A simple tranformation of copulas

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Abstract

We study how copulas properties are modified after some suitable transformations. In particular, we show that using appropriate transformations permits to fit the dependence structure in a better way.

1 Introduction

Copulas is one of the most promising tool for financial modelling. Bouyé, Durrleman, Nikeghbali, Riboulet and Roncalli [2000] review different financial problems and show how copulas could help to solve them. For example, they used copulas for operational risk measurement and the study of multidimensional stress scenarios. One of the difficulty is in general the choice of the copula. Durrleman, Nikeghbali and Roncalli [2000] present some procedures to find the 'optimal' copula in a given class $\mathcal{C}$. They are different methods to construct the copula function (see the chapter 3 of Nelsen [1998]). The most famous construction is based on the following composition

$$\mathbf{C}(u,v)=\varphi^{-1}(\varphi(u)+\varphi(v)) \tag{1}$$

with φ a C^2 function and $\varphi(1)=0$, $\varphi'(x)<0$ and $\varphi''(x)>0$ for all $0\leq x\leq 1$. Such copulas are called *Archimedean* copulas (Genest and MacKay [1986]). They play an important role, because they are very 'tractable'. Mikusiński and Taylor [2000] extend this construction to copulas with the form $\mathbf{C}(u,v)=\varphi^{-1}(\varphi(u)\oplus\varphi(v))$ where $\oplus$ denotes a continuous associative operation. The construction of new families of copulas is an important field of research.

The aim of this article is to introduce a simple transformation of copulas which permits to generate new families — **this transformation has been previously presented by Christian Genest in the conference "Distributions with Given Marginals and Statistical Modelling" (Barcelona, July 17-20, 2000).** In a first section, we find necessary and sufficient condition for the transformed copula being a copula too. In the second section, we explore the effect of the transformation on the dependence structure. And in the last section, we show how the transformation procedure could be used to fit the empirical dependence structure in a better way.

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2 How to generate new families of copulas?

In this section, we limit ourselves to the case of 2-dimensional copulas. Let $\mathbf{C}$ be a copula. Given a bijection $\gamma : [0,1] \rightarrow [0,1]$, we can now define $\mathbf{C}_\gamma$ on $[0,1]^2$ by

$$\mathbf{C}_\gamma (x,y) = \gamma^{-1} \left(\mathbf{C} \left(\gamma (x) , \gamma (y) \right) \right) \tag{2}$$

We can find necessary and sufficient conditions for $\mathbf{C}_\gamma (u,v)$ being a copula by introducing stronger conditions for γ.

Theorem 1 (Strong conditions) *Assume that γ be a concave C^1–diffeomorphism from $]0,1[$ onto $]0,1[$, twice derivable and continous from $[0,1]$ onto $[0,1]$ such that $\gamma(0) = 0$ and $\gamma(1) = 1$, then $\mathbf{C}_\gamma$ is a copula.*

Theorem 2 (Weak conditions) *Assume that γ be a C^1–diffeomorphism from $]0,1[$ onto $]0,1[$, twice derivable on $]0,1[$ and continous from $[0,1]$ onto $[0,1]$ such that $\gamma(0) = 0$ and $\gamma(1) = 1$, then $\mathbf{C}_\gamma$ is a copula if and only if*

$$\frac{\partial^2 \mathbf{C}}{\partial x \, \partial y} (u,v) \geq \frac{\gamma'' \left(\gamma^{-1} \left(\mathbf{C} (u,v) \right) \right)}{\left[\gamma' \left(\gamma^{-1} \left(\mathbf{C} (u,v) \right) \right) \right]^2} \frac{\partial \mathbf{C}}{\partial x} (u,v) \frac{\partial \mathbf{C}}{\partial y} (u,v) \tag{3}$$

for every (u,v) where the derivatives of $\mathbf{C}$ exist.

Proof. It is sufficient to prove the second result because in the case where γ is concave, we trivially have

$$\frac{\gamma'' \left(\gamma^{-1} \left(\mathbf{C} (u,v) \right) \right)}{\left[\gamma' \left(\gamma^{-1} \left(\mathbf{C} (u,v) \right) \right) \right]^2} \frac{\partial \mathbf{C}}{\partial x} (u,v) \frac{\partial \mathbf{C}}{\partial y} (u,v) \leq 0 \tag{4}$$

and therefore the condition is match as soon as $\mathbf{C}$ is a copula. In order to prove the second theorem, assume that $\mathbf{C}_\gamma$ is a copula. By definition, we have

$$\gamma \left(\mathbf{C}_\gamma (x,y) \right) = \mathbf{C} \left(\gamma (x) , \gamma (y) \right) \tag{5}$$

and by doing $(x,y) = (0,0)$ and $(x,y) = (1,1)$, we obtain the necessary condition that $\gamma(0) = 0$ and $\gamma(1) = 1$. The continuity of γ and the assumption that γ be a C^1–difféomorphism imply that $\gamma' > 0$. At every (u,v) where the derivatives of $\mathbf{C}$ exist, we compute the cross-derivative of $\mathbf{C}_\gamma$

$$\begin{aligned} \frac{\partial^2 \mathbf{C}_\gamma}{\partial x \, \partial y} (u,v) &= \frac{\gamma'(u) \gamma'(v)}{\gamma' \left(\gamma^{-1} \left(\mathbf{C} \left(\gamma(u) , \gamma(v) \right) \right) \right)} \times \\ & \left[\frac{\partial^2 \mathbf{C}}{\partial x \, \partial y} \left(\gamma(u) , \gamma(v) \right) - \frac{\gamma'' \left(\gamma^{-1} \left(\mathbf{C} \left(\gamma(u) , \gamma(v) \right) \right) \right)}{\left[\gamma' \left(\gamma^{-1} \left(\mathbf{C} \left(\gamma(u) , \gamma(v) \right) \right) \right) \right]^2} \frac{\partial \mathbf{C}}{\partial x} \left(\gamma(u) , \gamma(v) \right) \frac{\partial \mathbf{C}}{\partial y} \left(\gamma(u) , \gamma(v) \right) \right] \end{aligned} \tag{6}$$

The condition is then necessary. On the other hand, by integrating the cross-derivative between $0 < x_1 \leq x_2 < 1$ and $0 < y_1 \leq y_2 < 1$, we get

$$\mathbf{C}_\gamma (x_2, y_2) - \mathbf{C}_\gamma (x_1, y_2) - \mathbf{C}_\gamma (x_2, y_1) + \mathbf{C}_\gamma (x_1, y_1) \geq 0 \tag{7}$$

Since $\gamma(0) = 0$ and $\gamma(1) = 1$, we obtain

$$\begin{aligned} \mathbf{C}_\gamma (x,0) &= 0 \\ \mathbf{C}_\gamma (x,1) &= x \\ \mathbf{C}_\gamma (0,y) &= 0 \\ \mathbf{C}_\gamma (1,y) &= y \end{aligned} \tag{8}$$

this shows us that inequality (7) holds for every $0 \leq x_1 \leq x_2 \leq 1$ and $0 \leq y_1 \leq y_2 \leq 1$, and this completes the proof. ■

Remark 3 *Conditions on γ seem very complicated but they endow us, as we shall see in the sequel, with a large family of feasible functions γ such as $x \mapsto \sqrt[n]{x}$ or $x \mapsto \sin\left(\frac{\pi}{2}x\right)$. Moreover, we could deduce new functions from existing ones, because $\gamma(x) = (\gamma_1 \circ \gamma_2)(x)$ and $\gamma(x) = 1 - \gamma_1^{-1}(1-x)$ verify the assumptions of the theorem (1).*

