Financial Applications of Copulas

CREREG, Rennes, 11/16/00

Thierry Roncalli

Groupe de Recherche Opérationnelle
Crédit Lyonnais

Joint work with Eric Bouyé, Valdo Durrleman, Ashkan Nikeghbali and Gaël Riboulet.

The Working Paper "Copulas For Finance" is available on the web site: *http://www.gloriamundi.org/var/wps.html*

1 Introduction

Definition 1 *A copula function* $\mathbf{C}$ *is a multivariate* **uniform** *distribution.*

Theorem 1 *Let* $\mathbf{F}_1, \ldots, \mathbf{F}_N$ *be* N *univariate distributions. It comes that*

$$\mathbf{C}\left(\mathbf{F}_1\left(x_1\right), \ldots, \mathbf{F}_n\left(x_n\right), \ldots, \mathbf{F}_N\left(x_N\right)\right)$$

defines a multivariate distributions with margins $\mathbf{F}_1, \ldots, \mathbf{F}_N$ *(because the integral transforms are uniform distributions).*

$\Rightarrow$ Copulas are also a general tool to construct multivariate distributions, and so multivariate models.

2 The dependence function

- Canonical representation
- Corcondance order
- Measure of dependence

From 1958 to 1976, virtually all the results concerning copulas were obtained in connection with the study and development of the theory of probabilistic metric spaces (Schweizer [1991]).

$\Rightarrow$ Schweizer and Wolff [1976] = connection with rank statistics.

2.1 Canonical representation

Theorem 2 (Sklar's theorem) *Let $\mathbf{F}$ be a N-dimensional distribution function with continuous margins $\mathbf{F}_1, \ldots, \mathbf{F}_N$. Then $\mathbf{F}$ has a unique copula representation*

$$\mathbf{F}(x_1, \ldots, x_N) = \mathbf{C}(\mathbf{F}_1(x_1), \ldots, \mathbf{F}_N(x_N))$$

$\Rightarrow$ Copulas are also a powerful tool, because the modelling problem could be decomposed into two steps:

- Identification of the marginal distributions;
- Defining the appropriate copula function.

In terms of the density, we have the following canonical representation $f(x_1, \ldots, x_N) = c(\mathbf{F}_1(x_1), \ldots, \mathbf{F}_N(x_N)) \times \prod_{n=1}^{N} f_n(x_n)$.

The copula function of **random variables** $(X_1, \ldots, X_N)$ is **invariant** under strictly increasing transformations ($\partial_x h_n(x) > 0$):

$$\mathbf{C}_{X_1,\ldots,X_N} = \mathbf{C}_{h_1(X_1),\ldots,h_N(X_N)}$$

> *... the copula is invariant while the margins may be changed at will, it follows that is precisely the copula which captures those properties of the joint distribution which are invariant under a.s. strickly increasing transformations* (Schweizer and Wolff [1981]).

$\Rightarrow$ **Copula = dependence function of random variables.**

This property was <u>already</u> etablished by Deheuvels [1978,1979].

2.2 Examples

For the Normal copula, We have

$$\mathbf{C}(u_1, \ldots, u_N; \rho) = \Phi_\rho\left(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_N)\right)$$

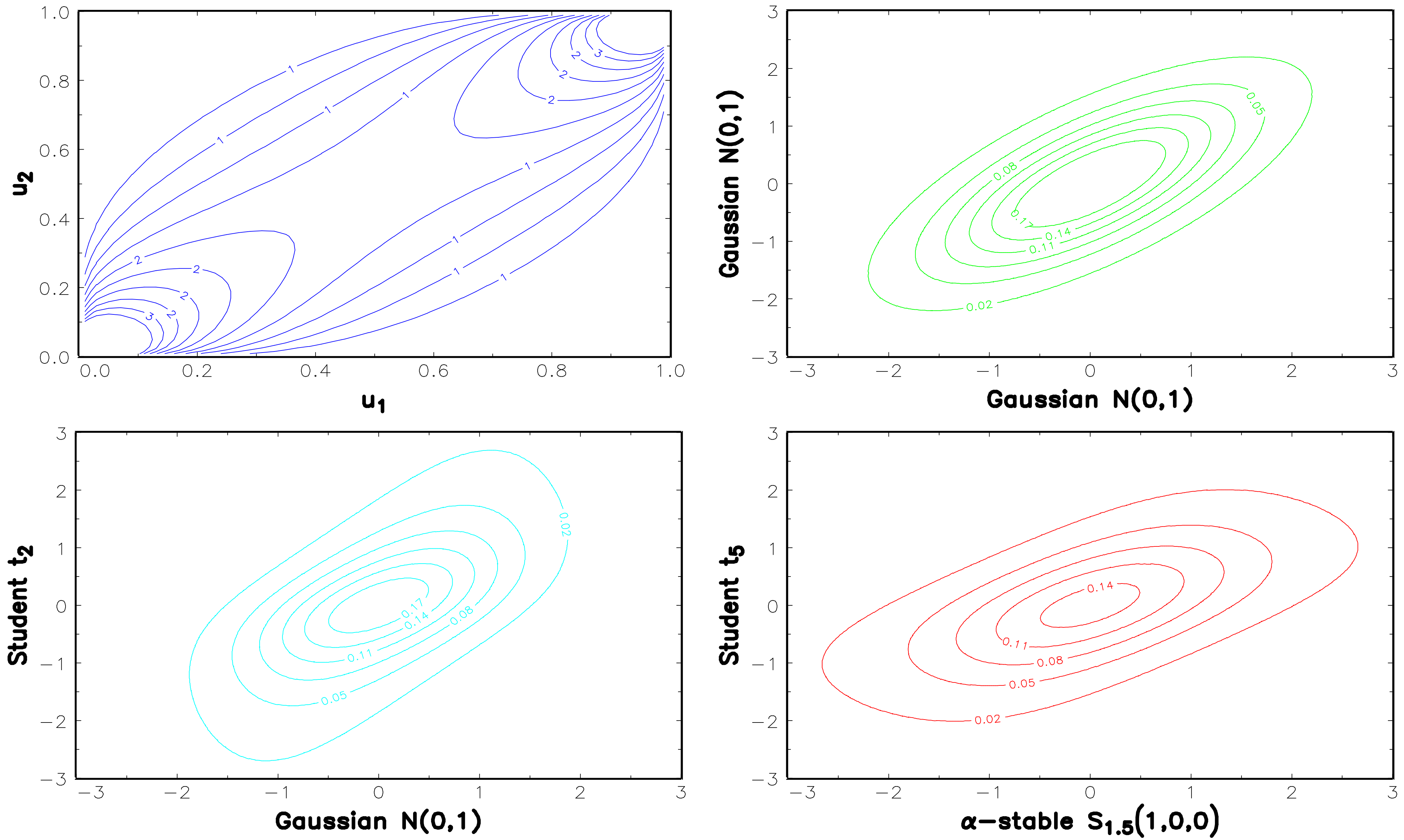
and

$$c(u_1, \ldots, u_N; \rho) = \frac{1}{|\rho|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\varsigma^\top\left(\rho^{-1} - \mathbb{I}\right)\varsigma\right)$$

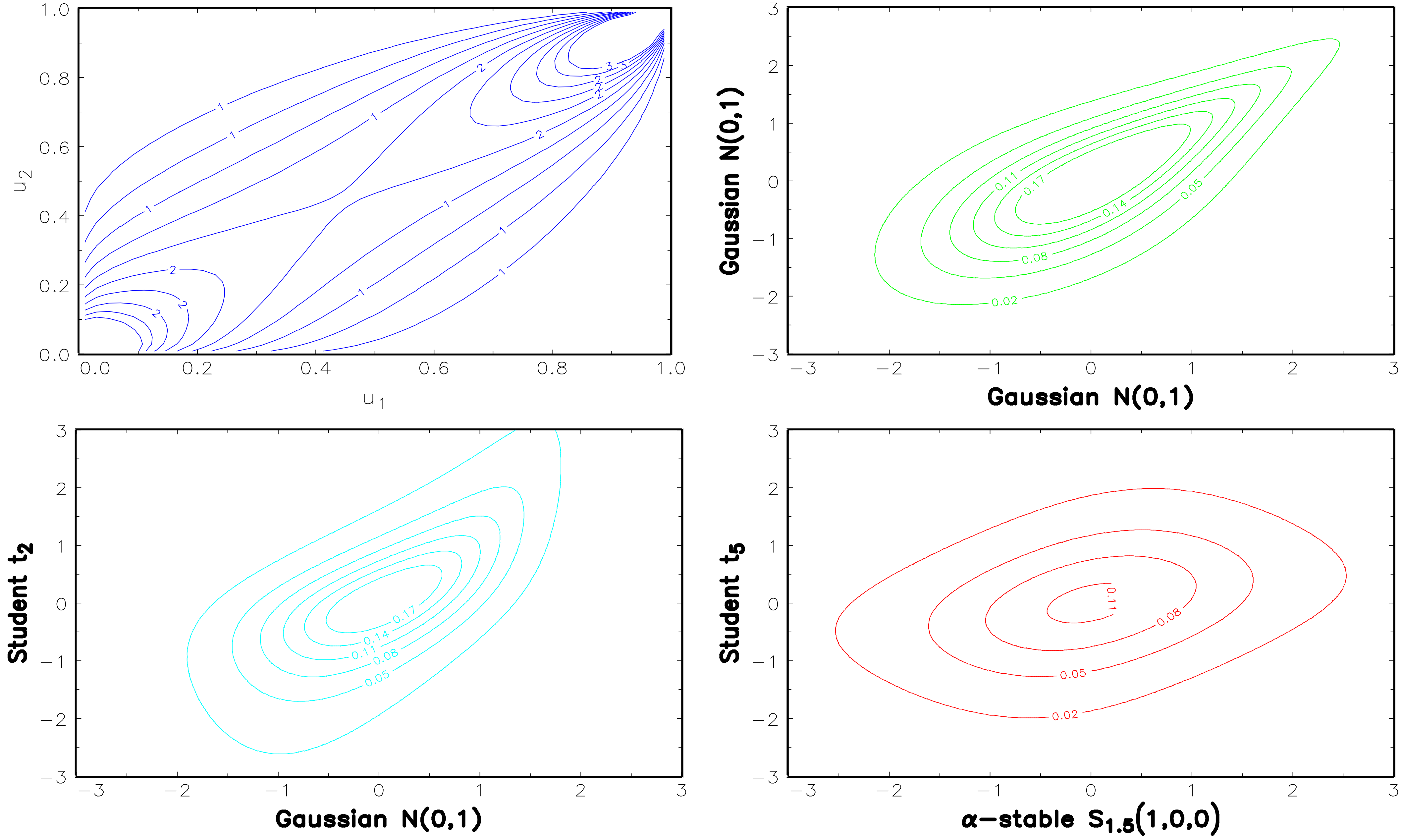
For the Gumbel copula, We have

$$\mathbf{C}(u_1, u_2) = \exp\left(-\left((-\ln u_1)^\delta + (-\ln u_2)^\delta\right)^{\frac{1}{\delta}}\right)$$

Other copulas: Archimedean, Plackett, Frank, Student, Clayton, etc.



Contours of density for Normal copula (Kendall's tau = 0.5)



Contours of density for Gumbel copula
(Kendall's tau = 0.5)

2.3 Concordance order

The copula $\mathbf{C}_1$ is **smaller** than the copula $\mathbf{C}_2$ ($\mathbf{C}_1 \prec \mathbf{C}_2$) if

$$\forall\,(u_1,\ldots,u_N) \in \mathbf{I}^N, \quad \mathbf{C}_1(u_1,\ldots,u_N) \leq \mathbf{C}_2(u_1,\ldots,u_N)$$

$\Rightarrow$ The lower and upper Fréchet bounds $\mathbf{C}^-$ and $\mathbf{C}^+$ are

$$\begin{aligned}
\mathbf{C}^-(u_1,\ldots,u_N) &= \max\left(\sum_{n=1}^{N} u_n - N + 1, 0\right) \\
\mathbf{C}^+(u_1,\ldots,u_N) &= \min(u_1,\ldots,u_N)
\end{aligned}$$

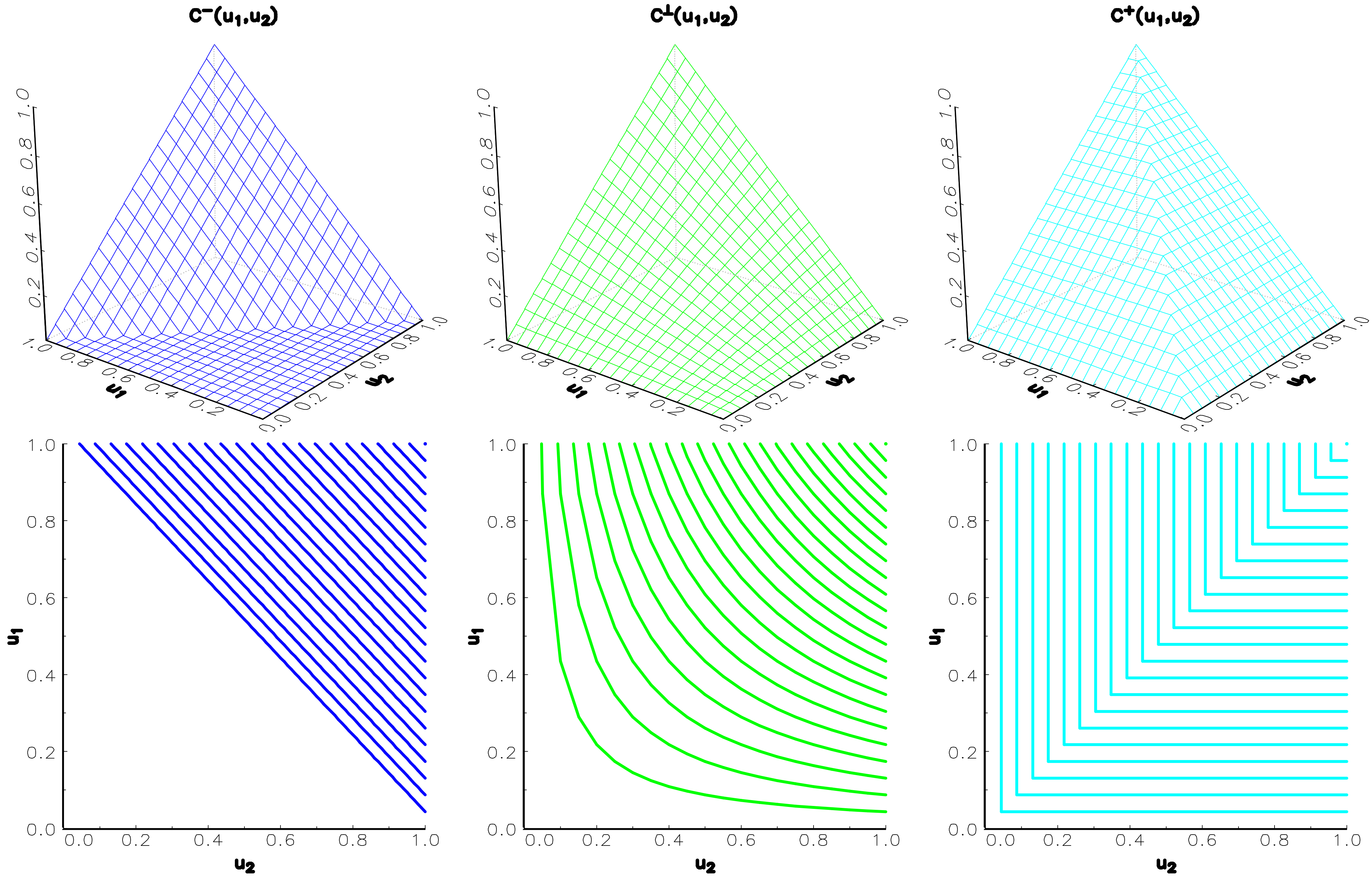
We can show that the following order holds for any copula $\mathbf{C}$:

$$\mathbf{C}^- \prec \mathbf{C} \prec \mathbf{C}^+$$

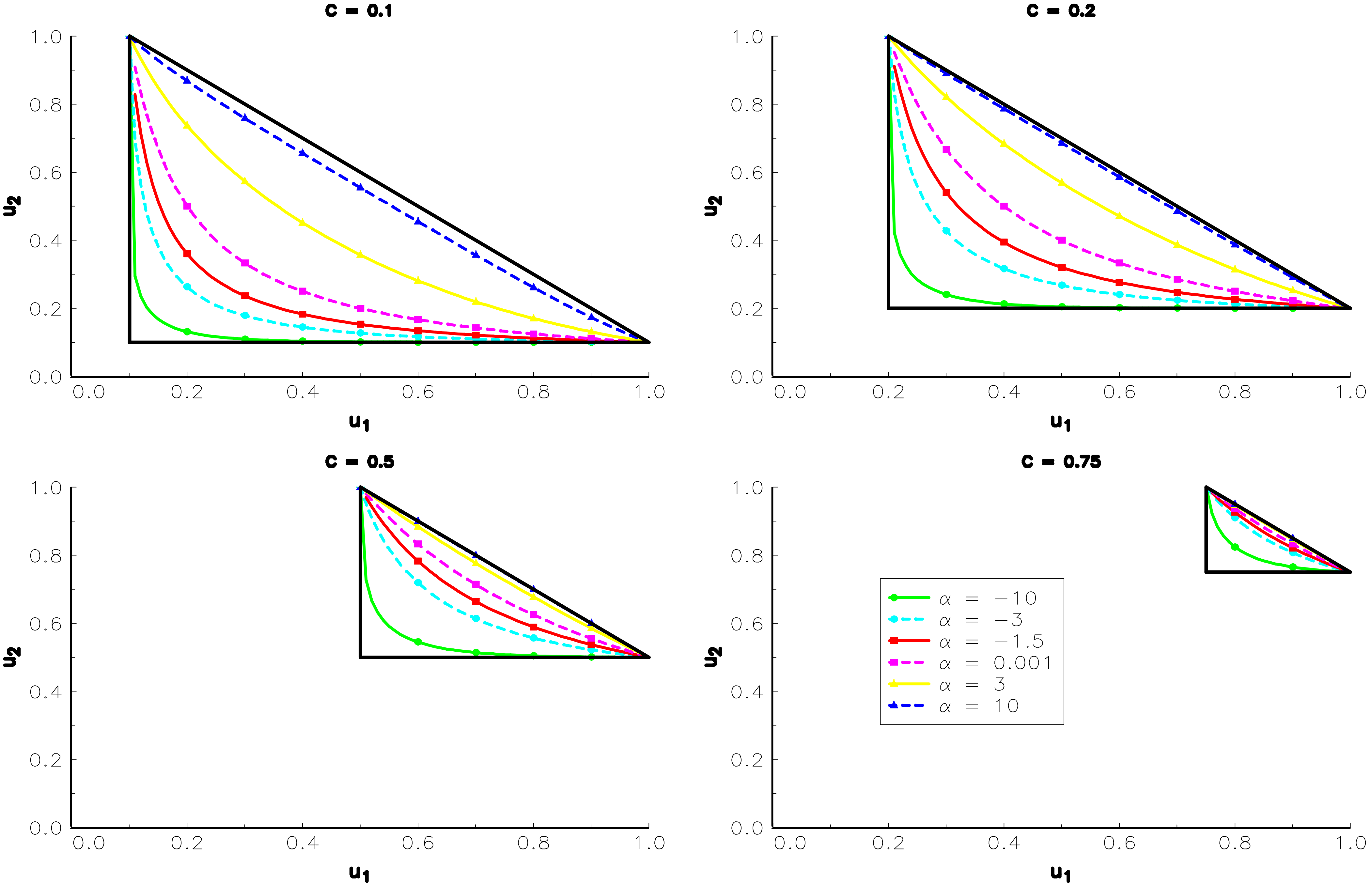
$\Rightarrow$ The minimal and maximal distributions of the Fréchet class $\mathcal{F}(\mathbf{F}_1,\mathbf{F}_2)$ are then $\mathbf{C}^-(\mathbf{F}_1(x_1),\mathbf{F}_2(x_2))$ and $\mathbf{C}^+(\mathbf{F}_1(x_1),\mathbf{F}_2(x_2))$.

