Chapter 11

Copulas and Dependence Modeling

One of the main challenges in risk management is the aggregation of individual risks. We can move the issue aside by assuming that the random variables modeling individual risks are independent or are only dependent by means of a common risk factor. The problem becomes much more involved when one wants to model fully dependent random variables. Again a classic solution is to assume that the vector of individual risks follows a multivariate normal distribution. However, all risks are not likely to be well described by a Gaussian random vector, and the normal distribution may fail to catch some features of the dependence between individual risks.

Copula functions are a statistical tool to solve the previous issue. A copula function is nothing else but the joint distribution of a vector of uniform random variables. Since it is always possible to map any random vector into a vector of uniform random variables, we are able to split the marginals and the dependence between the random variables. Therefore, a copula function represents the statistical dependence between random variables, and generalizes the concept of correlation when the random vector is not Gaussian.

11.1 Canonical representation of multivariate distributions

The concept of copula has been introduced by Sklar in 1959. During a long time, only a small number of people have used copula functions, more in the field of mathematics than this of statistics. The publication of Genest and MacKay (1986b) in the *American Statistician* marks a breakdown and opens areas of study in empirical modeling, statistics and econometrics. In what follows, we intensively use the materials developed in the books of Joe (1997) and Nelsen (2006).

11.1.1 Sklar's theorem

Nelsen (2006) defines a bi-dimensional copula (or a 2-copula) as a function $\mathbf{C}$ which satisfies the following properties:

1. $\operatorname{Dom} \mathbf{C} = [0,1] \times [0,1]$;
2. $\mathbf{C}(0,u) = \mathbf{C}(u,0) = 0$ and $\mathbf{C}(1,u) = \mathbf{C}(u,1) = u$ for all u in $[0,1]$;
3. $\mathbf{C}$ is 2-increasing:

$$\mathbf{C}(v_1,v_2) - \mathbf{C}(v_1,u_2) - \mathbf{C}(u_1,v_2) + \mathbf{C}(u_1,u_2) \geq 0$$

for all $(u_1,u_2) \in [0,1]^2$, $(v_1,v_2) \in [0,1]^2$ such that $0 \leq u_1 \leq v_1 \leq 1$ and $0 \leq u_2 \leq v_2 \leq 1$.

This definition means that $\mathbf{C}$ is a cumulative distribution function with uniform marginals:

$$\mathbf{C}(u_1, u_2) = \Pr\{U_1 \leq u_1, U_2 \leq u_2\}$$

where U_1 and U_2 are two uniform random variables.

Example 108 *Let us consider the function* $\mathbf{C}^{\perp}(u_1, u_2) = u_1 u_2$. *We have* $\mathbf{C}^{\perp}(0, u) = \mathbf{C}^{\perp}(u, 0) = 0$ *and* $\mathbf{C}^{\perp}(1, u) = \mathbf{C}^{\perp}(u, 1) = u$. *Since we have* $v_2 - u_2 \geq 0$ *and* $v_1 \geq u_1$, *it follows that* $v_1(v_2 - u_2) \geq u_1(v_2 - u_2)$ *and* $v_1 v_2 + u_1 u_2 - u_1 v_2 - v_1 u_2 \geq 0$. *We deduce that* $\mathbf{C}^{\perp}$ *is a copula function. It is called the product copula.*

Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be any two univariate distributions. It is obvious that $\mathbf{F}(x_1, x_2) = \mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2))$ is a probability distribution with marginals $\mathbf{F}_1$ and $\mathbf{F}_2$. Indeed, $u_i = \mathbf{F}_i(x_i)$ defines a uniform transformation ($u_i \in [0, 1]$). Moreover, we verify that $\mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(\infty)) = \mathbf{C}(\mathbf{F}_1(x_1), 1) = \mathbf{F}_1(x_1)$. Copulas are then a powerful tool to build a multivariate probability distribution when the marginals are given. Conversely, Sklar (1959) proves that any bivariate distribution $\mathbf{F}$ admits such a representation:

$$\mathbf{F}(x_1, x_2) = \mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)) \tag{11.1}$$

and that the copula $\mathbf{C}$ is unique provided the marginals are continuous. This result is important, because we can associate to each bivariate distribution a copula function. We then obtain a canonical representation of a bivariate probability distribution: on one side, we have the marginals or the univariate directions $\mathbf{F}_1$ and $\mathbf{F}_2$; on the other side, we have the copula $\mathbf{C}$ that links these marginals and gives the dependence between the unidimensional directions.

Example 109 *The Gumbel logistic distribution is the function* $\mathbf{F}(x_1, x_2) = \left(1 + e^{-x_1} + e^{-x_2}\right)^{-1}$ *defined on* $\mathbb{R}^2$. *We notice that the marginals are* $\mathbf{F}_1(x_1) \equiv \mathbf{F}(x_1, \infty) = \left(1 + e^{-x_1}\right)^{-1}$ *and* $\mathbf{F}_2(x_2) \equiv \left(1 + e^{-x_2}\right)^{-1}$. *The quantile functions are then* $\mathbf{F}_1^{-1}(u_1) = \ln u_1 - \ln(1 - u_1)$ *and* $\mathbf{F}_2^{-1}(u_2) = \ln u_2 - \ln(1 - u_2)$. *We finally deduce that:*

$$\mathbf{C}(u_1, u_2) = \mathbf{F}\left(\mathbf{F}_1^{-1}(u_1), \mathbf{F}_2^{-1}(u_2)\right) = \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2}$$

is the Gumbel logistic copula.

11.1.2 Expression of the copula density

If the joint distribution function $\mathbf{F}(x_1, x_2)$ is absolutely continuous, we obtain:

$$\begin{aligned} f(x_1, x_2) &= \partial_{1,2}\, \mathbf{F}(x_1, x_2) \\ &= \partial_{1,2}\, \mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)) \\ &= c(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)) \cdot f_1(x_1) \cdot f_2(x_2) \end{aligned} \tag{11.2}$$

where $f(x_1, x_2)$ is the joint probability density function, f_1 and f_2 are the marginal densities and c is the copula density:

$$c(u_1, u_2) = \partial_{1,2}\, \mathbf{C}(u_1, u_2)$$

We notice that the condition $\mathbf{C}(v_1, v_2) - \mathbf{C}(v_1, u_2) - \mathbf{C}(u_1, v_2) + \mathbf{C}(u_1, u_2) \geq 0$ is then equivalent to $\partial_{1,2}\, \mathbf{C}(u_1, u_2) \geq 0$ when the copula density exists.

Example 110 *In the case of the Gumbel logistic copula, we obtain* $c(u_1, u_2) = 2 u_1 u_2 / (u_1 + u_2 - u_1 u_2)^3$. *We easily verify the 2-increasing property.*

From Equation (11.2), we deduce that:

$$c(u_1,u_2) = \frac{f\left(\mathbf{F}_1^{-1}(u_1), \mathbf{F}_2^{-1}(u_2)\right)}{f_1\left(\mathbf{F}_1^{-1}(u_1)\right) \cdot f_2\left(\mathbf{F}_2^{-1}(u_2)\right)} \tag{11.3}$$

We obtain a second canonical representation based on density functions. For some copulas, there is no explicit analytical formula. This is the case of the Normal copula, which is equal to $\mathbf{C}(u_1,u_2;\rho) = \Phi\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho\right)$. Using Equation (11.3), we can however characterize its density function:

$$\begin{aligned} c(u_1,u_2;\rho) &= \frac{2\pi\left(1-\rho^2\right)^{-1/2}\exp\left(-\frac{1}{2(1-\rho^2)}\left(x_1^2+x_2^2-2\rho x_1 x_2\right)\right)}{(2\pi)^{-1/2}\exp\left(-\frac{1}{2}x_1^2\right)\cdot(2\pi)^{-1/2}\exp\left(-\frac{1}{2}x_2^2\right)} \\ &= \frac{1}{\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2}\frac{\left(x_1^2+x_2^2-2\rho x_1 x_2\right)}{(1-\rho^2)}+\frac{1}{2}\left(x_1^2+x_2^2\right)\right) \end{aligned}$$

where $x_1 = \mathbf{F}_1^{-1}(u_1)$ and $x_2 = \mathbf{F}_2^{-1}(u_2)$. It is then easy to generate bivariate non-normal distributions.

Example 111 *In Figure 11.1, we have built a bivariate probability distribution by considering that the marginals are an inverse Gaussian distribution and a beta distribution. The copula function corresponds to the Normal copula such that its Kendall's tau is equal to 50%.*

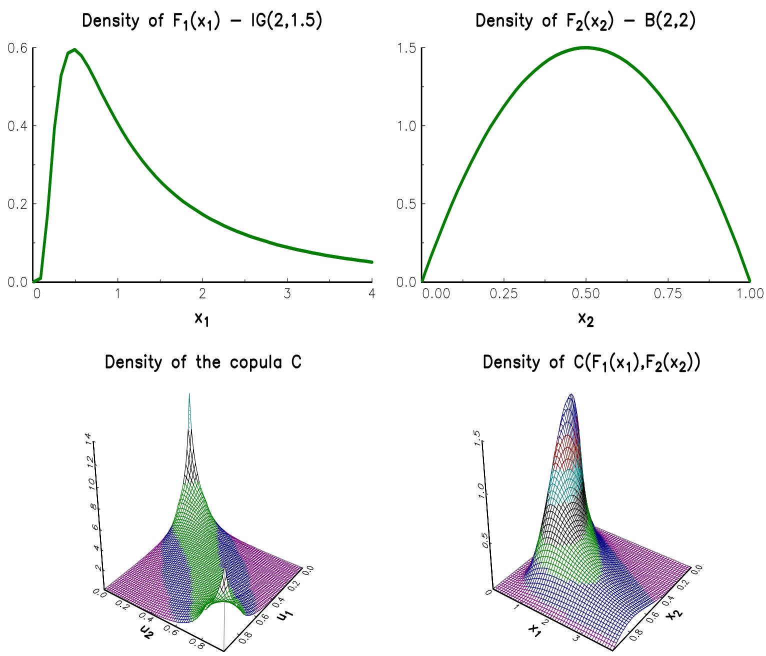

FIGURE 11.1: Example of a bivariate probability distribution with given marginals

11.1.3 Fréchet classes

The goal of Fréchet classes is to study the structure of the class of distributions with given marginals. These latter can be unidimensional, multidimensional or conditional. Let us consider the bivariate distribution functions $\mathbf{F}_{12}$ and $\mathbf{F}_{23}$. The Fréchet class $\mathcal{F}(\mathbf{F}_{12}, \mathbf{F}_{23})$ is the set of trivariate probability distributions that are compatible with the two bivariate marginals $\mathbf{F}_{12}$ and $\mathbf{F}_{23}$. In this handbook, we restrict our focus on the Fréchet class $\mathcal{F}(\mathbf{F}_1, \ldots, \mathbf{F}_n)$ with univariate marginals.

11.1.3.1 The bivariate case

Let us first consider the bivariate case. The distribution function $\mathbf{F}$ belongs to the Fréchet class $(\mathbf{F}_1, \mathbf{F}_2)$ and we note $\mathbf{F} \in \mathcal{F}(\mathbf{F}_1, \mathbf{F}_2)$ if and only if the marginals of $\mathbf{F}$ are $\mathbf{F}_1$ and $\mathbf{F}_2$, meaning that $\mathbf{F}(x_1, \infty) = \mathbf{F}_1(x_1)$ and $\mathbf{F}(\infty, x_2) = \mathbf{F}_2(x_2)$. Characterizing the Fréchet class $\mathcal{F}(\mathbf{F}_1, \mathbf{F}_2)$ is then equivalent to find the set $\mathcal{C}$ of copula functions:

$$\mathcal{F}(\mathbf{F}_1, \mathbf{F}_2) = \{\mathbf{F} : \mathbf{F}(x_1, x_2) = \mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)), \mathbf{C} \in \mathcal{C}\}$$

Therefore this problem does not depend on the marginals $\mathbf{F}_1$ and $\mathbf{F}_2$.

We can show that the extremal distribution functions $\mathbf{F}^-$ and $\mathbf{F}^+$ of the Fréchet class $\mathcal{F}(\mathbf{F}_1, \mathbf{F}_2)$ are:

$$\mathbf{F}^-(x_1, x_2) = \max(\mathbf{F}_1(x_1) + \mathbf{F}_2(x_2) - 1, 0)$$

and:

$$\mathbf{F}^+(x_1, x_2) = \min(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2))$$

$\mathbf{F}^-$ and $\mathbf{F}^+$ are called the Fréchet lower and upper bounds. We deduce that the corresponding copula functions are:

$$\mathbf{C}^-(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$$

and:

$$\mathbf{C}^+(u_1, u_2) = \min(u_1, u_2)$$

Example 112 *We consider the Fréchet class $\mathcal{F}(\mathbf{F}_1, \mathbf{F}_2)$ where $\mathbf{F}_1 \sim \mathcal{N}(0,1)$ and $\mathbf{F}_2 \sim \mathcal{N}(0,1)$. We know that the bivariate normal distribution with correlation ρ belongs to $\mathcal{F}(\mathbf{F}_1, \mathbf{F}_2)$. Nevertheless, a lot of bivariate non-normal distributions are also in this Fréchet class. For instance, this is the case of this probability distribution:*

$$\mathbf{F}(x_1, x_2) = \frac{\Phi(x_1)\Phi(x_2)}{\Phi(x_1) + \Phi(x_2) - \Phi(x_1)\Phi(x_2)}$$

We can also show that[1]*:*

$$\mathbf{F}^-(x_1, x_2) := \Phi(x_1, x_2; -1) = \max(\Phi(x_1) + \Phi(x_2) - 1, 0)$$

and:

$$\mathbf{F}^+(x_1, x_2) := \Phi(x_1, x_2; +1) = \min(\Phi(x_1), \Phi(x_2))$$

Therefore, the bounds of the Fréchet class $\mathcal{F}(\mathcal{N}(0,1), \mathcal{N}(0,1))$ correspond to the bivariate normal distribution, whose correlation is respectively equal to -1 and $+1$.

[1] We recall that:

$$\Phi(x_1, x_2; \rho) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \phi(y_1, y_2; \rho) \, \mathrm{d}y_1 \, \mathrm{d}y_2$$

11.1.3.2 The multivariate case

The extension of bivariate copulas to multivariate copulas is straightforward. Thus, the canonical decomposition of a multivariate distribution function is:

$$\mathbf{F}(x_1,\ldots,x_n)=\mathbf{C}(\mathbf{F}_1(x_1),\ldots,\mathbf{F}_n(x_n))$$

We note $\mathbf{C}_{\mathcal{E}}$ the sub-copula of $\mathbf{C}$ such that arguments that are not in the set $\mathcal{E}$ are equal to 1. For instance, with a dimension of 4, we have $\mathbf{C}_{12}(u,v)=\mathbf{C}(u,v,1,1)$ and $\mathbf{C}_{124}(u,v,w)=\mathbf{C}(u,v,1,w)$. Let us consider the 2-copulas $\mathbf{C}_1$ and $\mathbf{C}_2$. It seems logical to build a copula of higher dimension with copulas of lower dimensions. In fact, the function $\mathbf{C}_1(u_1,\mathbf{C}_2(u_2,u_3))$ is not a copula in most cases (Quesada Molina and Rodríguez Lallena, 1994). For instance, we have:

$$\begin{aligned}\mathbf{C}^-\left(u_1,\mathbf{C}^-(u_2,u_3)\right) &= \max\left(u_1+\max\left(u_2+u_3-1,0\right)-1,0\right)\\ &= \max\left(u_1+u_2+u_3-2,0\right)\\ &= \mathbf{C}^-(u_1,u_2,u_3)\end{aligned}$$

However, the function $\mathbf{C}^-(u_1,u_2,u_3)$ is not a copula.

In the multivariate case, we define:

$$\mathbf{C}^-(u_1,\ldots,u_n)=\max\left(\sum_{i=1}^n u_i-n+1,0\right)$$

and:

$$\mathbf{C}^+(u_1,\ldots,u_n)=\min(u_1,\ldots,u_n)$$

As discussed above, we can show that $\mathbf{C}^+$ is a copula, but $\mathbf{C}^-$ does not belong to the set $\mathcal{C}$. Nevertheless, $\mathbf{C}^-$ is the best-possible bound, meaning that for all $(u_1,\ldots,u_n)\in[0,1]^n$, there is a copula that coincide with $\mathbf{C}^-$ (Nelsen, 2006). This implies that $\mathcal{F}(\mathbf{F}_1,\ldots,\mathbf{F}_n)$ has a minimal distribution function if and only if $\max\left(\sum_{i=1}^n\mathbf{F}_i(x_i)-n+1,0\right)$ is a probability distribution (Dall'Aglio, 1972).

