Some Pratical Issues on Credit Risk

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Agenda

1. New Trends on the Market of Credit Risk
2. Credit Risk Modelling in the New Basle Capital Accord
3. Extending the Basle II model
4. The Measurement of Credit Risk
5. Credit Portfolio Management
6. The Time-inconsistency of the Copula Model

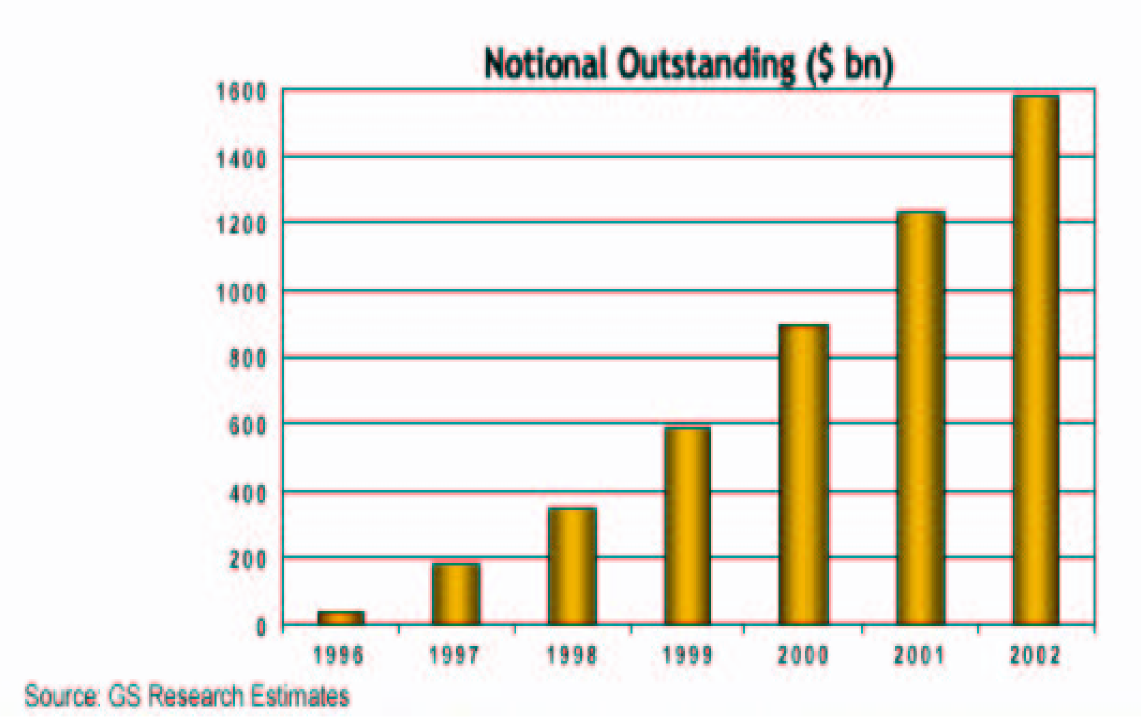
1 New Trends on the Market of Credit Risk

1.1 15th Annual Report of Moody's (2001)

This report is Moody's fifteenth annual study of corporate debt defaults. It comes a critical juncture for the capital markets worldwide. Record defaults — unreached in a number and dollar volume since the Great Depression — have culminated in the bankruptcies of well-known firms whose rapid collapse caught investors by surprise. In the wake of these failures, concern for credit quality has grown to a level not seen in seventy years.

- The default rate for all Moody's-rated corporate bond issuers ended 2001 at 3.7%. For speculative-grade rated issuers, the default rate reached 10.2%.
- Rating downgrades exceed rating upgrades 1.9 to 1 in 2001.
- The average recovery rate of defaulted bonds fell to a record low of 21% of par.

1.2 Credit Derivatives



CDO in Europe

Year	1996	1997	1998	1999	2000	2001	2002
Number	1	3	3	24	50	133	144
Volume ($ bn)	5	5.7	4.5	29.2	63.2	106	143.4

Source: Moody's Investor Service

2 Credit Risk Modelling in the New Basle Capital Accord

2.1 The New Basel Capital Accord

The 1988 Capital Accord concerns only credit risk (and market risk — Amendment of January 1996) $\Rightarrow$ the Cooke Ratio requires capital to be at least 8 percent of the "risk" of the bank.

- January 2001: proposal for a New Basel Capital Accord (credit risk measurement will be more risk sensitive + explicit capital calculations for operational risk)
- November 2002: QIS 3 (Quantitative Impact Study)

$\Rightarrow$ The objectives of the New Accord are the following:

1. Capital calculations will be more risk sensitive.
2. Convergence between economic capital (internal measure) and regulatory capital.

The McDonough ratio

It is defined as follows:

$$\frac{\text{Capital (Tier I + Tier II)}}{\text{credit risk} + \text{market risk} + \text{operational risk}} \geq 8\%$$

The aim of allocation for the industry is

Risk	January 2001	Now
Credit	75%	83%
Market	5%	5%
Operational	20%	12%

The measurement methods

Risk weighted assets are calculated as follows :

$$\text{RWA} = \text{EaD} \times \text{RW}$$

1. Standardized Approach **(SA)**
   The risk weights are based on external ratings:

Rating		AAA/AA-	A+/A-	BBB+/BBB-	BB+/B-	B-/C	non rated
Sovereign		0%	20%	50%	100%	150%	100%
Bank	1	20%	50%	100%	100%	150%	100%
	2	20%	50%	50%	100%	150%	50%
	2 (−3M)	20%	20%	20%	50%	150%	20%
Corporate		20%	50%	BBB+/BB- 100%		B+/C 150%	100%

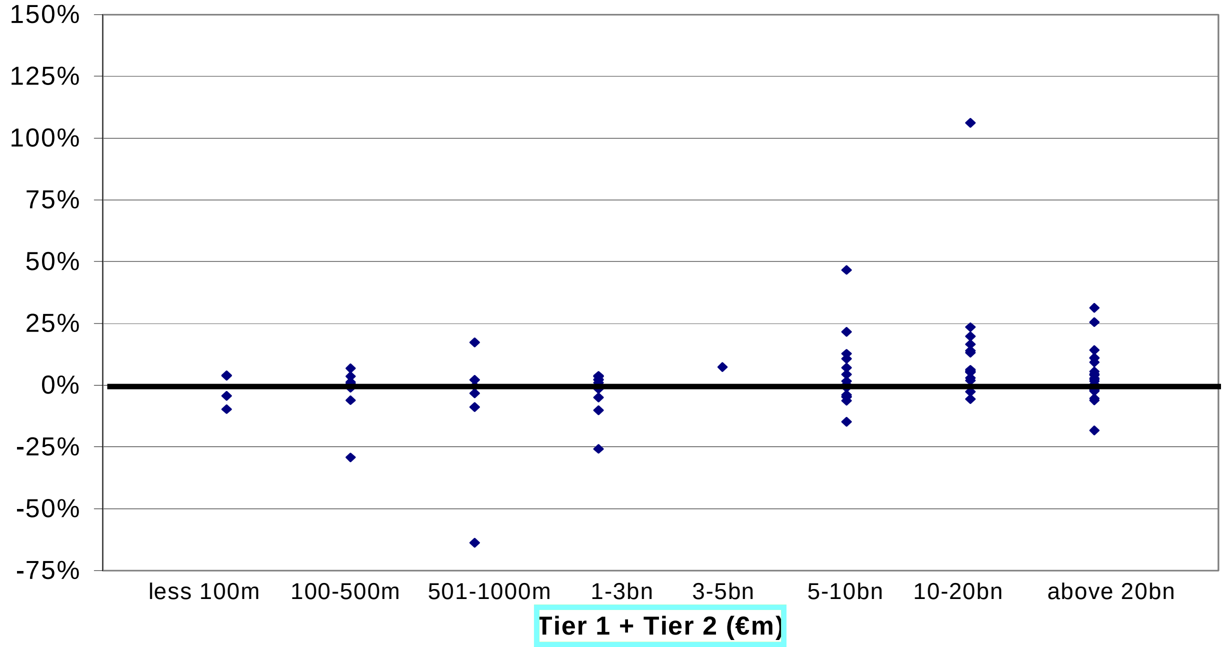
2. Internal Rating Based Approach (IRB)

$$\text{RW} = c \cdot \text{LGD} \cdot \text{RC}\,(\text{PD})$$

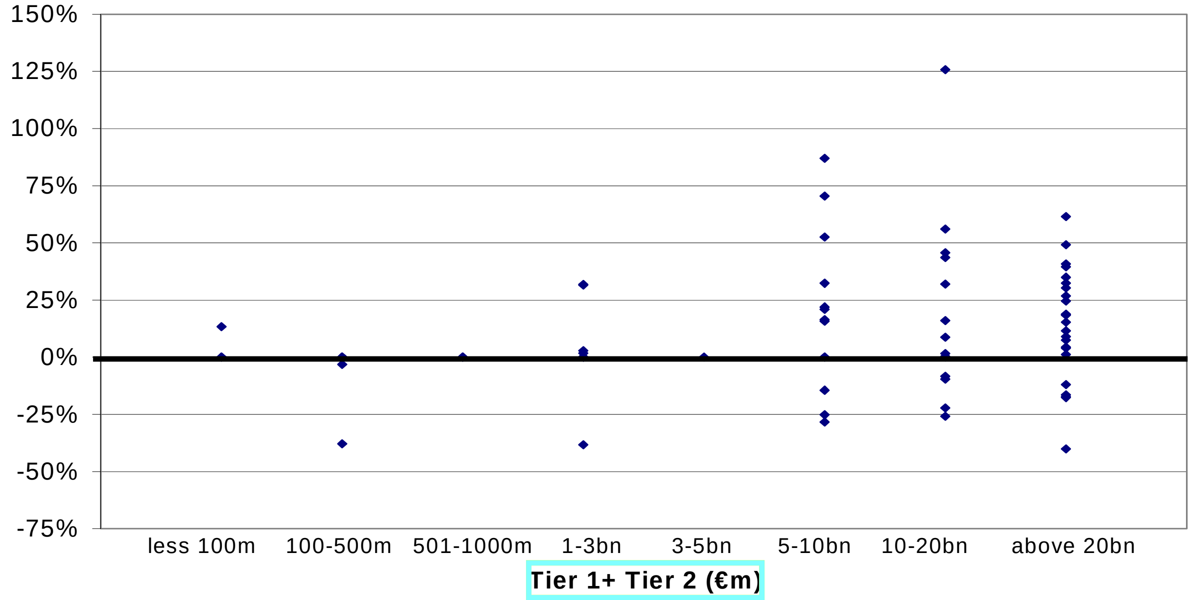
   1. *foundation approach*
   2. *advanced approach*

Percentage Changes in Capital Requirements for G10 Banks

(a) Standardised Approach



(b) IRB Foundation Approach



Source: QIS2, BCBS, BIS (November 2001)

2.2 The IRB Approach

The (original) IRB risk weights are

$$\mathrm{RW} = \min\left(\frac{\mathrm{LGD}}{50} \times \mathrm{BRW}\left(\mathrm{PD}\right), 12.5 \times \mathrm{LGD}\right)$$

where BRW is a benchmark function calibrated on a 50% LGD

$$\mathrm{BRW}\left(\mathrm{PD}\right) = 976.5 \times \Phi\left(1.118 \times \Phi^{-1}\left(\mathrm{PD}\right) + 1.288\right) \times \left(1 + 0.470 \times \frac{1 - \mathrm{PD}}{\mathrm{PD}^{0.44}}\right)$$

Infinitely fine-grained portfolio and risk contribution Let us consider a portfolio Π with I loans. The loss is

$$\mathbf{L} = \sum_{i=1}^{I} \mathrm{EaD}_i \cdot \mathrm{LGD}_i \cdot \mathbf{1}\left\{\tau_i \leq t_i\right\}$$

We assume that the defaut probability $P_i = \Pr\left\{\tau_i \leq t_i\right\}$ is $P_i\left(X\right)$ where X is the systematic factor with distribution $\mathbf{H}$. For the Infinitely fine-grained portfolio Π_∞ 'equivalent' to the original portfolio Π, we have

$$\Pr\left\{\mathbf{L}_\infty = \mathbb{E}\left[\mathbf{L}\right] \mid X\right\} = 1$$

If P_i are increasing functions with respect to X, the percentile α of the loss distribution is

$$\mathbf{F}_\infty^{-1}\left(\alpha\right) := \sum_{i=1}^{I} \underbrace{\mathrm{EaD}_i \cdot \mathbb{E}\left[\mathrm{LGD}_i\right] \cdot P_i\left(\mathbf{H}^{-1}\left(\alpha\right)\right)}_{\text{risk contribution of the loan } i}$$

IRB approach explained (from Wilde [2001])

Merton/Vasicek model

$$Z_i = \sqrt{\rho} X + \sqrt{1-\rho}\varepsilon_i$$

$$D_i = \mathbf{1}\{\tau_i \leq t_i\} \Leftrightarrow Z_i < B_i$$

P_i is the unconditional default probability

$$P_i(X) = \Pr\{D_i = 1 \mid X\} = \Phi\left(\frac{\Phi^{-1}(P_i) - \sqrt{\rho} X}{\sqrt{1-\rho}}\right)$$

$$\mathrm{RC}_i = \mathrm{EaD}_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot \Phi\left(\frac{\Phi^{-1}(P_i) - \sqrt{\rho}\Phi^{-1}(1-\alpha)}{\sqrt{1-\rho}}\right)$$

With $\alpha = 99.5\%$ and $\rho = 20\%$, we have

$$\mathrm{RC}_i = \mathrm{EaD}_i \cdot \mathbb{E}[\mathrm{LGD}_i] \cdot \Phi\left(1.118\Phi^{-1}(P_i) + 1.288\right)$$

If $\mathrm{RW}(\mathrm{PD}=0.7\%, \mathrm{LGD}=50\%)=100\%$, we have

$$\mathrm{BRW}(\mathrm{PD}) = \underbrace{619.59}_{\text{3Y scaling factor}} \times \underbrace{\Phi\left(1.118 \times \Phi^{-1}\left(1-(1-\mathrm{PD})^{3}\right)+1.288\right)}_{\text{3Y conditional default probability}}$$

<u>Basel's approximation formula:</u>

$$\begin{aligned}\mathrm{BRW}(\mathrm{PD}) &= \underbrace{976.5}_{\text{1Y scaling factor}} \times \underbrace{\Phi\left(1.118 \times \Phi^{-1}(\mathrm{PD})+1.288\right)}_{\text{1Y conditional default probability}} \\ &\times \underbrace{\left(1+0.470 \times \frac{1-\mathrm{PD}}{\mathrm{PD}^{0.44}}\right)}_{\text{3Y maturity adjustement}}\end{aligned}$$

Infinitely fine-grained portfolio by the example

Vasicek model — Rating CCC

$\text{EAD} = 1 - \text{LGD} \sim \mathcal{B}(3,3) - \text{PD} = 17.5\% - \rho = 20\%$

α	$\mathbb{E}[\mathbf{L}_n \mid X]$	VaR $[\mathbf{L}_n]$	rel. diff.	$\mathbb{E}[\mathbf{L}_n \mid X]$	VaR $[\mathbf{L}_n]$	rel. diff.
		$n = 5$			$n = 100$	
50%	0.370	0.286	29.502	7.402	7.361	0.553
75%	0.599	0.720	-16.853	11.979	12.146	-1.374
90%	0.858	1.213	-29.304	17.153	17.566	-2.349
95%	1.030	1.510	-31.788	20.599	21.175	-2.721
99%	1.368	2.114	-35.314	27.354	28.280	-3.275
99.5%	1.490	2.333	-36.121	29.800	30.820	-3.310
99.9%	1.729	2.763	-37.431	34.577	35.642	-2.989
		$n = 500$			$n = 5000$	
50%	37.008	37.101	-0.249	370.085	369.291	0.215
75%	59.895	60.112	-0.362	598.947	597.975	0.163
90%	85.765	86.164	-0.463	857.649	857.280	0.043
95%	102.993	103.579	-0.566	1029.929	1030.332	-0.039
99%	136.768	137.854	-0.788	1367.684	1367.906	-0.016
99.5%	149.000	150.168	-0.777	1490.005	1489.127	0.059
99.9%	172.884	174.333	-0.831	1728.844	1716.432	0.723

Vasicek model — Rating BBB

EAD = 1 − LGD $\sim \mathcal{B}(3,3)$ − PD = 0.20% − $\rho = 20\%$

α	$\mathbb{E}[\mathbf{L}_n \mid X]$	VaR $[\mathbf{L}_n]$	rel. diff.	$\mathbb{E}[\mathbf{L}_n \mid X]$	VaR $[\mathbf{L}_n]$	rel. diff.
		$n = 5$			$n = 100$	
50%	0.002	0.000		0.032	0.000	
75%	0.005	0.000		0.099	0.000	
90%	0.012	0.000		0.249	0.434	-42.604
95%	0.021	0.000		0.415	0.702	-40.856
99%	0.050	0.000		0.998	1.489	-32.992
99.5%	0.067	0.513	-86.924	1.340	1.912	-29.912
99.9%	0.118	0.782	-84.918	2.359	3.135	-24.733
		$n = 500$			$n = 5000$	
50%	0.161	0.000		1.614	1.574	2.583
75%	0.496	0.585	-15.185	4.961	5.036	-1.485
90%	1.245	1.442	-13.653	12.454	12.658	-1.616
95%	2.075	2.353	-11.827	20.750	20.913	-0.783
99%	4.988	5.447	-8.421	49.884	50.372	-0.969
99.5%	6.701	7.170	-6.530	67.014	67.108	-0.141
99.9%	11.797	12.491	-5.559	117.967	116.891	0.920

