Modelling dependence for credit derivatives with copulas*

J.-F. Jouanin, G. Rapuch, G. Riboulet & T. Roncalli†
Groupe de Recherche Opérationnelle, Crédit Lyonnais, France

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Abstract

In this paper, we address the problem of incorporating default dependency in intensity-based credit risk models. Following the works of Li [2000], Giesecke [2001] and Schönbucher and Schubert [2001], we use copulas to model the joint distribution of the default times. Two approaches are considered. The first one consists in modelling the joint survival function directly with survival copulas of default times, whereas in the second approach, copulas are used to correlate the threshold exponential random variables. We compare these two approaches and give some results about their relationships. Then we try some simulations of simple products, such as first-to-defaults. Finally, we discuss the calibration issue according to Moody's diversity score.

1 Introduction

This paper tackles the problem of modelling correlated default events, which is a first step towards the valuation of credit risk derivatives. Recently, many models of the dependency of firms' defaults have been proposed. Here we briefly review those models and apply them to the simulation of very simple products that depend only on the first default time of several firms.

In the absence of correlated defaults, intensity models are widespread and commonly used for modelling the default process of a single company. To the practitioner's viewpoint, intensity models provide a quite flexible framework and can be easily fitted to actual term structure of credit spreads. Besides, their most important mathematical properties can be quickly retrieved by the use of Cox processes.

When one wants to incorporate default correlation mechanism in these models, many different ways have been tried. A naive idea is to correlate the intensity processes of different firms, which presents the advantage of keeping the models unchanged. But, an explicit derivation of the implied copula of the default times will show that high correlations cannot be attained in this framework. Many refinements of this approach exist, such as the infection models of Jarrow and Yu [1999].

Li [2000] proposes to use survival copulas to define the joint survival function of default times. In this approach, the specification of the stochastic dependence function is independent of the marginals of survival times. The calibration could be somewhat easy if information about historical default times is available (Hamilton, James and Webber [2001]). Following other ideas introduced by Giesecke [2001] and Schönbucher and Schubert [2001], we may also add some dependence between the triggers of the firms. More precisely, as a default occurs when the intensity process of a firm reaches a pre-specified trigger, which is an exponential random variable, the trick is to link the different thresholds with a copula. This approach has the advantage of splitting the distribution of each single intensity process and the joint law of the default triggers, in such a way that the calibration of individual intensities to term structures remains easy. More intricate will be the calibration of the parameters of the copula which models the dependence between all triggers.

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†*Corresponding author:* Groupe de Recherche Opérationnelle, Bercy-Expo — Immeuble Bercy Sud — 4è étage, 90 quai de Bercy — 75613 Paris Cedex 12 — France; *E-mail:* thierry.roncalli@creditlyonnais.fr

The rest of the paper is organized as follows. After recalling some well-known results about Cox processes and their use in intensity models, we try different methods to correlate the default events. We conclude with some simulation results and briefly discuss the calibration procedure.

2 Modelling defaults with Cox processes

For bold readers who are not scared of the stuff of 'the general theory of stochastic processes' such as dual predictable projections (we send the interested reader for example to Dellacherie and Meyer [1980] or Rogers and Williams [2000] for details and to Elliott, Jeanblanc and Yor [2000] for a corresponding account of intensity models), Cox processes will seem desperately useless to the understanding of default random times. Indeed the default time is often defined by

$$\tau := \inf \left\{ t : \int_0^t \lambda_s \, \mathrm{d}s \geq \theta \right\} \tag{1}$$

where θ is an exponential r.v. of parameter 1 and λ a nonnegative process called the intensity process. But it happens that Cox processes allow to retrieve the most important properties of default times in a very simple and somewhat elegant way. Considering a Cox process $\tilde{N}$, the default time is just

$$\tau := \inf \left\{ t \geq 0 : \tilde{N}_t > 0 \right\} \tag{2}$$

And the two definitions match as soon as λ is chosen to be the intensity of the Cox process $\tilde{N}$. Among many papers dealing with Cox processes, we only quote here Lando [1998] and Schönbucher [2000], without of course forgetting the seminal work of Brémaud [1981].

2.1 From Poisson towards Cox processes

First, we recall the definition of a standard Poisson process. In all the paper, we are given a complete filtered probability space $(\Omega, \mathcal{B}, (\mathcal{F}_t), \mathbb{P})$.

Definition 1 *After drawing a sequence (θ_n) of independent exponential r.v. of parameter 1, we let T_n be the partial sum of the first n terms of the sequence*

$$T_n = \sum_{i=1}^{n} \theta_i \tag{3}$$

and define the stochastic process

$$N_t := \sum_{n=1}^{\infty} \mathbf{1}_{\{T_n \leq t\}} \tag{4}$$

This process is a standard Poisson process of parameter 1.

It is not hard to check that N is a process with independent and stationary increments and such that for all t, N_t follows a Poisson distribution of parameter t.

The two properties of the Poisson process which are commonly used are the following: on the one hand, the process $M_t := N_t - t$ is a martingale (easily seen) called the *compensated Poisson process*; and, on the other, if we define a *counting process* as an RCLL integer-valued and non-decreasing process, N is a Poisson process *iff* it is a counting process which is also a Lévy process, the jumps of which are a.s. equal to 1 — see Revuz and Yor [1999].

Now we pick an RCLL non-decreasing function Λ such that $\Lambda_0 = 0$, $\Lambda_t < \infty$ for all t and $\Lambda_\infty = \infty$, and we consider the time-changed Poisson process

$$\bar{N}_t = N_{\Lambda_t} \tag{5}$$

This new process is still a process with independent increments, but here the law of $\bar{N}_t - \bar{N}_s$ is a Poisson distribution with parameter $\Lambda_t - \Lambda_s$ $(s \leq t)$. It is called an *inhomogeneous Poisson process* and Λ is called the *intensity*. The most often, we will assume that Λ admits a density λ, so that $\Lambda_t = \int_0^t \lambda_s \, \mathrm{d}s$. In that case, what we call the intensity is simply the density λ.

Finally we let the intensity Λ be stochastic and will thus obtain the so-called *Cox process* — former also known as *doubly stochastic Poisson process*. Once a sample of the process $\Lambda(\cdot, \omega)$ has been drawn, we draw an independent standard Poisson process and then build the subordinated process like above. Formally, conditionally to the knowledge of the intensity — that is the $\sigma-$field $\mathcal{F}_\infty^\Lambda = \sigma(\Lambda_t, \ t \geq 0)$ — the Cox Process $\tilde{N}$ is an inhomogeneous Poisson process of intensity Λ. For first reading, one can even assume that (θ_n) defining the Poisson process are independent from (Λ_t). It is worth noticing that the intensity can be recovered from the Cox process just by taking its expectation. In the remaining of this paper, we impose the following assumption:

Assumption 1 *We take an $(\mathcal{F}_t)-$adapted, non-negative and continuous process λ and set $\Lambda_t := \int_0^t \lambda_s \, \mathrm{d}s$ with a.s. $\int_0^t \lambda_s \, \mathrm{d}s < \infty$ for all $t \geq 0$ and $\Lambda_\infty = \infty$.*

We have reached the point where we can introduce the default time τ. Considering a Cox process $\tilde{N}$ with an intensity λ, we set

$$\tau := \inf \left\{ t \geq 0 : \tilde{N}_t > 0 \right\} \tag{6}$$

τ is a stopping time with respect to the filtration generated by the Cox process $\tilde{N}$ (but in almost cases it will not be a stopping time relatively to $(\mathcal{F}_t)$, so one has to be cautious when conditioning). In general, τ is no longer an exponential r.v. but we still have, using the independence of increments of N and conditioning on $\mathcal{F}_\infty \vee \mathcal{H}_t$ with $\mathcal{H}_t = \sigma(H_s, \ s \leq t)$ the filtration of the survival process $H_t = \mathbf{1}_{\{\tau > t\}}$:

$$\mathbf{1}_{\{\tau > t\}} \mathbb{P}(\tau > T \mid \mathcal{F}_\infty \vee \mathcal{H}_t) = \mathbf{1}_{\{\tau > t\}} \exp\left(- \int_t^T \lambda_s \, \mathrm{d}s \right) \tag{7}$$

The default process $\mathbf{1}_{\{\tau \leq t\}}$ is then nothing else but the Cox process stopped at τ, the compensated process of which reads:

$$L_t := \mathbf{1}_{\{\tau \leq t\}} - \int_0^{\tau \wedge t} \lambda_s \, \mathrm{d}s \tag{8}$$

is a martingale. We have thus easily retrieved the most important properties of the default time and process.

2.2 Pricing credit derivatives within intensity models

As an application of the intensity framework, we recall the well-known result for the pricing of a derivative security, when there is only one defaultable firm. We here borrow from Lando [1998].

We denote by $(\mathcal{F}_t)$ the filtration generated by all state variables (economic variables, interest rates, currencies, etc.) including the intensity process (λ_t). For example, when using a (multi-) factor interest rate model, we can use the factor(s) for also driving the intensity process λ in order to provide correlations between interest rates and the default process. Drawing now a Cox process of intensity λ and the default time τ, we define the 'true' public information at time t as the enlarged $\sigma-$field (made right-continuous if necessary)

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\tau > s, s \leq t) = \mathcal{F}_t \vee \mathcal{H}_t \tag{9}$$

A contingent claim of maturity T is thus a r.v. X (positive or bounded, so that all expectations will exist) measurable with respect to $\mathcal{G}_T$. Assuming here that $\mathbb{P}$ can be chosen as a martingale probability measure, the arbitrage price at time t of the claim X is $\mathbb{E}\left[\exp\left(- \int_t^T r_s \, \mathrm{d}s \right) X \mid \mathcal{G}_t \right]$. Basic example claims are a payment Y if no default has occurred before maturity, a flow of payments at some rate until T, or a recovery payment in case of default. We focus here on the first case, that is $X := Y\mathbf{1}_{\{\tau>T\}}$. We state here without proof the very important fact that, for any $\mathcal{G}_T-$measurable r.v. Y, we have $\mathbf{1}_{\{\tau>T\}}Y = \mathbf{1}_{\{\tau>T\}}\tilde{Y}$ with $\tilde{Y}$ an $\mathcal{F}_T-$measurable r.v. — see RUTKOWSKI [2000] for details. Thus we are allowed to suppose that Y is itself $\mathcal{F}_T-$measurable.