11.1.3.3 Concordance ordering

Using the result of the previous paragraph, we have:

$$\mathbf{C}^-(u_1,u_2)\leq\mathbf{C}(u_1,u_2)\leq\mathbf{C}^+(u_1,u_2)$$

for all $\mathbf{C}\in\mathcal{C}$. For a given value $\alpha\in[0,1]$, the level curves of $\mathbf{C}$ are then in the triangle defined as follows:

$$\left\{(u_1,u_2):\max(u_1+u_2-1,0)\leq\alpha,\min(u_1,u_2)\geq\alpha\right\}$$

An illustration is shown in Figure 11.2. In the multidimensional case, the region becomes a n-volume.

We now introduce a stochastic ordering on copulas. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two copula functions. We say that the copula $\mathbf{C}_1$ is smaller than the copula $\mathbf{C}_2$ and we note $\mathbf{C}_1\prec\mathbf{C}_2$ if we verify that $\mathbf{C}_1(u_1,u_2)\leq\mathbf{C}_2(u_1,u_2)$ for all $(u_1,u_2)\in[0,1]^2$. This stochastic ordering is called the concordance ordering and may be viewed as the first order of the stochastic dominance on probability distributions.

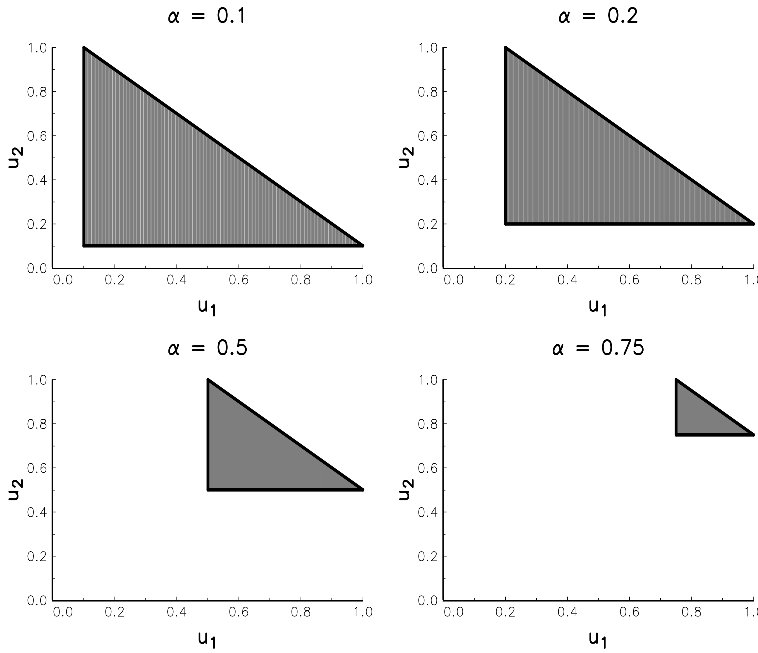

FIGURE 11.2: The triangle region of the contour lines $\mathbf{C}(u_1, u_2) = \alpha$

Example 113 *This ordering is partial because we cannot compare all copula functions. Let us consider the cubic copula defined by* $\mathbf{C}(u_1, u_2; \theta) = u_1 u_2 + \theta [u(u-1)(2u-1)][v(v-1)(2v-1)]$ *where* $\theta \in [-1, 2]$. *If we compare it to the product copula* $\mathbf{C}^{\perp}$, *we have:*

$$\mathbf{C}\left(\frac{3}{4}, \frac{3}{4}; 1\right) = 0.5712 \geq \mathbf{C}^{\perp}\left(\frac{3}{4}, \frac{3}{4}\right) = 0.5625$$

but:

$$\mathbf{C}\left(\frac{3}{4}, \frac{1}{4}; 1\right) = 0.1787 \leq \mathbf{C}^{\perp}\left(\frac{3}{4}, \frac{1}{4}\right) = 0.1875$$

Using the Fréchet bounds, we always have $\mathbf{C}^{-} \prec \mathbf{C}^{\perp} \prec \mathbf{C}^{+}$. A copula $\mathbf{C}$ has a positive quadrant dependence (PQD) if it satisfies the inequality $\mathbf{C}^{\perp} \prec \mathbf{C} \prec \mathbf{C}^{+}$. In a similar way, $\mathbf{C}$ has a negative quadrant dependence (NQD) if it satisfies the inequality $\mathbf{C}^{-} \prec \mathbf{C} \prec \mathbf{C}^{\perp}$. As it is a partial ordering, there exist copula functions $\mathbf{C}$ such that $\mathbf{C} \not\succ \mathbf{C}^{\perp}$ and $\mathbf{C} \not\prec \mathbf{C}^{\perp}$. A copula function may then have a dependence structure that is neither positive or negative. This is the case of the cubic copula given in the previous example. In Figure 11.3, we report the cumulative distribution function (above panel) and its contour lines (right panel) of the three copula functions $\mathbf{C}^{-}$, $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$, which plays an important role to understand the dependance between unidimensional risks.

Let $\mathbf{C}_{\theta}(u_1, u_2) = \mathbf{C}(u_1, u_2; \theta)$ be a family of copula functions that depends on the parameter θ. The copula family $\{\mathbf{C}_{\theta}\}$ is totally ordered if, for all $\theta_2 \geq \theta_1$, $\mathbf{C}_{\theta_2} \succ \mathbf{C}_{\theta_1}$ (positively ordered) or $\mathbf{C}_{\theta_2} \prec \mathbf{C}_{\theta_1}$ (negatively ordered). For instance, the Frank copula defined by:

$$\mathbf{C}(u_1, u_2; \theta) = -\frac{1}{\theta} \ln\left(1 + \frac{\left(e^{-\theta u_1} - 1\right)\left(e^{-\theta u_2} - 1\right)}{e^{-\theta} - 1}\right)$$

where $\theta \in \mathbb{R}$ is a positively ordered family (see Figure 11.4).

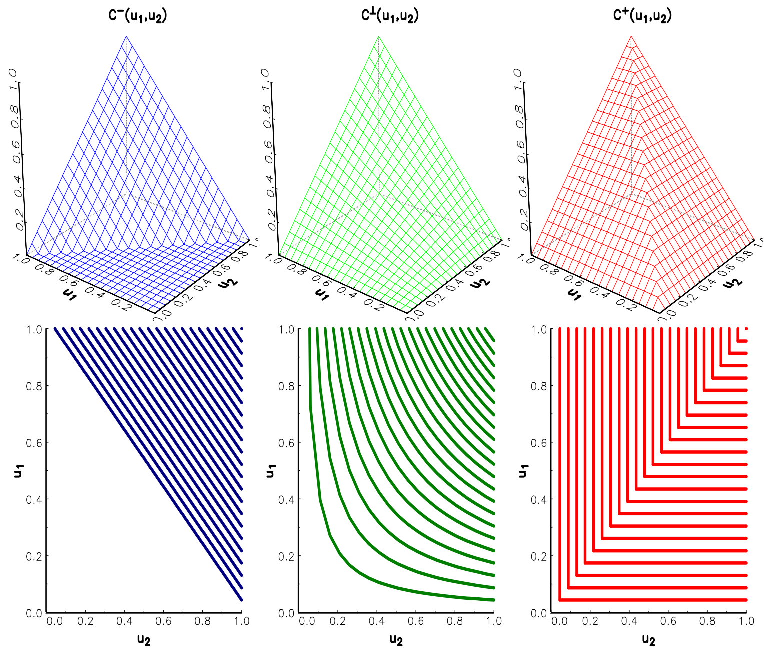

FIGURE 11.3: The three copula functions $\mathbf{C}^-$, $\mathbf{C}^\perp$ and $\mathbf{C}^+$

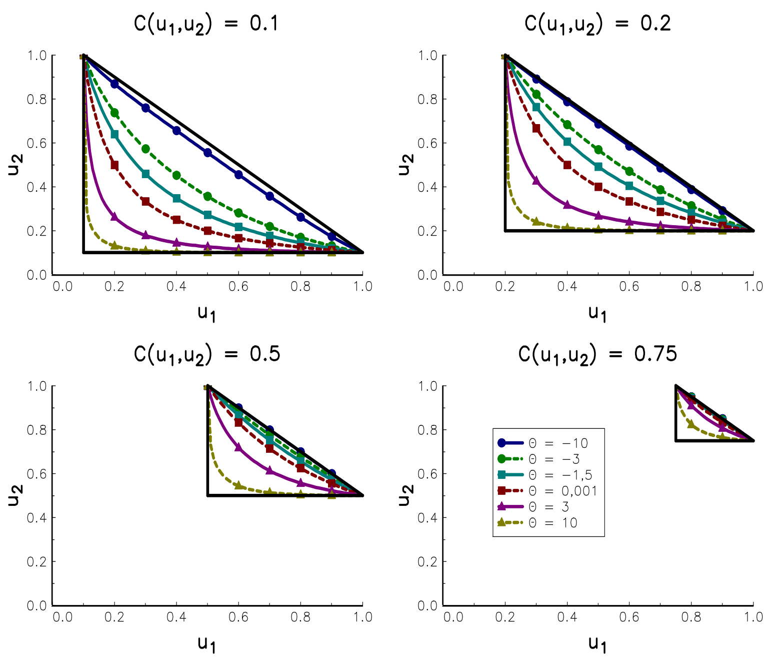

FIGURE 11.4: Concordance ordering of the Frank copula

Example 114 *Let us consider the copula function $\mathbf{C}_\theta = \theta \cdot \mathbf{C}^- + (1-\theta) \cdot \mathbf{C}^+$ where $0 \leq \theta \leq 1$. This copula is a convex sum of the extremal copulas $\mathbf{C}^-$ and $\mathbf{C}^+$. When $\theta_2 \geq \theta_1$, we have:*

$$\begin{aligned}
\mathbf{C}_{\theta_2}(u_1,u_2) &= \theta_2 \cdot \mathbf{C}^-(u_1,u_2) + (1-\theta_2) \cdot \mathbf{C}^+(u_1,u_2) \\
&= \mathbf{C}_{\theta_1}(u_1,u_2) - (\theta_2-\theta_1) \cdot \left(\mathbf{C}^+(u_1,u_2) - \mathbf{C}^-(u_1,u_2)\right) \\
&\leq \mathbf{C}_{\theta_1}(u_1,u_2)
\end{aligned}$$

We deduce that $\mathbf{C}_{\theta_2} \prec \mathbf{C}_{\theta_1}$. This copula family is negatively ordered.

11.2 Copula functions and random vectors

Let $X = (X_1, X_2)$ be a random vector with distribution $\mathbf{F}$. We define the copula of (X_1, X_2) by the copula of $\mathbf{F}$:

$$\mathbf{F}(x_1,x_2) = \mathbf{C}\langle X_1, X_2\rangle (\mathbf{F}_1(x_1), \mathbf{F}_2(x_2))$$

In what follows, we give the main results on the dependence of the random vector X found in Deheuvels (1978), Schweizer and Wolff (1981), and Nelsen (2006).

11.2.1 Countermonotonicity, comonotonicity and scale invariance property

We give here a probabilistic interpretation of the three copula functions $\mathbf{C}^-$, $\mathbf{C}^\perp$ and $\mathbf{C}^+$:

- X_1 and X_2 are countermonotonic – or $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^-$ – if there exists a random variable X such that $X_1 = f_1(X)$ and $X_2 = f_2(X)$ where f_1 and f_2 are respectively decreasing and increasing functions[2];
- X_1 and X_2 are independent if the dependence function is the product copula $\mathbf{C}^\perp$;
- X_1 are X_2 are comonotonic – or $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^+$ – if there exists a random variable X such that $X_1 = f_1(X)$ and $X_2 = f_2(X)$ where f_1 and f_2 are both increasing functions[3].

Let us consider a uniform random vector (U_1, U_2). We have $U_2 = 1 - U_1$ when $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^-$ and $U_2 = U_1$ when $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^+$. In the case of a standardized Gaussian random vector, we obtain $X_2 = -X_1$ when $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^-$ and $X_2 = X_1$ when $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^+$. If the marginals are log-normal, it follows that $X_2 = X_1^{-1}$ when $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^-$ and $X_2 = X_1$ when $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^+$. For these three examples, we verify that X_2 is a decreasing (resp. increasing) function of X_1 if the copula function $\mathbf{C}\langle X_1, X_2\rangle$ is $\mathbf{C}^-$ (resp. $\mathbf{C}^+$). The concepts of counter- and comonotonicity concepts generalize the cases where the linear correlation of a Gaussian vector is equal to -1 or $+1$. Indeed, $\mathbf{C}^-$ and $\mathbf{C}^+$ define respectively perfect negative and positive dependence.

[2] We also have $X_2 = f(X_1)$ where $f = f_2 \circ f_1^{-1}$ is a decreasing function.
[3] In this case, $X_2 = f(X_1)$ where $f = f_2 \circ f_1^{-1}$ is an increasing function.

We now give one of the most important theorems on copulas. Let (X_1, X_2) be a random vectors, whose copula is $\mathbf{C}\langle X_1, X_2\rangle$. If h_1 and h_2 are two increasing functions on $\operatorname{Im} X_1$ and $\operatorname{Im} X_2$, then we have:

$$\mathbf{C}\langle h_1(X_1), h_2(X_2)\rangle = \mathbf{C}\langle X_1, X_2\rangle$$

This means that copula functions are invariant under strictly increasing transformations of the random variables. To prove this theorem, we note $\mathbf{F}$ and $\mathbf{G}$ the probability distributions of the random vectors (X_1, X_2) and $(Y_1, Y_2) = (h_1(X_1), h_2(X_2))$. The marginals of $\mathbf{G}$ are:

$$\begin{aligned}
\mathbf{G}_1(y_1) &= \Pr\{Y_1 \leq y_1\} \\
&= \Pr\{h_1(X_1) \leq y_1\} \\
&= \Pr\left\{X_1 \leq h_1^{-1}(y_1)\right\} \quad \text{(because } h_1 \text{ is strictly increasing)} \\
&= \mathbf{F}_1\left(h_1^{-1}(y_1)\right)
\end{aligned}$$

and $\mathbf{G}_2(y_2) = \mathbf{F}_2\left(h_2^{-1}(y_2)\right)$. We deduce that $\mathbf{G}_1^{-1}(u_1) = h_1\left(\mathbf{F}_1^{-1}(u_1)\right)$ and $\mathbf{G}_2^{-1}(u_2) = h_2\left(\mathbf{F}_2^{-1}(u_2)\right)$. By definition, we have:

$$\mathbf{C}\langle Y_1, Y_2\rangle(u_1, u_2) = \mathbf{G}\left(\mathbf{G}_1^{-1}(u_1), \mathbf{G}_2^{-1}(u_2)\right)$$

Moreover, it follows that:

$$\begin{aligned}
\mathbf{G}\left(\mathbf{G}_1^{-1}(u_1), \mathbf{G}_2^{-1}(u_2)\right) &= \Pr\left\{Y_1 \leq \mathbf{G}_1^{-1}(u_1), Y_2 \leq \mathbf{G}_2^{-1}(u_2)\right\} \\
&= \Pr\left\{h_1(X_1) \leq \mathbf{G}_1^{-1}(u_1), h_2(X_2) \leq \mathbf{G}_2^{-1}(u_2)\right\} \\
&= \Pr\left\{X_1 \leq h_1^{-1}\left(\mathbf{G}_1^{-1}(u_1)\right), X_2 \leq h_2^{-1}\left(\mathbf{G}_2^{-1}(u_2)\right)\right\} \\
&= \Pr\left\{X_1 \leq \mathbf{F}_1^{-1}(u_1), X_2 \leq \mathbf{F}_2^{-1}(u_2)\right\} \\
&= \mathbf{F}\left(\mathbf{F}_1^{-1}(u_1), \mathbf{F}_2^{-1}(u_2)\right)
\end{aligned}$$

Because we have $\mathbf{C}\langle X_1, X_2\rangle(u_1, u_2) = \mathbf{F}\left(\mathbf{F}_1^{-1}(u_1), \mathbf{F}_2^{-1}(u_2)\right)$, we deduce that $\mathbf{C}\langle Y_1, Y_2\rangle = \mathbf{C}\langle X_1, X_2\rangle$.