Vasicek model — Rating AA

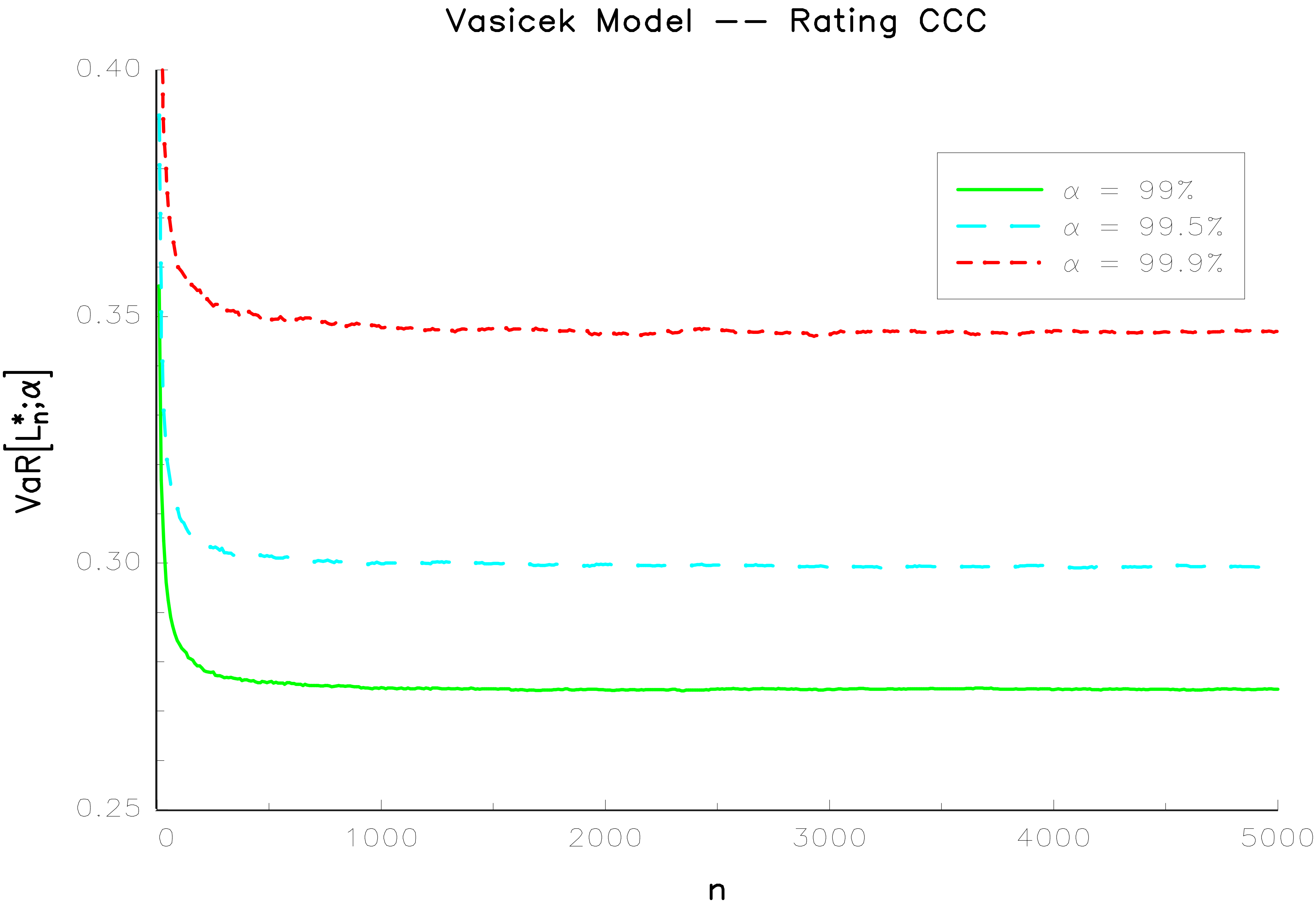
EAD = 1 − LGD $\sim \mathcal{B}(3,3)$ − PD = 0.03% − ρ = 20%

α	$\mathbb{E}[\mathbf{L}_n \mid X]$	VaR $[\mathbf{L}_n]$	rel. diff.	$\mathbb{E}[\mathbf{L}_n \mid X]$	VaR $[\mathbf{L}_n]$	rel. diff.
	$n = 5$			$n = 100$		
50%	0.000	0.000		0.003	0.000	
75%	0.001	0.000		0.012	0.000	
90%	0.002	0.000		0.035	0.000	
95%	0.003	0.000		0.064	0.000	
99%	0.009	0.000		0.188	0.591	-68.253
99.5%	0.014	0.000		0.270	0.732	-63.097
99.9%	0.027	0.459	-94.030	0.548	1.180	-53.538
	$n = 500$			$n = 5000$		
50%	0.016	0.000		0.156	0.000	
75%	0.058	0.000		0.583	0.658	-11.471
90%	0.174	0.132	31.750	1.743	1.922	-9.330
95%	0.322	0.591	-45.531	3.220	3.504	-8.102
99%	0.938	1.349	-30.465	9.383	9.776	-4.018
99.5%	1.351	1.814	-25.487	13.514	13.781	-1.937
99.9%	2.741	3.373	-18.718	27.415	26.581	3.138

Vasicek model — Rating BBB

EAD = 1 − LGD $\sim \mathcal{B}(3,3)$ − PD = 0.20% − $\rho = 80\%$

α	$\mathbb{E}[\mathbf{L}_n \mid X]$	$\mathrm{VaR}[\mathbf{L}_n]$	rel. diff.	$\mathbb{E}[\mathbf{L}_n \mid X]$	$\mathrm{VaR}[\mathbf{L}_n]$	rel. diff.
		$n = 5$			$n = 100$	
50%	0.000	0.000		0.000	0.000	
75%	0.000	0.000		0.000	0.000	
90%	0.000	0.000		0.003	0.000	
95%	0.002	0.000		0.041	0.000	
99%	0.093	0.000		1.864	2.015	-7.457
99.5%	0.249	0.372	-33.079	4.978	4.979	-0.021
99.9%	0.998	1.241	-19.548	19.962	19.540	2.159
		$n = 500$			$n = 5000$	
50%	0.000	0.000		0.000	0.000	
75%	0.000	0.000		0.000	0.000	
90%	0.013	0.000		0.135	0.000	
95%	0.207	0.000		2.069	2.047	1.060
99%	9.322	9.356	-0.362	93.219	92.604	0.664
99.5%	24.888	25.179	-1.155	248.881	242.187	2.764
99.9%	99.811	92.704	7.667	998.114	1028.333	-2.939


Vasicek Model -- Rating CCC
VaR[L_n^*;α]
n
0.40
0.35
0.30
0.25
0
1000
2000
3000
4000
5000
α = 99%
α = 99.5%
α = 99.9%

3 Extending Basel II model

3.1 A new formulation of the Basle II model

$$Z_i = \sqrt{\rho} X + \sqrt{1-\rho}\varepsilon_i$$

$$P_i(t, X) = \Phi\left(\frac{\Phi^{-1}(1 - \mathbf{S}_i(t)) - \sqrt{\rho}X}{\sqrt{1-\rho}}\right)$$

$Z = (Z_1, \ldots, Z_I)$ is a Gaussian vector with a covariance matrix $\Sigma = C_I(\rho)$ which is equal to

$$\Sigma = \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & & \vdots \\ \vdots & & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{pmatrix}$$

The joint default probability is

$$\begin{aligned} P_{1,\ldots,I} &= \Pr\{D_1 = 1, \ldots, D_I = 1\} \\ &= \Pr\{Z_1 \leq B_1, \ldots, Z_I \leq B_I\} \\ &= \Phi(B_1, \ldots, B_I; \Sigma) \\ &= \Phi\left(\Phi^{-1}(P_1), \ldots, \Phi^{-1}(P_I); \Sigma\right) \\ &= \mathbf{C}(P_1, \ldots, P_I; \Sigma) \end{aligned}$$

with $\mathbf{C}$ the Normal copula with the matrix of canonical correlations $C_I(\rho)$.

If we now consider the joint survival function of default times, we have

$$\begin{aligned} \mathbf{S}(t_1, \ldots, t_I) &= \Pr\{\tau_1 > t_1, \ldots, \tau_I > t_I\} \\ &= \Pr\left\{Z_1 > \Phi^{-1}(P_1(t_1)), \ldots, Z_I > \Phi^{-1}(P_I(t_I))\right\} \\ &= \mathbf{C}(1 - P_1(t_1), \ldots, 1 - P_I(t_I); \Sigma) \\ &= \mathbf{C}(\mathbf{S}_1(t_1), \ldots, \mathbf{S}_I(t_I); \Sigma) \end{aligned}$$

3.2 The loss distribution

If we consider ‘zero coupon’ loans, we have

$$\mathbf{L} = \sum_{i=1}^{I} x_i \cdot (1 - R_i) \cdot \mathbf{1}\{\tau_i \le t_i\}$$

where x_i is the notional of the loan and R_i and τ_i are the recovery rate and the default time of the firm. The random variables are $R_1, \ldots, R_I$ and $\tau_1, \ldots, \tau_I$.

Assumptions:

1. The distributions of these random variables are given (because of internal credit rating system).
2. $R_i \perp \tau_i$
3. We have informations about default correlations between ‘sectors’.

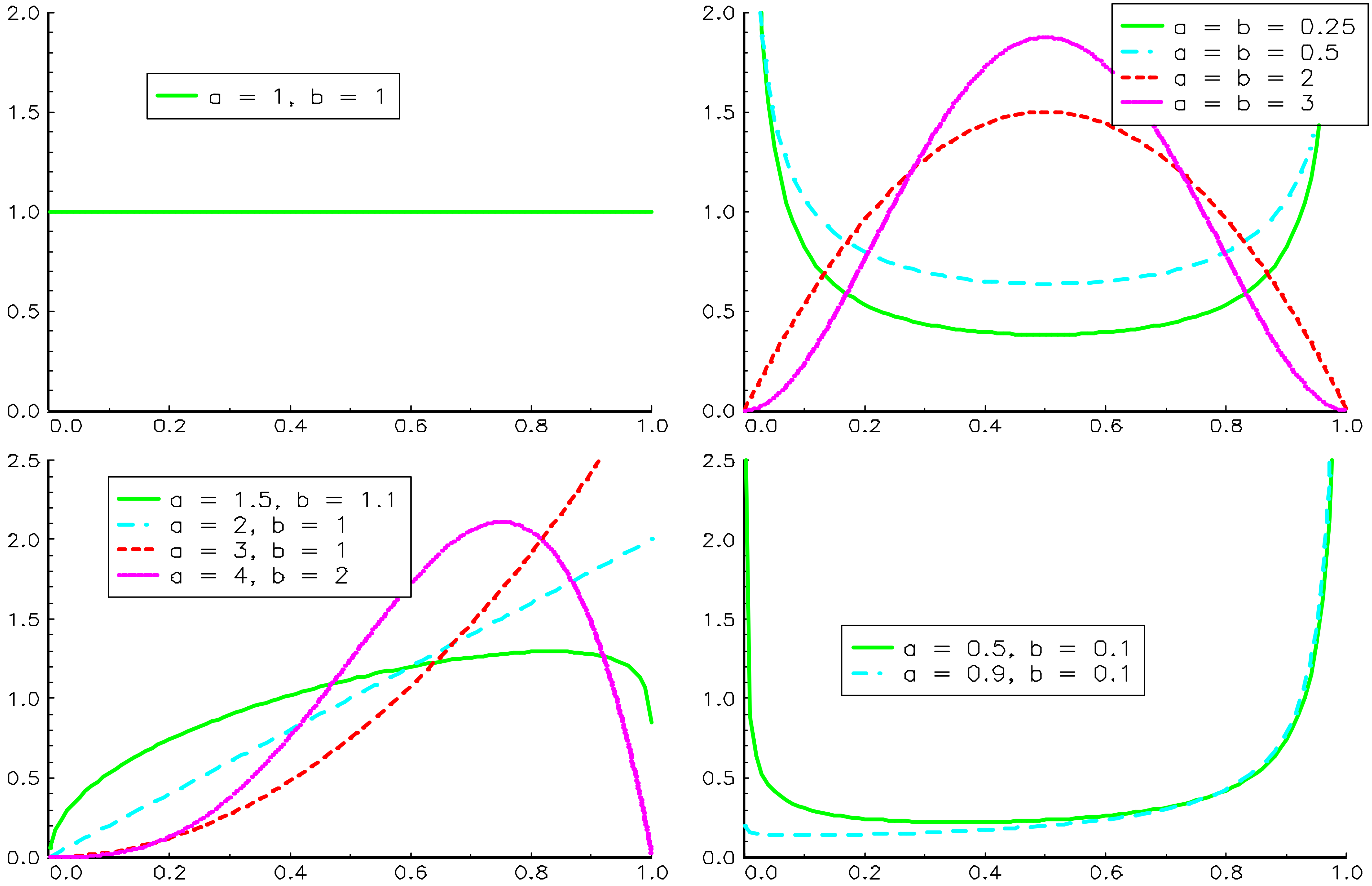
Introducing stochastic recovery rate

The standard of the industry is the Beta distribution:

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{\int_0^1 x^{a-1}(1-x)^{b-1}\, dx}$$

Given the first two moments $\mu(R)$ and $\sigma(R)$ of the recovery rate, we may estimate the parameters by the method of moments:

$$\begin{aligned} a &= \frac{\mu^2(R)(1-\mu(R))}{\sigma^2(R)} - \mu(R) \\ b &= \frac{\mu^2(R)(1-\mu(R))^2}{\mu(R)\sigma^2(R)} - (1-\mu(R)) \end{aligned}$$

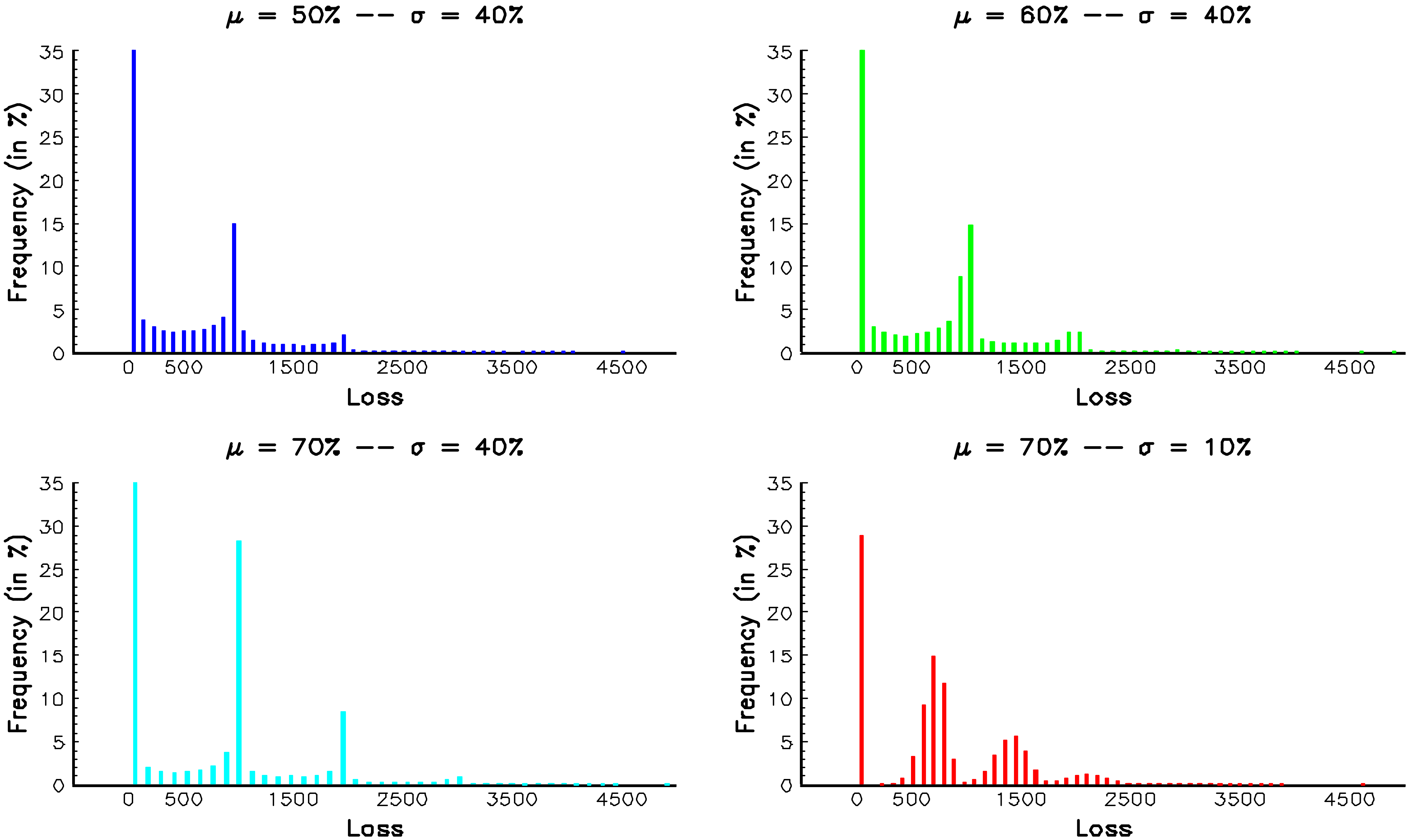


Density of the Beta Distribution

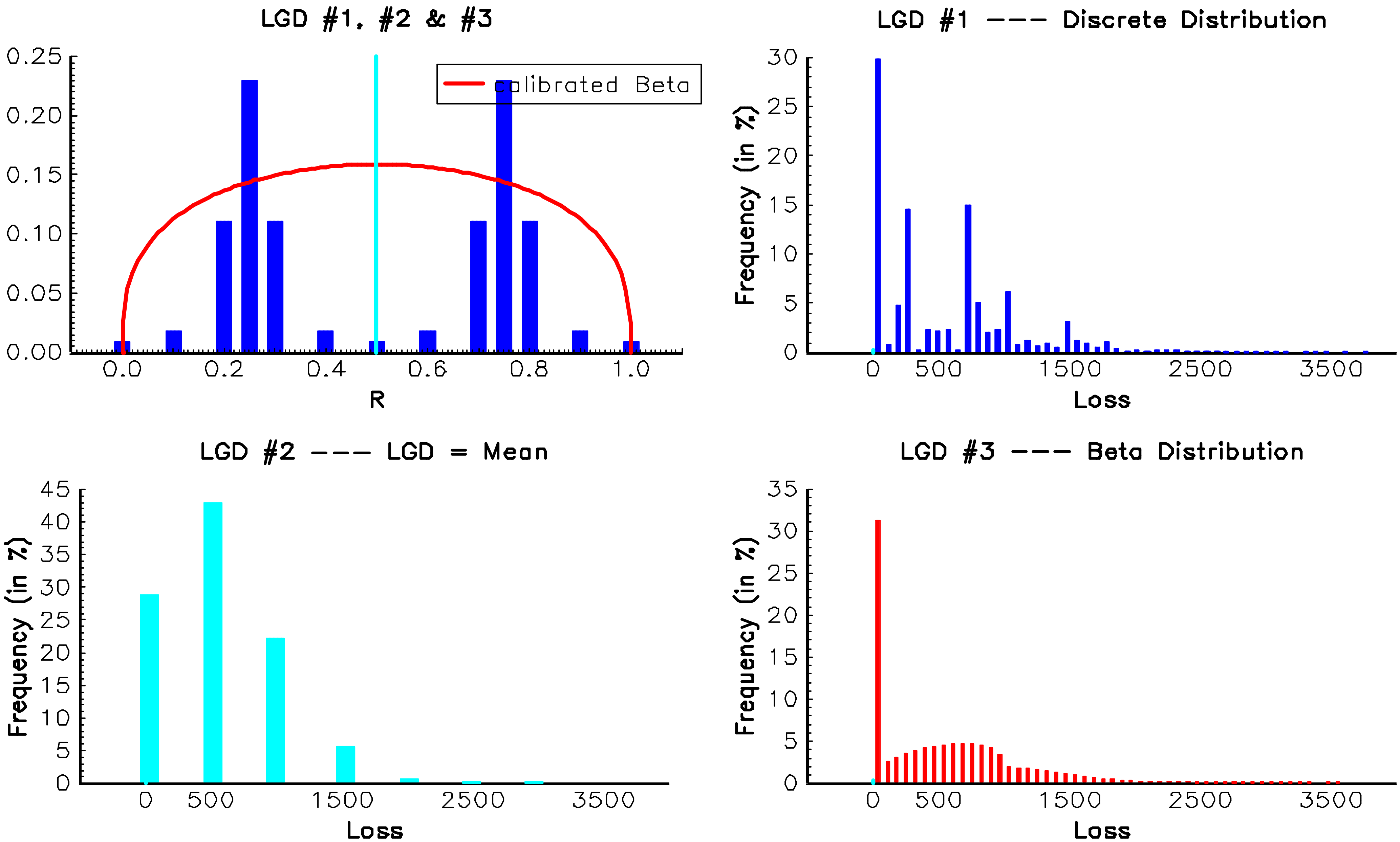
Proposition 1 *Given a random variable U in $[0,1]$, there exists (almost) always a random variable B with a Beta distribution such that $\mathbb{E}[B] = \mathbb{E}[U]$ and $\sigma[B] = \sigma[U]$.*

The idea of the proof is the following. Because $\mathbb{E}\left[U^2\right] \leq \mathbb{E}[U]$, we have $\sigma[U] \leq \sigma^{+}(\mathbb{E}[U]) = \sqrt{\mathbb{E}[U](1-\mathbb{E}[U])}$. For the Beta distribution, because $a > 0$ and $b > 0$, we have

$$\sigma(B) < \sqrt{\mathbb{E}[B](1-\mathbb{E}[B])}$$



Influence of the LGD Distribution on
a Portfolio of 10 Loans with 5Y Maturity



Influence of the LGD distribution
The Non Granularity Case

Modelling dependence of default times

We assume that Z_i depends on **one** factor :

$$Z_i = \sum_{j=1}^{J} \beta_{i,j} X_j + \varepsilon_i \qquad \text{with} \quad \sum_{j=1}^{J} \mathbf{1}\left\{\beta_{i,j} = 0\right\} = J - 1$$

with $X_j \perp \varepsilon_i$, but X_{j_1} and X_{j_2} are not necessarily independent.