Proposition 1 *For a positive or bounded $\mathcal{F}_T-$measurable r.v. Y, we have on $\{\tau > t\}$:*

$$\mathbb{E}\left[\exp\left(-\int_t^T r_s\,\mathrm{d}s\right)\mathbf{1}_{\{\tau>T\}}Y \mid \mathcal{G}_t\right] = \mathbb{E}\left[\exp\left(-\int_t^T (r_s+\lambda_s)\,\mathrm{d}s\right)Y \mid \mathcal{F}_t\right] \tag{10}$$

Proof. We begin by using the inclusion $\mathcal{G}_t \subset \mathcal{F}_\infty \vee \mathcal{H}_t$ and have:

$$\mathbb{E}\left[\exp\left(-\int_t^T r_s\,\mathrm{d}s\right)\mathbf{1}_{\{\tau>T\}}Y \mid \mathcal{G}_t\right] = \mathbb{E}\left[\exp\left(-\int_t^T r_s\,\mathrm{d}s\right)\mathbb{E}\left[\mathbf{1}_{\{\tau>T\}} \mid \mathcal{F}_\infty \vee \mathcal{H}_t\right]Y \mid \mathcal{G}_t\right]$$

We then find using expression (7)

$$\mathbb{E}\left[\exp\left(-\int_t^T r_s\,\mathrm{d}s\right)\mathbf{1}_{\{\tau>T\}}Y \mid \mathcal{G}_t\right] = \mathbf{1}_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^T (r_s+\lambda_s)\,\mathrm{d}s\right)Y \mid \mathcal{G}_t\right] \tag{11}$$

Now it remains to replace conditioning on $\mathcal{G}_t$ with $\mathcal{F}_t$. Recall that τ is defined by $\tau = \inf\left\{t : \int_0^t \lambda_s\,\mathrm{d}s \geq \theta\right\}$ with θ an exponential r.v. of parameter 1 that happens to be independent of $\mathcal{F}_t$, we have

$$\mathbb{E}\left[\exp\left(-\int_t^T (r_s+\lambda_s)\,\mathrm{d}s\right)Y \mid \mathcal{F}_t \vee \sigma(\theta)\right] = \mathbb{E}\left[\exp\left(-\int_t^T (r_s+\lambda_s)\,\mathrm{d}s\right)Y \mid \mathcal{F}_t\right] \tag{12}$$

Finally noticing that we have the inclusions $\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{F}_t \vee \sigma(\theta)$ we get the desired result on re-conditioning last equation with respect to $\mathcal{G}_t$. ■

Similar equations for the cases of flows of payments or recovery are available in LANDO [1998]. We point out that it would be very unsatisfactory if we could not replace the conditioning on $\mathcal{G}_t$ with $\mathcal{F}_t$ because we do not know the dynamics of the state variables r_t, λ_t, etc. in the filtration $(\mathcal{G}_t)$ — this problem could be solved by using the theory of (progressive) enlargement of filtration, but we can afford disregarding it here.

2.3 A simple example with two firms

The preceding framework is readily generalized to the case of I defaultable firms. We consider thus I intensity processes λ^i, that may be correlated, and define the corresponding default times:

$$\tau_i := \inf\left\{t \geq 0 : \int_0^t \lambda_s^i\,\mathrm{d}s \geq \theta_i\right\} \tag{13}$$

where θ_i are independent exponential r.v. of parameter 1. We will show on a simple example that we cannot produce high correlations within this framework. Yet this example will be used in the following as a benchmark for comparing the different models with more dependence between defaults.

We will choose quadratic intensities $\lambda_t^i := (W_t^i)^2$ where $\mathbf{W} = \left(W^1, \ldots, W^I\right)$ is a vector of I correlated $(\mathcal{F}_t)$ Brownian motions — we shall note ρ^W for the correlation matrix. We also assume that there are no interest rates. Noting τ_i for the time of default of firm i, we are interested with the first default, that is the stopping time $\tau = \bigwedge_{i=1}^I \tau_i$. We will derive a closed-form formula for a product that pays one unit of money in case that $\tau > T$, that is no defaults have occurred. The price of this product is simply the survival probability of τ.

Proposition 2 *The joint survival function of the default times* $\boldsymbol{\tau} = (\tau_1, \tau_2)$ *is given by, for* $t_1 \geqslant t_2$ *:*

$$\begin{aligned}\mathbf{S}(t_1, t_2) &:= \mathbb{P}(\tau_1 > t_1, \tau_2 > t_2) \\ &= \left[\cosh\left(\sqrt{2}(t_1 - t_2)\right)\det\left(\cosh\left(t_2\sqrt{2\mathbf{D}}\right) + \Psi(t_1, t_2)(2\mathbf{D})^{-1/2}\sinh\left(t_2\sqrt{2\mathbf{D}}\right)\right)\right]^{-1/2} \quad (14)\end{aligned}$$

where we introduce the matrices $\mathbf{D} := \begin{pmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{pmatrix}$ *and* $\mathbf{Q} := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ *which are the matrices of eigenvalues and eigenvectors of* ρ^W *and* $\Psi(t_1, t_2) := \sqrt{2}\tanh\left(\sqrt{2}(t_1 - t_2)\right)\mathbf{D}^{1/2}\mathbf{Q}^\top\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{Q}\mathbf{D}^{1/2}$.

The rather technical proof is to be found in Appendix A. This formula enables us to perform numerical computations rather than time-consuming Monte Carlo simulations for the calculation of the correlations. For the margins, one has[1]

$$\mathbb{P}(\tau_1 > t) = \mathbb{P}(\tau_2 > t) = \frac{1}{\sqrt{\cosh\left(t\sqrt{2}\right)}} \quad (16)$$

and in the case $t_1 = t_2 = t$, we get:

$$\mathbf{S}_\tau(t) := \mathbb{P}(\tau_1 > t, \tau_2 > t) = \frac{1}{\sqrt{\cosh\left(t\sqrt{2(1-\rho)}\right)\left(\cosh\left(t\sqrt{2(1+\rho)}\right)\right)}} \quad (17)$$

Assuming that there are no interest rates, we have hence derived a closed-form formula for a product that pays one unit of money in case that $\tau_1 \wedge \tau_2 > T$, that is no defaults have occurred. It can be chosen that the joint distribution does not much depend on ρ. In Figure 1, we have represented the survival function $\mathbf{S}_\tau$ and the corresponding density f_τ:

$$f_\tau(t) = \frac{1}{2}\mathbf{S}_\tau^3(t)\left(\xi(t, \rho) + \xi(t, -\rho)\right) \quad (18)$$

where

$$\xi(t, \rho) = \sqrt{2(1-\rho)}\sinh\left(t\sqrt{2(1-\rho)}\right)\cosh\left(t\sqrt{2(1+\rho)}\right) \quad (19)$$

We also have reported two correlation measures. The first one is the *discrete default corrlation* which corresponds to $\operatorname{cor}(\vartheta_1(t), \vartheta_2(t))$ where $\vartheta_i(t) = \mathbf{1}_{\{\tau_i > t\}}$ whereas the second one is the correlation between the two random survival times $\operatorname{cor}(\tau_1, \tau_2)$, called the *survival time correlation* by Li [2000]. We remark that this simple model does not suffice to produce significant correlations between defaults[2]. We shall try other ways to incorporate more dependency in these models, in the next section.

3 Introducing dependence into intensity models

Now that we have recalled some basic results on intensity models and that we thus are able to model the default distribution of a single counterpart, we would like to model the dependence structure of the defaults of I firms. Copulas are user-friendly tools for modelling dependence and turn to be widespread used in finance. Two different approaches are proposed: putting the copula either directly on the single default times τ_i or on the exponential r.v. defining the underlying Poisson default processes, which will now called the thresholds or triggers.

[1] In this case and the next one, the result stems immediately from the Cameron-Martin formula:

$$\mathbb{E}\left[\exp\left(-\frac{1}{2}\alpha^2\int_0^t B_s^2\,\mathrm{d}s\right)\right] = (\cosh\alpha t)^{-1/2} \quad (15)$$

[2] See Appendix B for more significant results about the stochastic dependence of (τ_1, τ_2).

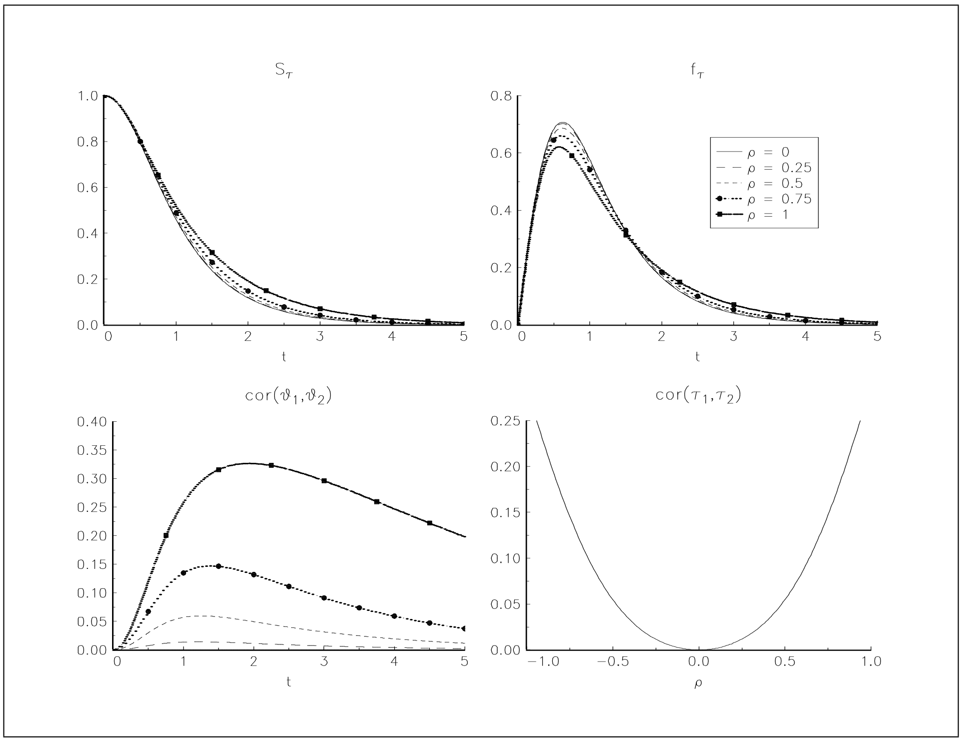

Figure 1: Influence of the correlation parameter on the first default time

3.1 The survival approach

Let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_I)$ be the random vector of default times. We define the survival function $\mathbf{S}_i$ of the default time τ_i as follows:

$$\mathbf{S}_i(t_i) = \mathbb{P}(\tau_i > t_i) \tag{20}$$

with

$$\tau_i := \inf\left\{t \geq 0 : \int_0^t \lambda_s^i \,\mathrm{d}s \geq \theta_i\right\} \tag{21}$$

