Copulas, multivariate risk-neutral distributions and implied dependence functions*

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Abstract

In this paper, we use copulas to define multivariate risk-neutral distributions. We can then derive general pricing formulas for multi-asset options and best possible bounds with given volatility smiles. Finally, we apply the copula framework to define 'forward-looking' indicators of the dependence function between asset returns.

1 Introduction

Copulas have been introduced in finance for risk management purposes. For derivatives pricing, Rosenberg [1999] proposes to use Plackett distributions for the following reason:

> *[...] This method allows for completely general marginal risk-neutral densities and is compatible with all univariate risk-neutral density estimation techniques. Multivariate contingent claim prices using this method are consistent with current market prices of univariate contigent claims.*

A Plackett distribution is actually a special case of the copula construction of multidimensional probability distribution. Cherubini and Luciano [2000] extend then Rosenberg's original work by using general copula functions. At the same time, Bikos [2000] uses the same framework to estimate multivariate RND for monetary policy purposes.

Our paper follows these previous works. After defining multivariate risk-neutral distributions with copulas, we derive pricing formulas for some multi-asset options. We study then best possible bounds with given volatility smiles. Finally, we apply the copula framework to define 'forward-looking' indicators of the dependence function between asset returns.

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2 Multivariate risk-neutral distributions

2.1 The multivariate factor model

In the M-factor arbitrage model which satisfies the standard regularity conditions, the price of the financial asset[1] $P(t) = P(t, X(t))$ satisfies the following partial differential equation (PDE):

$$\begin{cases} \frac{1}{2}\operatorname{trace}\left(\Sigma(t,X)^\top \partial_X^2 P(t,X)\,\Sigma(t,X)\,\rho\right) + \left[\mu(t,X)^\top - \lambda(t,X)^\top \Sigma(t,X)^\top\right] P_X(t,X) \\ +P_t(t,X) - r(t,X)\,P(t,X) + g(t,X) = 0 \\ P(T) = G(T, X(T)) \end{cases} \tag{1}$$

The M-dimensional state vector X is a Markov diffusion process taking values in $\mathcal{R}_X \subset \mathbb{R}^M$ defined by the following stochastic differential equation (SDE):

$$\begin{cases} \mathrm{d}X(t) &= \mu(t, X(t))\,\mathrm{d}t + \Sigma(t, X(t))\,\mathrm{d}W(t) \\ X(t_0) &= X_0 \end{cases} \tag{2}$$

where $W(t)$ is a N-dimensional Wiener process defined on the fundamental probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the covariance matrix

$$\mathbb{E}\left[W(t)\,W(t)^\top\right] = \rho t \tag{3}$$

The solution of the equation (1) with the terminal value $P(T) = G(T, X(T))$ is then given by the Feynman-Kac representation theorem (Friedman [1975]):

$$P(t_0) = \mathbb{E}^{\mathbb{Q}}\left[G(T, X(T))\exp\left(-\int_{t_0}^{T} r(t, X(t))\,\mathrm{d}t\right) + \int_{t_0}^{T} g(t, X(t))\exp\left(-\int_{t_0}^{t} r(s, X(s))\,\mathrm{d}s\right)\mathrm{d}t \,\middle|\, \mathcal{F}_{t_0}\right] \tag{4}$$

with $\mathbb{Q}$ the *martingale* probability measure obtained with the Girsanov theorem. The pricing of *European* options could be done by solving the PDE (1) or by integrating the formula (4).

2.2 From the multivariate RND to the risk-neutral copula

One of the main result is the following proposition.

Proposition 1 *The margins of the risk-neutral distribution $\mathbb{Q}$ are necessarily the univariate risk-neutral distributions $\mathbb{Q}_n$.*

Proof. This statement is obvious. Nevertheless, we give here some mathematical justifications. We assume that the assets follow the Black-Scholes model. Under the probability $\mathbb{P}$, we have

$$\mathrm{d}S_n(t) = \mu_n S_n(t)\,\mathrm{d}t + \sigma_n S_n(t)\,\mathrm{d}W_n(t) \tag{5}$$

for $n = 1, \ldots, N$. Using Girsanov theorem, the density of $\mathbb{Q}$ with respect to $\mathbb{P}$ is

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(-\int_{t_0}^{t}\langle\lambda(s), \mathrm{d}W(s)\rangle - \frac{1}{2}\int_{t_0}^{t}\langle\lambda(s), \lambda(s)\rangle\,\mathrm{d}s\right) \tag{6}$$

with $\lambda(t) = \begin{bmatrix} \lambda_1 & \cdots & \lambda_N \end{bmatrix}^\top$ and $\lambda_n = \sigma_n^{-1}(\mu_n - r)$. In order to simplify the calculation, we consider the two-dimensional case and $\mathbb{E}[W_1(t)\,W_2(t)] = 0$. To get the density of the margins, we just have to calculate the mean in Girsanov's formula while fixing the Brownian motion corresponding to the margin we want to derive:

$$\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}_1} = \mathbb{E}^{\mathbb{P}}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \mid W_1(t)\right] \tag{7}$$

[1] The maturity date of the asset is T. The delivery value G depends on the values taken by the state variables at the maturity date $G = P(T) = G(T, X(T))$ and the asset pays a continuous dividend g which is a function of the state vector $g = g(t, X(t))$.

It comes that

$$\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}_1} = \exp\left(-\lambda_1\left(W_1(t) - W_1(t_0)\right) - \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2\right)(t - t_0)\right)\mathbb{E}^{\mathbb{P}_1}\left[e^{-\lambda_2\left(W_2(t) - W_2(t_0)\right)}\right] \tag{8}$$

Using the Laplace transform of a gaussian random variable, we finally obtain

$$\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}_1} = \exp\left(-\lambda_1\left(W_1(t) - W_1(t_0)\right) - \frac{1}{2}\lambda_1^2(t - t_0)\right) \tag{9}$$

In the case where $\mathbb{E}\left[W_1(t)\,W_2(t)\right] = \rho t$, we have the same calculus using the orthogonal transformation $W_2(t) = \rho W_1(t) + \sqrt{1-\rho^2}W_2^\star(t)$. ■

Using Sklar's theorem, it comes that the RND $\mathbb{Q}^t$ at time t has the following canonical representation:

$$\mathbb{Q}^t(x_1, \ldots, x_N) = \mathbf{C}_t^{\mathbb{Q}}\left(\mathbb{Q}_1^t(x_1), \ldots, \mathbb{Q}_N^t(x_N)\right) \tag{10}$$

$\mathbf{C}_t^{\mathbb{Q}}$ is called the risk-neutral copula (RNC) (Rosenberg [2000]). It is the dependence function between the risk-neutral random variables. Because we have also $\mathbb{P}^t(x_1, \ldots, x_N) = \mathbf{C}_t^{\mathbb{P}}\left(\mathbb{P}_1^t(x_1), \ldots, \mathbb{P}_N^t(x_N)\right)$, we have this proposition.