Example of the bivariate Normal copula ($\mathbf{C}^\perp(u_1,u_2) = u_1 u_2$):

$$\mathbf{C}^- = \mathbf{C}_{-1} \prec \mathbf{C}_{\rho<0} \prec \mathbf{C}_0 = \mathbf{C}^\perp \prec \mathbf{C}_{\rho>0} \prec \mathbf{C}_1 = \mathbf{C}^+$$



Lower Frechet, product and upper Frechet copulas



Level curves of the Frank copula

Mikusiński, Sherwood and Taylor [1991] give the following interpretation of the three copulas $\mathbf{C}^{-}$, $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$:

- Two random variables X_1 and X_2 are **countermonotonic** — or $\mathbf{C} = \mathbf{C}^{-}$ — if there exists a r.v. X such that $X_1 = f_1(X)$ and $X_2 = f_2(X)$ with f_1 non-increasing and f_2 non-decreasing;
- Two random variables X_1 and X_2 are **independent** if the dependence structure is the product copula $\mathbf{C}^{\perp}$;
- Two random variables X_1 and X_2 are **comonotonic** — or $\mathbf{C} = \mathbf{C}^{+}$ — if there exists a random variable X such that $X_1 = f_1(X)$ and $X_2 = f_2(X)$ where the functions f_1 and f_2 are non-decreasing;

2.4 Measures of association or dependence

If κ is a *measure of concordance*, it satisfies the properties:
$-1 \leq \kappa_{\mathbf{C}} \leq 1$; $\mathbf{C}_1 \prec \mathbf{C}_2 \Rightarrow \kappa_{\mathbf{C}_1} \leq \kappa_{\mathbf{C}_2}$; etc.

Schweizer and Wolff [1981] show that Kendall's tau and Spearman's rho can be (re)formulated in terms of copulas

$$
\begin{aligned}
\tau &= 4 \iint_{\mathbf{I}^2} \mathbf{C}(u_1, u_2)\, \mathrm{d}\mathbf{C}(u_1, u_2) - 1 \\
\varrho &= 12 \iint_{\mathbf{I}^2} u_1 u_2 \, \mathrm{d}\mathbf{C}(u_1, u_2) - 3
\end{aligned}
$$

$\Rightarrow$ The linear (or Pearson) correlation is not a measure of dependence.

2.5 Some misinterpretations of the correlation

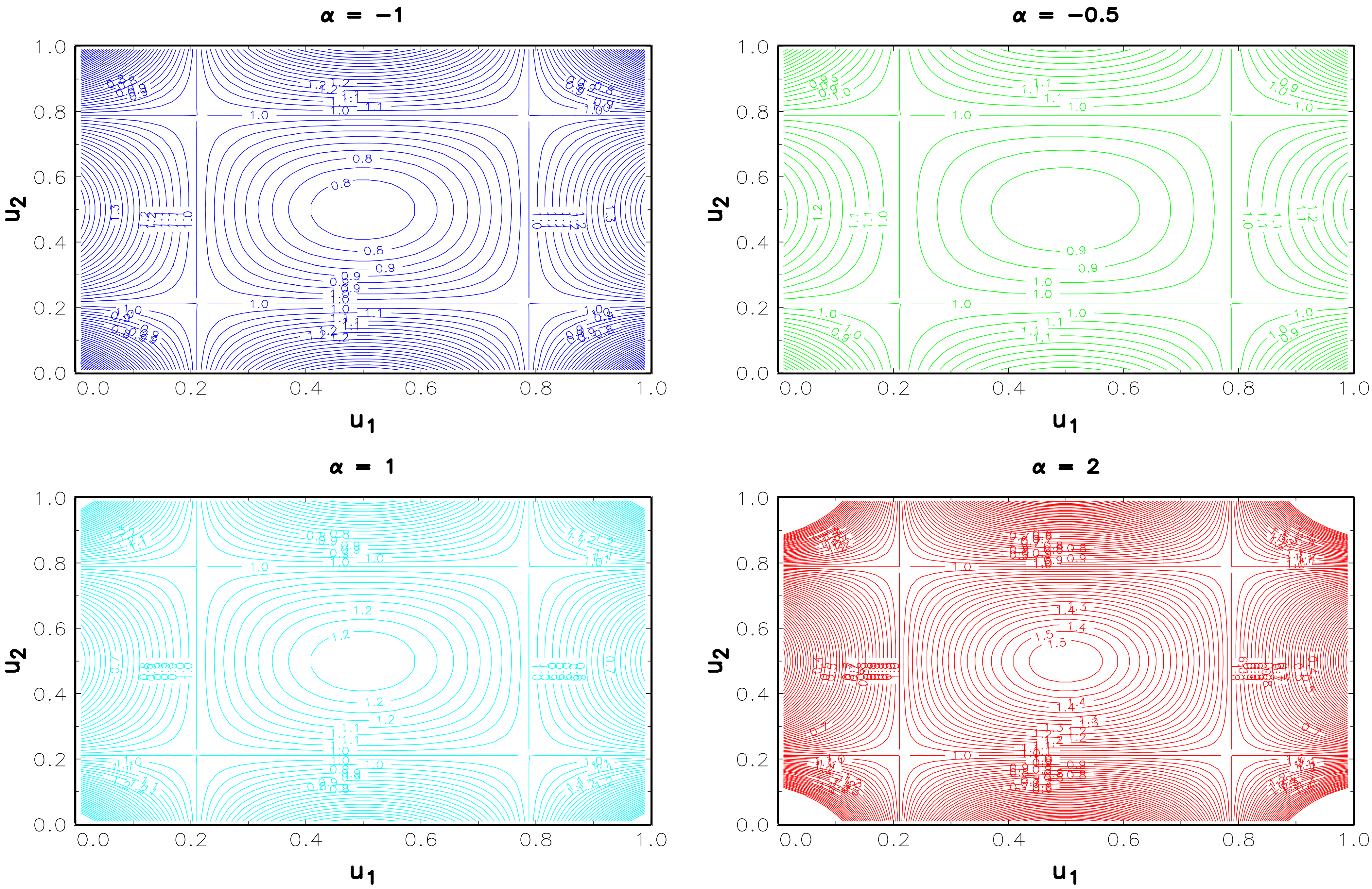
The following statements are **false**:

1. X_1 and X_2 are independent **if and only if** $\rho(X_1, X_2) = 0$;
2. For given margins, the permissible range of $\rho(X_1, X_2)$ is $[-1, 1]$;
3. $\rho(X_1, X_2) = 0$ means that there are no relationship between X_1 and X_2.

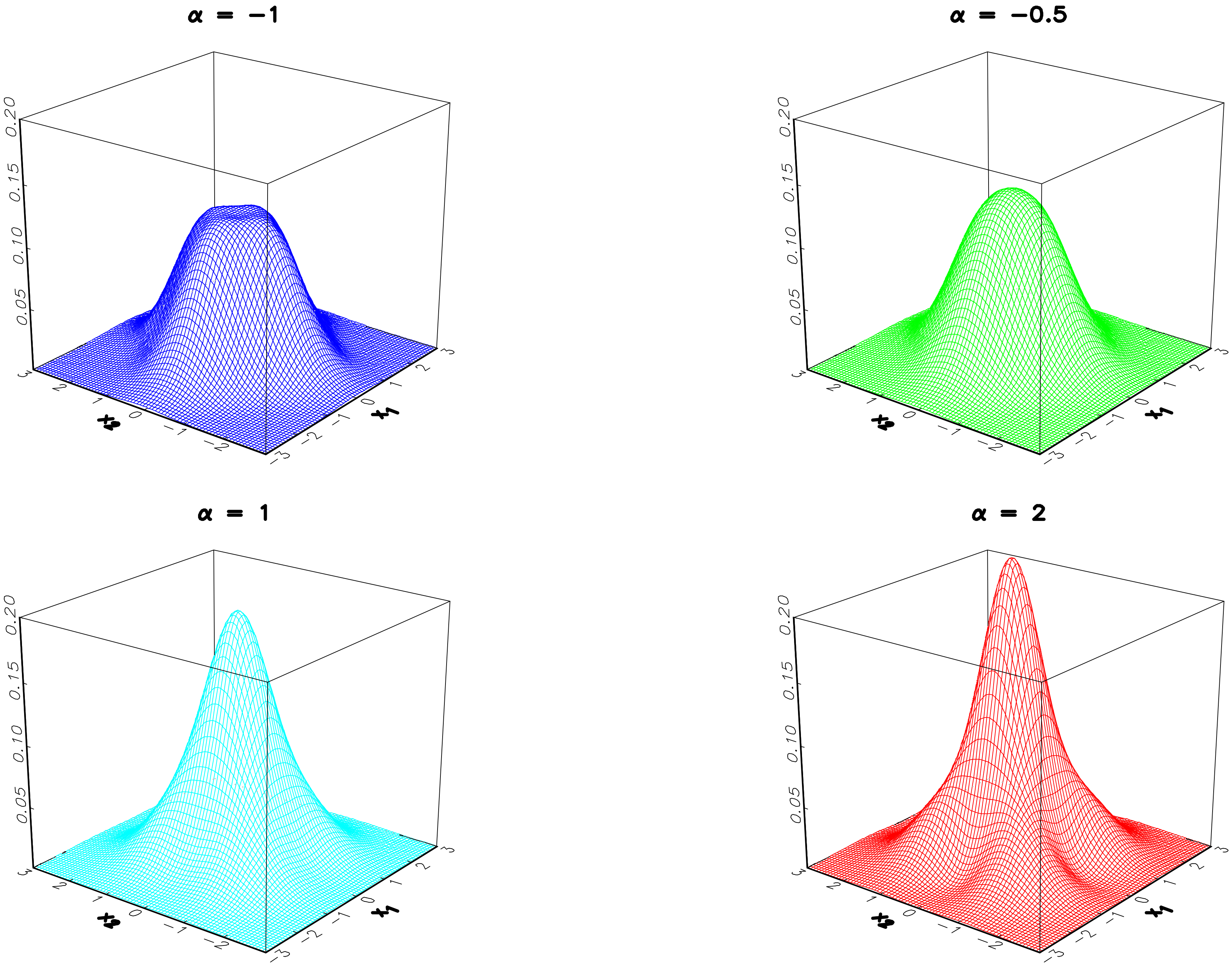
- We consider the cubic copula of Durrleman, Nikeghbali and Roncalli [2000]

$$\mathbf{C}(u_1, u_2) = u_1 u_2 + \alpha [u_1(u_1 - 1)(2u_1 - 1)] [u_2(u_2 - 1)(2u_2 - 1)]$$

with $\alpha \in [-1, 2]$. If the margins $\mathbf{F}_1$ and $\mathbf{F}_2$ are continous and symmetric, the Pearson correlation is zero. Moreover, if $\alpha \neq 0$, the random variables X_1 and X_2 are not independent.



Contours of density for the cubic copula



PDF of the cubic copula with Gaussian margins

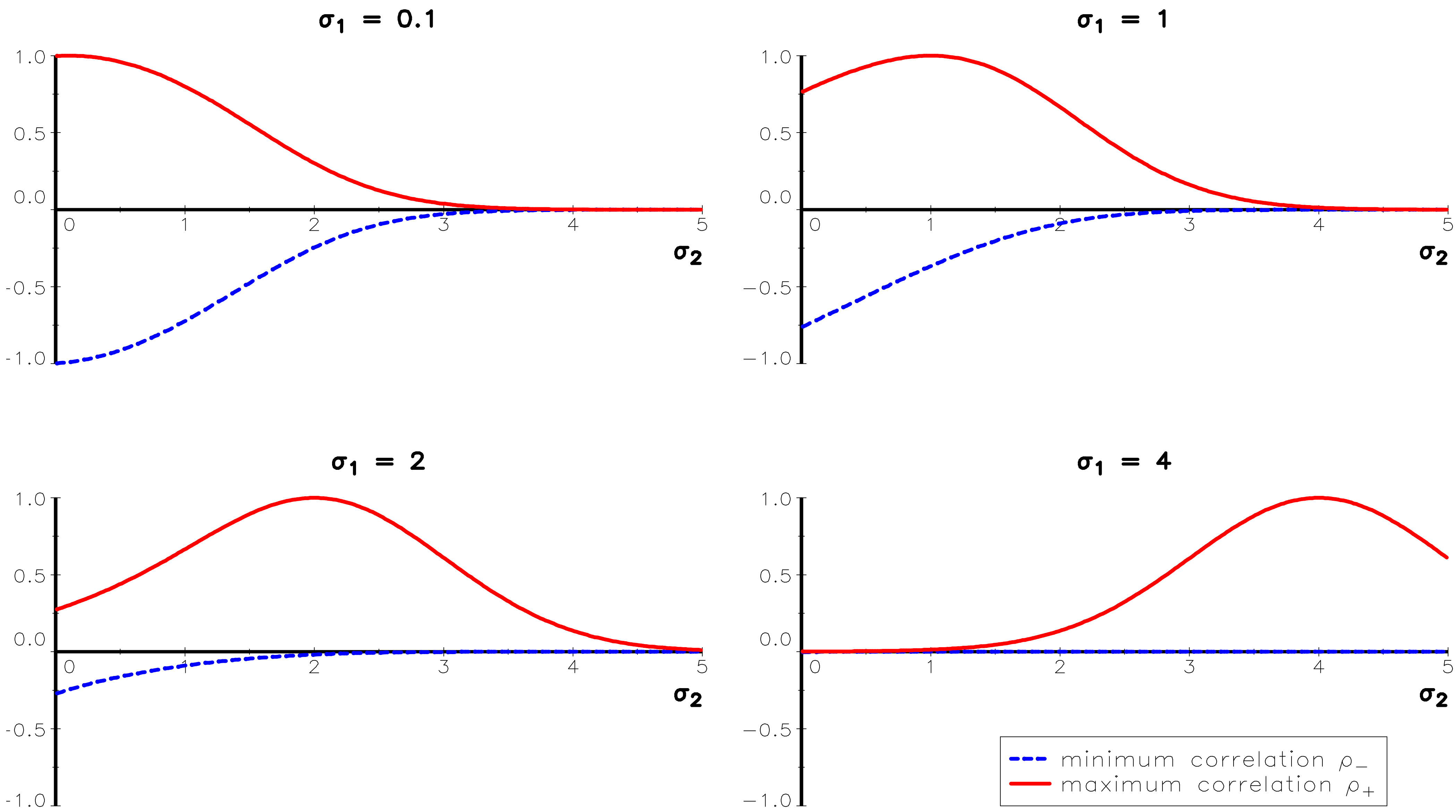
- Wang [1997] shows that the min. and max. correlations of $X_1 \sim \mathcal{LN}(\mu_1, \sigma_1)$ and $X_2 \sim \mathcal{LN}(\mu_2, \sigma_2)$ are

$$\rho_- = \frac{e^{-\sigma_1\sigma_2} - 1}{\left(e^{\sigma_1^2} - 1\right)^{\frac{1}{2}} \left(e^{\sigma_2^2} - 1\right)^{\frac{1}{2}}} \leq 0$$

$$\rho_+ = \frac{e^{\sigma_1\sigma_2} - 1}{\left(e^{\sigma_1^2} - 1\right)^{\frac{1}{2}} \left(e^{\sigma_2^2} - 1\right)^{\frac{1}{2}}} \geq 0$$

ρ_- and ρ_+ are not necessarily equal to -1 and 1. Example with $\sigma_1 = 1$ and $\sigma_2 = 3$:

Copula	$\rho(X_1, X_2)$	$\tau(X_1, X_2)$	$\varrho(X_1, X_2)$
$\mathbf{C}^-$	-0.008	-1	-1
$\rho = -0.7$	$\simeq 0$	-0.49	-0.68
$\mathbf{C}^\perp$	0	0	0
$\rho = 0.7$	$\simeq 0.10$	0.49	0.68
$\mathbf{C}^+$	0.16	1	1



Permissible range of $\rho(X_1,X_2)$
when X_1 and X_2 are two LN random variables

- Using an idea of Ferguson [1994], Nelsen [1998] defines the following copula

$$\mathbf{C}(u_1, u_2) = \begin{cases} u_1 & 0 \le u_1 \le \frac{1}{2}u_2 \le \frac{1}{2} \\ \frac{1}{2}u_2 & 0 \le \frac{1}{2}u_2 \le u_1 \le 1 - \frac{1}{2}u_2 \\ u_1 + u_2 - 1 & \frac{1}{2} \le 1 - \frac{1}{2}u_2 \le u_1 \le 1 \end{cases}$$

  We have $\operatorname{cov}(U_1, U_2) = 0$, but $\Pr\{U_2 = 1 - |2U_1 - 1|\} = 1$, i.e. "*the two random variables can be uncorrelated although one can be predicted perfectly from the other*".