Remark 4 *Thanks to the expression (6), we deduce that the density function $c_\gamma(u,v)$ associated to the copula $\mathbf{C}_\gamma$ is*

$$c_\gamma(u,v) = \frac{\gamma'(u)\gamma'(v)}{\gamma'(\mathbf{C}_\gamma(u,v))}\left[c(\gamma(u),\gamma(v)) - \frac{\gamma''(\mathbf{C}_\gamma(u,v))}{[\gamma'(\mathbf{C}_\gamma(u,v))]^2}\frac{\partial \mathbf{C}}{\partial x}(\gamma(u),\gamma(v))\frac{\partial \mathbf{C}}{\partial y}(\gamma(u),\gamma(v))\right] \tag{9}$$

where $c(u,v)$ is the density of C. Note also that the conditional distributions have the following form

$$\Pr\{V \leq v \mid U = u\} = \frac{\gamma'(u)}{\gamma'(\gamma^{-1}(\mathbf{C}(\gamma(u),\gamma(v))))}\frac{\partial \mathbf{C}}{\partial x}(\gamma(u),\gamma(v)) \tag{10}$$

Simulation of variates with distribution $\mathbf{C}_\gamma$ is then straightforward thanks to a numerical root finding procedure.

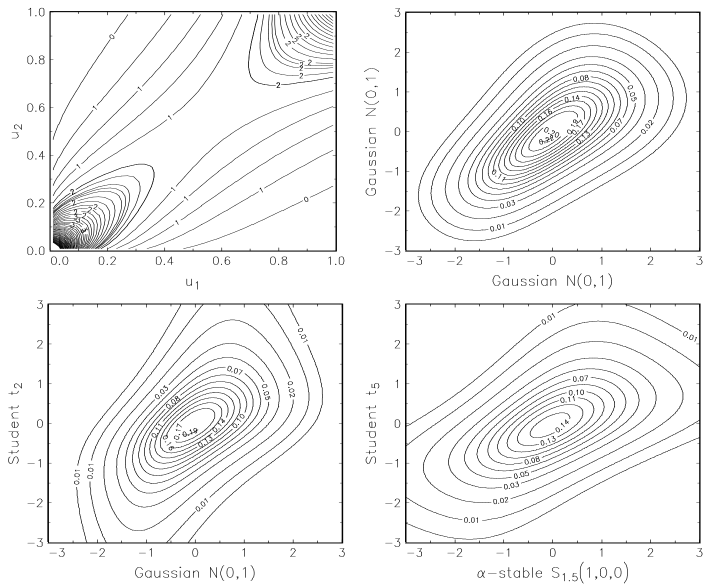

Figure 1: Contours of density for Frank copula

Example 5 (Frank copula) *The Frank copula has the following form*

$$\mathbf{C}(u,v) = -\alpha^{-1}\ln\left(\frac{1}{1-e^{-\alpha}}\left[\left(1-e^{-\alpha}\right) - \left(1-e^{-\alpha u}\right)\left(1-e^{-\alpha v}\right)\right]\right)$$

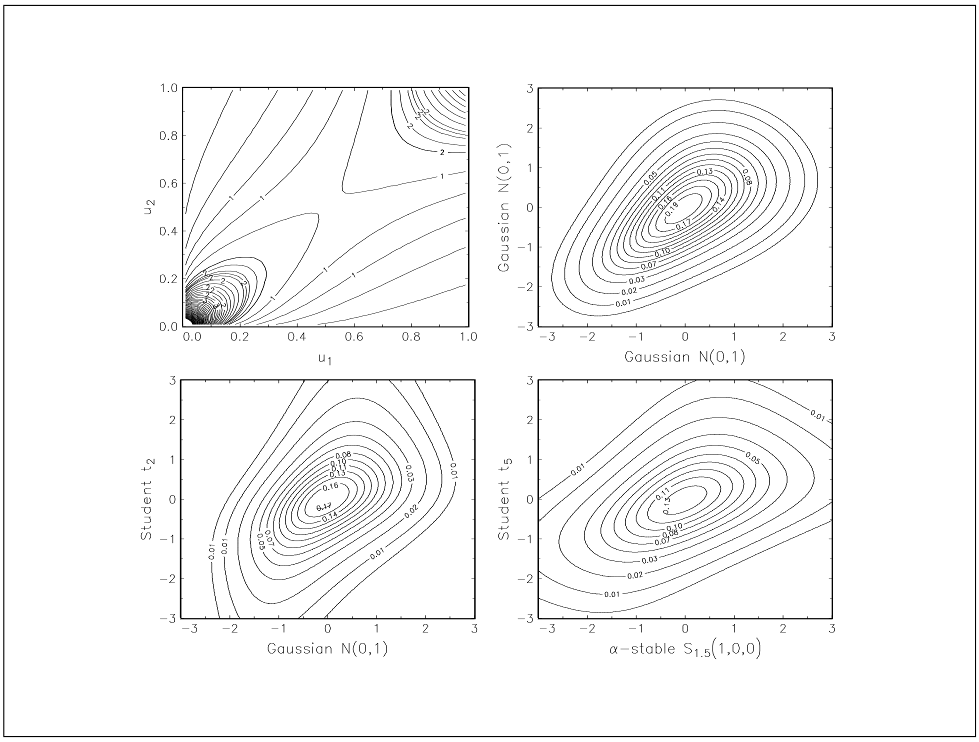

Figure 2: Contours of density for the transfomed Frank copula with $\beta = 3$

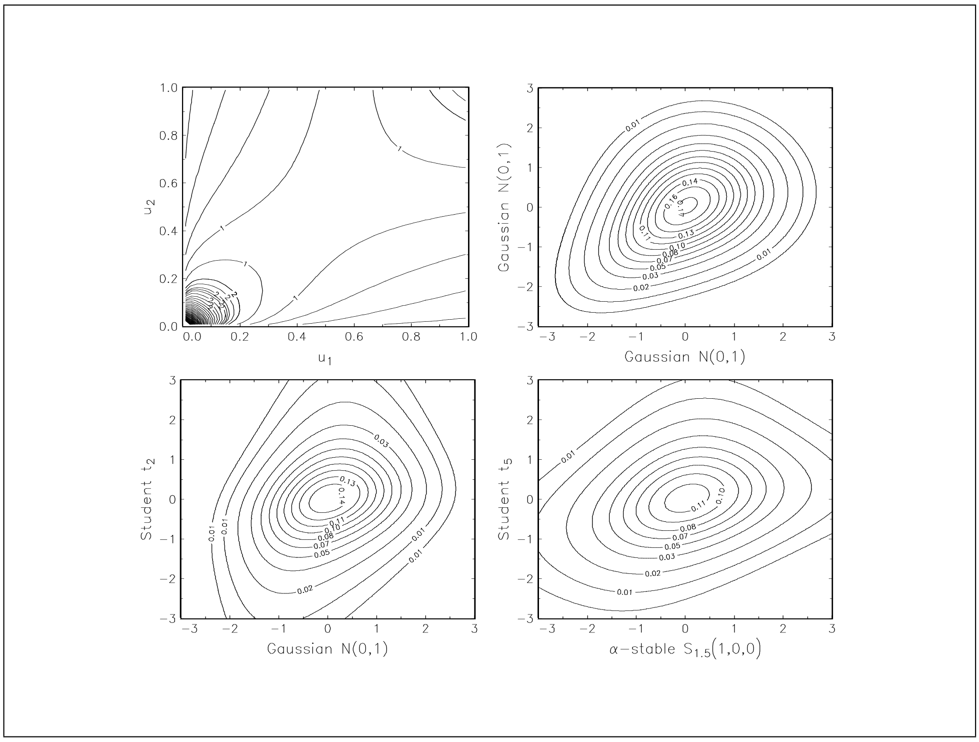

Figure 3: Contours of density for the transfomed Frank copula with $\beta = 7$

Assume that $\gamma(x) = x^{\frac{1}{\beta}}$ *with* $\beta \geq 1$. γ *verifies the theorem (1) and it comes that* $\mathbf{C}_\gamma(u,v)$ *is a copula. The corresponding density function is then*

