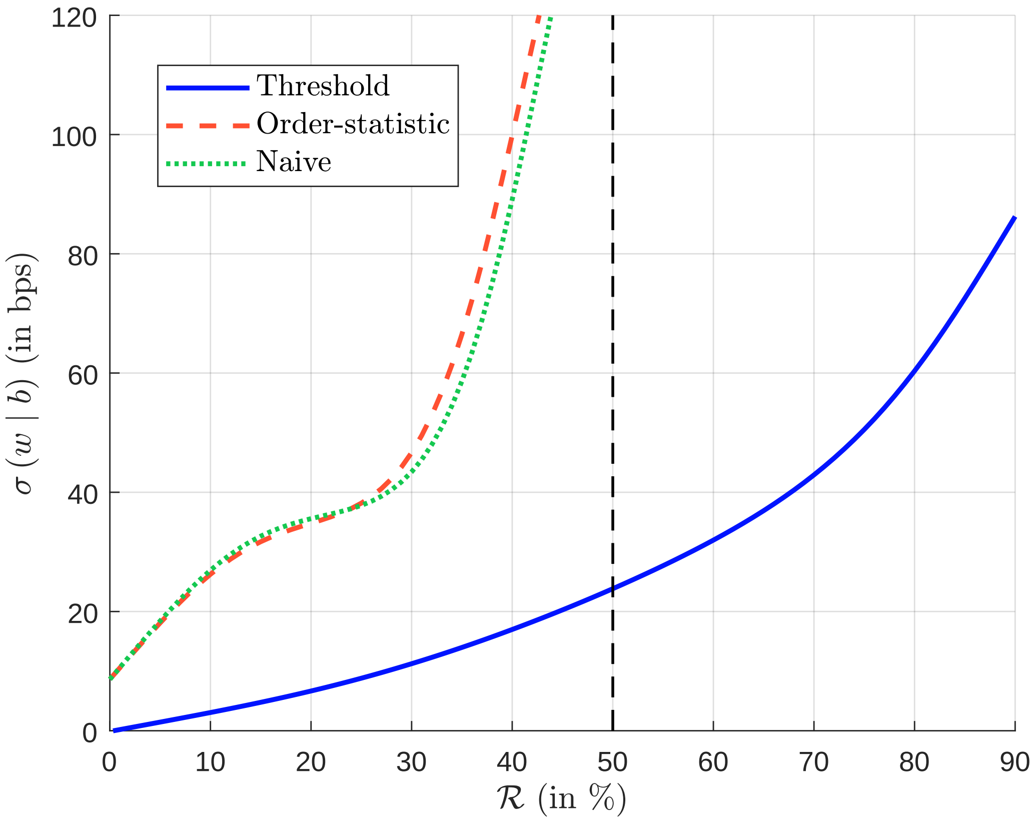
Figure 12: Tracking error volatility (MSCI World, June 2022, $\mathcal{SC}_{1-3}$, first ordering method)



Source: MSCI (2022), Trucost (2022) & Author's calculations

Equity portfolios

Figure 13: Tracking error volatility (MSCI World, June 2022, $\mathcal{SC}_{1-3}$, second ordering method)



Source: MSCI (2022), Trucost (2022) & Author's calculations

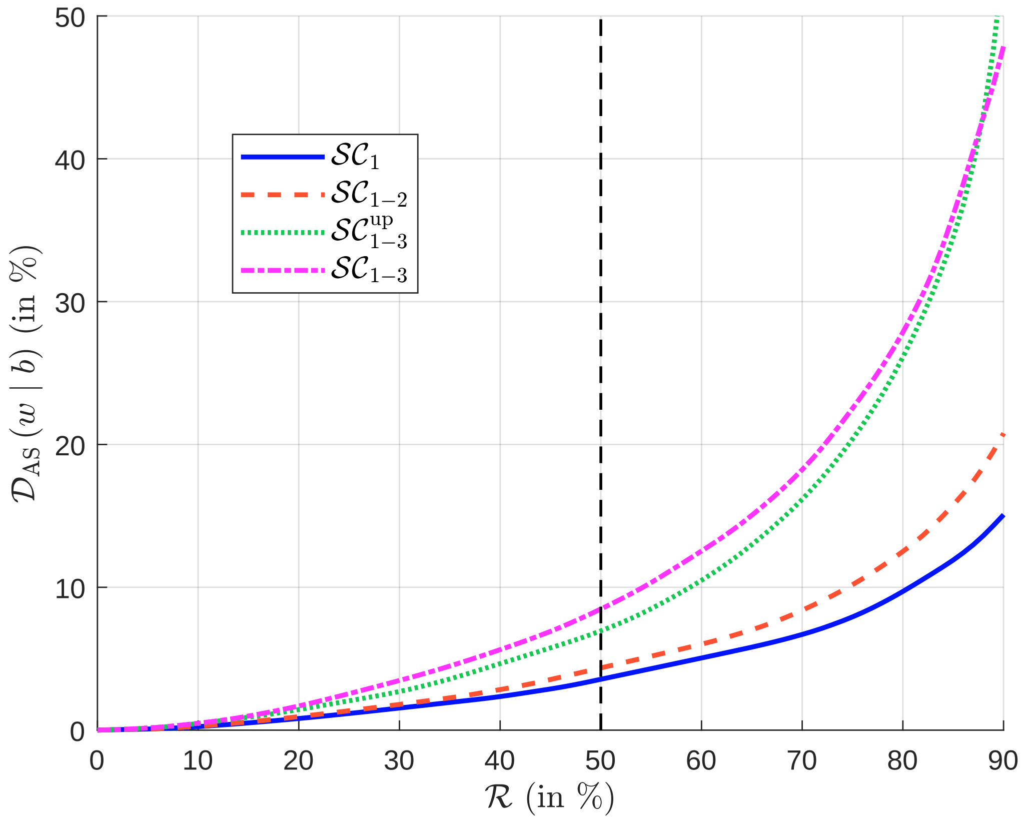
Bond portfolios

The optimization problem is:

$$
\begin{array}{rcl}
w^{\star} & = & \arg\min \dfrac{1}{2}\sum_{i=1}^{n}\left|w_i - b_i\right| + 50\sum_{j=1}^{n_{\mathcal{S}ector}}\left|\sum_{i\in\mathcal{S}ector_j}\left(w_i - b_i\right)\mathrm{DTS}_i\right| \\
& \text{s.t.} & \left\{\begin{array}{l}\mathcal{CI}\left(w\right) \leq \left(1-\mathcal{R}\right)\mathcal{CI}\left(b\right) \\ w \in \mathcal{C}_0 \cap \mathcal{C}'_1 \cap \mathcal{C}'_3 \cap \mathcal{C}'_4\end{array}\right.
\end{array}
$$

Bond portfolios

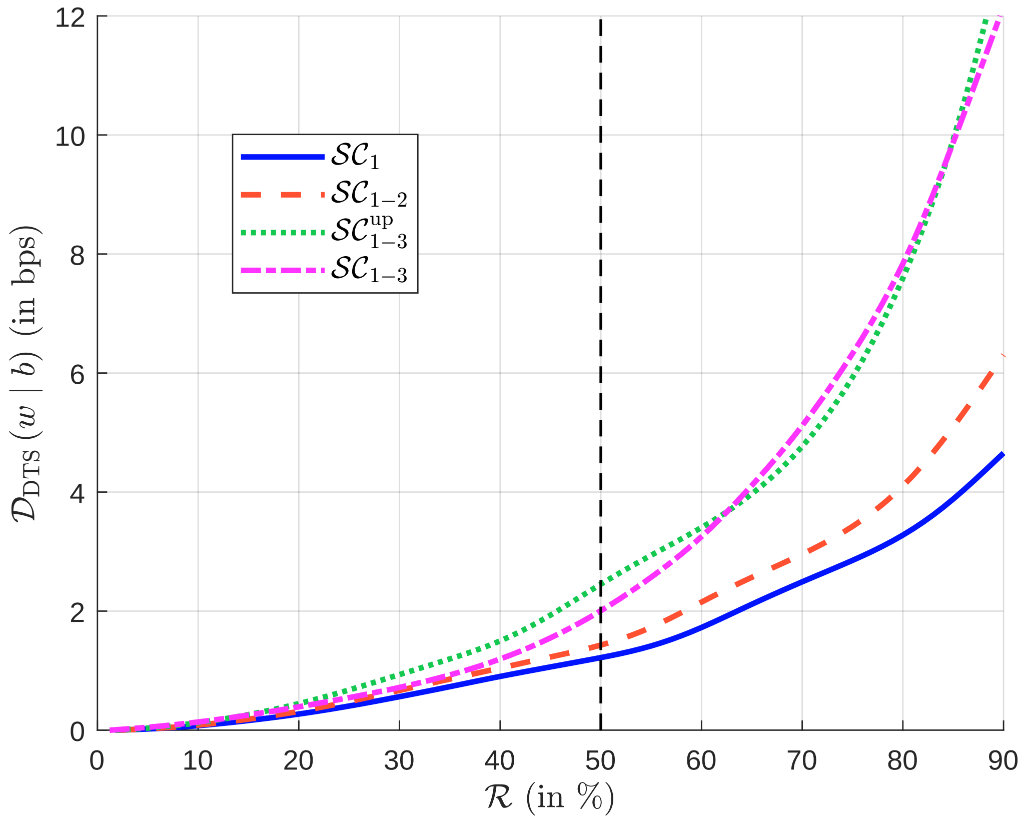
Figure 14: Impact of the carbon scope on the active share in % (ICE Global Corp., June 2022)



Source: ICE (2022), Trucost (2022) & Barahhou *et al.* (2022)

Bond portfolios

Figure 15: Impact of the carbon scope on the DTS risk in bps (ICE Global Corp., June 2022)



Source: ICE (2022), Trucost (2022) & Barahhou *et al.* (2022)

Bond portfolios

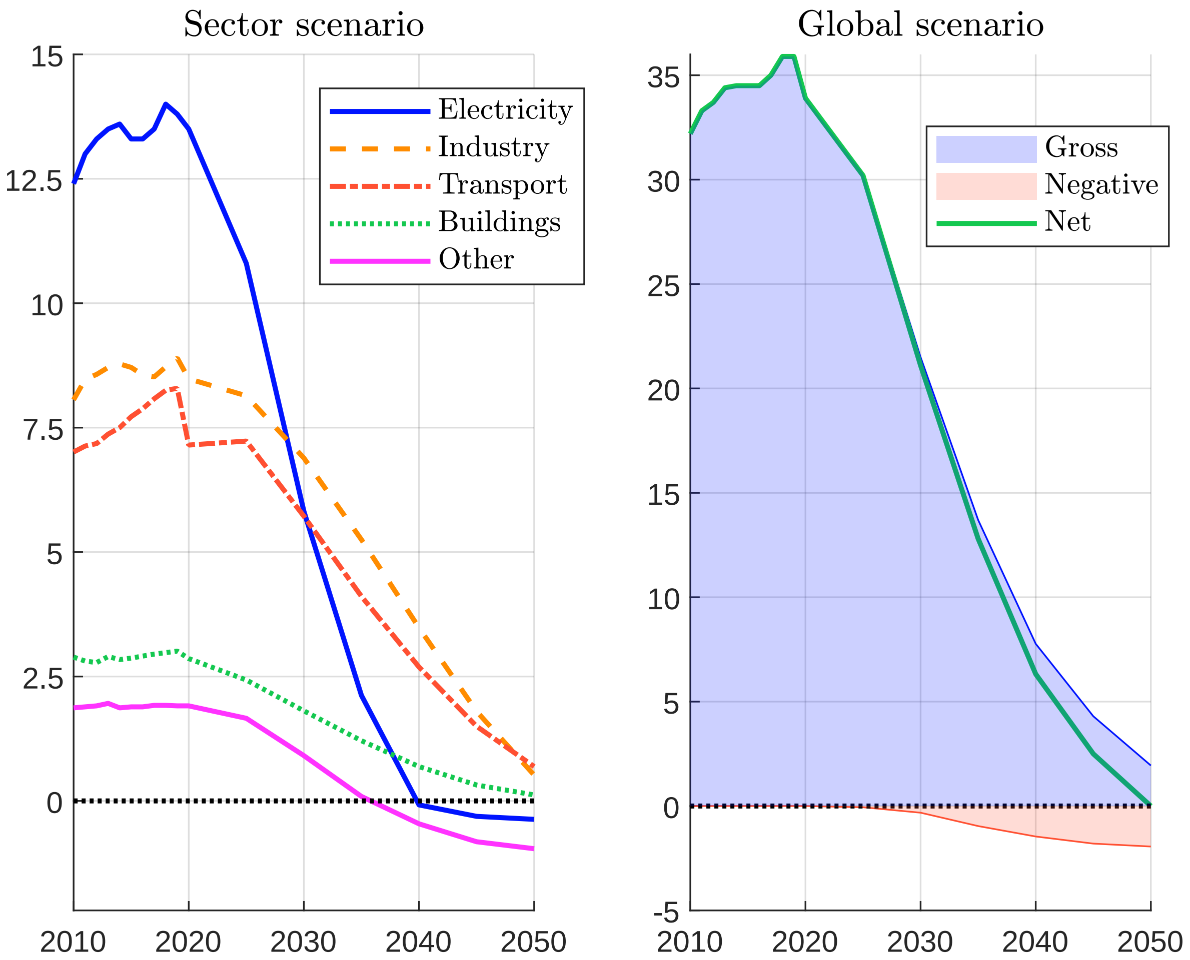
Table 16: Sector allocation in % (ICE Global Corp., June 2022, scope $\mathcal{SC}_{1-3}$)

Sector	Index	Reduction rate $\mathcal{R}$ 30%	40%	50%	60%	70%	80%	90%
Communication Services	7.34	7.35	7.34	7.37	7.43	7.43	7.31	7.30
Consumer Discretionary	5.97	5.97	5.96	5.94	5.93	5.46	4.48	3.55
Consumer Staples	6.04	6.04	6.04	6.04	6.04	6.02	5.39	4.06
Energy	6.49	5.49	4.42	3.84	3.69	3.23	2.58	2.52
Financials	33.91	34.64	35.66	35.96	36.09	37.36	38.86	39.00
Health Care	7.50	7.50	7.50	7.50	7.50	7.50	7.52	7.48
Industrials	8.92	9.38	9.62	10.19	11.34	12.07	13.55	18.13
Information Technology	5.57	5.57	5.59	5.59	5.60	5.60	5.52	5.27
Materials	3.44	3.43	3.31	3.18	3.12	2.64	2.25	1.86
Real Estate	4.76	4.74	4.74	4.74	4.74	4.66	4.61	3.93
Utilities	10.06	9.89	9.82	9.64	8.52	8.04	7.92	6.88

Source: ICE (2022), Trucost (2022) & Barahhou *et al.* (2022)

Choice of the decarbonization scenario

Figure 16: CO_2 emissions by sector in the IEA NZE scenario (in $\mathrm{GtCO_2e}$)



Source: International Energy Agency (2021).