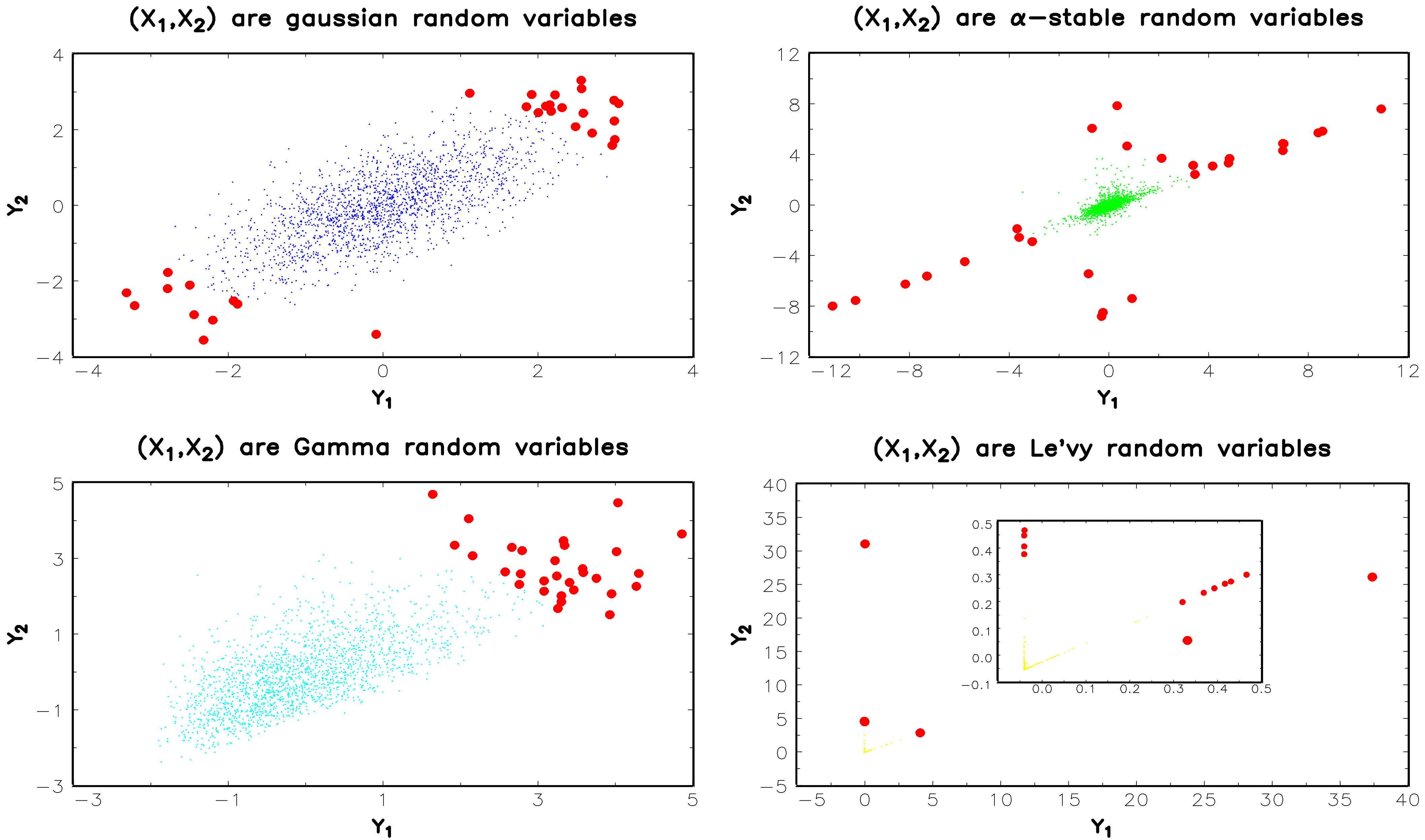
3 An open field for risk management

- Economic capital adequacy
- Market risk
- Credit risk
- Operational risk

3.1 Economic capital adequacy

With copulas, it appears that the risk can be splitted into two parts: the individual risks and the dependence structure between them.

- **Coherent multivariate statistical model = Coherent model for individual risks + coherent dependence function**
- **Coherent model for individual risks** = taking into account fat-tailed distributions, etc.
- **coherent dependence function** = understanding the aggregation of quantiles of the individual risks.



Bivariate distributions with same first and second moments
= Gaussian VaRs are equal

⇒ The influence of margins

Rating	*VaR*	BBB	A	AA	AAA
α	99%	99.75%	99.9%	99.95%	99.97%
Return time	100 days	400 days	4 years	8 years	13 years
$\frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(0.99)}$	1	1.20	1.33	1.41	1.48
$\frac{t_4^{-1}(\alpha)}{t_4^{-1}(0.99)}$	1	1.49	1.91	2.30	2.62

⇒ The influence of the dependence function: If a bivariate copula $\mathbf{C}$ is such that*

$$\lim_{u\to 1} \frac{\bar{\mathbf{C}}(u,u)}{1-u} = \lambda$$

exists, then $\mathbf{C}$ has **upper tail dependence** for $\lambda \in (0,1]$ and no upper tail dependence for $\lambda = 0$.

*$\bar{\mathbf{C}}$ is the joint survival function, that is

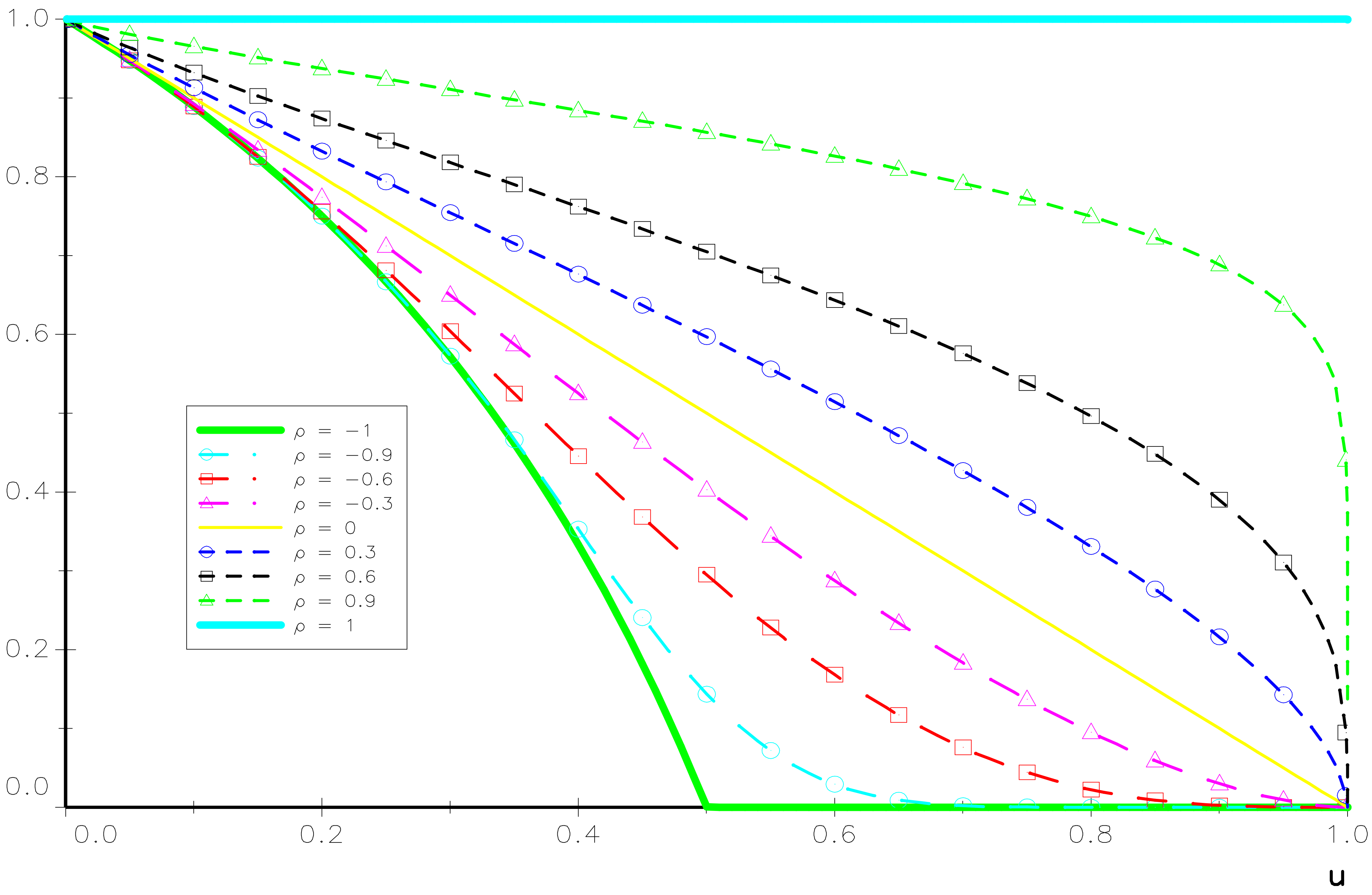
$$\bar{\mathbf{C}}(u_1,u_2) = 1 - u_1 - u_2 + \mathbf{C}(u_1,u_2)$$

Remark 1 *The measure λ is the probability that one variable is extreme given that the other is extreme.*

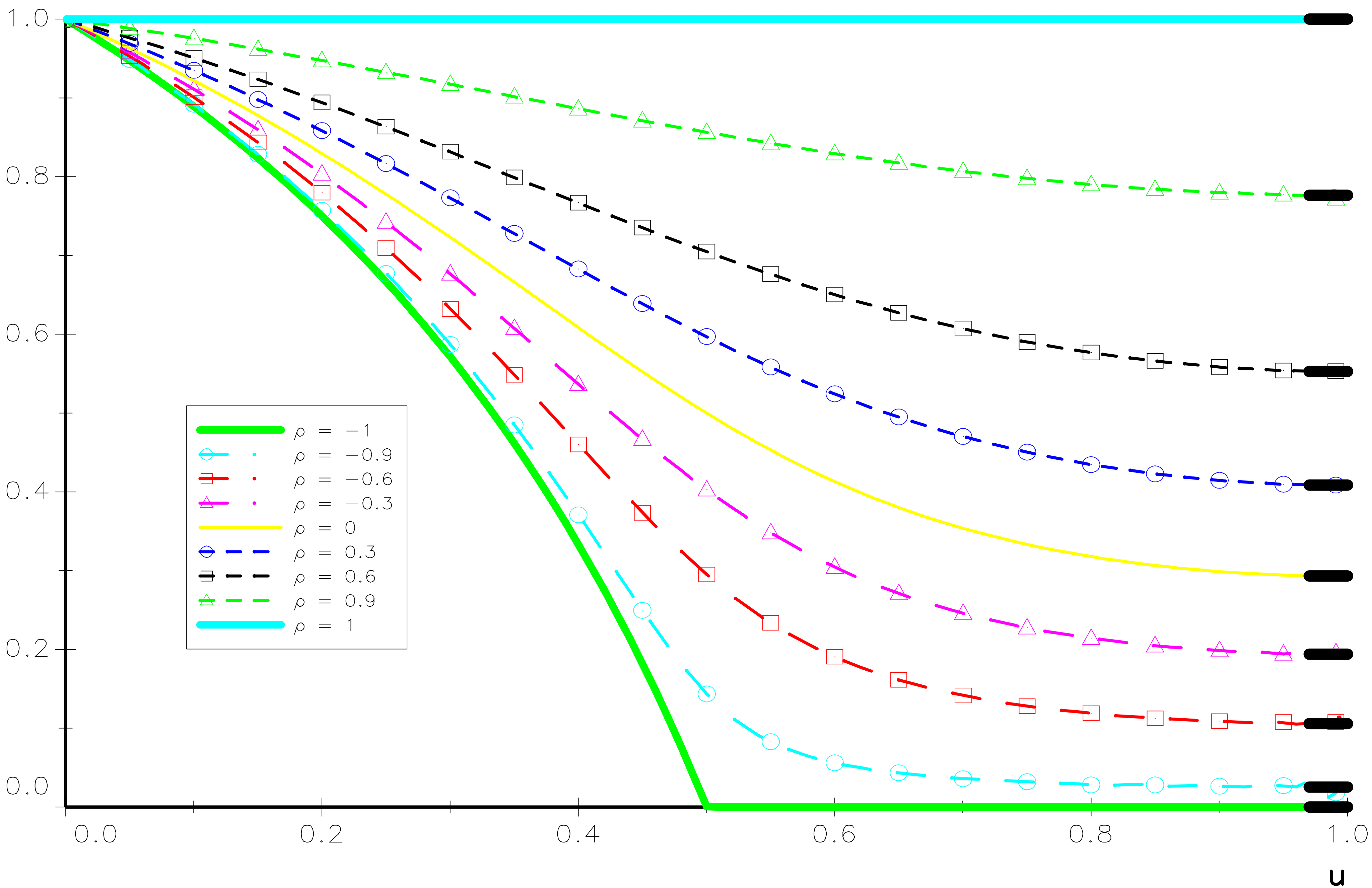
$\Rightarrow$ Coles, Currie and Tawn [1999] define the quantile-dependent measure of dependence as follows

$$\lambda(u) = \Pr\{U_1 > u | U_2 > u\} = \frac{\bar{\mathbf{C}}(u,u)}{1-u}$$

1. Normal copula $\Rightarrow$ extremes are asymptotically independent for $\rho \neq 1$, i.e $\lambda = 0$ for $\rho < 1$.
2. Student copula $\Rightarrow$ extremes are asymptotically dependent for $\rho \neq -1$.



Quantile-dependent measure λ(u) for the Normal copula



λ(u) for the Student copula and $\nu = 1$

3.2 Market risk

Copulas = a powerful tool for market risk measurement.

Copulas have been already incorporated in some software solutions:

- SAS Risk Dimensions
- Palisade @Risk

3.2.1 Value-at-Risk

LME example:

	AL	AL-15	CU	NI	PB
P_1	1	1	1	1	1
P_2	-1	-1	-1	1	1
P_3	2	1	-3	4	5

- Gaussian margins and Normal copula

	90%	95%	99%	99.5%	99.9%
P_1	7.26	9.33	13.14	14.55	17.45
P_2	4.04	5.17	7.32	8.09	9.81
P_3	13.90	17.82	25.14	27.83	33.43

- Student margins ($\nu = 4$) and Normal copula

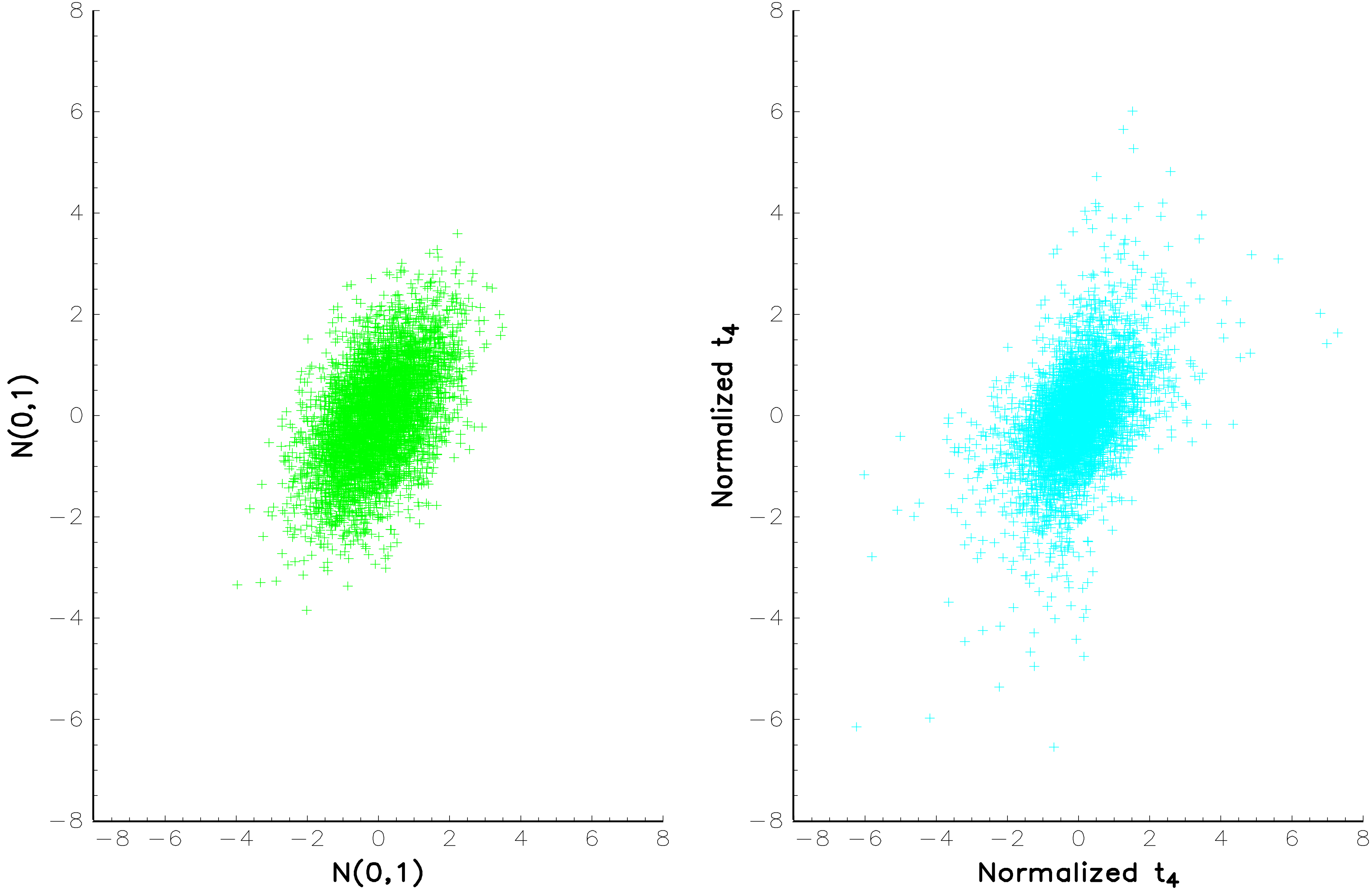
	90%	95%	99%	99.5%	99.9%
P_1	9.20	12.48	20.16	23.95	34.07
P_2	5.33	7.08	11.16	13.17	19.17
P_3	18.04	24.11	38.90	46.45	69.51

- Gaussian margins and Student copula ($\nu = 1$)