$$\begin{aligned} c_\gamma(u,v) &= \frac{\bar{u}\bar{v}\mathbf{C}_\gamma(u,v)\,e^{-\alpha(\bar{u}+\bar{v})}}{\beta uv\left(\bar{\mathbf{C}}_\gamma(u,v)\right)^2}\left(\frac{1}{(1-e^{-\alpha})\exp\left(-\alpha\bar{\mathbf{C}}_\gamma(u,v)\right)}\right)^2\times \\ &\quad \left(\alpha\left(1-e^{-\alpha}\right)\bar{\mathbf{C}}_\gamma(u,v)-(1-\beta)\left(1-e^{-\alpha\bar{v}}\right)\left(1-e^{-\alpha\bar{u}}\right)\right) \end{aligned} \tag{11}$$

with

$$\mathbf{C}_\gamma(u,v) = \left(-\alpha^{-1}\ln\left(\frac{1}{1-e^{-\alpha}}\left[\left(1-e^{-\alpha}\right)-\left(1-e^{-\alpha\bar{u}}\right)\left(1-e^{-\alpha\bar{v}}\right)\right]\right)\right)^\beta \tag{12}$$

where $\bar{u} = u^{\frac{1}{\beta}}$, $\bar{v} = v^{\frac{1}{\beta}}$ *and* $\bar{\mathbf{C}}_\gamma(u,v) = \mathbf{C}_\gamma(u,v)^{\frac{1}{\beta}}$. *In the figure 1, we have represented the density contours of different multivariate distributions generated by the Frank Copula. We have set* α *equal to 5.7363 (the corresponding Kendall's tau is then equal to 0.5). When the margins are uniform (left and top quadrant), we obtain directly the density of the copula. In the other quadrant, the margins are gaussian, Student or* α*–stable distributions. We have reported the contour plots of the transformed copula in the figure 2 and 3 in the cases* $\beta = 3$ *and* $\beta = 7$. *We remark clearly that this transform function has an important impact on the dependence structure. Note moreover that the transformed copula belongs then to a two-parameter family. In the appendix, we present another transform functions and illustrate graphically the impact on the dependence structure.*

We introduce now the notations $\mathcal{G}$ and $\mathcal{G}^\star$ which represent the sets of the functions γ that verify respectiveley the hypothesis of the theorem (1) and (2). Before studying the impact on the dependence measures, we can try to know more globally about this transformation. This seems to be a difficult issue. We are only able to state the following remarks.

Theorem 6 $\gamma \in \mathcal{G}$ *but we also assume that* γ *is a* $C^2([0,1],[0,1])$ *function then* $\mathbf{C}$ *is an Archimedean copula if and only if* $\mathbf{C}^\gamma$ *is an Archimedean copula.*

Proof. Suppose $\mathbf{C}$ is an *Archimedean* copula, then $\mathbf{C}(u,v) = \varphi^{-1}(\varphi(u)+\varphi(v))$ for a given φ. We have

$$\mathbf{C}_\gamma(u,v) = (\varphi\circ\gamma)^{-1}\left((\varphi\circ\gamma)(u)+(\varphi\circ\gamma)(v)\right) \tag{13}$$

Because $\gamma \in \mathcal{G}$, $\varphi\circ\gamma$ has the same properties as φ (φ is a C^2 function with $\varphi(1) = 0$, $\varphi'(x) < 0$ and $\varphi''(x) > 0$ for all $0 \leq x \leq 1$). It comes that $\varphi\circ\gamma$ is the *Archimedean* generator of $\mathbf{C}_\gamma$. On the other hand suppose that $\mathbf{C}_\gamma$ is an *Archimedean* copula, then for all $(u,v) \in [0,1]$,

$$\gamma^{-1}\left(\mathbf{C}\left(\gamma(u),\gamma(v)\right)\right) = \varphi^{-1}\left(\varphi(u)+\varphi(v)\right) \tag{14}$$

Because γ is a bijection for all $(u,v) \in [0,1]$, it comes that

$$\mathbf{C}(u,v) = \left(\varphi\circ\gamma^{-1}\right)^{-1}\left(\left(\varphi\circ\gamma^{-1}\right)(u)+\left(\varphi\circ\gamma^{-1}\right)(v)\right) \tag{15}$$

and $\mathbf{C}$ is an *Archimedean* copula with the generator $\varphi\circ\gamma^{-1}$. ■

Corollary 7 *Any Archimedean copula* $\mathbf{C}(u,v) = \varphi^{-1}(\varphi(u)+\varphi(v))$ *can be rewritten as a transformed copula of the independant copula* $\mathbf{C}^\perp$ *with* $\gamma = \exp\circ\left(-\varphi^{-1}\right)$.

Corollary 8 *Let* $\mathbf{C}$ *be a copula. Define* $\mathcal{C}(\mathbf{C}) = \left\{\mathbf{C}_\gamma \mid \gamma \in \mathcal{G} \text{ and } \gamma \text{ is a } C^2([0,1],[0,1]) \text{ function}\right\}$. *If* $\mathbf{C}$ *is an Archimedean copula, then* $\mathcal{C}(\mathbf{C})$ *is the set of all Archimedean copulas.*

Lemma 9 *Let* $\mathbf{C}^+$ *be the upper Fréchet Bound. Then,* $\mathcal{C}(\mathbf{C}^+) = \{\mathbf{C}^+\}$.

Proof. This result is obvious because of the monotonicity of γ — $\min(\gamma(u),\gamma(v)) = \gamma(\min(u,v))$. ■

3 Dependence properties of the transformed copula

We are now interested in knowing more about the tranformed copula. Let us look first at the Kendall's tau of the transformed copula.

Theorem 10 *$\gamma \in \mathcal{G}$ and assume that $\frac{\partial \mathbf{C}}{\partial x}\frac{\partial \mathbf{C}}{\partial y}$ is integrable and that for all $x \in [0,1]$, $0 < \alpha \le \gamma'(x) \le \beta < +\infty$, then we have*

$$1 + \frac{\tau - 1}{\alpha^2} \le \tau_\gamma \le 1 + \frac{\tau - 1}{\beta^2} \tag{16}$$

Proof. We use the following definition of Kendall's tau

$$\tau = 4 \int\int_{[0,1]^2} \mathbf{C}(u,v) \frac{\partial^2 \mathbf{C}}{\partial x\, \partial y}(u,v)\, \mathrm{d}u\, \mathrm{d}v - 1 \tag{17}$$

Thanks to the Green's formula to obtain another tractable expression for τ

$$\tau = 4 \left(\int_{\partial[0,1]^2} \mathbf{C}(u,v) \frac{\partial \mathbf{C}}{\partial x}(u,v)\, \mathrm{d}\sigma - \int\int_{[0,1]^2} \frac{\partial \mathbf{C}}{\partial x}(u,v) \frac{\partial \mathbf{C}}{\partial y}(u,v)\, \mathrm{d}u\, \mathrm{d}v \right) - 1 \tag{18}$$

and the first term can be evaluated (see Nelsen [1998])

$$\tau = 1 - 4 \int\int_{[0,1]^2} \frac{\partial \mathbf{C}}{\partial x}(u,v) \frac{\partial \mathbf{C}}{\partial y}(u,v)\, \mathrm{d}u\, \mathrm{d}v \tag{19}$$