The carbon emissions/intensity approach

A decarbonization scenario is defined as a function that relates a decarbonization rate to a time index t:

$$\begin{aligned} f \quad : \quad & \mathbb{R}^+ \longrightarrow [0,1] \\ & t \longmapsto \mathcal{R}(t_0, t) \end{aligned}$$

where t_0 is the base year and $\mathcal{R}\left(t_0, t_0^-\right) = 0$

Two choices

1. Carbon emissions

$$\mathcal{CE}(t) = (1 - \mathcal{R}(t_0, t))\,\mathcal{CE}(t_0)$$

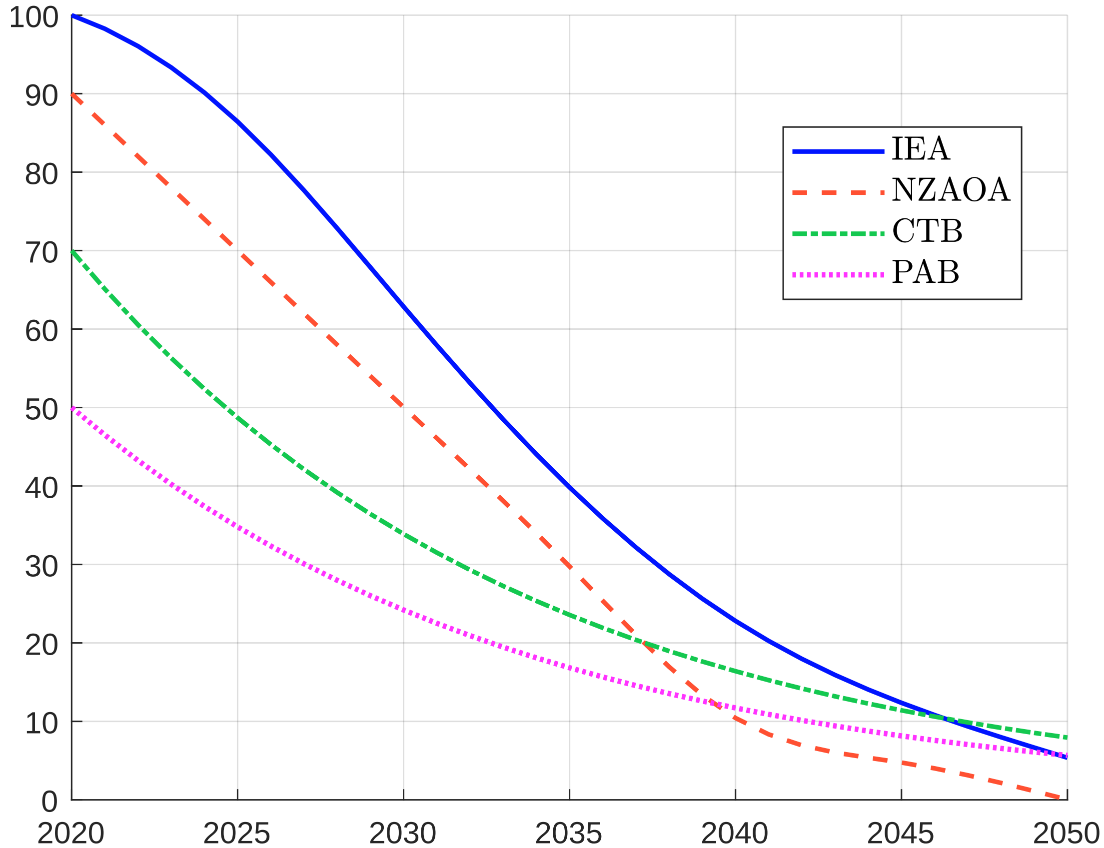
2. Carbon intensity (CTB/PAB)

$$\mathcal{CI}(t) = (1 - \Delta\mathcal{R})^{t-t_0}\left(1 - \mathcal{R}^-\right)\mathcal{CI}(t_0)$$

where $\Delta\mathcal{R} = 7\%$ and $\mathcal{R}^-$ takes the values 30%/50% (CTB/PAB)

The carbon emissions/intensity approach

Figure 17: IEA, NZAOA, CTB and PAB decarbonization pathways



IEA = International Energy Agency, NZAOA = Net Zero Asset Owners Alliance, CTB = Climate Transition Benchmark, PAB = Paris Aligned Benchmark

The carbon emissions/intensity approach

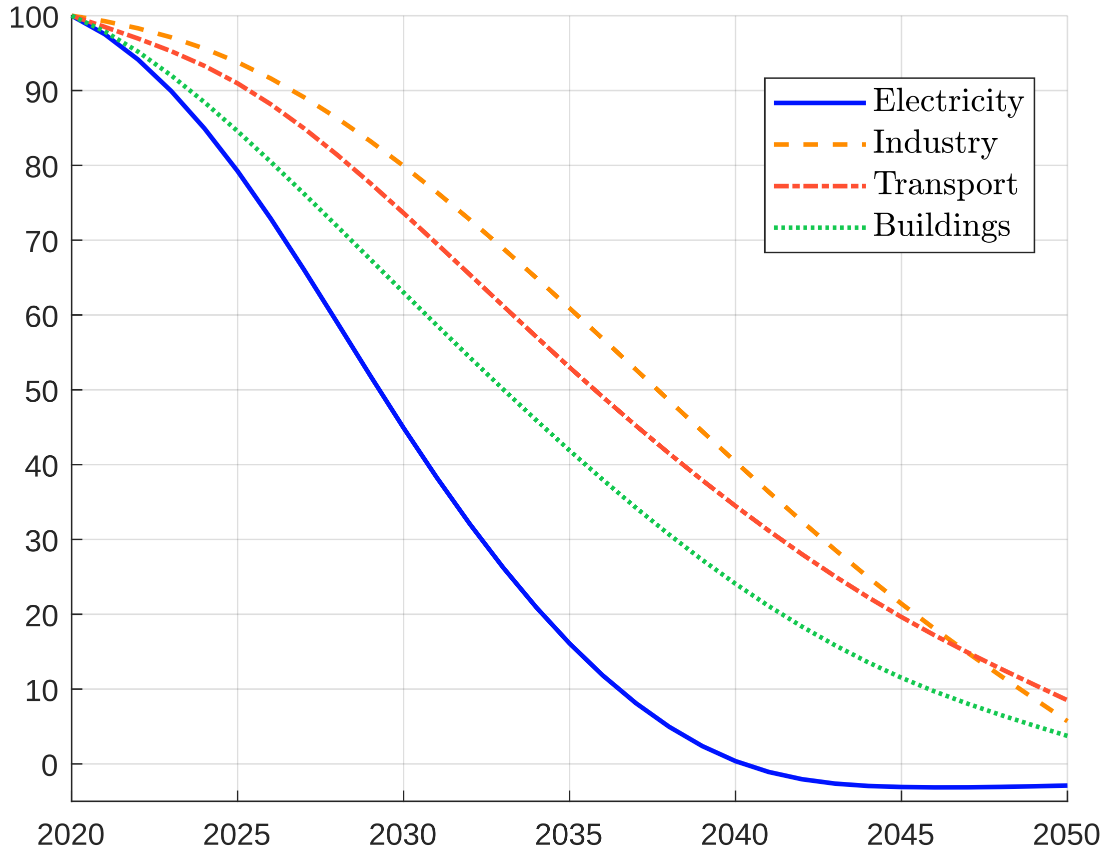
Table 17: IEA, NZAOA, CTB and PAB decarbonization rates (baseline = 2020)

Year $\mathcal{R}^-$ $\Delta\mathcal{R}$	CTB 30% 7%	PAB 50% 7%	NZE IEA Scenario	NZAOA Average Scenario
2020	30.0%	50.0%	0.0%	10.0%
2021	34.9%	53.5%	1.7%	14.0%
2022	39.5%	56.8%	3.9%	18.0%
2023	43.7%	59.8%	6.7%	22.0%
2024	47.6%	62.6%	9.9%	26.0%
2025	51.3%	65.2%	13.6%	30.0%
2026	54.7%	67.7%	17.8%	34.0%
2027	57.9%	69.9%	22.3%	38.0%
2028	60.8%	72.0%	27.2%	42.0%
2029	63.6%	74.0%	32.1%	46.0%
2030	66.1%	75.8%	37.1%	50.0%
2035	76.4%	83.2%	60.2%	70.3%
2040	83.6%	88.3%	77.2%	89.6%
2045	88.6%	91.9%	87.6%	95.2%
2050	92.1%	94.3%	94.6%	100.0%

Source: Ben Slimane *et al.* (2023).

The carbon emissions/intensity approach

Figure 18: Sectoral decarbonization pathways



Electricity $\succ$ Buildings $\succ$ Transport $\succ$ Industry

The carbon budget approach

A NZE scenario is defined by the following constraints:

$$\begin{cases} \mathcal{CB}(t_0, 2050) \leq \mathcal{CB}^+ \text{ GtCO}_2\text{e} \\ \mathcal{CE}(2050) \approx 0 \text{ GtCO}_2\text{e} \end{cases}$$

where t_0 is the base date and $\mathcal{CB}^+$ is the maximum carbon budget

IPCC SR15

- $t_0 = 2019$ and $\mathcal{CB}^+ = 580\,\text{GtCO}_2\text{e}$: there is a 50% probability of limiting the global warning to $1.5°\text{C}$
- $t_0 = 2019$ and $\mathcal{CB}^+ = 420\,\text{GtCO}_2\text{e}$: the probability is 66%
- $t_0 = 2019$ and $\mathcal{CB}^+ = 300\,\text{GtCO}_2\text{e}$: the probability is 83%

The carbon budget approach

If we have:

$$\mathcal{CE}(t) = (1 - \Delta\mathcal{R})^{t-t_0} (1 - \mathcal{R}^-) \mathcal{CE}(t_0)$$

we obtain:

$$\mathcal{CB}(t_0, t) = \left(\frac{(1 - \Delta\mathcal{R})^{t-t_0} - 1}{\ln(1 - \Delta\mathcal{R})} \right) (1 - \mathcal{R}^-) \mathcal{CE}(t_0)$$

Table 18: Carbon budget $\mathcal{CB}(2020, 2050)$ (in $\mathrm{GtCO_2e}$) when defining the decarbonization pathway of carbon emissions and assuming that $\mathcal{CE}(2020) = 36\,\mathrm{GtCO_2e}$

$\mathcal{R}^-$		0%	10%	20%	30%	50%	75%
$\Delta\mathcal{R}$	5%	551	496	441	386	276	138
	6%	491	442	393	344	245	123
	7%	440	396	352	308	220	110
	8%	396	357	317	277	198	99
	9%	359	323	287	251	180	90
	10%	327	294	262	229	164	82

Dynamic decarbonization and portfolio alignment

We have:

$$\mathcal{CI}(t, w) \leq (1 - \mathcal{R}(t_0, t))\,\mathcal{CI}(t_0, b(t_0))$$

where:

- t_0 is the base year
- $\mathcal{R}(t_0, t)$ is the decarbonization pathway of the NZE scenario
- $\mathcal{CI}(t_0, b(t_0))$ is the carbon intensity of the benchmark at time t_0

Dynamic decarbonization and portfolio alignment

Some properties:

- Decarbonizing the aligned portfolio becomes easier over time as the benchmark decarbonizes itself:

$$\mathcal{CI}\left(t, b\left(t\right)\right) \ll \mathcal{CI}\left(t_0, b\left(t_0\right)\right) \qquad \text{for } t > t_0$$

- Decarbonizing the aligned portfolio becomes more difficult over time as the benchmark carbonizes itself:

$$\mathcal{CI}\left(t, b\left(t\right)\right) \gg \mathcal{CI}\left(t_0, b\left(t_0\right)\right) \qquad \text{for } t > t_0$$

- The aligned portfolio matches the benchmark portfolio if the benchmark is sufficiently decarbonized:

$$\mathcal{CI}\left(t, b\left(t\right)\right) \leq \left(1 - \mathcal{R}\left(t_0, t\right)\right) \mathcal{CI}\left(t_0, b\left(t_0\right)\right)$$

Equity portfolios

The optimization problem becomes:

$$
\begin{array}{rcl}
w^{\star}(t) & = & \arg\min \dfrac{1}{2}\left(w - b(t)\right)^{\top} \Sigma(t) \left(w - b(t)\right) \\
& & \text{s.t.} \quad \left\{ \begin{array}{l} \boldsymbol{\mathcal{CI}}(t, w) \leq \left(1 - \boldsymbol{\mathcal{R}}(t_0, t)\right) \boldsymbol{\mathcal{CI}}(t_0, b(t_0)) \\ w \in \Omega_0 \cap \Omega \end{array} \right.
\end{array}
$$

where:

- $\Omega_0 = \mathcal{C}_0 = \left\{ w : \mathbf{1}_n^{\top} w = 1, \mathbf{0}_n \leq w \leq \mathbf{1}_n \right\}$ defines the long-only constraint
- Ω is the set of additional constraints

Equity portfolios

Example #7

We consider Example #5. We want to align the portfolio with respect to the CTB scenario. To compute the optimal portfolio $w^{\star}(t)$ where $t = t_0 + h$ and $h = 0, 1, 2, \ldots$ years, we assume that the benchmark $b(t)$, the covariance matrix $\Sigma(t)$, and the vector $\mathcal{CI}(t)$ of carbon intensities do not change over time.

Equity portfolios

1. First, we compute the mapping function between the time t and the decarbonization rate $\mathcal{R}(t_0, t)$:

$$\mathcal{R}(t_0, t) = 1 - (1 - 30\%) \times (1 - 7\%)^h$$

   We get $\mathcal{R}(t_0, t_0) = 30\%$, $\mathcal{R}(t_0, t_0 + 1) = 34.90\%$, $\mathcal{R}(t_0, t_0 + 2) = 39.46\%$, and so on

2. Second, we solve the optimization problem for the different values of time t

Equity portfolios

Table 19: Equity portfolio alignment (Example #7)

t	$b(t_0)$	t_0	t_0+1	t_0+2	t_0+3	t_0+4	t_0+5	t_0+10
$w_1^\star$	20.00	21.86	22.21	22.54	22.84	23.02	22.92	8.81
$w_2^\star$	19.00	18.70	18.41	18.15	17.90	17.58	17.04	0.00
$w_3^\star$	17.00	8.06	5.69	3.48	1.43	0.00	0.00	0.00
$w_4^\star$	13.00	8.74	7.66	6.65	5.72	4.56	2.70	0.00
$w_5^\star$	12.00	13.07	13.29	13.51	13.70	13.91	14.18	21.22
$w_6^\star$	8.00	22.57	25.59	28.39	31.00	33.39	35.54	62.31
$w_7^\star$	6.00	7.00	7.15	7.29	7.42	7.53	7.63	7.66
$w_8^\star$	5.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\sigma(w^\star \mid b(t))$	0.01	104.10	126.22	147.14	166.79	185.24	203.51	352.42
$\mathcal{CI}(t,w)$	160.57	112.40	104.53	97.22	90.41	84.08	78.20	54.40
$\mathcal{R}(w \mid b(t_0))$	0.00	30.00	34.90	39.46	43.70	47.64	51.30	66.12

The reduction rate and weights are expressed in %, while the tracking error volatility is measured in bps.