Let $j = m(i)$ be the mapping function between the loan i and its sector j.

The survival time copula $(\tau_1, \ldots, \tau_I)$ is the Normal copula with the following matrix of canonical correlations :

$$\Sigma = \begin{pmatrix} 1 & \rho(m(1), m(2)) & \cdots & \rho(m(1), m(I)) \\ & 1 & & \vdots \\ & & & \rho(m(I-1), m(I)) \\ & & & 1 \end{pmatrix}$$

Consider the example with 4 sectors

Sector	1	2	3	4
1	**30%**	*20%*	*10%*	*0%*
2		**40%**	*30%*	*20%*
3			**50%**	*10%*
4				**60%**

and 7 loans

i	1	2	3	4	5	6	7
$j = m(i)$	1	1	2	3	3	3	4

The matrix of canonical correlations is then

$$\Sigma = \begin{pmatrix} 1.00 & 0.30 & 0.20 & 0.10 & 0.10 & 0.10 & 0.00 \\ & 1.00 & 0.20 & 0.10 & 0.10 & 0.10 & 0.00 \\ & & 1.00 & 0.30 & 0.30 & 0.30 & 0.20 \\ & & & 1.00 & 0.50 & 0.50 & 0.10 \\ & & & & 1.00 & 0.50 & 0.10 \\ & & & & & 1.00 & 0.10 \\ & & & & & & 1.00 \end{pmatrix}$$

The fast [Sloane] algorithm (from Riboulet and Roncalli [2002])

We want to simulate r.v. $(u_1, \ldots, u_I)$ from the **Normal** copula.

The [CHOL] algorithm is

$$\begin{aligned} P &= \operatorname{chol}(\Sigma) \\ \mathbf{z} &= P\varepsilon \text{ with } \varepsilon_{i_1} \perp \varepsilon_{i_2} \\ u_i &= \Phi(z_i) \end{aligned}$$

This algorithm is time-consuming and memory-consuming :

I	100	1000	10000
Memory size	78.125 Kb	7.629 Mb	762.94 Mb

If Σ is $C_I(\rho)$, the $[\sqrt{\rho}]$ algorithm is more efficient:

$$\begin{aligned} z_i &= \sqrt{\rho}x + \sqrt{1-\rho}\varepsilon_i \text{ with } x \perp \varepsilon_{i_1} \perp \varepsilon_{i_2} \\ u_i &= \Phi(z_i) \end{aligned}$$

Let $\rho^\star$ be the symmetric matrix with $\rho^\star_{j,j}$ the intra-sector canonical correlations and $\rho^\star_{j_1,j_2}$ the inter-sector canonical correlations.
$\rho^\star$ is not a correlation matrix.

The [Sloane] algorithm is the following:

$$
\begin{aligned}
\rho^\star &= V^\star \Lambda^\star V^{\star\top} \text{ (eigensystem)} \\
A^\star &= V^\star_\bullet \left(\Lambda^\star\right)^{\frac{1}{2}} \text{ } (V^\star_\bullet \text{ is the } L_2\text{-normalized matrix of } V^\star) \\
z_i &= \sum_{j=1}^{J} A^\star_{m(i),j} x_j + \sqrt{1-\rho^\star\left(m\left(i\right), m\left(i\right)\right)}\varepsilon_i \text{ with } x_{j_1} \perp x_{j_2} \perp \varepsilon_{i_1} \perp \varepsilon_{i_2} \\
u_i &= \Phi\left(z_i\right)
\end{aligned}
$$

If $J = 1$, [Sloane] $=$ [$\sqrt{\rho}$].

Proposition 2 *If the eigenvalues λ^*_j are positive, then Σ is a correlation matrix.*

The algorithm order of [CHOL] is I^2.

The algorithm order of [Sloane] is I (because J is fixed).

	Dimension of the matrix	Number of random variates	Number of $+$ operations	Number of $\times$ operations
[CHOL]	$I \times I$	I	$I \times (I-1)$	$I \times I$
[Sloane]	$J \times J$	$I + J$	$I \times J$	$I \times J$
	10000 loans + 20 sectors			
[CHOL]	10^8	10000	$\simeq 10^8$	10^8
[Sloane]	400	10020	2×10^5	2×10^5

3.3 An example

500 loans, 5Y maturity, EaD $= 1000$, $\mu(R) = 50\%$, $\sigma(R) = 20\%$.

The $\rho^{\star}$ matrix is

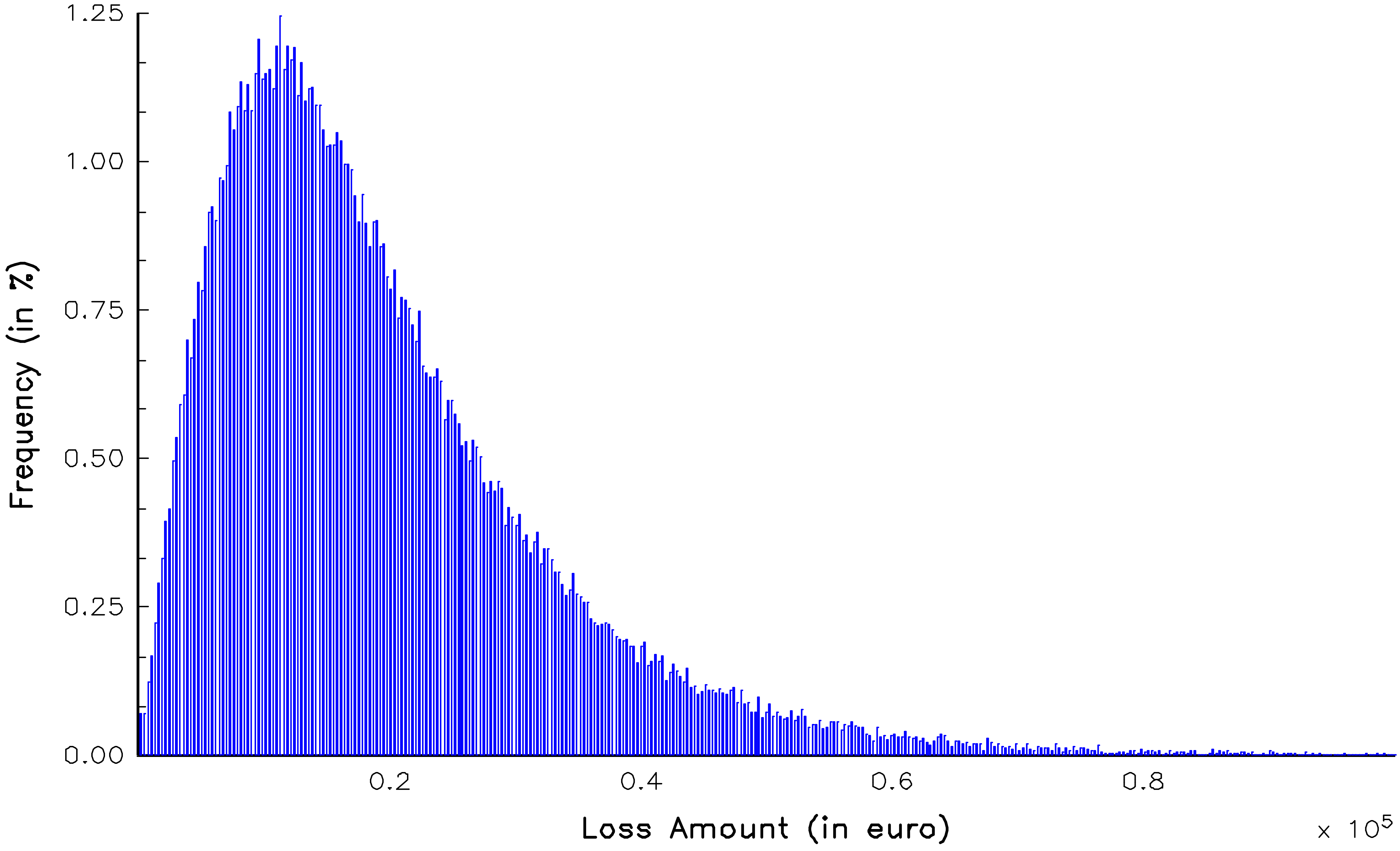
Sector	1	2	3	4
1	**30%**	*20%*	*10%*	*0%*
2		**40%**	*30%*	*20%*
3			**50%**	*10%*
4				**60%**

The repartition by ratings is

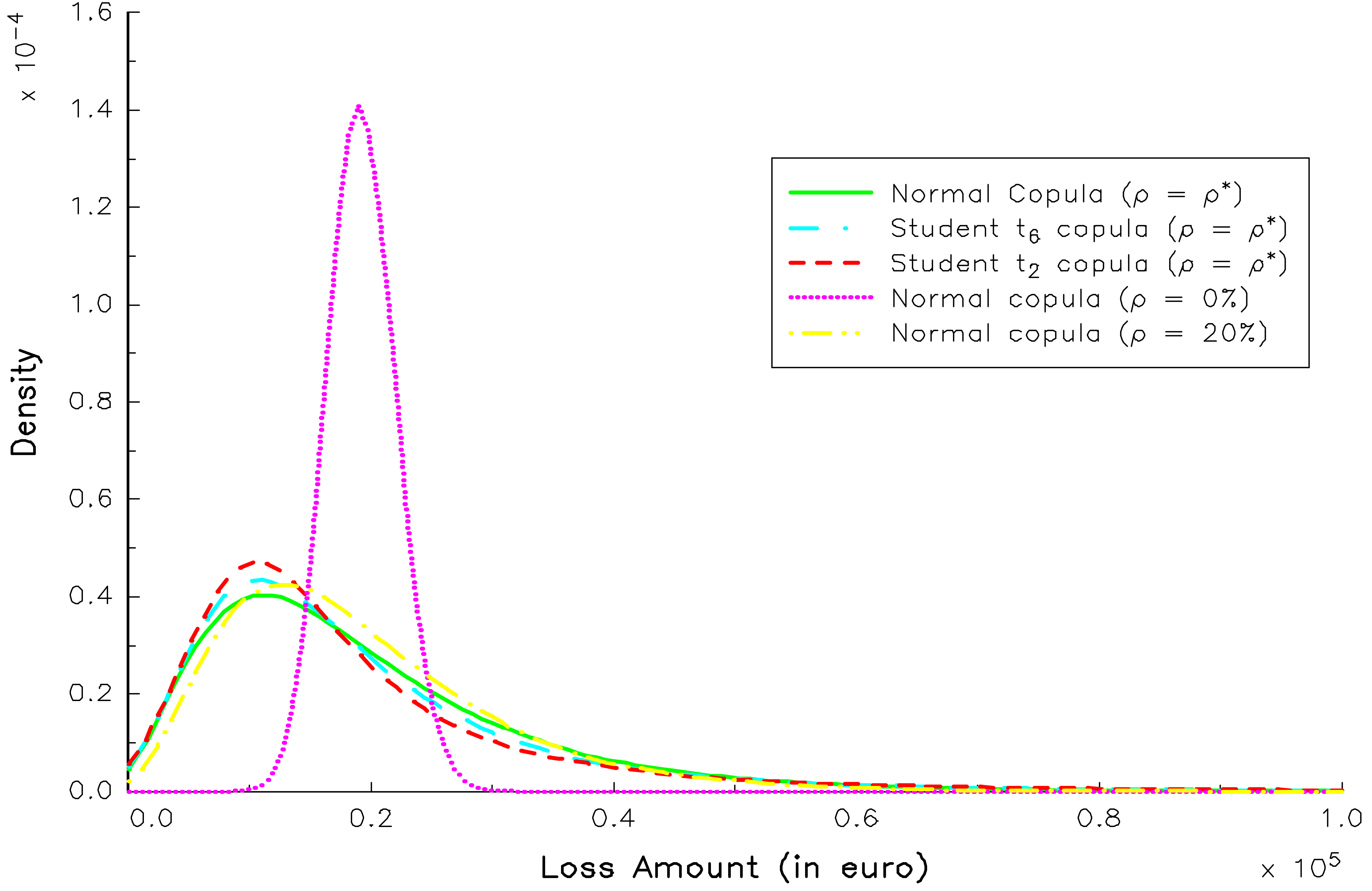
Rating	AAA	AA	A	BBB	BB	B	CCC
Number of loans	5%	15%	20%	30%	15%	10%	5%

The repartition by sectors is

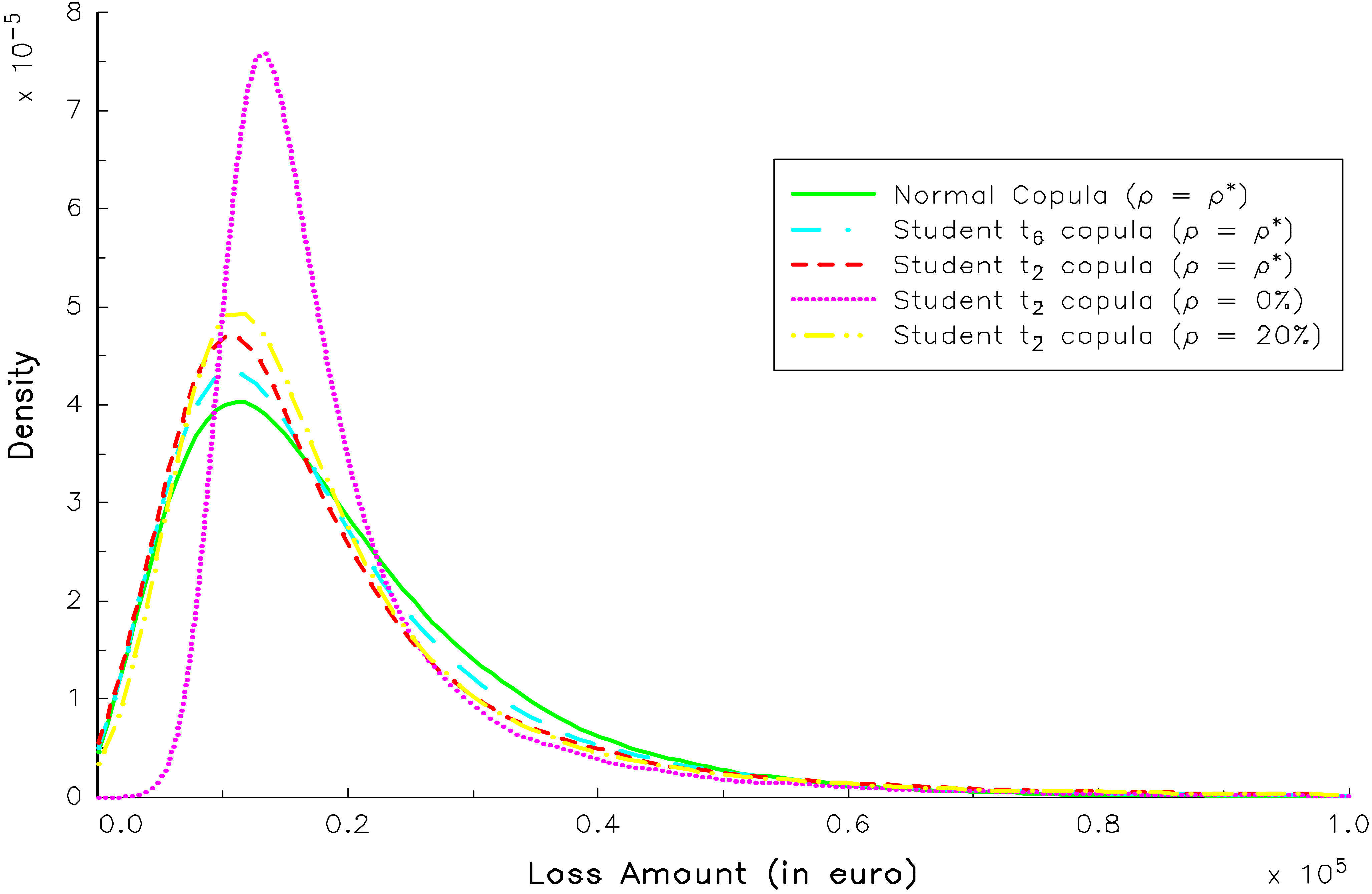
Sector	1	2	3	4
Number of loans	20%	30%	10%	40%



Frequency of the Loss Distribution
(Normal Copula/Sector Correlations)



Density of the Loss Distribution



Density of the Loss Distribution

4 The Measurement of Credit Risk

4.1 The Credit Risk Measure

1. **Value-at-Risk**

$$\text{CreditVaR}\,(\alpha) = \inf\left\{L : \Pr\left\{L\,(t) \leq L\right\} \geq \alpha\right\}$$