With this expression we can easily derive an expression for τ_γ by the the change of variable formula

$$\tau_\gamma = 1 - 4 \int\int_{[0,1]^2} \frac{\frac{\partial \mathbf{C}}{\partial x}(u,v) \frac{\partial \mathbf{C}}{\partial y}(u,v)}{\left[\gamma'\left(\gamma^{-1}\left(\mathbf{C}(u,v)\right)\right)\right]^2}\, \mathrm{d}u\, \mathrm{d}v \tag{20}$$

Under the assumptions on γ, we have the following inequalities for all $x \in [0,1]$,

$$\frac{1}{\beta} \le \frac{1}{\gamma'\left(\gamma^{-1}(x)\right)} \le \frac{1}{\alpha} \tag{21}$$

$$\frac{x}{\beta} \le \gamma^{-1}(x) \le \frac{x}{\alpha} \tag{22}$$

and the result is then easily derived. ■

Theorem 11 *$\gamma \in \mathcal{G}$ and assume that for all $x \in [0,1]$, $0 < \alpha \le \gamma'(x) \le \beta < +\infty$, then we have*

$$\frac{\rho + 3}{\beta^3} - 3 \le \rho_\gamma \le \frac{\rho + 3}{\alpha^3} - 3 \tag{23}$$

Proof. We use the following definition of Spearman's rho

$$\rho = 12 \int\int_{[0,1]^2} \mathbf{C}(u,v)\, \mathrm{d}u\, \mathrm{d}v - 3 \tag{24}$$

With this expression we can easily derive an expression for ρ_γ by the change of variable formula

$$\rho_\gamma = 12 \int\int_{[0,1]^2} \frac{\gamma^{-1}\left(\mathbf{C}(u,v)\right)}{\gamma'\left(\gamma^{-1}(u)\right)\gamma'\left(\gamma^{-1}(v)\right)}\, \mathrm{d}u\, \mathrm{d}v - 3 \tag{25}$$

Using the inequalities (21) and (22), we obtain

$$\frac{1}{\beta^3}(\rho+3) \leq \rho_\gamma + 3 \leq \frac{1}{\alpha^3}(\rho+3) \tag{26}$$

and this completes the proof. ■

One interesting feature of our transformation is that it does not change the tail dependences of the initial copula $\mathbf{C}$. So, we can concentrate upon modifying the Kendall's tau or the Spearman's rho without changing the tail dependences. Let's first recall the definition of the upper tail dependence measure (Joe [1997]). Let $\mathbf{C}$.be a copula and $\lambda(\mathbf{C})$ be defined as

$$\begin{aligned} \lambda(\mathbf{C}) &= \lim_{u\to 1^-} \frac{\bar{\mathbf{C}}(u,u)}{1-u} \\ &= -\lim_{u\to 1^-} \frac{\mathrm{d}}{\mathrm{d}u}\bar{\mathbf{C}}(u,u) \end{aligned} \tag{27}$$

where $\bar{\mathbf{C}}(u,v) = 1-u-v+\mathbf{C}(u,v)$. $\lambda(\mathbf{C})$ is called the upper tail dependence measure of $\mathbf{C}$ and $\mathbf{C}$ is said to have upper tail dependence if $\lambda(\mathbf{C})$ exists and is strictly greater than zero, this to say $\lambda(\mathbf{C}) \in\,]0,1]$. If $\lambda(\mathbf{C}) = 0$, we say that $\mathbf{C}$ has no upper tail dependence. Let's denote $\lambda(\mathbf{C}_\gamma)$ the upper tail dependence of the transformed copula $\mathbf{C}_\gamma(x,y) = \gamma^{-1}(\mathbf{C}(\gamma(x),\gamma(y)))$. We have the following interesting result[1]:

Theorem 12 *If the limit $\lambda(\mathbf{C})$ exists, then we have*

$$\lambda(\mathbf{C}_\gamma) = \lambda(\mathbf{C}) \tag{28}$$

Proof. We have

$$\lambda(\mathbf{C}) = 2 - \lim_{u\to 1^-} \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{C}(u,u) \tag{29}$$

and

$$\lambda(\mathbf{C}_\gamma) = 2 - \lim_{u\to 1^-} \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{C}_\gamma(u,u)$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathbf{C}(u,u) = \frac{\partial}{\partial u}\mathbf{C}(u,u) + \frac{\partial}{\partial v}\mathbf{C}(u,u) \tag{30}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathbf{C}_\gamma(u,u) = \frac{\gamma'(u)}{\gamma'(\gamma^{-1}(\mathbf{C}(u,u)))}\left[\frac{\partial}{\partial u}\mathbf{C}(\gamma(u),\gamma(u)) + \frac{\partial}{\partial v}\mathbf{C}(\gamma(u),\gamma(u))\right] \tag{31}$$

So, as we assumed that $\lim_{u\to 1^-} \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{C}(u,u)$ exists and as $\gamma^{-1}(1) = \gamma(1) = 1$, we have by the classical theorem of limit composition

$$\lim_{u\to 1^-} \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{C}_\gamma(u,u) = \lim_{u\to 1^-} \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{C}(u,u) \tag{32}$$

hence we obtain the desired result. ■

To illustrate these properties, we consider the two-dimensional student copula (Bouyé, Durrleman, Nikeghbali, Riboulet and Roncalli [2000]). Let us denote ρ and ν the parameters. We could then show that (Embrechts, McNeil and Straumann [1999])

$$\lambda(\mathbf{C}) = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right) \tag{33}$$

[1] We also have a similar result for the lower tail dependence measure, which is invariant under the γ-transformation of the copula $\mathbf{C}$.

where $\bar{t}$ is the tail of a univariate student distribution. We have represented in the figure 4 the corresponding upper tail dependence and the function $\lambda(u) = \Pr\{U_1 > u | U_2 > u\} = (1-u)^{-1} \bar{\mathbf{C}}(u,u)$ for the student copula with $\rho = 0$ and $\nu = 1$. For the transformed student copula, we have taken $\gamma_1(x) = \sqrt{x}$, $\gamma_2(x) = \sqrt[5]{x}$ and $\gamma_3(x) = \sin\left(\frac{\pi}{2}x\right)$. We remark effectively that the limits are the same (but its convergence to this limit is very different[2]).