Bond portfolios

For bonds, the tracking error volatility is replaced by the active risk function:

$$\mathcal{D}(w \mid b) = \underbrace{\varphi \sum_{s=1}^{n_{\mathcal{S}ector}} \left| \sum_{i \in s} (w_i - b_i) \, \mathrm{DTS}_i \right|}_{\text{DTS component}} + \underbrace{\frac{1}{2} \sum_{i \in b} |w_i - b_i|}_{\text{AS component}} + \underbrace{\mathbb{1}_{\Omega_{\mathrm{MD}}}(w)}_{\text{MD component}}$$

where:

- DTS_i and MD_i are the duration-times-spread and modified duration factors
- $\Omega_{\mathrm{MD}} = \left\{ w : \sum_{i=1}^{n} (w_i - b_i) \, \mathrm{MD}_i = 0 \right\}$
- $\mathbb{1}_{\Omega}(w)$ is the convex indicator function

Bond portfolios

The optimization problem becomes then:

$$
\begin{array}{rcl}
w^{\star}(t) & = & \arg\min \mathcal{D}(w \mid b(t)) \\
& \text{s.t.} & \left\{ \begin{array}{l} \mathcal{CI}(t, w) \leq (1 - \mathcal{R}(t_0, t)) \mathcal{CI}(t_0, b(t_0)) \\ w \in \Omega_0 \cap \Omega \end{array} \right.
\end{array}
$$

Bond portfolios

Example #8

We consider Example #6. We want to align the portfolio with respect to the CTB scenario. To compute the optimal portfolio $w^{\star}(t)$ where $t = t_0 + h$ and $h = 0, 1, 2, \ldots$ years, we assume that the benchmark, the modified duration and the duration-times-spread factors do not change over time.

Bond portfolios

The corresponding LP problem is:

$$\begin{aligned} x^{\star} & = & \arg\min c^{\top} x \\ & \text{s.t.} & \left\{ \begin{array}{l} Ax = B \\ Cx \leq D \\ x^{-} \leq x \leq x^{+} \end{array} \right. \end{aligned}$$

where:

- $x = (w, \tau_w, \tau_{\mathrm{DTS}})$ is a 18×1 vector
- The 18×1 vector c is equal to $\left(\mathbf{0}_8, \dfrac{1}{2}\mathbf{1}_8, \varphi\mathbf{1}_2\right)$
- The equality constraint includes the convex indicator function $\mathbb{1}_{\Omega_{\mathrm{MD}}}(w)$ and is defined by:

$$Ax = B \Leftrightarrow \left(\begin{array}{ccc} \mathbf{1}_8^{\top} & \mathbf{0}_8^{\top} & \mathbf{0}_2^{\top} \\ \mathrm{MD}^{\top} & \mathbf{0}_8^{\top} & \mathbf{0}_2^{\top} \end{array} \right) x = \left(\begin{array}{c} 1 \\ 5.476 \end{array} \right)$$

Bond portfolios

- The inequality constraints are:

$$Cx \leq D \Leftrightarrow \begin{pmatrix} I_8 & -I_8 & \mathbf{0}_{8,2} \\ -I_8 & -I_8 & \mathbf{0}_{8,2} \\ C_{\text{DTS}} & \mathbf{0}_{2,8} & -I_2 \\ -C_{\text{DTS}} & \mathbf{0}_{2,8} & -I_2 \\ \mathcal{CI}(t)^\top & \mathbf{0}_{1,8} & 0 \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \\ 192.68 \\ 108.37 \\ -192.68 \\ -108.37 \\ 160.574 \times (1 - \mathcal{R}(t_0, t)) \end{pmatrix}$$

where:

$$C_{\text{DTS}} = \begin{pmatrix} 100 & 0 & 575 & 436 & 0 & 0 & 0 & 365 \\ 0 & 155 & 0 & 0 & 159 & 145 & 804 & 0 \end{pmatrix}$$

- Finally, the bounds are $x^- = \mathbf{0}_{18}$ and $x^+ = \infty \cdot \mathbf{1}_{18}$

Bond portfolios

Table 20: Bond portfolio alignment (Example #8)

t	$b(t_0)$	t_0	t_0+1	t_0+2	t_0+3	t_0+4	t_0+5	t_0+10
$w_1^\star$	20.00	20.00	20.00	20.00	13.98	17.64	16.02	5.02
$w_2^\star$	19.00	13.99	17.79	19.00	19.00	19.00	19.00	19.00
$w_3^\star$	17.00	25.43	20.96	17.78	17.00	13.64	11.65	4.61
$w_4^\star$	13.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$w_5^\star$	12.00	28.97	30.71	35.84	43.52	48.80	53.33	71.37
$w_6^\star$	8.00	8.00	8.00	5.67	6.46	0.92	0.00	0.00
$w_7^\star$	6.00	3.61	2.53	1.70	0.04	0.00	0.00	0.00
$w_8^\star$	5.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\mathrm{AS}(w)$	0.00	25.40	22.68	24.62	31.52	36.80	41.33	59.37
$\mathrm{MD}(w)$	5.48	5.48	5.48	5.48	5.48	5.48	5.48	5.48
$\mathrm{DTS}(w)$	301.05	274.61	248.91	230.60	220.10	204.46	197.26	174.46
$\mathcal{D}(w \mid b)$	0.00	0.39	0.49	0.60	0.72	0.85	0.99	1.57
$\mathcal{CI}(w)$	160.57	112.40	104.53	97.22	90.41	84.08	78.20	54.40
$\mathcal{R}(w \mid b)$	0.00	30.00	34.90	39.46	43.70	47.64	51.30	66.12

The reduction rate, weights, and active share metrics are expressed in %, the MD metrics are measured in years, and the DTS metrics are calculated in bps.

Defining a net-zero investment policy

General framework

The set of constraints to be applied must include the transition dimension:

$$\Omega = \Omega_{\text{alignment}} \cap \Omega_{\text{transition}}$$

where:

$$\Omega_{\text{alignment}} = \left\{ w : \mathcal{CI}\left(t, w\right) \leq \left(1 - \mathcal{R}\left(t_0, t\right)\right) \mathcal{CI}\left(t_0, b\left(t_0\right)\right) \right\}$$

and:

$$\Omega_{\text{transition}} = \Omega_{\text{self-decarbonization}} \cap \Omega_{\text{greenness}} \cap \Omega_{\text{exclusion}}$$

Self-decarbonization and endogeneity of the decarbonization pathway

	Bad case	Mixed case	Good case
Effective decarbonization			
at the beginning of the year t	30%	30%	30%
at the end of the year t	25%	33%	36%
Self-decarbonization	0%	3%	6%
Relabancing requirement	10%	2%	0%

We can specify the self-decarbonization constraint as follows:

$$\Omega_{\text{self-decarbonization}} = \{w : \mathcal{CM}(t, w) \leq \mathcal{CM}^{\star}(t)\}$$

where:

- $\mathcal{CM}(t, w)$ is the carbon momentum of the portfolio w at time t
- $\mathcal{CM}^{\star}(t)$ is the self-decarbonization minimum threshold

Green footprint

The greenness constraint can be written as follows:

$$\Omega_{\text{greenness}} = \left\{ w : \mathcal{GI}\left(t, w\right) \geq \mathcal{GI}^{\star}\left(t\right) \right\}$$

where:

- $\mathcal{GI}\left(t, w\right)$ is the green intensity of the portfolio w at time t
- $\mathcal{GI}^{\star}\left(t\right)$ is the minimum threshold

Remark

In general, the absolute measure $\mathcal{GI}^{\star}\left(t\right)$ is expressed as a relative value with respect to the benchmark:

$$\mathcal{GI}^{\star}\left(t\right) = \left(1 + \mathcal{G}\right) \mathcal{GI}\left(t, b\left(t\right)\right)$$

where $\mathcal{G}$ is the minimum growth value. For example, if $\mathcal{G} = 100\%$, we want to improve the green footprint of the benchmark so that the green intensity of the portfolio is at least twice the green intensity of the benchmark

Net-zero exclusion policy

- Net-zero enemies
- Temperature score (Implied Temperature Rating or ITR)
- Barahhou *et al.* (2022) suggest excluding issuers whose carbon momentum is greater than a threshold $\mathcal{CM}^{+}$:

$$\Omega_{\text{exclusion}} = \left\{ w : \mathcal{CM}_i \geq \mathcal{CM}^{+} \Rightarrow w_i = 0 \right\}$$

Equity portfolios

The optimization problem becomes:

$$
\begin{aligned}
w^{\star}(t) &= \arg\min \frac{1}{2}\left(w - b(t)\right)^{\top} \Sigma(t) \left(w - b(t)\right) \\
\text{s.t.} & \left\{
\begin{array}{ll}
\mathcal{CI}(t, w) \leq \left(1 - \mathcal{R}(t_0, t)\right) \mathcal{CI}(t_0, b(t_0)) & \leftarrow \text{Alignment} \\
\mathcal{CM}(t, w) \leq \mathcal{CM}^{\star}(t) & \leftarrow \text{Self-decarbonization} \\
\mathcal{GI}(t, w) \geq (1 + \mathcal{G}) \mathcal{GI}(t, b(t)) & \leftarrow \text{Greenness} \\
0 \leq w_i \leq \mathbb{1}\left\{\mathcal{CM}_i(t) \leq \mathcal{CM}^{+}\right\} & \leftarrow \text{Exclusion} \\
w \in \Omega_0 \cap \Omega & \leftarrow \text{Other constraints}
\end{array}
\right.
\end{aligned}
$$

Equity portfolios

We deduce that the quadratic form is $Q = \Sigma(t)$, $R = \Sigma(t)\, b(t)$, $A = \mathbf{1}_n^\top$, $B = 1$, $w^- = \mathbf{0}_n$, $w^+ = \mathbf{1}\left\{\mathcal{CM}(t) \leq \mathcal{CM}^+\right\}$.

- If we assume that the carbon momentum function is a linear function:

$$\mathcal{CM}(t, w) = w^\top \mathcal{CM}(t) = \sum_{i=1}^{n} w_i \mathcal{CM}_i(t)$$

where $\mathcal{CM}(t) = (\mathcal{CM}_1(t), \ldots, \mathcal{CM}_n(t))$ is the carbon momentum vector, we get:

$$Cw \leq D \Leftrightarrow \left(\begin{array}{c} \mathcal{CI}(t)^\top \\ \mathcal{CM}(t)^\top \\ -\mathcal{GI}(t)^\top \end{array}\right) w \leq \left(\begin{array}{c} (1 - \mathcal{R}(t_0, t))\, \mathcal{CI}(t_0, b(t_0)) \\ \mathcal{CM}^\star(t) \\ -(1 + \mathcal{G})\, \mathcal{GI}(t, b(t)) \end{array}\right)$$

Equity portfolios

- If we use an exact calculation of the carbon momentum at the portfolio level, we get:

$$Cw \leq D \Leftrightarrow \left(\begin{array}{c} \mathcal{CI}(t)^\top \\ \zeta^\top \\ -\mathcal{GI}(t)^\top \end{array} \right) w \leq \left(\begin{array}{c} (1 - \mathcal{R}(t_0, t))\, \mathcal{CI}(t_0, b(t_0)) \\ 0 \\ -(1 + \mathcal{G})\, \mathcal{GI}(t, b(t)) \end{array} \right)$$

where $\zeta = (\zeta_1, \ldots, \zeta_n)$ and $\zeta_i = \mathcal{CI}_i(t)\left(\mathcal{CM}_i(t) - \mathcal{CM}^\star(t)\right)$

Equity portfolios

Example #9

We consider Example #7. The carbon momentum values are equal to -3.1%, -1.2%, -5.8%, -1.4%, $+7.4\%$, -2.6%, $+1.2\%$, and -8.0%. We measure the green intensity by the green revenue share. Its values are equal to 10.2%, 45.3%, 7.5%, 0%, 0%, 35.6%, 17.8% and 3.0%. The net-zero investment policy imposes to follow the CTB decarbonization pathway with a self-decarbonization of 3%, and to improve the green intensity of the benchmark by 100%

Equity portfolios

Table 21: Net-zero equity portfolio (Example #9)

t	$b(t_0)$	t_0	t_0+1	t_0+2	t_0+3	t_0+4	t_0+5	t_0+10
$w_1^\star$	20.00	5.26	3.51	1.49	0.00	0.02		
$w_2^\star$	19.00	20.96	17.27	13.00	8.82	4.16		
$w_3^\star$	17.00	3.35	7.27	11.82	15.02	14.32		
$w_4^\star$	13.00	0.00	0.00	0.00	0.00	0.00	No feasible solution	
$w_5^\star$	12.00	0.00	0.00	0.00	0.00	0.00		
$w_6^\star$	8.00	60.06	64.69	70.05	75.37	81.51		
$w_7^\star$	6.00	0.00	0.00	0.00	0.00	0.00		
$w_8^\star$	5.00	10.37	7.25	3.64	0.79	0.00		
$\sigma(w^\star \mid b(t))$	0.00	370.16	376.38	398.30	430.94	472.44		
$\mathcal{CI}(t,w)$	160.57	110.85	104.53	97.22	90.41	84.08		
$\mathcal{R}(w \mid b(t_0))$	0.00	30.96	34.90	39.46	43.70	47.64		
$\mathcal{CM}(t,w)$	−1.66	−3.00	−3.00	−3.00	−3.00	−3.00		
$\mathcal{GI}(t,w)$	15.99	31.98	31.98	31.98	31.98	31.98		

The reduction rate, weights, carbon momentum and green intensity are expressed in %, while the tracking error volatility is measured in bps.