The joint survival function $\mathbf{S}$ of the random vector $\boldsymbol{\tau}$ corresponds to

$$\mathbf{S}(t_1, \ldots, t_I) = \mathbb{P}(\tau_1 > t_1, \ldots, \tau_I > t_I) \tag{22}$$

If the intensity processes are independent, we obtain

$$\mathbf{S}(t_1, \ldots, t_I) = \prod_{i=1}^{I} \mathbf{S}_i(t_i) \tag{23}$$

If they are not independent, the previous equality does not hold. Using Sklar's theorem (see for example [8]), $\mathbf{S}$ has a copula representation

$$\mathbf{S}(t_1, \ldots, t_I) = \breve{\mathbf{C}}(\mathbf{S}_1(t_1), \ldots, \mathbf{S}_I(t_I)) \tag{24}$$

with $\breve{\mathbf{C}}$ a survival copula function. The main idea of Li [2000] is then to introduce directly the copula into the previous representation. The relevant filtration now to be considered for pricing securities is defined by:

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\tau_i > s, s \leq t, i = 1, \ldots, I) = \mathcal{F}_t \vee \mathcal{H}_t \tag{25}$$

For computing the price of a contingent claim, we are not able to simplify the conditional expectation, so that we have to proceed with Monte Carlo simulations.

3.2 The threshold approach

As above, we consider a family of I exponential random variables θ_i, and the time of default for firm i is defined as:

$$\tau_i := \inf \left\{ t \geq 0 : \int_0^t \lambda_s^i \, \mathrm{d}s \geq \theta_i \right\} \tag{26}$$

where λ^i is the intensity process of firm i. Here, for the dependency between the defaults, we directly link the distributions of the thresholds $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_I)$ with a copula $\mathbf{C}$. Let $\mathbf{F}$ be the joint distribution of $(\theta_1, \ldots, \theta_I)$ and let us denote $\mathbf{F}_i$ the marginals. We have

$$\mathbf{F}(x_1, \ldots, x_I) = \mathbf{C}(\mathbf{F}_1(x_1), \ldots, \mathbf{F}_I(x_I)) \tag{27}$$

Schönbucher and Schubert [2001] introduce the default countdown process γ_t^i as follows

$$\gamma_t^i := \exp\left(- \int_0^t \lambda_s^i \, \mathrm{d}s \right) \tag{28}$$

The time of default for firm i is then defined as:

$$\tau_i := \inf \left\{ t \geq 0 : \gamma_t^i \leq U_i \right\} \tag{29}$$

where $\mathbf{U} = (U_1, \ldots, U_I)$ are distributed according to the copula $\mathbf{C}'$. Remark that our framework is exactly the same as in Schönbucher and Schubert and we have $\breve{\mathbf{C}} = \mathbf{C}'$ because $\theta_i = -\ln U_i$. It comes that the copula $\mathbf{C}$ corresponds to the survival copula of $\mathbf{C}'$. Besides Assumption 1, we state the following:

Assumption 2 *$\boldsymbol{\theta}$ is independent from $\mathcal{F}_\infty$.*

As Assumption 1, this one will be useful, but there are indeed some minor drawbacks: we cannot generate an infection model as in Jarrow and Yu [1999] where e.g. the intensity of firm 2 depends on the default time of firm 1.

The relevant filtration now to be considered for pricing securities is again defined by:

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\tau_i > s, s \leq t, i = 1, \ldots, I) = \mathcal{F}_t \vee \mathcal{H}_t \tag{30}$$

In order to find the $(\mathcal{G}_t)$ −intensity of the default process, Schönbucher and Schubert [2001] also suppose that the copula $\mathbf{C}$ is twice differentiable. If we will not compute this intensity, the former hypthesis is not required. Within the threshold framework, we can restate Proposition 1 for the pricing of a credit derivative depending on firm i's default. We note $\tau = \tau_1 \wedge \ldots \wedge \tau_I$ the first-to-default and $B_t = \exp\left(\int_0^t r_s \, \mathrm{d}s\right)$ for the saving account.

Proposition 3 *For a positive or bounded $\mathcal{F}_T$−measurable r.v. Y, we have on $\{\tau > t\}$:*

$$\mathbb{E}\left[\frac{B_t}{B_T} \mathbf{1}_{\{\tau_i > T\}} Y \mid \mathcal{G}_t\right] = \mathbb{E}\left[\frac{B_t}{B_T} \frac{\breve{\mathbf{C}}\left(\exp\left(-\int_0^t \lambda_s^1 \, \mathrm{d}s\right), \ldots, \exp\left(-\int_0^T \lambda_s^i \, \mathrm{d}s\right), \ldots, \exp\left(-\int_0^t \lambda_s^I \, \mathrm{d}s\right)\right)}{\breve{\mathbf{C}}\left(\exp\left(-\int_0^t \lambda_s^1 \, \mathrm{d}s\right), \ldots, \exp\left(-\int_0^t \lambda_s^i \, \mathrm{d}s\right), \ldots, \exp\left(-\int_0^t \lambda_s^I \, \mathrm{d}s\right)\right)} Y \mid \mathcal{F}_t\right] \tag{31}$$

When the thresholds are independent — that is $\mathbf{C} := \mathbf{C}^\perp$ — we get the same formula as in Proposition 1 because $\breve{\mathbf{C}}^\perp = \mathbf{C}^\perp$.

Proof. Whatever the pricing formula looks unpleasant, the proof is quite the same than for Proposition 1. What changes is just the conditioning on $\mathcal{F}_\infty \vee \mathcal{H}_t$. Here we have:

$$\breve{\mathbf{C}}\left(\exp\left(-\int_0^t \lambda_s^1 \, \mathrm{d}s\right), \ldots, \exp\left(-\int_0^T \lambda_s^i \, \mathrm{d}s\right), \ldots, \exp\left(-\int_0^t \lambda_s^I \, \mathrm{d}s\right)\right) = \mathbb{P}(\tau_1 > t, \ldots, \tau_i > T, \ldots, \tau_I > t \mid \mathcal{F}_\infty) \tag{32}$$

and together we have

$$\mathbb{P}\left(\tau_1 > t, \dots, \tau_i > T, \dots, \tau_I > t \mid \mathcal{F}_\infty\right) = \mathbb{E}\left[\mathbf{1}_{\{\tau_1 > t\}} \times \cdots \times \mathbb{P}\left(\tau_i > T \mid \mathcal{F}_\infty \vee \mathcal{H}_t\right) \times \cdots \times \mathbf{1}_{\{\tau_I > t\}} \mid \mathcal{F}_\infty\right] \tag{33}$$

Then one uses again (32) with $T = t$ to get $\mathbb{P}\left(\tau_i > T \mid \mathcal{F}_\infty \vee \mathcal{H}_t\right)$ and the desired result. ■

3.3 Comparison between the two approaches

In the sequel of the paper, we always use the model of Section 2.3. To avoid misunderstanding, we denote now $\mathbf{C}^{\boldsymbol{\tau}}$ and $\mathbf{C}^{\boldsymbol{\theta}}$ respectively for the copula of the default times and the copula of the thresholds. In all examples, the Normal copula is used and the matrices of parameters are noted $\rho^{\mathbf{S}}$ and $\rho^{\boldsymbol{\theta}}$. $\rho^{\mathbf{W}}$ is the correlation matrix of the Brownian motions. In the case of two firms, the same notations $\rho^{\mathbf{S}}$, $\rho^{\boldsymbol{\theta}}$ and $\rho^{\mathbf{W}}$ are used for the (1,2) element of the corresponding matrices.

3.3.1 Relationships between the survival copula of the default times and the threshold copula

As in Gesiecke [2001], we can express $\mathbf{C}^{\boldsymbol{\tau}}$ as a function of $\mathbf{C}^{\boldsymbol{\theta}}$.

Proposition 4 *The relationship between $\breve{\mathbf{C}}^{\boldsymbol{\tau}}$ and $\breve{\mathbf{C}}^{\boldsymbol{\theta}}$ is given by*

$$\breve{\mathbf{C}}^{\boldsymbol{\tau}}\left(\mathbf{S}_1\left(t_1\right), \dots, \mathbf{S}_I\left(t_I\right)\right) = \mathbb{E}\left[\breve{\mathbf{C}}^{\boldsymbol{\theta}}\left(\exp\left(-\int_0^{t_1} \lambda_s^1 \,\mathrm{d}s\right), \dots, \exp\left(-\int_0^{t_I} \lambda_s^I \,\mathrm{d}s\right)\right)\right] \tag{34}$$

Proof. We have

$$\begin{aligned}
\mathbb{P}\left(\tau_1 > t_1, \dots, \tau_I > t_I \mid \mathcal{F}_0\right) &= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\tau_1 > t_1, \dots, \tau_I > t_I\}} \mid \mathcal{F}_\infty\right] \mid \mathcal{F}_0\right] \\
&= \mathbb{E}\left[\mathbb{P}\left(\tau_1 > t_1, \dots, \tau_I > t_I \mid \mathcal{F}_\infty\right) \mid \mathcal{F}_0\right] \\
&= \mathbb{E}\left[\mathbb{P}\left(\theta_1 > \int_0^{t_1} \lambda_s^1 \,\mathrm{d}s, \dots, \theta_I > \int_0^{t_I} \lambda_s^I \,\mathrm{d}s\right) \mid \mathcal{F}_0\right] \\
&= \mathbb{E}\left[\breve{\mathbf{C}}^{\boldsymbol{\theta}}\left(\exp\left(-\int_0^{t_1} \lambda_s^1 \,\mathrm{d}s\right), \dots, \exp\left(-\int_0^{t_I} \lambda_s^I \,\mathrm{d}s\right)\right)\right]
\end{aligned} \tag{35}$$

■

We notice that if the intensities are all deterministic and constant, the two copulas are equal. In this case, the univariate survival functions are given by

$$\mathbf{S}_i\left(t_i\right) = \mathbb{P}\left(\lambda^i t_i < \theta_i \mid \mathcal{F}_0\right) = \int_{\lambda^i t_i}^{\infty} \exp\left(-s\right) \,\mathrm{d}s = \exp\left(-\lambda^i t_i\right) \tag{36}$$

Incorporating the marginals into expression (24) we get

$$\mathbf{S}\left(t_1, \dots, t_I\right) = \breve{\mathbf{C}}^{\boldsymbol{\tau}}\left(\exp\left(-\lambda^1 t_1\right), \dots, \exp\left(-\lambda^I t_I\right)\right) \tag{37}$$

In the threshold approach, we have

$$\begin{aligned}
\Pr\left\{\tau_1 > t_1, \dots, \tau_I > t_I \mid \mathcal{F}_0\right\} &= \Pr\left\{\lambda^1 t_1 < \theta_1, \dots, \lambda^I t_I < \theta_I \mid \mathcal{F}_0\right\} \\
&= \breve{\mathbf{C}}^{\boldsymbol{\theta}}\left(\exp\left(-\lambda^1 t_1\right), \dots, \exp\left(-\lambda^I t_I\right)\right)
\end{aligned} \tag{38}$$

So, we verify that $\breve{\mathbf{C}}^{\boldsymbol{\tau}} = \breve{\mathbf{C}}^{\boldsymbol{\theta}}$ and $\mathbf{C}^{\boldsymbol{\tau}} = \mathbf{C}^{\boldsymbol{\theta}}$.