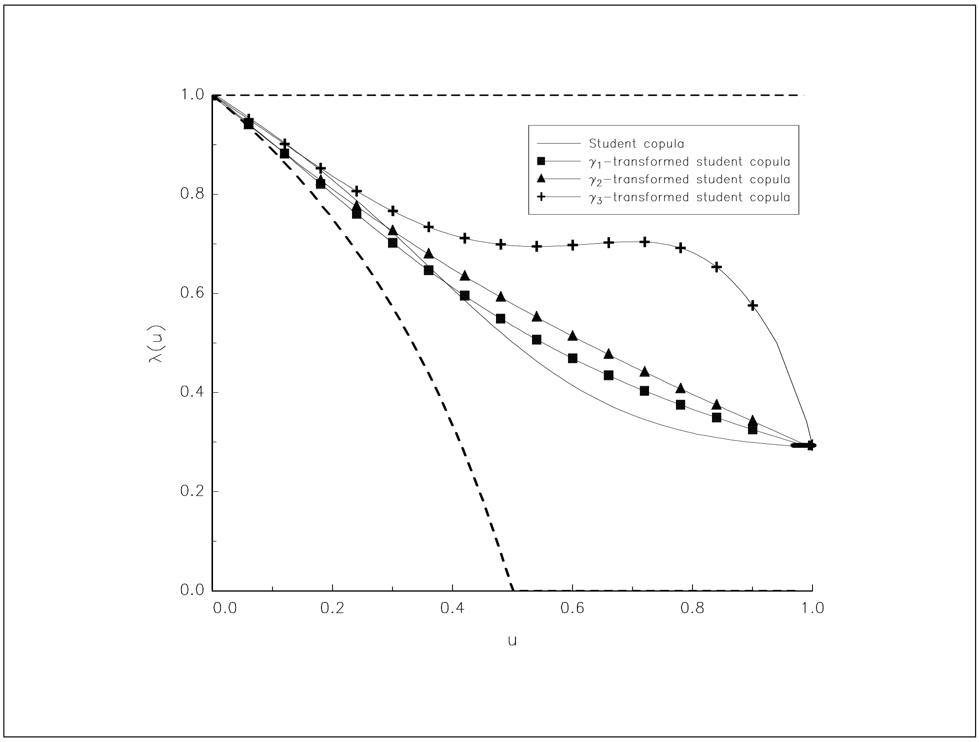

Figure 4: Quantile-dependent measure $\lambda(u)$

4 Fitting the dependence structure with the transformed copula

In one previous work (Durrleman, Nikeghbali and Roncalli [2000]), we consider the problem to find the 'optimal' copula in a given class $\mathcal{C}$. The idea is the following. Let $\hat{\mathbf{C}}$ be the empirical copula (Deheuvels [1979]). Let $\tilde{\mathcal{C}}$ be finite subset of copulas ($\tilde{\mathcal{C}} \subset \mathcal{C}$), then we are interested in knowing which one of the copulas in $\tilde{\mathcal{C}}$ fits best the dependence structure of data in the sense of a given distance. The problem here is different. Suppose that we have found a copula $\mathbf{C}$ which could be considered as a good candidate to match the dependence structure of the data. How to find a transformed function γ such that we could improve the adequacy of $\mathbf{C}_\gamma$ to the empirical copula $\hat{\mathbf{C}}$. We illustrate this problem by two examples.

4.1 The $\tau - \varrho$ example

Nelsen [1998] presents some relationships between the measures τ and ϱ. They could be summarized by a bounding region $\mathcal{B}(\tau, \varrho)$ defined by

$$(\tau, \varrho) \in \mathcal{B}(\tau, \varrho) \Longleftrightarrow \begin{cases} \frac{3\tau - 1}{2} \leq \varrho \leq \frac{1 + 2\tau - \tau^2}{2} & \tau \geq 0 \\ \frac{\tau^2 + 2\tau - 1}{2} \leq \varrho \leq \frac{1 + 3\tau}{2} & \tau \leq 0 \end{cases} \tag{34}$$

[2]We have indicated with the dotted lines the quantile-dependent measure for the Fréchet bounds.

We have represented this region in the figure 5. Moreover, NELSEN [1998] presents some arguments to show that we could think that the bounds could not be improved. However, when we compute the $\tau - \varrho$ region of a large number of families, it appears that it is difficult to attain any points of the boundary region. For example, we have reported in the figure the $\tau - \varrho$ region of the 10 copula families[3] presented in JOE [1997]. These families are the most popular and the most used in copula modelling. Nevertheless, we remark that we could not find a copula which belongs to these families such that τ and ϱ corresponds to the points A, B, C or D. In fact, the attainable region is very thin.

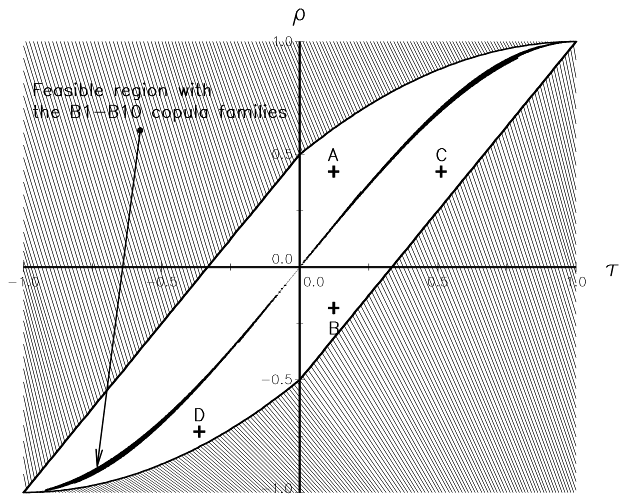

Figure 5: Bounding region for τ and ϱ

The question is now the following: could we find a copula $\mathbf{C}_\gamma(u,v)$ such that we attain these points. The answer is partially yes[4]. For example, if we use the gaussian copula with $\gamma(x) = \sin\left(\frac{\pi}{2}x^{\frac{1}{\beta}}\right)$ and $\beta \in [1, +\infty[$, the $\tau - \varrho$ region is shifted to the left and to the top (see the figure 6). We then obtain some copulas that verify $\tau \leq 0$ and $\varrho > 0$.

4.2 The $\tau / \varrho - \lambda$ example

Let $\hat{\tau}$ and $\hat{\lambda}$ be the empirical measures associated to the data. Imagine that we want to fit both the Kendall's tau and the upper tail dependence. We could define a finite subset of copulas $\tilde{\mathcal{C}}$ and find the copula which matches the two observed measures. However, that requires a lot of computations. Another possibility is to find a copula $\mathbf{C}$ such that $\lambda = \hat{\lambda}$ and then find the function $\gamma \in \mathcal{G}$ such that we fit the observed Kendall's tau. This two steps modelling is possible because the transformation preserves the upper tail dependence ($\lambda(\mathbf{C}_\gamma) = \lambda(\mathbf{C})$).

[3] except the family B9.

[4] In fact, there exist yet copulas which attain the boundary regions. They are the shuffles of Min (MIKUSIŃSKI, SHERWOOD and TAYLOR [1992]). Nevertheless, these copulas are very special because the support consists of line segments. Moreover, they are not parametric.