Bond portfolios

The optimization problem becomes:

$$
\begin{array}{rcl}
w^{\star}(t) & = & \arg\min \mathcal{D}(w \mid b(t)) \\
 & & \text{s.t.} \left\{ \begin{array}{ll}
\mathcal{CI}(t,w) \leq (1-\mathcal{R}(t_0,t))\,\mathcal{CI}(t_0,b(t_0)) & \leftarrow \text{Alignment} \\
\mathcal{CM}(t,w) \leq \mathcal{CM}^{\star}(t) & \leftarrow \text{Self-decarbonization} \\
\mathcal{GI}(t,w) \geq (1+\mathcal{G})\,\mathcal{GI}(t,b(t)) & \leftarrow \text{Greenness} \\
0 \leq w_i \leq \mathbb{1}\left\{\mathcal{CM}_i(t) \leq \mathcal{CM}^{+}\right\} & \leftarrow \text{Exclusion} \\
w \in \Omega_0 \cap \Omega & \leftarrow \text{Other constraints}
\end{array} \right.
\end{array}
$$

Bond portfolios

We get the same LP form except for the set of inequality constraints $Cx \leq D$:

$$\left(\begin{array}{ccc} I_n & -I_n & \mathbf{0}_{n,n_{\mathcal{S}ector}} \\ -I_n & -I_n & \mathbf{0}_{n,n_{\mathcal{S}ector}} \\ C_{\mathrm{DTS}} & \mathbf{0}_{n_{\mathcal{S}ector},n} & -I_{n_{\mathcal{S}ector}} \\ -C_{\mathrm{DTS}} & \mathbf{0}_{n_{\mathcal{S}ector},n} & -I_{n_{\mathcal{S}ector}} \\ \mathcal{CI}(t)^\top & \mathbf{0}_{1,n} & \mathbf{0}_{1,n_{\mathcal{S}ector}} \\ \mathcal{CM}(t)^\top & \mathbf{0}_{1,n} & \mathbf{0}_{1,n_{\mathcal{S}ector}} \\ -\mathcal{GI}(t)^\top & \mathbf{0}_{1,n} & \mathbf{0}_{1,n_{\mathcal{S}ector}} \end{array}\right) x \leq \left(\begin{array}{c} b \\ -b \\ \mathrm{DTS}^\star \\ -\mathrm{DTS}^\star \\ (1-\mathcal{R}(t_0,t))\,\mathcal{CI}(t_0,b(t_0)) \\ \mathcal{CM}^\star(t) \\ -(1+\mathcal{G})\,\mathcal{GI}(t,b(t)) \end{array}\right)$$

and the upper bound:

$$x^+ = \left(\mathbb{1}\left\{\mathcal{CM}(t) \leq \mathcal{CM}^+\right\}, \infty \cdot \mathbf{1}_n, \infty \cdot \mathbf{1}_{n_{\mathcal{S}ector}}\right)$$

Bond portfolios

Example #10

We consider Example #8. The carbon momentum values are equal to -3.1%, -1.2%, -5.8%, -1.4%, $+7.4\%$, -2.6%, $+1.2\%$, and -8.0%. We measure the green intensity by the green revenue share. Its values are equal to 10.2%, 45.3%, 7.5%, 0%, 0%, 35.6%, 17.8% and 3.0%. The net-zero investment policy imposes to follow the CTB decarbonization pathway with a self-decarbonization of 2%, and to improve the green intensity of the benchmark by 100%.

Bond portfolios

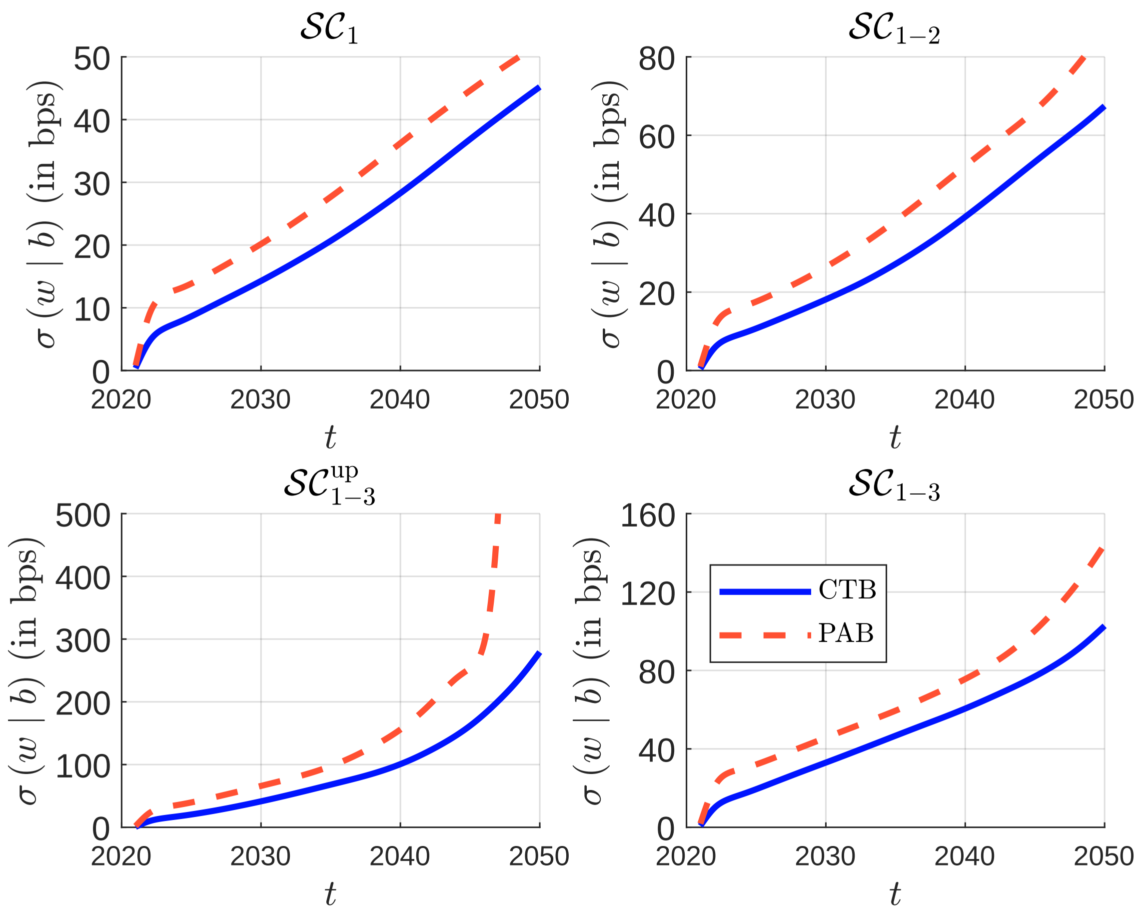
Table 22: Net-zero bond portfolio (Example #10)

t	$b(t_0)$	t_0	t_0+1	t_0+2	t_0+3	t_0+4	t_0+5	t_0+10
$w_1^\star$	20.00	4.28	13.80	20.48	26.34	19.02		
$w_2^\star$	19.00	34.78	38.94	42.72	46.23	49.01		
$w_3^\star$	17.00	21.03	13.86	7.73	2.11	0.00		
$w_4^\star$	13.00	0.00	0.00	0.00	0.00	0.00	No feasible solution	
$w_5^\star$	12.00	0.00	0.00	0.00	0.00	0.00		
$w_6^\star$	8.00	39.91	33.40	29.07	25.32	31.97		
$w_7^\star$	6.00	0.00	0.00	0.00	0.00	0.00		
$w_8^\star$	5.00	0.00	0.00	0.00	0.00	0.00		
$\mathrm{AS}(w)$	0.00	51.72	45.34	45.27	50.89	53.98		
$\mathrm{MD}(w)$	5.48	5.48	5.48	5.48	5.48	5.48		
$\mathrm{DTS}(w)$	301.05	236.99	202.30	173.29	146.83	141.34		
$\mathcal{D}(w \mid b)$	0.00	0.87	0.95	1.09	1.28	1.48		
$\mathcal{CI}(w)$	160.57	112.40	104.53	97.22	90.41	84.08		
$\mathcal{R}(w \mid b)$	0.00	30.00	34.90	39.46	43.70	47.64		
$\mathcal{CM}(t, w)$	−1.66	−2.81	−2.57	−2.35	−2.15	−2.01		
$\mathcal{GI}(t, w)$	15.99	31.98	31.98	32.37	32.80	35.52		

The reduction rate, weights, carbon momentum, green intensity and active share metrics are expressed in %, the MD metrics are measured in years, and the DTS metrics are

Empirical results

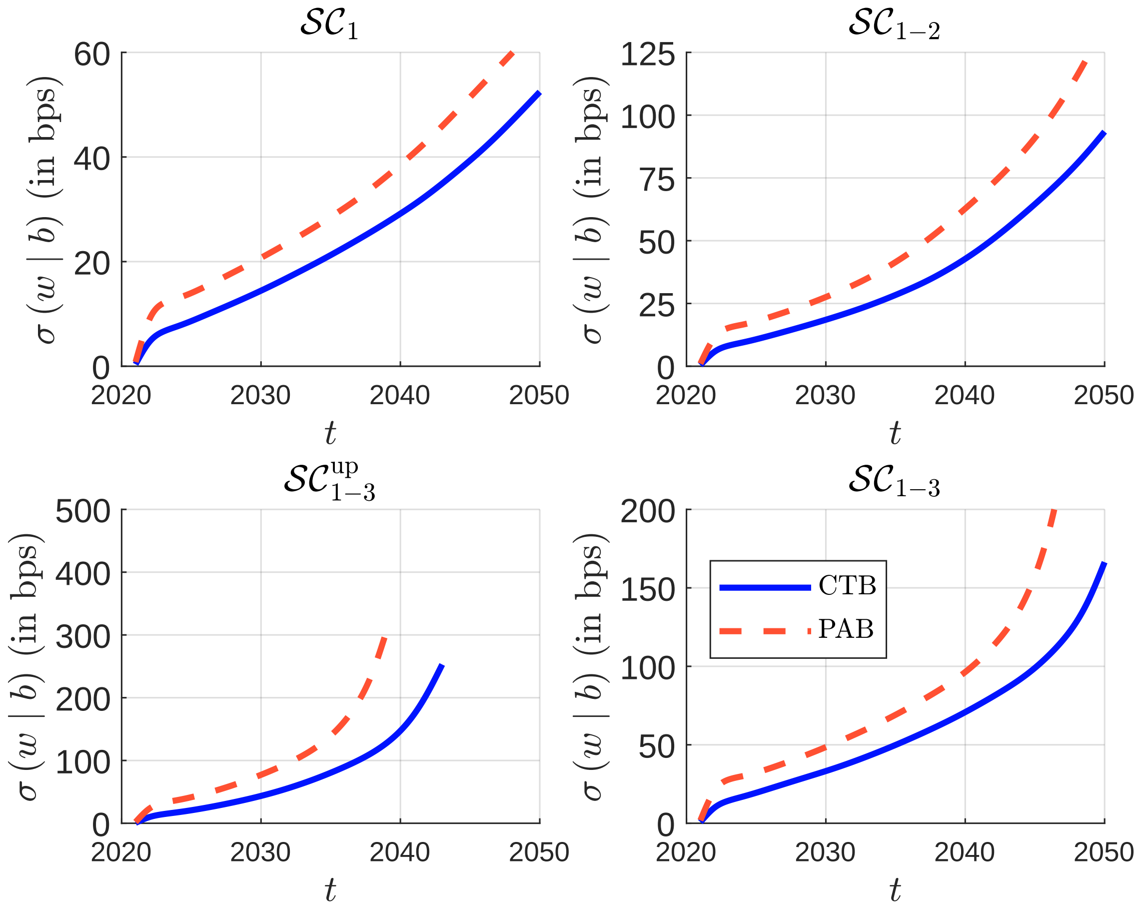
Figure 19: Tracking error volatility of dynamic decarbonized portfolios (MSCI World, June 2022, $\mathcal{C}_0$ constraint)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022).

Empirical results

Figure 20: Tracking error volatility of dynamic decarbonized portfolios (MSCI World, June 2022, $\mathcal{C}_3(0, 10, 2)$ constraint)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022).

Empirical results

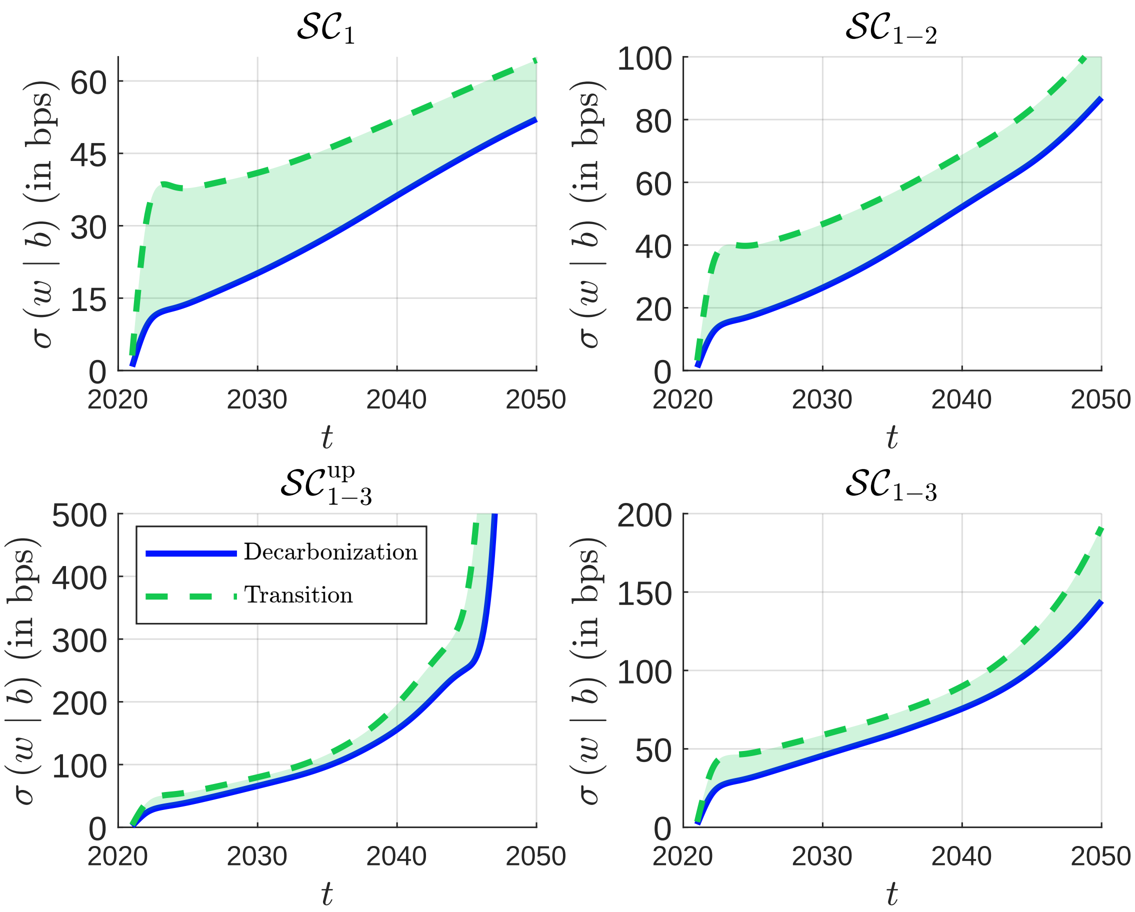
The previous analysis deals only with the decarbonization dimension. Barahhou *et al.* (2022) then introduced the transition dimension and solved the following optimization problem:

$$
\begin{aligned}
w^{\star}(t) &= \arg\min \frac{1}{2}(w - b(t))^{\top} \Sigma(t)(w - b(t)) \\
\text{s.t.} & \left\{
\begin{array}{l}
\mathcal{CI}(t, w) \leq (1 - \mathcal{R}(t_0, t))\,\mathcal{CI}(t_0, b(t_0)) \\
\mathcal{CM}(t, w) \leq \mathcal{CM}^{\star}(t) \\
\mathcal{GI}(t, w) \geq (1 + \mathcal{G})\,\mathcal{GI}(t, b(t)) \\
w \in \mathcal{C}_0 \cap \mathcal{C}_3(0, 10, 2)
\end{array}
\right.
\end{aligned}
$$

where $\mathcal{CM}^{\star}(t) = -5\%$ and $\mathcal{G} = 100\%$

Empirical results

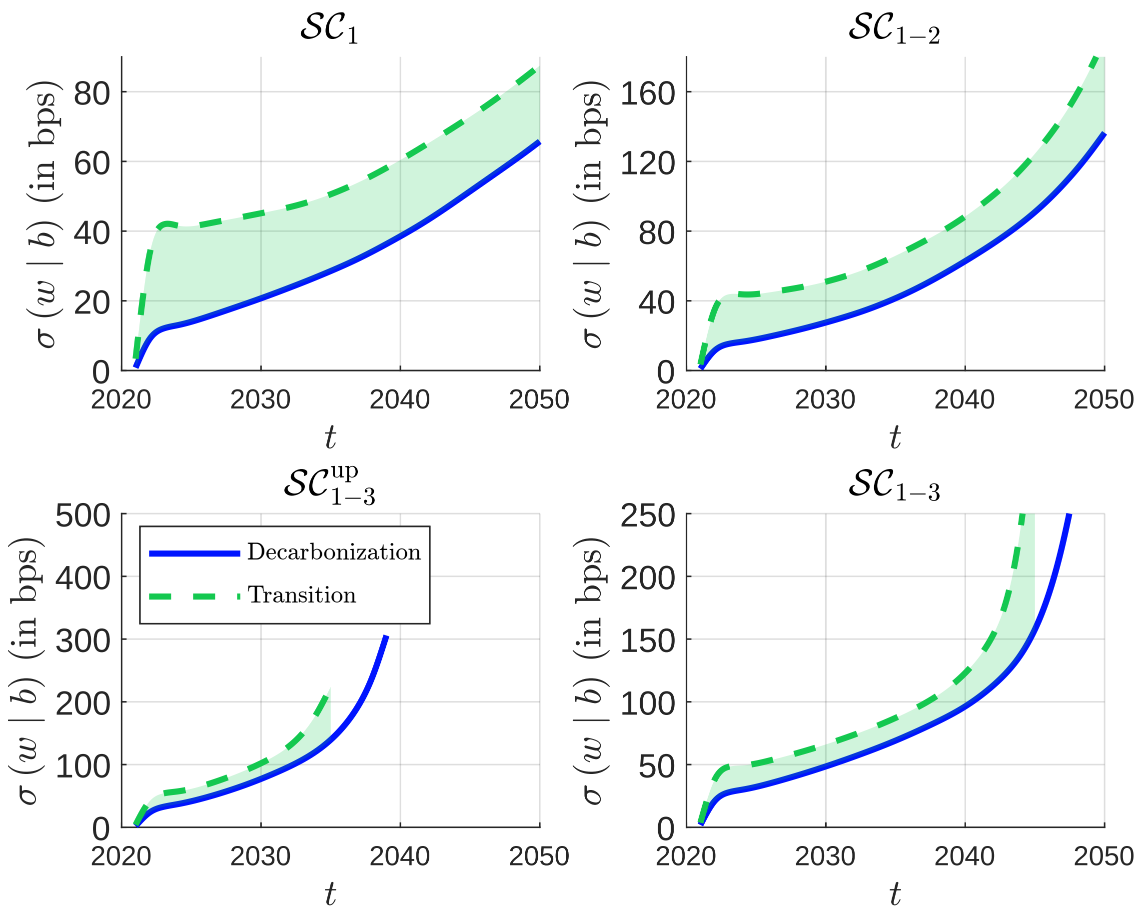
Figure 21: Tracking error volatility of net-zero portfolios (MSCI World, June 2022, $\mathcal{C}_0$ constraint, $\mathcal{G} = 100\%$, $\mathcal{CM}^{\star} = -5\%$, PAB)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022).

Empirical results

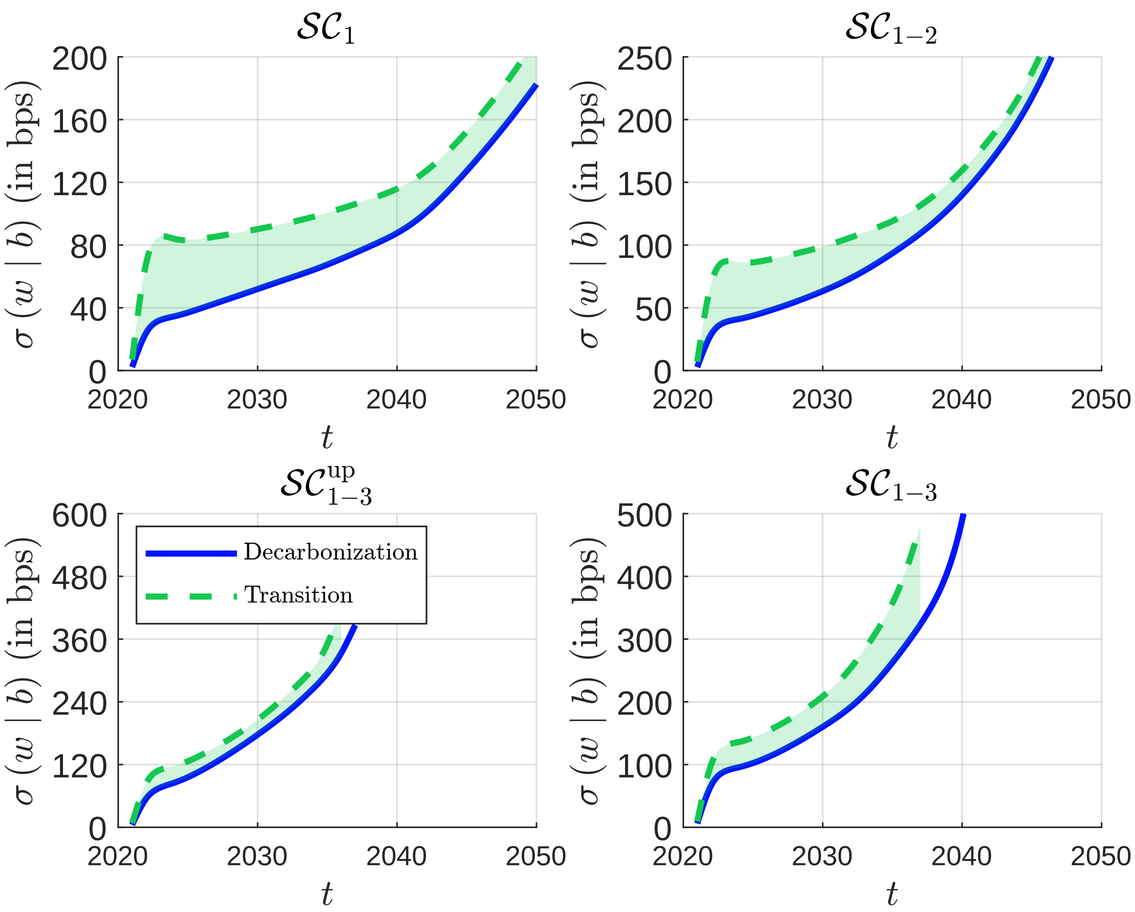
Figure 22: Tracking error volatility of net-zero portfolios (MSCI World, June 2022, $\mathcal{C}_3(0, 10, 2)$ constraint, $\mathcal{G} = 100\%$, $\mathcal{CM}^\star = -5\%$, PAB)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022).

Empirical results

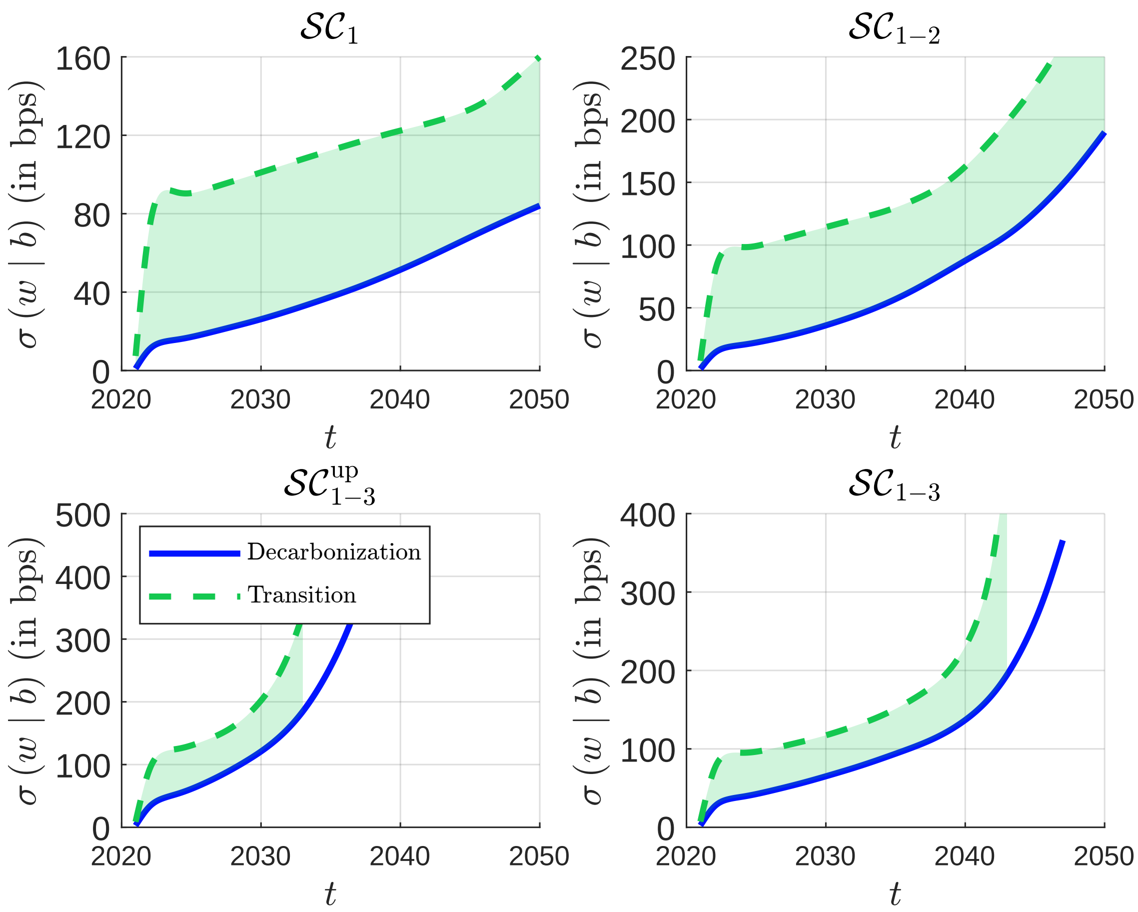
Figure 23: Tracking error volatility of net-zero portfolios (MSCI EMU, June 2022, $\mathcal{C}_3\,(0,10,2)$ constraint, $\mathcal{G} = 100\%$, $\mathcal{CM}^\star = -5\%$, PAB)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022).

Empirical results

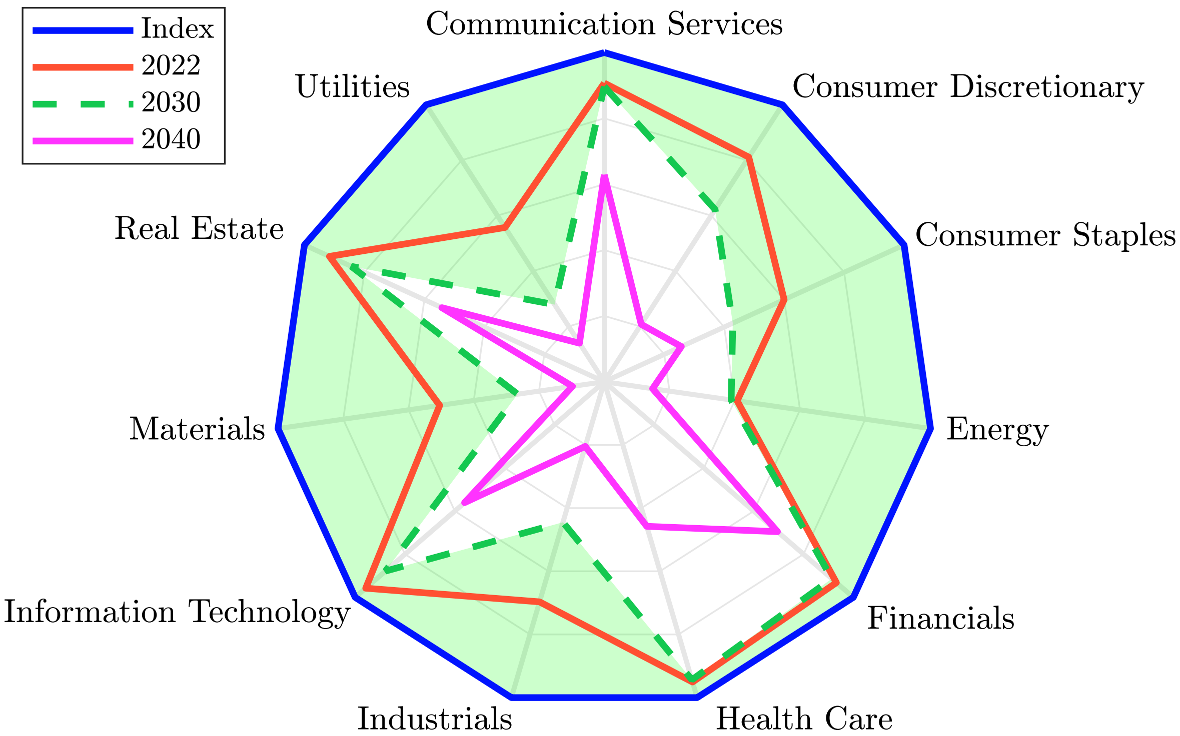
Figure 24: Tracking error volatility of net-zero portfolios (MSCI USA, Jun. 2022, $\mathcal{C}_3(0, 10, 2)$ constraint, $\mathcal{G} = 100\%$, $\mathcal{CM}^\star = -5\%$, PAB)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022).