Under $\mathcal{F}_\infty$, a satisfactory feature is that the concordance order of the survival threshold copula implies the concordance order of the survival copula of default times:

$$\breve{\mathbf{C}}_1^{\boldsymbol{\theta}} \succ \breve{\mathbf{C}}_2^{\boldsymbol{\theta}} \Rightarrow \breve{\mathbf{C}}_1^{\boldsymbol{\tau}} \succ \breve{\mathbf{C}}_2^{\boldsymbol{\tau}} \tag{39}$$

However, we recall that the concordance order is not necessarily respected when mapping a copula to its corresponding survival copula except for some special cases (Georges, Lamy, Nicolas, Quibel and Roncalli [2001]). But we could also prove that

$$\mathbf{C}_1^{\boldsymbol{\theta}} \succ \mathbf{C}_2^{\boldsymbol{\theta}} \Rightarrow \mathbf{C}_1^{\boldsymbol{\tau}} \succ \mathbf{C}_2^{\boldsymbol{\tau}} \tag{40}$$

In the general case, we remark that if $\mathbf{C}^{\boldsymbol{\theta}} = \mathbf{C}^{\perp}$, $\mathbf{C}^{\boldsymbol{\tau}}$ is the product copula if and only if the intensity processes are uncorrelated. One of the main difference between the two approaches is that there are two sources of correlation in the threshold approach: correlation between the intensity processes and correlation between the random thresholds. To distinguish between them, Gesiecke [2001] calls them *macro-correlation* and *micro-correlation*.

Remark 1 *In the survival approach, correlation between intensity processes does not influence the joint survival function. So, it is sufficient to use independent intensity processes.*

3.3.2 Computational algorithms

The two approaches are very different in terms of computational techniques. We first consider the problem of the simulation of random times. In the case of the threshold approach, we have the following straightforward algorithm:

1. Simulate $(u_1, \ldots, u_I)$ from the copula $\mathbf{C}^{\boldsymbol{\theta}}$;
2. Compute $\theta_i = -\ln u_i$;
3. For each firm, simulate the intensity processes $\left(\lambda^i\right)$ to compute Λ_t^i. Stop when $\Lambda_t^i \geq \theta_i$ and take $\tau_i = t$.

In the case of the survival approach, we assume that the analytical expression of the margins $\mathbf{S}_i$ is known. The numerical algorithm is then very simple:

1. Simulate $(u_1, \ldots, u_I)$ from the survival copula $\breve{\mathbf{C}}^{\boldsymbol{\tau}}$;
2. Compute $\tau_i = \mathbf{S}_i^{(-1)}(u_i)$.

However, in most cases, the analytical expression of the margins $\mathbf{S}_i$ is unknown. That's why we have to modify the step 2:

2a. For each firm, simulate the intensity processes $\left(\lambda^i\right)$ to compute Λ_t^i. Stop when $\Lambda_t^i \geq \theta_i$ and take $\tau_i' = t$.

2b. Repeat (2a) n times and estimate the *empirical survival function* $\mathbb{S}_{i,n}$.

2c. Compute $\tau_i = \mathbb{S}_{i,n}^{(-1)}(u_i)$.

The key point of the convergence of the previous algorithm is of course the convergence rate of the empirical survival process $\sqrt{n}\left[\mathbb{S}_{i,n}(t) - \mathbf{S}_i(t)\right]$. In the general case, we can show that $\sup_t \left|\mathbb{S}_{i,n}(t) - \mathbf{S}_i(t)\right| \overset{\text{a.s.}}{\to} 0$ as $n \to 0$ (Shorack and Wellner [1986]). The loss of efficiency of the second algorithm depends on the regularity of the survival function and is sensitive to the tails.

In order to illustrate these algorithms, we set $I = 2$ and assume that the intensity processes are uncorrelated ($\rho^{\mathbf{W}} = 0$). The parameter of the Normal copula is equal to 50%. Figure 2 is a scatterplot of the simulated default times of the first firm in both approaches. In order to compare the two simulated series, we use the same uniform random numbers. We remark that the two methods give very different simulated default times, even if the margins are the same.

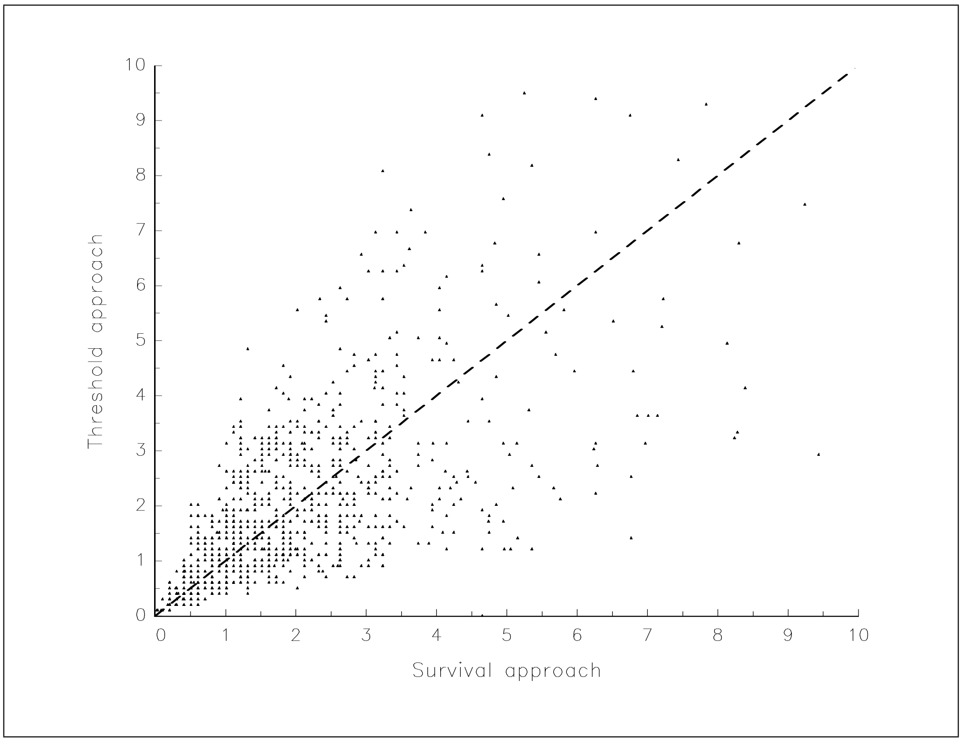

Figure 2: Simulation of the default time τ_1

3.3.3 Some numerical illustrations

In order to show that the two default correlation mechanisms are different, we compare the survival time correlations $\mathrm{cor}\,(\tau_1, \tau_2)$. Figure 3 represents the link between the survival time correlation and the parameter of the survival copula $\rho^{\mathbf{S}}$. In the survival approach, the joint survival function does not depend on the correlation mechanism of the intensity processes. So, we use independent intensity processes. In our example, we remark that $\mathrm{cor}\,(\tau_1, \tau_2)$ is close to the copula parameter $\rho^{\mathbf{S}}$ only for positive survival time correlations (the line with short dashes represents the equation $\mathrm{cor}\,(\tau_1, \tau_2) = \rho^{\mathbf{S}}$). Because the Normal 2-copula is a positively ordered family with respect to its parameter, we also retreive the well known results of the general case (Georges, Lamy, Nicolas, Quibel and Roncalli [2001]):

1. $\mathbf{C}_1^{\boldsymbol{\tau}} \succ \mathbf{C}_2^{\boldsymbol{\tau}} \Rightarrow \mathrm{cor}\,(\tau_1, \tau_2; \mathbf{C}_1^{\boldsymbol{\tau}}) \geq \mathrm{cor}\,(\tau_1, \tau_2; \mathbf{C}_2^{\boldsymbol{\tau}})$;
2. $\mathbf{C}^{\boldsymbol{\tau}} = \mathbf{C}^{\perp} \Rightarrow \mathrm{cor}\,(\tau_1, \tau_2) = 0$;
3. $\mathbf{C}^{\boldsymbol{\tau}} \succ \mathbf{C}^{\perp} \Rightarrow \mathrm{cor}\,(\tau_1, \tau_2) \geq 0$;
4. $\mathbf{C}^{\boldsymbol{\tau}} \prec \mathbf{C}^{\perp} \Rightarrow \mathrm{cor}\,(\tau_1, \tau_2) \leq 0$;
5. $\mathbf{C}^{\boldsymbol{\tau}} = \mathbf{C}^{-} \Rightarrow \mathrm{cor}\,(\tau_1, \tau_2) > -1$;

In our example, we notice that even if $\mathbf{C}^{\boldsymbol{\tau}}$ is the lower Fréchet copula (the survival times are said countermonotonic), the survival time correlation is far from -1. In Figure 3, we consider the threshold method. In this case, both the threshold copula $\mathbf{C}^{\boldsymbol{\theta}}$ and the stochastic dependence between the intensity processes influence the copula $\mathbf{C}^{\boldsymbol{\tau}}$ of the survival times. In our case, the dependence between the intensities is introduced by correlating the Brownian motions. It results that the previous assertions are not necessarily true if we replace $\mathbf{C}^{\boldsymbol{\tau}}$ by $\mathbf{C}^{\boldsymbol{\theta}}$. In the general case, only the first and fifth assertion holds — $\mathbf{C}_1^{\boldsymbol{\theta}} \succ \mathbf{C}_2^{\boldsymbol{\theta}} \Rightarrow \mathrm{cor}\left(\tau_1, \tau_2; \mathbf{C}_1^{\boldsymbol{\theta}}\right) \geq \mathrm{cor}\left(\tau_1, \tau_2; \mathbf{C}_2^{\boldsymbol{\theta}}\right)$ and $\mathbf{C}^{\boldsymbol{\theta}} = \mathbf{C}^{-} \Rightarrow \mathrm{cor}\,(\tau_1, \tau_2) > -1$.