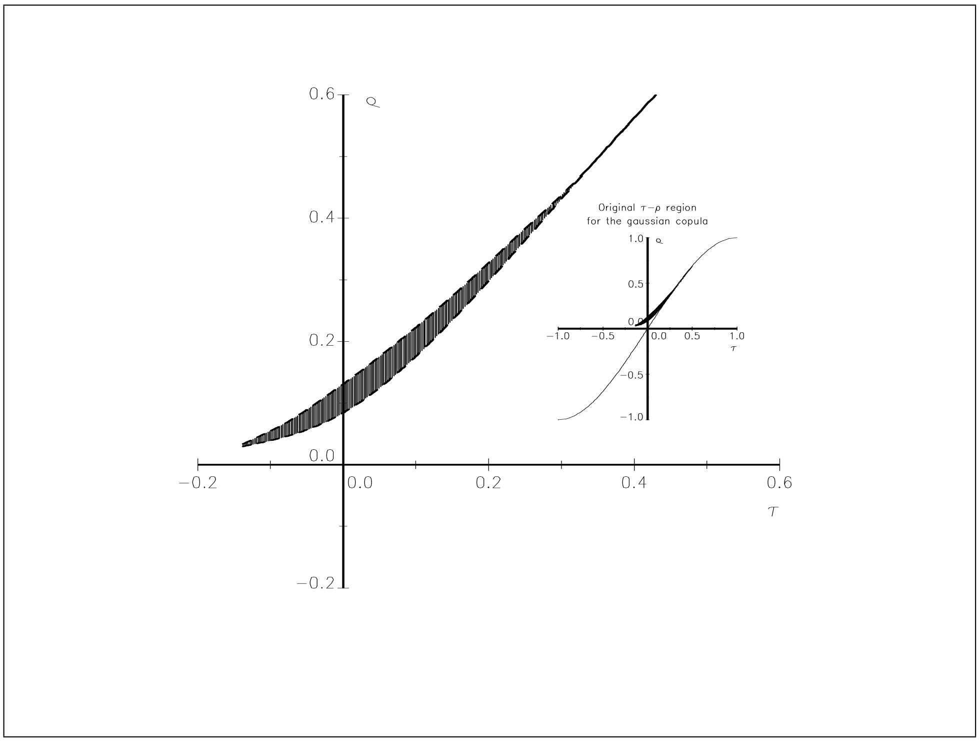

Figure 6: $\tau - \varrho$ region of the transformed Normal copula

We consider the example of the Marshall-Olkin copula, which is defined in the following way

$$\mathbf{C}(u,v) = u^{1-\alpha_1} v^{1-\alpha_2} \min\left(u^{\alpha_1}, v^{\alpha_2}\right) \tag{35}$$

with $(\alpha_1, \alpha_2) \in [0,1]^2$. We have $\tau = \alpha_1\alpha_2 \left(\alpha_1 - \alpha_1\alpha_2 + \alpha_2\right)^{-1}$, $\varrho = 3\alpha_1\alpha_2 \left(2\alpha_1 - \alpha_1\alpha_2 + 2\alpha_2\right)^{-1}$ and $\lambda = \min(\alpha_1, \alpha_2)$ (Lindskog [2000]). We have represented in the figure 7 the corresponding $\tau - \varrho - \lambda$ region and the transformed region with $\gamma(x) = \sin\left(\frac{\pi}{2}x\right)$. For $\tau \geq \tau_\gamma^0 \left(= \frac{1}{3}\right)$ and $\varrho \geq \varrho_\gamma^0 \left(\simeq \frac{1}{2}\right)$, we could now find a copula which gives the same upper tail dependence, but with a smaller τ or a bigger ϱ.

5 Conclusion

In this paper, we have investigated a new approach to construct copula families. This approach has an interesting property, because it changes dependence measures like the Kendall's tau or the Spearman's rho, without changing the upper (or lower) tail dependence. This γ-transformation could then be used in a $\tau/\,\varrho - \lambda$ problematic, for example in financial modelling of asset returns.

References

[1] Bouyé, E., V. Durrleman, A. Nikeghbali, G. Riboulet and T. Roncalli [2000], Copulas for finance — A reading guide and some applications, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*

[2] Deheuvels, P. [1979], La fonction de dépendance empirique et ses propriétés — Un test non paramétrique d'indépendance, *Académie Royale de Belgique – Bulletin de la Classe des Sciences – 5e Série*, **65**, 274-292

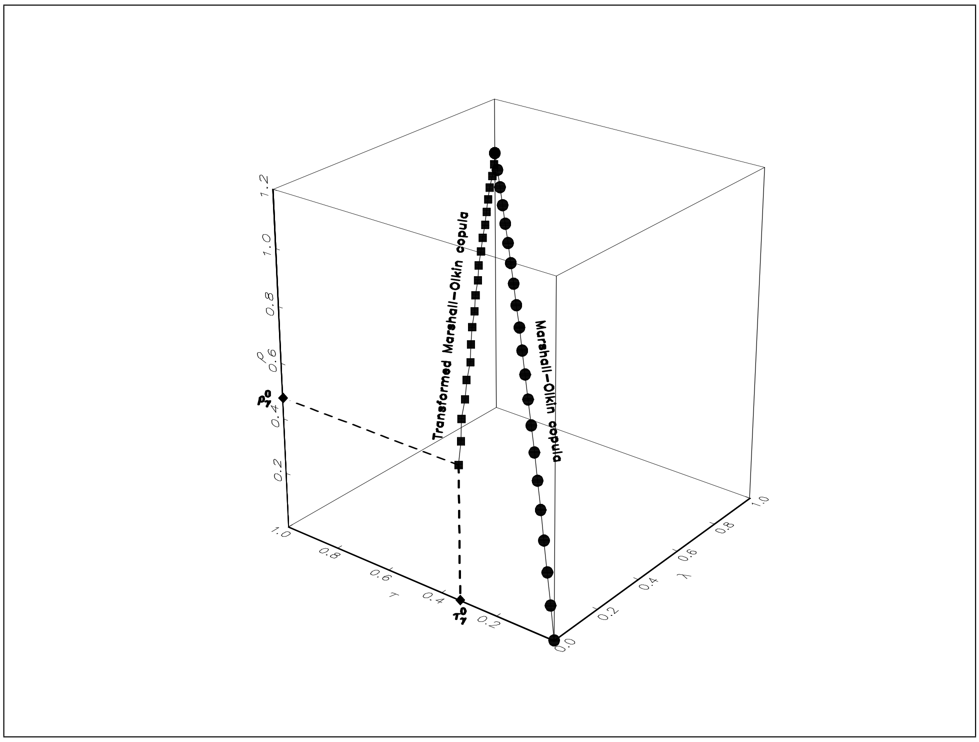

Figure 7: $\tau - \varrho - \lambda$ region of the Marshall-Olkin and the γ -transformed copulas

[3] Durrleman, V., A. Nikeghbali, and T. Roncalli [2000], Which copula is the right one?, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*

[4] Embrechts, P., McNeil, A.J. and D. Straumann [1999], Correlation and dependency in risk management : properties and pitfalls, Departement of Mathematik, ETHZ, Zürich, *Working Paper*

[5] Genest, C., K. Ghoudi and L.P. Rivest [1998], Discussion of "Understanding relationships using copulas, by Edward Frees and Emiliano Valdez", *North American Actuarial Journal*, **3**, 143-149.

[6] Genest, C. and J. MacKay [1986], The joy of copulas: Bivariate distributions with uniform marginals, *American Statistician*, **40**, 280-283

[7] Joe, H. [1997], Multivariate Models and Dependence Concepts, *Monographs on Statistics and Applied Probability*, **73**, Chapmann & Hall, London

[8] Lindskog, F. [2000], Modelling dependence with copulas and applications to risk management, *RiskLab Research Paper*

[9] Mikusiński, P., H. Sherwood and M.D. Taylor [1992], Shuffles of min, *Stochastica*, **13**, 61-74

[10] Mikusiński, P. and M.D. Taylor [2000], A remark on associative copulas, forthcoming in *Commentationes Mathematicae Universitatis Carolinae*

[11] Nelsen, R.B. [1998], An Introduction to Copulas, *Lectures Notes in Statistics*, **139**, Springer Verlag, New York

Appendix — Some special functions $\gamma(x)$

We present now some functions that belong to $\mathcal{G}$. Then, we show graphically how these functions transform the repartition of the probability masses of the copula.