Example 115 *If X_1 and X_2 are two positive random variables, the previous theorem implies that:*

$$\begin{aligned}
\mathbf{C}\langle X_1, X_2\rangle &= \mathbf{C}\langle \ln X_1, X_2\rangle \\
&= \mathbf{C}\langle \ln X_1, \ln X_2\rangle \\
&= \mathbf{C}\langle X_1, \exp X_2\rangle \\
&= \mathbf{C}\left\langle \sqrt{X_1}, \exp X_2\right\rangle
\end{aligned}$$

Applying an increasing transformation does not change the copula function, only the marginals. Thus, the copula of the multivariate log-normal distribution is the same than the copula of the multivariate normal distribution.

The scale invariance property is perhaps not surprising if we consider the canonical decomposition of the bivariate probability distribution. Indeed, the copula $\mathbf{C}\langle U_1, U_2\rangle$ is equal to the copula $\mathbf{C}\langle X_1, X_2\rangle$ where $U_1 = \mathbf{F}_1(X_1)$ and $U_2 = \mathbf{F}_2(X_2)$. In some sense, Sklar's theorem is an application of the scale invariance property by considering $h_1(x_1) = \mathbf{F}_1(x_1)$ and $h_2(x_2) = \mathbf{F}_2(x_2)$.

Example 116 *We assume that $X_1 \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right)$ and $X_2 \sim \mathcal{N}\left(\mu_2, \sigma_2^2\right)$. If the copula of (X_1, X_2) is $\mathbf{C}^-$, we have $U_2 = 1 - U_1$. This implies that:*

$$\begin{aligned}
\Phi\left(\frac{X_2 - \mu_2}{\sigma_2}\right) &= 1 - \Phi\left(\frac{X_1 - \mu_1}{\sigma_1}\right) \\
&= \Phi\left(-\frac{X_1 - \mu_1}{\sigma_1}\right)
\end{aligned}$$

We deduce that X_1 and X_2 are countermonotonic if:

$$X_2 = \mu_2 - \frac{\sigma_2}{\sigma_1}(X_1 - \mu_1)$$

By applying the same reasoning to the copula function $\mathbf{C}^+$, we show that X_1 and X_2 are comonotonic if:

$$X_2 = \mu_2 + \frac{\sigma_2}{\sigma_1}(X_1 - \mu_1)$$

We now consider the log-normal random variables $Y_1 = \exp(X_1)$ and $Y_2 = \exp(X_2)$. For the countermonotonicity case, we obtain:

$$\ln Y_2 = \mu_2 - \frac{\sigma_2}{\sigma_1}(\ln Y_1 - \mu_1)$$

or:

$$Y_2 = \exp\left(\mu_2 + \frac{\sigma_2}{\sigma_1}\mu_1\right) \cdot Y_1^{-\sigma_2/\sigma_1}$$

For the comonotonicity case, the relationship becomes:

$$Y_2 = \exp\left(\mu_2 - \frac{\sigma_2}{\sigma_1}\mu_1\right) \cdot Y_1^{\sigma_2/\sigma_1}$$

If we assume that $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2$, the log-normal random variables Y_1 and Y_2 are countermonotonic if $Y_2 = Y_1^{-1}$ and comonotonic if $Y_2 = Y_1$.

11.2.2 Dependence measures

We can interpret the copula function $\mathbf{C}\langle X_1, X_2\rangle$ as a standardization of the joint distribution after eliminating the effects of marginals. Indeed, it is a comprehensive statistic of the dependence function between X_1 and X_2. Therefore, a non-comprehensive statistic will be a dependence measure if it can be expressed using $\mathbf{C}\langle X_1, X_2\rangle$.

11.2.2.1 Concordance measures

Following Nelsen (2006), a numeric measure m of association between X_1 and X_2 is a measure of concordance if it satisfies the following properties:

1. $-1 = m\langle X, -X\rangle \leq m\langle \mathbf{C}\rangle \leq m\langle X, X\rangle = 1$;
2. $m\left\langle \mathbf{C}^\perp \right\rangle = 0$;
3. $m\langle -X_1, X_2\rangle = m\langle X_1, -X_2\rangle = -m\langle X_1, X_2\rangle$;
4. if $\mathbf{C}_1 \prec \mathbf{C}_2$, then $m\langle \mathbf{C}_1\rangle \leq m\langle \mathbf{C}_2\rangle$;

Using this last property, we have: $\mathbf{C} \prec \mathbf{C}^\perp \Rightarrow m\langle \mathbf{C}\rangle < 0$ and $\mathbf{C} \succ \mathbf{C}^\perp \Rightarrow m\langle \mathbf{C}\rangle > 0$. The concordance measure can then be viewed as a generalization of the linear correlation when the dependence function is not normal. Indeed, a positive quadrant dependence (PQD) copula will have a positive concordance measure whereas a negative quadrant dependence (NQD) copula will have a negative concordance measure. Moreover, the bounds -1 and $+1$ are reached when the copula function is countermonotonic and comonotonic.

Among the several concordance measures, we find Kendall's tau and Spearman's rho, which play an important role in non-parametric statistics. Let us consider a sample of n observations $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ of the random vector (X, Y). Kendall's tau is the probability of concordance – $(X_i - X_j) \cdot (Y_i - Y_j) > 0$ – minus the probability of discordance – $(X_i - X_j) \cdot (Y_i - Y_j) < 0$:

$$\tau = \Pr\{(X_i - X_j) \cdot (Y_i - Y_j) > 0\} - \Pr\{(X_i - X_j) \cdot (Y_i - Y_j) < 0\}$$

Spearman's rho is the linear correlation of the rank statistics $(X_{i:n}, Y_{i:n})$. We can also show that Spearman's rho has the following expression:

$$\varrho = \frac{\operatorname{cov}\left(\mathbf{F}_X(X), \mathbf{F}_Y(Y)\right)}{\sigma\left(\mathbf{F}_X(X)\right) \cdot \sigma\left(\mathbf{F}_Y(Y)\right)}$$

Schweizer and Wolff (1981) showed that Kendall's tau and Spearman's rho are concordance measures and have the following expressions:

$$\begin{aligned}
\tau &= 4 \iint_{[0,1]^2} \mathbf{C}(u_1, u_2)\, \mathrm{d}\mathbf{C}(u_1, u_2) - 1 \\
\varrho &= 12 \iint_{[0,1]^2} u_1 u_2\, \mathrm{d}\mathbf{C}(u_1, u_2) - 3
\end{aligned}$$

From a numerical point of view, the following formulas should be preferred (Nelsen, 2006):

$$\begin{aligned}
\tau &= 1 - 4 \iint_{[0,1]^2} \partial_{u_1}\mathbf{C}(u_1, u_2)\, \partial_{u_2}\mathbf{C}(u_1, u_2)\, \mathrm{d}u_1\, \mathrm{d}u_2 \\
\varrho &= 12 \iint_{[0,1]^2} \mathbf{C}(u_1, u_2)\, \mathrm{d}u_1\, \mathrm{d}u_2 - 3
\end{aligned}$$

For some copulas, we have analytical formulas. For instance, we have:

Copula	ϱ	τ
Normal	$6\pi^{-1} \arcsin(\rho/2)$	$2\pi^{-1} \arcsin(\rho)$
Gumbel	✓	$(\theta - 1)/\theta$
FGM	$\theta/3$	$2\theta/9$
Frank	$1 - 12\theta^{-1}(\mathbf{D}_1(\theta) - \mathbf{D}_2(\theta))$	$1 - 4\theta^{-1}(1 - \mathbf{D}_1(\theta))$

where $\mathbf{D}_k(x)$ is the Debye function. The Gumbel (or Gumbel-Hougaard) copula is equal to:

$$\mathbf{C}(u_1, u_2; \theta) = \exp\left(-\left[(-\ln u_1)^\theta + (-\ln u_2)^\theta\right]^{1/\theta}\right)$$

for $\theta \geq 1$, whereas the expression of the Farlie-Gumbel-Morgenstern (or FGM) copula is:

$$\mathbf{C}(u_1, u_2; \theta) = u_1 u_2 (1 + \theta(1 - u_1)(1 - u_2))$$

for $-1 \leq \theta \leq 1$.

For illustration, we report in Figures 11.5, 11.6 and 11.7 the level curves of several density functions built with Normal, Frank and Gumbel copulas. In order to compare them, the parameter of each copula is calibrated such that Kendall's tau is equal to 50%. This means that these 12 distributions functions have the same dependence with respect to Kendall's tau. However, the dependence is different from one figure to another, because their copula function is not the same. This is why Kendall's tau is not an exhaustive statistic of the dependence between two random variables.

We could build bivariate probability distributions, which are even less comparable. Indeed, the set of these three copula families (Normal, Frank and Gumbel) is very small

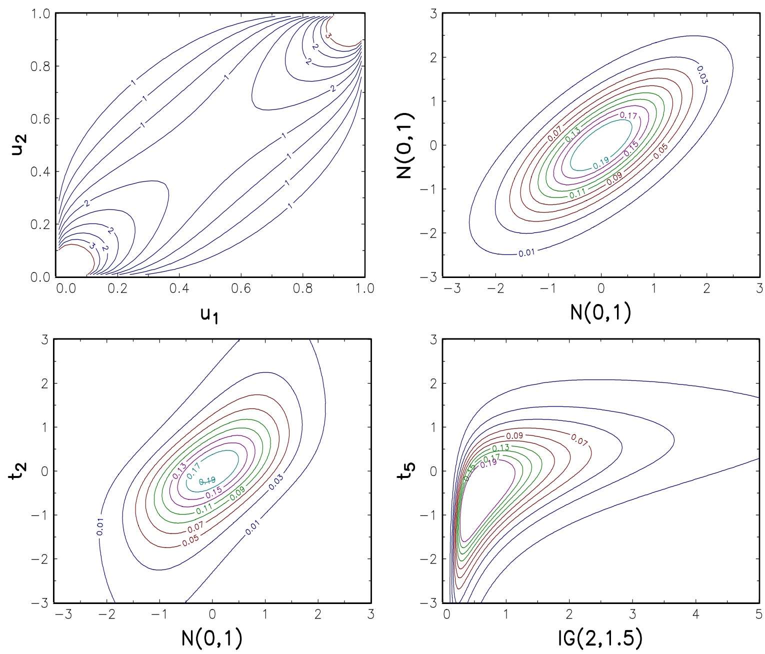

FIGURE 11.5: Contour lines of bivariate densities (Normal copula)

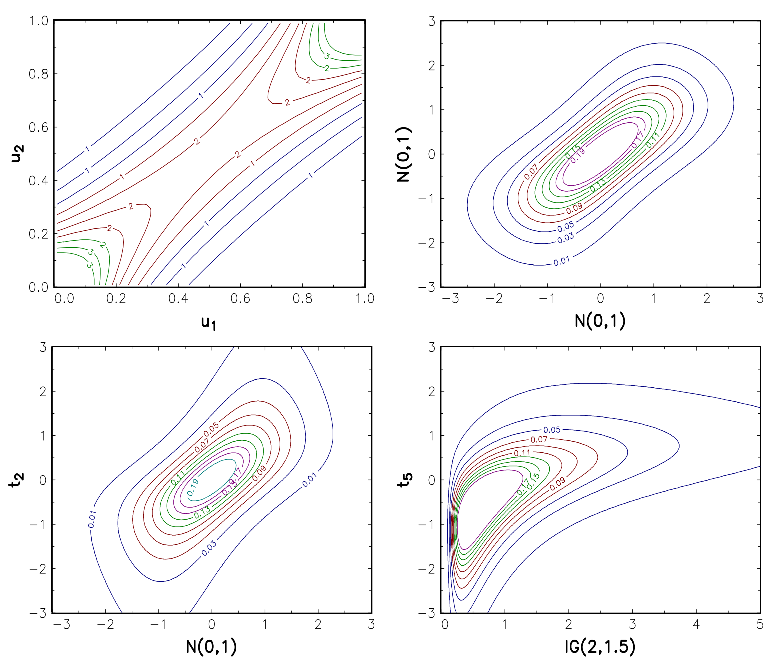

FIGURE 11.6: Contour lines of bivariate densities (Frank copula)

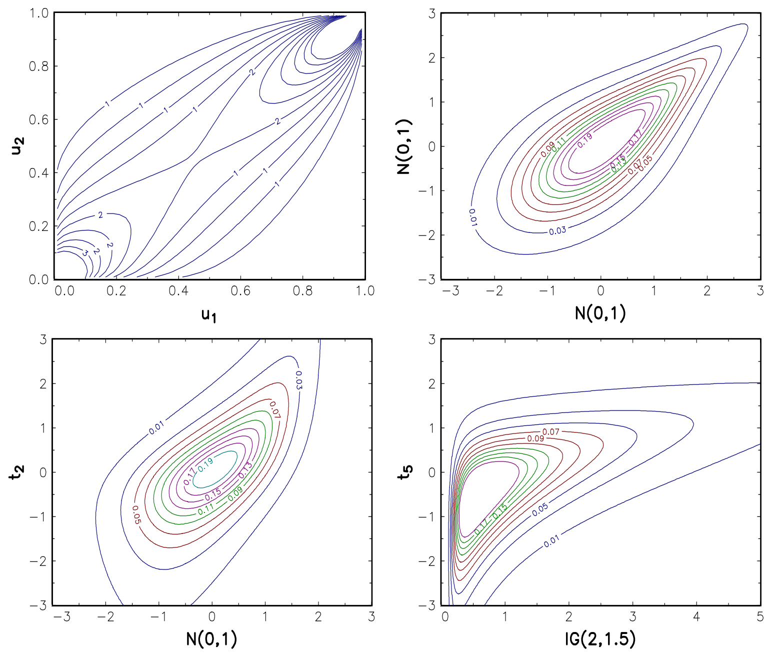

FIGURE 11.7: Contour lines of bivariate densities (Gumbel copula)

compared to the set $\mathcal{C}$ of copulas. However, there exist other dependence functions that are very far from the previous copulas. For instance, we consider the region $\mathcal{B}(\tau, \varrho)$ defined by:

$$(\tau, \varrho) \in \mathcal{B}(\tau, \varrho) \Leftrightarrow \begin{cases} (3\tau - 1)/2 \leq \varrho \leq \left(1 + 2\tau - \tau^2\right)/2 & \text{if} \quad \tau \geq 0 \\ \left(\tau^2 + 2\tau - 1\right)/2 \leq \varrho \leq (1 + 3\tau)/2 & \text{if} \quad \tau \leq 0 \end{cases}$$

Nelsen (2006) shows that these bounds cannot be improved and there is always a copula function that corresponds to a point of the boundary $\mathcal{B}(\tau, \varrho)$. In Figure 11.8 we report the bounds $\mathcal{B}(\tau, \varrho)$ and the area reached by 8 copula families (Normal, Plackett, Frank, Clayton, Gumbel, Galambos, Hüsler-Reiss, FGM). These copulas covered a small surface of the $\tau - \varrho$ region. These copula families are then relatively similar if we consider these concordance measures. Obtaining copulas that have a different behavior requires that the dependence is not monotone[4] on the whole domain $[0, 1]^2$.

11.2.2.2 Linear correlation

We recall that the linear correlation (or Pearson's correlation) is defined as follows:

$$\rho \langle X_1, X_2 \rangle = \frac{\mathbb{E}[X_1 \cdot X_2] - \mathbb{E}[X_1] \cdot \mathbb{E}[X_2]}{\sigma(X_1) \cdot \sigma(X_2)}$$

Tchen (1980) showed the following properties of this measure:

- if the dependence of the random vector (X_1, X_2) is the product copula $\mathbf{C}^{\perp}$, then $\rho \langle X_1, X_2 \rangle = 0$;
- ρ is an increasing function with respect to the concordance measure:

$$\mathbf{C}_1 \succ \mathbf{C}_2 \Rightarrow \rho_1 \langle X_1, X_2 \rangle \geq \rho_2 \langle X_1, X_2 \rangle$$

[4]For instance, the dependence can be positive in one region and negative in another region.