Proposition 2 *If the functions μ, σ and λ are non-stochastic, the risk-neutral copula $\mathbf{C}^{\mathbb{Q}}$ and the objective copula $\mathbf{C}^{\mathbb{P}}$ are the same.*

Proof. By solving the SDE, we obtain

$$X_n(t) = X_n(t_0)\exp\left(\int_{t_0}^t\left(\mu_n(s) - \frac{1}{2}\sigma_n^2(s)\right)\mathrm{d}s + \int_{t_0}^t \sigma_n(s)\,\mathrm{d}W_n(s)\right) \tag{11}$$

under $\mathbb{P}$ and

$$X_n(t) = X_n(t_0)\exp\left(\int_{t_0}^t\left(\mu_n(s) - \frac{1}{2}\sigma_n^2(s) - \lambda_n(s)\right)\mathrm{d}s + \int_{t_0}^t \sigma_n(s)\,\mathrm{d}W_n^{\mathbb{Q}}(s)\right) \tag{12}$$

under $\mathbb{Q}$. Using the Girsanov theorem, we remark that the objective expression of $X_n(t)$ and the corresponding risk-neutral expression are both strictly increasing functions of $\int_{t_0}^t \sigma_n(s)\,\mathrm{d}W_n(s)$ and $\int_{t_0}^t \sigma_n(s)\,\mathrm{d}W_n^{\mathbb{Q}}(s)$. Thus the copula of $X(t)$ is the same that the one of $\left(\int_{t_0}^t \sigma_1(s)\,\mathrm{d}W_1(s), \ldots, \int_{t_0}^t \sigma_N(s)\,\mathrm{d}W_N(s)\right)$ under $\mathbb{P}$ and the same that the one of $\left(\int_{t_0}^t \sigma_1(s)\,\mathrm{d}W_1^{\mathbb{Q}}(s), \ldots, \int_{t_0}^t \sigma_N(s)\,\mathrm{d}W_N^{\mathbb{Q}}(s)\right)$ under $\mathbb{Q}$. Both of these copulas are Normal, we just have to verify that they have the same matrix of parameters. We have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\int_{t_0}^t \sigma_i(s)\,\mathrm{d}W_i(s)\cdot\int_{t_0}^t \sigma_j(s)\,\mathrm{d}W_j(s)\right] &= \int_{t_0}^t \sigma_i(s)\,\sigma_j(s)\,\mathrm{d}\left\langle W_i(s), W_j(s)\right\rangle \\
&= \int_{t_0}^t \sigma_i(s)\,\sigma_j(s)\,\mathrm{d}\left\langle W_i^{\mathbb{Q}}(s), W_j^{\mathbb{Q}}(s)\right\rangle \\
&= \mathbb{E}^{\mathbb{Q}}\left[\int_{t_0}^t \sigma_i(s)\,\mathrm{d}W_i^{\mathbb{Q}}(s)\cdot\int_{t_0}^t \sigma_j(s)\,\mathrm{d}W_j^{\mathbb{Q}}(s)\right]
\end{aligned}$$

This completes the proof. ■

The property that λ is non-stochastic is not a sufficient condition, except to some special cases. For example, let consider the bivariate extension of the Vasicek model. Under $\mathbb{P}$, we have

$$\mathrm{d}X_n(t) = \kappa_n\left(\theta_n - X_n(t)\right)\mathrm{d}t + \sigma_n\,\mathrm{d}W_n(t) \tag{13}$$

for $n = 1, 2$ and $\mathbb{E}\left[W_1(t)\,W_2(t)\right] = \rho t$. The diffusion representation is then

$$X_n(t) = X_n(t_0)\,e^{-\kappa_n(t-t_0)} + \theta_n\left(1 - e^{-\kappa_n(t-t_0)}\right) + \sigma_n\int_{t_0}^t e^{\kappa_n(s-t)}\,\mathrm{d}W_n(s) \tag{14}$$

It comes that the distribution of $(X_1(t), X_2(t))$ is gaussian. Using properties of copulas, we have

$$\mathbf{C}^{\mathbb{P}}\left\langle X_1(t), X_2(t)\right\rangle = \mathbf{C}\left\langle \int_{t_0}^{t} e^{\kappa_1(s-t)}\,\mathrm{d}W_1(s), \int_{t_0}^{t} e^{\kappa_2(s-t)}\,\mathrm{d}W_2(s)\right\rangle \tag{15}$$

and

$$\mathbf{C}^{\mathbb{Q}}\left\langle X_1(t), X_2(t)\right\rangle = \mathbf{C}\left\langle \int_{t_0}^{t} e^{\kappa_1(s-t)}\,\mathrm{d}W_1^{\mathbb{Q}}(s), \int_{t_0}^{t} e^{\kappa_2(s-t)}\,\mathrm{d}W_2^{\mathbb{Q}}(s)\right\rangle \tag{16}$$

because λ only affects the parameters θ_n. We deduce that the objective and risk-neutral copulas are Normal with parameter $\rho_{\mathbf{C}}$ defined as follows

$$\rho_{\mathbf{C}} = 2\rho\frac{\sqrt{\kappa_1\kappa_2}}{\kappa_1+\kappa_2}\frac{1-e^{-(\kappa_1+\kappa_2)(t-t_0)}}{\sqrt{1-e^{-2\kappa_1(t-t_0)}}\sqrt{1-e^{-2\kappa_2(t-t_0)}}} \tag{17}$$

2.3 The change of numéraire in the Black-Scholes model

In this section we interpret the change of numéraire in terms of copulas. The purpose of the change of numéraire is to reduce the number of assets. We will see how it affects the copula in the same framework.

In the Black-scholes model, the dynamics of the asset prices are under $\mathbb{Q}$

$$\mathrm{d}S_n(t) = rS_n(t)\,\mathrm{d}t + \sigma_n S_n(t)\,\mathrm{d}W_n(t) \tag{18}$$

where $W = (W_1, \ldots, W_N)$ is a vector of N correlated brownian motions with $\mathbb{E}\left[W(t)\,W(t)^{\top}\right] = \rho t$. The copula of $(S_1(t), \ldots, S_N(t))$ is a Normal copula with matrix of parameters ρ. In the S_N−numéraire, the copula of $\left(\frac{S_1(t)}{S_N(t)}, \ldots, \frac{S_{N-1}(t)}{S_N(t)}\right)$ is also the Normal copula under an equivalent probability $\tilde{\mathbb{Q}}$.

Proof. Let $X_n(t)$ be the asset price $S_n(t)$ divided by $S_N(t)$. Under $\mathbb{Q}$, we have

$$\frac{\mathrm{d}X_n(t)}{X_n(t)} = \sigma_n\left[\mathrm{d}W_n(t) - \rho_{n,N}\sigma_N\,\mathrm{d}t\right] - \sigma_N\left[\mathrm{d}W_N(t) - \sigma_N\,\mathrm{d}t\right] \tag{19}$$

It comes that $(X_1(t), \ldots, X_N(t))$ has a Normal copula the matrix of parameters of which equals $\tilde{\rho}$:

$$\tilde{\rho}_{i,j} = \frac{\rho_{i,j}\sigma_i\sigma_j - \rho_{i,N}\sigma_i\sigma_N - \rho_{j,N}\sigma_j\sigma_N + \sigma_N^2}{\tilde{\sigma}_{i,N}\tilde{\sigma}_{j,N}} \tag{20}$$

with $\tilde{\sigma}_{n,N}^2 = \sigma_n^2 + \sigma_N^2 - 2\rho_{n,N}\sigma_n\sigma_N$. We now prove that $X_n(t)$ is a GBM process and a martingale under $\tilde{\mathbb{Q}}$. We intoduce here A the Cholesky reduction of the correlation matrix $\rho = AA^{\top}$. We know that there exists independent Brownian motions $\tilde{W}_n$ such that $W_n(t) = \sum_j A_{n,j}\tilde{W}_j(t)$. Because A is invertible, we can find a unique vector $\boldsymbol{\lambda}$ which solves the linear system $\sum_j \lambda_j A_{n,j} = -\rho_{n,N}\sigma_N$ for all n. Then, using Girsanov's theorem, we define the probability $\tilde{\mathbb{Q}}$ with its Radon-Nikodym derivative:

$$\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} = \exp\left(\sum_{j=1}^{N} -\lambda_j\tilde{W}_j(t) - \frac{1}{2}\lambda_j^2 t\right) \tag{21}$$

$\mathrm{d}\tilde{B}_j(t) = \mathrm{d}\tilde{W}_j(t) + \lambda_j\,\mathrm{d}t$ defines a Brownian motion under $\tilde{\mathbb{Q}}$. Let us define B_n as follow $\mathrm{d}B_n(t) = \sum_j A_{n,j}\mathrm{d}\tilde{B}_j(t) = \mathrm{d}W_n(t) - \rho_{n,N}\sigma_N\,\mathrm{d}t$. $B = (B_1, \ldots, B_N)$ is then a Brownian motion under $\tilde{\mathbb{Q}}$ with the same matrix of correlation than W under $\mathbb{Q}$. We can rewrite the dynamics of $X_n(t)$ in the following manner

$$\frac{\mathrm{d}X_n(t)}{X_n(t)} = \sigma_n\,\mathrm{d}B_n(t) - \sigma_N\,\mathrm{d}B_N(t) \tag{22}$$