	Parameters	$\gamma(x)$	$\gamma^{-1}(x)$
1	$\beta \geq 1$	$x^{\frac{1}{\beta}}$	x^{β}
2	✓	$\sin\ \frac{\pi}{2}x$	$\frac{2}{\pi}\arcsin(x)$
3	✓	$\frac{4}{\pi}\arctan(x)$	$\tan\ \frac{\pi}{4}x$
4	$(\beta_1, \beta_2) \in \mathbb{R}_+^2$	$\hbar(x) = (\beta_1 + \beta_2)\, x\, (\beta_1 x + \beta_2)^{-1}$	$\beta_1 x\, (\beta_1 + \beta_2 - \beta_1 x)$
5	$f \in L^1(]0,1[)\,,\ f(x) \geq 0,\ f'(x) \leq 0$	$\left(\int_0^1 f(t)\ \mathrm{d}t\right)^{-1} \int_0^x f(t)\ \mathrm{d}t$	✓

Note that we could obtain other functions by convex combination. For example, $\gamma \in \mathcal{G}$ with $\gamma(x) = (\gamma_4 \circ \gamma_1)(x) = (\beta_1 + \beta_2)\, x^{\frac{1}{\beta_3}} \left(\beta_1 x^{\frac{1}{\beta_3}} + \beta_2\right)^{-1}$. We have reported in the figures 8–14 the contour plots of the transformed Frank copula and the transformed distribution[5]. We use the first example of the second section. We remark that the repartition of the mass probability changes consequently thanks to the choice of the function γ. We could explain that because the transformations will introduce some distortions. For example, they will put more masses on the lower or upper corner.

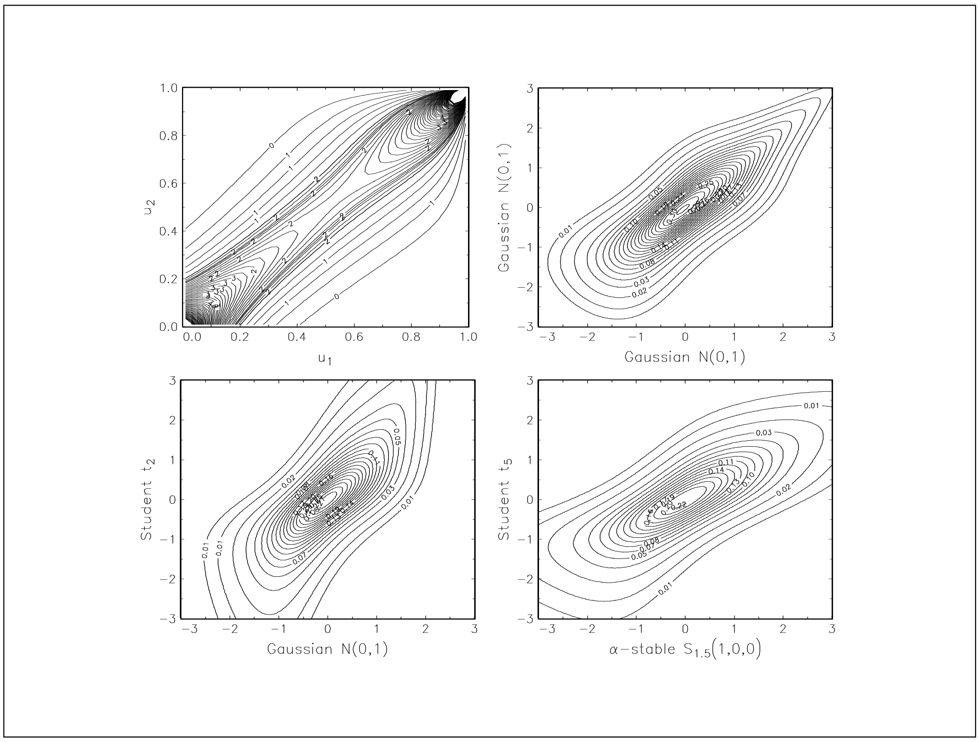

Figure 8: Contours of density for the transfomed Frank copula with $\varrho(x) = \sin\left(\frac{\pi}{2}x\right)$

[5]The parameters β_1 and β_2 are set to 1 and 0.025 in figure 13, and 0.5 and 3 in figure 14.

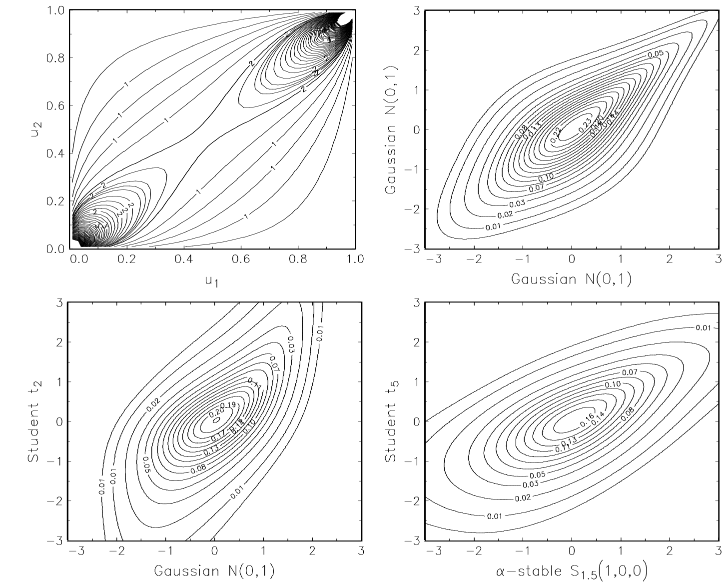

Figure 9: Contours of density for the transfomed Frank copula with $\sin\left(\frac{\pi}{2}\sqrt[5]{x}\right)$

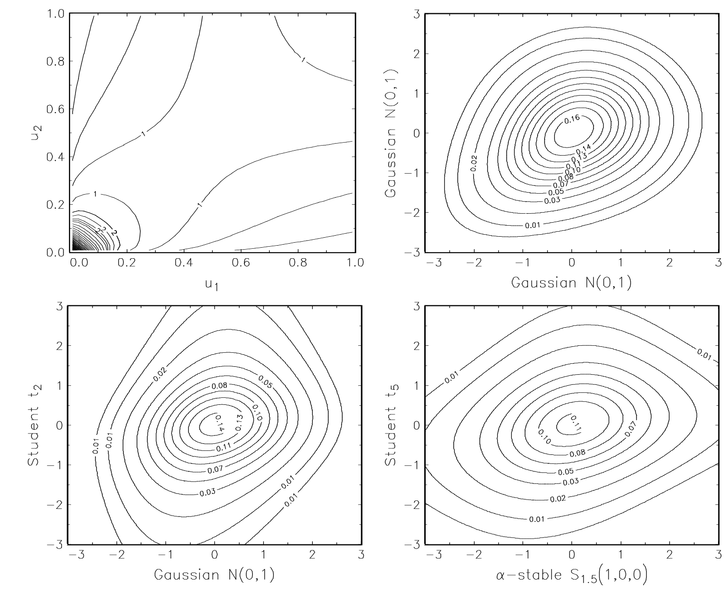

Figure 10: Contours of density for the transfomed Frank copula with $\sqrt[8]{\frac{4}{\pi}\arctan\left(\sqrt[2]{x}\right)}$