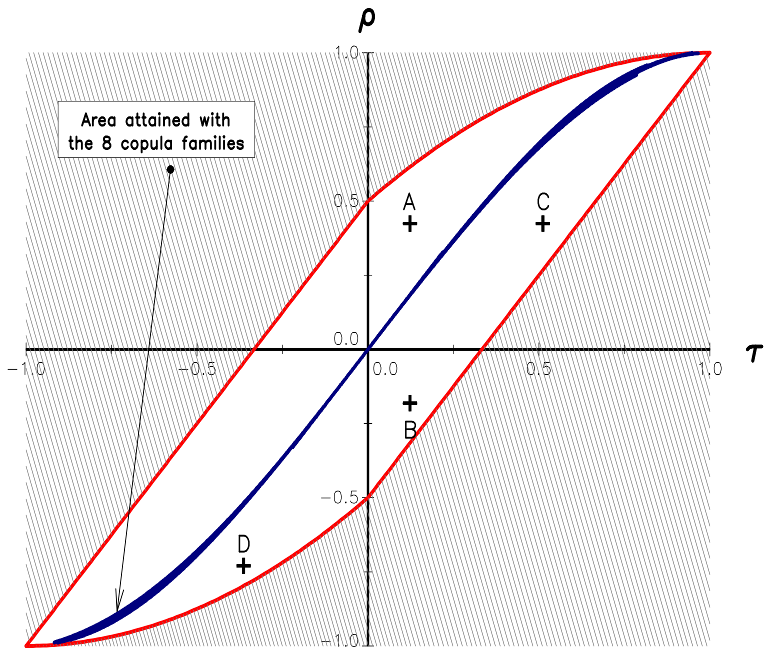

FIGURE 11.8: Bounds of (τ, ϱ) statistics

- $\rho \langle X_1, X_2 \rangle$ is bounded:

$$\rho^- \langle X_1, X_2 \rangle \leq \rho \langle X_1, X_2 \rangle \leq \rho^+ \langle X_1, X_2 \rangle$$

  and the bounds are reached for the Fréchet copulas $\mathbf{C}^-$ and $\mathbf{C}^+$.

However, a concordance measure must satisfy $m \langle \mathbf{C}^- \rangle = -1$ and $m \langle \mathbf{C}^+ \rangle = +1$. If we use the stochastic representation of Fréchet bounds, we have:

$$\rho^- \langle X_1, X_2 \rangle = \rho^+ \langle X_1, X_2 \rangle = \frac{\mathbb{E}\left[f_1(X) \cdot f_2(X)\right] - \mathbb{E}\left[f_1(X)\right] \cdot \mathbb{E}\left[f_2(X)\right]}{\sigma\left(f_1(X)\right) \cdot \sigma\left(f_2(X)\right)}$$

The solution of the equation $\rho^- \langle X_1, X_2 \rangle = -1$ is $f_1(x) = a_1 x + b_1$ and $f_2(x) = a_2 x + b_2$ where $a_1 a_2 < 0$. For the equation $\rho^+ \langle X_1, X_2 \rangle = +1$, the condition becomes $a_1 a_2 > 0$. Except for Gaussian random variables, there are few probability distributions that can satisfy these conditions. Moreover, if the linear correlation is a concordance measure, it is an invariant measure by increasing transformations:

$$\rho \langle X_1, X_2 \rangle = \rho \langle f_1(X_1), f_2(X_2) \rangle$$

Again, the solution of this equation is $f_1(x) = a_1 x + b_1$ and $f_2(x) = a_2 x + b_2$ where $a_1 a_2 > 0$. We now have a better understanding why we say that this dependence measure is linear. In summary, the copula function generalizes the concept of linear correlation in a non-Gaussian non-linear world.

Example 117 *We consider the bivariate log-normal random vector* (X_1, X_2) *where* $X_1 \sim \mathcal{LN}\left(\mu_1, \sigma_1^2\right)$*,* $X_2 \sim \mathcal{LN}\left(\mu_2, \sigma_2^2\right)$ *and* $\rho = \rho \langle \ln X_1, \ln X_2 \rangle$*.*

We can show that:

$$\mathbb{E}\left[X_1^{p_1} \cdot X_2^{p_2}\right] = \exp\left(p_1 \mu_1 + p_2 \mu_2 + \frac{p_1^2 \sigma_1^2 + p_2^2 \sigma_2^2}{2} + p_1 p_2 \rho \sigma_1 \sigma_2\right)$$

It follows that:

$$\rho\langle X_1, X_2\rangle = \frac{\exp(\rho\sigma_1\sigma_2) - 1}{\sqrt{\exp(\sigma_1^2) - 1} \cdot \sqrt{\exp(\sigma_2^2) - 1}}$$

We deduce that $\rho\langle X_1, X_2\rangle \in [\rho^-, \rho^+]$, but the bounds are not necessarily -1 and $+1$. For instance, when we use the parameters $\sigma_1 = 1$ and $\sigma_2 = 3$, we obtain the following results:

Copula	$\rho\langle X_1, X_2\rangle$	$\tau\langle X_1, X_2\rangle$	$\varrho\langle X_1, X_2\rangle$
$\mathbf{C}^-$	-0.008	-1.000	-1.000
$\rho = -0.7$	-0.007	-0.494	-0.683
$\mathbf{C}^\perp$	0.000	0.000	0.000
$\rho = 0.7$	0.061	0.494	0.683
$\mathbf{C}^+$	0.162	1.000	1.000

When the copula function is $\mathbf{C}^-$, the linear correlation takes a value close to zero! In Figure 11.9, we show that the bounds ρ^- and ρ^+ of $\rho\langle X_1, X_2\rangle$ are not necessarily -1 and $+1$. When the marginals are log-normal, the upper bound $\rho^+ = +1$ is reached only when $\sigma_1 = \sigma_2$ and the lower bound $\rho^- = -1$ is never reached. This poses a problem to interpret the value of a correlation. Let us consider two random vectors (X_1, X_2) and (Y_1, Y_2). What could we say about the dependence function when $\rho\langle X_1, X_2\rangle \geq \rho\langle Y_1, Y_2\rangle$? The answer is nothing if the marginals are not Gaussian. Indeed, we have seen previously that a 70% linear correlation between two Gaussian random vectors becomes a 6% linear correlation if we apply an exponential transformation. However, the two copulas of (X_1, X_2) and (Y_1, Y_2) are exactly the same. In fact, the drawback of the linear correlation is that this measure depends on the marginals and not only on the copula function.

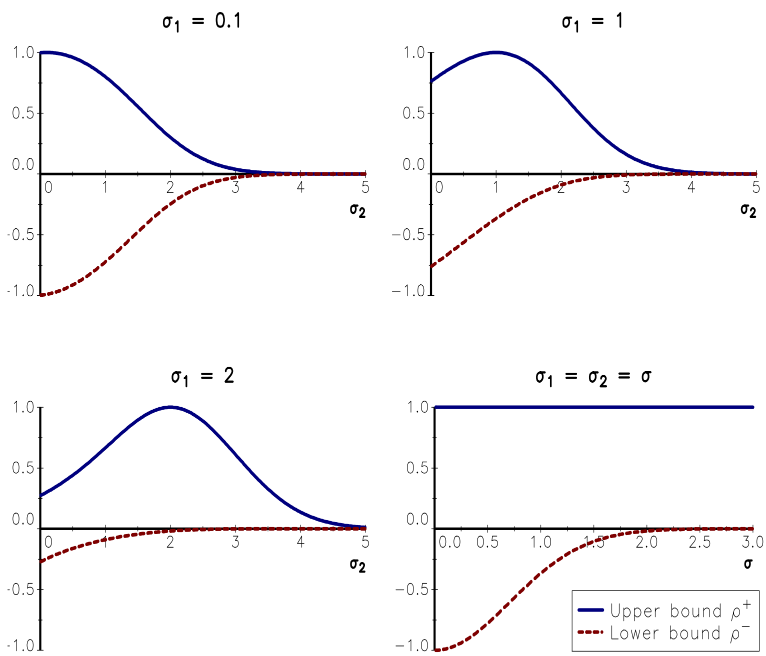

FIGURE 11.9: Bounds of the linear correlation between two log-normal random variables

11.2.2.3 Tail dependence

Contrary to concordance measures, tail dependence is a local measure that characterizes the joint behavior of the random variables X_1 and X_2 at the extreme points $x^- = \inf \{x : \mathbf{F}(x) > 0\}$ and $x^+ = \sup \{x : \mathbf{F}(x) < 1\}$. Let $\mathbf{C}$ be a copula function such that the following limit exists:

$$\lambda^+ = \lim_{u \to 1^-} \frac{1 - 2u + \mathbf{C}(u, u)}{1 - u}$$

We say that $\mathbf{C}$ has an upper tail dependence when $\lambda^+ \in (0, 1]$ and $\mathbf{C}$ has no upper tail dependence when $\lambda^+ = 0$ (Joe, 1997). For the lower tail dependence λ^-, the limit becomes:

$$\lambda^- = \lim_{u \to 0^+} \frac{\mathbf{C}(u, u)}{u}$$

We notice that λ^+ and λ^- can also be defined as follows:

$$\lambda^+ = \lim_{u \to 1^-} \Pr \{U_2 > u \mid U_1 > u\}$$

and:

$$\lambda^- = \lim_{u \to 0^+} \Pr \{U_2 < u \mid U_1 < u\}$$

To compute the upper tail dependence, we consider the joint survival function $\bar{\mathbf{C}}$ defined by:

$$\begin{aligned}
\bar{\mathbf{C}}(u_1, u_2) &= \Pr \{U_1 > u_1, U_2 > u_2\} \\
&= 1 - u_1 - u_2 + \mathbf{C}(u_1, u_2)
\end{aligned}$$

The expression of the upper tail dependence is then equal to:

$$\begin{aligned}
\lambda^+ &= \lim_{u \to 1^-} \frac{\bar{\mathbf{C}}(u, u)}{1 - u} \\
&= -\lim_{u \to 1^-} \frac{\mathrm{d}\bar{\mathbf{C}}(u, u)}{\mathrm{d}u} \\
&= -\lim_{u \to 1^-} \left(-2 + \partial_1 \mathbf{C}(u, u) + \partial_2 \mathbf{C}(u, u)\right) \\
&= \lim_{u \to 1^-} \left(\Pr \{U_2 > u \mid U_1 = u\} + \Pr \{U_1 > u \mid U_2 = u\}\right)
\end{aligned}$$

By assuming that the copula is symmetric, we finally obtain:

$$\begin{aligned}
\lambda^+ &= 2 \lim_{u \to 1^-} \Pr \{U_2 > u \mid U_1 = u\} \\
&= 2 - 2 \lim_{u \to 1^-} \Pr \{U_2 < u \mid U_1 = u\} \\
&= 2 - 2 \lim_{u \to 1^-} \mathbf{C}_{2|1}(u, u)
\end{aligned} \tag{11.4}$$

In a similar way, we find that the lower tail dependence of a symmetric copula is equal to:

$$\lambda^- = 2 \lim_{u \to 0^+} \mathbf{C}_{2|1}(u, u) \tag{11.5}$$

For the copula functions $\mathbf{C}^-$ and $\mathbf{C}^\perp$, we have $\lambda^- = \lambda^+ = 0$. For the copula $\mathbf{C}^+$, we obtain $\lambda^- = \lambda^+ = 1$. However, there exist copulas such that $\lambda^- \neq \lambda^+$. This is the case of the

Gumbel copula $\mathbf{C}(u_1, u_2; \theta) = \exp\left(-\left[(-\ln u_1)^\theta + (-\ln u_2)^\theta\right]^{1/\theta}\right)$, because we have $\lambda^- = 0$ and $\lambda^+ = 2 - 2^{1/\theta}$. The Gumbel copula has then an upper tail dependence, but no lower tail dependence. If we consider the Clayton copula $\mathbf{C}(u_1, u_2; \theta) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}$, we obtain $\lambda^- = 2^{-1/\theta}$ and $\lambda^+ = 0$.

Coles *et al.* (1999) define the quantile-quantile dependence function as follows:

$$\lambda^+(\alpha) = \Pr\left\{X_2 > \mathbf{F}_2^{-1}(\alpha) \mid X_1 > \mathbf{F}_1^{-1}(\alpha)\right\}$$

It is the conditional probability that X_2 is larger than the quantile $\mathbf{F}_2^{-1}(\alpha)$ given that X_1 is larger than the quantile $\mathbf{F}_1^{-1}(\alpha)$. We have:

$$\begin{aligned}
\lambda^+(\alpha) &= \Pr\left\{X_2 > \mathbf{F}_2^{-1}(\alpha) \mid X_1 > \mathbf{F}_1^{-1}(\alpha)\right\} \\
&= \frac{\Pr\left\{X_2 > \mathbf{F}_2^{-1}(\alpha), X_1 > \mathbf{F}_1^{-1}(\alpha)\right\}}{\Pr\left\{X_1 > \mathbf{F}_1^{-1}(\alpha)\right\}} \\
&= \frac{1 - \Pr\left\{X_1 \leq \mathbf{F}_1^{-1}(\alpha)\right\} - \Pr\left\{X_2 \leq \mathbf{F}_2^{-1}(\alpha)\right\}}{1 - \Pr\left\{X_1 \leq \mathbf{F}_1^{-1}(\alpha)\right\}} + \\
&\quad \frac{\Pr\left\{X_2 \leq \mathbf{F}_2^{-1}(\alpha), X_1 \leq \mathbf{F}_1^{-1}(\alpha)\right\}}{1 - \Pr\left\{\mathbf{F}_1(X_1) \leq \alpha\right\}} \\
&= \frac{1 - 2\alpha + \mathbf{C}(\alpha, \alpha)}{1 - \alpha}
\end{aligned}$$

The tail dependence λ^+ is then the limit of the conditional probability $\lambda^+(\alpha)$ when the confidence level α tends to 1. It is also the probability of one variable being extreme given that the other is extreme. Because $\lambda^+(\alpha)$ is a probability, we verify that $\lambda^+ \in [0, 1]$. If the probability is zero, the extremes are independent. If λ^+ is equal to 1, the extremes are perfectly dependent. To illustrate the measures[5] $\lambda^+(\alpha)$ and $\lambda^-(\alpha)$, we represent their values for the Gumbel and Clayton copulas in Figure 11.10. The parameters are calibrated with respect to Kendall's tau.

Remark 138 *We consider two portfolios, whose losses correspond to the random variables L_1 and L_2 with probability distributions $\mathbf{F}_1$ and $\mathbf{F}_2$. The probability that the loss of the second portfolio is larger than its value-at-risk knowing that the value-at-risk of the first portfolio is exceeded is exactly equal to the quantile-quantile dependence measure $\lambda^+(\alpha)$:*

$$\begin{aligned}
\lambda^+(\alpha) &= \Pr\left\{L_2 > \mathbf{F}_2^{-1}(\alpha) \mid L_1 > \mathbf{F}_1^{-1}(\alpha)\right\} \\
&= \Pr\left\{L_2 > \mathrm{VaR}_\alpha(L_2) \mid L_1 > \mathrm{VaR}_\alpha(L_1)\right\}
\end{aligned}$$

11.3 Parametric copula functions

In this section, we study the copula families, which are commonly used in risk management. They are parametric copulas, which depend on a set of parameters. Statistical inference, in particular parameter estimation, is developed in the next section.

[5] We have $\lambda^-(\alpha) = \Pr\left\{X_2 < \mathbf{F}_2^{-1}(\alpha) \mid X_1 < \mathbf{F}_1^{-1}(\alpha)\right\}$ and $\lim_{\alpha \to 0} \lambda^-(\alpha) = \lambda^-$.