2. **Expected Regret**

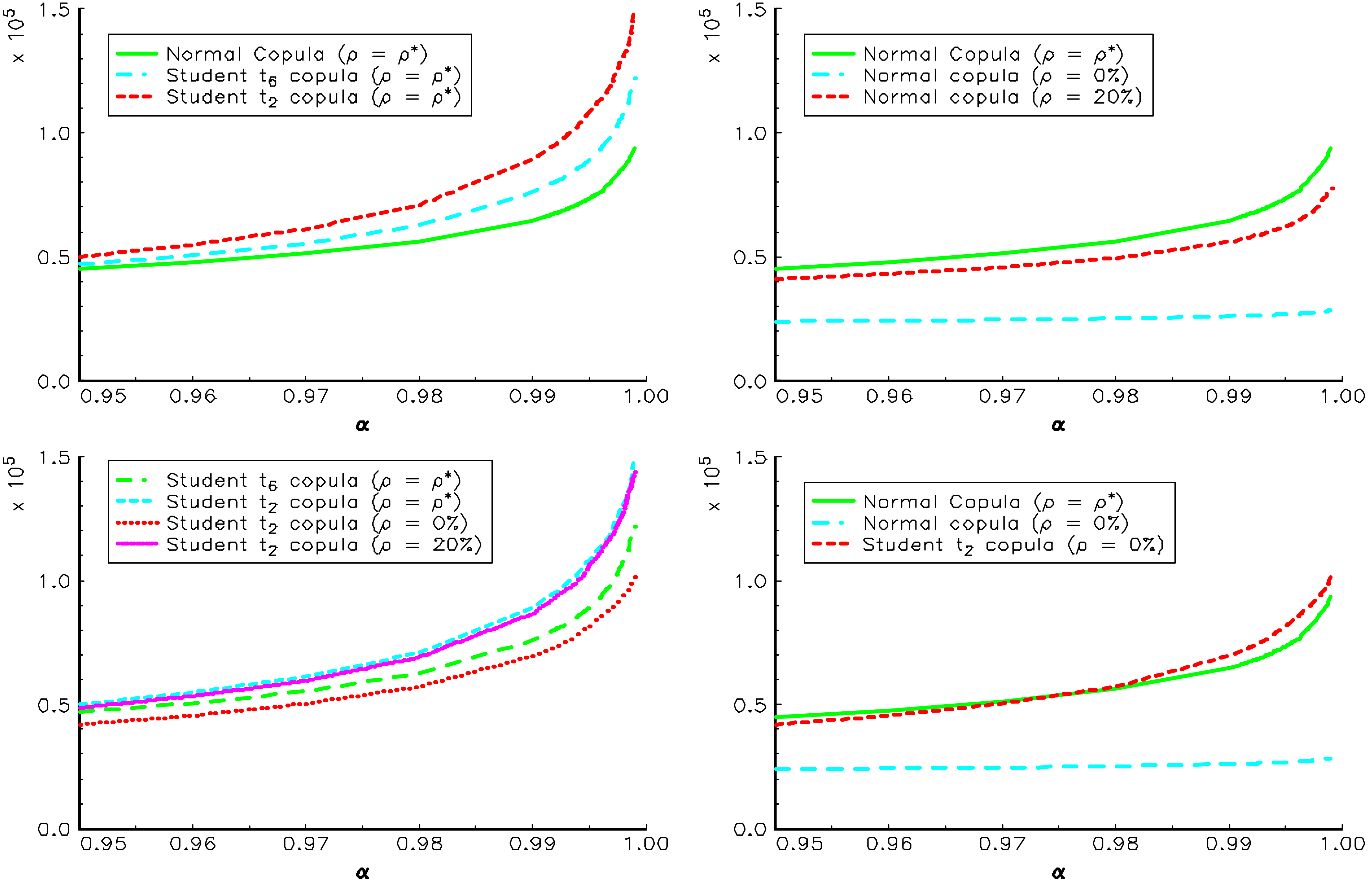
$$\text{ER}\,(\bar{L}) = \mathbb{E}\left[L\,(t) \mid L\,(t) \geq \bar{L}\right]$$

3. **Expected Shortfall**

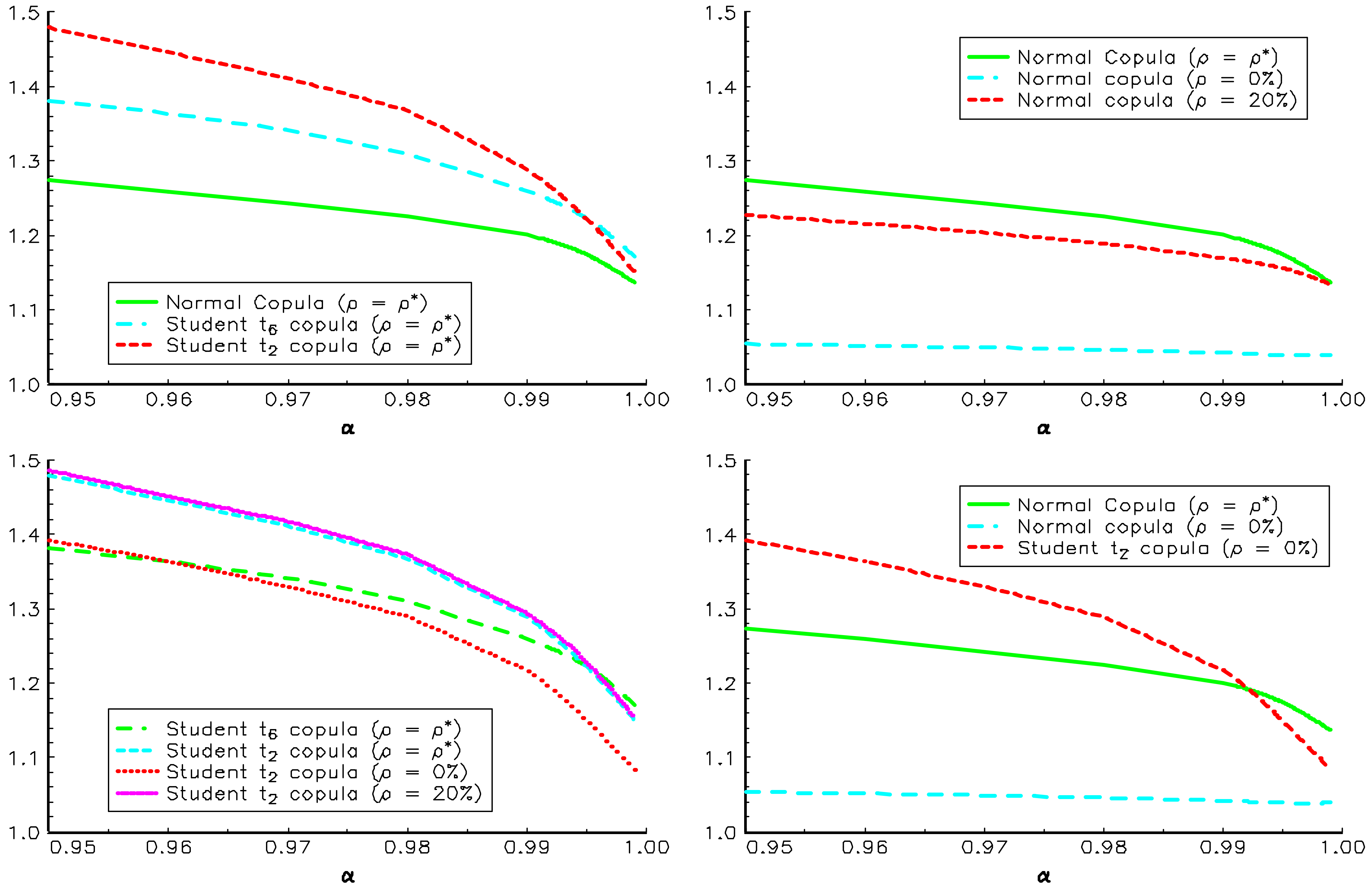
$$\text{ES}\,(\alpha) = \mathbb{E}\left[L\,(t) \mid L\,(t) \geq \text{CreditVaR}\,(\alpha)\right]$$

4. **Unexpected Loss**

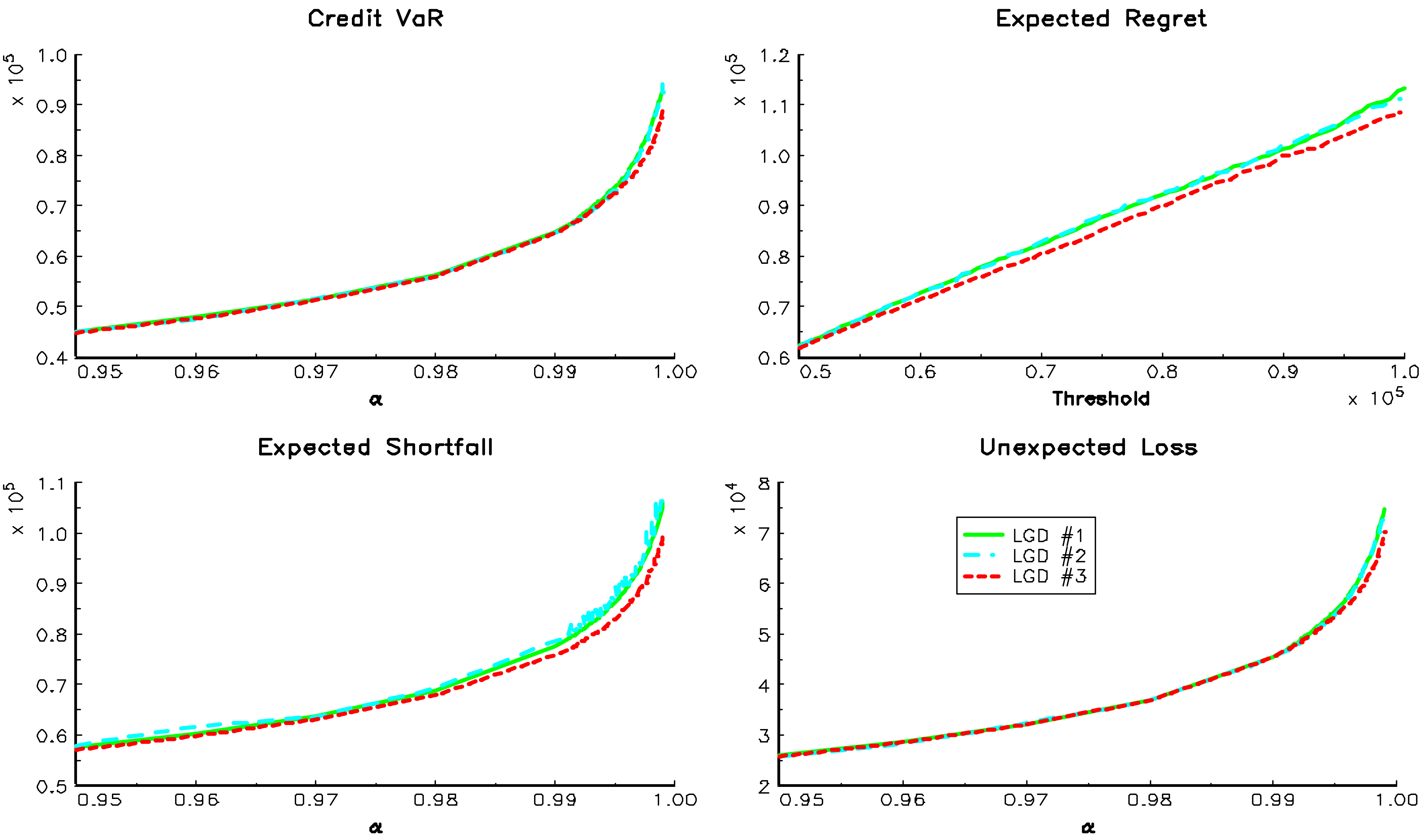
$$\text{UL}\,(\alpha) = \text{CreditVaR}\,(\alpha) - \mathbb{E}\left[L\,(t)\right]$$



Portfolio Credit VaR



Ratio Expected Shortfall / Credit VaR



Influence of the LGD distribution
The Granularity Case

4.2 The Risk Contribution

The *discrete marginal contribution* is defined as follows:

$$\text{RC}(i) = \text{Risk}\left(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_I\right) - \text{Risk}\left(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_I\right)$$

We have

$$\text{Risk} \neq \sum_{i=1}^{I} \text{RC}(i)$$

In the following table, we report the values of $\sum_{i=1}^{I} \text{RC}(i) \Big/ \text{Risk}$:

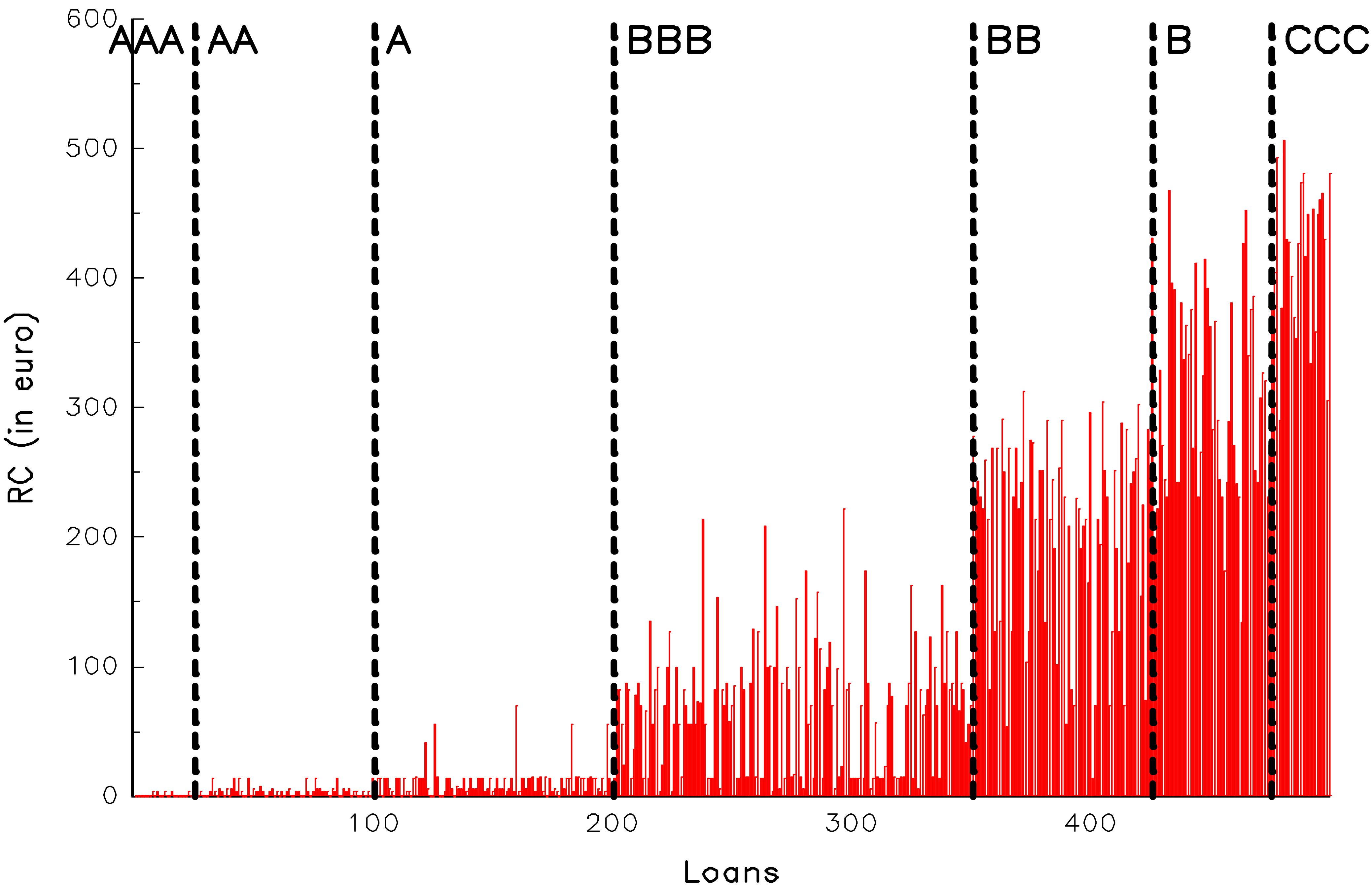
α	CreditVaR Normal copula	ES Normal copula	CreditVaR t_6 copula
95%	88.7%	99.0%	99.0%
99%	81.1%	99.1%	92.4%
99.9%	79.4%	99.4%	135%

For a set $\mathcal{A}$ of loans, we have

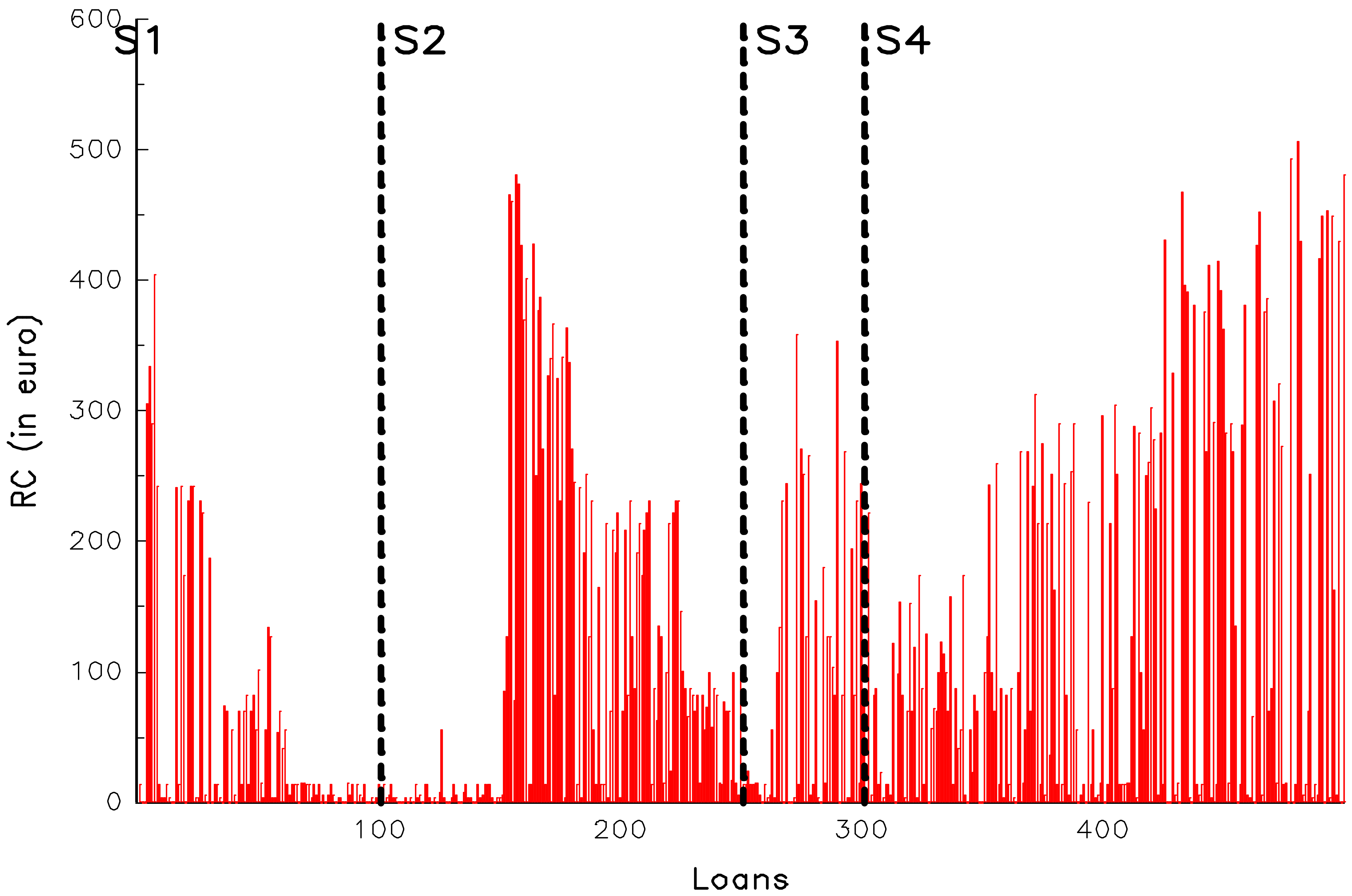
$$\mathsf{RC}\,(\mathcal{A}) = \mathsf{Risk} - \mathsf{Risk}\,(x_i \notin \mathcal{A})$$

