Financial Risk Management

Tutorial Class — Session 2

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2 Credit Risk

2.1 Single and multi-name credit default swaps

1. We assume that the default time $\boldsymbol{\tau}$ follows an exponential distribution with parameter λ. Write the cumulative distribution function $\mathbf{F}$, the survival function $\mathbf{S}$ and the density function f of the random variable $\boldsymbol{\tau}$. How do we simulate this default time?

2. We consider a CDS 3M with two-year maturity and \$1 mn notional principal. The recovery rate $\boldsymbol{\mathcal{R}}$ is equal to 40% whereas the spread $\boldsymbol{s}$ is equal to 150 bps at the inception date. We assume that the protection leg is paid at the default time.

   (a) Give the cash flow chart. What is the P&L of the protection seller A if the reference entity does not default? What is the PnL of the protection buyer B if the reference entity defaults in one year and two months?

   (b) What is the relationship between $\boldsymbol{s}$, $\boldsymbol{\mathcal{R}}$ and λ? What is the implied one-year default probability at the inception date?

   (c) Seven months later, the CDS spread has increased and is equal to 450 bps. Estimate the new default probability. The protection buyer B decides to realize his P&L. For that, he reassigns the CDS contract to the counterparty C. Explain the offsetting mechanism if the risky PV01 is equal to 1.189.

3. We consider the following CDS spread curves for three reference entities:

Maturity	#1	#2	#3
6M	130 bps	1 280 bps	30 bps
1Y	135 bps	970 bps	35 bps
3Y	140 bps	750 bps	50 bps
5Y	150 bps	600 bps	80 bps

   (a) Define the notion of credit curve. Comment the previous spread curves.

   (b) Using the Merton Model, we estimate that the one-year default probability is equal to 2.5% for #1, 5% for #2 and 2% for #3 at a five-year horizon time. Which arbitrage position could we consider about the reference entity #2?

4. We consider a basket of n single-name CDS.

   (a) What is a first-to-default (FtD), a second-to-default (StD) and a last-to-default (LtD)?

   (b) Define the notion of default correlationẆhat is its impact on three previous spreads?

(c) We assume that $n = 3$. Show the following relationship:

$$\mathcal{s}_1^{\mathrm{CDS}} + \mathcal{s}_2^{\mathrm{CDS}} + \mathcal{s}_3^{\mathrm{CDS}} = \mathcal{s}^{\mathrm{FtD}} + \mathcal{s}^{\mathrm{StD}} + \mathcal{s}^{\mathrm{LtD}}$$

where $\mathcal{s}_1^{\mathrm{CDS}}$ is the CDS spread of the i^{th} reference entity.

(d) Many professionals and academics believe that the subprime crisis is due to the use of the Normal copula. Using the results of the previous question, what could you conclude?

2.2 Risk contribution in the Basel II model

1. We note L the portfolio loss of n credit and w_i the exposure at default of the i^{th} credit. We have:

$$L(w) = w^\top \varepsilon = \sum_{i=1}^{n} w_i \times \varepsilon_i \tag{1}$$

where ε_i is the unit loss of the i^{th} credit. Let $\mathbf{F}$ be the cumulative distribution function of $L(w)$.

(a) We assume that $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \sim \mathcal{N}(\mathbf{0}, \Sigma)$. Compute the value-at-risk $\mathrm{VaR}_\alpha(w)$ of the portfolio when the confidence level is equal to α.

(b) Deduce the marginal value-at-risk of the i^{th} credit. Define then the risk contribution $\mathcal{RC}_i$ of the i^{th} credit.

(c) Check that the marginal value-at-risk is equal to:

$$\frac{\partial\, \mathrm{VaR}_\alpha(w)}{\partial\, w_i} = \mathbb{E}\left[\varepsilon_i \mid L(w) = \mathbf{F}^{-1}(\alpha)\right]$$

Comment on this result.

2. We consider the Basel II model of credit risk and the value-at-risk risk measure. The expression of the portfolio loss is given by:

$$L = \sum_{i=1}^{n} \mathrm{EAD}_i \times \mathrm{LGD}_i \times \mathbb{1}\{\boldsymbol{\tau}_i < M_i\} \tag{2}$$

(a) Define the different parameters EAD_i, LGD_i, $\boldsymbol{\tau}_i$ and M_i. Show that Model (2) can be written as Model (1) by identifying w_i and ε_i.

(b) What are the necessary assumptions $(\mathcal{H})$ to obtain this result:

$$\mathbb{E}\left[\varepsilon_i \mid L = \mathbf{F}^{-1}(\alpha)\right] = \mathbb{E}[\mathrm{LGD}_i] \times \mathbb{E}\left[D_i \mid L = \mathbf{F}^{-1}(\alpha)\right]$$

with $D_i = \mathbb{1}\{\boldsymbol{\tau}_i < M_i\}$.

(c) Deduce the risk contribution $\mathcal{RC}_i$ of the i^{th} credit and the value-at-risk of the credit portfolio.