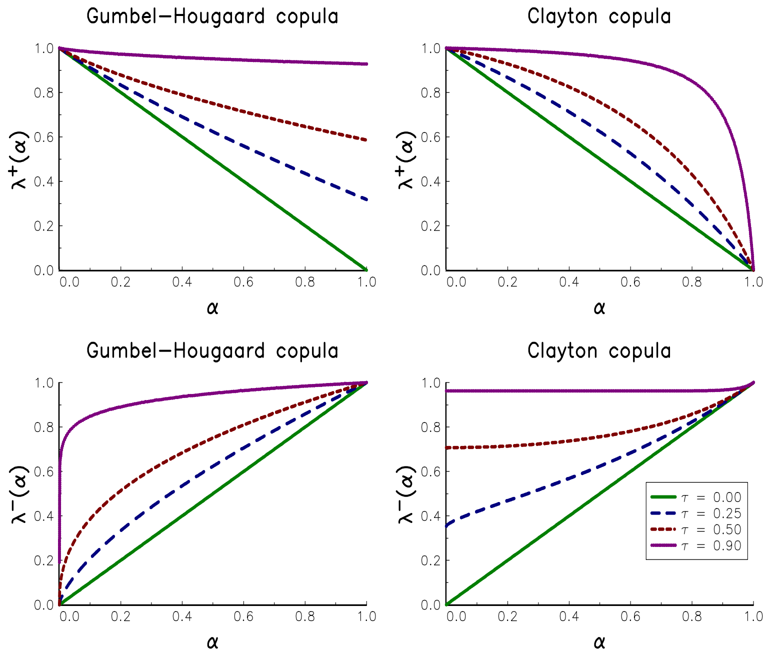

FIGURE 11.10: Quantile-quantile dependence measures $\lambda^{+}(\alpha)$ and $\lambda^{-}(\alpha)$

11.3.1 Archimedean copulas

11.3.1.1 Definition

Genest and MacKay (1986b) define Archimedean copulas as follows:

$$\mathbf{C}(u_1, u_2) = \begin{cases} \varphi^{-1}(\varphi(u_1) + \varphi(u_2)) & \text{if } \varphi(u_1) + \varphi(u_2) \leq \varphi(0) \\ 0 & \text{otherwise} \end{cases}$$

where φ a C^2 is a function which satisfies $\varphi(1) = 0$, $\varphi'(u) < 0$ and $\varphi''(u) > 0$ for all $u \in [0, 1]$. $\varphi(u)$ is called the generator of the copula function. If $\varphi(0) = \infty$, the generator is said to be strict. Genest and MacKay (1986a) link the construction of Archimedean copulas to the independence of random variables. Indeed, by considering the multiplicative generator $\lambda(u) = \exp(-\varphi(u))$, the authors show that:

$$\mathbf{C}(u_1, u_2) = \lambda^{-1}(\lambda(u_1)\lambda(u_2))$$

This means that:

$$\lambda(\Pr\{U_1 \leq u_1, U_2 \leq u_2\}) = \lambda(\Pr\{U_1 \leq u_1\}) \times \lambda(\Pr\{U_2 \leq u_2\})$$

In this case, the random variables (U_1, U_2) become independent when the scale of probabilities has been transformed.

Example 118 *If $\varphi(u) = u^{-1} - 1$, we have $\varphi^{-1}(u) = (1 + u)^{-1}$ and:*

$$\mathbf{C}(u_1, u_2) = \left(1 + \left(u_1^{-1} - 1 + u_2^{-1} - 1\right)\right)^{-1} = \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2}$$

The Gumbel logistic copula is then an Archimedean copula.

Example 119 *The product copula* $\mathbf{C}^{\perp}$ *is Archimedean and the associated generator is* $\varphi(u) = -\ln u$. *Concerning Fréchet copulas, only* $\mathbf{C}^{-}$ *is Archimedean with* $\varphi(u) = 1 - u$.

In Table 11.1, we provide another examples of Archimedean copulas[6].

TABLE 11.1: Archimedean copula functions

Copula	$\varphi(u)$	$\mathbf{C}(u_1, u_2)$
$\mathbf{C}^{\perp}$	$-\ln u$	$u_1 u_2$
Clayton	$u^{-\theta} - 1$	$\left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}$
Frank	$-\ln \dfrac{e^{-\theta u} - 1}{e^{-\theta} - 1}$	$-\dfrac{1}{\theta} \ln \left(1 + \dfrac{\left(e^{-\theta u_1} - 1\right)\left(e^{-\theta u_2} - 1\right)}{e^{-\theta} - 1}\right)$
Gumbel	$(-\ln u)^{\theta}$	$\exp\left(-\left(\tilde{u}_1^{\theta} + \tilde{u}_2^{\theta}\right)^{1/\theta}\right)$
Joe	$-\ln\left(1 - (1-u)^{\theta}\right)$	$1 - \left(\bar{u}_1^{\theta} + \bar{u}_2^{\theta} - \bar{u}_1^{\theta}\bar{u}_2^{\theta}\right)^{1/\theta}$

11.3.1.2 Properties

Archimedean copulas play an important role in statistics, because they present many interesting properties, for example:

- $\mathbf{C}$ is symmetric, meaning that $\mathbf{C}(u_1, u_2) = \mathbf{C}(u_2, u_1)$;
- $\mathbf{C}$ is associative, implying that $\mathbf{C}(u_1, \mathbf{C}(u_1, u_3)) = \mathbf{C}(\mathbf{C}(u_1, u_2), u_3)$;
- the diagonal section $\delta(u) = \mathbf{C}(u, u)$ satisfies $\delta(u) < u$ for all $u \in (0, 1)$;
- if a copula $\mathbf{C}$ is associative and $\delta(u) < u$ for all $u \in (0, 1)$, then $\mathbf{C}$ is Archimedean.

Genest and MacKay (1986a) also showed that the expression of Kendall's tau is:

$$\tau \langle \mathbf{C} \rangle = 1 + 4 \int_0^1 \frac{\varphi(u)}{\varphi'(u)} \, \mathrm{d}u$$

whereas the copula density is:

$$c(u_1, u_2) = -\frac{\varphi''(\mathbf{C}(u_1, u_2))\, \varphi'(u_1)\, \varphi'(u_2)}{[\varphi'(\mathbf{C}(u_1, u_2))]^3}$$

Example 120 *With the Clayton copula, we have* $\varphi(u) = u^{-\theta} - 1$ *and* $\varphi'(u) = -\theta u^{-\theta-1}$. *We deduce that:*

$$\begin{aligned} \tau &= 1 + 4 \int_0^1 \frac{1 - u^{-\theta}}{\theta u^{-\theta-1}} \, \mathrm{d}u \\ &= \frac{\theta}{\theta + 2} \end{aligned}$$

[6] We use the notations $\bar{u} = 1 - u$ and $\tilde{u} = -\ln u$.

11.3.1.3 Two-parameter Archimedean copulas

Nelsen (2006) showed that if $\varphi(t)$ is a strict generator, then we can build two-parameter Archimedean copulas by considering the following generator:

$$\varphi_{\alpha,\beta}(t) = (\varphi(t^{\alpha}))^{\beta}$$

where $\alpha > 0$ and $\beta > 1$. For instance, if $\varphi(t) = t^{-1} - 1$, the two-parameter generator is $\varphi_{\alpha,\beta}(t) = (t^{-\alpha} - 1)^{\beta}$. Therefore, the corresponding copula function is defined by:

$$\mathbf{C}(u_1, u_2) = \left(\left[\left(u_1^{-\alpha} - 1\right)^{\beta} + \left(u_2^{-\alpha} - 1\right)^{\beta}\right]^{1/\beta} + 1\right)^{-1/\alpha}$$

This is a generalization of the Clayton copula, which is obtained when the parameter β is equal to 1.

11.3.1.4 Extension to the multivariate case

We can build multivariate Archimedean copulas in the following way:

$$\mathbf{C}(u_1, \ldots, u_n) = \varphi^{-1}(\varphi(u_1) + \ldots + \varphi(u_n))$$

However, $\mathbf{C}$ is a copula function if and only if the function $\varphi^{-1}(u)$ is completely monotone (Nelsen, 2006):

$$(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}u^k}\varphi^{-1}(u) \geq 0 \qquad \forall\, k \geq 1$$

For instance, the multivariate Gumbel copula is defined by:

$$\mathbf{C}(u_1, \ldots, u_n) = \exp\left(-\left((-\ln u_1)^{\theta} + \ldots + (-\ln u_n)^{\theta}\right)^{1/\theta}\right)$$

The previous construction is related to an important class of multivariate distributions, which are called frailty models (Oakes, 1989). Let $\mathbf{F}_1, \ldots, \mathbf{F}_n$ be univariate distribution functions, and let $\mathbf{G}$ be an n-variate distribution function with univariate marginals $\mathbf{G}_i$, such that $\bar{\mathbf{G}}(0, \ldots, 0) = 1$. We denote by ψ_i the Laplace transform of $\mathbf{G}_i$. Marshall and Olkin (1988) showed that the function defined by:

$$\mathbf{F}(x_1, \ldots, x_n) = \int \cdots \int \mathbf{C}\left(\mathbf{H}_1^{t_1}(x_1), \ldots, \mathbf{H}_n^{t_n}(x_n)\right) \mathrm{d}\mathbf{G}(t_1, \ldots, t_n)$$

is a multivariate probability distribution with marginals $\mathbf{F}_1, \ldots, \mathbf{F}_n$ if $\mathbf{H}_i(x) = \exp\left(-\psi_i^{-1}(\mathbf{F}_i(x))\right)$. If we assume that the univariate distributions $\mathbf{G}_i$ are the same and equal to $\mathbf{G}_1$, $\mathbf{G}$ is the upper Fréchet bound and $\mathbf{C}$ is the product copula $\mathbf{C}^{\perp}$, the previous expression becomes:

$$\begin{aligned}
\mathbf{F}(x_1, \ldots, x_n) &= \int \prod_{i=1}^{n} \mathbf{H}_i^{t_1}(x_i)\, \mathrm{d}\mathbf{G}_1(t_1) \\
&= \int \exp\left(-t_1 \sum_{i=1}^{n} \psi^{-1}(\mathbf{F}_i(x_i))\right) \mathrm{d}\mathbf{G}_1(t_1) \\
&= \psi\left(\psi^{-1}(\mathbf{F}_1(x_1)) + \ldots + \psi^{-1}(\mathbf{F}_n(x_n))\right)
\end{aligned}$$

The corresponding copula is then given by:

$$\mathbf{C}(u_1, \ldots, u_n) = \psi\left(\psi^{-1}(u_1) + \ldots + \psi^{-1}(u_n)\right)$$

This is a special case of Archimedean copulas where the generator φ is the inverse of a Laplace transform. For instance, the Clayton copula is a frailty copula where $\psi(x) = (1+x)^{-1/\theta}$ is the Laplace transform of a Gamma random variable. The Gumbel-Hougaard copula is frailty too and we have $\psi(x) = \exp\left(-x^{1/\theta}\right)$. This is the Laplace transform of a positive stable distribution.

For frailty copulas, Joe (1997) showed that upper and lower tail dependence measures are given by:

$$\lambda^+ = 2 - 2\lim_{x\to 0}\frac{\psi'(2x)}{\psi'(x)}$$

and:

$$\lambda^- = 2\lim_{x\to\infty}\frac{\psi'(2x)}{\psi'(x)}$$

Example 121 *In the case of the Clayton copula, the Laplace transform is* $\psi(x) = (1+x)^{-1/\theta}$. *We have:*

$$\frac{\psi'(2x)}{\psi'(x)} = \frac{(1+2x)^{-1/\theta-1}}{(1+x)^{-1/\theta-1}}$$

We deduce that:

$$\begin{aligned}\lambda^+ &= 2 - 2\lim_{x\to 0}\frac{(1+2x)^{-1/\theta-1}}{(1+x)^{-1/\theta-1}} \\ &= 2-2 \\ &= 0\end{aligned}$$

and:

$$\begin{aligned}\lambda^- &= 2\lim_{x\to\infty}\frac{(1+2x)^{-1/\theta-1}}{(1+x)^{-1/\theta-1}} \\ &= 2\times 2^{-1/\theta-1} \\ &= 2^{-1/\theta}\end{aligned}$$

11.3.2 Normal copula

The Normal copula is the dependence function of the multivariate normal distribution with a correlation matrix ρ:

$$\mathbf{C}(u_1,\ldots,u_n;\rho) = \Phi_n\left(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_n);\rho\right)$$

By using the canonical decomposition of the multivariate density function:

$$f(x_1,\ldots,x_n) = c(\mathbf{F}_1(x_1),\ldots,\mathbf{F}_n(x_n))\prod_{i=1}^{n} f_i(x_i)$$

we deduce that the probability density function of the Normal copula is:

$$c(u_1,\ldots,u_n,;\rho) = \frac{1}{|\rho|^{\frac{1}{2}}}\exp\left(-\frac{1}{2}x^\top\left(\rho^{-1}-I_n\right)x\right)$$

where $x_i = \Phi^{-1}(u_i)$. In the bivariate case, we obtain[7]:

$$c(u_1, u_2; \rho) = \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{2(1-\rho^2)} + \frac{x_1^2 + x_2^2}{2}\right)$$

It follows that the expression of the bivariate Normal copula function is also equal to:

$$\mathbf{C}(u_1, u_2; \rho) = \int_{-\infty}^{\boldsymbol{\Phi}^{-1}(u_1)} \int_{-\infty}^{\boldsymbol{\Phi}^{-1}(u_2)} \phi_2(x_1, x_2; \rho)\, \mathrm{d}x_1\, \mathrm{d}x_2 \tag{11.6}$$

where $\phi_2(x_1, x_2; \rho)$ is the bivariate normal density:

$$\phi_2(x_1, x_2; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{2(1-\rho^2)}\right)$$

Example 122 *Let (X_1, X_2) be a standardized Gaussian random vector, whose cross-correlation is ρ. Using the Cholesky decomposition, we write X_2 as follows:*

$$X_2 = \rho X_1 + \sqrt{1-\rho^2} X_3$$

where $X_3 \sim \mathcal{N}(0,1)$ is independent from X_1 and X_2. We have:

$$\begin{aligned}
\Phi_2(x_1, x_2; \rho) &= \Pr\{X_1 \leq x_1, X_2 \leq x_2\} \\
&= \mathbb{E}\left[\Pr\left\{X_1 \leq x_1, \rho X_1 + \sqrt{1-\rho^2} X_3 \leq x_2 \mid X_1\right\}\right] \\
&= \int_{-\infty}^{x_1} \Phi\left(\frac{x_2 - \rho x}{\sqrt{1-\rho^2}}\right) \phi(x)\, \mathrm{d}x
\end{aligned}$$

It follows that:

$$\mathbf{C}(u_1, u_2; \rho) = \int_{-\infty}^{\Phi^{-1}(u_1)} \Phi\left(\frac{\Phi^{-1}(u_2) - \rho x}{\sqrt{1-\rho^2}}\right) \phi(x)\, \mathrm{d}x$$

We finally obtain that the bivariate Normal copula function is equal to:

$$\mathbf{C}(u_1, u_2; \rho) = \int_0^{u_1} \Phi\left(\frac{\Phi^{-1}(u_2) - \rho \Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right) \mathrm{d}u \tag{11.7}$$

This expression is more convenient to use than Equation (11.6).

Like the normal distribution, the Normal copula is easy to manipulate for computational purposes. For instance, Kendall's tau and Spearman's rho are equal to:

$$\tau = \frac{2}{\pi} \arcsin \rho$$

and:

$$\varrho = \frac{6}{\pi} \arcsin \frac{\rho}{2}$$

[7] In the bivariate case, the parameter ρ is the cross-correlation between X_1 and X_2, that is the element $(1,2)$ of the correlation matrix.

The conditional distribution $\mathbf{C}_{2|1}(u_1, u_2)$ has the following expression:

$$\begin{aligned}\mathbf{C}_{2|1}(u_1, u_2) &= \partial_1 \mathbf{C}(u_1, u_2) \\ &= \Phi\left(\frac{\Phi^{-1}(u_2) - \rho\Phi^{-1}(u_1)}{\sqrt{1-\rho^2}}\right)\end{aligned}$$

To compute the tail dependence, we apply Equation (11.4) and obtain:

$$\begin{aligned}\lambda^+ &= 2 - 2\lim_{u\to 1^-} \Phi\left(\frac{\Phi^{-1}(u) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right) \\ &= 2 - 2\lim_{u\to 1^-} \Phi\left(\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\Phi^{-1}(u)\right)\end{aligned}$$

We finally deduce that:

$$\lambda^+ = \lambda^- = \begin{cases} 0 & \text{if } \rho < 1 \\ 1 & \text{if } \rho = 1 \end{cases}$$

In Figure 11.11, we have represented the quantile-quantile dependence measure $\lambda^+(\alpha)$ for several values of the parameter ρ. When ρ is equal to 90% and α is close to one, we notice that $\lambda^+(\alpha)$ dramatically decreases. This means that even if the correlation is high, the extremes are independent.