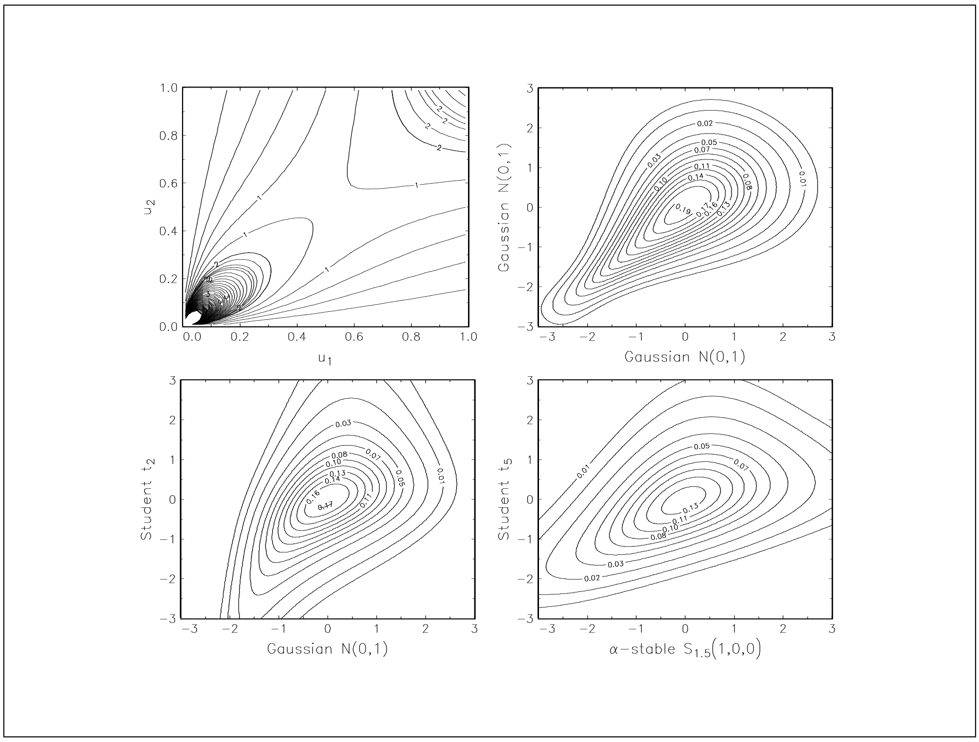

Figure 11: Contours of density for the transfomed Frank copula with $\varrho(x) = \hbar(x)$ ($\beta_1 = 1$, $\beta_2 = 0.025$

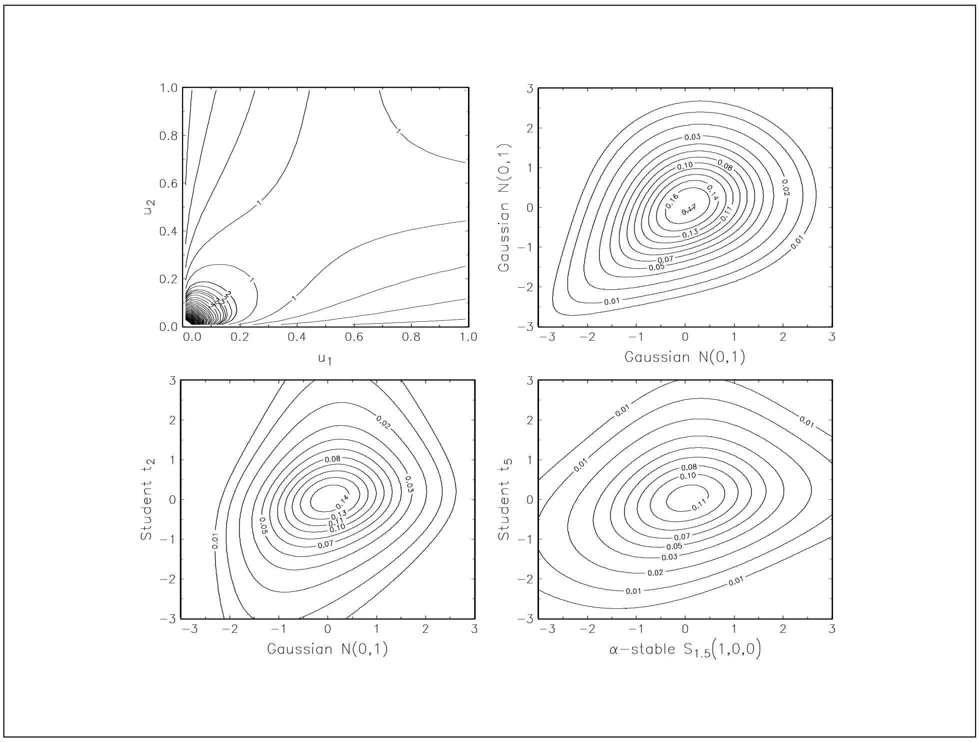

Figure 12: Contours of density for the transfomed Frank copula with $\varrho(x) = \sqrt{\hbar(\sqrt{x})}$ ($\beta_1 = 1$, $\beta_2 = 0.025$

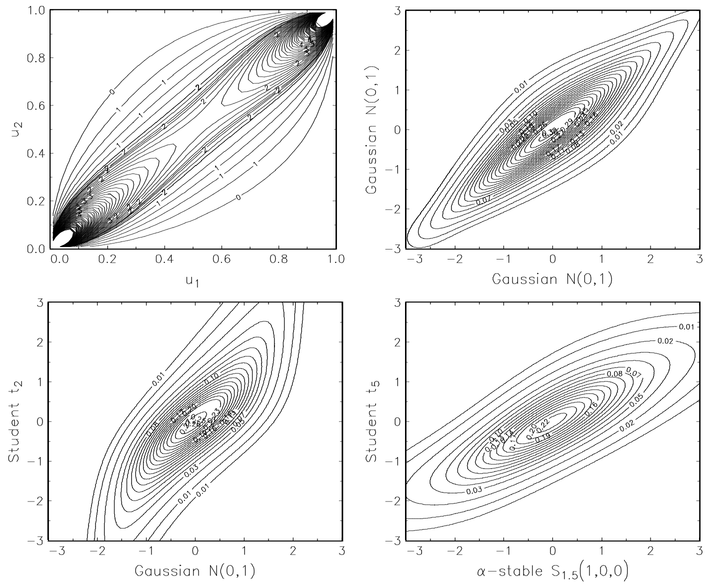

Figure 13: Contours of density for the transfomed Frank copula with $\varrho(x) = \hbar\left(\sin\left(\frac{\pi}{2}x\right)\right)$

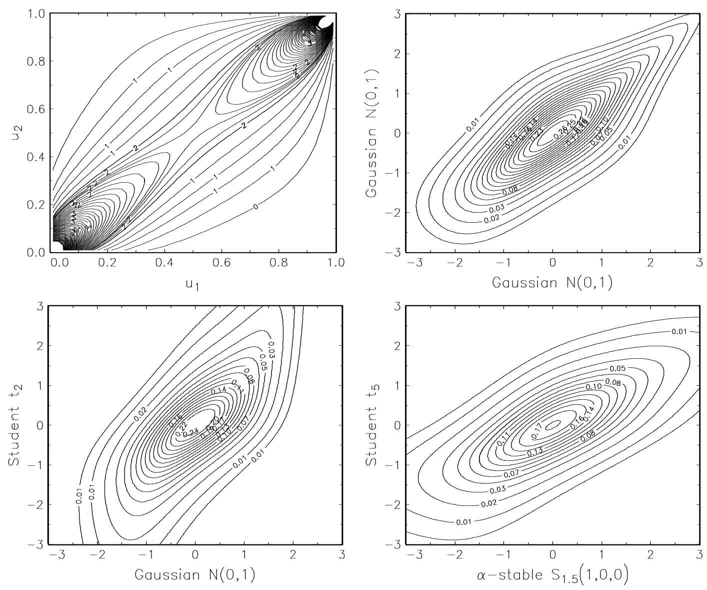

Figure 14: Contours of density for the transfomed Frank copula with $\varrho(x) = \sqrt{\hbar\left(\sin\left(\frac{\pi}{2}x\right)\right)}$