Empirical results

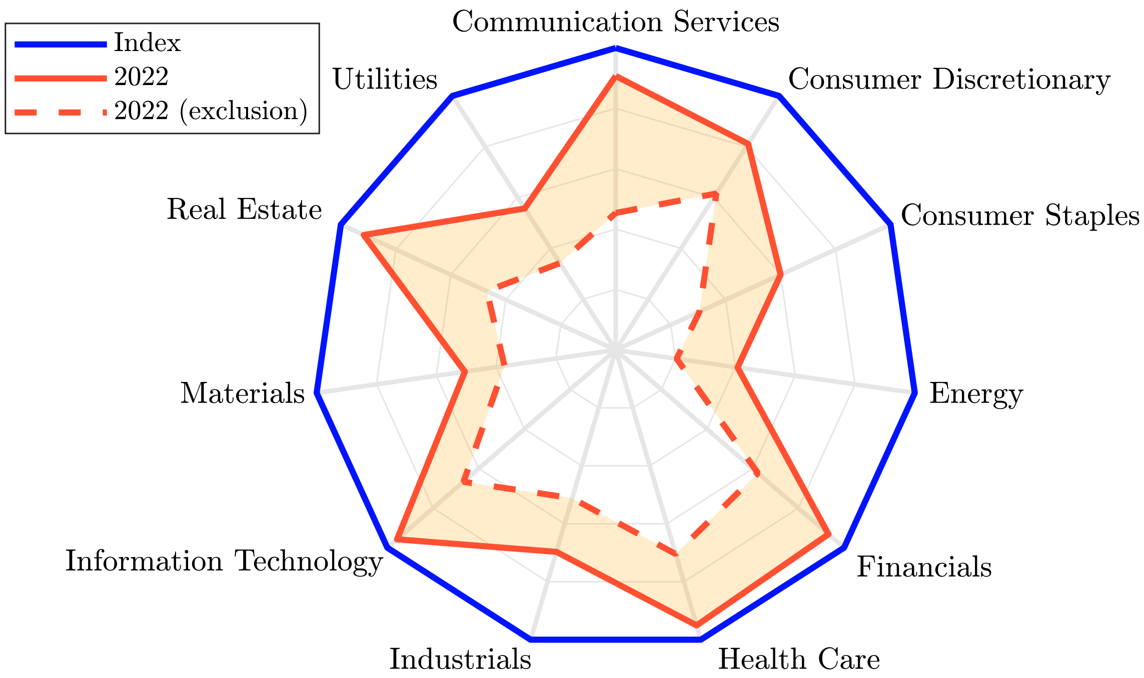
Figure 25: Radar chart of investment universe shrinkage (MSCI World, June 2022, $\mathcal{C}_3\left(0, 10, 2\right)$ constraint, $\mathcal{G} = 100\%$, $\mathcal{CM}^{\star} = -5\%$, PAB, Scope $\mathcal{SC}_{1-3}$)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022).

Empirical results

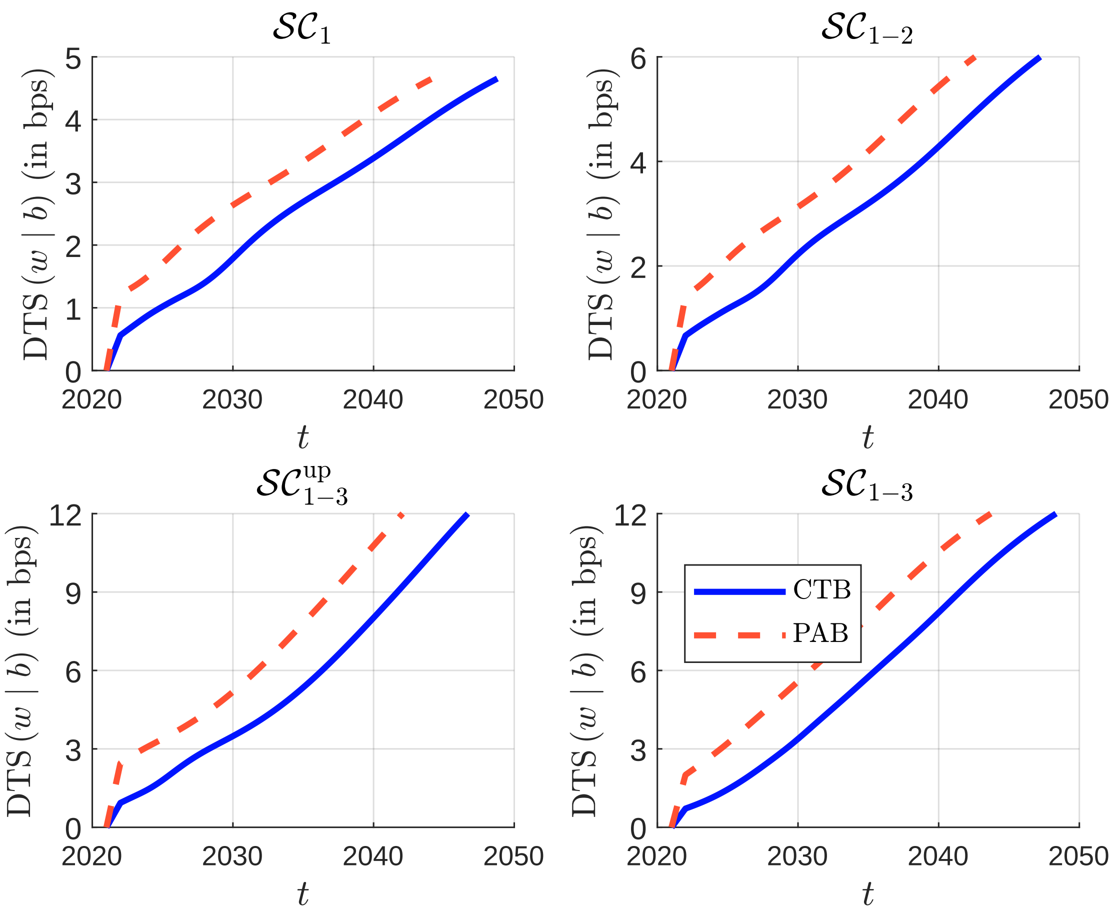
Figure 26: Impact of momentum exclusion on universe shrinkage (MSCI World, June 2022, $\mathcal{C}_3(0, 10, 2)$ constraint, $\mathcal{G} = 100\%$, $\mathcal{CM}^\star = -5\%$, PAB, Scope $\mathcal{SC}_{1-3}$, $\mathcal{CM}^+ = 0\%$)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022).

Empirical results

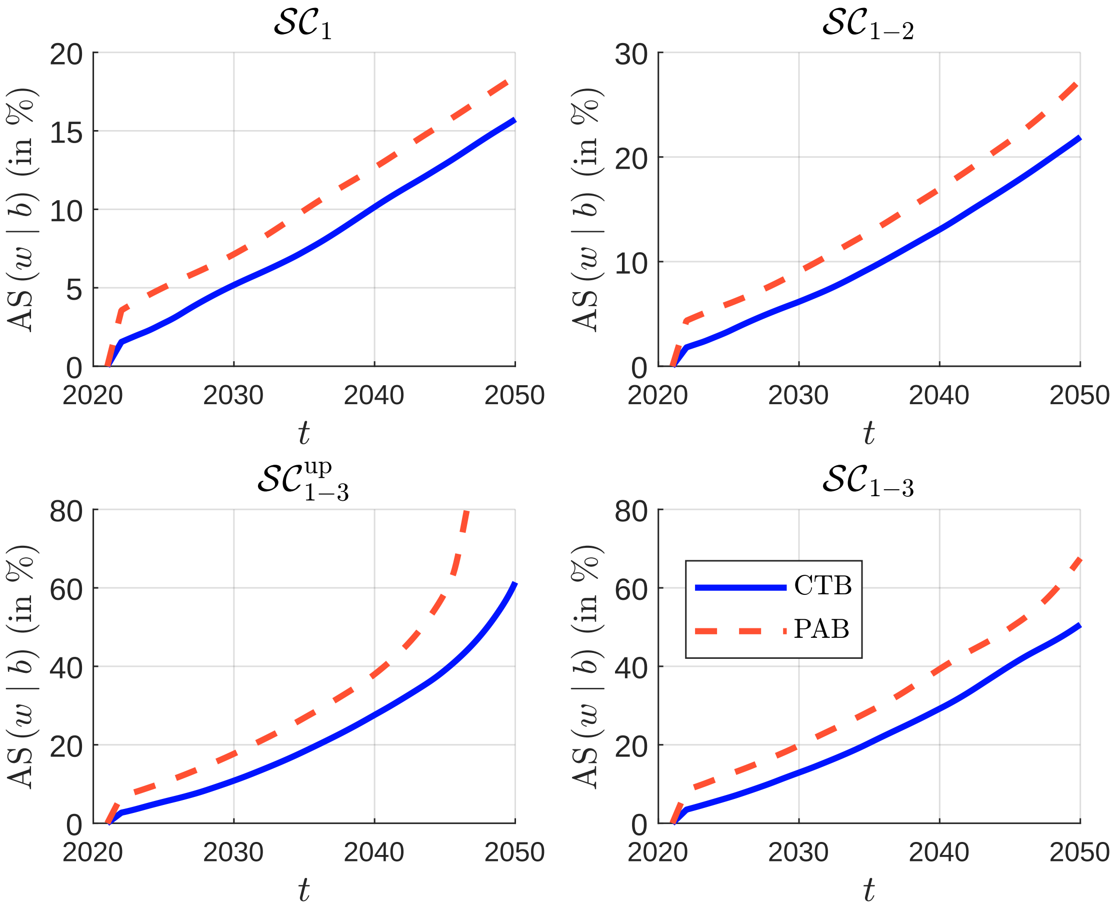
Figure 27: Duration-times-spread cost of dynamically decarbonized portfolios (Global Corporate, June 2022)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022).

Empirical results

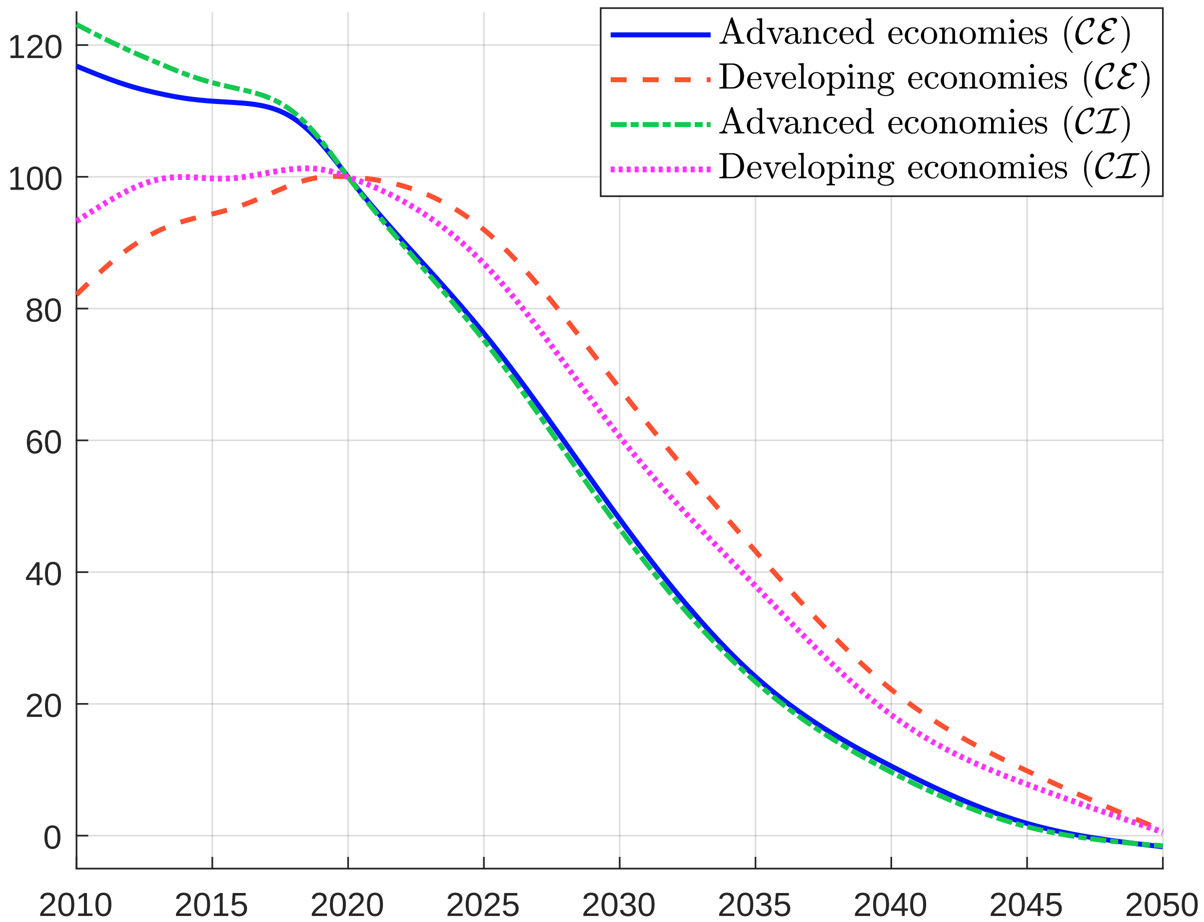
Figure 28: Active share of dynamically decarbonized portfolios (Global Corporate, June 2022)



Source: MSCI (2022), Trucost (2022) & Barahhou *et al.* (2022).

Empirical results

Figure 29: IEA decarbonization pathways



Empirical results

Table 23: First year of country exit from the NZE investment portfolio (GHG/GDP intensity metric)

Australia	2025	Finland	2029	Lithuania	2025	Romania	2029
Austria	2029	France	2029	Luxembourg	2029	Singapore	2029
Belgium	2028	Germany	2029	Mexico	2029	Slovakia	2025
Canada	2024	Hong Kong	2029	Malaysia	2028	Slovenia	2028
Chile	2029	Hungary	2029	Malta	2029	South Korea	2024
China	2028	Indonesia	2024	Netherlands	2029	Spain	2028
Colombia	2029	Ireland	2029	Norway	2029	Switzerland	2029
Cyprus	2029	Israel	2029	New Zealand	2024	Sweden	2029
Czechia	2024	Italy	2029	Peru	2029	Thailand	2025
Denmark	2029	Japan	2029	Poland	2029	United Kingdom	2029
Estonia	2025	Latvia	2028	Portugal	2028	United States	2028

Source: Barahhou *et al.* (2023, Table 9, page 26.

Empirical results

Table 24: Country exclusion year by intensity metric

Metric	$\frac{GHG}{GDP}$	$\frac{GHG}{Population}$	$\frac{CO_2 \text{ (production)}}{GDP}$	$\frac{CO_2 \text{ (consumption)}}{Population}$
China	2028	2031	2027	2031
France	2029	2032	2027	2031
Indonesia	2024	2032	2024	2031
Ireland	2029	2030	2027	2030
Japan	2029	2032	2027	2031
United States	2028	2030	2026	2029
United Kingdom	2029	2032	2027	2031
Sweden	2029	2032	2027	2031

Source: Barahhou *et al.* (2023, Table 14, page 31.