A straightforward calculation of the correlation matrix completes the proof. ■

Let us consider an example. We investigate the case of a payoff function of the form $(S_1(T) - S_2(T) - S_3(T))^+$. In the S_3-numéraire, we get

$$P(t_0) = S_3(t_0)\,\mathbb{E}_{t_0}^{\tilde{\mathbb{Q}}}\left[\left(\frac{S_1(T)}{S_3(T)} - \frac{S_2(T)}{S_3(T)} - 1\right)^+\right] \tag{23}$$

We retrieve the price of a spread option between $\frac{S_1}{S_3}$ and $\frac{S_2}{S_3}$, which are geometric Brownian motions under $\tilde{\mathbb{Q}}$ with volatilities $\tilde{\sigma}_{1,3}^2 = \sigma_1^2 + \sigma_3^2 - 2\rho_{1,3}\sigma_1\sigma_3$ and $\tilde{\sigma}_{2,3}^2 = \sigma_2^2 + \sigma_3^2 - 2\rho_{2,3}\sigma_2\sigma_3$. The strike is one and there is no spot rate. Moreover, the copula is a Normal 2-copula with parameter $\tilde{\rho}_{1,2}$. We can then use the spread option formula of the next section to price this option. With the same method, we can have formulas for every kind of basket option with no strike in dimension 3.

2.4 The RND copula and the risk-neutral assumption

In the Black-Scholes model, the asset prices follow GBM processes under the martingale probability measure. However, when we compute implied volatilities from the market prices, we remark that they are not constant and they depend on the strike. This is the smile effect or the volatility smile. So banks have developped more satisfactory models to take into account this smile effect. But, in the case of multi-assets options, the Black-Scholes model is often used because it is very tractable, and because there does not really exist satisfactory multivariate models.

Suppose that the bank uses two models: a model $\mathcal{M}$ for one-asset options and the Black-Scholes model for multi-asset options. In this case, the marginals of the multivariate RND are not the univariate RND. We have then two problems:

1. First, it could involve arbitrage opportunities;
2. Secondly, the price of the multi-asset options is not necessarily the cost of the hedging strategy.

For example, let us consider a portfolio of long position with a Max call option and a Min call option on the same two assets and with the same strike K. The value of this portfolio π is equal to $P_{\max}^c(t_0) + P_{\min}^c(t_0)$. Simple calculus show that π is also the sum of the two call options with strike K. So, we have

$$\pi = \begin{cases} P_{\max}^c(t_0) + P_{\min}^c(t_0) & \text{(Black-Scholes model)} \\ P_1^c(t_0) + P_2^c(t_0) & \text{(model } \mathcal{M}) \end{cases}$$

Because the two models are not the same, $P_{\max}^c(t_0) + P_{\min}^c(t_0) - P_1^c(t_0) - P_2^c(t_0)$ could be different from zero. So, it is easy to build arbitrage opportunities. Moreover, if $P_{\max}^c(t_0) + P_{\min}^c(t_0) \neq P_1^c(t_0) + P_2^c(t_0)$ and if we assume that the "true" model is $\mathcal{M}$, then the price of the multi-asset option is not a risk-neutral price.

Using copulas, the bank could extend the model $\mathcal{M}$ in the multivariate case in a natural way for the calibration issues. Nevertheless, the corresponding multivariate distribution of the asset prices is not necessarily "risk-neutral". But we could hope that this way to build the multivariate model gives better prices and greeks. Moreover, we do not think that the market is *fully* risk-neutral. We believe that the market is risk-neutral **only** in some directions and for some maturities. Imposing these restrictions could be done in the copula framework.

Let us consider the simple example where the model $\mathcal{M}$ corresponds to the following SDE[2]:

$$\begin{cases} \mathrm{d}S_n(t) &= \sigma_n^{\mathcal{M}}\left(S_n(t) + \delta_n^{\mathcal{M}}\right)\mathrm{d}W_n^{\mathbb{Q}}(t) \\ S_n(t_0) &= S_{n,0} \end{cases} \tag{24}$$

The model $\mathcal{M}$ is used to price one-asset options. To price the previous portfolio π, we consider the Black-Scholes model. In Figure 1, we have represented $|\pi_{BS} - \pi_{\mathcal{M}}|$ with respect to the strike[3]. We assume that the BS model is calibrated using either ATM vanilla options ($K = 100\%$) or OTM vanilla options ($K = 105\%$). We verify that arbitrage opportunities exist.

[2] For convenience, cost-of-carry parameters are set to 0.

[3] The parameters are the following: $S_{n,0} = 1$, $\sigma_n^{\mathcal{M}} = 10\%$, $\delta_n^{\mathcal{M}} = 100\%$. The maturity is one year and the interest rate is equal to 5%.

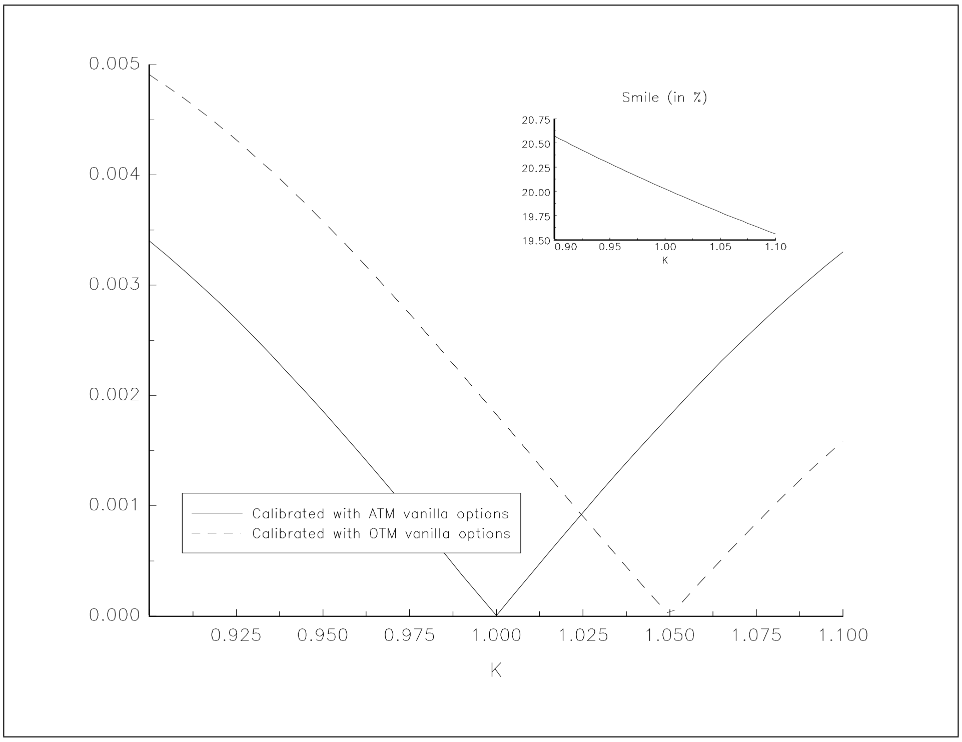

Figure 1: Absolute difference between π_{BS} and $\pi_{\mathcal{M}}$

In the previous example, the dependence function has no influence on the portfolio. Let us now take the example of the spread option $(S_2(T) - S_1(T) - K)^+$. We assume that the model $\mathcal{M}$ is the stochastic volatility model of Heston [1993]:

$$\left\{ \begin{array}{lcl} \mathrm{d}S_n(t) & = & \mu_n S_n(t)\,\mathrm{d}t + \sqrt{V_n(t)} S_n(t)\,\mathrm{d}W_n^1(t) \\ \mathrm{d}V_n(t) & = & \kappa_n (V_n(\infty) - V_n(t))\,\mathrm{d}t + \sigma_n \sqrt{V_n(t)}\,\mathrm{d}W_n^2(t) \end{array} \right. \tag{25}$$

with $\mathbb{E}\left[W_n^1(t)\,W_n^2(t) \mid \mathcal{F}_{t_0}\right] = \rho_n^W (t - t_0)$, $\kappa_n > 0$, $V_n(\infty) > 0$ and $\sigma_n > 0$. The market prices of risk processes are $\lambda_n^1(t) = (\mu_n - r) / \sqrt{V_n(t)}$ and $\lambda_n^2(t) = \lambda_n \sigma_n^{-1} \sqrt{V_n(t)}$. Figure 2 shows the impact of the parameter ρ_n^W on the volatility smile[4]. To compute prices of spread options, we consider that the RNC $\mathbf{C}^{\mathbb{Q}} \langle S_1(T), S_2(T) \rangle$ is the Normal copula with parameter ρ. We have reported the density of the risk-neutral distribution of the spread $\Delta = S_2(T) - S_1(T)$ in Figures 3 and 4. We compare the Heston model with the implied BS model. In this last case, parameters of the BS model are calibrated using ATM options. It is obvious that we do not obtain the same densities. As a result, the option prices may be very different.

3 Pricing and bounds of multi-asset options

3.1 Pricing formulas

Cherubini and Luciano [2000] shows that the price of the double digital call is

$$P^c(t_0) = e^{-r(T-t_0)} \mathbf{C}^{\mathbb{Q}} (\mathbf{F}_1(K_1), \mathbf{F}_2(K_2)) \tag{26}$$

We extend now the previous result to other two-asset options.

[4] The numerical values are $S_n(t_0) = 100$, $V_n(t_0) = V_n(\infty) = \sqrt{20\%}$, $\kappa_n = 0.5$, $\sigma_n = 90\%$ and $\lambda_n = 0$. The interest rate is equal to 5% and the maturity option is one month.

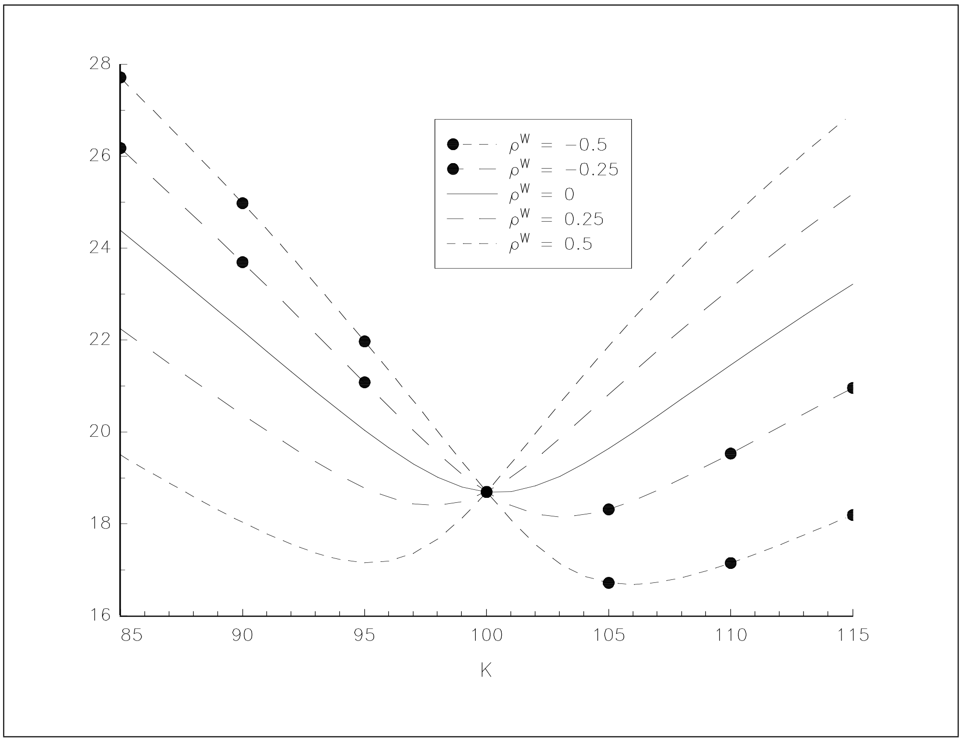

Figure 2: Volatility smile of the Heston model (in %)

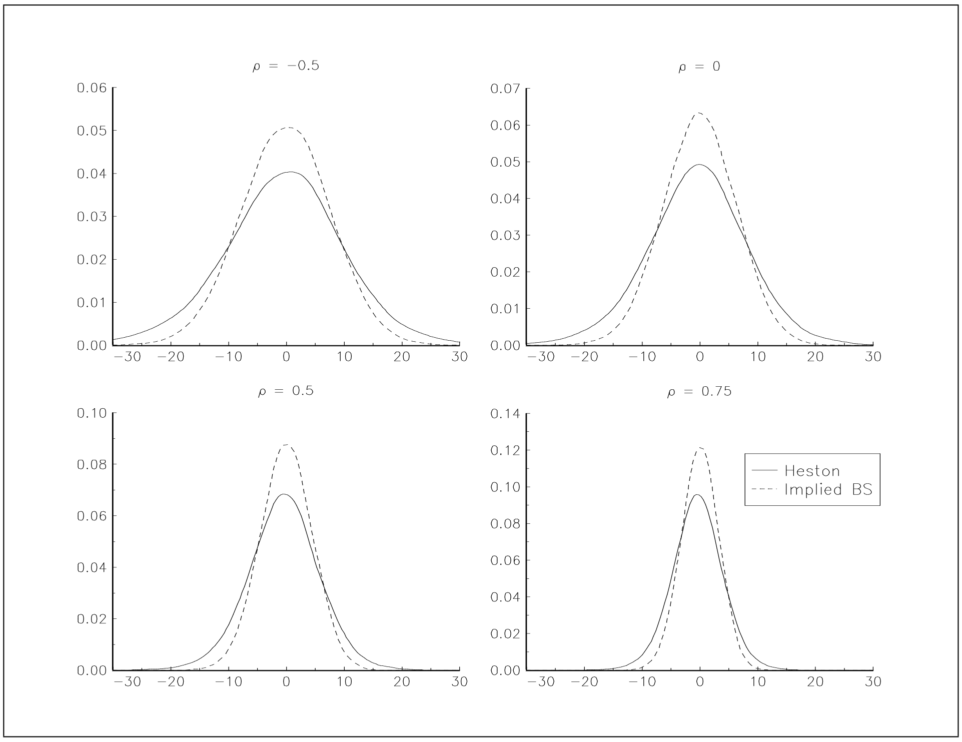

Figure 3: Density of the RND of the spread ($\rho_1^W = -0.75$, $\rho_2^W = -0.50$,)

