Financial Risk Management

Tutorial Class — Session 4

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4 Operational Risk

4.1 Estimation of the loss severity distribution

We consider a sample of n individual losses $\{x_1, \ldots, x_n\}$. We assume that they can be described by different probability distributions:

(i) X follows a log-normal distribution $\mathcal{LN}\left(\mu, \sigma^2\right)$.

(ii) X follows a Pareto distribution $\mathcal{P}\left(\alpha, x^-\right)$ defined by:

$$\Pr\left\{X \leq x\right\} = 1 - \left(\frac{x}{x_-}\right)^{-\alpha}$$

with $x \geq x_-$ and $\alpha > 0$.

(iii) X follows a gamma distribution $\Gamma\left(\alpha, \beta\right)$ defined by:

$$\Pr\left\{X \leq x\right\} = \int_0^x \frac{\beta^\alpha t^{\alpha-1} e^{-\beta t}}{\Gamma\left(\alpha\right)} \,\mathrm{d}t$$

with $x \geq 0$, $\alpha > 0$ and $\beta > 0$.

(iv) The natural logarithm of the loss X follows a gamma distribution: $\ln X \sim \Gamma\left(\alpha; \beta\right)$.

1. We consider the case *(i)*.

   (a) Show that the probability density function is:

   $$f\left(x\right) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right)$$

   (b) Calculate the two first moments of X. Deduce the orthogonal conditions of the generalized method of moments.

   (c) Find the maximum likelihood estimators $\hat{\mu}$ and $\hat{\sigma}$.

2. We consider the case *(ii)*.

   (a) Calculate the two first moments of X. Deduce the GMM conditions for estimating the parameter α.

   (b) Find the maximum likelihood estimator $\hat{\alpha}$.

3. We consider the case *(iii)*. Write the log-likelihood function associated to the sample of individual losses $\{x_1, \ldots, x_n\}$. Deduce the first-order conditions of the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$.

4. We consider the case *(iv)*. Show that the probability density function of X is:

$$f(x) = \frac{\beta^{\alpha} (\ln x)^{\alpha-1}}{\Gamma(\alpha) x^{\beta+1}}$$

   What is the support of this probability density function? Write the log-likelihood function associated to the sample of individual losses $\{x_1, \ldots, x_n\}$.

5. We now assume that the losses $\{x_1, \ldots, x_n\}$ have been collected beyond a threshold H meaning that $X \geq H$.

   (a) What becomes the generalized method of moments in the case *(i)*.

   (b) Calculate the maximum likelihood estimator $\hat{\alpha}$ in the case *(ii)*.

   (c) Write the log-likelihood function in the case *(iii)*.

4.2 Estimation of the loss frequency distribution

We consider a dataset of individual losses $\{x_1, \ldots, x_n\}$ corresponding to a sample of T annual loss numbers $\{N_{Y_1}, \ldots, N_{Y_T}\}$. This implies that:

$$\sum_{t=1}^{T} N_{Y_t} = n$$

If we measure the number of losses per quarter $\{N_{Q_1}, \ldots, N_{Q_{4T}}\}$, we use the notation:

$$\sum_{t=1}^{4T} N_{Q_t} = n$$

1. We assume that the annual number of losses follows a Poisson distribution $\mathcal{P}(\lambda_Y)$. Calculate the maximum likelihood estimator $\hat{\lambda}_Y$ associated to the sample $\{N_{Y_1}, \ldots, N_{Y_T}\}$.

2. We assume that the quarterly number of losses follows a Poisson distribution $\mathcal{P}(\lambda_Q)$. Calculate the maximum likelihood estimator $\hat{\lambda}_Q$ associated to the sample $\{N_{Q_1}, \ldots, N_{Q_{4T}}\}$.

3. What is the impact of considering a quarterly or annual basis on the computation of the capital charge?

4. What does this result become if we consider a method of moments based on the first moment?

5. Same question if we consider a method of moments based on the second moment.

5 Asset Liability Management Risk

5.1 Computation of the amortization functions $\mathbf{S}(t, u)$ and $\mathbf{S}^{\star}(t, u)$

In what follows, we consider a debt instrument, whose remaining maturity is equal to m. We note t the current date and $T = t + m$ the maturity date.

1. We consider a bullet repayment debt. Define its amortization function $\mathbf{S}(t, u)$. Calculate the survival function $\mathbf{S}^{\star}(t, u)$ of the stock. Show that:

$$\mathbf{S}^{\star}(t, u) = \mathbb{1}\{t \leq u < t + m\} \cdot \left(1 - \frac{u - t}{m}\right)$$

   in the case where the new production is constant. Comment on this result.

2. Same question if we consider a debt instrument, whose amortization rate is constant.

3. Same question if we assume[1] that the amortization function is exponential with parameter λ.

4. Find the expression of $\mathcal{D}^\star(t)$ when the new production is constant.

5. Calculate the durations $\mathcal{D}(t)$ and $\mathcal{D}^\star(t)$ for the three previous cases.

6. Calculate the corresponding dynamics $\mathrm{d}N(t)$.

5.2 Impact of prepayment on the amortization scheme of the CAM

We recall that the outstanding balance at time t is given by:

$$N(t) = \mathbf{1}\{t < m\} \cdot N_0 \frac{1 - e^{-i(m-t)}}{1 - e^{-im}}$$

1. Find the dynamics $\mathrm{d}N(t)$.

2. We note $\tilde{N}(t)$ the modified outstanding balance that takes into account the prepayment risk. Let $\lambda_p(t)$ be the prepayment rate at time t. Write the dynamics of $\tilde{N}(t)$.

3. Show that $\tilde{N}(t) = N(t)\,\mathbf{S}_p(t)$ where $\mathbf{S}_p(t)$ is the prepayment-based survival function.

4. Calculate the liquidity duration $\tilde{\mathcal{D}}(t)$ associated to the outstanding balance $\tilde{N}(t)$ when the hazard rate of prepayments is constant and equal to λ_p.

[1]By definition of the exponential amortization, we have $m = +\infty$.