The core-satellite approach

The two building block approach

Decarbonizing the portfolio

- Net-zero decarbonization portfolio
- Net-zero transition portfolio
- Dynamic low-carbon portfolio

Financing the transition

- Net-zero contribution portfolio
- Net-zero funding portfolio
- Net-zero transformation portfolio

The core-satellite approach

The core-satellite approach

Decarbonized portfolio

- Carbon intensity
- Decarbonization pathway(**s**)
- Top-down approach
- Portfolio construction
- Net-zero **carbon** metrics

$1 - \alpha(t)$

$+$

Transition portfolio

- Green intensity
- Financing the transition
- Bottom-up approach
- Security selection
- Net-zero **transition** metrics

$\alpha(t)$

Core portfolio

A typical program for the equity bucket looks like this:

$$
\begin{aligned}
w^{\star}(t) & = \arg\min \frac{1}{2}(w - b(t))^{\top} \Sigma(t)(w - b(t)) \\
& \text{s.t.} \left\{
\begin{array}{l}
\mathcal{CI}(t, w) \leq (1 - \mathcal{R}(t_0, t)) \mathcal{CI}(t_0, b(t_0)) \\
\mathcal{CM}(t, w) \leq \mathcal{CM}^{\star}(t) \\
0 \leq w_i \leq \mathbb{1}\left\{\mathcal{CM}_i(t) \leq \mathcal{CM}^{+}\right\} \\
w \in \Omega_0 \cap \Omega
\end{array}
\right.
\end{aligned}
$$

Core portfolio

For the bond bucket, we get a similar optimization problem:

$$
\begin{array}{rcl}
w^{\star}(t) & = & \arg\min \mathcal{D}\left(w \mid b(t)\right) \\
& & \text{s.t.} \left\{
\begin{array}{l}
\mathcal{CI}(t, w) \leq \left(1 - \mathcal{R}(t_0, t)\right) \mathcal{CI}(t_0, b(t_0)) \\
\mathcal{CM}(t, w) \leq \mathcal{CM}^{\star}(t) \\
0 \leq w_i \leq \mathbb{1}\left\{\mathcal{CM}_i(t) \leq \mathcal{CM}^{+}\right\} \\
w \in \Omega_0 \cap \Omega
\end{array}
\right.
\end{array}
$$

The electricity sector scenario in the core portfolio

The constraint to meet a reduction rate for a given sector $\mathcal{S}ector_j$ is:

$$\frac{\sum_{i=1}^{n} \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\} w_i \mathcal{CI}_i}{\sum_{i=1}^{n} \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\} w_i} = \mathcal{CI}\left(\mathcal{S}ector_j, \mathcal{R}_j\right)$$

where $\mathcal{CI}\left(\mathcal{S}ector_j, \mathcal{R}_j\right)$ is the carbon intensity target for the given sector:

$$\mathcal{CI}\left(\mathcal{S}ector_j, \mathcal{R}_j\right) = \left(1 - \mathcal{R}_j\right) \frac{\sum_{i=1}^{n} \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\} b_i \mathcal{CI}_i}{\sum_{i=1}^{n} \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\} b_i}$$

We deduce that:

$$\sum_{i=1}^{n} \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\} w_i \mathcal{CI}_i = \mathcal{CI}\left(\mathcal{S}ector_j, \mathcal{R}_j\right) \sum_{i=1}^{n} \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\} w_i$$

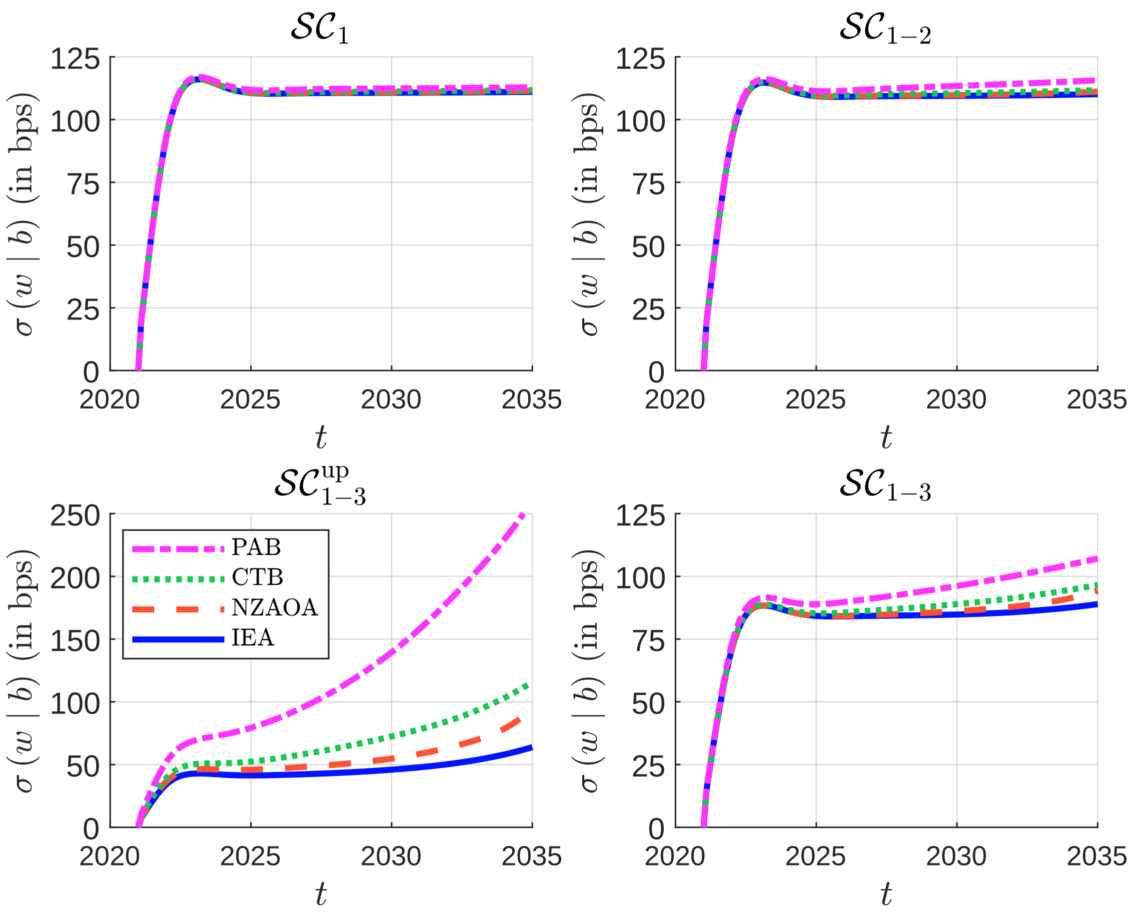
which is equivalent to the following constraint:

$$\sum_{i=1}^{n} \mathbb{1}\left\{i \in \mathcal{S}ector_j\right\} w_i \left(\mathcal{CI}_i - \mathcal{CI}\left(\mathcal{S}ector_j, \mathcal{R}_j\right)\right) = 0 \Leftrightarrow \left(\boldsymbol{s}_j \circ \left(\mathcal{CI}_i - \mathcal{CI}_j^{\star}\right)\right)^{\top} w = 0$$

where $\mathcal{CI}_j^{\star} = \mathcal{CI}\left(\mathcal{S}ector_j, \mathcal{R}_j\right)$

Core portfolio

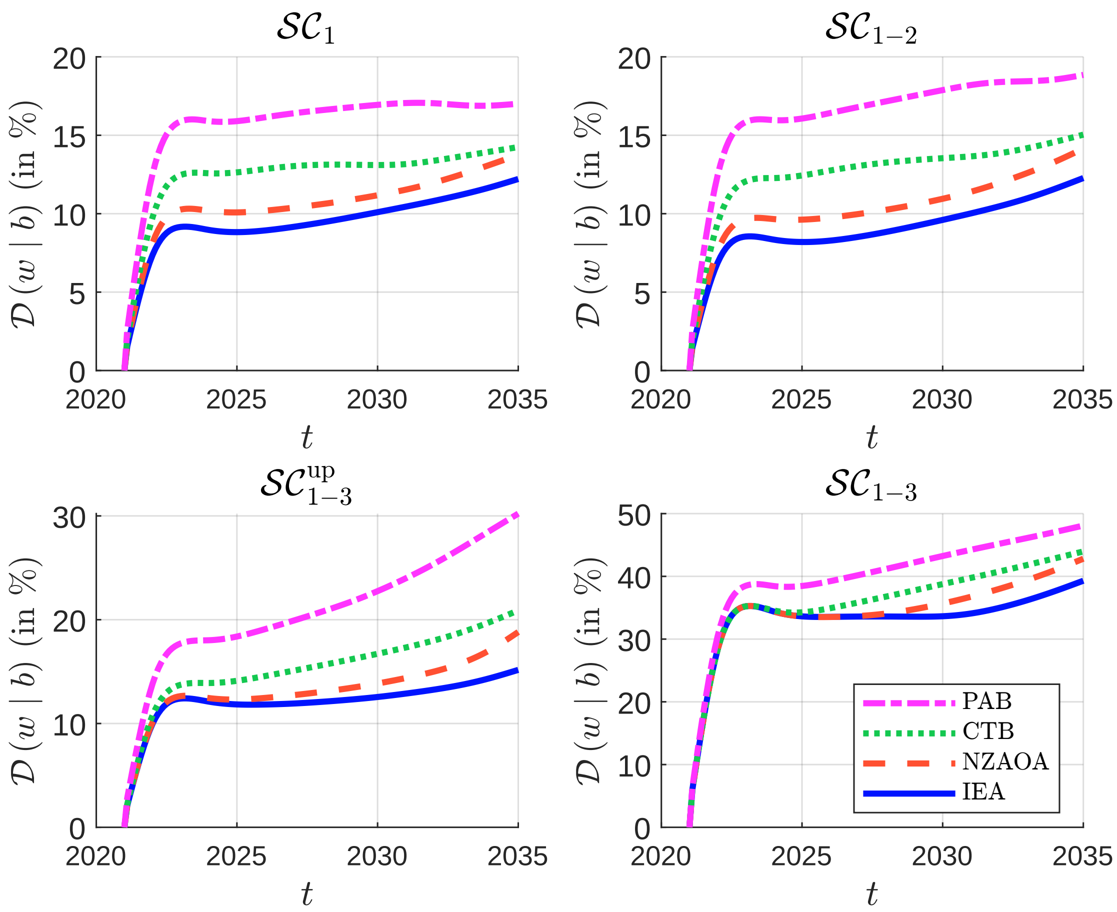
Figure 30: TE volatility of decarbonized portfolios (MSCI World, December 2021, $\mathcal{CM}^{\star} = -3.5\%$, $\mathcal{CM}^{+} = 10\%$, IEA NZE electricity sector scenario)



Source: Ben Slimane *et al.* (2023).

Core portfolio

Figure 31: Active risk of decarbonized portfolios (Global Corporate, December 2021, $\mathcal{CM}^{\star} = -3.5\%$, $\mathcal{CM}^{+} = 10\%$, IEA NZE electricity sector scenario)



Source: Ben Slimane *et al.* (2023).

Satellite portfolio

- Green, sustainability and sustainability-linked bonds
- Green stocks
- Green infrastructure
- Sustainable real estate

Satellite portfolio

Figure 32: Narrow specification of the satellite investment universegeneral]GICS

Source: Ben Slimane *et al.* (2023).

Green bonds

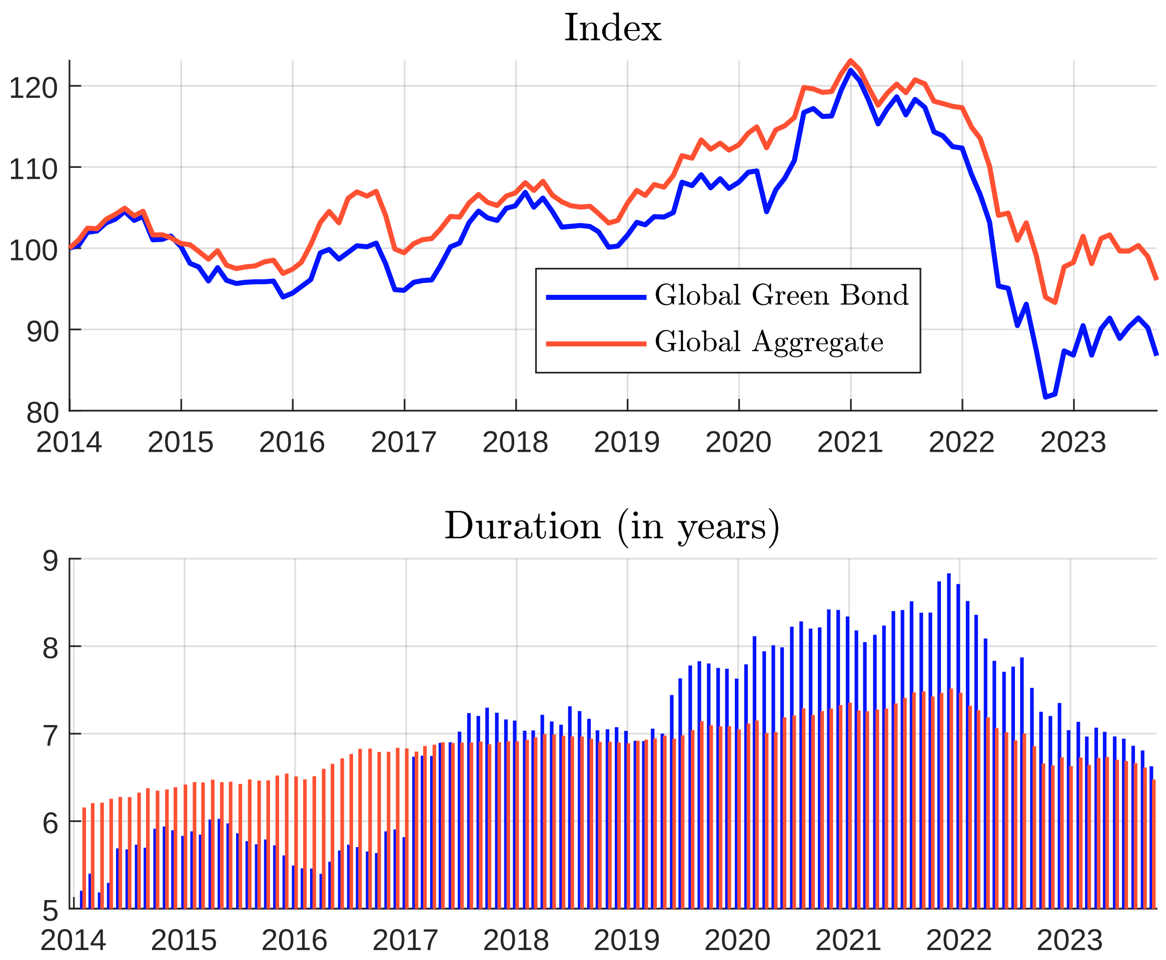
Table 25: GSS+ bond issuance

Year	Green		Social		Sustainability		SLB	
	#	$ bn	#	$ bn	#	$ bn	#	$ bn
2022	1 784	531.6	542	152.8	614	174.8	382	144.3
2021	1 971	686.1	554	242.1	646	233.2	343	161.5
2020	1 076	291.2	273	172.0	308	154.8	47	16.5
2019	877	268.0	99	22.2	333	85.2	18	8.9
2018	582	165.3	48	16.5	52	22.1	1	2.2
2017	472	160.9	46	11.8	17	9.2	1	0.2
2016	285	99.7	14	2.2	16	6.6	0	0.0

Source: Bloomberg (2023), GSS+ Instrument Indicator & Author's calculations.