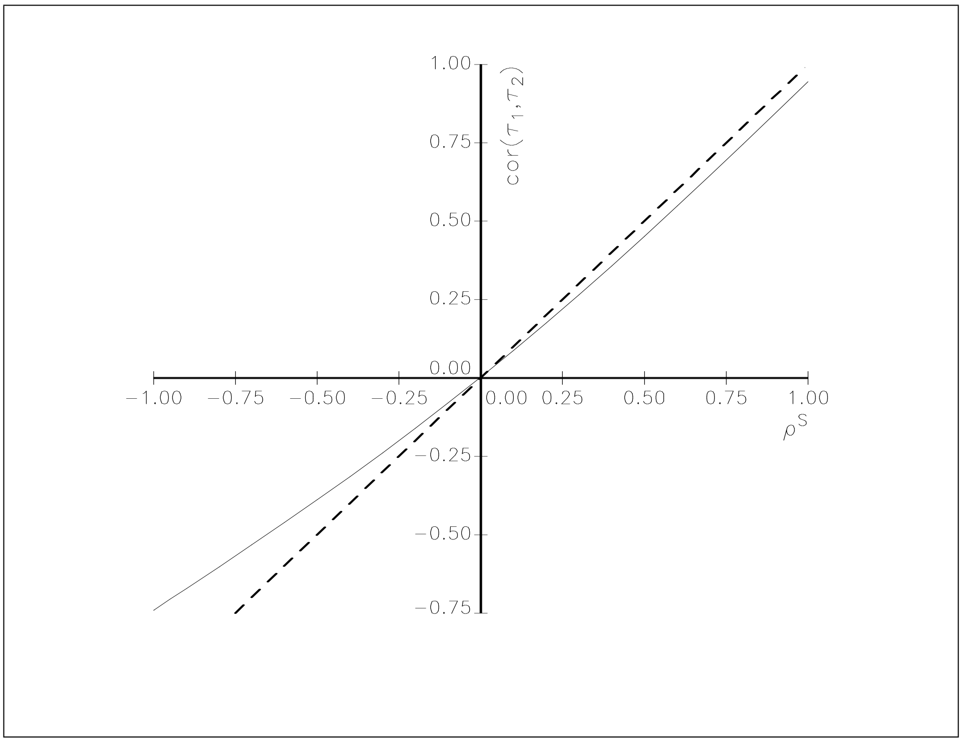

Figure 3: Relationship between the parameter $\rho^{\mathbf{S}}$ and the survival time correlation

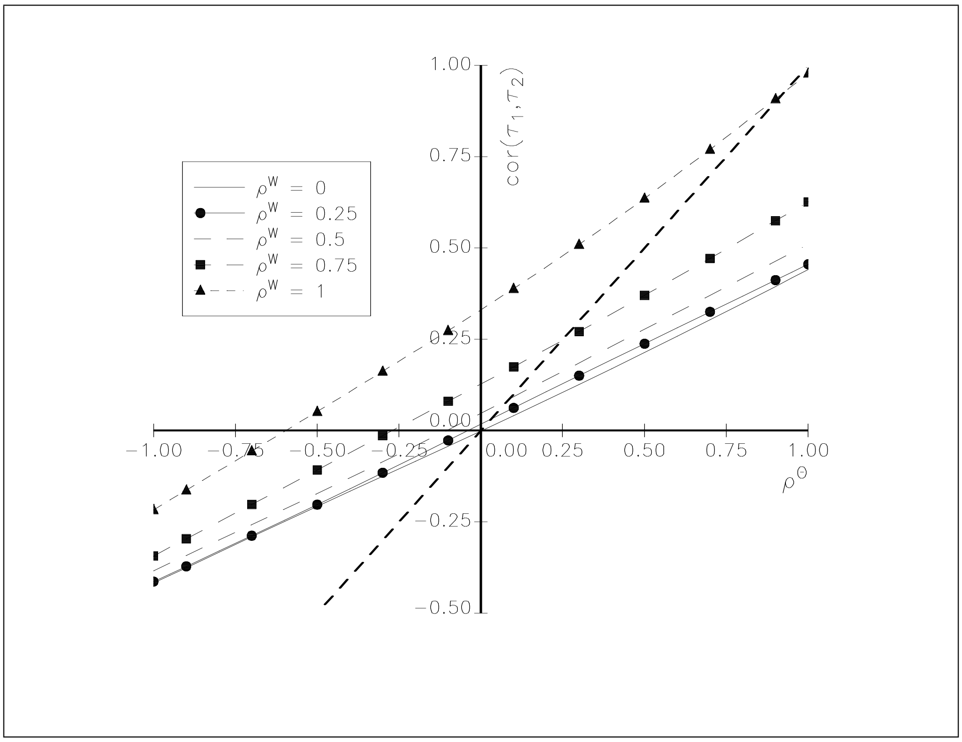

Figure 4: Relationship between the parameter $\rho^{\boldsymbol{\theta}}$ and the survival time correlation

4 Simulation results

We carry out here some simulations of very simple multi-firm credit derivatives. They will only depend on the time of the first default τ — if any — before T. We compute a first-to-default, which is a product that pays one unity of cash at maturity in case of a default.

4.1 Payoff at maturity

From the definition of the first-to-default, we write for the price at time $t < T$ on $\{\tau > t\}$:

$$\mathbb{E}\left[\exp\left(-\int_t^T r_s\,\mathrm{d}s\right)\mathbf{1}_{\{\tau\leq T\}}\mid\mathcal{G}_t\right] = B\left(t,T\right) - \mathbb{E}\left[\exp\left(-\int_t^T r_s\,\mathrm{d}s\right)\mathbf{1}_{\{\tau>T\}}\mid\mathcal{G}_t\right] \tag{41}$$

where $B\left(t,T\right)$ is the default-free zero-coupon price of maturity T at time t. We therefore only focus on the last expectation. In the case of the survival approach, we cannot get simpler expressions. In the case of the threshold approach, we can simplify that expectation. Adapting the proof of Proposition 3, we find, still on $\{\tau > t\}$:

$$\mathbb{E}\left[\frac{B_t}{B_T}\mathbf{1}_{\{\tau>T\}}\mid\mathcal{G}_t\right] = \mathbb{E}\left[\frac{B_t}{B_T}\frac{\breve{\mathbf{C}}^{\boldsymbol{\theta}}\left(\exp\left(-\int_0^T\lambda_s^1\,\mathrm{d}s\right),\ldots,\exp\left(-\int_0^T\lambda_s^I\,\mathrm{d}s\right)\right)}{\breve{\mathbf{C}}^{\boldsymbol{\theta}}\left(\exp\left(-\int_0^t\lambda_s^1\,\mathrm{d}s\right),\ldots,\exp\left(-\int_0^t\lambda_s^I\,\mathrm{d}s\right)\right)}\mid\mathcal{F}_t\right] \tag{42}$$

To show the importance of the dependence on the price of the first-to-default, we have computed the bounds of the option prices and reported them in Figure 5. We point out that depending on the maturity, the range of the price may be very large.

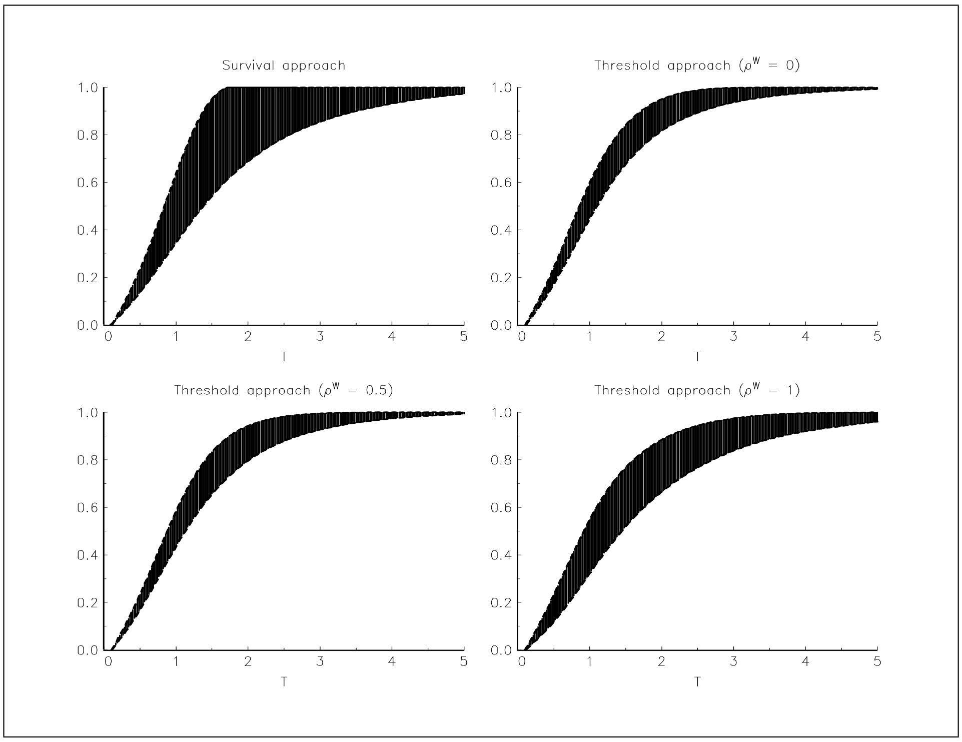

Figure 5: Bounds of the first-to-default price (payoff at maturity)