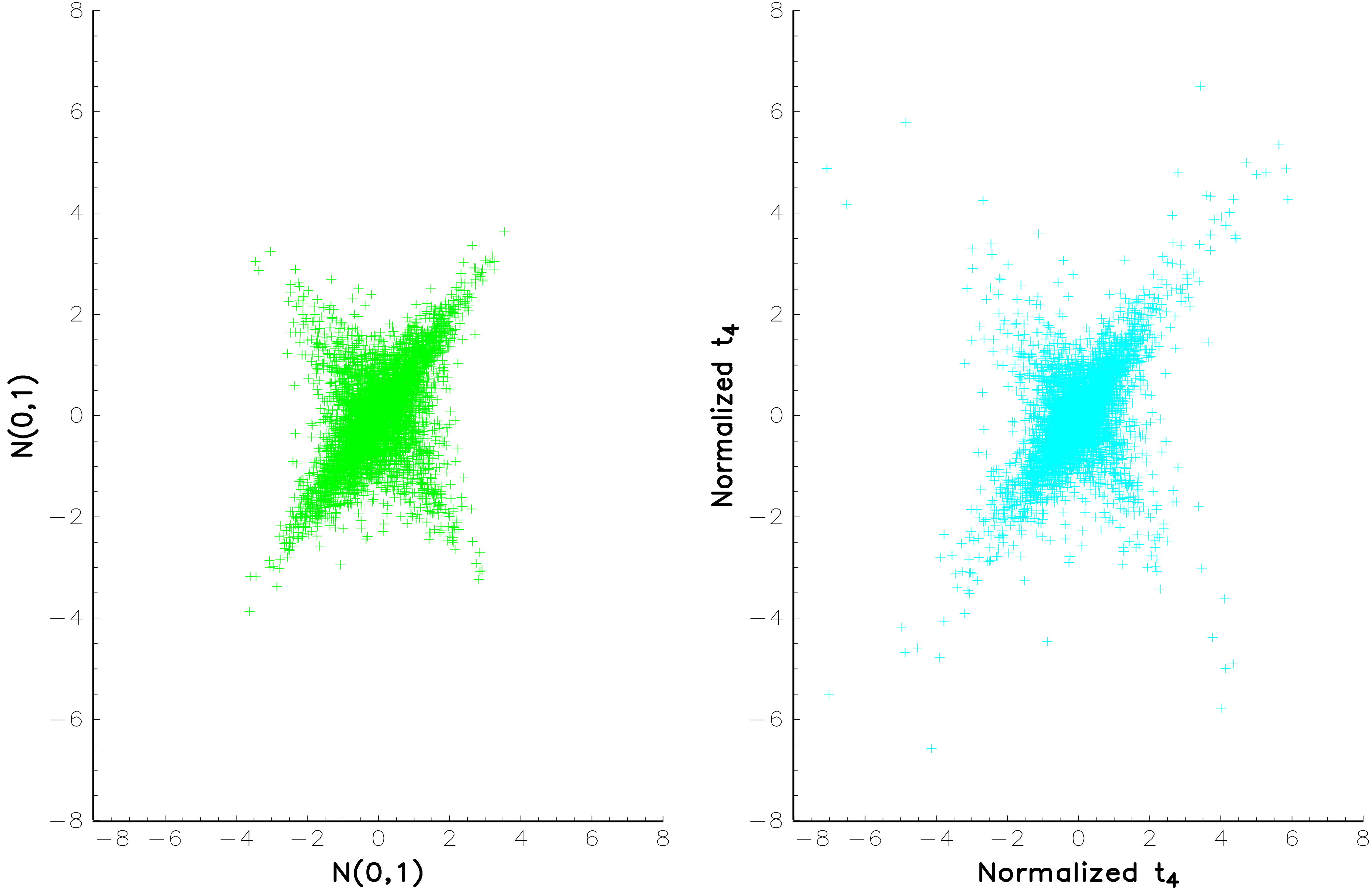
	90%	95%	99%	99.5%	99.9%
P_1	6.49	8.94	14.48	16.67	21.11
P_2	3.45	5.08	9.17	11.03	15.77
P_3	11.99	17.53	31.88	37.94	51.66

Value-at-risk based on Student margins and a Normal copula (Gauss software, Pentium III 550 Mhz, 100000 simulations)

Number of assets	Computational time
2	0.1 sc
10	24.5 sc
100	4 mn 7 sc
500	33 mn 22 sc
1000	1 hr 44 mn 45 sc



5000 simulations with a Normal copula ($\rho = 0.5$)



5000 simulations with a Student copula ($\rho = 0.5$, $\nu = 1$)

3.2.2 Stress testing

Stress testing program = what are the larger risks in the portfolio?
$\Rightarrow$ **Extreme value theory** allows to model the maxima or minima of a distribution and to apply stress scenarios to a portfolio.

Problem: **multivariate** stress scenarios.

Multivariate extreme value theory An extreme value copula satisfy the following condition

$$\mathbf{C}\left(u_1^t, \ldots, u_N^t\right) = \mathbf{C}^t\left(u_1, \ldots, u_N\right) \quad \forall\, t > 0$$

For example, the Gumbel copula is an extreme value copula:

$$\begin{aligned}\mathbf{C}\left(u_1^t, u_2^t\right) &= \exp\left(-\left[\left(-\ln u_1^t\right)^{\alpha} + \left(-\ln u_2^t\right)^{\alpha}\right]^{\frac{1}{\alpha}}\right) \\ &= \left[\exp\left(-\left[(-\ln u_1)^{\alpha} + (-\ln u_1)^{\alpha}\right]^{\frac{1}{\alpha}}\right)\right]^t = \mathbf{C}^t\left(u_1, u_2\right)\end{aligned}$$

What is the link between extreme value copulas and the multivariate extreme value theory? **The joint limit distribution $\mathbf{G}$ of multivariate extremes is of the type**

$$\mathbf{G}\left(\chi_1^+,\ldots,\chi_N^+\right)=\mathbf{C}_\star\left(\mathbf{G}_1\left(\chi_1^+\right),\ldots,\mathbf{G}_N\left(\chi_N^+\right)\right)$$

where $\mathbf{C}_\star$ is an extreme value copula and $\mathbf{G}_n$ a non-degenerate univariate extreme value distribution.

Univariate theory $\Rightarrow$ Fisher-Tippet theorem.

Multivariate theory $\Rightarrow$ the class of multivariate extreme value distribution is the class of **extreme value copulas** with nondegenerate marginals.

Let $\mathbf{D}$ be a multivariate distribution with unit exponential survival margins and $\mathbf{C}$ an extreme value copula. Using the relation

$$\mathbf{C}(u_1, \ldots, u_N) = \mathbf{C}\left(\mathrm{e}^{-\tilde{u}_1}, \ldots, \mathrm{e}^{-\tilde{u}_N}\right) = \mathbf{D}(\tilde{u}_1, \ldots, \tilde{u}_N)$$

we have $\mathbf{D}^t(\tilde{\mathbf{u}}) = \mathbf{D}(t\tilde{\mathbf{u}})$ and then $\mathbf{D}$ is a *min-stable multivariate exponential* (MSMVE) distribution.

Theorem 3 (Pickands representation of MSMVE distributions) *Let $\mathbf{D}(\tilde{\mathbf{u}})$ be a survival function with exponential margins. $\mathbf{D}$ satisfies*

$$\begin{aligned} \mathbf{D}(\tilde{\mathbf{u}}) &= \exp\left[-\left(\sum_{n=1}^{N} \tilde{u}_n\right) B(w_1, \ldots, w_N)\right] \\ B(\mathbf{w}) &= \int \cdots \int_{\mathcal{S}_N} \max_{1 \leq n \leq N} (q_n w_n)\, \mathrm{d}S(\mathbf{q}) \end{aligned}$$

with $w_n = \tilde{u}_n / \sum_1^N \tilde{u}_n$ and where $\mathcal{S}_N$ is the N-dimensional unit simplex and S a finite measure on $\mathcal{S}_N$. B is a convex function and $\max(w_1, \ldots, w_N) \leq B(\mathbf{w}) \leq 1$.

It comes necessarily that an extreme value copula verifies

$$\mathbf{C}^{\perp} \prec \mathbf{C} \prec \mathbf{C}^{+}$$

Maximum domain of attraction: $\mathbf{F} \in \mathsf{MDA}(\mathbf{G})$ iff

1. $\mathbf{F}_n \in \mathsf{MDA}(\mathbf{G}_n)$ for all $n = 1 \ldots, N$;
2. $\mathbf{C} \in \mathsf{MDA}(\mathbf{C}_\star)$.

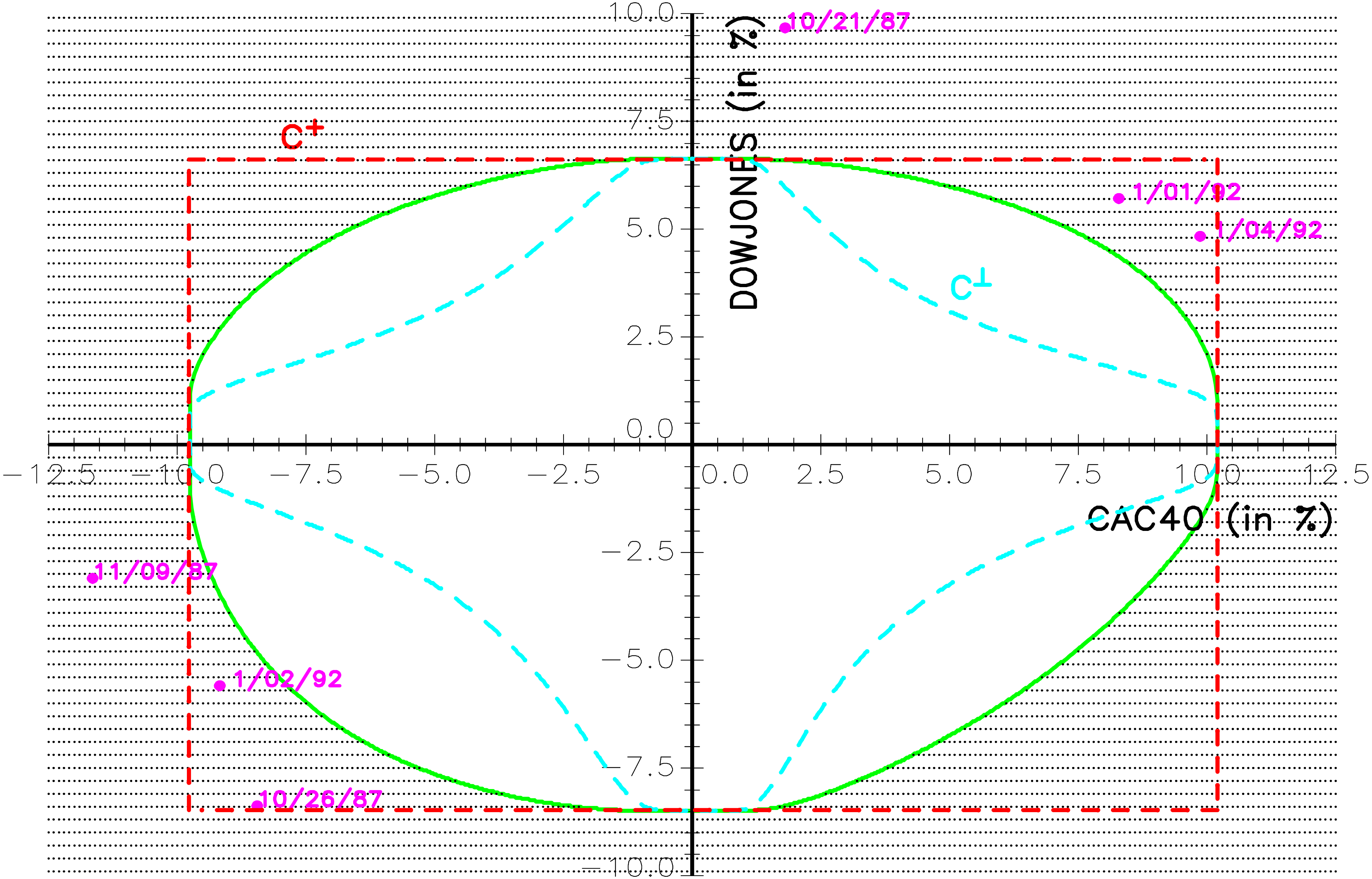
Bivariate stress testing A *failure area* = set of values $\left(\chi_1^+, \chi_2^+\right)$ such that

$$\Pr\left\{\chi_1^+ > \chi_1, \chi_2^+ > \chi_2\right\} = 1 - \mathbf{G}_1(\chi_1) - \mathbf{G}_2(\chi_2) + \mathbf{C}(\mathbf{G}_1(\chi_1), \mathbf{G}_2(\chi_2))$$

equals a given level of probability.

Return time of the CAC40/DowJones example of Costinot, Roncalli and Teïletche [2000]:

Date	CAC40	DowJones	EVT	Gaussian hyp.
10/19/1987	−10.14%	−25.63%	105.79	1.44×10^{14}
10/21/1987	+1.80%	+9.67%	18.14	2.88×10^{14}
10/26/1987	−8.45%	−8.38%	9.18	1.80×10^{13}
11/09/1987	−11.65%	−3.10%	8.12	2.30×10^{9}
01/01/1992	+8.28%	+5.71%	6.85	1.66×10^{8}
01/02/1992	−9.18%	−5.59%	6.39	2.96×10^{9}
01/04/1992	+9.87%	+4.83%	7.06	2.05×10^{9}



Failure area for a 5 years waiting time

Multivariate stress testing In the multivariate case, the *failure area* is defined as the following set

$$\left\{ (\chi_1, \ldots, \chi_N) \in \mathbb{R}^N \mid u_1 = \mathbf{F}_1(\chi_1), \ldots, u_N = \mathbf{F}_N(\chi_N), \bar{\mathbf{C}}(u_1, \ldots, u_N) < \frac{1}{\mathfrak{t}} \right\}$$

with

$$\bar{\mathbf{C}}(u_1, \ldots, u_n, \ldots, u_N) = \sum_{n=0}^{N} \left[(-1)^n \sum_{\mathbf{u} \in \mathcal{Z}(N-n,N)} \mathbf{C}(\mathbf{u}) \right]$$

where $\mathcal{Z}(M, N)$ denote the set $\left\{ \mathbf{u} \in [0,1]^N \mid \sum_{n=1}^{N} \mathcal{X}_{\{1\}}(u_n) = M \right\}$. It is also possible to compute the **implied** *return period* $\mathfrak{t}$ for a given vector $(\chi_1, \ldots, \chi_n, \ldots, \chi_N)$. We have then

$$\mathfrak{t}(\chi_1, \ldots, \chi_n, \ldots, \chi_N) = \bar{\mathbf{C}}^{-1}(\mathbf{F}_1(\chi_1), \ldots, \mathbf{F}_n(\chi_n), \ldots, \mathbf{F}_N(\chi_N))$$

The strength of a crisis $(\chi_1, \ldots, \chi_N)$ is generally a subjective notion.
$\Rightarrow$ The **implied** return period $\simeq$ measure of the severity.
$\Rightarrow$ it can be used to quantify the stress tests provided by the economists for the stress testing program of a bank and to verify their coherence.

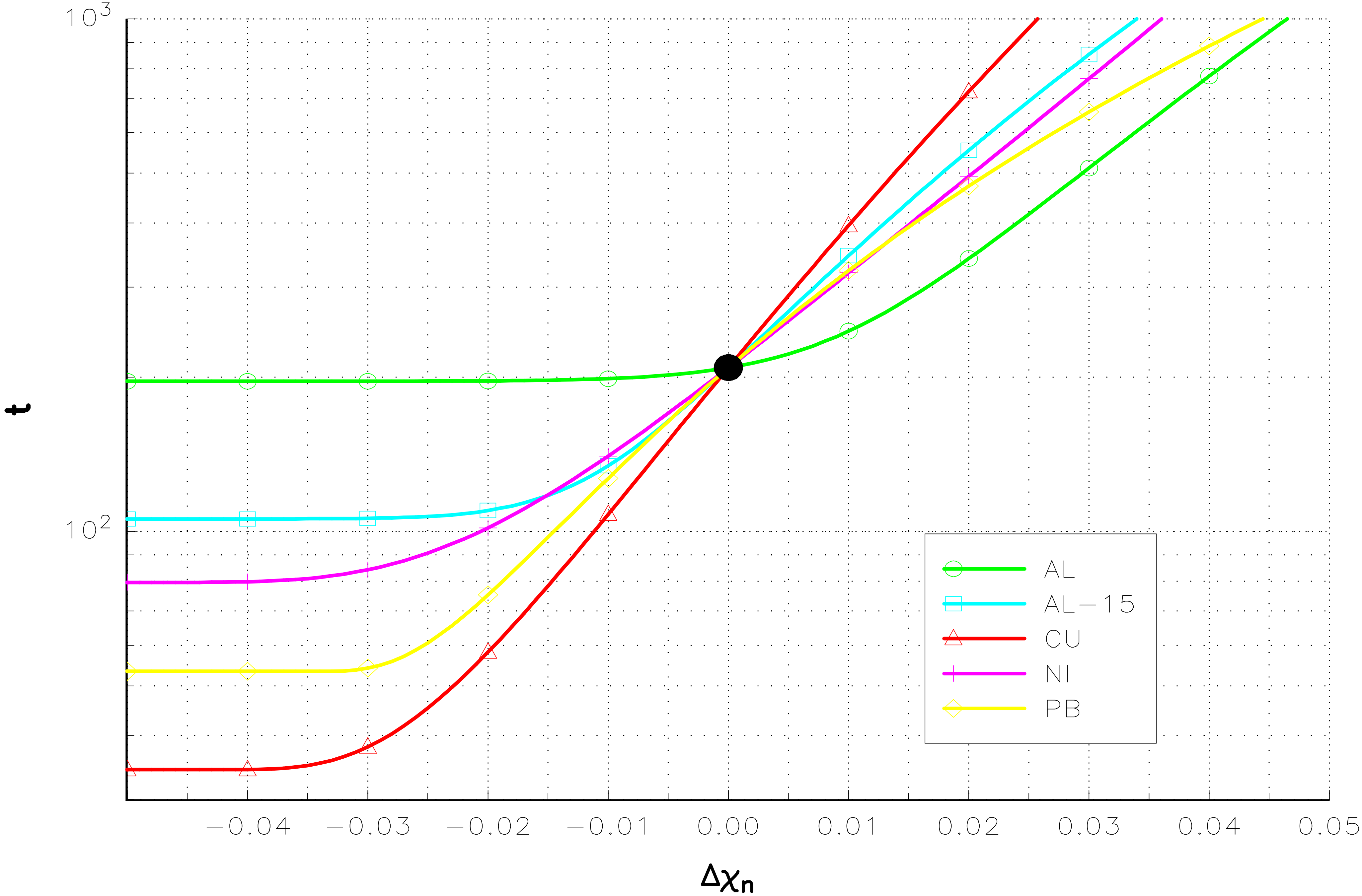
LME example of Bouyé, Durrleman, Nikeghbali, Riboulet and Roncalli [2000]: $\chi^{(1)} = \begin{bmatrix} 0.05 \\ 0.05 \\ 0.05 \\ 0.05 \\ 0.05 \end{bmatrix}$ $\chi^{(2)} = \begin{bmatrix} 0.02 \\ 0.03 \\ 0.03 \\ 0.01 \\ 0.10 \end{bmatrix}$ $\chi^{(3)} = \begin{bmatrix} 0.0696 \\ 0.0579 \\ 0.0753 \\ 0.0846 \\ 0.1113 \end{bmatrix}$

$\Rightarrow$ $\mathfrak{t}^{\aleph}_{(1)} = 209$, $\mathfrak{t}^{\perp}_{(1)} = 2317$, and $\mathfrak{t}^{+}_{(1)} = 3$.
$\Rightarrow$ $\mathfrak{t}^{\aleph}_{(2)} = 27$, $\mathfrak{t}^{\perp}_{(2)} = 34$, and $\mathfrak{t}^{+}_{(2)} = 4$.
$\Rightarrow$ $\mathfrak{t}^{\aleph}_{(3)} = 49939$, $\mathfrak{t}^{\perp}_{(3)} = 3247832$ and $\mathfrak{t}^{+}_{(3)} = 5$.