For example, with the CreditVaR measure, we have

α	95%		99%		99.9%	
Sector	$\sum_{i\in\mathcal{A}} \mathsf{RC}\,(i)$	$\mathsf{RC}\,(\mathcal{A})$	$\sum_{i\in\mathcal{A}} \mathsf{RC}\,(i)$	$\mathsf{RC}\,(\mathcal{A})$	$\sum_{i\in\mathcal{A}} \mathsf{RC}\,(i)$	$\mathsf{RC}\,(\mathcal{A})$
1	5075	5208	5295	5721	5829	4836
2	12317	12908	16158	16168	28797	20356
3	4440	4495	4484	5046	8086	6153
4	18118	17195	26448	24683	31600	36756
$\sum_{\mathcal{A}}$	39950	39806	52384	51619	74314	68100
Risk	45063		64592		93581	



Risk Contribution (CreditVaR 99%)



Risk Contribution (CreditVaR 99%)

4.3 The Risk Sensitivity

We have

$$\mathsf{DR}\,(i)=\frac{\partial\,\mathsf{Risk}\,(x_1,\ldots,x_I)}{\partial\,x_i}$$

For example, with the ER measure, we have

$$\mathsf{DR}\,(i)=\frac{\mathbb{E}\left[(1-R_i)\cdot\mathbf{1}\left\{\tau_i\leq t_i\right\}\cdot\mathbf{1}\left\{L\,(t)\geq\bar{L}\right\}\right]}{\Pr\left\{L\,(t)\geq\bar{L}\right\}}$$

and

$$\sum_{i=1}^{I}\mathsf{cRC}\,(i)=\sum_{i=1}^{I}x_i\cdot\mathsf{DR}\,(i)=\mathsf{ER}\left(\bar{L}\right)$$

$\mathsf{cRC}\,(i)$ is the *continuous marginal contribution*.

The CreditVaR sensitivity

Theoretical result of Gouriéroux, Laurent and Scaillet [2000]

Theorem 1 *Let* $(\varepsilon_1, \ldots, \varepsilon_I)$ *be a random vector and* $(x_1, \ldots, x_I)$ *a vector in* $\mathbb{R}^I$. *We consider the loss* L *defined by*

$$L = \sum_{i=1}^{I} x_i \cdot \varepsilon_i$$

Let $Q(L; \alpha)$ *the percentile* α *of* L. *We have*

$$\frac{\partial\, Q(L; \alpha)}{\partial\, x_i} = \mathbb{E}\left[\varepsilon_i \mid L = Q(L; \alpha)\right]$$

The Gaussian case $L=\sum_{i=1}^{I}x_i\cdot\varepsilon_i$ with $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_I)\sim\mathcal{N}(\mu,\Sigma)$. We have $L\sim\mathcal{N}\left(\mathbf{x}^\top\mu,\mathbf{x}^\top\Sigma\mathbf{x}\right)$ and $Q(L;\alpha)=\mathbf{x}^\top\mu+\Phi^{-1}(\alpha)\sqrt{\mathbf{x}^\top\Sigma\mathbf{x}}$. The derivatives are

$$\frac{\partial\,Q(L;\alpha)}{\partial\,\mathbf{x}}=\mu+\Phi^{-1}(\alpha)\frac{\Sigma\mathbf{x}}{\sqrt{\mathbf{x}^\top\Sigma\mathbf{x}}}$$

We remark that

$$\begin{pmatrix}\varepsilon\\ L\end{pmatrix}\sim\mathcal{N}\left(\begin{pmatrix}\mu\\ \mathbf{x}^\top\mu\end{pmatrix},\begin{pmatrix}\Sigma & \Sigma\mathbf{x}\\ \mathbf{x}^\top\Sigma & \mathbf{x}^\top\Sigma\mathbf{x}\end{pmatrix}\right)$$

It comes that $\varepsilon\mid L=\ell\sim\mathcal{N}\left(\mu_{\varepsilon|L},\Sigma_{\varepsilon|L}\right)$ with $\mu_{\varepsilon|L}=\mu+\Sigma\mathbf{x}\left(\mathbf{x}^\top\Sigma\mathbf{x}\right)^{-1}\left(\ell-\mathbf{x}^\top\mu\right)$ and $\Sigma_{\varepsilon|L}=\Sigma-\Sigma\mathbf{x}\left(\mathbf{x}^\top\Sigma\mathbf{x}\right)^{-1}\mathbf{x}^\top\Sigma$. We deduce that

$$\begin{aligned}
\mathbb{E}\left[\varepsilon\mid L=Q(L;\alpha)\right] &= \mathbb{E}\left[\varepsilon\mid L=\mathbf{x}^\top\mu+\Phi^{-1}(\alpha)\sqrt{\mathbf{x}^\top\Sigma\mathbf{x}}\right]\\
&= \mu+\Sigma\mathbf{x}\left(\mathbf{x}^\top\Sigma\mathbf{x}\right)^{-1}\left(\mathbf{x}^\top\mu+\Phi^{-1}(\alpha)\sqrt{\mathbf{x}^\top\Sigma\mathbf{x}}-\mathbf{x}^\top\mu\right)\\
&= \mu+\Phi^{-1}(\alpha)\Sigma\mathbf{x}\frac{\sqrt{\mathbf{x}^\top\Sigma\mathbf{x}}}{\left(\mathbf{x}^\top\Sigma\mathbf{x}\right)^{-1}}\\
&= \frac{\partial\,Q(L;\alpha)}{\partial\,\mathbf{x}}
\end{aligned}$$

Application to the credit loss

We have

$$\frac{\partial \text{ CreditVaR}\,(\alpha)}{\partial\, x_k} = \mathbb{E}\left[(1 - R_k) \cdot \mathbf{1}\left\{\tau_k \leq t_k\right\} \mid \sum_{i=1}^{I} x_i \cdot (1 - R_i) \cdot \mathbf{1}\left\{\tau_i \leq t_i\right\} = \text{CreditVaR}\,(\alpha)\right]$$

Numerical computation

$$\text{CreditVaR}(\alpha) = L_{n\alpha:n}$$

If $n\alpha = \lfloor n\alpha \rfloor$, we have $\text{CreditVaR}(\alpha) = L_{\kappa_{n\alpha}}$ and

$$\frac{\partial \, \text{CreditVaR}(\alpha)}{\partial \, x_i} = \left(1 - R_{i,\kappa_{n\alpha}}\right) \cdot \mathbf{1}\left\{\tau_{i,\kappa_{n\alpha}} \leq t_i\right\}$$

If $n\alpha > \lfloor n\alpha \rfloor$, we use the linear interpolation

$$\begin{aligned} \text{CreditVaR}(\alpha) &= (1 - n\alpha + \lfloor n\alpha \rfloor) L_{\kappa_{\lfloor n\alpha \rfloor}} + (n\alpha - \lfloor n\alpha \rfloor) L_{\kappa_{\lfloor n\alpha \rfloor + 1}} \\ &= L_{\kappa_{\lfloor n\alpha \rfloor}} + (n\alpha - \lfloor n\alpha \rfloor)\left(L_{\kappa_{\lfloor n\alpha \rfloor + 1}} - L_{\kappa_{\lfloor n\alpha \rfloor}}\right) \end{aligned}$$

We have

$$\begin{aligned} \frac{\partial \, \text{CreditVaR}(\alpha)}{\partial \, x_i} &= (1 - n\alpha + \lfloor n\alpha \rfloor)\left(\left(1 - R_{i,\kappa_{\lfloor n\alpha \rfloor}}\right) \cdot \mathbf{1}\left\{\tau_{i,\kappa_{\lfloor n\alpha \rfloor}} \leq t_i\right\}\right) + \\ &\quad (n\alpha - \lfloor n\alpha \rfloor)\left(\left(1 - R_{i,\kappa_{\lfloor n\alpha \rfloor + 1}}\right) \cdot \mathbf{1}\left\{\tau_{i,\kappa_{\lfloor n\alpha \rfloor + 1}} \leq t_i\right\}\right) \end{aligned}$$

$\Rightarrow$ large variance of estimates.

The localization method We suppose that

$$\text{CreditVaR}(\alpha) = \sum_{m \in \mathcal{M}} p_m L_m$$

where $\sum_{m \in \mathcal{M}} p_m = 1$. Under the measure of probability $\{p_m, m \in \mathcal{M}\}$, we have

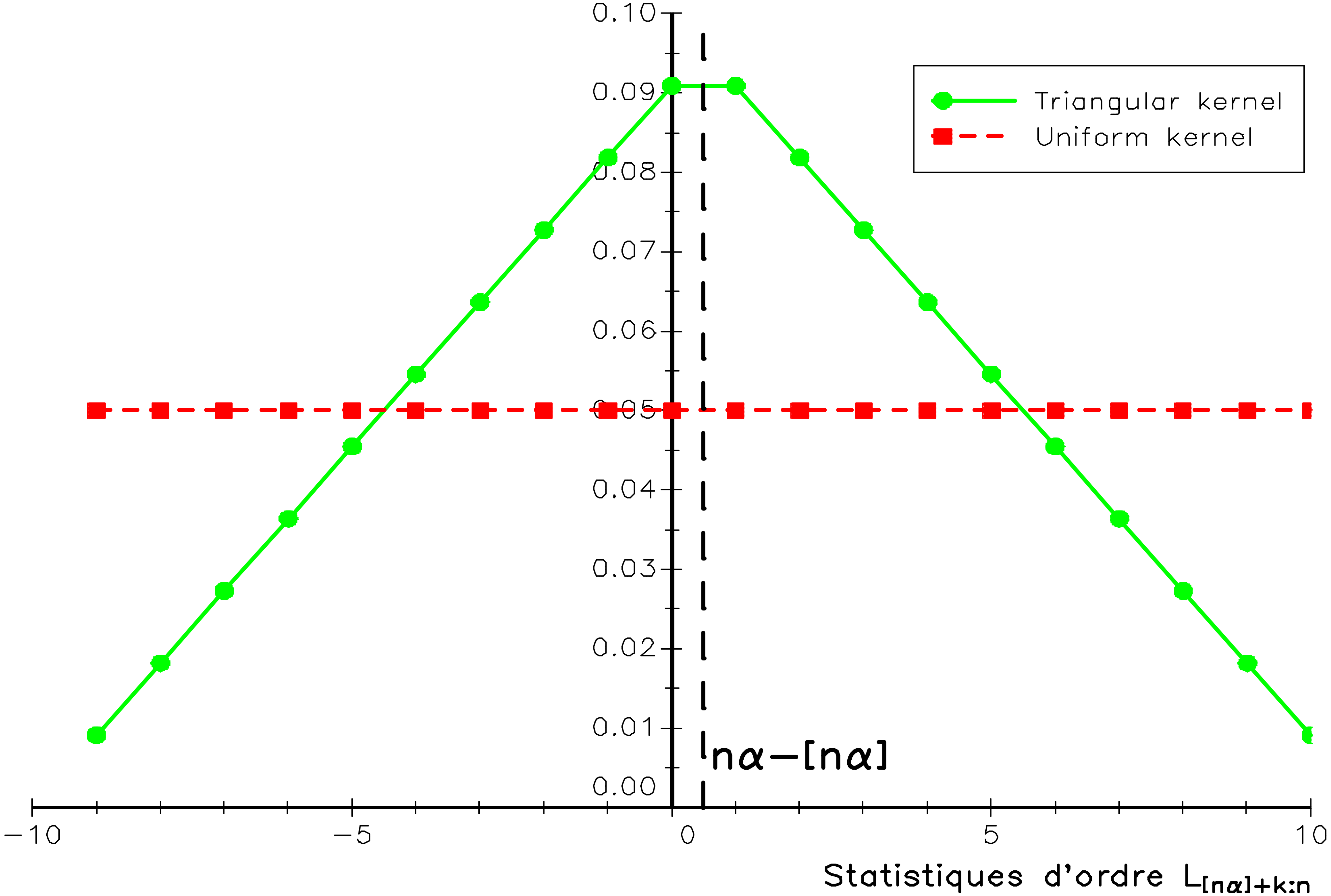
$$\frac{\partial\, \text{CreditVaR}(\alpha)}{\partial\, x_i} = \sum_{m \in \mathcal{M}} p_m \frac{L_{i,m}(t)}{x_i}$$

We pose $\mathcal{M} = \left\{\kappa_{\lfloor n\alpha \rfloor - h}, \ldots, \kappa_{\lfloor n\alpha \rfloor}, \kappa_{\lfloor n\alpha \rfloor + 1}, \ldots, \kappa_{\lfloor n\alpha \rfloor + h}\right\}$ with a triangular kernel:

$$p_{\kappa_{\lfloor n\alpha \rfloor + k}} = \begin{cases} \frac{h+1-k}{h+1-(n\alpha - \lfloor n\alpha \rfloor)} & \text{if } k > 0 \\ \frac{h+k}{h+(n\alpha - \lfloor n\alpha \rfloor)} & \text{if } k \leq 0 \end{cases}$$

or a uniform kernel:

$$p_{\kappa_{\lfloor n\alpha \rfloor + k}} = \frac{1}{2h}$$



Triangular and Uniform Kernels

Numerical experiments

$$L = \sum_{i=1}^{2} x_i \cdot \varepsilon_i$$

with

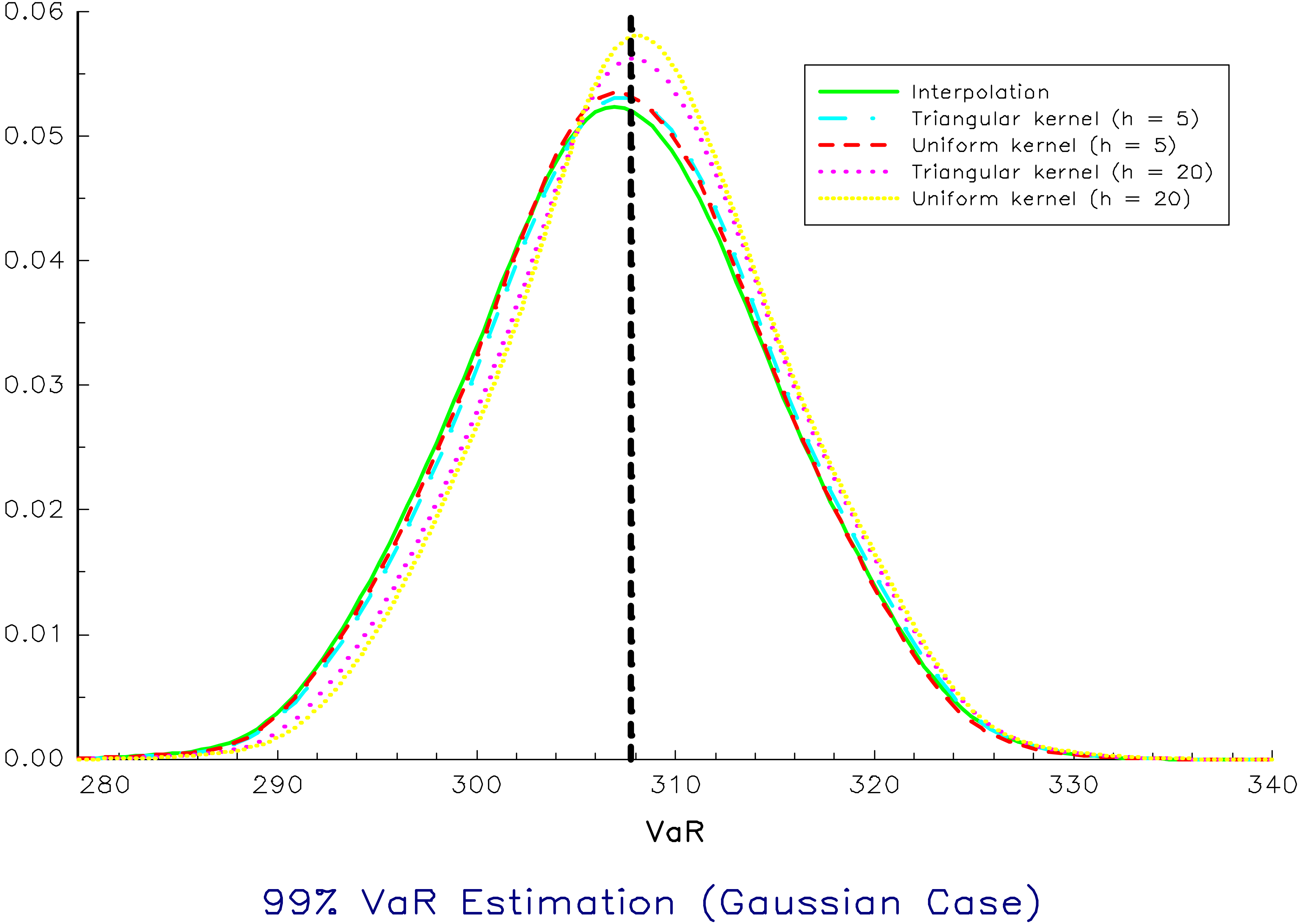
$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}\right)$$