4.2 Payoff at default

For the threshold method, in the case of a payment at default, we need first to find the conditional density of τ given $\mathcal{F}_\infty \vee \mathcal{H}_t$, that is

$$\zeta_t(\mathrm{d}v) = -\frac{\partial}{\partial v}\mathbb{P}(\tau > v \mid \mathcal{F}_\infty \vee \mathcal{H}_t)\,\mathrm{d}v = -\frac{\partial}{\partial v}\frac{\breve{\mathbf{C}}^{\boldsymbol{\theta}}\left(\exp\left(-\int_0^v \lambda_s^1\,\mathrm{d}s\right),\ldots,\exp\left(-\int_0^v \lambda_s^I\,\mathrm{d}s\right)\right)}{\breve{\mathbf{C}}^{\boldsymbol{\theta}}\left(\exp\left(-\int_0^t \lambda_s^1\,\mathrm{d}s\right),\ldots,\exp\left(-\int_0^t \lambda_s^I\,\mathrm{d}s\right)\right)}\,\mathrm{d}v \tag{43}$$

Then we have on $\{\tau > t\}$

$$\begin{aligned}\mathbb{E}\left[\exp\left(-\int_t^\tau r_s\,\mathrm{d}s\right)\mathbf{1}_{\{\tau\leqslant T\}} \mid \mathcal{G}_t\right] &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(-\int_t^\tau r_s\,\mathrm{d}s\right)\mathbf{1}_{\{\tau\leqslant T\}} \mid \mathcal{F}_\infty \vee \mathcal{H}_t\right] \mid \mathcal{G}_t\right] \\ &= \mathbb{E}\left[\int_t^T \zeta_t(\mathrm{d}v)\exp\left(-\int_t^v r_s\,\mathrm{d}s\right) \mid \mathcal{F}_t\right] \end{aligned}\tag{44}$$

In the case of the product copula, one has for example:

$$\mathbb{E}\left[\exp\left(-\int_t^\tau r_s\,\mathrm{d}s\right)\mathbf{1}_{\{\tau\leqslant T\}} \mid \mathcal{G}_t\right] = \mathbb{E}\left[\int_t^T \left(\lambda_v^1 + \ldots + \lambda_v^I\right)\exp\left(-\int_t^v \left(r_s + \lambda_s^1 + \ldots + \lambda_s^I\right)\mathrm{d}s\right)\mathrm{d}v \mid \mathcal{F}_t\right] \tag{45}$$

Figure 6 represents the bounds of the first-to-default in the case where the intensities are independent[3]. Moreover, we assume a constant deterministic interest rate r. As previously, the impact of the dependence could be important on the prices.

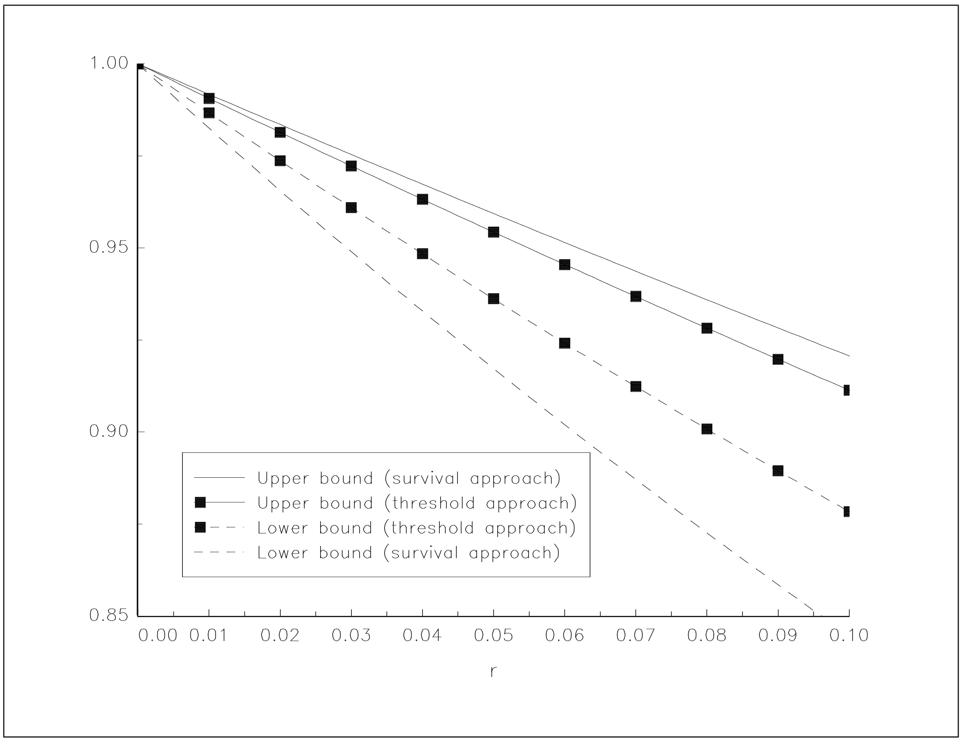

Figure 6: Bounds of the first-to-default price (payoff at default)

[3] T is set to infinity.

5 Calibration issues

Although it seems obvious to calibrate the dependence function between intensities using CDS or risky bonds prices, useful and reliable market information about joint thresholds distribution is too much rare. That is why practicioners assume a pre-specified contagion mechanism which enables them to cope with joint default probability issue. Those *a priori* models may rely either on economic intuition or statistical arguments. For further information, interested readers could see DAVIS and LO [2000], AVALLANEDA and WU [2000].

5.1 Using survival time correlations

We first assume that intensity processes have been previously estimated. The problem now is the calibration of the stochastic dependence function, that is the copula $\mathbf{C}^{\boldsymbol{\tau}}$ or $\mathbf{C}^{\boldsymbol{\theta}}$. For that, we could use historical default probabilities. For example, one may first think to calibrate the copula using discrete default correlations $\operatorname{cor}\left(\mathbf{1}_{\{\tau_i>T\}}, \mathbf{1}_{\{\tau_j>T\}}\right)$. This method has severe shortcomings because the margins are Bernoulli (see MARSHALL [1996] for a discussion). More satisfactory is the use of survival time correlations, which can be estimated by aggregating data (HAMILTON, JAMES and WEBBER [2001]). For example, if we use the previous model, the calibration of given survival time correlations is reported in Figure 7 — we use again a Normal copula. We remark that the calibration step may fail. For example, if we assume that $\operatorname{cor}(\tau_1, \tau_2)$ is equal to 50%, there is no corresponding copula function if the intensity processes are independent.

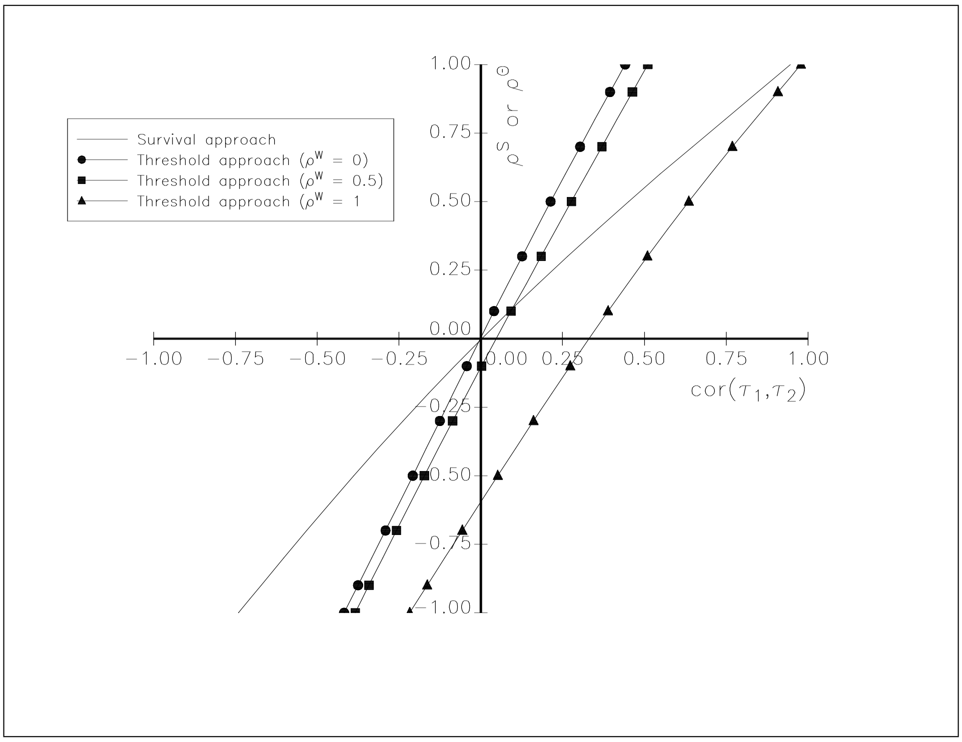

Figure 7: Calibration of the survival time correlations

We may have to find original methods that are based on the practice of the credit market rather than mimicking statistical methods that are never used by the practitioners. In the next paragraph, we propose one method based on the diversity score of Moody's.

5.2 Using Moody's diversity score

Recent waves of securitization on credit market may look like an efficient source of information. Indeed, returns of the different *tranches* of CDO are relevant to the default contagion in a pool of credits. Each *tranche* is noted following the Moody's *Binomial Expansion Technique*. That approach is based on the assumption that the distribution of the number of defaults among I risky issuers of the same sector could be summarized using only D independant issuers, i.e.

$$\sum_{i=1}^{I} \mathbf{1}_{\{\tau_i \leq t\}} \overset{law}{=} \frac{I}{D} \sum_{d=1}^{D} \mathbf{1}_{\{\bar{\tau}_i \leq t\}} \tag{46}$$

where $\left(\mathbf{1}_{\{\tau_1 \leq t\}}, \ldots, \mathbf{1}_{\{\tau_I \leq t\}}\right)$ are dependent Bernoulli random variables with parameters p_i ($i \in \{1, \ldots, I\}$) and $\left(\mathbf{1}_{\{\bar{\tau}_1 \leq t\}}, \ldots, \mathbf{1}_{\{\bar{\tau}_D \leq t\}}\right)$ are *i.i.d.* Bernoulli random variables with parameter p. According to Moody's, D is computed in order to match the two first moments on both sides of (46) on empirical observations. One may present in Table 1 the correspondence between the number of firms in a same sector and the *diversity score* D. Thus, in a first step, it may appear quite legitimate to calibrate the copula which linked

Number of firms	Diversity score D
1	1.00
2	1.50
3	2.00
4	2.33
5	2.67
6	3.00
7	3.25
8	3.50
9	3.75
10	4.00
>10	evaluated on a case-by-case basis

Table 1: Moody's diversity score

the thresholds to that market consensus. Since Moody's technique is based on the knowledge of the default probability one has to condition the expectations by $\mathcal{F}_\infty$ during calibration procedure[4]. The two first moments procedure induces the following equalities

$$Ip = \mathbb{E}\left[\sum_{i=1}^{I} \mathbf{1}_{\{\tau_i \leq t\}} \mid \mathcal{F}_\infty\right] = \sum_{i=1}^{I} p_i \tag{48}$$

and

$$\begin{aligned} \frac{I^2 p(1+(D-1)\,p)}{D} &= \mathbb{E}\left[\left(\sum_{i=1}^{I} \mathbf{1}_{\{\tau_i \leq t\}}\right)^2 \mid \mathcal{F}_\infty\right] \\ &= \sum_{i=1}^{I} p_i + 2\sum_{i<j} \mathbf{C}^{\boldsymbol{\theta}}\left(1, \ldots, p_i, \ldots, p_j, \ldots, 1\right) \end{aligned} \tag{49}$$

Here, one may remark that for Normal copula (like for most of copula functions) $\mathbf{C}^{\boldsymbol{\theta}}\left(1, \ldots, p_i, \ldots, p_j, \ldots, 1\right)$ with the matrix of parameters $\boldsymbol{\rho}$ is simply the Normal 2-copula $\mathbf{C}^{\boldsymbol{\theta}}\left(p_i, p_j\right)$ with parameter $\rho_{i,j}$.