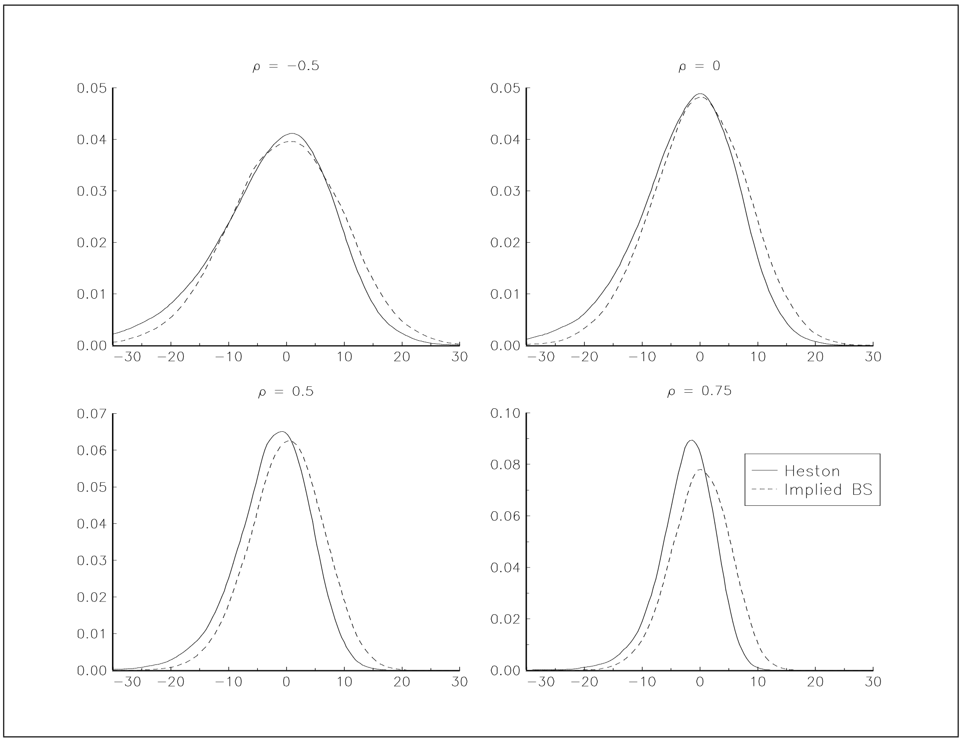

Figure 4: Density of the RND of the spread ($\rho_1^W = -0.75$, $\rho_2^W = 0.50$,)

The spread option has been studied by DURRLEMAN [2001]. We have

$$\begin{aligned}\Pr\{S_2(T) - S_1(T) \leq y\} &= \mathbb{E}^{\mathbb{Q}}\left[\Pr\{S_2(T) \leq x + y\} \mid S_1(T) = x\right] \\ &= \int_0^{+\infty} f_1(x)\partial_1 \mathbf{C}^{\mathbb{Q}}\left(\mathbf{F}_1(x), \mathbf{F}_2(x+y)\right) \mathrm{d}x\end{aligned}$$

Since we have $\Pr\{S_2(T) - S_1(T) \leq y\} = 1 + e^{r(T-t_0)}\partial_K P^c(T, y) = e^{r(T-t_0)}\partial_K P^p(T, y)$, an integration calculus and the call-put parity give the general pricing formula using copulas

$$\int_{-\infty}^{K}\left[1 + e^{r(T-t_0)}\partial_K P(T, y)\right] \mathrm{d}y = \int_0^{+\infty}\int_{-x}^{K} f_1(x)\partial_1 \mathbf{C}^{\mathbb{Q}}\left(\mathbf{F}_1(x), \mathbf{F}_2(x+y)\right) \mathrm{d}y\,\mathrm{d}x \tag{27}$$

Proposition 3 *The price of the spread option is given by*

$$\begin{aligned}P^c(t_0) &= S_2(t_0) - S_1(t_0) - Ke^{-r(T-t_0)} + \\ &\quad e^{-r(T-t_0)}\int_0^{+\infty}\int_{-x}^{K} f_1(x)\partial_1 \mathbf{C}^{\mathbb{Q}}\left(\mathbf{F}_1(x), \mathbf{F}_2(x+y)\right) \mathrm{d}x\,\mathrm{d}y\end{aligned} \tag{28}$$

Let P_1^c and P_2^c be the prices of the corresponding European option. This last expression can be rewritten using the vanilla prices

$$\begin{aligned}P^c(t_0) &= S_2(t_0) - S_1(t_0) - Ke^{-r(T-t_0)} + \\ &\quad e^{-r(T-t_0)}\int_{-x}^{K}\int_0^{+\infty} \partial_K^2 P_1^c(T, x)\partial_1 \mathbf{C}^{\mathbb{Q}}\left(1 + e^{r(T-t_0)}\partial_K P_1^c(T, x), 1 + e^{r(T-t_0)}\partial_K P_2^c(T, x+y)\right) \mathrm{d}x\,\mathrm{d}y\end{aligned} \tag{29}$$

Proof. We have to verify that $\lim_{K \to -\infty} P^p(t_0) = 0$. This is straightforward using the monotone convergence theorem for the risk neutral expression of the put price. Then, we use the call-put parity relationship $P^c(t_0) - P^p(t_0) = S_2(t_0) - S_1(t_0) - Ke^{-r(T-t_0)}$. ■

The same method can be used as well for other options such as basket, Max, BestOf, WorstOf, etc. For the Basket or the Spread, we have as many integrals as assets but for the Max option, we only have one integral whatever the number of assets is.

Proposition 4 *The price of the Basket option is*

$$\begin{aligned} P^c(t_0) &= S_1(t_0) + S_2(t_0) - Ke^{-r(T-t_0)} + \\ & e^{-r(T-t_0)} \int_0^{+\infty} \int_x^K f_1(x) \partial_1 \mathbf{C}^{\mathbb{Q}} (\mathbf{F}_1(x), \mathbf{F}_2(y-x)) \, \mathrm{d}x \, \mathrm{d}y \end{aligned} \tag{30}$$

In fact, this method cannot always give a formula for call options without involving the price for a strike $K = 0$. In the example of Max/Min options, we can obtain an interesting formula for the put because the price for $K = 0$ is 0. We have then

$$\Pr\{\max(S_1(T), S_2(T)) \le y\} = e^{r(T-t_0)} \partial_K P^p(T, y) \tag{31}$$

Thus,

$$P^p_{\max}(t_0) = e^{-r(T-t_0)} \int_0^K \mathbf{C}^{\mathbb{Q}}(\mathbf{F}_1(y), \mathbf{F}_2(y)) \, \mathrm{d}y \tag{32}$$

We can do the same for the Min option or use directly the relation $\min(S_1, S_2) = S_1 + S_2 - \max(S_1, S_2)$:

$$\begin{aligned} P^p_{\min}(t_0) &= Ke^{-r(T-t_0)} - e^{-r(T-t_0)} \int_0^K \breve{\mathbf{C}}^{\mathbb{Q}}(\mathbf{1} - \mathbf{F}_1(y), \mathbf{1} - \mathbf{F}_2(y)) \, \mathrm{d}y \\ &= P^p_1(t_0) + P^p_2(t_0) - P^p_{\max}(t_0) \end{aligned} \tag{33}$$

We can notice that these formulas are also available for higher dimensions.

Remark 5 *We retrieve the fact that the Max (or Min) put option is monotone with respect to the copula order (which is equivalent to the concordance order in two dimensions).*

Remark 6 *It is not always possible to derive the call formulas without knowing the price for $K = 0$. The derivation of the risk neutral expression with respect to the strike provides us the following expression*

$$e^{r(T-t_0)} \partial_K P^p_{\max}(T, y) = 1 + e^{r(T-t_0)} \partial_K P^c_{\max}(T, y) \tag{34}$$

So, it comes that

$$P^c_{\max}(T, K) - P^p_{\max}(T, K) = P^c_{\max}(T, 0) - Ke^{-r(T-t_0)} \tag{35}$$

Therefore, we have to calculate $P^c_{\max}(T, 0)$. The problem is that the discounted process $\max(S_1, S_2)$ is a submartingale[5] *(and not a martingale). However, we can use the equality $\max(S_1, S_2) = S_1 + (S_2 - S_1)^+$. Therefore, using the results on the spread option, we obtain*

$$P^c_{\max}(T, 0) = S_1(t_0) + e^{-r(T-t_0)} \int_0^{+\infty} \int_{-x}^0 f_1(x) \partial_1 \mathbf{C}^{\mathbb{Q}}(\mathbf{F}_1(x), \mathbf{F}_2(x+y)) \, \mathrm{d}x \, \mathrm{d}y \tag{38}$$