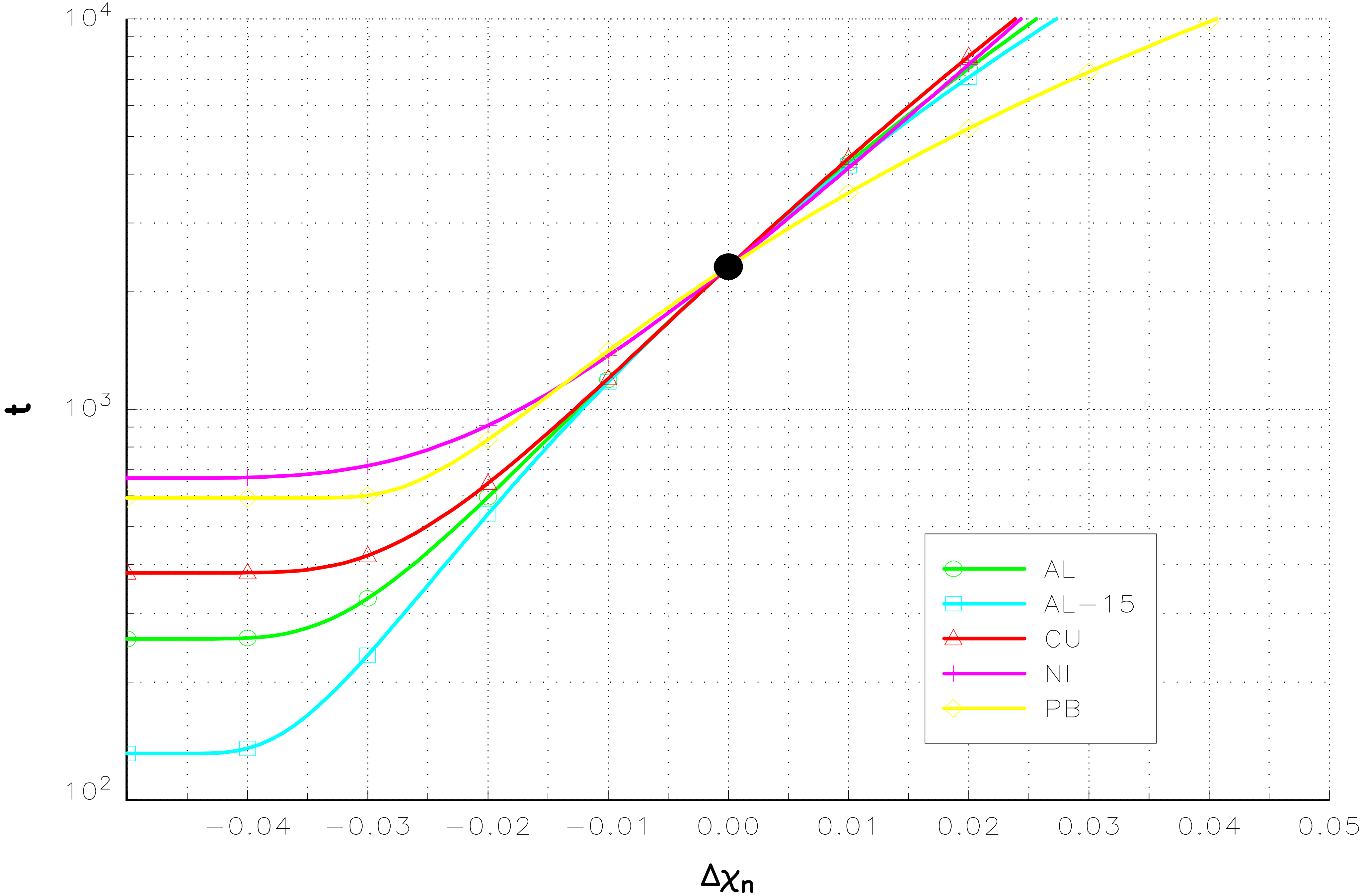
Sensitivity analysis

	AL	AL-15	CU	NI	PB
$\mathfrak{t}\left(\chi_1,\ldots,\chi_n=-\infty,\ldots,\chi_N\right)$	196	106	34	80	53

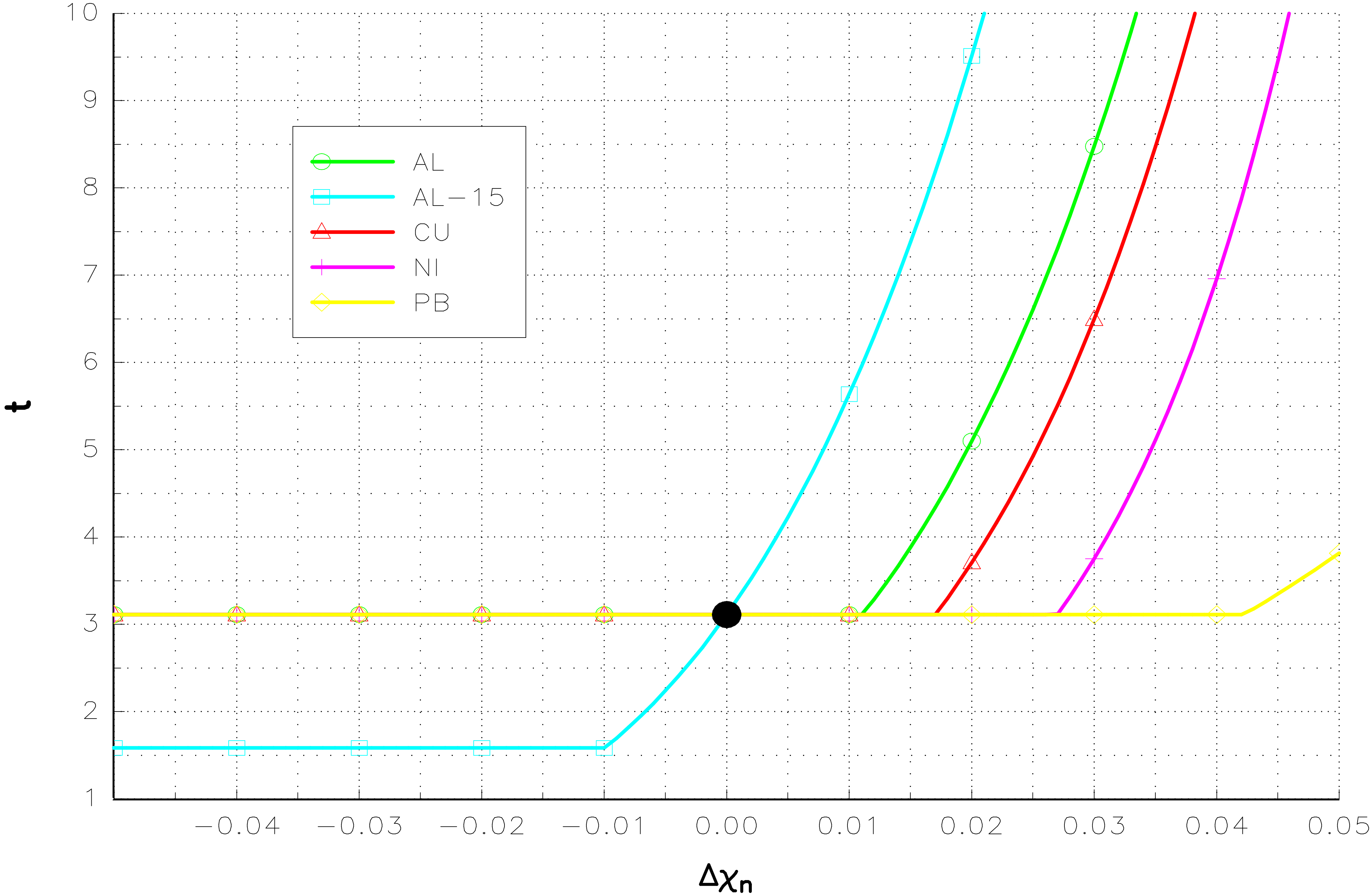
A deeper analysis could show that the higher value of 209 years is explained by the dependence structure of CU and PB — $\mathfrak{t}^{\aleph}\left(0.05, 0.05, -\infty, 0.05, -\infty\right)$ is equal to 9 years!



Univariate stress scenario contribution with the copula $C^{\#}$



Univariate stress scenario contribution with the copula $C^{\perp}$



Univariate stress scenario contribution with the copula C^{+}

3.2.3 Quantile aggregation

Makarov inequalities Let L denotes a two-place function (for example, the four arithmetic operators $+$, $-$, $\times$ and $\div$). The *supremal convolution* $\tau_{\mathbf{C},L}(\mathbf{F}_1, \mathbf{F}_2)$ is

$$\tau_{\mathbf{C},L}(\mathbf{F}_1, \mathbf{F}_2)(x) = \sup_{L(x_1,x_2)=x} \mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2))$$

whereas the *infimal convolution* $\rho_{\mathbf{C},L}(\mathbf{F}_1, \mathbf{F}_2)$ corresponds to

$$\rho_{\mathbf{C},L}(\mathbf{F}_1, \mathbf{F}_2)(x) = \inf_{L(x_1,x_2)=x} \tilde{\mathbf{C}}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2))$$

with $\tilde{\mathbf{C}}$ the dual of the copula $\mathbf{C}$.

Frank, Nelsen and Schweizer [1987] and Williamson [1989] show that the distribution $\mathbf{G}$ of $X = L(X_1, X_2)$ is contained within the bounds $\mathbf{G}_{\vee}(x) \leq \mathbf{G}(x) \leq \mathbf{G}_{\wedge}(x)$ with $\mathbf{G}_{\vee}(x) = \tau_{\mathbf{C}_-,L}(\mathbf{F}_1, \mathbf{F}_2)(x)$ and $\mathbf{G}_{\wedge}(x) = \rho_{\mathbf{C}_-,L}(\mathbf{F}_1, \mathbf{F}_2)(x)$. **These bounds are the pointwise <u>best</u> possible.**

Dependency bounds of the VaR Using the **duality** theorem of Frank and Schweizer [1979], it comes that if $\mathbf{C}_{-} = \mathbf{C}^{-}$ and L is the operation $+$, we have

$$\mathbf{G}_{\vee}^{(-1)}(u) = \inf_{\max(u_1+u_2-1,0)=u} \mathbf{F}_1^{(-1)}(u_1) + \mathbf{F}_2^{(-1)}(u_2)$$

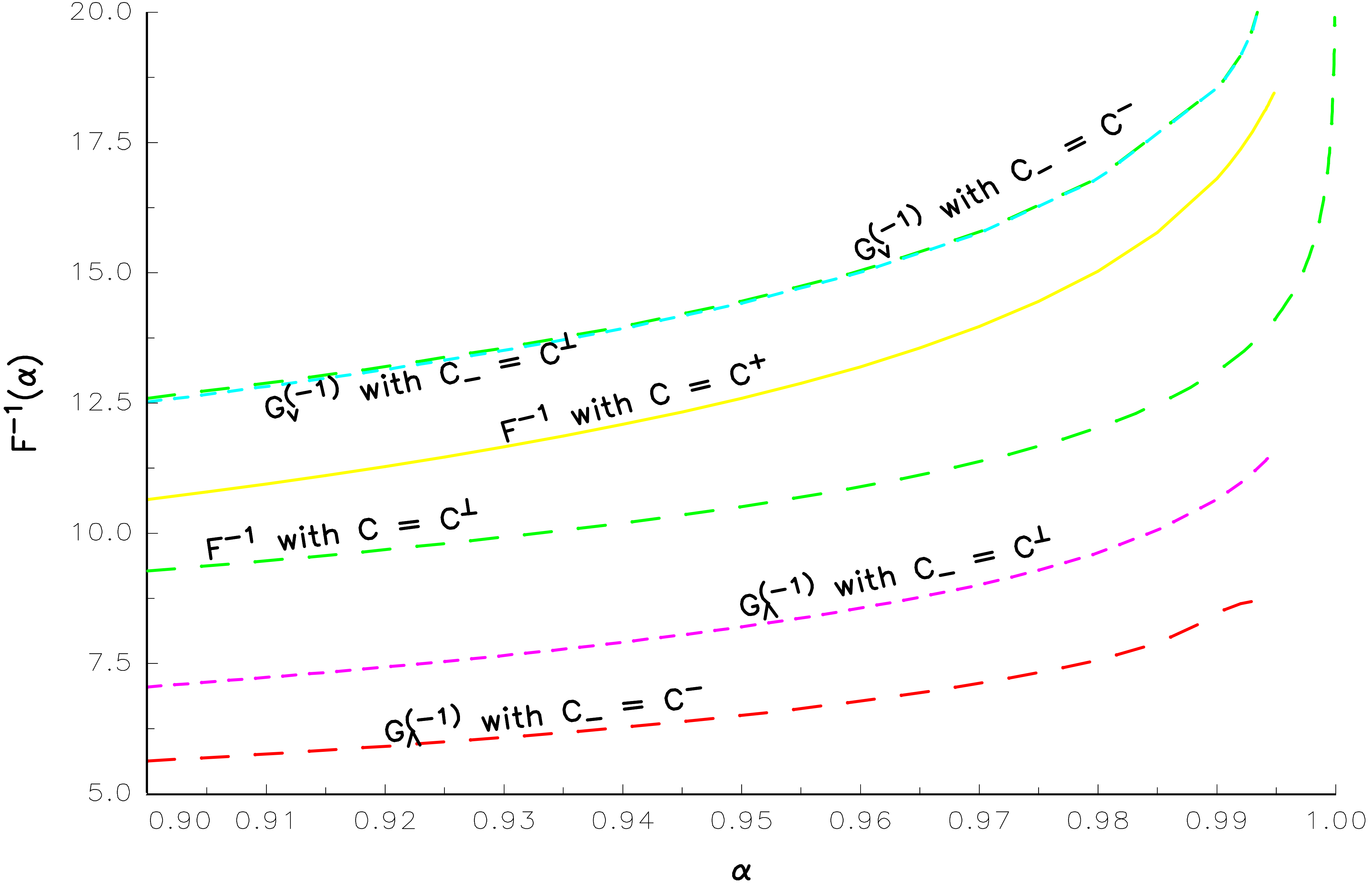
and

$$\mathbf{G}_{\wedge}^{(-1)}(u) = \sup_{\min(u_1+u_2,1)=u} \mathbf{F}_1^{(-1)}(u_1) + \mathbf{F}_2^{(-1)}(u_2)$$

We recall that $\text{VaR}_{\alpha}(X) = \mathbf{F}^{-1}(\alpha)$. The corresponding dependency bounds are then

$$\mathbf{G}_{\wedge}^{(-1)}(\alpha) \leq \text{VaR}_{\alpha}(X_1 + X_2) \leq \mathbf{G}_{\vee}^{(-1)}(\alpha)$$

Numerical algorithms to compute the dependency bounds exist (for example the *uniform quantisation* method of Williamson [1989]).



Dependency bounds for VaR with Gamma margins

The problem of the definition of the diversification effect If we define the diversification effect as follows

$$\mathsf{D} = \frac{\mathsf{VaR}_\alpha(X_1) + \mathsf{VaR}_\alpha(X_2) - \mathsf{VaR}_\alpha(X_1 + X_2)}{\mathsf{VaR}_\alpha(X_1) + \mathsf{VaR}_\alpha(X_2)}$$

there are situations where $\mathsf{VaR}_\alpha(X_1 + X_2) > \mathsf{VaR}_\alpha(X_1) + \mathsf{VaR}_\alpha(X_2)$. A more appropriate definition is then

$$\bar{\mathsf{D}} = \frac{\mathbf{G}_\vee^{(-1)}(\alpha) - \mathsf{VaR}_\alpha(X_1 + X_2)}{\mathbf{G}_\vee^{(-1)}(\alpha)}$$

It comes that $\bar{\mathsf{D}} = \chi\left(\mathbf{C}_\vee^{(\alpha)}, \mathbf{C}^+; \alpha\right) + \left[1 - \chi\left(\mathbf{C}_\vee^{(\alpha)}, \mathbf{C}^+; \alpha\right)\right]\mathsf{D}$ with

$$\chi\left(\mathbf{C}_\vee^{(\alpha)}, \mathbf{C}^+; \alpha\right) = \frac{\mathbf{G}_\vee^{(-1)}(\alpha) - \mathsf{VaR}_\alpha(X_1) + \mathsf{VaR}_\alpha(X_2)}{\mathbf{G}_\vee^{(-1)}(\alpha)}$$

Embrechts, McNeil and Straumann [1999] interpret $\chi\left(\mathbf{C}_\vee^{(\alpha)}, \mathbf{C}^+; \alpha\right)$ as "*the amount by which VaR fails to be subadditive*".

VaR aggregation in practice LME example of Durrleman, Nikeghbali and Roncalli [2000]:

	AL	AL-15	CU	NI	PB
P_1	5	3			
P_2			5	2	-3

	Analytical *VaR*	Historical *VaR*
P_1	363.05	445.74
P_2	1026.03	1274.64

Here are the values of $\mathbf{G}_{\vee}^{(-1)}(\alpha)$ for α equal to 99%:

	P_1 Analytical *VaR*	P_1 Historical *VaR*
P_2 Analytical *VaR*	1507.85	1680.77
P_2 Historical *VaR*	1930.70	2103.67

3.3 Credit risk

Problem: joint default distribution

A default is generally described by a *survival* function $\mathbf{S}(t) = \Pr\{T > t\}$. Let $\breve{\mathbf{C}}$ be a *survival* copula. A multivariate survival distributions $\mathbf{S}$ can be defined as follows

$$\mathbf{S}(t_1, \ldots, t_N) = \breve{\mathbf{C}}(\mathbf{S}_1(t_1), \ldots, \mathbf{S}_N(t_N))$$

where $(\mathbf{S}_1, \ldots, \mathbf{S}_N)$ are the marginal survival functions. Nelsen [1998] notices that "*$\bar{C}$ couples the joint survival function to its univariate margins in a manner completely analogous to the way in which a copula connects the joint distribution function to its margins*".