Let us consider some illustrations. We first suppose that $\mathbf{C}^{\boldsymbol{\theta}}$ is a Normal copula with a constant matrix

[4] In this case, we have

$$p_i = 1 - \exp\left(-\int_0^t \lambda_s^i \, \mathrm{d}s\right) \tag{47}$$

where t is the time horizon.

of parameters $\boldsymbol{\rho}$

$$\boldsymbol{\rho} = \begin{bmatrix} 1 & \rho & \cdots & \rho & \rho \\ & 1 & \ddots & \vdots & \vdots \\ & & 1 & \rho & \rho \\ & & & 1 & \rho \\ & & & & 1 \end{bmatrix}$$

and that the vectors of default probabilities $\mathbf{p}$ is a constant vector. In Figure 8, we have represented the relationship between the parameter ρ and the implied diversity score D computed thanks to equations (48) and (49). We notice that the link between $\frac{D}{I}$ and ρ is highly dependent on the size of the credit portfolio. Given diversity scores, we can compute the implied parameter ρ of the threshold copula $\mathbf{C}^{\boldsymbol{\theta}}$ (see Figures 9 and 10). In Figure 11, we compare the diversity score with these computed using Moody's model:

$$D_{\text{Moody's}} = \frac{-1 + \sqrt{1 + 8I}}{2} \tag{50}$$

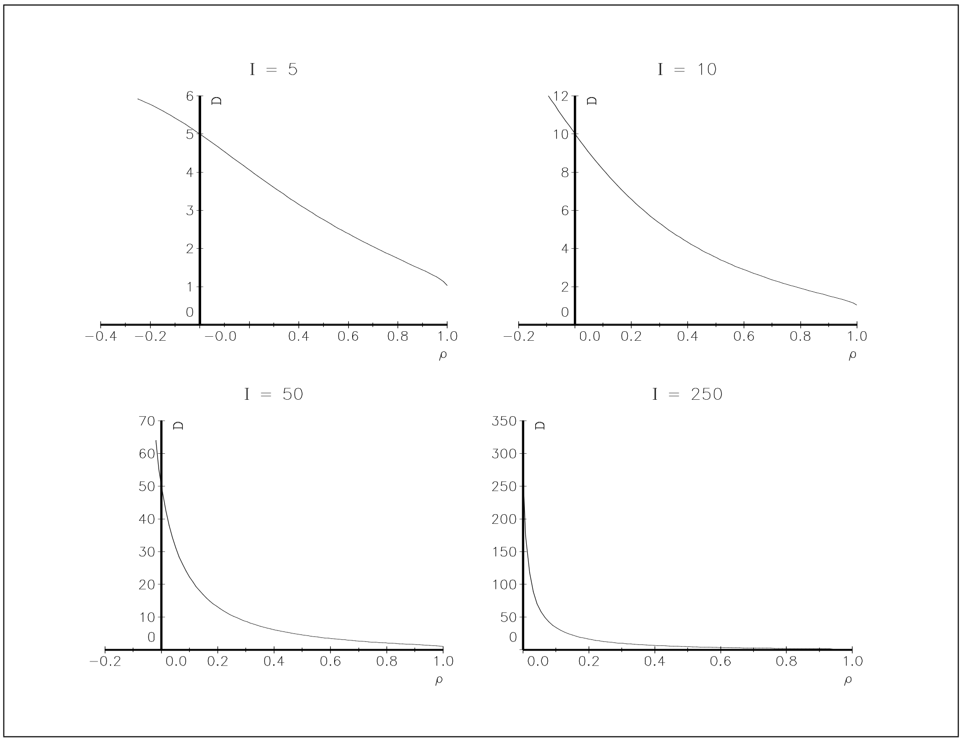

Figure 8: Relationship between ρ and D ($p_i = 5\%$)

We remark that three elements influence the diversity score: the choice of the copula $\mathbf{C}^{\boldsymbol{\theta}}$, the default probabilities p_i and the number of firms I. To illustrate the impact of the copula $\mathbf{C}^{\boldsymbol{\theta}}$, we have represented the diversity score for the previous Normal copula and the following Cook-Johnson copula:

$$\mathbf{C}(u_1, \ldots, u_I; \alpha) = \left(\sum_{i=1}^{I} u_i^{-\alpha} - (N+1) \right)^{-\alpha^{-1}} \tag{51}$$

In order to compare the results, we consider the concordance measure Kendall's tau. In Figure 12, we remark that even if the two copulas have the same Kendall's tau, diversity scores may be very different.

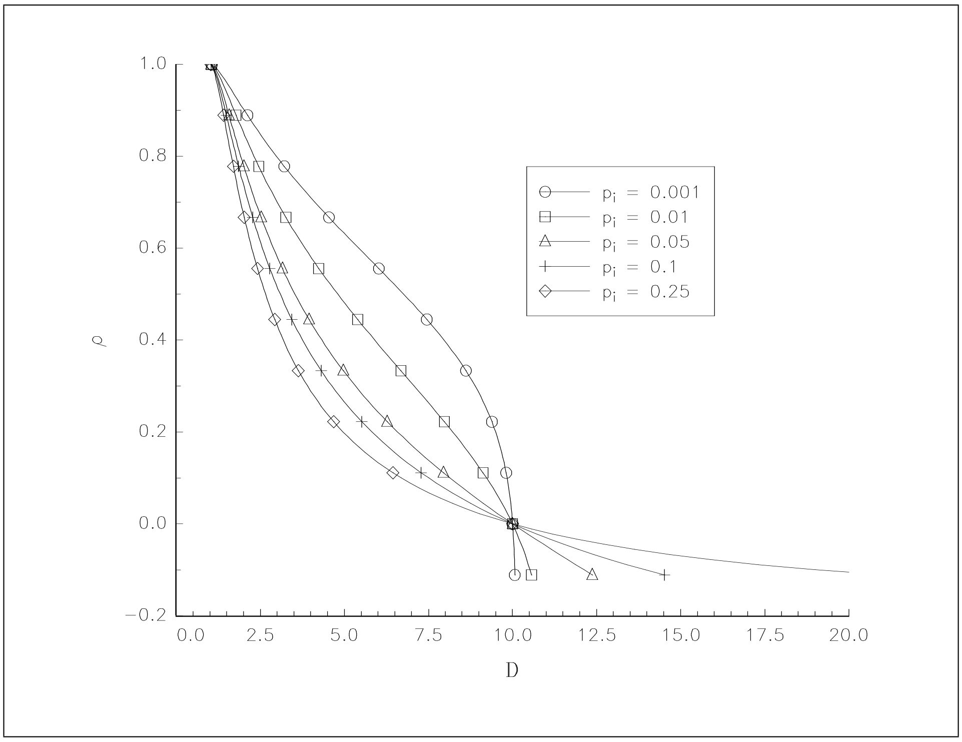

Figure 9: Implied parameter ρ to diversity score D ($I = 10$)

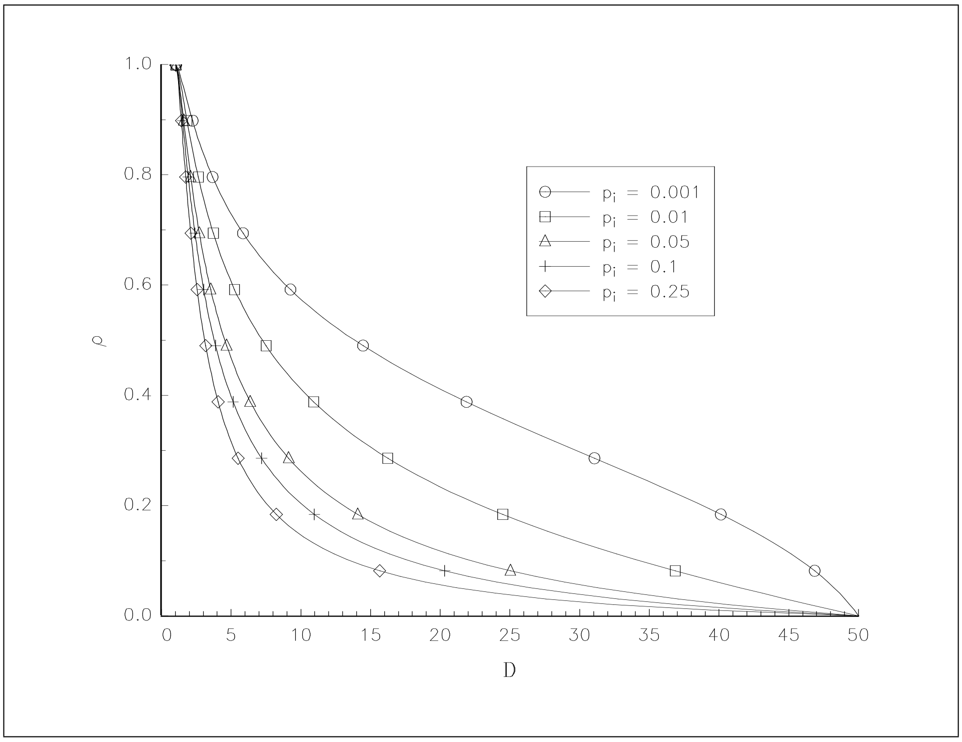

Figure 10: Implied parameter ρ to diversity score D ($I = 50$)