[5] For every $t \ge s$, we have

$$\begin{aligned} \mathbb{E}[\max(S_1(t), S_2(t)) | \mathcal{F}_s] &= \mathbb{E}[S_1(t) | \mathcal{F}_s] + \mathbb{E}[\max(S_1(t), S_2(t)) - S_1(t) | \mathcal{F}_s] \\ &\ge e^{r(t-s)} S_1(s) \end{aligned} \tag{36}$$

And the same holds with S_2. So, we finally obtain

$$e^{-r(t-s)} \mathbb{E}[\max(S_1(t), S_2(t)) | \mathcal{F}_s] \ge \max(S_1(s), S_2(s)) \tag{37}$$

We now consider a BestOf put/put option which payoff is $\max\left((K_1 - S_1(T))^+, (K_2 - S_2(T))^+\right)$. We differentiate with respect to K_1

$$\begin{aligned} e^{r(T-t_0)}\partial_{K_1} P^{p,p}_{\mathrm{BestOf}}(T,y,K_2) &= P\left\{S_1(T) \le \max(S_2(T) + y - K_2, y)\right\} \\ &= \int_0^{+\infty} f_2(x)\,\partial_2 \mathbf{C}^{\mathbb{Q}}\left(\mathbf{F}_1(\max(y+x-K_2,x)), \mathbf{F}_2(x)\right) \mathrm{d}x \end{aligned} \tag{39}$$

Finally, we have

$$P^{p,p}_{\mathrm{BestOf}}(T,K_1,K_2) = P^p_2(t_0,K_2) + e^{-r(T-t_0)} \int_0^{K_1}\int_0^{+\infty} f_2(x)\,\partial_2 \mathbf{C}^{\mathbb{Q}}\left(\mathbf{F}_1(\max(y+x-K_2,x)), \mathbf{F}_2(x)\right) \mathrm{d}x\,\mathrm{d}y \tag{40}$$

In the case of the BestOf put/call option which payoff is $\max\left((K_1 - S_1(T))^+, (S_2(T) - K_2)^+\right)$, we have

$$\begin{aligned} e^{r(T-t_0)}\partial_{K_1} P^{p,c}_{\mathrm{BestOf}}(T,y,K_2) &= P\left\{S_1(T) \le \max(-S_2(T) + y - K_2, y)\right\} \\ &= \int_0^{+\infty} f_2(x)\,\partial_2 \mathbf{C}^{\mathbb{Q}}\left(\mathbf{F}_1(\max(y-x-K_2,x)), \mathbf{F}_2(x)\right) \mathrm{d}x \end{aligned} \tag{41}$$

and

$$P^{p,c}_{\mathrm{BestOf}}(T,K_1,K_2) = P^c_2(t_0,K_2) + e^{-r(T-t_0)} \int_0^{K_1}\int_0^{+\infty} f_2(x)\,\partial_2 \mathbf{C}^{\mathbb{Q}}\left(\mathbf{F}_1(\max(y-x-K_2,x)), \mathbf{F}_2(x)\right) \mathrm{d}x\,\mathrm{d}y \tag{42}$$

3.2 Bounds of contingent claim prices

Let us introduce the lower and upper Fréchet copulas $\mathbf{C}^-(u_1,u_2) = \max(u_1+u_2-1,0)$ and $\mathbf{C}^+(u_1,u_2) = \min(u_1,u_2)$. We can prove that for any copula $\mathbf{C}$, we have $\mathbf{C}^- \prec \mathbf{C} \prec \mathbf{C}^+$. For any distribution $\mathbf{F}$ with given marginals $\mathbf{F}_1$ and $\mathbf{F}_2$, it comes that $\mathbf{C}^-(\mathbf{F}_1(x_1),\mathbf{F}_2(x_2)) \le \mathbf{F}(x_1,x_2) \le \mathbf{C}^+(\mathbf{F}_1(x_1),\mathbf{F}_2(x_2))$ for all $(x_1,x_2) \in \mathbb{R}^2_+$. Let $P^-(S_1,S_2,t)$ and $P^+(S_1,S_2,t)$ be respectively the lower and upper bounds and G be the payoff fuction. We can now recall the following proposition.

Proposition 7 *If $\partial^2_{1,2}G$ is a nonpositive (resp. nonnegative) measure then $P^-(S_1,S_2,t)$ and $P^+(S_1,S_2,t)$ correspond to the cases $\mathbf{C} = \mathbf{C}^+$ (resp. $\mathbf{C} = \mathbf{C}^-$) and $\mathbf{C} = \mathbf{C}^-$ (resp. $\mathbf{C} = \mathbf{C}^+$).*

Proof. see Rapuch and Roncalli [2001]. ■

In the multivariate case (more than two assets), we are not able to obtain similar results, except in the case of the Black-Scholes model. Note that the previous bounds for two-asset options are the best possible. Moreover, under the previous assumptions, given a price between these bounds $P^-(S_1,S_2,t)$ and $P^+(S_1,S_2,t)$, there exist copula functions (not necessarily unique) such that this price is reached. Let us consider the previous example with the Heston model. In Figure 5, we have reported the bounds of the Spread put option. Remark that these bounds are computed for given univariate risk-neutral distributions (or equivalently for given volatility smiles). That explains the difference between the bounds of the Heston model and the bounds of the implied BS model — the bounds of the Heston model correspond to solid lines. This simple example raises a huge problem. If we assume that the Heston model is the market model, there are market prices that are not reached with the implied BS model (for example the option price A). Moreover, we remark that for a given strike, there exist option prices that are smaller than the lower Heston bound and bigger than the upper BS bound (for example the option price A). In these cases, the implied BS model underestimates systematically the spread put option.

4 Implied dependence functions of asset returns

There is a large litterature on estimating risk-neutral distributions for monetary policy in order to obtain 'forward-looking' indicators[6] (see for example Bahra [1996], Bates [1991] or Söderlind and Svensson

[6] Option prices incorporate market expectations over the maturity of the option, they may also provide interesting additional information not contained in the historical data.

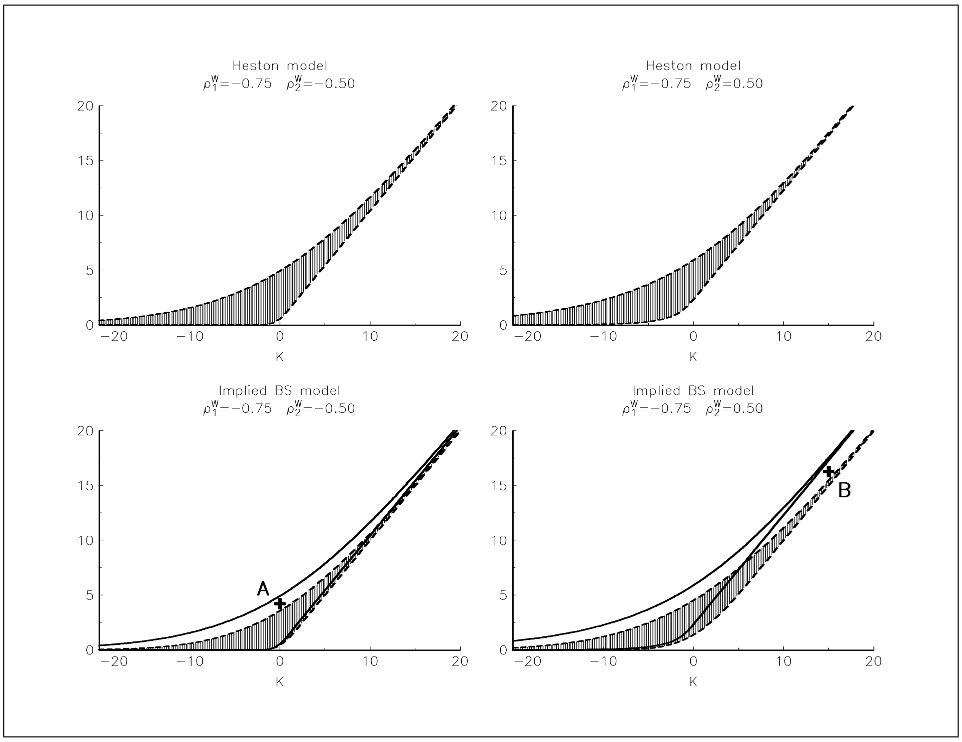

Figure 5: Bounds of the Spread put option

[1997]). However, all the works done until now only consider the univariate case. Bikos [2000] discusses "potential applications of estimating multivariate RND to extract forward information about co-movements of different asset returns". A straightforward solution may be to estimate the multivariate RND from multi-asset options prices. As we have seen above, the marginals of the multivariate RND are the univariate RND. So, informations provided by vanilla options are certainly more pertinent to estimate these univariate RND. Multi-asset options contain this information too, but this information is already known. **Consider the trader's point of view. When the trader says that he buys or sells volatility on the vanilla markets, it means that he "bets" on the risk-neutral distribution. In the market of multi-asset options, the trader does not bet on the volatility anymore, but correlations. In other words, he bets on the dependence function between the asset prices. That characterizes the difference between the market of Vanilla options and the market of multi-asset options.** A similar point of view is given by Bikos [2000] for monetary policy:

> *[The Bank of England] currently produces univariate implied PDF from option prices for a wide variety of underlying assets. In building a multivariate model we would like to use these implied densities as inputs. In other words we would like our univariate/marginal implied PDF to be consistent with, and derivable from the multivariate model. [...] the statistical tool that naturally deals with this type of problem is known as a copula function.*

To compute "forward-looking" indicators for co-movements of asset returns, Bikos [2000] suggests then the following method:

1. Estimate the univariate RND $\hat{\mathbb{Q}}_n$ using Vanilla options;
2. Estimate the copula $\hat{\mathbf{C}}$ using multi-asset options by imposing that $\mathbb{Q}_n = \hat{\mathbb{Q}}_n$;
3. Derive "forward-looking" indicators directly from $\hat{\mathbf{C}}$.

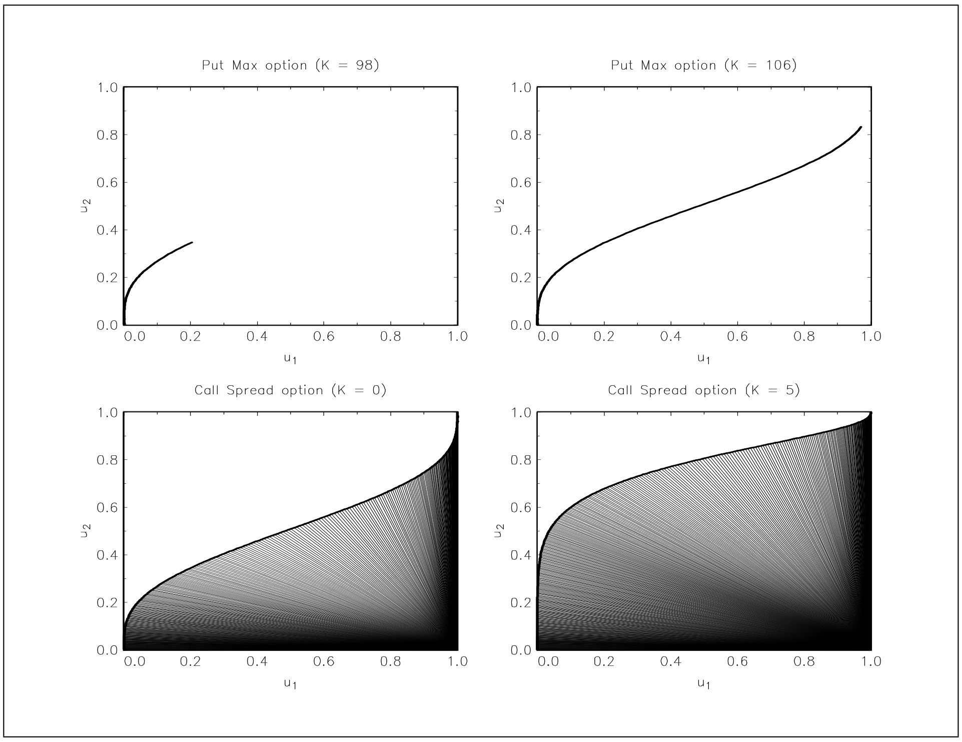

Figure 6: Information on the copula used by multi-asset options

This estimation method is consistent with our framework and seems to be the more relevant regarding informations available in options markets (vanilla options and multi-asset options).

Estimating parametric or non-parametric copula functions using multi-asset options could be viewed as a statistical problem. This is generally an optimisation problem, which could be solved in a classical way. But we must choose carefully the calibration set of multi-asset options. Indeed, a multi-asset option price uses only partial information about the dependence structure. For example, we use only a portion of the diagonal section of the multivariate risk-neutral distribution to price a Max or a Min option. In Figure 6, we have reported the set of points involved in pricing a Put Max option and a Spread option[7]. We remark that the set for the Spread option is much more larger than for the Max option. As a consequence, it seems more relevant to calibrate copulas with Spread options than Max options. However, the unit square is never fully used.

We finally give an example to show that mispecifications of the univariate RND could lead to a misinterpretation of the anticipation of the market. The set of calibration consists of 6 Vanilla options for each asset and one Spread option (ATM call). The maturity of the options is three months. We consider two models. The first one is the BS model which is calibrated using ATM Vanilla options, whereas the second model is the Bahra model developped at the Bank of England. As for the BS model, we use a Normal copula. We have reported the univariate RND in Figures 7 and 8 and the multivariate RND in Figures 9 and 10. The parameter of the Normal copula is negative for the BS model, but positive for the Bahra model. So, the market anticipation corresponds to a negative dependence between asset returns for the BS model, whereas the market anticipation corresponds to a positive dependence for the Bahra model.

Remark 8 *It is necessary to verify that given a two-asset option price, there is only one corresponding parameter of copula to define an implied parameter. This is the case for the Spread option (* Rapuch *and* Roncalli *[2001]). Counterexamples like WorstOf call/put options are given in the same paper.*

[7] The margins correspond to the BS RND for a maturity of one month. The parameters are the following:: $S_1(t_0) = S_2(t_0) = 100$, $r = 5\%$, $\sigma_1 = 10\%$ and $\sigma_2 = 20\%$.