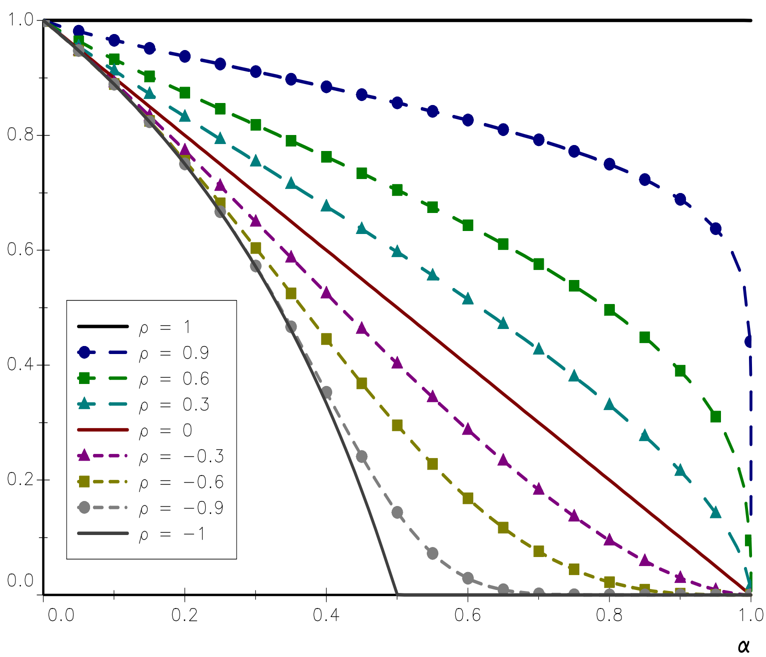

FIGURE 11.11: Tail dependence $\lambda^+(\alpha)$ for the Normal copula

11.3.3 Student's t copula

In a similar way, the Student's t copula is the dependence function associated with the multivariate Student's t probability distribution:

$$\mathbf{C}(u_1, \ldots, u_n; \rho, \nu) = \mathbf{T}_n\left(\mathbf{T}_\nu^{-1}(u_1), \ldots, \mathbf{T}_\nu^{-1}(u_n); \rho, \nu\right)$$

By using the definition of the cumulative distribution function:

$$\mathbf{T}_n(x_1,\ldots,x_n;\rho,\nu)=\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_n}\frac{\Gamma\left(\frac{\nu+n}{2}\right)|\rho|^{-\frac{1}{2}}}{\Gamma\left(\frac{\nu}{2}\right)(\nu\pi)^{\frac{n}{2}}}\left(1+\frac{1}{\nu}x^\top\rho^{-1}x\right)^{-\frac{\nu+n}{2}}\,\mathrm{d}x$$

we can show that the copula density is then:

$$c(u_1,\ldots,u_n,;\rho,\nu)=|\rho|^{-\frac{1}{2}}\frac{\Gamma\left(\frac{\nu+n}{2}\right)\left[\Gamma\left(\frac{\nu}{2}\right)\right]^n}{\left[\Gamma\left(\frac{\nu+1}{2}\right)\right]^n\Gamma\left(\frac{\nu}{2}\right)}\frac{\left(1+\frac{1}{\nu}x^\top\rho^{-1}x\right)^{-\frac{\nu+n}{2}}}{\prod_{i=1}^n\left(1+\frac{x_i^2}{\nu}\right)^{-\frac{\nu+1}{2}}}$$

where $x_i=\mathbf{T}_\nu^{-1}(u_i)$. In the bivariate case, we deduce that the t copula has the following expression:

$$\begin{aligned}\mathbf{C}(u_1,u_2;\rho,\nu) &= \int_{-\infty}^{\mathbf{T}_\nu^{-1}(u_1)}\int_{-\infty}^{\mathbf{T}_\nu^{-1}(u_2)}\frac{1}{2\pi\sqrt{1-\rho^2}}\cdot \\ &\quad \left(1+\frac{x_1^2+x_2^2-2\rho x_1x_2}{\nu(1-\rho^2)}\right)^{-\frac{\nu+2}{2}}\,\mathrm{d}x_1\,\mathrm{d}x_2\end{aligned}$$

Like the Normal copula, we can obtain another expression, which is easier to manipulate. Let (X_1,X_2) be a random vector whose probability distribution is $\mathbf{T}_2(x_1,x_2;\rho,\nu)$. Conditionally to $X_1=x_1$, we have:

$$\left(\frac{\nu+1}{\nu+x_1^2}\right)^{1/2}\frac{X_2-\rho x_1}{\sqrt{1-\rho^2}}\sim\mathbf{T}_{\nu+1}$$

The conditional distribution $\mathbf{C}_{2|1}(u_1,u_2)$ is then equal to:

$$\mathbf{C}_{2|1}(u_1,u_2;\rho,\nu)=\mathbf{T}_{\nu+1}\left(\left(\frac{\nu+1}{\nu+\left[\mathbf{T}_\nu^{-1}(u_1)\right]^2}\right)^{1/2}\frac{\mathbf{T}_\nu^{-1}(u_2)-\rho\mathbf{T}_\nu^{-1}(u_1)}{\sqrt{1-\rho^2}}\right)$$

We deduce that:

$$\mathbf{C}(u_1,u_2;\rho,\nu)=\int_0^{u_1}\mathbf{C}_{2|1}(u,u_2;\rho,\nu)\,\mathrm{d}u$$

We can show that the expression of Kendall's tau for the t copula is the one obtained for the Normal copula. In the case of Spearman's rho, there is no analytical expression. We denote by $\varrho_t(\rho,\nu)$ and $\varrho_n(\rho)$ the values of Spearman's rho for Student's t and Normal copulas with same parameter ρ. We can show that $\varrho_t(\rho,\nu)>\varrho_n(\rho)$ for negative values of ρ and $\varrho_t(\rho,\nu)<\varrho_n(\rho)$ for positive values of ρ. In Figure 11.12, we report the relationship between τ and ϱ for different degrees of freedom ν.

Because the t copula is symmetric, we can apply Equation (11.4) and obtain:

$$\begin{aligned}\lambda^+ &= 2-2\lim_{u\to1^-}\mathbf{T}_{\nu+1}\left(\left(\frac{\nu+1}{\nu+\left[\mathbf{T}_\nu^{-1}(u)\right]^2}\right)^{1/2}\frac{\mathbf{T}_\nu^{-1}(u)-\rho\mathbf{T}_\nu^{-1}(u)}{\sqrt{1-\rho^2}}\right) \\ &= 2-2\cdot\mathbf{T}_{\nu+1}\left(\left(\frac{(\nu+1)(1-\rho)}{(1+\rho)}\right)^{1/2}\right)\end{aligned}$$

We finally deduce that:

$$\lambda^+=\begin{cases}0 & \text{if } \rho=-1\\ >0 & \text{if } \rho>-1\end{cases}$$

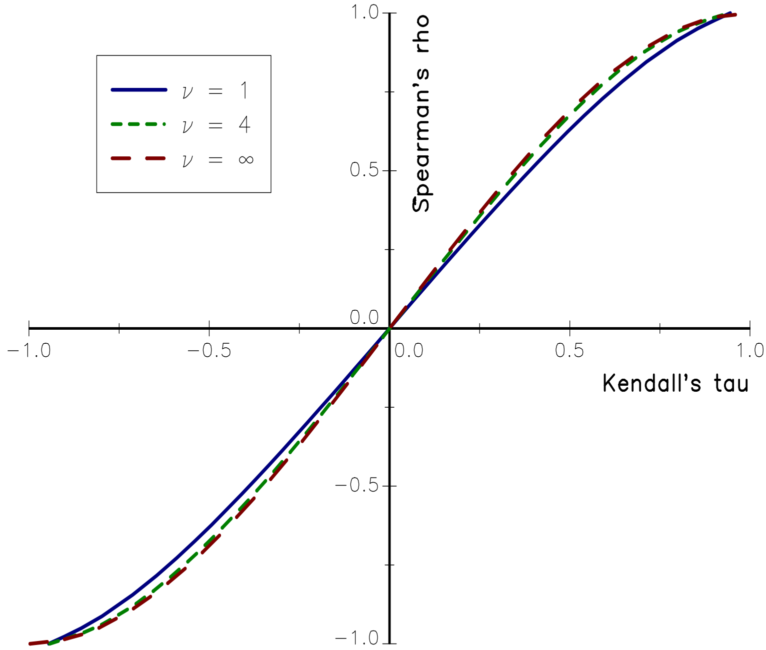

FIGURE 11.12: Relationship between τ and ϱ of the Student's t copula

Contrary to the Normal copula, the t copula has an upper tail dependence. In Figures 11.13 and 11.14, we represent the quantile-quantile dependence measure $\lambda^+(\alpha)$ for two degrees of freedom ν. We observe that the behavior of $\lambda^+(\alpha)$ is different than the one obtained in Figure 11.11 with the Normal copula. In Table 11.2, we give the numerical values of the coefficient λ^+ for various values of ρ and ν. We notice that it is strictly positive for small degrees of freedom even if the parameter ρ is negative. For instance, λ^+ is equal to 13.40% when ν and ρ are equal to 1 and −50%. We also observe that the convergence to the Gaussian case is low when the parameter ρ is positive.

TABLE 11.2: Values in % of the upper tail dependence λ^+ for the Student's t copula

ν	Parameter ρ (in %)					
	−70.00	−50.00	0.00	50.00	70.00	90.00
1	7.80	13.40	29.29	50.00	61.27	77.64
2	2.59	5.77	18.17	39.10	51.95	71.77
3	0.89	2.57	11.61	31.25	44.81	67.02
4	0.31	1.17	7.56	25.32	39.07	62.98
6	0.04	0.25	3.31	17.05	30.31	56.30
10	0.00	0.01	0.69	8.19	19.11	46.27
∞	0.00	0.00	0.00	0.00	0.00	0.00

Remark 139 *The Normal copula is a particular case of the Student's t copula when ν tends to ∞. This is why these two copulas are often compared for a given value of ρ. However, we must be careful because the previous analysis of the tail dependence has shown that these two copulas are very different. Let us consider the bivariate case. We can write the Student's t*

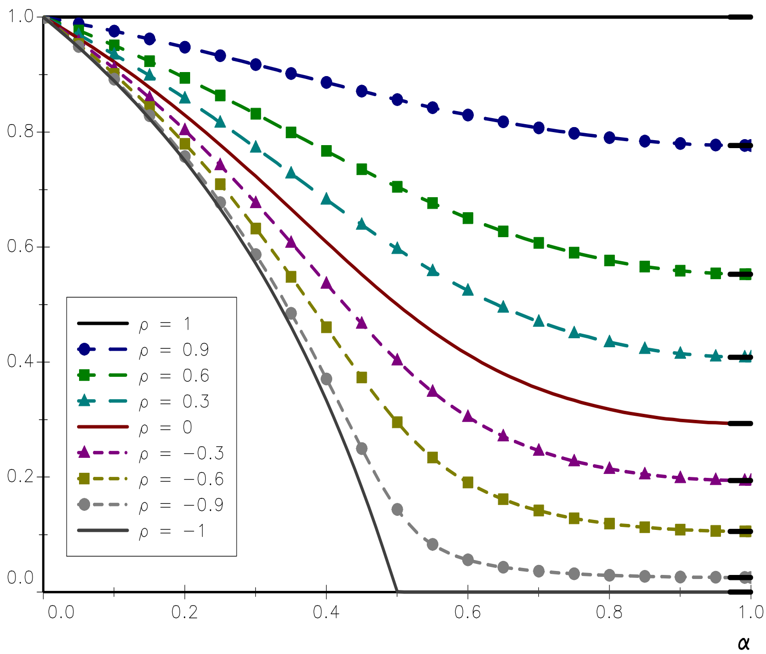

FIGURE 11.13: Tail dependence $\lambda^+(\alpha)$ for the Student's t copula ($\nu = 1$)

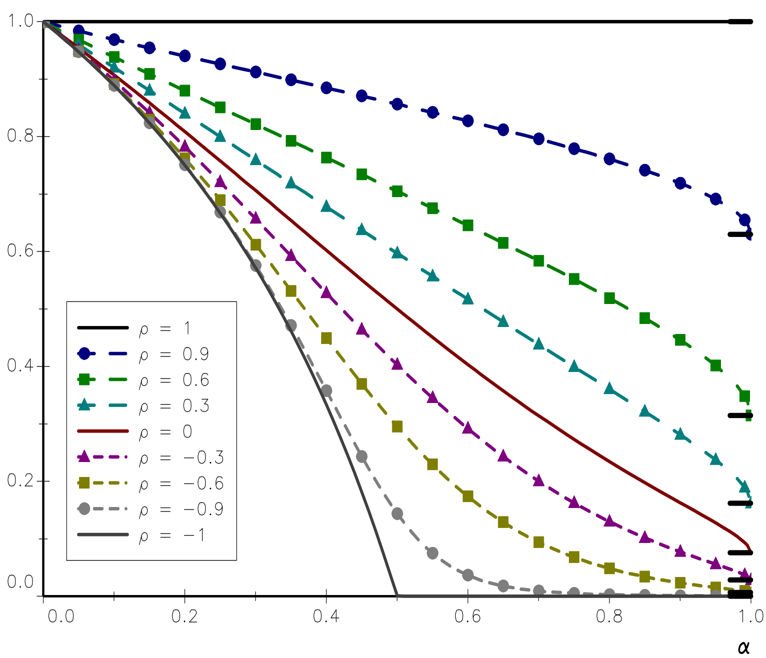

FIGURE 11.14: Tail dependence $\lambda^+(\alpha)$ for the Student's t copula ($\nu = 4$)

random vector (T_1, T_2) *as follows:*

$$\begin{aligned}(T_1, T_2) &= \frac{(N_1, N_2)}{\sqrt{X/\nu}} \\ &= \left(\frac{N_1}{\sqrt{X/\nu}}, \rho\frac{N_1}{\sqrt{X/\nu}} + \sqrt{1-\rho^2}\frac{N_3}{\sqrt{X/\nu}}\right)\end{aligned}$$

where N_1 *and* N_3 *are two independent Gaussian random variables and* X *is a random variable, whose probability distribution is* $\chi^2(\nu)$. *This is the introduction of the random variable* X *that produces a strong dependence between* T_1 *and* T_2, *and correlates the extremes. Even if the parameter* ρ *is equal to zero, we obtain:*

$$(T_1, T_2) = \left(\frac{N_1}{\sqrt{X/\nu}}, \frac{N_3}{\sqrt{X/\nu}}\right)$$

This implies that the product copula $\mathbf{C}^{\perp}$ *can never be attained by the* t *copula.*

11.4 Statistical inference and estimation of copula functions

We now consider the estimation problem of copula functions. We first introduce the empirical copula, which may viewed as a non-parametric estimator of the copula function. Then, we discuss the method of moments to estimate the parameters of copula functions. Finally, we apply the method of maximum likelihood and show the different forms of implementation.