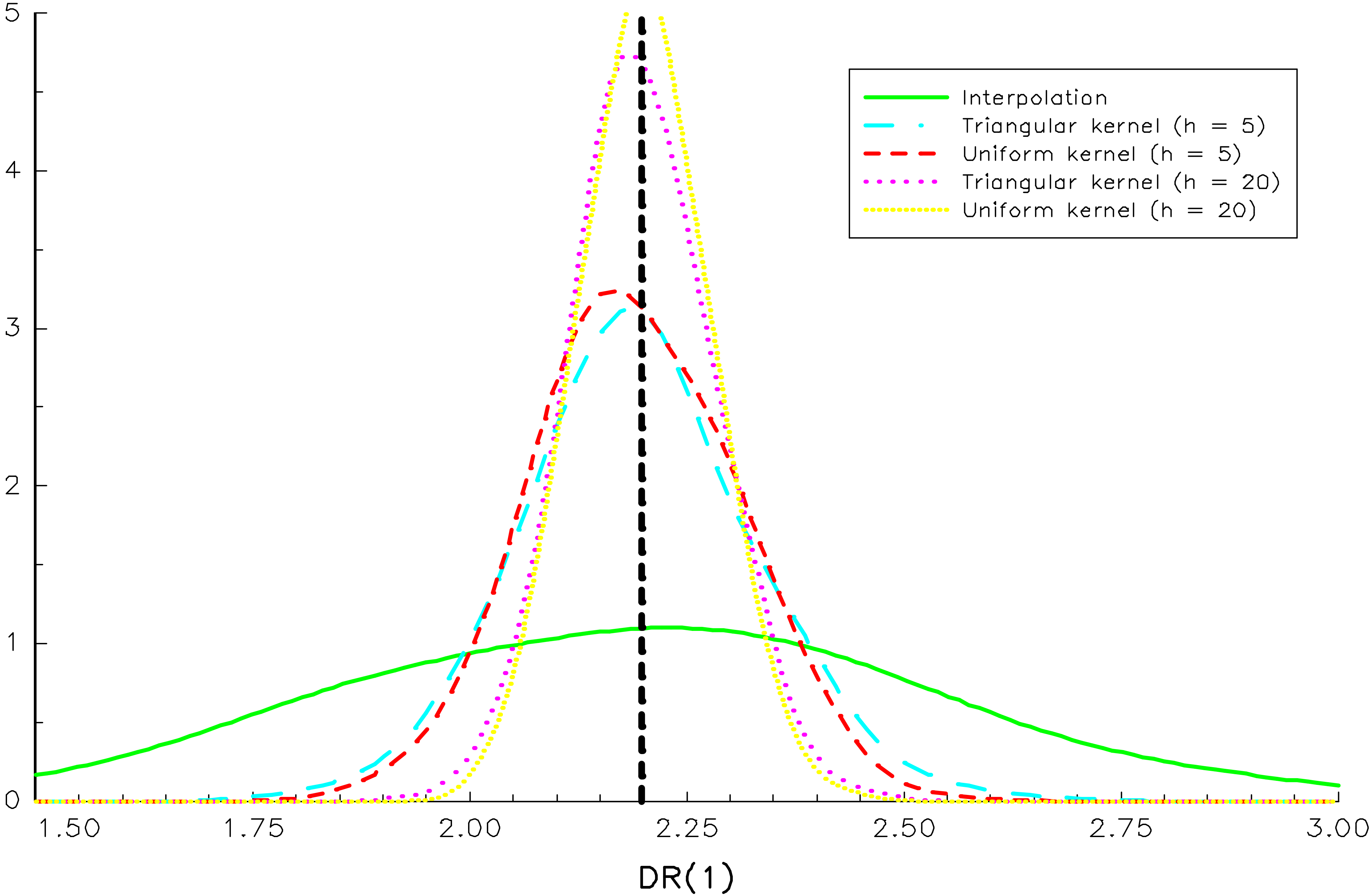
and $x_1 = 100$ and $x_2 = 50$.

Analytical calculus gives $\text{CreditVaR}\,(99\%) = 307.7469$, $\text{DR}\,(1) = 2.1981921$ and $\text{DR}\,(2) = 1.7585537$.

We remark that

$$307.7469 = x_1 \times 2.1981921 + x_2 \times 1.7585537$$


0.06
0.05
0.04
0.03
0.02
0.01
0.00
280
290
300
310
320
330
340
VaR
Interpolation
Triangular kernel (h = 5)
Uniform kernel (h = 5)
Triangular kernel (h = 20)
Uniform kernel (h = 20)
99% VaR Estimation (Gaussian Case)



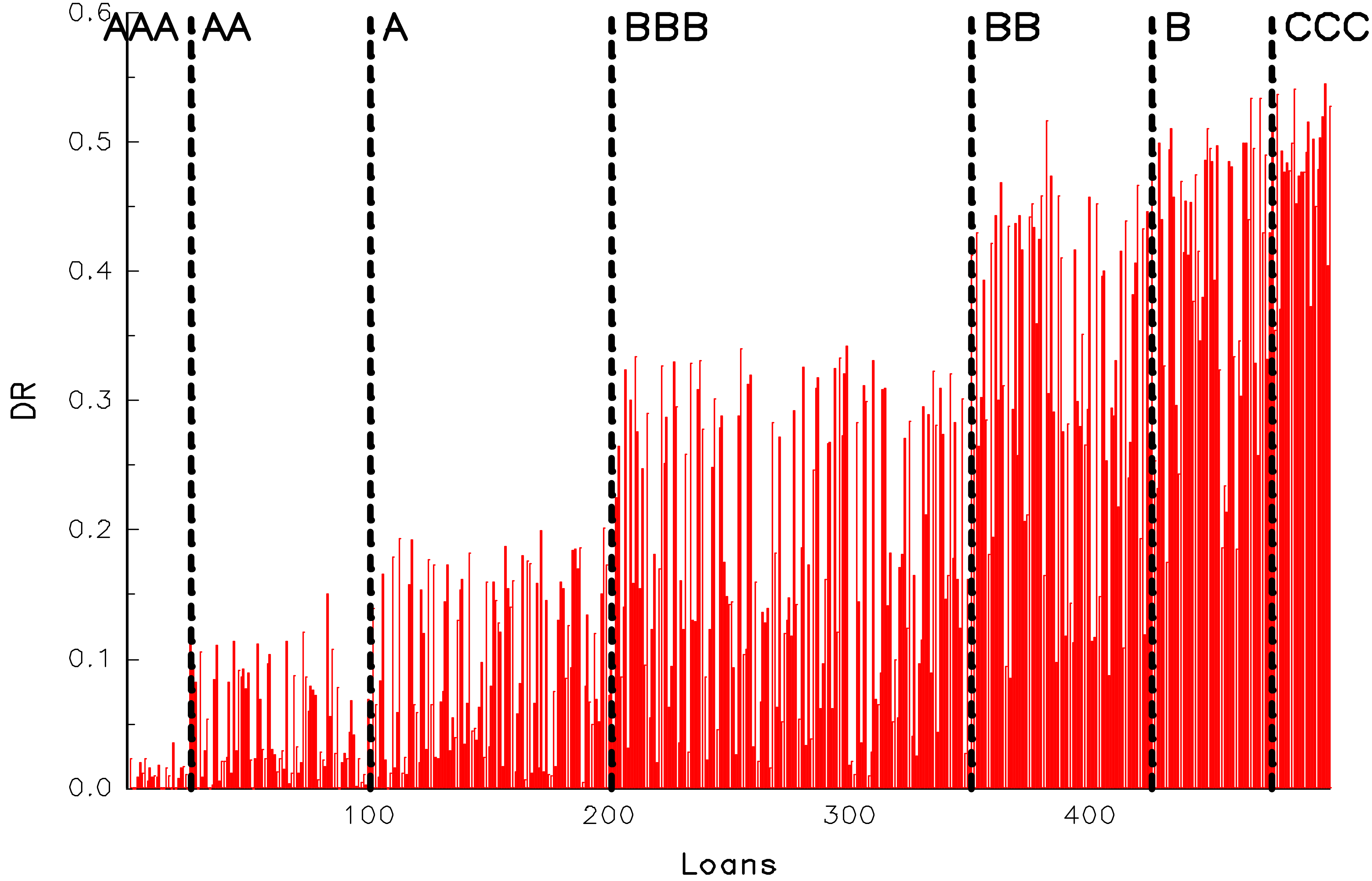
Risk Sensitivity (Gaussian Case -- CreditVaR 99%)

Main result

Proposition 3 *Because the* CreditVaR *is expressed in terms of order statistics, we have*

$$\text{CreditVaR}(\alpha) = \sum_{i=1}^{I} x_i \frac{\partial\, \text{CreditVaR}(\alpha)}{\partial\, x_i}$$

rating/sector	1	2	3	4	Total by rating
AAA	0	81	13	170	264
AA	40	725	137	2752	3654
A	328	1849	199	7061	9437
BBB	1308	6718	1430	16661	26117
BB	1362	6988	1592	13488	23430
B	2275	4211	3019	10323	19827
CCC	1502	4983	902	4561	11948
Total by sector	6816	25554	7291	55015	94676 = CreditVaR



Risk Sensitivity

5 Credit Portfolio Management

5.1 The pair Risk/return

We define the Risk Adjusted Performance measure by

$$\mathrm{RAPM} = \frac{(\mathrm{Euribor} + \mathrm{Sp})}{\mathrm{Risk}}$$

For a loan, we have

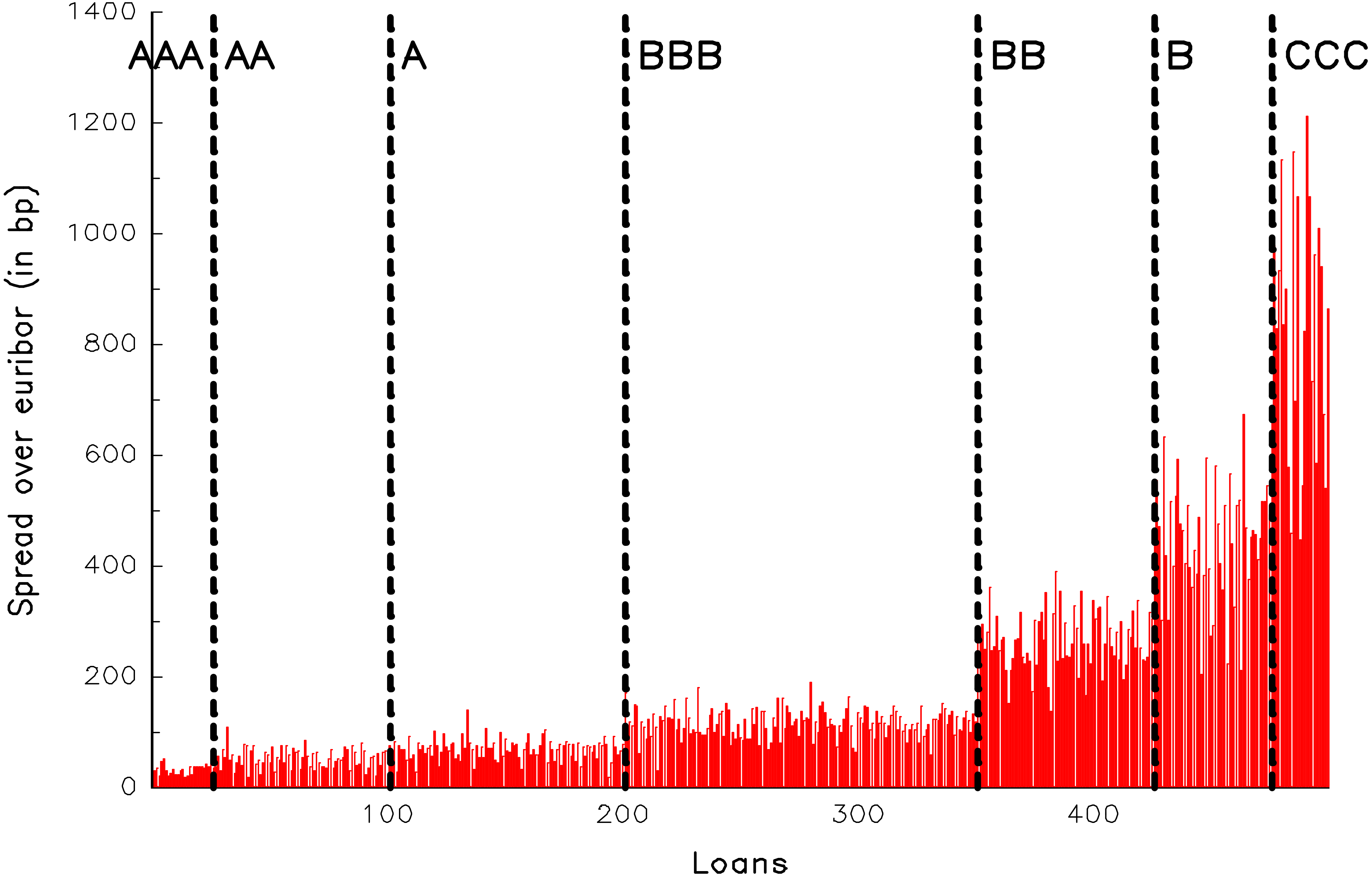
$$\mathrm{RAPM}\,(i) = \frac{x_i \cdot (\mathrm{Euribor} + \mathrm{Sp}\,(i))}{\mathrm{Risk}\,(i)}$$

For a portfolio, we have

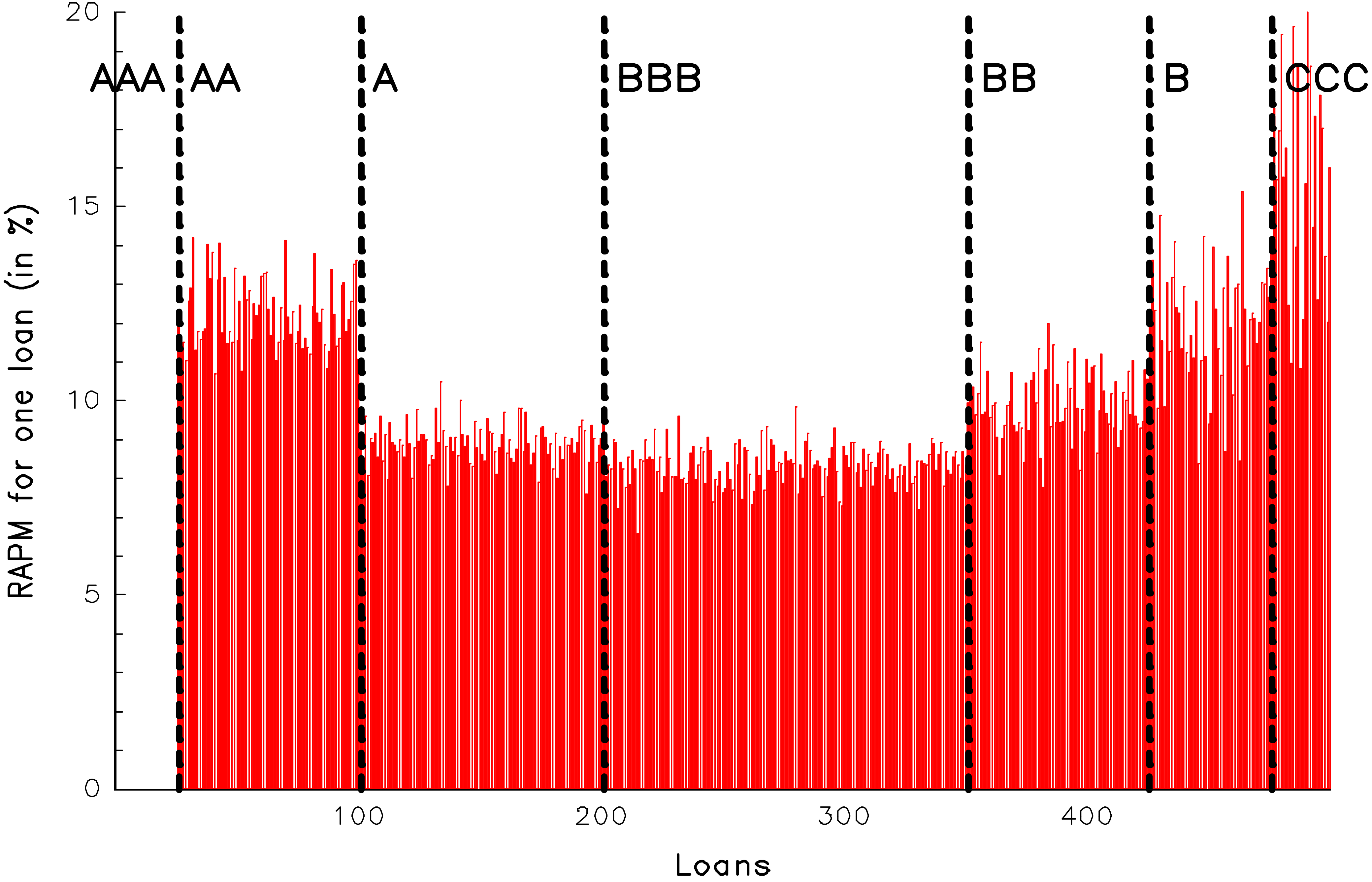
$$\mathrm{RAPM}\,(x_1, \ldots, x_I) = \frac{\sum_{i=1}^{I} x_i \cdot (\mathrm{Euribor} + \mathrm{Sp}\,(i))}{\mathrm{Risk}\,(x_1, \ldots, x_I)}$$