$\Rightarrow$ Introducing correlation between defaultable securities can then be done using the copula framework.

3.3.1 The VaR of a portfolio

⇒ Li [2000] introduces the notion of **credit curve** (= distribution of survival time for a given credit).
⇒ The *VaR* is computed by combining the credit curve and the payment schedule and the recovery rate of the credit.

Other approach: Independent default risk based on component shock Poisson model and an extended Marshall-Olkin copula (Lindskog [2000]) = similar to the Duffie and Singleton [1999] model.

Interpreting CreditMetrics joint default probability *(not yet done)*

An illustration *(not yet done)*

3.3.2 Pricing of credit derivatives

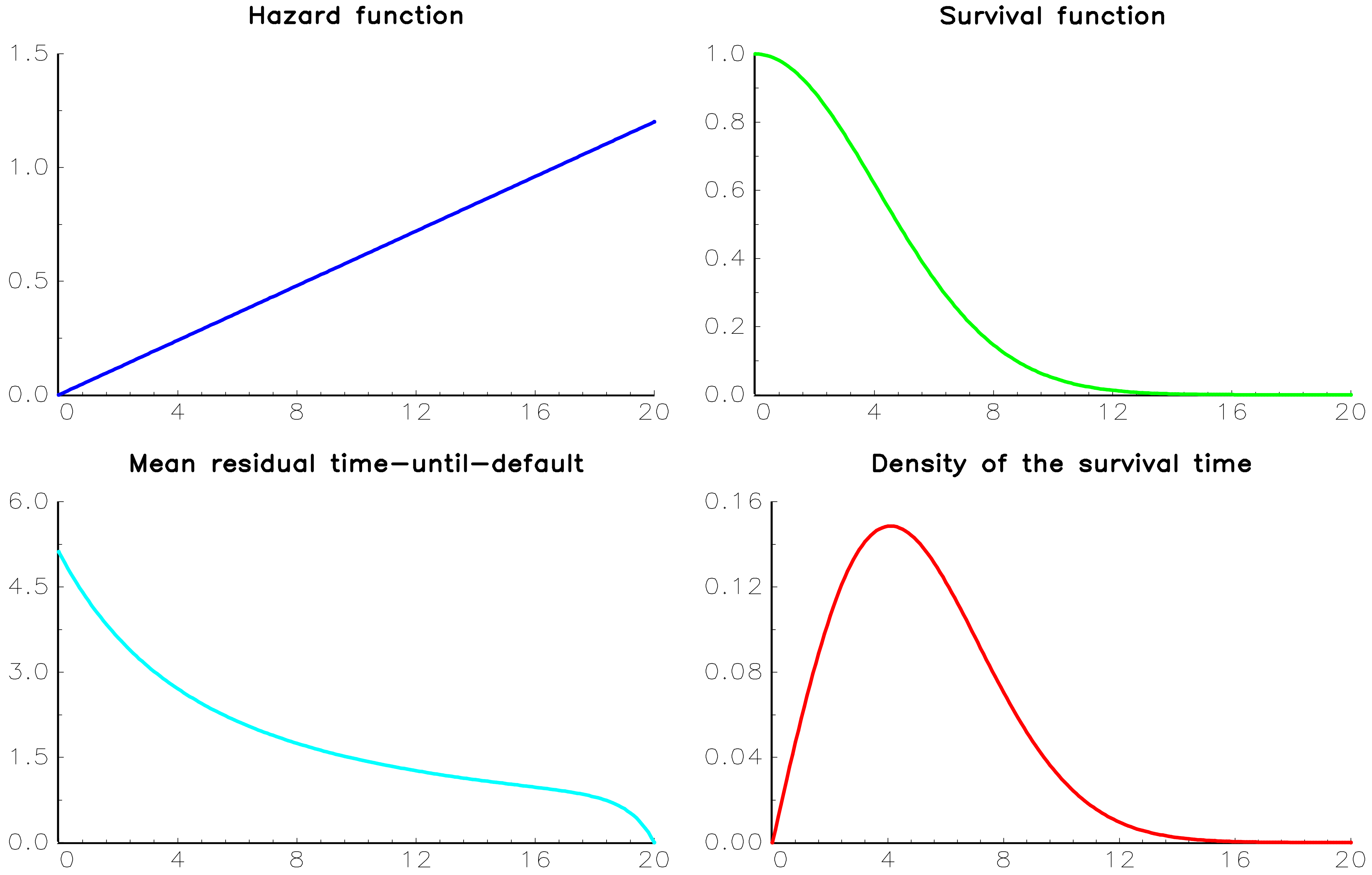
First-to-default example of Bouyé, Durrleman, Nikeghbali, Riboulet and Roncalli [2000]: Let us define the first-to-default τ as follows

$$\tau = \min \left(T_1, \ldots, T_N \right)$$

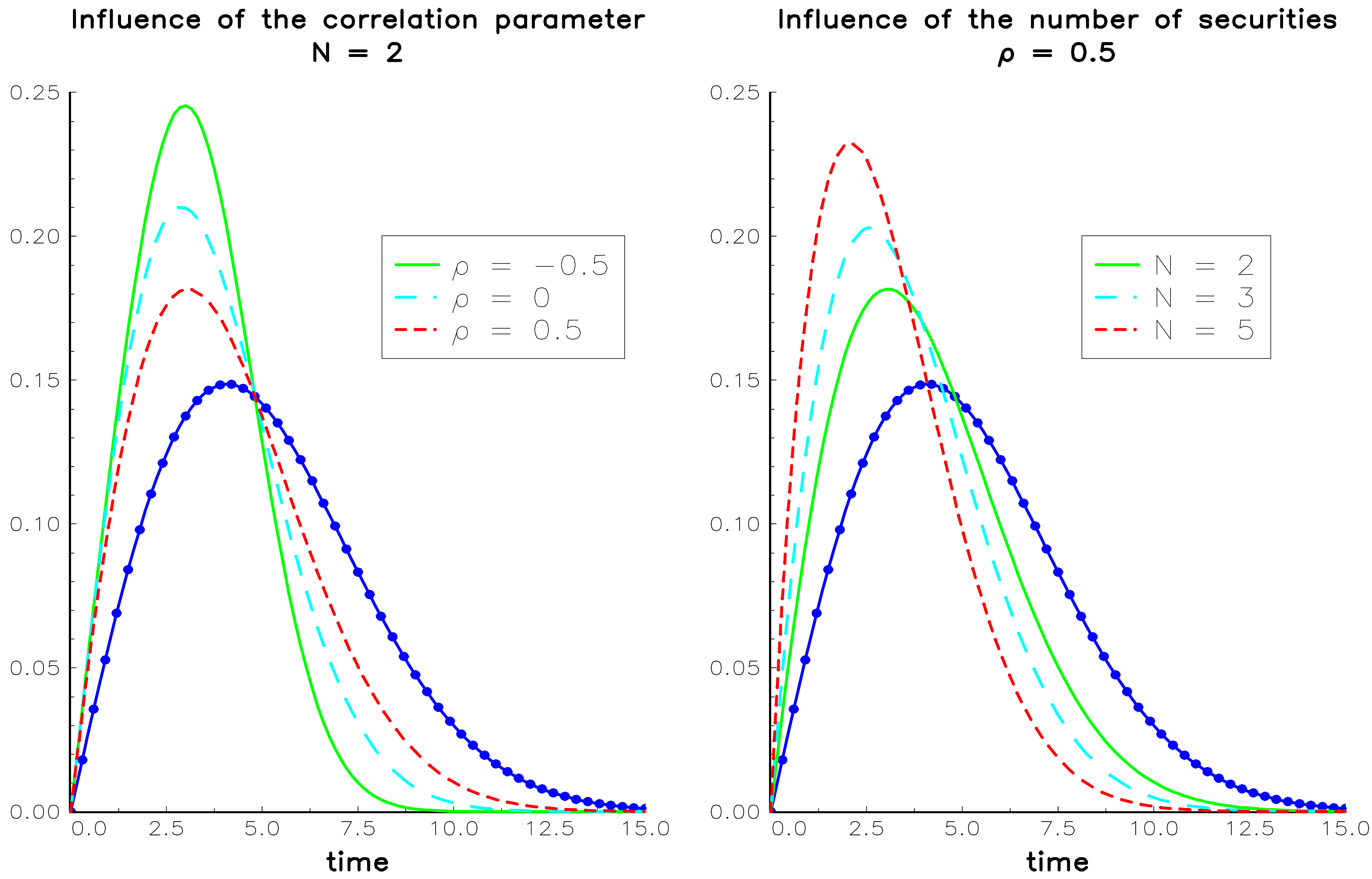
Nelsen [1998] shows that the survival function of τ is given by the *diagonal section* of the survival copula.

⇒ Theory of competing risks, multiple decrement theory.

⇒ Example: N credit events, default of each credit event given by a Weibull survival function (the baseline hazard is constant and equal to 3% per year and the Weibull parameter is 2).



Weibull survival time



Density of the first-to-default τ

3.4 Operational risk

- The standard statistical method
- Bivariate Poisson distribution
- Multivariate Poisson distribution based on copulas
- The impact of the "correlated frequencies" on the diversification effect

3.4.1 The standard statistical method

- Let ζ be the random variable that describes the severity of loss. We define also $\zeta^k(t)$ as the random process of ζ for each operational risk k $(k = 1, \ldots, K)$.
- For each risk, we assume that the number of events at time t is a random variable $N_k(t)$.
- The loss process $\varrho(t)$ is also defined as

$$\varrho(t) = \sum_{k=1}^{K} \varrho^k(t) = \sum_{k=1}^{K} \sum_{j=1}^{N_k(t)} \zeta_j^k(t)$$

- The *Economic Capital* with an α confidence level is usually defined as

$$\text{EC} = \mathbf{F}^{-1}(\alpha)$$

3.4.2 Bivariate Poisson distribution

Let $N_1 = N_{11} + N_{12}$ and $N_2 = N_{22} + N_{12}$ be the sum of three independent Poisson variates. It comes that

$$N_1 \sim \mathcal{P}\left(\lambda_1 = \lambda_{11} + \lambda_{12}\right)$$

$$N_2 \sim \mathcal{P}\left(\lambda_2 = \lambda_{22} + \lambda_{12}\right)$$

The Pearson correlation between N_1 and N_2 is $\rho = \lambda_{12}\left[\left(\lambda_{11} + \lambda_{12}\right)\left(\lambda_{22} + \lambda_{12}\right)\right]^{-\frac{1}{2}}$ and it comes that

$$\rho \in \left[0, \min\left(\sqrt{\frac{\lambda_{11} + \lambda_{12}}{\lambda_{22} + \lambda_{12}}}, \sqrt{\frac{\lambda_{22} + \lambda_{12}}{\lambda_{11} + \lambda_{12}}}\right)\right]$$

$\Rightarrow$ **only positive dependence**.

3.4.3 Multivariate Poisson distribution based on copulas

Computational problems with the extension to multivariate case (Johnson, Kotz and Balakrishnan [1997]).

⇒ Song [2000] proposes then to use the Normal copula. The cumulative distribution function is given by*

$$\mathbf{F}(n_1, \ldots, n_K) = \mathbf{C}(\mathbf{F}_1(n_1), \ldots, \mathbf{F}_K(n_K))$$

whereas the probability mass function is given by the Randon-Nikodym density of the distribution function:

$$\Pr\{N_1 = n_1, \ldots, N_K = n_K\} = \sum_{i_1=1}^{2} \cdots \sum_{i_K=1}^{2} (-1)^{i_1+\cdots+i_K} \mathbf{C}\left(\sum_{n=0}^{n_1} \frac{\lambda_1^{n+1-i_1} e^{-\lambda_1}}{n!}, \ldots, \sum_{n=0}^{n_K} \frac{\lambda_K^{n+1-i_K} e^{-\lambda_K}}{n!}\right)$$

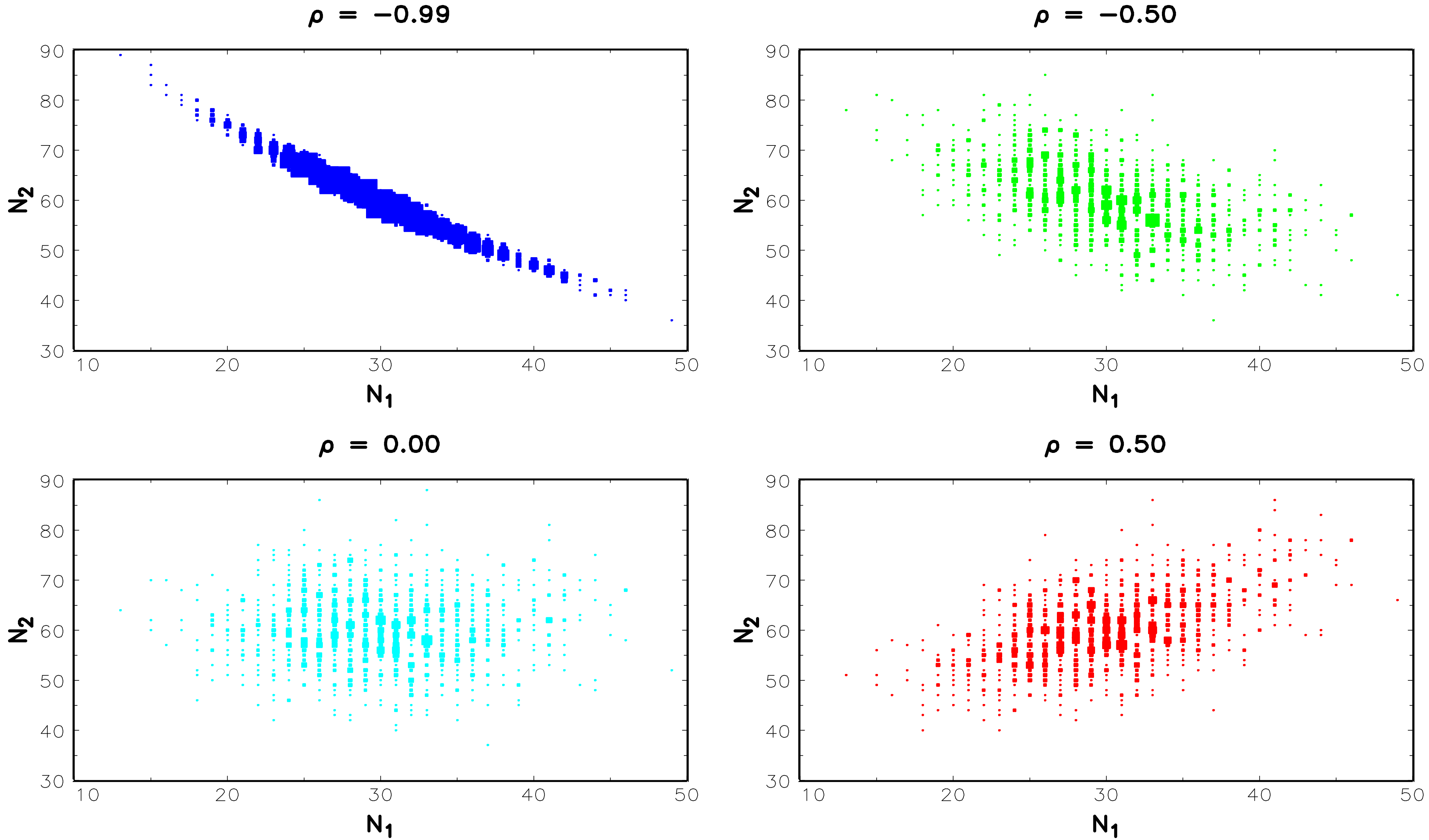
*In this case, the copula is not uniquely defined and the dependence function is determined by the subcopula (see Marshall [1996]).

The next table contains the probability mass function $p_{i,j} = \Pr\{N_1 = i, N_2 = j\}$ of the bivariate Poisson distribution $P(\lambda_1 = 1, \lambda_2 = 1, \rho = 0.5)$.

$p_{i,,j}$	0	1	2	3	4	5	$\cdots$	$p_{i,\cdot}$
0	0.0945	0.133	0.0885	0.0376	0.0114	0.00268		0.368
1	0.0336	0.1	0.113	0.0739	0.0326	0.0107		0.368
2	0.00637	0.0312	0.0523	0.0478	0.0286	0.0123		0.184
3	0.000795	0.00585	0.0137	0.0167	0.013	0.0071		0.0613
4	7.28E-005	0.000767	0.00241	0.00381	0.00373	0.00254		0.0153
5	5.21E-006	7.6E-005	0.000312	0.000625	0.000759	0.000629		0.00307
$\vdots$								
$p_{\cdot,j}$	0.135	0.271	0.271	0.18	0.0902	0.0361		1

If $\rho = -0.5$, we obtain the following values for $p_{i,j}$.

$p_{i,,j}$	0	1	2	3	4	5	$\cdots$	$p_{i,\cdot}$
0	0.0136	0.0617	0.101	0.0929	0.058	0.027		0.368
1	0.0439	0.112	0.111	0.0649	0.026	0.00775		0.368
2	0.0441	0.0683	0.0458	0.0188	0.00548	0.00121		0.184
3	0.0234	0.0229	0.0109	0.00331	0.000733	0.000126		0.0613
4	0.00804	0.00505	0.00175	0.000407	7.06E-005	9.71E-006		0.0153
5	0.002	0.00081	0.000209	3.79E-005	5.26E-006	5.89E-007		0.00307
$\vdots$								
$p_{\cdot,j}$	0.135	0.271	0.271	0.18	0.0902	0.0361		1



Random generation of bivariate Poisson variables
P(30) and P(60) with Normal copula

3.4.4 The impact of the "Correlated frequencies" on the diversification effect

very big problems

$\Rightarrow$ For high quantiles (and if the σ values are high), there are no impact of the "*Correlated* frequencies".

$\Rightarrow$ Computing the Economic Capital seems independent of the correlation matrix.

The problem comes from the copula function, which is Normal.