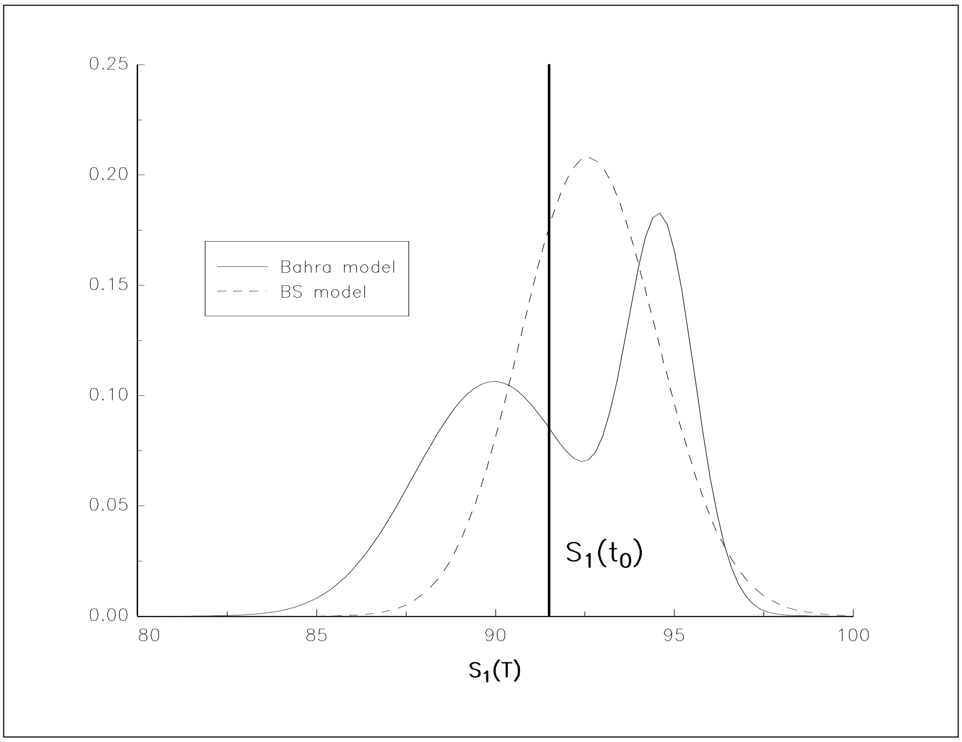

Figure 7: Calibrated RND for the first asset

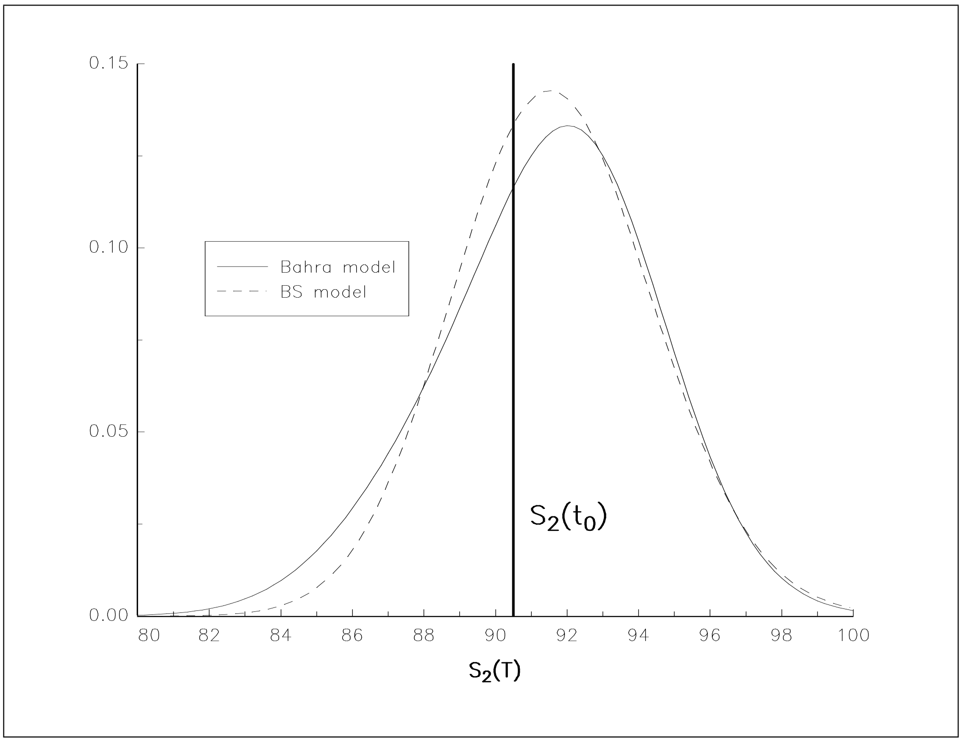

Figure 8: Calibrated RND for the second asset

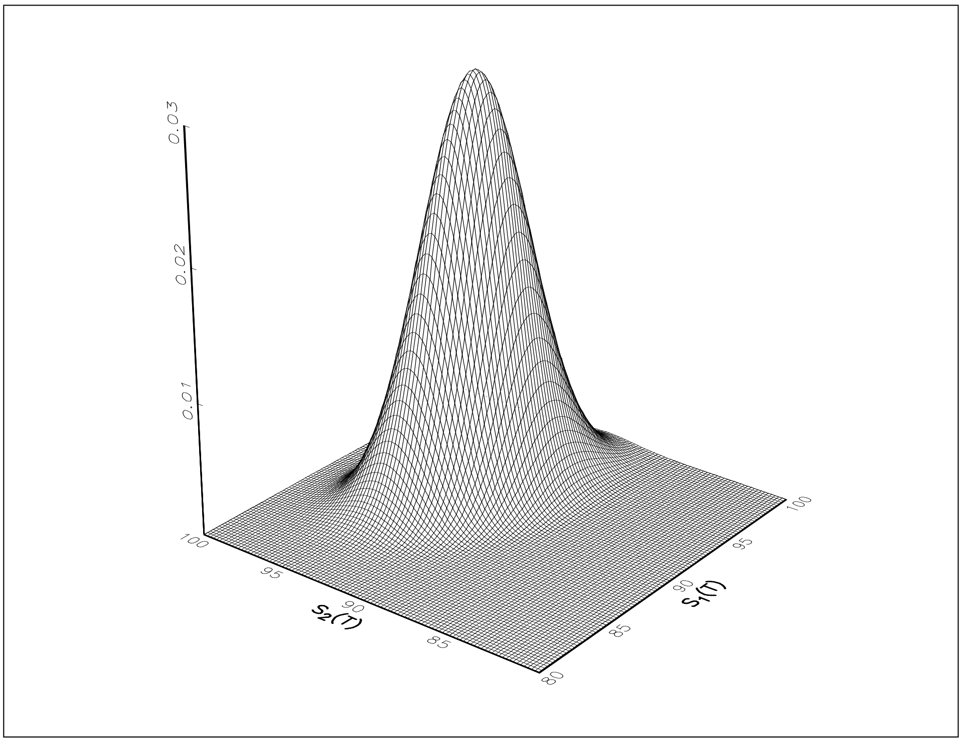

Figure 9: Calibrated multivariate RND with univariate BS margins

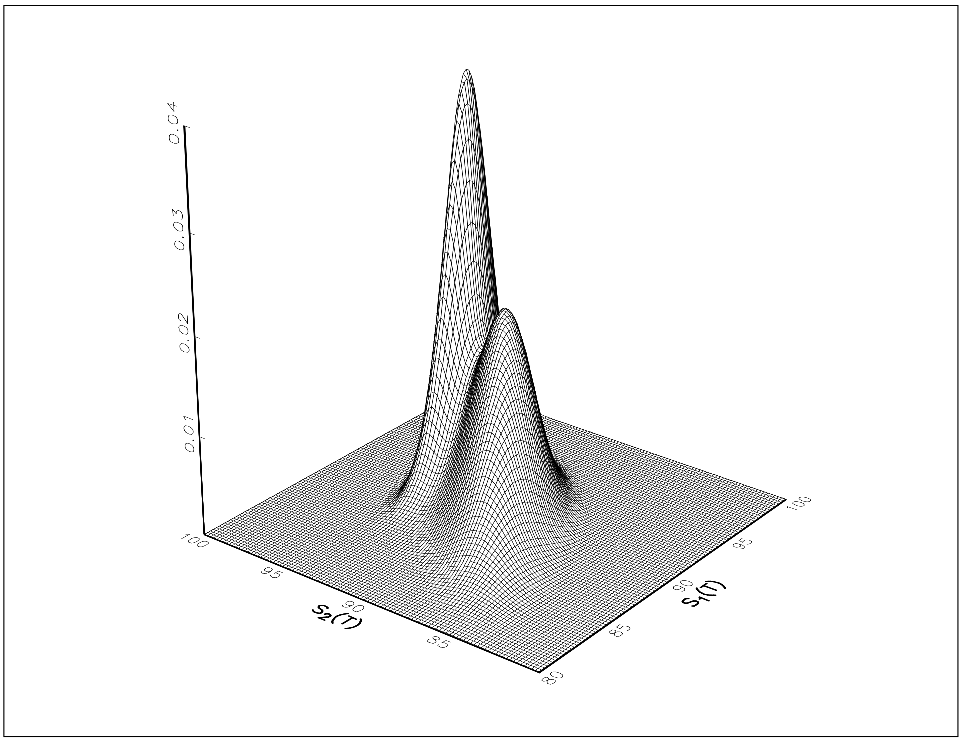

Figure 10: Calibrated multivariate RND with univariate Bahra margins

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