For a loan in a portfolio, we have

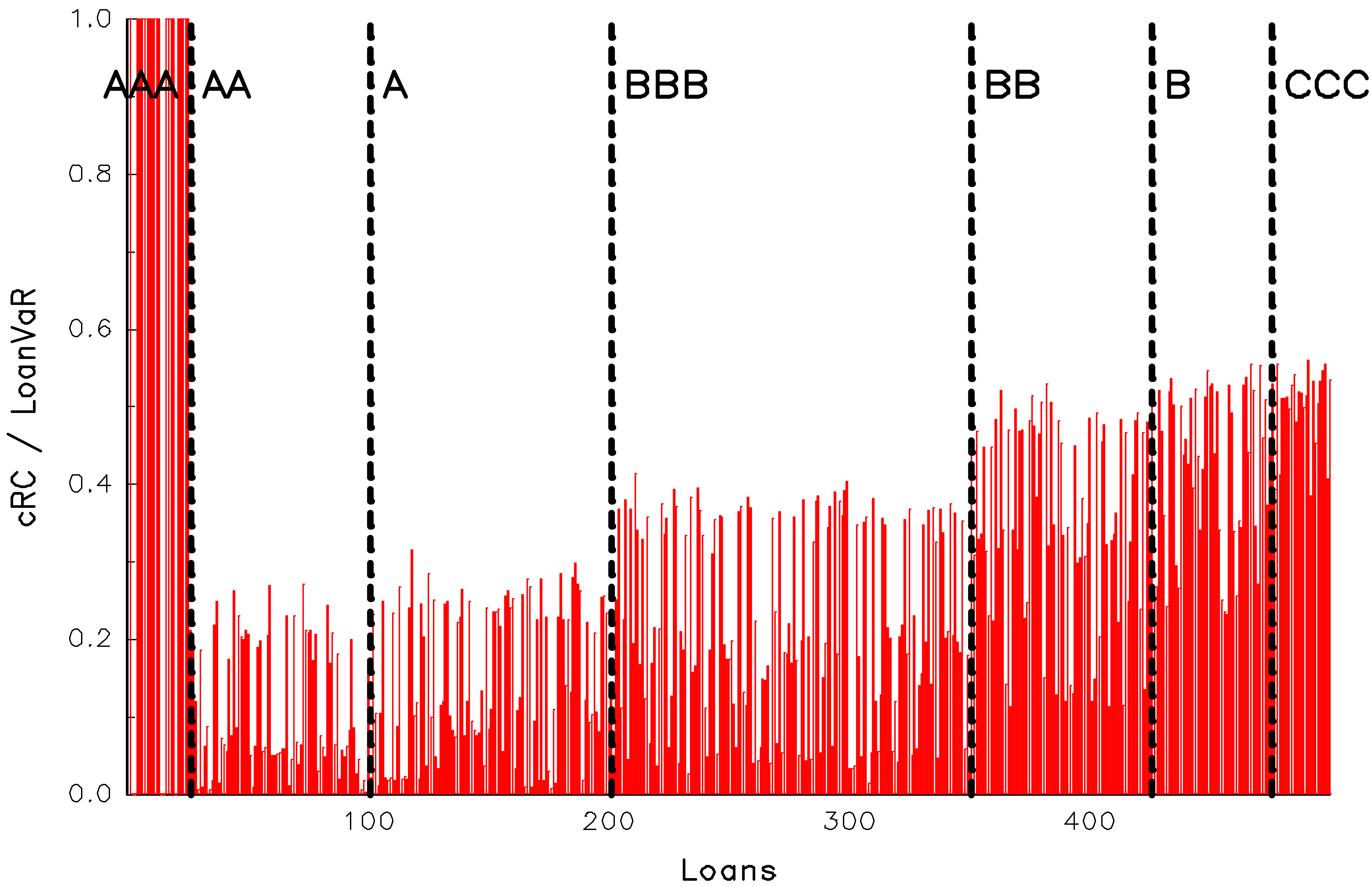
$$\mathrm{RAPM}\,(i; x_1, \ldots, x_I) = \frac{x_i \cdot (\mathrm{Euribor} + \mathrm{Sp}\,(i))}{\mathrm{RC}\,(i)}$$



Annual Spread of the Loans



RAPM of Individual Loans



RC(i) / LoanVaR(i)

5.2 The Efficient Frontier

The problem is ($C \in \mathbb{R}_+$ and $\mathbf{x} \in \Omega$)

$$\begin{array}{ll} \max & \text{ExReturn}\,(x_1,\ldots,x_I) \\ \text{u.c.} & \text{Risk}\,(x_1,\ldots,x_I) \leq C \end{array}$$

The simulation method Naive algorithm / Frontier-based algorithm

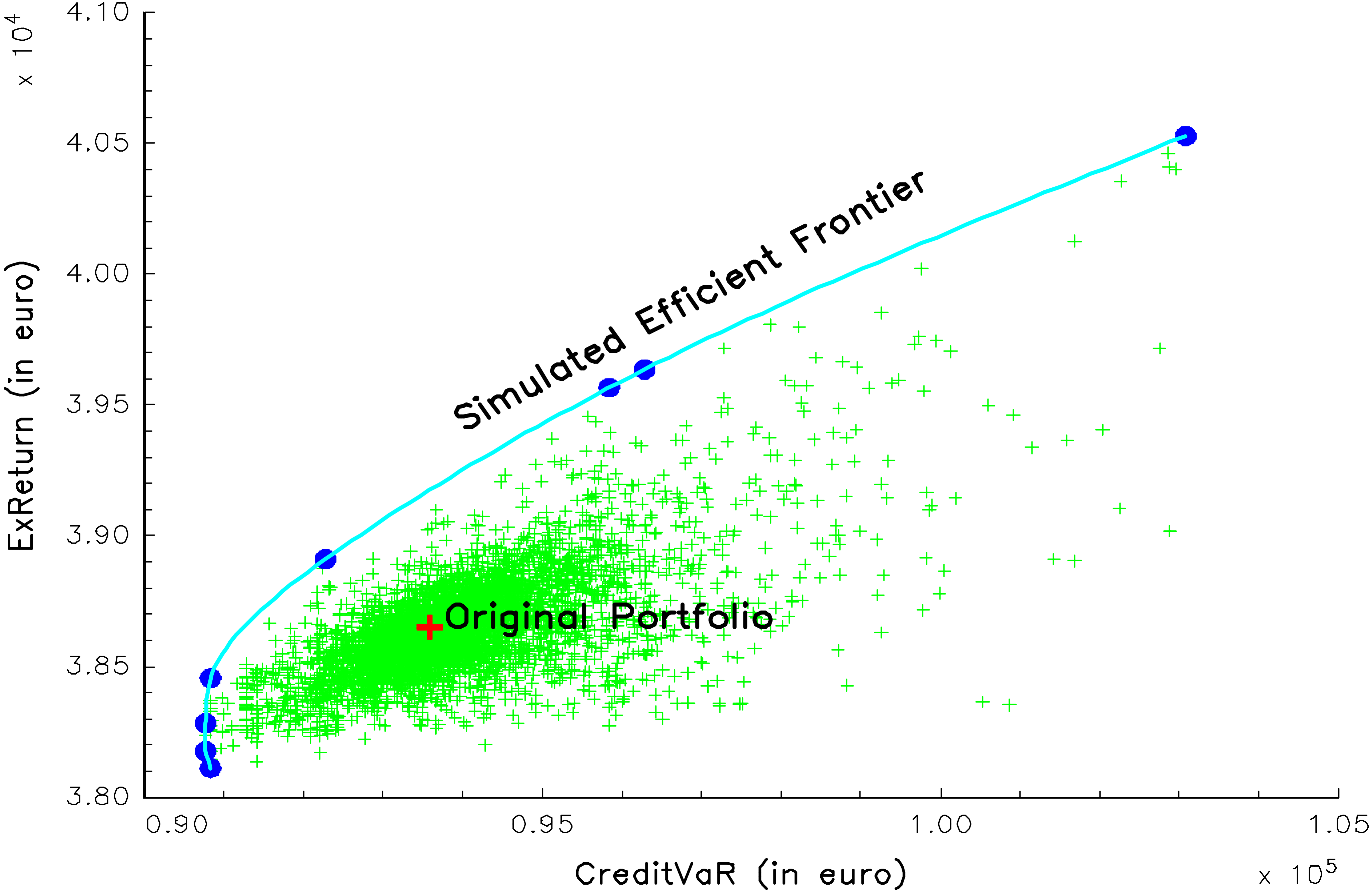
The optimisation method The ES problem

$$\begin{array}{ll} \min & \text{ES}\,(x_1,\ldots,x_I) \\ \text{u.c.} & \text{ExReturn}\,(x_1,\ldots,x_I) \geq C \end{array}$$

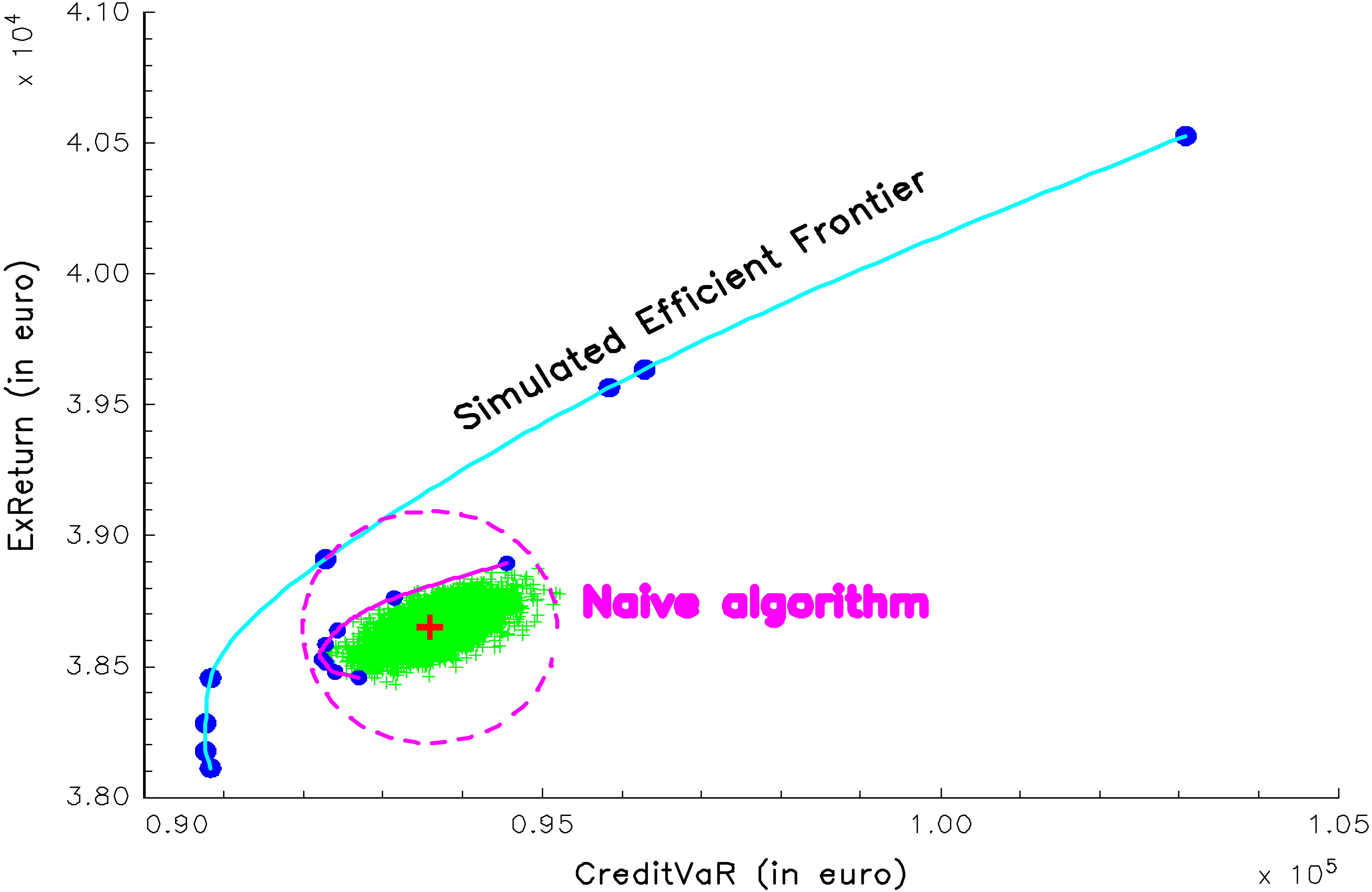
may be solved by LP technique:

$$\begin{array}{ll} \min & \Psi + (1-\alpha)^{-1}\frac{1}{S}\sum_{s=1}^{S} z_s \\ \text{u.c.} & \text{ExReturn}\,(x_1,\ldots,x_I) \geq C \\ & \mathbf{x} \in \Omega \\ & z_s \geq \sum_{i=1}^{I} x_i R_i^s D_i^s - \Psi \\ & z_s \geq 0 \end{array}$$

Buiding the CreditVaR frontier with the ES/ER optimisation problem



The Frontier-Based Simulation Algorithm



The Naive Simulation Algorithm

5.3 Other Techniques

The method of contributions

The method of Lagrange multipliers

6 The Time-inconsistency of the Copula Model

The Stationarity of the Default Probability Let τ_1 and τ_2 be two default times with the joint survival function :

$$\mathbf{S}(t_1, t_2) = \breve{\mathbf{C}}(\mathbf{S}_1(t_1), \mathbf{S}_2(t_2))$$

We have

$$\mathbf{S}_1(t \mid \tau_2 = t^\star) = \partial_2 \breve{\mathbf{C}}(\mathbf{S}_1(t), \mathbf{S}_2(t^\star))$$

If $\mathbf{C} \neq \mathbf{C}^\perp$, the probability of default of one firm changes when another firm defaults (Schmidt and Ward [2002]).

Remark 1 *Next computations are performed with the generator Λ of the Markov chain associated with the annual S&P TM. Let K be the state of default and i the initial rating of the firm. We have*

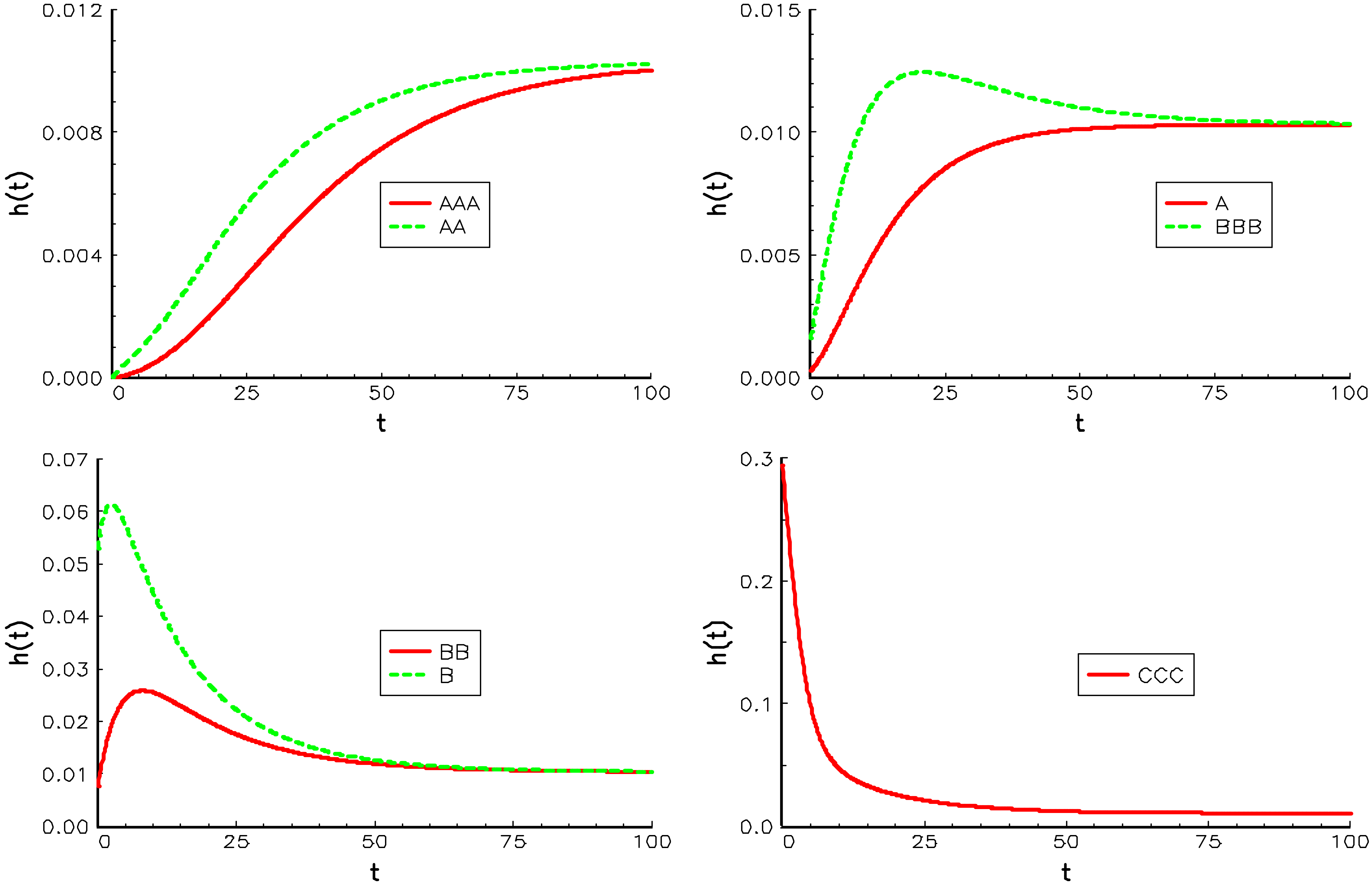
$$\mathbf{S}_i(t) = 1 - \mathbf{e}_i^\top \exp(t\Lambda)\, \mathbf{e}_K$$

The hasard rate is defined by

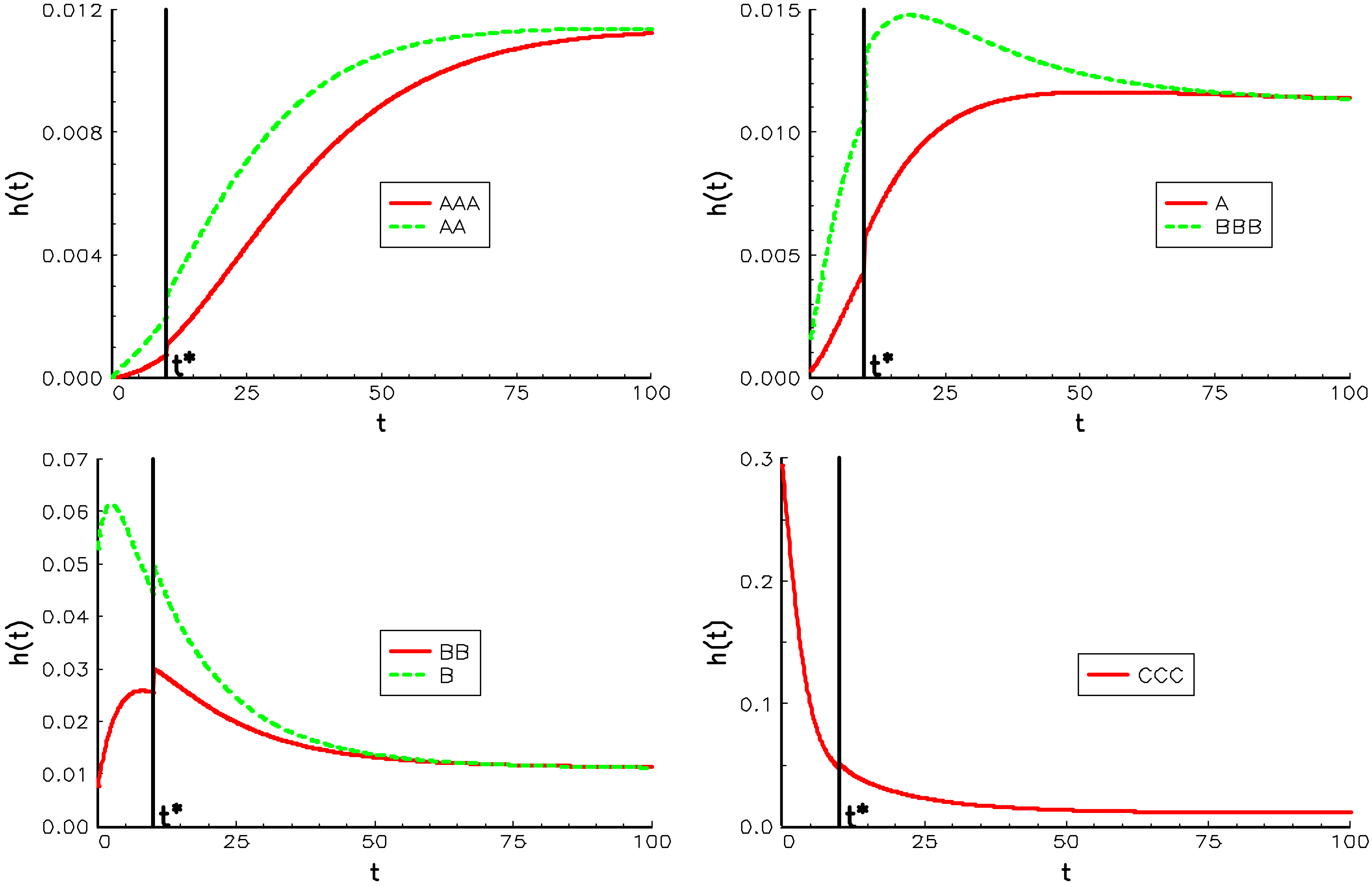
$$\begin{aligned} \lambda(t) &= \lim_{\Delta \to 0^+} \frac{1}{\Delta} \Pr\{t \leq \tau \leq t + \Delta \mid \tau \geq t\} \\ &= \frac{f(t)}{\mathbf{S}(t)} \end{aligned}$$