11.4.1 The empirical copula

Let $\hat{\mathbf{F}}$ be the empirical distribution associated to a sample of T observations of the random vector $(X_1, \ldots, X_n)$. Following Deheuvels (1979), any copula $\hat{\mathbf{C}} \in \mathcal{C}$ defined on the lattice $\mathfrak{L}$:

$$\mathfrak{L} = \left\{\left(\frac{t_1}{T}, \ldots, \frac{t_n}{T}\right) : 1 \le j \le n, t_j = 0, \ldots, T\right\}$$

by the function:

$$\hat{\mathbf{C}}\left(\frac{t_1}{T}, \ldots, \frac{t_n}{T}\right) = \frac{1}{T}\sum_{t=1}^{T}\prod_{i=1}^{n}\mathbb{1}\left\{\mathfrak{R}_{t,i} \le t_i\right\}$$

is an empirical copula. Here $\mathfrak{R}_{t,i}$ is the rank statistic of the random variable X_i meaning that $X_{\mathfrak{R}_{t,i}:T,i} = X_{t,i}$. We notice that $\hat{\mathbf{C}}$ is the copula function associated to the empirical distribution $\hat{\mathbf{F}}$. However, $\hat{\mathbf{C}}$ is not unique because $\hat{\mathbf{F}}$ is not continuous. In the bivariate case, we obtain:

$$\begin{aligned}\hat{\mathbf{C}}\left(\frac{t_1}{T}, \frac{t_2}{T}\right) &= \frac{1}{T}\sum_{t=1}^{T}\mathbb{1}\left\{\mathfrak{R}_{t,1} \le t_1, \mathfrak{R}_{t,2} \le t_2\right\} \\ &= \frac{1}{T}\sum_{t=1}^{T}\mathbb{1}\left\{x_{t,1} \le x_{t_1:T,1}, x_{t,2} \le x_{t_2:T,2}\right\}\end{aligned}$$

where $\{(x_{t,1}, x_{t,2}), t = 1, \ldots, T\}$ denotes the sample of (X_1, X_2). Nelsen (2006) defines the empirical copula frequency function as follows:

$$\begin{aligned}\hat{c}\left(\frac{t_1}{T}, \frac{t_2}{T}\right) &= \hat{\mathbf{C}}\left(\frac{t_1}{T}, \frac{t_2}{T}\right) - \hat{\mathbf{C}}\left(\frac{t_1 - 1}{T}, \frac{t_2}{T}\right) - \\ &\quad \hat{\mathbf{C}}\left(\frac{t_1}{T}, \frac{t_2 - 1}{T}\right) + \hat{\mathbf{C}}\left(\frac{t_1 - 1}{T}, \frac{t_2 - 1}{T}\right) \\ &= \frac{1}{T}\sum_{t=1}^{T} \mathbb{1}\left\{x_{t,1} = x_{t_1:T,1}, x_{t,2} = x_{t_2:T,2}\right\}\end{aligned}$$

We have then:

$$\hat{\mathbf{C}}\left(\frac{t_1}{T}, \frac{t_2}{T}\right) = \sum_{j_1=1}^{t_1}\sum_{j_2=1}^{t_2} \hat{c}\left(\frac{j_1}{T}, \frac{j_2}{T}\right)$$

We can interpret $\hat{c}$ as the probability density function of the sample.

Example 123 *We consider the daily returns of European (EU) and American (US) MSCI equity indices from January 2006 to December 2015. In Figure 11.15, we represent the level lines of the empirical copula and compare them with the level lines of the Normal copula. For this copula function, the parameter ρ is estimated by the linear correlation between the daily returns of the two MSCI equity indices. We notice that the Normal copula does not exactly fit the empirical copula.*

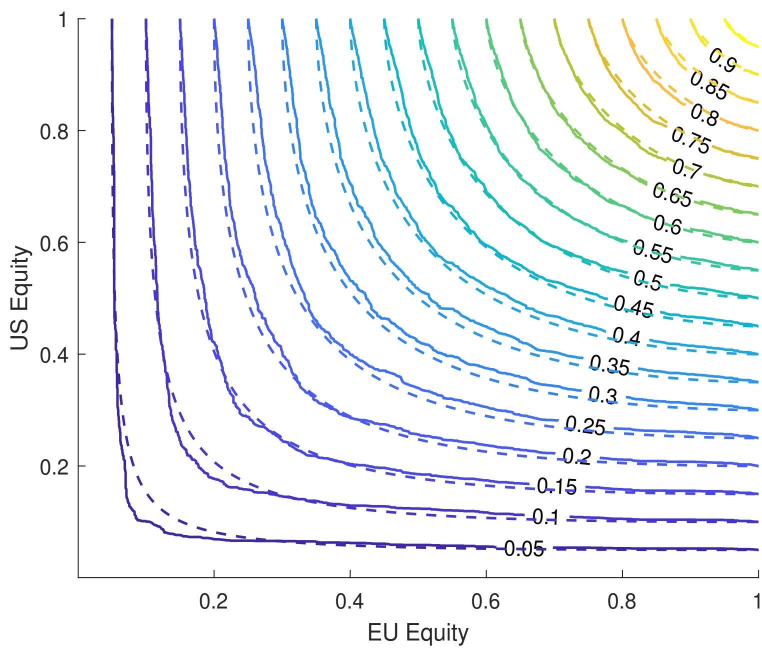

FIGURE 11.15: Comparison of the empirical copula (solid line) and the Normal copula (dashed line)

Like the histogram of the empirical distribution function $\hat{\mathbf{F}}$, it is difficult to extract information from $\hat{\mathbf{C}}$ or $\hat{c}$, because these functions are not smooth[8]. It is better to use a dependogram. This representation has been introduced by Deheuvels (1981), and consists in transforming the sample $\{(x_{t,1}, x_{t,2}), t = 1, \ldots, T\}$ of the random vector (X_1, X_2) into a sample $\{(u_{t,1}, u_{t,2}), t = 1, \ldots, T\}$ of uniform random variables (U_1, U_2) by considering the rank statistics:

$$u_{t,i} = \frac{1}{T}\mathfrak{R}_{t,i}$$

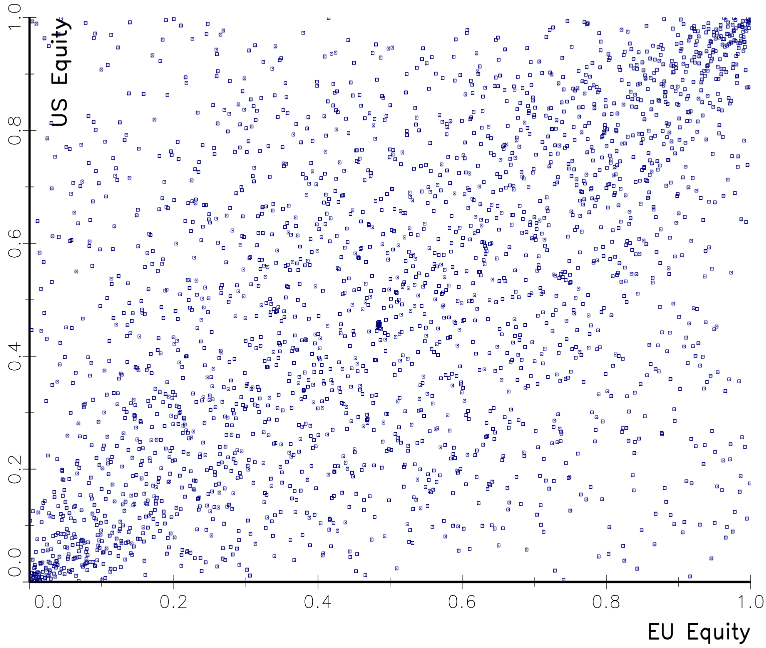

FIGURE 11.16: Dependogram of EU and US equity returns

The dependogram is then the scatter plot between $u_{t,1}$ and $u_{t,2}$. For instance, Figure 11.16 shows the dependogram of EU and US equity returns. We can compare this figure with the one obtained by assuming that equity returns are Gaussian. Indeed, Figure 11.17 shows the dependogram of a simulated bivariate Gaussian random vector when the correlation is equal to 57.8%, which is the estimated value between EU and US equity returns during the study period.

11.4.2 The method of moments

When it is applied to copulas, this method is different than the one presented in Chapter 10. Indeed, it consists in estimating the parameters θ of the copula function from the population version of concordance measures. For instance, if $\tau = f_\tau(\theta)$ is the relationship between θ and Kendall's tau, the MM estimator is simply the inverse of this relationship:

$$\hat{\theta} = f_\tau^{-1}(\hat{\tau})$$

[8]This is why they are generally coupled with approximation methods based on Bernstein polynomials (Sancetta and Satchell, 2004).

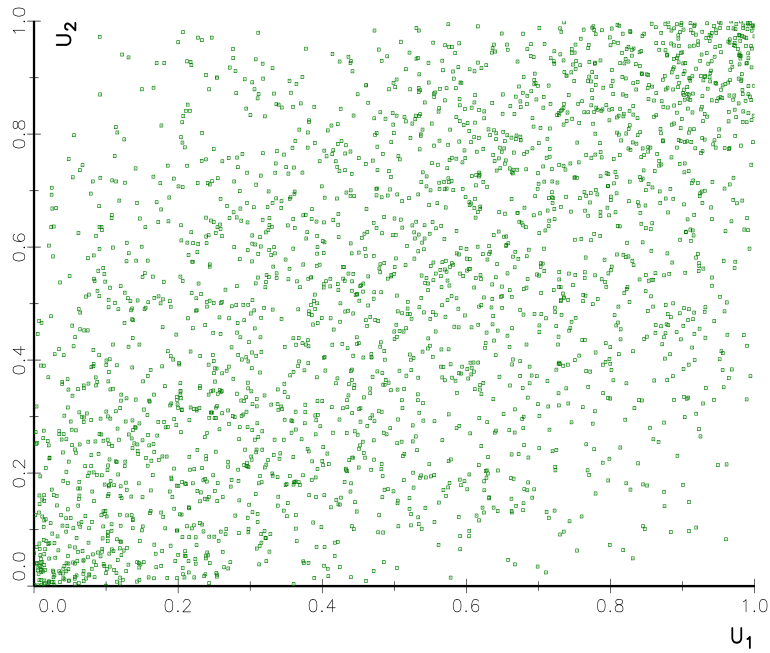

FIGURE 11.17: Dependogram of simulated Gaussian returns

where $\hat{\tau}$ is the estimate of Kendall's tau based on the sample[9]. For instance, in the case of the Gumbel copula, we obtain:

$$\hat{\theta} = \frac{1}{1 - \hat{\tau}}$$

Remark 140 *This approach is also valid for other concordance measures like Spearman's rho. We have then:*

$$\hat{\theta} = f_{\varrho}^{-1}(\hat{\varrho})$$

where $\hat{\varrho}$ is the estimate[10] of Spearman's rho and f_{ϱ} is the theoretical relationship between θ and Spearman's rho.

Example 124 *We consider the daily returns of 5 asset classes from January 2006 to December 2015. These asset classes are represented by the European MSCI equity index, the American MSCI equity index, the Barclays sovereign bond index, the Barclays corporate investment grade bond index and the Bloomberg commodity index. In Table 11.3, we report the correlation matrix. In Tables 11.4 and 11.5, we assume that the dependence function is a Normal copula and give the matrix $\hat{\rho}$ of estimated parameters using the method of moments based on Kendall's tau and Spearman's rhorho. We notice that these two matrices are very close, but we also observe some important differences with the correlation matrix reported in Table 11.3.*

[9] We have:

$$\hat{\tau} = \frac{c - d}{c + d}$$

where c and d are respectively the number of concordant and discordant pairs.

[10] It is equal to the linear correlation between the rank statistics.

TABLE 11.3: Matrix of linear correlations $\hat{\rho}_{i,j}$

	EU Equity	US Equity	Sovereign	Credit	Commodity
EU Equity	100.0				
US Equity	57.8	100.0			
Sovereign	−34.0	−32.6	100.0		
Credit	−15.1	−28.6	69.3	100.0	
Commodity	51.8	34.3	−22.3	−14.4	100.0

TABLE 11.4: Matrix of parameters $\hat{\rho}_{i,j}$ estimated using Kendall's tau

	EU Equity	US Equity	Sovereign	Credit	Commodity
EU Equity	100.0				
US Equity	57.7	100.0			
Sovereign	−31.8	−32.1	100.0		
Credit	−17.6	−33.8	73.9	100.0	
Commodity	43.4	30.3	−19.6	−15.2	100.0

TABLE 11.5: Matrix of parameters $\hat{\rho}_{i,j}$ estimated using Spearman's rho

	EU Equity	US Equity	Sovereign	Credit	Commodity
EU Equity	100.0				
US Equity	55.4	100.0			
Sovereign	−31.0	−31.3	100.0		
Credit	−17.1	−32.7	73.0	100.0	
Commodity	42.4	29.4	−19.2	−14.9	100.0

11.4.3 The method of maximum likelihood

Let us denote by $\{(x_{t,1},\ldots,x_{t,n}),t=1\ldots,T\}$ the sample of the random vector $(X_1,\ldots,X_n)$, whose multivariate distribution function has the following canonical decomposition:

$$\mathbf{F}(x_1,\ldots,x_n)=\mathbf{C}(\mathbf{F}_1(x_1;\theta_1),\ldots,\mathbf{F}_n(x_n;\theta_n);\theta_c)$$

This means that this statistical model depends on two types of parameters:

- the parameters $(\theta_1,\ldots,\theta_n)$ of univariate distribution functions;
- the parameters θ_c of the copula function.

The expression of the log-likelihood function is:

$$\begin{aligned}\boldsymbol{\ell}(\theta_1,\ldots,\theta_n,\theta_c) &= \sum_{t=1}^{T}\ln c(\mathbf{F}_1(x_{t,1};\theta_1),\ldots,\mathbf{F}_n(x_{t,n};\theta_n);\theta_c)+\\ &\quad \sum_{t=1}^{T}\sum_{i=1}^{n}\ln f_i(x_{t,i};\theta_i)\end{aligned}$$

where c is the copula density and f_i is the probability density function associated to $\mathbf{F}_i$. The ML estimator is then defined as follows:

$$\left(\hat{\theta}_1,\ldots,\hat{\theta}_n,\hat{\theta}_c\right)=\arg\max\boldsymbol{\ell}(\theta_1,\ldots,\theta_n,\theta_c)$$

The estimation by maximum likelihood method can be time-consuming when the number of parameters is large. However, the copula approach suggests a two-stage parametric method (Shih and Louis, 1995):

1. the first stage involves maximum likelihood from univariate marginals, meaning that we estimate the parameters $\theta_1, \ldots, \theta_n$ separately for each marginal:

$$\hat{\theta}_i = \arg\max \sum_{t=1}^{T} \ln f_i\left(x_{t,i}; \theta_i\right)$$

2. the second stage involves maximum likelihood of the copula parameters θ_c with the univariate parameters $\hat{\theta}_1, \ldots, \hat{\theta}_n$ held fixed from the first stage:

$$\hat{\theta}_c = \arg\max \sum_{t=1}^{T} \ln c\left(\mathbf{F}_1\left(x_{t,1}; \hat{\theta}_1\right), \ldots, \mathbf{F}_n\left(x_{t,n}; \hat{\theta}_n\right); \theta_c\right)$$

This approach is known as the method of inference functions for marginals or IFM (Joe, 1997). Let $\hat{\theta}_{\mathrm{IFM}}$ be the IFM estimator obtained with this two-stage procedure. We have:

$$T^{1/2}\left(\hat{\theta}_{\mathrm{IFM}} - \theta_0\right) \rightarrow \mathcal{N}\left(\mathbf{0}, \mathcal{V}^{-1}\left(\theta_0\right)\right)$$

where $\mathcal{V}\left(\theta_0\right)$ is the Godambe matrix (Joe, 1997).

Genest *et al.* (1995) propose a third estimation method, which consists in estimating the copula parameters θ_c by considering the non-parametric estimates of the marginals $\mathbf{F}_1, \ldots, \mathbf{F}_n$:

$$\hat{\theta}_c = \arg\max \sum_{t=1}^{T} \ln c\left(\hat{\mathbf{F}}_1\left(x_{t,1}\right), \ldots, \hat{\mathbf{F}}_n\left(x_{t,n}\right); \theta_c\right)$$

In this case, $\hat{\mathbf{F}}_i\left(x_{t,i}\right)$ is the normalized rank $\mathfrak{R}_{t,i}/T$. This estimator called omnibus or OM is then the ML estimate applied to the dependogram.