4 Statistical modelling

- Estimation methods
- Simulation methods
- The Normal copula
- Extending univariate statistical models
- Scoring functions

4.1 Estimation methods

1. Maximum likelihood
2. Inference for margins
3. Canonical maximum likelihood (or omnibus method)

4.1.1 Maximum likelihood method

Let $\mathcal{X} = \left\{\left(x_1^t, \ldots, x_N^t\right)\right\}_{t=1}^T$ denote a sample, f_n the density of the margin $\mathbf{F}_n$ and c the density of the copula

$$c\left(u_1, \ldots, u_n, \ldots, u_N\right) = \frac{\partial\, \mathbf{C}\left(u_1, \ldots, u_n, \ldots, u_N\right)}{\partial\, u_1 \cdots \partial\, u_n \cdots \partial\, u_N}$$

The expression of the log-likelihood is then

$$\ell\left(\theta\right) = \sum_{t=1}^T \ln c\left(\mathbf{F}_1\left(x_1^t\right), \ldots, \mathbf{F}_n\left(x_n^t\right), \ldots, \mathbf{F}_N\left(x_N^t\right)\right) + \sum_{t=1}^T \sum_{n=1}^N \ln f_n\left(x_n^t\right)$$

4.1.2 Inference for margins (IFM) method

The log-likelihood could be written as (Joe and Xu [1996])

$$
\begin{aligned}
\ell(\theta) = & \sum_{t=1}^{T} \ln c\left(\mathbf{F}_1\left(x_1^t;\theta_1\right),\ldots,\mathbf{F}_n\left(x_n^t;\theta_n\right),\ldots,\mathbf{F}_N\left(x_N^t;\theta_N\right);\alpha\right) + \\
& \sum_{t=1}^{T}\sum_{n=1}^{N} \ln f_n\left(x_n^t;\theta_n\right)
\end{aligned}
$$

$\Rightarrow$ θ_n and α are the vectors of parameters of the parametric marginal distribution $\mathbf{F}_n$ and the copula $\mathbf{C}$.

We could also perform the estimation of the univariate marginal distributions in a first time

$$\hat{\theta}_n = \arg\max \sum_{t=1}^{T} \ln f_n\left(x_n^t; \theta_n\right)$$

and then estimate α given the previous estimates

$$\hat{\alpha} = \arg\max \sum_{t=1}^{T} \ln c\left(\mathbf{F}_1\left(x_1^t; \hat{\theta}_1\right), \ldots, \mathbf{F}_n\left(x_n^t; \hat{\theta}_n\right), \ldots, \mathbf{F}_N\left(x_N^t; \hat{\theta}_N\right); \alpha\right)$$

4.1.3 Canonical maximum likelihood

The method consists in transforming the data $\left(x_1^t, \ldots, x_N^t\right)$ into uniform variates $\left(\hat{u}_1^t, \ldots, \hat{u}_N^t\right)$ — using the empirical distributions — and then estimate the parameters in the following way:

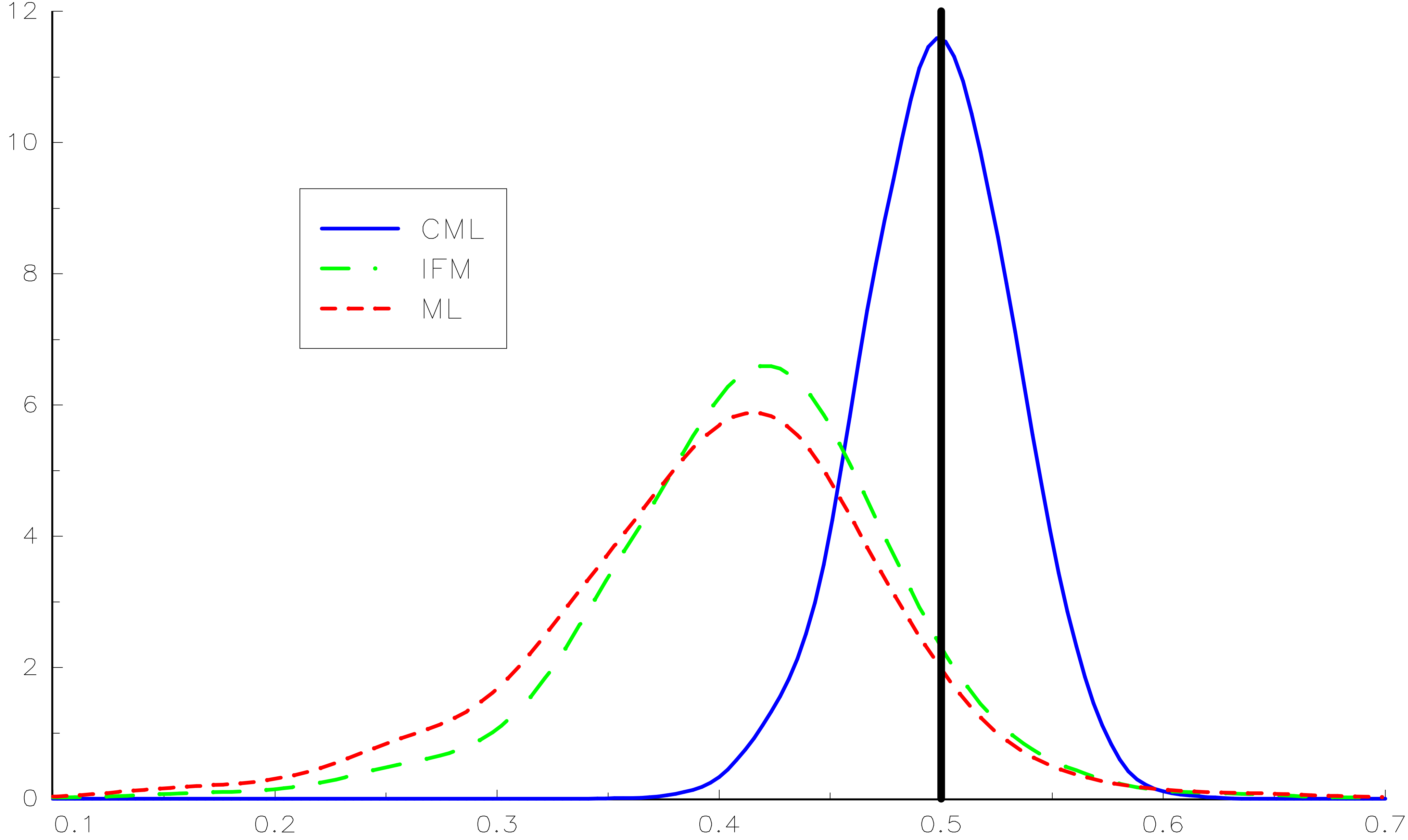
$$\hat{\alpha} = \arg\max \sum_{t=1}^{T} \ln c\left(\hat{u}_1^t, \ldots, \hat{u}_n^t, \ldots, \hat{u}_N^t; \alpha\right)$$

4.1.4 A Monte Carlo study

$\Rightarrow$ CML is the best estimator, because there are no assumptions on the margins.

Problem: If we use wrong margins, MLE and IFM will 'modify' the dependence function.

$\Rightarrow$ Example with a bivariate distribution $\mathbf{F}$ with Normal copula ($\rho = 0.5$) and two margins $\mathbf{F}_1$ and $\mathbf{F}_2$ which are student distributions ($\mathbf{F}_1 = t_2$ and $\mathbf{F}_2 = t_3$). We fit the distribution with a Normal copula and two gaussian margins.



Comparison of the density of the CML, IFM and ML estimators when the margins are wrong

4.2 Simulation methods

1. Simulate the random vector $\mathbf{u}$ from the multivariate uniform distribution $\mathbf{C}$.
2. Use the **inversion** method to obtain the desired random vector $\mathbf{x}$ with

$$x_n = \mathbf{F}_n^{-1}(u_n)$$

For the first step, there exist specific algorithms (Devroye [1986], Marshall and Olkin [1988], etc.).

4.3 Working with the Normal copula

Remark 2 *The multivariate normal distribution is very tractable. It is very easy to estimate the parameters and simulation is straightforward. Moreover, this distribution has nice properties and most of tractable statistical methods (linear regression, factor analysis, etc.) assume the normality.*

Is it always the case for the Normal copula?

4.3.1 The Ψ transform

We define the operator Ψ as follows

$$\begin{aligned} \Psi[\mathbf{F}] \; : \; & \mathbb{R} \longrightarrow \mathbb{R} \\ & x \longmapsto \Psi[\mathbf{F}](x) = \Phi^{-1}(\mathbf{F}(x)) \end{aligned}$$

We note also Ψ^{-1} the (left) inverse operator ($\Psi^{-1} \circ \Psi = 1$), i.e. $\Psi^{-1}[\mathbf{F}](x) = \mathbf{F}^{[-1]}(\Phi(x))$.

4.3.2 Estimation

If we assume uniform margins ($\mathbf{F} = U_{[0,1]}$), the log-likelihood function is

$$\ell(\theta) = -\frac{T}{2} \ln |\rho| - \frac{1}{2} \sum_{t=1}^{T} \varsigma_t^\top \left(\rho^{-1} - \mathbb{I}\right) \varsigma_t$$

with $\varsigma_t = \left(\Psi\left[\mathbf{F}_1\right]\left(x_1^t\right), \ldots, \Psi\left[\mathbf{F}_N\right]\left(x_N^t\right)\right)$ and the ML estimate of ρ is also

$$\hat{\rho}_{\mathrm{ML}} = \frac{1}{T} \sum_{t=1}^{T} \varsigma_t^\top \varsigma_t$$

1. IFM estimate: $\mathbf{F}_n =$ MLE of the n^{th} marginal distribution.
2. CML estimate : $\mathbf{F}_n = n^{\text{th}}$ empirical distribution.

Example of Costinot, Roncalli and Teïletche [2000]:

- Normal copula + Gaussian marginals = Gaussian distribution.

$$\hat{\rho} = \begin{bmatrix} 1 & 0.158 & 0.175 \\ & 1 & 0.0589 \\ & & 1 \end{bmatrix}$$

It means that

$$\mathbf{C}_{(\mathsf{CAC40,NIKKEI})} \succ \mathbf{C}_{(\mathsf{CAC40,DowJones})} \succ \mathbf{C}_{(\mathsf{NIKKEI,DowJones})}$$

- Normal copula + Empirical marginals.

$$\hat{\rho} = \begin{bmatrix} 1 & 0.207 & 0.157 \\ & 1 & 0.0962 \\ & & 1 \end{bmatrix}$$

In this case, we verify that

$$\mathbf{C}_{(\mathsf{CAC40,DowJones})} \succ \mathbf{C}_{(\mathsf{CAC40,NIKKEI})} \succ \mathbf{C}_{(\mathsf{NIKKEI,DowJones})}$$

4.3.3 Simulation

- Generate N independent gaussian random variables $\mathbf{x} = (x_1, \ldots, x_N)$;
- Create a new dependent vector $\mathbf{y} = \mathbb{P}\mathbf{x}$ with $\rho = \mathbb{P}\mathbb{P}^{\mathsf{T}}$ ($\mathbb{P}$ is the lower Choleski decomposition of ρ);
- The resulting random numbers are

$$\mathbf{z} = \left(\Psi^{-1}\left[\mathbf{F}_1\right](y_1), \ldots, \Psi^{-1}\left[\mathbf{F}_N\right](y_N)\right)$$

4.3.4 Quantile regression

Costinot, Roncalli and Teiletche [2000] show that

$$\frac{\partial}{\partial u_1}\mathbf{C}\left(u_1,u_2\right)=\Phi\left(\varsigma\right)$$

with

$$\varsigma=\frac{\Phi^{-1}\left(u_2\right)-\beta\Phi^{-1}\left(u_1\right)}{\sqrt{1-\beta^2}}$$

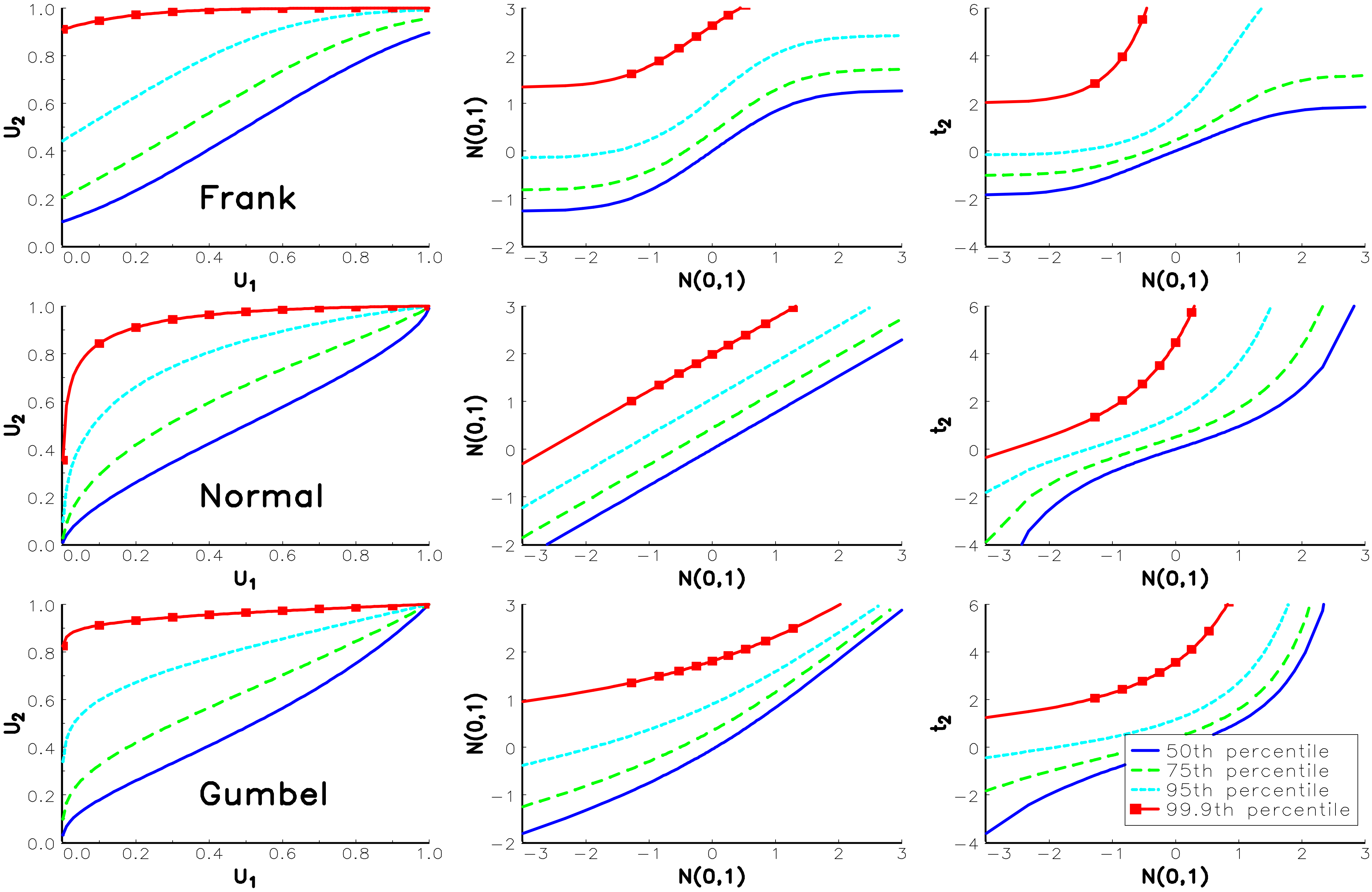
The expression of the function $u_2=\mathbf{q}^{\star}\left(u_1;\alpha\right)$ is also

$$u_2=\Phi\left(\beta\Phi^{-1}\left(u_1\right)+\sqrt{1-\beta^2}\Phi^{-1}\left(\alpha\right)\right)$$

If the margins are gaussians, we obtain the well-known curve

$$X_2=\left[\mu_2-\beta\frac{\sigma_2}{\sigma_1}\mu_1+\sqrt{1-\beta^2}\Phi^{-1}\left(\alpha\right)\right]+\beta\frac{\sigma_2}{\sigma_1}X_1$$

We remark that the relationship is **linear**. When the margins are not gaussians, the relationship is linear in the Ψ projection space.



Regression quantiles with different copula functions

Remark 3 *If we assume that the dependence function is Normal, we can use the Portnoy-Koenker algorithm with the transformed variables* $Y_i = \Psi[\mathbf{F}_i](X_i)$. *Let* $\hat{a}$ *and* $\hat{b}$ *be the estimates of the linear quantile regression*

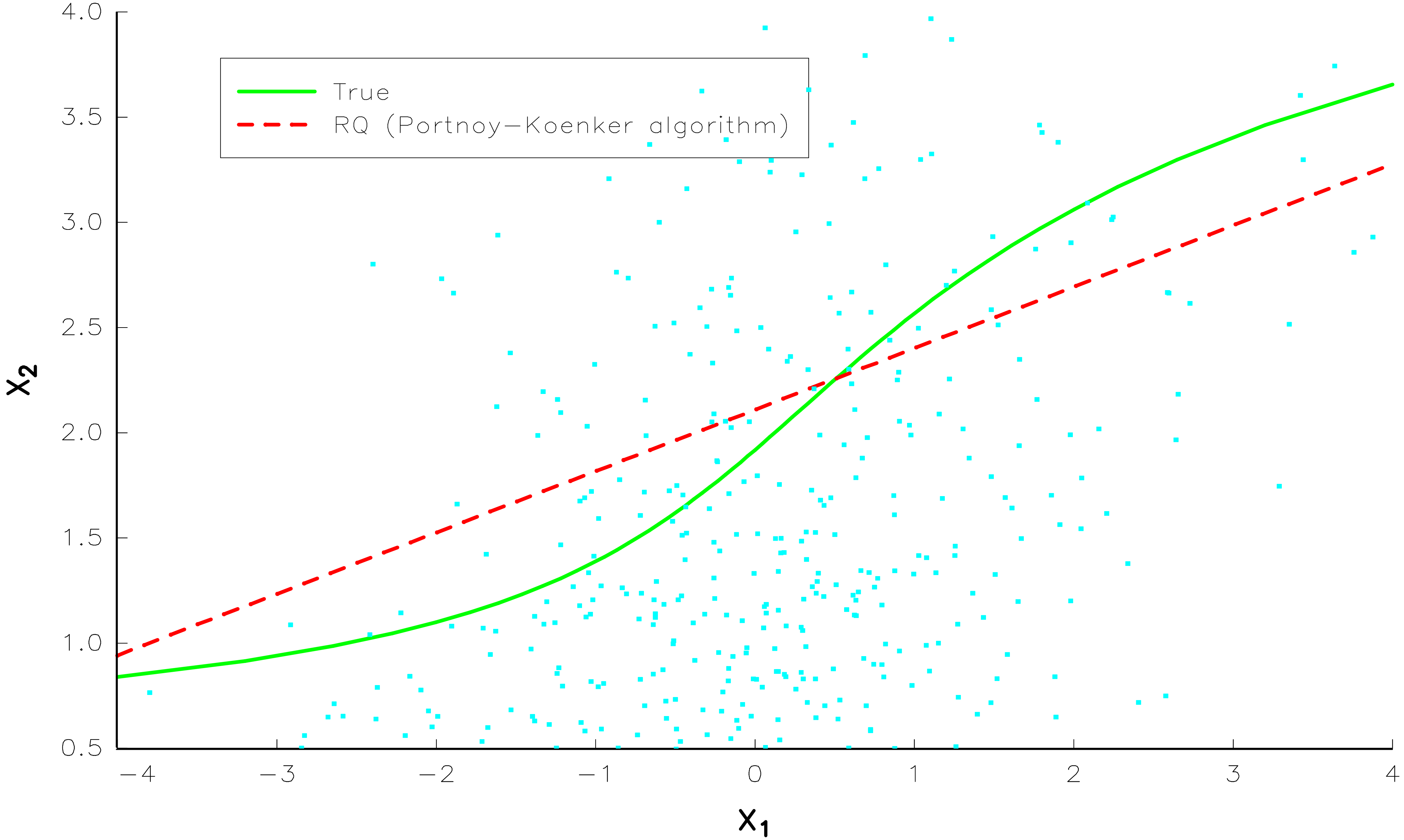
$$\begin{cases} Y_2 = a + bY_1 + U \\ \Pr\{Y_2 \leq y_2 \mid Y_1 = y_1\} = \alpha \end{cases}$$

The quantile regression curve of X_2 *on* X_1 *is then obtained as follows*

$$X_2 = \Psi^{-1}[\mathbf{F}_2]\left(\hat{a} + \hat{b}\Psi[\mathbf{F}_1](X_1)\right)$$

Linearity = Normality

Can we extend the previous analysis to other statistical models (linear regression, factor analysis, etc.)?