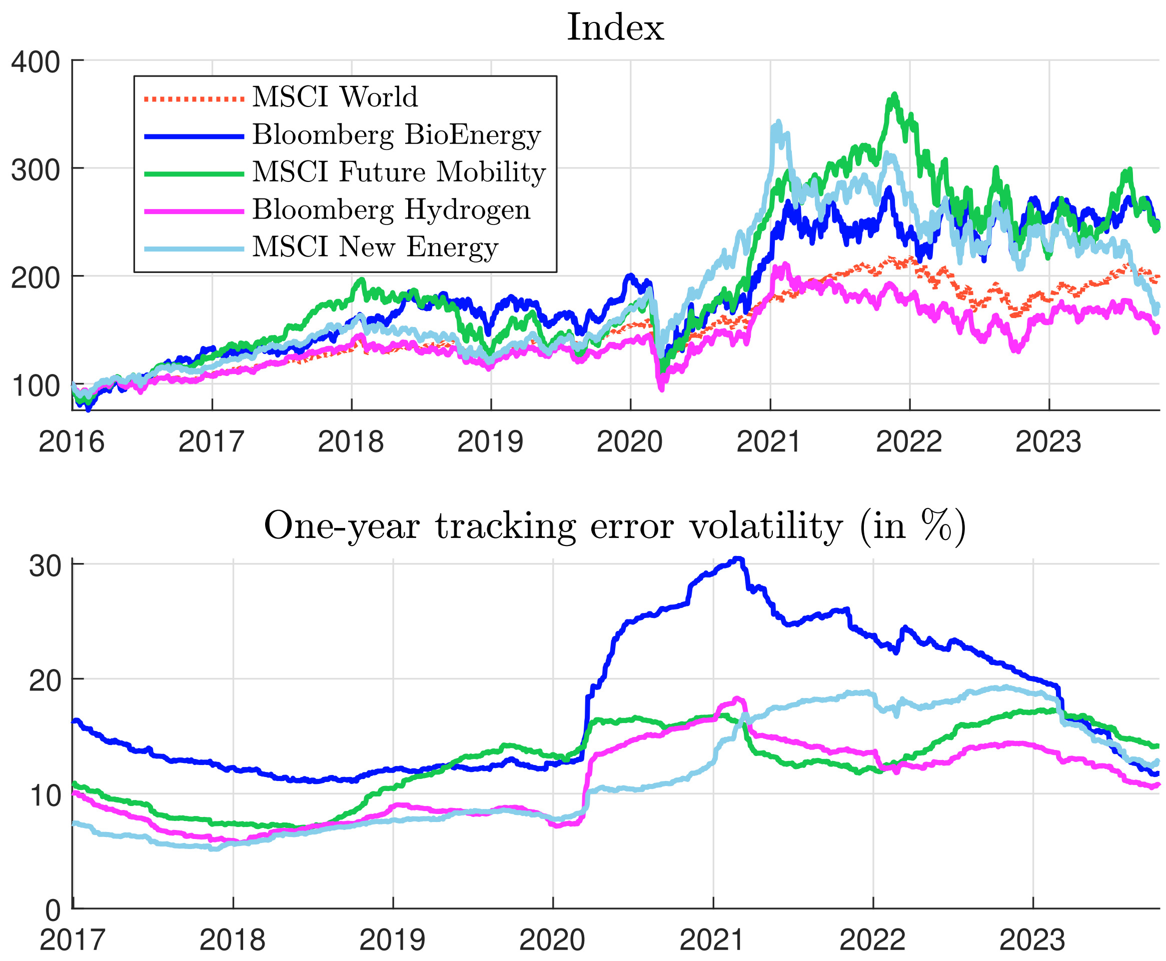
Green bonds

Figure 33: Performance and duration of the Bloomberg Global Green Bond and Aggregate indices



Green stocks

Figure 34: Performance and tracking error volatility of thematic equity indices



Green infrastructure

The European Commission defines green infrastructure as "*a strategically planned network of natural and semi-natural areas with other environmental features, designed and managed to deliver a wide range of ecosystem services, while also enhancing biodiversity*". Green infrastructure is implemented in a variety of sectors, from energy through energy transmission infrastructure, water through natural water retention measures or sustainable urban drainage systems, to the urban landscape with street trees to help sequester carbon or green roofs to help regulate the temperature of buildings. The cost of implementing green infrastructure is in the identification, mapping, planning and creation of the infrastructure, but the environmental, economic and social benefits make it worthwhile. Funds that assess infrastructure needs are emerging in the market and typically invest in owners of sustainable infrastructure assets as well as companies that are leaders in infrastructure investment. In addition to infrastructure funds, investors are also considering direct investments such as green car parks, water infrastructure and flood defences.

Sustainable real estate

- CRREM (Carbon Risk Real Estate Monitor) $\Rightarrow$ whole-building approach for in-use emissions
- GRESB $\Rightarrow$ GHG Protocol principles to the real estate industry (corporate approach)
- SBTi Building Guidelines
- PCAF/CRREM/GRESB joint technical guidance $\Rightarrow$ Accounting and reporting of financed GHG emissions from real estate operations (GHG Protocol)

Allocation process

The stock/bond mix allocation

- Let α_{equity} and α_{bond} be the proportions of stocks and bonds in the multi-asset portfolio
- Let $\alpha^{\text{satellite}}$ be the weight of the satellite portfolio
- The core allocation is given by the vector $\left(\alpha_{\text{equity}}^{\text{core}}, \alpha_{\text{bond}}^{\text{core}}\right)$, while the satellite allocation is defined by $\left(\alpha_{\text{equity}}^{\text{satellite}}, \alpha_{\text{bond}}^{\text{satellite}}\right)$
- We have the following identities:

$$\left\{\begin{array}{l}\alpha_{\text{equity}} = \left(1 - \alpha^{\text{satellite}}\right) \alpha_{\text{equity}}^{\text{core}} + \alpha^{\text{satellite}} \alpha_{\text{equity}}^{\text{satellite}} \\ \alpha_{\text{bond}} = \left(1 - \alpha^{\text{satellite}}\right) \alpha_{\text{bond}}^{\text{core}} + \left(1 - \alpha^{\text{satellite}}\right) \alpha_{\text{bond}}^{\text{satellite}}\end{array}\right.$$

Allocation process

The stock/bond mix allocation

- In general, the fund manager targets a strategic asset allocation at the portfolio level, *i.e.* the proportions α_{equity} and α_{bond} are given
- For example, a defensive portfolio corresponds to a 20/80 constant mix strategy, while the 50/50 allocation is known as a balanced portfolio. Another famous allocation rule is the 60/40 portfolio, which is 60% in stocks and 40% in bonds.
- The solution is to calculate the proportion of bonds in the core portfolio relative to the proportion of bonds in the satellite portfolio:

$$\alpha_{\text{bond}}^{\text{core}} = \frac{\alpha_{\text{bond}} - \alpha^{\text{satellite}} \alpha_{\text{bond}}^{\text{satellite}}}{1 - \alpha^{\text{satellite}}}$$

Allocation process

The stock/bond mix allocation

Example #11

We consider a 60/40 constant mix strategy. The satellite portfolio represents 10% of the net zero investments. We assume that the satellite portfolio has 70% exposure to green bonds.

Allocation process

The stock/bond mix allocation

We have $\alpha_{\text{equity}} = 60\%$, $\alpha_{\text{bond}} = 40\%$, $\alpha^{\text{core}} = 90\%$, $\alpha^{\text{satellite}} = 10\%$ and $\alpha_{\text{bond}}^{\text{satellite}} = 70\%$. We deduce that:

$$\alpha_{\text{bond}}^{\text{core}} = \frac{0.40 - 0.10 \times 0.70}{1 - 0.10} = \frac{33}{90} = 36.67\%$$

The core allocation is then $(63.33\%, 36.67\%)$, while the satellite allocation is $(30\%, 70\%)$. We check that:

$$\left\{\begin{array}{l} \alpha_{\text{equity}} = 0.90 \times \left(1 - \dfrac{33}{90}\right) + 0.10 \times 0.30 = 60\% \\ \alpha_{\text{bond}} = 0.90 \times \dfrac{33}{90} + 0.10 \times 0.70 = 40\% \end{array}\right.$$

Allocation process

The stock/bond mix allocation

Table 26: Calculating the bond allocation in the core portfolio ($\alpha_{\text{bond}}^{\text{core}}$ in %)

Strategy		60/40			50/50			20/80		
$\alpha_{\text{bond}}^{\text{satellite}}$		70.0	80.0	90.0	70.0	80.0	90.0	70.0	80.0	90.0
$\alpha^{\text{satellite}}$	0%	40.0	40.0	40.0	50.0	50.0	50.0	80.0	80.0	80.0
	1%	39.7	39.6	39.5	49.8	49.7	49.6	80.1	80.0	79.9
	5%	38.4	37.9	37.4	48.9	48.4	47.9	80.5	80.0	79.5
	10%	36.7	35.6	34.4	47.8	46.7	45.6	81.1	80.0	78.9
	15%	34.7	32.9	31.2	46.5	44.7	42.9	81.8	80.0	78.2
	20%	32.5	30.0	27.5	45.0	42.5	40.0	82.5	80.0	77.5
	25%	30.0	26.7	23.3	43.3	40.0	36.7	83.3	80.0	76.7

Allocation process

Tracking error risk of the core-satellite portfolio

The tracking error volatility of the core-satellite portfolio has the following expression:

$$\sigma\left(w \mid b\right) = \sqrt{\tilde{\boldsymbol{\alpha}}^\top \tilde{\boldsymbol{\Sigma}}\left(w \mid b\right) \tilde{\boldsymbol{\alpha}}} = \sqrt{\left(\tilde{\boldsymbol{\alpha}} \circ \tilde{\boldsymbol{\sigma}}\left(w \mid b\right)\right)^\top \tilde{\boldsymbol{\rho}}\left(w \mid b\right) \left(\tilde{\boldsymbol{\alpha}} \circ \tilde{\boldsymbol{\sigma}}\left(w \mid b\right)\right)}$$

where:

- $\tilde{\boldsymbol{\alpha}}$ is the vector of allocation:

$$\tilde{\boldsymbol{\alpha}} = \begin{pmatrix} \left(1 - \alpha^{\text{satellite}}\right) \alpha^{\text{core}}_{\text{equity}} \\ \left(1 - \alpha^{\text{satellite}}\right) \alpha^{\text{core}}_{\text{bond}} \\ \alpha^{\text{satellite}} \alpha^{\text{satellite}}_{\text{equity}} \\ \alpha^{\text{satellite}} \alpha^{\text{satellite}}_{\text{bond}} \end{pmatrix}$$

- $\tilde{\boldsymbol{\rho}}\left(w \mid b\right)$ is the correlation matrix of $R\left(w\right) - R\left(b\right)$
- $\tilde{\boldsymbol{\sigma}}\left(w \mid b\right)$ is the vector of tracking error volatilities:

$$\tilde{\boldsymbol{\sigma}}\left(w \mid b\right) = \begin{pmatrix} \sigma\left(w^{\text{core}}_{\text{equity}} \mid b_{\text{equity}}\right) \\ \sigma\left(w^{\text{core}}_{\text{bond}} \mid b_{\text{bond}}\right) \\ \sigma\left(w^{\text{satellite}}_{\text{equity}} \mid b_{\text{equity}}\right) \\ \sigma\left(w^{\text{satellite}}_{\text{bond}} \mid b_{\text{bond}}\right) \end{pmatrix}$$

Allocation process

Tracking error risk of the core-satellite portfolio

Example #12

The tracking error volatilities are 2% for the core equity portfolio, 25 bps for the core bond portfolio, 20% for the satellite equity portfolio, and 3% for the satellite bond portfolio. To define the correlation matrix $\tilde{\rho}(w \mid b)$, we assume an 80% correlation between the two equity baskets, a 50% correlation between the two bond baskets, and a 0% correlation between the equity and bond baskets. We consider a 60/40 constant mix strategy. The satellite portfolio represents 10% of the net zero portfolio and has 70% exposure to green bonds

Allocation process

Tracking error risk of the core-satellite portfolio

We compute the tracking error covariance matrix $\tilde{\boldsymbol{\Sigma}}(w \mid b)$ as follows:

- The tracking error variance for the core equity portfolio is $\tilde{\boldsymbol{\Sigma}}_{1,1}(w \mid b) = 0.02^2$
- The tracking error variance for the satellite equity portfolio is $\tilde{\boldsymbol{\Sigma}}_{3,3}(w \mid b) = 0.20^2$
- The tracking error covariance for the two core portfolios is $\tilde{\boldsymbol{\Sigma}}_{1,2}(w \mid b) = 0 \times 0.02 \times 0.0025$
- The tracking error covariance for the core equity portfolio and the satellite equity portfolio is $\tilde{\boldsymbol{\Sigma}}_{1,3}(w \mid b) = 0.80 \times 0.02 \times 0.20$
- Etc.

Finally, we get:

$$\tilde{\boldsymbol{\Sigma}}(w \mid b) = \begin{pmatrix} 4 & 0 & 32 & 0 \\ 0 & 0.0625 & 0 & 0.375 \\ 32 & 0 & 400 & 0 \\ 0 & 0.375 & 0 & 9 \end{pmatrix} \times 10^{-4}$$

and $\sigma(w \mid b) = 1.68\%$ because $\tilde{\boldsymbol{\alpha}} = (57\%, 33\%, 3\%, 7\%)$

Allocation process

Tracking error risk of the core-satellite portfolio

Table 27: Estimation of the tracking error volatility of the core-satellite portfolio (in %)

	$\alpha^{\text{satellite}}$	Bond	Defensive	Balanced	60/40	Dynamic	Equity
Lower bound	10%	0.38	0.62	1.36	1.62	2.15	2.69
	20%	0.63	1.00	2.18	2.60	3.45	4.31
	30%	0.92	1.43	3.11	3.71	4.93	6.16
Upper bound	10%	0.53	1.18	2.16	2.49	3.15	3.80
	20%	0.80	1.76	3.20	3.68	4.64	5.60
	30%	1.07	2.34	4.24	4.87	6.13	7.40

Allocation process

Tracking error risk of the core-satellite portfolio

$$\frac{\partial \alpha^{\text{satellite}}(t)}{\partial t} \geq 0$$