Example 125 *Let us assume that the dependence function of asset returns (X_1, X_2) is the Frank copula whereas the marginals are Gaussian. The log-likelihood function for observation t is then equal to:*

$$\begin{aligned}
\boldsymbol{\ell}_t &= \ln\left(\theta_c\left(1 - e^{-\theta_c}\right) e^{-\theta_c\left(\Phi\left(y_{t,1}\right)+\Phi\left(y_{t,2}\right)\right)}\right) - \\
&\quad \ln\left(\left(1 - e^{-\theta_c}\right) - \left(1 - e^{-\theta_c \Phi\left(y_{t,1}\right)}\right)\left(1 - e^{-\theta_c \Phi\left(y_{t,2}\right)}\right)\right)^2 - \\
&\quad \left(\frac{1}{2}\ln 2\pi + \frac{1}{2}\ln \sigma_1^2 + \frac{1}{2}y_{t,1}^2\right) - \\
&\quad \left(\frac{1}{2}\ln 2\pi + \frac{1}{2}\ln \sigma_2^2 + \frac{1}{2}y_{t,2}^2\right)
\end{aligned}$$

where $y_{t,i} = \sigma_i^{-1}\left(x_{t,i} - \mu_i\right)$ is the standardized return of asset i for the observation t. The vector of parameters to estimate is $\theta = \left(\mu_1, \sigma_1, \mu_2, \sigma_2, \theta_c\right)$. In the case of the IFM approach, the parameters $\left(\mu_1, \sigma_1, \mu_2, \sigma_2\right)$ are estimated in a first step. Then, we estimate the copula parameter θ_c by considering the following log-likelihood function:

$$\begin{aligned}
\boldsymbol{\ell}_t &= \ln\left(\theta_c\left(1 - e^{-\theta_c}\right) e^{-\theta_c\left(\Phi\left(\hat{y}_{t,1}\right)+\Phi\left(\hat{y}_{t,2}\right)\right)}\right) - \\
&\quad \ln\left(\left(1 - e^{-\theta_c}\right) - \left(1 - e^{-\theta_c \Phi\left(\hat{y}_{t,1}\right)}\right)\left(1 - e^{-\theta_c \Phi\left(\hat{y}_{t,2}\right)}\right)\right)^2
\end{aligned}$$

where $\hat{y}_{t,i}$ is equal to $\hat{\sigma}_i^{-1}(x_{t,i} - \hat{\mu}_i)$. Finally, the OM approach uses the uniform variates $u_{t,i} = \Re_{t,i}/T$ in the expression of the log-likelihood function function:

$$\begin{aligned} \boldsymbol{\ell}_t &= \ln\left(\theta_c\left(1-e^{-\theta_c}\right)e^{-\theta_c(u_{t,1}+u_{t,2})}\right) - \\ &\quad \ln\left(\left(1-e^{-\theta_c}\right)-\left(1-e^{-\theta_c u_{t,1}}\right)\left(1-e^{-\theta_c u_{t,2}}\right)\right)^2 \end{aligned}$$

Using the returns of MSCI Europe and US indices for the last 10 years, we obtain the following results for the parameter θ_c of the Frank copula:

	ML	IFM	OM	Method of Moments Kendall	Method of Moments Spearman
$\hat{\theta}_c$	6.809	6.184	4.149	3.982	3.721
$\hat{\tau}$	0.554	0.524	0.399	0.387	0.367
$\hat{\varrho}$	0.754	0.721	0.571	0.555	0.529

We obtain $\hat{\theta}_c = 6.809$ for the method of maximum likelihood and $\hat{\theta}_c = 6.184$ for the IFM approach. These results are very close, that is not the case with the omnibus approach where we obtain $\hat{\theta}_c = 4.149$. This means that the assumption of Gaussian marginals is far to be verified. The specification of wrong marginals in ML and IFM approaches induces then a bias in the estimation of the copula parameter. With the omnibus approach, we do not face this issue because we consider non-parametric marginals. This explains that we obtain a value, which is close to the MM estimates (Kendall's tau and Spearman's rho).

For IFM and OM approaches, we can obtain a semi-analytical expression of $\hat{\theta}_c$ for some specific copula functions. In the case of the Normal copula, the matrix ρ of the parameters is estimated with the following algorithm:

1. we first transform the uniform variates $u_{t,i}$ into Gaussian variates:
$$n_{t,i} = \Phi^{-1}(u_{t,i})$$
2. we then calculate the correlation matrix of the Gaussian variates $n_{t,i}$.

For the Student's t copula, Bouyé *et al.* (2000) suggest the following algorithm:

1. let $\hat{\rho}_0$ be the estimated value of ρ for the Normal copula;
2. $\hat{\rho}_{k+1}$ is obtained using the following equation:
$$\hat{\rho}_{k+1} = \frac{1}{T}\sum_{t=1}^{T}\frac{(\nu+n)\,\varsigma_t\varsigma_t^\top}{\nu+\varsigma_t^\top\hat{\rho}_k^{-1}\varsigma_t}$$
where:
$$\varsigma_t = \begin{pmatrix} \mathbf{t}_\nu^{-1}(u_{t,1}) \\ \vdots \\ \mathbf{t}_\nu^{-1}(u_{t,n}) \end{pmatrix}$$
3. repeat the second step until convergence: $\hat{\rho}_{k+1} = \hat{\rho}_k := \hat{\rho}_\infty$.

Let us consider Example 124. We have estimated the parameter matrix ρ of Normal and Student's t copulas using the omnibus approach. Results are given in Tables 11.6, 11.7 and 11.8. We notice that these matrices are different than the correlation matrix calculated in Table 11.3. The reason is that we have previously assumed that the marginals were Gaussian. In this case, the ML estimate introduced a bias in the copula parameter in order to compensate the bias induced by the wrong specification of the marginals.

TABLE 11.6: Omnibus estimate $\hat{\rho}$ (Normal copula)

	EU Equity	US Equity	Sovereign	Credit	Commodity
EU Equity	100.0				
US Equity	56.4	100.0			
Sovereign	−32.5	−32.1	100.0		
Credit	−16.3	−30.3	70.2	100.0	
Commodity	46.5	30.7	−21.1	−14.7	100.0

TABLE 11.7: Omnibus estimate $\hat{\rho}$ (Student's t copula with $\nu = 1$)

	EU Equity	US Equity	Sovereign	Credit	Commodity
EU Equity	100.0				
US Equity	47.1	100.0			
Sovereign	−20.3	−18.9	100.0		
Credit	−9.3	−22.1	57.6	100.0	
Commodity	28.0	17.1	−7.4	−6.2	100.0

TABLE 11.8: Omnibus estimate $\hat{\rho}$ (Student's t copula with $\nu = 4$)

	EU Equity	US Equity	Sovereign	Credit	Commodity
EU Equity	100.0				
US Equity	59.6	100.0			
Sovereign	−31.5	−31.9	100.0		
Credit	−18.3	−32.9	71.3	100.0	
Commodity	43.0	30.5	−17.2	−13.4	100.0

Remark 141 *The discrepancy between the ML or IFM estimate and the OM estimate is interesting information for knowing if the specification of the marginals are right or not. In particular, a large discrepancy indicates that the estimated marginals are far from the empirical marginals.*

11.5 Exercises

11.5.1 Gumbel logistic copula

1. Calculate the density of the Gumbel logistic copula.

2. Show that it has a lower tail dependence, but no upper tail dependence.

11.5.2 Farlie-Gumbel-Morgenstern copula

We consider the following function:

$$\mathbf{C}(u_1, u_2) = u_1 u_2 (1 + \theta (1 - u_1)(1 - u_2)) \tag{11.8}$$

1. Show that $\mathbf{C}$ is a copula function for $\theta \in [-1, 1]$.

2. Calculate the tail dependence coefficient λ, the Kendall's τ statistic and the Spearman's ϱ statistic.

3. Let $X = (X_1, X_2)$ be a bivariate random vector. We assume that $X_1 \sim \mathcal{N}\left(\mu, \sigma^2\right)$ and $X_2 \sim \mathcal{E}(\lambda)$. Propose an algorithm to simulate (X_1, X_2) when the copula is the function (11.8).

4. Calculate the log-likelihood function of the sample $\left\{(x_{1,i}, x_{2,i})_{i=1}^{i=n}\right\}$.

11.5.3 Survival copula

Let $\mathbf{S}$ be the bivariate function defined by:

$$\mathbf{S}(x_1, x_2) = \exp\left(-\left(x_1 + x_2 - \theta \frac{x_1 x_2}{x_1 + x_2}\right)\right)$$

with $\theta \in [0, 1]$, $x_1 \geq 0$ et $x_2 \geq 0$.

1. Verify that $\mathbf{S}$ is a survival function.

2. Define the survival copula associated to $\mathbf{S}$.

11.5.4 Method of moments

Let (X_1, X_2) be a bivariate random vector such that $X_1 \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right)$ and $X_2 \sim \mathcal{N}\left(\mu_2, \sigma_2^2\right)$. We consider that the dependence function is given by the following copula:

$$\mathbf{C}(u_1, u_2) = \theta \cdot \mathbf{C}^{-}(u_1, u_2) + (1 - \theta) \cdot \mathbf{C}^{+}(u_1, u_2)$$

where $\theta \in [0, 1]$ is the copula parameter.

1. We assume that $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$. Find the parameter θ such that the linear correlation of X_1 and X_2 is equal to zero. Show that there exists a function f such that $X_1 = f(X_2)$. Comment on this result.

2. Calculate the linear correlation of X_1 and X_2 as a function of the parameters μ_1, μ_2, σ_1, σ_2 and θ.

3. Propose a method of moments to estimate θ.

11.5.5 Correlated loss given default rates

We assume that the probability distribution of the (annual) loss given default rate associated to a risk class $\mathcal{C}$ is given by:

$$\begin{aligned} \mathbf{F}(x) &= \Pr\{\text{LGD} \leq x\} \\ &= x^{\gamma} \end{aligned}$$

1. Find the conditions on the parameter γ that are necessary for $\mathbf{F}$ to be a probability distribution.

2. Let $\{x_1, \ldots, x_n\}$ be a sample of loss given default rates. Calculate the log-likelihood function and deduce the ML estimator $\hat{\gamma}_{\text{ML}}$.

3. Calculate the first moment $\mathbb{E}[\text{LGD}]$. Then find the method of moments estimator $\hat{\gamma}_{\text{MM}}$.

4. We assume that $x_i = 50\%$ for all i. Calculate the numerical values taken by $\hat{\gamma}_{\text{ML}}$ and $\hat{\gamma}_{\text{MM}}$. Comment on these results.

5. We now consider two risk classes $\mathcal{C}_1$ and $\mathcal{C}_2$ and note LGD_1 and LGD_2 the corresponding LGD rates. We assume that the dependence function between LGD_1 and LGD_2 is given by the Gumbel-Barnett copula:

$$\mathbf{C}(u_1, u_2) = u_1 u_2 e^{-\theta \ln u_1 \ln u_2}$$

where θ is the copula parameter. Show that the density function of the copula is equal to:

$$c(u_1, u_2; \theta) = \left(1 - \theta - \theta \ln(u_1 u_2) + \theta^2 \ln u_1 \ln u_2\right) e^{-\theta \ln u_1 \ln u_2}$$

6. Deduce the log-likelihood function of the historical sample $\left\{(x_i, y_i)_{i=1}^{i=n}\right\}$.

7. We note $\hat{\gamma}_1$, $\hat{\gamma}_2$ and $\hat{\theta}$ the ML estimators of the parameters γ_1 (risk class $\mathcal{C}_1$), γ_2 (risk class $\mathcal{C}_2$) and θ (copula parameter). Why the ML estimator $\hat{\gamma}_1$ does not correspond to the ML estimator $\hat{\gamma}_{\text{ML}}$ except in the case $\hat{\theta} = 0$? Illustrate with an example.

11.5.6 Calculation of correlation bounds

1. Give the mathematical definition of the copula functions $\mathbf{C}^-$, $\mathbf{C}^\perp$ and $\mathbf{C}^+$. What is the probabilistic interpretation of these copulas?

2. We note $\boldsymbol{\tau}$ and LGD the default time and the loss given default of a counterparty. We assume that $\boldsymbol{\tau} \sim \mathcal{E}(\lambda)$ and $\text{LGD} \sim \mathcal{U}_{[0,1]}$.

   (a) Show that the dependence between $\boldsymbol{\tau}$ and LGD is maximum when the following equality holds:
   $$\text{LGD} + e^{-\lambda \boldsymbol{\tau}} - 1 = 0$$

   (b) Show that the linear correlation $\rho(\boldsymbol{\tau}, \text{LGD})$ verifies the following inequality:
   $$|\rho\langle \boldsymbol{\tau}, \text{LGD}\rangle| \leq \frac{\sqrt{3}}{2}$$

   (c) Comment on these results.

3. We consider two exponential default times $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ with parameters λ_1 and λ_2.

   (a) We assume that the dependence function between $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ is $\mathbf{C}^+$. Demonstrate that the following relationship is true:
   $$\boldsymbol{\tau}_1 = \frac{\lambda_2}{\lambda_1} \boldsymbol{\tau}_2$$

   (b) Show that there exists a function f such that $\boldsymbol{\tau}_2 = f(\boldsymbol{\tau}_2)$ when the dependence function is $\mathbf{C}^-$.

   (c) Show that the lower and upper bounds of the linear correlation satisfy the following relationship:
   $$-1 < \rho\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2\rangle \leq 1$$

(d) In the more general case, show that the linear correlation of a random vector (X_1, X_2) cannot be equal to -1 if the support of the random variables X_1 and X_2 is $[0, +\infty]$.

4. We assume that (X_1, X_2) is a Gaussian random vector where $X_1 \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right)$, $X_2 \sim \mathcal{N}\left(\mu_2, \sigma_2^2\right)$ and ρ is the linear correlation between X_1 and X_2. We note $\theta = (\mu_1, \sigma_1, \mu_2, \sigma_2, \rho)$ the set of parameters.

   (a) Find the probability distribution of $X_1 + X_2$.

   (b) Then show that the covariance between $Y_1 = e^{X_1}$ and $Y_2 = e^{X_2}$ is equal to:

   $$\operatorname{cov}\left(Y_1, Y_2\right) = e^{\mu_1 + \frac{1}{2}\sigma_1^2} \cdot e^{\mu_2 + \frac{1}{2}\sigma_2^2} \cdot \left(e^{\rho\sigma_1\sigma_2} - 1\right)$$

   (c) Deduce the correlation between Y_1 and Y_2.

   (d) For which values of θ does the equality $\rho\left\langle Y_1, Y_2\right\rangle = +1$ hold? Same question when $\rho\left\langle Y_1, Y_2\right\rangle = -1$.

   (e) We consider the bivariate Black-Scholes model:

   $$\left\{\begin{array}{l} \mathrm{d}S_1(t) = \mu_1 S_1(t)\,\mathrm{d}t + \sigma_1 S_1(t)\,\mathrm{d}W_1(t) \\ \mathrm{d}S_2(t) = \mu_2 S_2(t)\,\mathrm{d}t + \sigma_2 S_2(t)\,\mathrm{d}W_2(t) \end{array}\right.$$

   with $\mathbb{E}\left[W_1(t)\,W_2(t)\right] = \rho t$. Deduce the linear correlation between $S_1(t)$ and $S_2(t)$. Find the limit case $\lim_{t\to\infty} \rho\left\langle S_1(t), S_2(t)\right\rangle$.

   (f) Comment on these results.

11.5.7 The bivariate Pareto copula

We consider the bivariate Pareto distribution:

$$\begin{aligned} \mathbf{F}(x_1, x_2) &= 1 - \left(\frac{\theta_1 + x_1}{\theta_1}\right)^{-\alpha} - \left(\frac{\theta_2 + x_2}{\theta_2}\right)^{-\alpha} + \\ &\quad \left(\frac{\theta_1 + x_1}{\theta_1} + \frac{\theta_2 + x_2}{\theta_2} - 1\right)^{-\alpha} \end{aligned}$$

where $x_1 \geq 0$, $x_2 \geq 0$, $\theta_1 > 0$, $\theta_2 > 0$ and $\alpha > 0$.

1. Show that the marginal functions of $\mathbf{F}(x_1, x_2)$ correspond to univariate Pareto distributions.

2. Find the copula function associated to the bivariate Pareto distribution.

3. Deduce the copula density function.

4. Show that the bivariate Pareto copula function has no lower tail dependence, but an upper tail dependence.

5. Do you think that the bivariate Pareto copula family can reach the copula functions $\mathbf{C}^-$, $\mathbf{C}^\perp$ and $\mathbf{C}^+$? Justify your answer.

6. Let X_1 and X_2 be two Pareto distributed random variables, whose parameters are (α_1, θ_1) and (α_2, θ_2).

(a) Show that the linear correlation between X_1 and X_2 is equal to 1 if and only if the parameters α_1 and α_2 are equal.

(b) Show that the linear correlation between X_1 and X_2 can never reached the lower bound -1.

(c) Build a new bivariate Pareto distribution by assuming that the marginal distributions are $\mathcal{P}(\alpha_1, \theta_1)$ and $\mathcal{P}(\alpha_2, \theta_2)$ and the dependence function is a bivariate Pareto copula with parameter α. What is the relevance of this approach for building bivariate Pareto distributions?