Linear quantile regression with
Normal copula and Student/Gamma margins

4.4 Extending univariate statistical models

- Multivariate survival modelling
- Multivariate count processes
- Multivariate time series

Some examples:

1. The Cox Proportional Hazard model has been extended to the bivariate case by Clayton [1978].
2. Frailty models (Hougaard [1986], Oakes [1989]) = Archimedean copulas where the generator is a Laplace transform.
3. Bouyé, Gaussel and Salmon [2000] extend the ARMA processes when the margins are not gaussians.
4. Song [2000] proposes generalization of dispertion models (which include binary regression, longitudinal model, etc.).

4.5 Scoring functions

We consider an individual i with p specifications $(x_1, \ldots, x_N)$. The scoring function for this individual is

$$S = S(x_1, \ldots, x_N)$$

We then define a rule based on a target $S^\star$. The area $\mathcal{A}$ of acceptation is the set

$$\mathcal{A} = \left\{ i : S^{(i)} \geq S^\star \right\}$$

S is a random variable, but only the order statistic is important.

Some example of scoring functions: credit scoring, individual customer risk score, insurance scoring, etc.

4.5.1 New statistical models

(not yet done)

4.5.2 Dependence between scoring functions

Let S_1 and S_2 be two scoring functions. They are equivalent if there exist an increasing function f such that $S_2(x) = f(S_1(x))$. Using the Deheuvels's characterization of copula in terms of rank statistics, it comes that the dependence function between S_1 and S_2 is the upper Fréchet copula.
$\Rightarrow$ Copulas can be used to compare scoring functions.

Application: Let S_1 and S_2 be a risk scoring function and a profitability scoring function. How to construct a new scoring function $S = g(S_1, S_2)$ which is a risk oriented scoring function, but take into account profitability?
$\Rightarrow$ We need a distance between S_1 and S_2, that is a distance on the copula (Alsina [1984]).

5 The mathematical machinery of copulas

Howard Sherwood in the AMS-IMS-SIAM Conference of 1993:

> *The subject matter of these conference proceedings comes in many guises. Some view it as the study of* **probability distributions with fixed marginals***; those coming to the subject from probabilistic geometry see it as the study of* **copulas***; experts in real analysis think of it as the study of* **doubly stochastic measures***; functional analysts think of it as the study of* **Markov operators***; and statisticians say it is the study of possible* **dependence** *relations between pairs of random variables. All are right since all these topics are isomorphic.*

5.1 Probabilistic metric spaces

Let $\mathbf{F}_{p,q}(x)$ be the probability that the distance between p and q is less than x. Menger [1942] proposes to replace the triangle inequality of metric spaces by

$$\mathbf{F}_{p,q}(x+y) \geq \mathbf{T}\left(\mathbf{F}_{p,q}(x), \mathbf{F}_{p,q}(y)\right)$$

Schweizer and Sklar [1958] define $\mathbf{T}$ as a *t-norm*.

$\Rightarrow$ $\mathbf{T}$ must be associative

$$\mathbf{T}(u, \mathbf{T}(v, w)) = \mathbf{T}(\mathbf{T}(u, v), w)$$

Theorem 4 *A 2-copula is a t-norm if and only if it is associative. A t-norm is a 2-copula if and only if it satisfies the Lipschitz condition.*

Most important results on copulas obtained in the study of PMS:

- Archimedean copulas (Ling [1965], Genest and MacKay [1986]);
- Frank copula (Frank [1979]);
- Characterization of associativity of convolutions;
- Makarov inequalities (Moynihan, Schweizer and Sklar [1978]).

5.2 Markov operators

Definition 2 (Olsen, Darsow and Nguyen (1996)) *Let* $(\Omega, \mathcal{F}, P)$ *be a probabilistic space. A linear operator* $\mathbf{T} : L^{\infty}(\Omega) \rightarrow L^{\infty}(\Omega)$ *is a Markov operator if*

1. $\mathbf{T}$ *is positive i.e.* $\mathbf{T}[f] \geq 0$ *whenever* $f \geq 0$*;*
2. 1 *is a fixed point of* $\mathbf{T}$*.*
3. $\mathbb{E}[\mathbf{T}[f]] = \mathbb{E}[f]$ *for every* $f \in L^{\infty}(\Omega)$*.*

$\Rightarrow$ Brown [1966] shows that "*the set* $\mathcal{M}$ *of all Markov operators is convex... $\mathcal{M}$ may be identified with the set of doubly stochastic measures...*".

Darsow, Nguyen and Olsen [1992] define the product of $\mathbf{C}_1$ and $\mathbf{C}_2$ by the following function

$$\begin{array}{rcl} \mathbf{I}^2 & \longrightarrow & \mathbf{I} \\ (u_1, u_2) & \longmapsto & (\mathbf{C}_1 * \mathbf{C}_2)(u_1, u_2) = \int_0^1 \partial_2 \mathbf{C}_1 (u_1, u)\, \partial_1 \mathbf{C}_2 (u, u_2)\, \mathrm{d}u \end{array}$$

where $\partial_1 \mathbf{C}$ and $\partial_2 \mathbf{C}$ represent the first-order partial derivatives with respect to the first and second variable.

Theorem 5 (Olsen, Darsow and Nguyen (1996)) *The set of copulas under the* $*$ *product is isomorphic to the set of Markov operators on* $L^\infty([0,1])$ *under composition, via the correspondance*

$$\begin{aligned} \mathbf{T}[f](u) &= \frac{\mathrm{d}}{\mathrm{d}u} \int_0^1 \partial_2 \mathbf{C}(u, v)\, f(v)\, \mathrm{d}v \\ \mathbf{C}(u_1, u_2) &= \int_0^{u_1} \mathbf{T}\left[\mathbf{1}_{[0,u_2]}\right](u)\, \mathrm{d}u \end{aligned}$$

5.2.1 Uniform convergence vs strong convergence

Kimeldorf and Sampson [1978] show that one can pass from stochastic dependence to complete dependence in the natural sense of weak convergence. This can be done using the theorem of Vitale [1991]:

Theorem 6 *Let U_1 and U_2 be uniformly distributed variables. There is a sequence of cyclic permutations $T_1, T_2, \ldots$ such that $(U_1, T_n U_1)$ converges in distribution to (U_1, U_2) as $n \to \infty$.*

... with respect to uniform convergence, it is essentially impossible to distinguish between situations in which one random variable completely determines another and a situation in which a pair of random variables is independent (Li, Mikusiński and Taylor [2000]).

⇒ Li, Mikusiński, Sherwood and Taylor [1998] introduced strong convergence of copulas, which is defined to be strong convergence of the corresponding Markov operators.

⇒ Using strong convergence, Li, Mikusiński and Taylor [2000] show that

$$\int_{\mathbf{I}^2} f(\mathbf{C}_1)\,\mathrm{d}\mathbf{C}_2 = \int_{\mathbf{I}} f(u)\,\mathrm{d}u - \int_{\mathbf{I}^2} f'(\mathbf{C}_1)\,\mathrm{d}_2\mathbf{C}_1\,\mathrm{d}_1\mathbf{C}_2$$

5.2.2 New families of copulas

Definition 3 $\left\{\phi_{1,\ldots,}\phi_n\right\} \in L^{\infty}([0,1])$ *is called a partition of unity if it satifies the following statements*

1. $\phi_i(x) \geq 0$;
2. $\int_0^1 \phi_i(x)\,\mathrm{d}x = \frac{1}{n}$;
3. $\sum_{i=1}^{n} \phi_i(x) = 1$ *for all* $x \in [0,1]$.

Theorem 7 *Let* $\left\{\phi_{1,\ldots,}\phi_n\right\} \in L^{\infty}([0,1])$ *be a partition of unity and* $\mathfrak{P}_n(\mathbf{C})$ *the two place function defined by*

$$\mathfrak{P}_n(\mathbf{C})(u,v) = n^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \Delta_{i,j}(\mathbf{C}) \int_0^u \phi_i(x)\ \mathrm{d}x \int_0^v \phi_j(x)\ \mathrm{d}y$$

where $\Delta_{i,j}(\mathbf{C}) = \mathbf{C}\left(\frac{i}{n},\frac{j}{n}\right) - \mathbf{C}\left(\frac{i-1}{n},\frac{j}{n}\right) - \mathbf{C}\left(\frac{i}{n},\frac{j-1}{n}\right) + \mathbf{C}\left(\frac{i-1}{n},\frac{j-1}{n}\right)$.
Then $\mathfrak{P}_n(\mathbf{C})$ *is a copula.*

The proof of this theorem has been etablished by Li, Mikusiński, Sherwood and Taylor [1997] and Kulpa [1998] using Markov operators.

⇒ Partition of unity have been used by Durrleman, Nikeghbali and Roncalli [2000] to generate new families of copulas.

5.2.3 Markov processes

Darsow, Nguyen and Olsen [1992] prove the following theorem:

Theorem 8 *Let* $X = \{X_t, \mathcal{F}_t; t \geq 0\}$ *be a stochastic process and let* $\mathbf{C}_{s,t}$ *denote the copula of the random variables* X_s *and* X_t*. Then the following are equivalent*

(i) The transition probabilities $P_{s,t}(x, \mathcal{A}) = \Pr\{X_t \in \mathcal{A} \mid X_s = x\}$ *satisfy the Chapman-Kolmogorov equations*

$$P_{s,t}(x, \mathcal{A}) = \int_{-\infty}^{\infty} P_{s,\theta}(x, \mathrm{d}y) \, P_{\theta,t}(y, \mathcal{A})$$

for all $s < \theta < t$ *and almost all* $x \in \mathbb{R}$*.*

(ii) For all $s < \theta < t$*,*

$$\mathbf{C}_{s,t} = \mathbf{C}_{s,\theta} * \mathbf{C}_{\theta,t} \tag{1}$$

In the conventional approach, one specifies a Markov process by giving the initial distribution μ and a family of transition probabilities $P_{s,t}(x, \mathcal{A})$ satisfying the Chapman-Kolmogorov equations. In our approach, one specifies a Markov process by giving all of the marginal distributions and a family of 2-copulas satisfying (1). Ours is accordingly an alternative approach to the study of Markov processes which is different in principle from the conventional one. Holding the transition probabilities of a Markov process fixed and varying the initial distribution necessarily varies all of the marginal distributions, but holding the copulas of the process fixed and varying the initial distribution does not affect any other marginal distribution (Darsow, Nguyen and Olsen [1992]).

The Brownian copula

$$\mathbf{C}_{s,t}(u_1, u_2) = \int_0^{u_1} \Phi\left(\frac{\sqrt{t}\Phi^{-1}(u_2) - \sqrt{s}\Phi^{-1}(u)}{\sqrt{t-s}}\right) \mathrm{d}u$$

Understanding the temporal dependence structure of diffusion processes

The copula of a Geometric Brownian motion is the Brownian copula.

The Ornstein-Uhlenbeck copula is

$$\mathbf{C}_{s,t}(u_1,u_2) = \int_0^{u_1} \Phi\left(\frac{\hbar(t_0,s,t)\,\Phi^{-1}(u_2) - \hbar(t_0,s,s)\,\Phi^{-1}(u)}{\hbar(s,s,t)}\right) \mathrm{d}u$$

with

$$\hbar(t_0,s,t) = \sqrt{e^{2a(t-s)} - e^{-2a(s-t_0)}}$$

Remark 4 *A new interpretation of the parameter a follows. For physicists, a is the mean-reverting coefficient. From a copula point of view, this parameter measures the dependence between the random variables of the diffusion process. The bigger this parameter, the less dependent the random variables.*

5.3 Quasi-copulas

The class of quasi-copulas was introduced by Alsina, Nelsen and Schweizer [1993] in order to characterize the class of binary operations ψ on distribution functions which are induced pointwise **and derivable from functions on random variables**.

Example: Mixtures are induced pointwise, convolutions are derivable. Mixtures are not derivable, convolutions are not induced pointwise.

> *The distinction between working directly with distributions functions and working with them indirectly, via random variables, is intrinsic and not just a matter of taste. The classical model for probability theory — which is based on random variables defined on a common probability space — has its limitations* (Alsina, Nelsen and Schweizer [1993]).

5.3.1 Characterization of quasi-copulas

The original definition of quasi-copulas is not tractable ("every track in the unit square coincides with a copula function").

Genest, Quesada Molina, Rodríguez Lallena and Sempi [1999] prove the following two theorems:

Theorem 9 *A function* $\mathbf{Q} : \mathbf{I}^2 \rightarrow \mathbf{I}$ *is a quasi-copula if and only if*

1. $\mathbf{Q}(0, u) = \mathbf{Q}(u, 0) = 0$ *and* $\mathbf{Q}(1, u) = \mathbf{Q}(u, 1) = 1$*;*
2. $\mathbf{Q}$ *is non-decreasing in each of its arguments;*
3. $\mathbf{Q}$ *satisfies Lipschitz's condition*

$$|\mathbf{Q}(u_2, v_2) - \mathbf{Q}(u_1, v_1)| \leq |u_2 - u_1| + |v_2 - v_1|$$

Theorem 10 *A function* $\mathbf{Q} : \mathbf{I}^2 \to \mathbf{I}$ *is a quasi-copula if and only if*

1. $\mathbf{Q}(0, u) = \mathbf{Q}(u, 0) = 0$ *and* $\mathbf{Q}(1, u) = \mathbf{Q}(u, 1) = 1$*;*
2. *if* $0 \leq u_1 \leq u_2 \leq 1$ *and* $0 \leq v_1 \leq v_2 \leq 1$*,*

$$\mathbf{Q}(u_2, v_2) + \mathbf{Q}(u_1, v_1) \geq \mathbf{Q}(u_2, v_1) + \mathbf{Q}(u_1, v_2)$$

holds true whenever at least one of the coordinates is either equal to 0 or to 1.

Nelsen, Quesada Molina, Rodríguez Lallena and Ùbeda Flores [2000] give a fourth characterization of quasi-copulas:

Theorem 11 *A function* $\mathbf{Q} : \mathbf{I}^2 \to \mathbf{I}$ *is a quasi-copula if and only if*

1. $\mathbf{Q}(0, u) = \mathbf{Q}(u, 0) = 0$ *and* $\mathbf{Q}(1, u) = \mathbf{Q}(u, 1) = 1$*;*
2. $\mathbf{Q}$ *is absolutely continuous in each variable;*
3. *the partial derivatives* $\partial_1 \mathbf{Q}$ *and* $\partial_2 \mathbf{Q}$ *exist for almost all* u *and* v*, and for such* u *and* v $0 \leq \partial_1 \mathbf{Q}(u, v) \leq 1$ *and* $0 \leq \partial_2 \mathbf{Q}(u, v) \leq 1$*.*

Remark 5 *All these characterizations are related to properties of copulas (see theorem 2.2.4, definition 2.1.1 and theorem 2.2.7 of* Nelsen *[1998]). But a quasi-copula does* **not** *satisfy necessarily the property of* N*-increasing.*

Remark 6 *Every copula is a quasi-copula.*

5.3.2 A first application of quasi-copulas

Alsina [2000] induces in $[0,1]$ families of metrics based on quasi-copulas. Let us define the co-quasi-copula as follows

$$\mathbf{Q}^{*}(u,v) = 1 - \mathbf{Q}(1-u, 1-v)$$

Theorem 12 $d_{\mathbf{Q}_2^{*},\mathbf{Q}_1}(u,v) = 1_{[u \neq v]}\left(\mathbf{Q}_2^{*}(u,v) - \mathbf{Q}_1(u,v)\right)$ *is a distance in [0,1].*

Geometric interpretation of a bivariate distribution:

$$\mathbf{F}(x_1,x_2) = \frac{\mathbf{F}_1(x_1) + \mathbf{F}_2(x_2)}{2} - \frac{1}{2} d_{\mathbf{C}^{*},\mathbf{C}}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2))$$

5.3.3 A second application of quasi-copulas

"Nelsen, Quesada Molina, Rodríguez Lallena and Ùbeda Flores [2000] have developped a method to find best-possible bounds on bivariate distribution functions with fixed marginals, when additional information of a distribution-free nature is known, **by using quasi-copulas**."

6 Conclusion

COPULAS = AN OPEN FIELD FOR FINANCE

FINANCE = Risk, Financial Econometrics, Derivatives Pricing, etc.

⇒ see for example the works of Bikos [2000], Cherubini and Luciano [2000] and Rosenberg [2000] on pricing of multivariate contingent claims.

Paradox: There were no people of financial institutions — except Crédit Lyonnais — and no people of financial academic research centres in the Barcelona conference (July 2000).

COPULAS = A FASHION IN FINANCE?

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