Using a Normal copula, we have

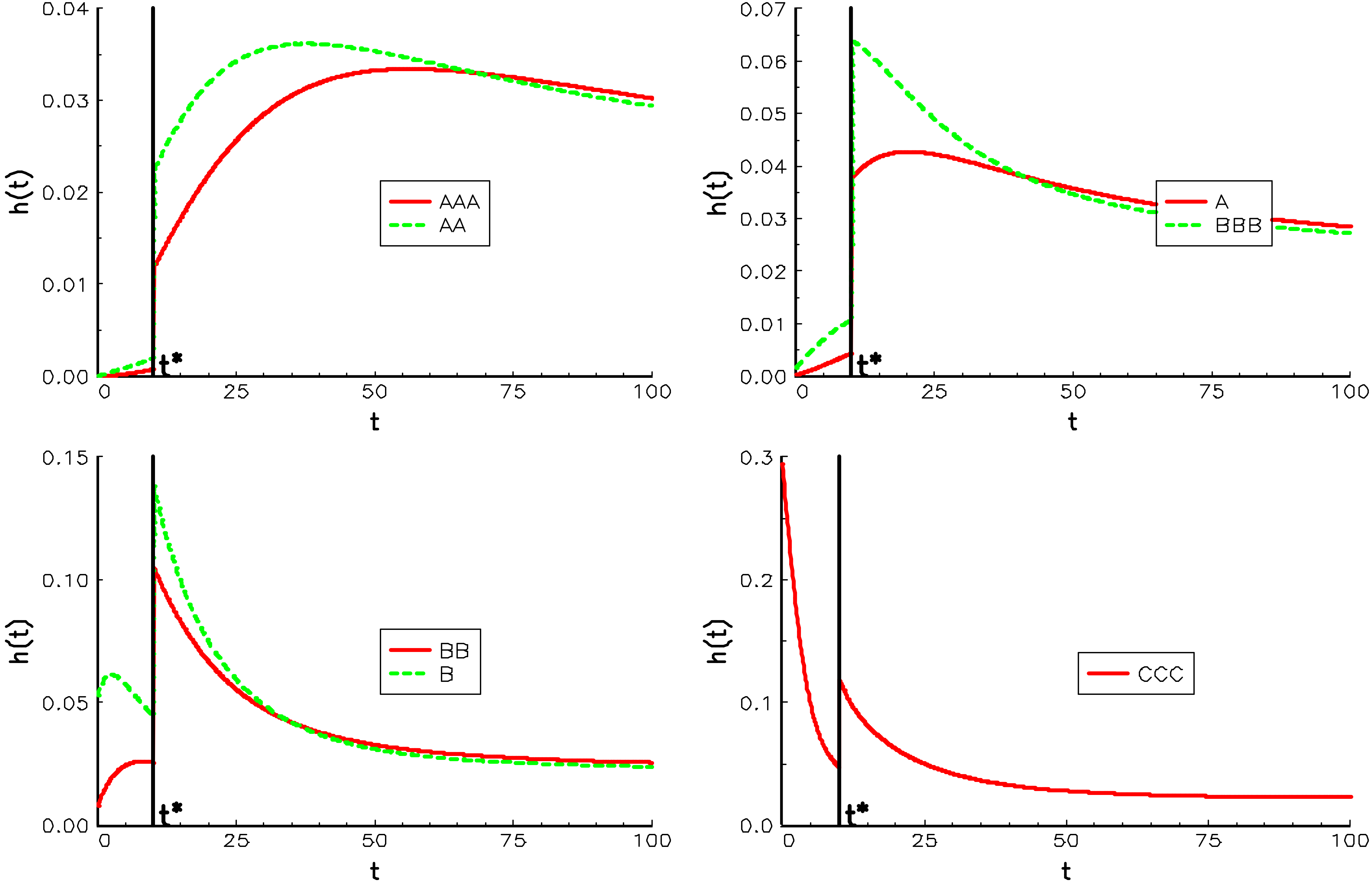
$$\mathbf{S}_{i_1}(t \mid \tau_{i_2} = t^\star) = \Phi\left(\frac{\Phi^{-1}(\mathbf{S}_{i_1}(t)) - \rho\Phi^{-1}(\mathbf{S}_{i_2}(t^\star))}{\sqrt{1-\rho^2}}\right)$$



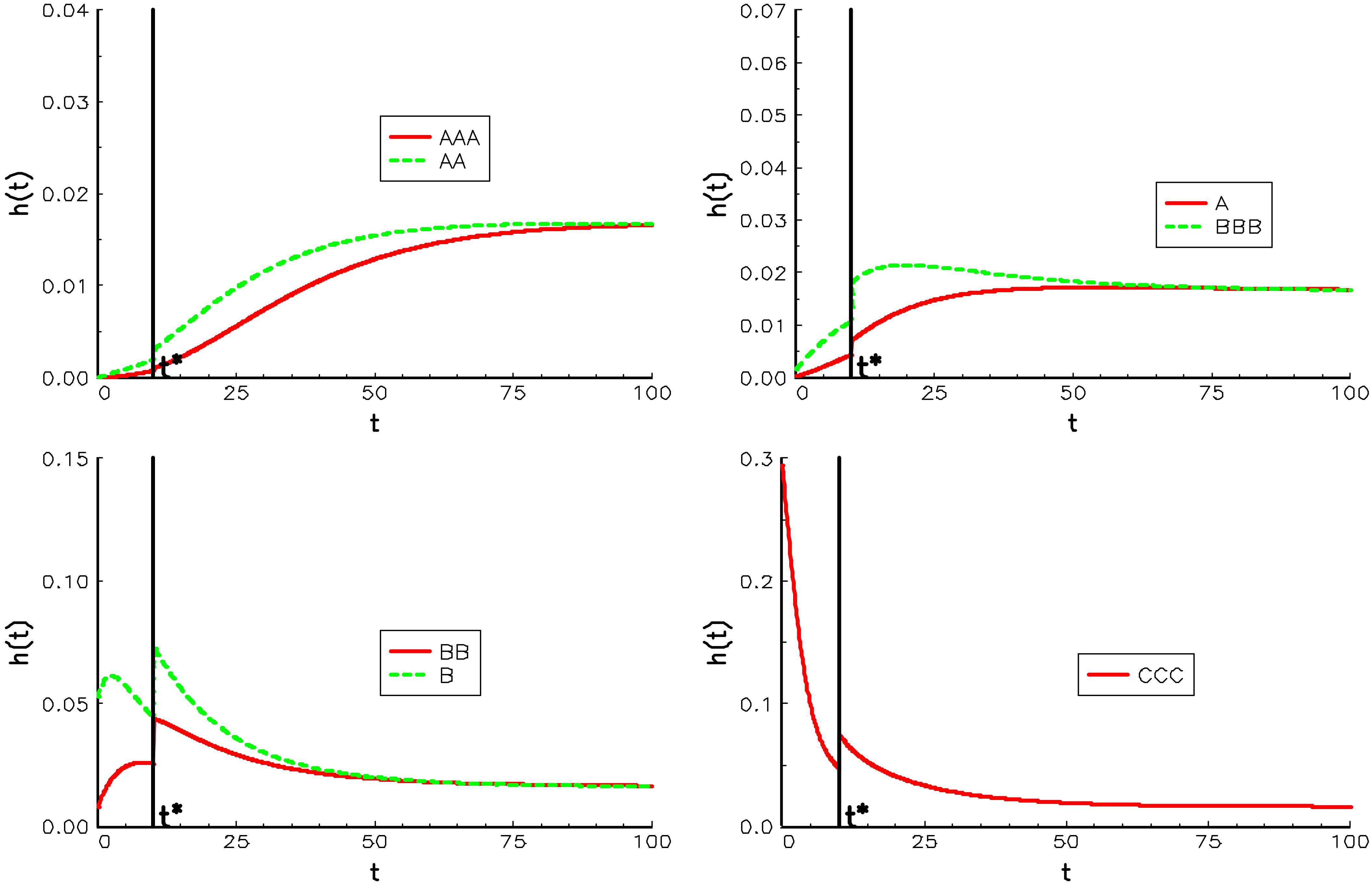
Hazard rate of the ratings



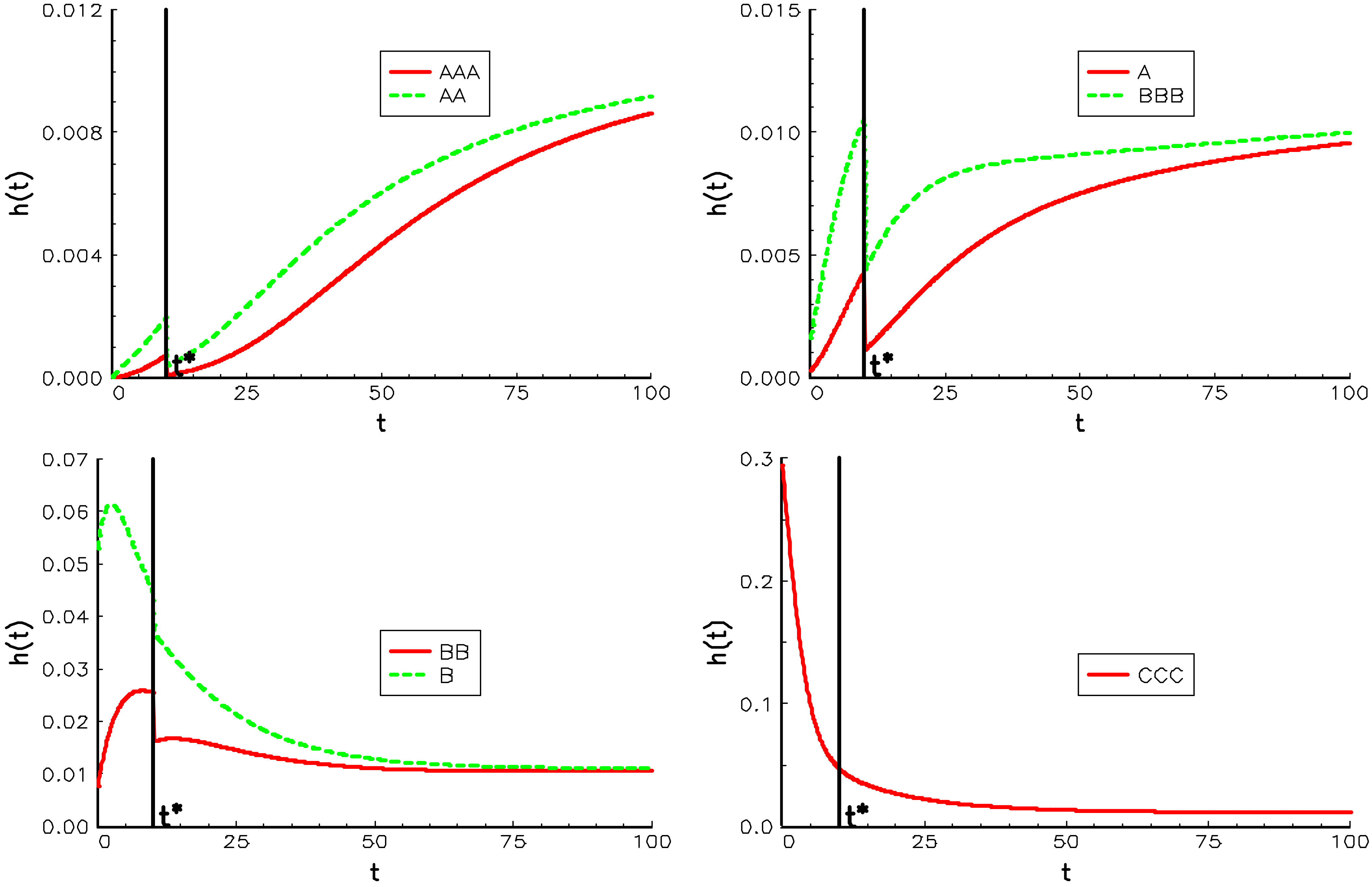
A firm rated AAA defaults – ρ = 5%



A firm rated AAA defaults – ρ = 50%



A firm rated BB defaults – $\rho = 50\%$



A firm rated CCC defaults – ρ = 50%

The Stationarity of the Survival Copula (from Jouanin [2002])

If the survival copula at time t_0 is $\breve{\mathbf{C}}$, and if no defaults occur between t_0 and t, the conditional survival copula at time t is not necessarily $\breve{\mathbf{C}}$ (Giesecke [2000]).

We have

$$\mathbf{S}\left(t_1, t_2 \mid \tau_1, \tau_2 > t\right) = \frac{\breve{\mathbf{C}}(\mathbf{S}_1\left(t_1\right), \mathbf{S}_2\left(t_2\right))}{\breve{\mathbf{C}}(\mathbf{S}_1\left(t\right), \mathbf{S}_2\left(t\right))}$$

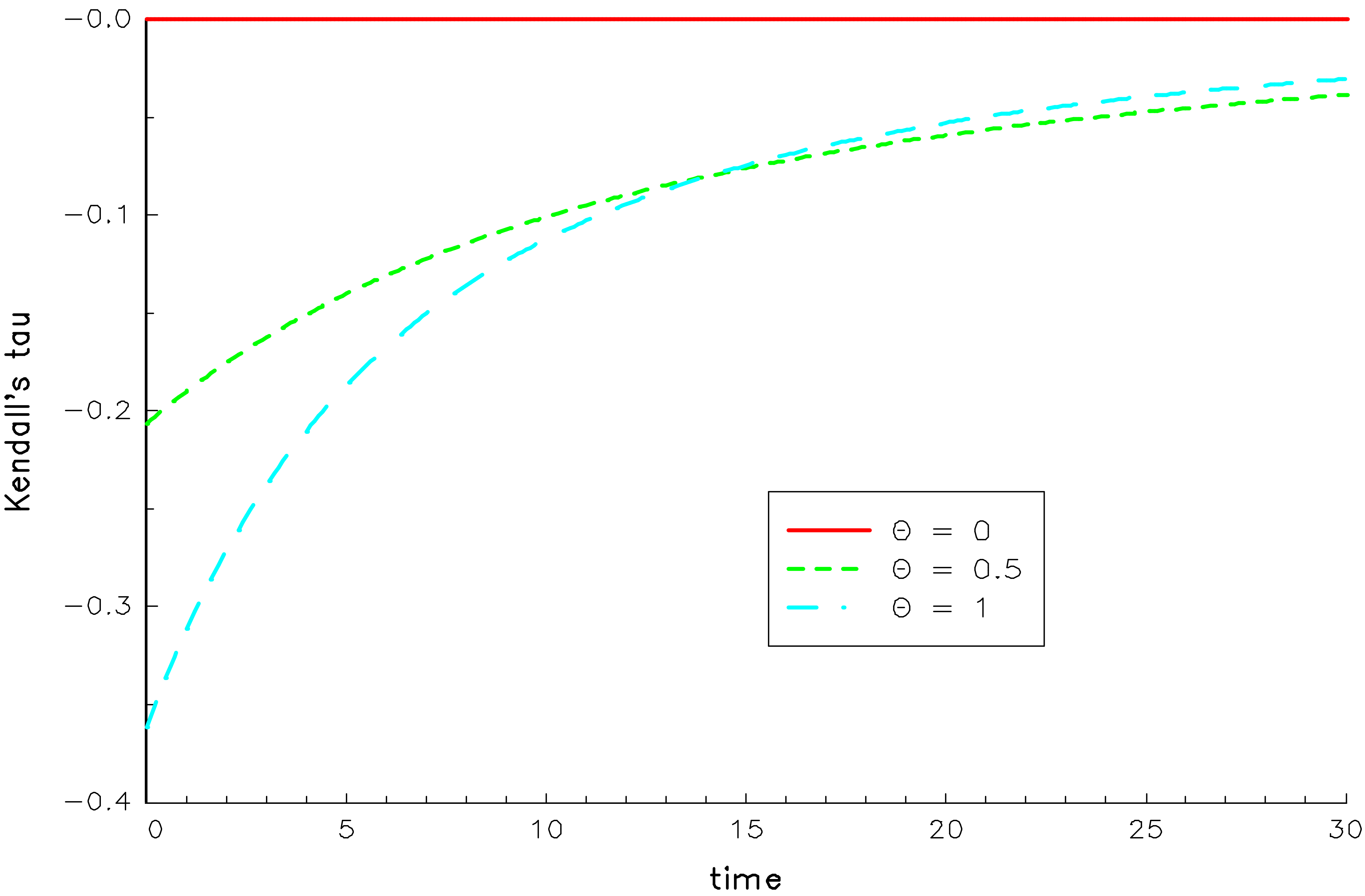
and we would like to have

$$\mathbf{S}\left(t_1, t_2 \mid \tau_1, \tau_2 > t\right) = \breve{\mathbf{C}}(\mathbf{S}_1\left(t_1 \mid \tau_1, \tau_2 > t\right), \mathbf{S}_2\left(t_2 \mid \tau_1, \tau_2 > t\right))$$

If $\breve{\mathbf{C}} = \mathbf{C}^{\perp}$, this property is verified.

To overcome the lack of Markov property, we may look for a copula family such that the conditional survival copula belongs to the same family. With exponential survival times, one solution is the Gumbel-Barnett copula. Let θ be the copula parameter at time t_0. The copula parameter at time t is

$$\theta\left(t\right) = \frac{\theta}{\left(1 + \theta\lambda_1 t\right)\left(1 + \theta\lambda_2 t\right)}$$



Kendall's tau of the conditional 'Markov' copula

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