(d) We assume that the credit i defaults before the maturity M_i if a latent variable Z_i goes below a barrier B_i:

$$\boldsymbol{\tau}_i \le M_i \Leftrightarrow Z_i \le B_i$$

We consider that $Z_i = \sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_i$ where Z_i, X and ε_i are three independent Gaussian variables $\mathcal{N}(0,1)$. X is the factor (or the systematic risk) and ε_i is the idiosyncratic risk.

i. Interpret the parameter ρ.

ii. Calculate the unconditional default probability:

$$p_i = \Pr\{\boldsymbol{\tau}_i \le M_i\}$$

iii. Calculate the conditional default probability:

$$p_i(x) = \Pr\{\boldsymbol{\tau}_i \leq M_i \mid X = x\}$$

(e) Show that, under the previous assumptions $(\mathcal{H})$, the risk contribution $\mathcal{RC}_i$ of the i^{th} credit is:

$$\mathcal{RC}_i = \mathrm{EAD}_i \times \mathbb{E}[\mathrm{LGD}_i] \times \Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right) \tag{3}$$

when the risk measure is the value-at-risk.

3. We now assume that the risk measure is the expected shortfall:

$$\mathrm{ES}_\alpha(w) = \mathbb{E}[L \mid L \geq \mathrm{VaR}_\alpha(w)]$$

(a) In the case of the Basel II framework, show that we have:

$$\mathrm{ES}_\alpha(w) = \sum_{i=1}^{n} \mathrm{EAD}_i \times \mathbb{E}[\mathrm{LGD}_i] \times \mathbb{E}\left[p_i(X) \mid X \leq \Phi^{-1}(1-\alpha)\right]$$

(b) By using the following result:

$$\int_{-\infty}^{c} \Phi(a+bx)\phi(x)\,\mathrm{d}x = \Phi_2\left(c, \frac{a}{\sqrt{1+b^2}}; \frac{-b}{\sqrt{1+b^2}}\right)$$

where $\Phi_2(x, y; \rho)$ is the cdf of the bivariate Gaussian distribution with correlation ρ on the space $[-\infty, x] \times [-\infty, y]$, deduce that the risk contribution $\mathcal{RC}_i$ of the i^{th} credit in the Basel II model is:

$$\mathcal{RC}_i = \mathrm{EAD}_i \times \mathbb{E}[\mathrm{LGD}_i] \times \frac{\mathbf{C}\left(1-\alpha, p_i; \sqrt{\rho}\right)}{1-\alpha} \tag{4}$$

when the risk measure is the expected shortfall. Here $\mathbf{C}(u_1, u_2; \theta)$ is the Normal copula with parameter θ.

(c) What do the results (3) and (4) become if the correlation ρ is equal to zero? Same question if $\rho = 1$.

4. The risk contributions (3) and (4) were obtained considering the assumptions $(\mathcal{H})$ and the default model defined in Question 2(d). What are the implications in terms of Pillar 2?

2.3 Modeling loss given default

1. What is the difference between the recovery rate and the loss given default?

2. We consider a bank that grants 250 000 credits per year. The average amount of a credit is equal to \$50 000. We estimate that the average default probability is equal to 1% and the average recovery rate is equal to 65%. The total annual cost of the litigation department is equal to \$12.5 mn. Give an estimation of the loss given default?

3. The probability density function of the beta probability distribution $\mathcal{B}(a, b)$ is:

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{\mathbf{B}(a,b)}$$

where $\mathbf{B}(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\,\mathrm{d}u$.

(a) Why is the beta probability distribution a good candidate to model the loss given default? Which parameter pair (a, b) correspond to the uniform probability distribution?

(b) Let us consider a sample $(x_1, \ldots, x_n)$ of n losses in case of default. Write the log-likelihood function. Deduce the first-order conditions of the maximum likelihood estimator.

(c) We recall that the first two moments of the beta probability distribution are:

$$\begin{aligned} \mathbb{E}[X] &= \frac{a}{a+b} \\ \sigma^2(X) &= \frac{ab}{(a+b)^2(a+b+1)} \end{aligned}$$

Find the method of moments estimator.

4. We consider a risk class $\mathcal{C}$ corresponding to a customer/product segmentation specific to retail banking. A statistical analysis of 1 000 loss data available for this risk class gives the following results:

LGD_k	0%	25%	50%	75%	100%
n_k	100	100	600	100	100

where n_k is the number of data corresponding to LGD_k.

(a) We consider a portfolio of 100 homogeneous credits, which belong to the risk class $\mathcal{C}$. The notional is \$10 000 whereas the annual default probability is equal to 1%. Calculate the expected loss of this credit portfolio with a one-year horizon time if we use the previous empirical distribution to model the LGD parameter.

(b) We assume that the LGD parameter follows a beta distribution $\mathcal{B}(a, b)$. Calibrate the parameters a and b with the method of moments.

(c) We assume that the Basel II model is valid. We consider the portfolio described in Question 4(a) and calculate the unexpected loss. What is the impact if we use a uniform probability distribution instead of the calibrated beta probability distribution? Why does this result hold even if we consider different factors to model the default time?