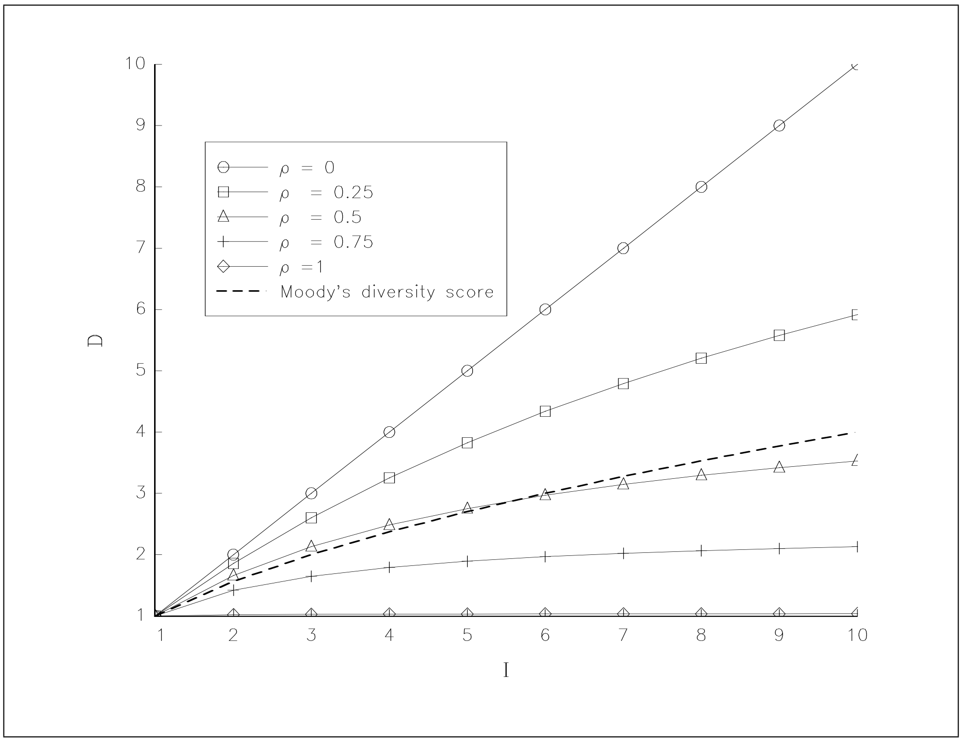

Figure 11: Relationship between the diversity score D and the number of firms I ($p_i = 5\%$)

For example, if Kendall's tau is equal to 20%, we get $D = 3.24$ for the Cook-Johnson copula whereas D is 7.31 for the Normal copula. It stems from the fact that $\mathbf{C}^{\boldsymbol{\theta}}(p_i, p_j)$ is equal to 0.729% for the Normal copula whereas it is 1.581% for the Cook-Johnson copula. If we use bigger default probabilities ($p_i > 0.05$), the difference between the diversity scores is smaller. But if the default probabilities are very small, the difference becomes very large. For example, if Kendall's tau is equal to 50%, D now equals 3.49 and 18.47 in the case $p_i = 0.1\%$. Indeed these results are related to the lower tail dependence of the copula function $\lambda_L^{\boldsymbol{\theta}}$. Let $\lambda_L^{\boldsymbol{\theta}}(u) = \mathbf{C}^{\boldsymbol{\theta}}(u,u)/u$ be the quantile dependent measure (Coles, Currie and Tawn [1999]). If we assume that the default probabilities p_i are the same, we have the following relationship

$$D = \frac{I(1-p_i)}{(1-Ip_i) + (I-1)\lambda_L^{\boldsymbol{\theta}}(p_i)} \tag{52}$$

The limit case ($p_i \to 0$) is then

$$D = \frac{I}{1 + (I-1)\lambda_L^{\boldsymbol{\theta}}} \tag{53}$$

If $\lambda_L^{\boldsymbol{\theta}}$ is equal respectively either to 0 or 1, we retrieve the well-known results: $D = I$ and $D = 1$. If the size of the credit portfolio is large ($I \to \infty$), D is equal to $1/\lambda_L^{\boldsymbol{\theta}}$. These results explain the big impact of the choice of the copula on the calibration issues for rare credit events (for example, for bonds which are rated AAA or AA). **The choice of the Normal copula is also not relevant with such default probabilities**.

Let us now study the impact of the default probabilities. We suppose that $p_i = 5\%$ for $i = 1, \ldots, I-1$ and evaluate the impact of the default probability p_I of the last issuer on the diversity score D. Because we have here a constant matrix of parameters for the Normal copula, we obtain understandable results[5] (Figure 13).

[5] Remark that the minimum is not reached at $p_I = 5\%$ because $\mathbf{C}^{\boldsymbol{\theta}} \neq \mathbf{C}^{\perp}$.

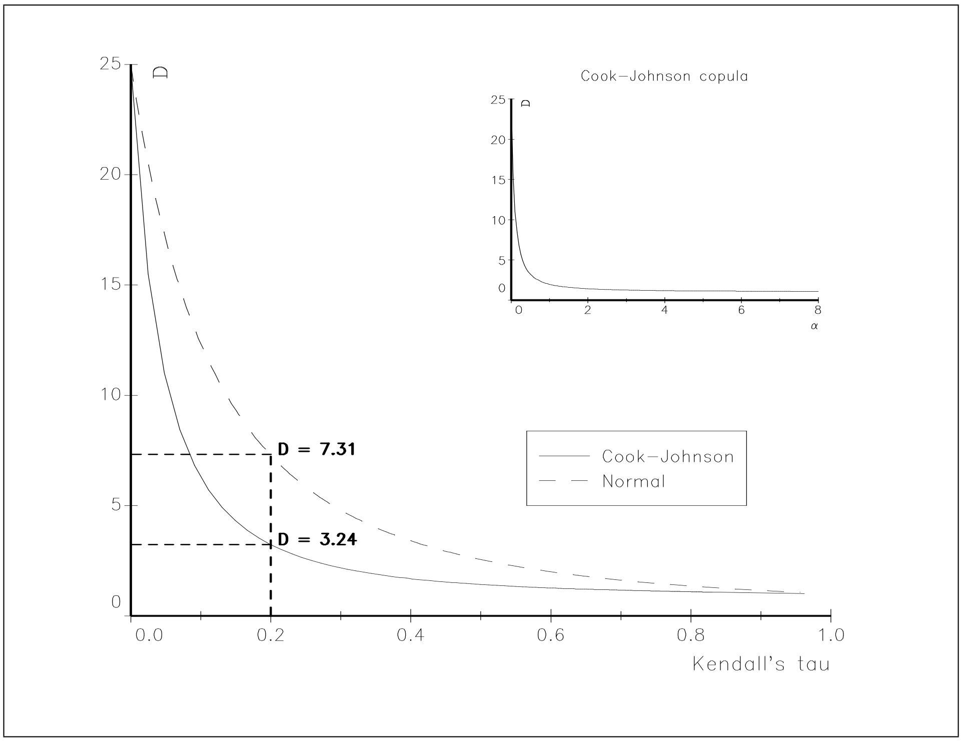

Figure 12: Difference between the Normal copula and the Cook-Johnson copula ($p_i = 5\%$, $I = 25$)

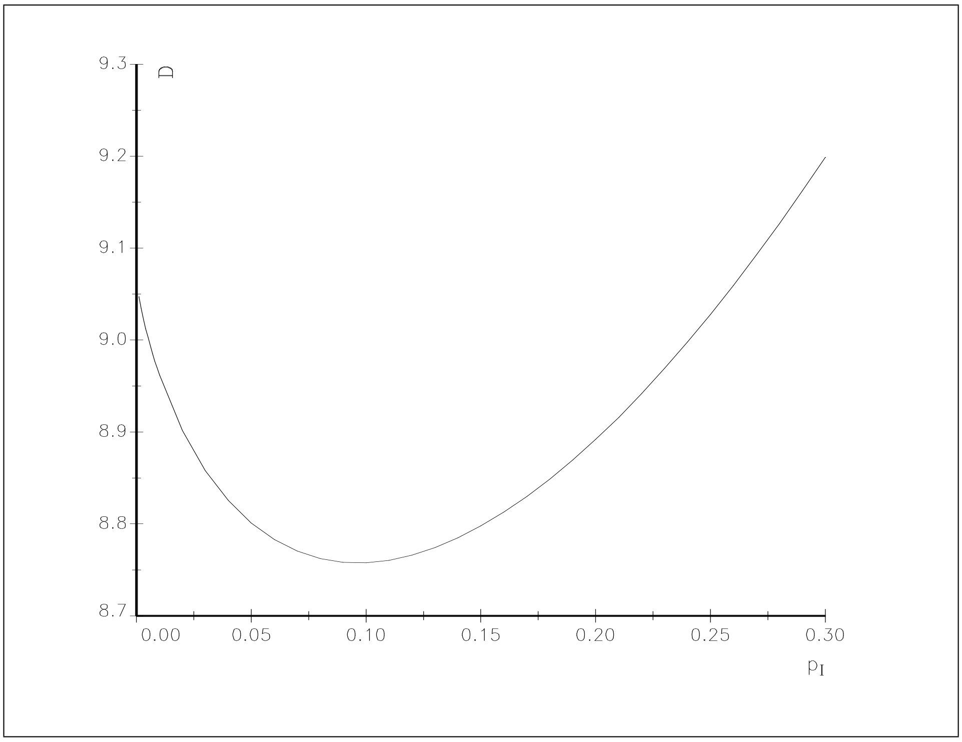

Figure 13: Influence of the default probability p_I ($p_i = 5\%$, $I = 25$, $\rho = 25\%$)

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A Proof of Proposition 2

We compute here the joint survival function of the default times (τ_1, τ_2) given in Proposition 2. Conditioning on $\mathcal{F}_\infty$ and using the independence of the exponentials r.v. (θ_1, θ_2), we have

$$\mathbb{P}(\tau_1 > t_1, \tau_2 > t_2) = \mathbb{E}\left[\mathbb{P}(\tau_1 > t_1, \tau_2 > t_2 \mid \mathcal{F}_\infty)\right] = \mathbb{E}\left[\exp\left(-\int_0^{t_1} \left(W_s^1\right)^2 \mathrm{d}s - \int_0^{t_2} \left(W_s^2\right)^2 \mathrm{d}s\right)\right] \tag{54}$$

As the default times have the same margins, we can assume without any restriction that $t_1 \geqslant t_2$. First, we condition on $\mathcal{F}_{t_2}$ and, together with the Markov property for W^1, use the following generalization of the Cameron-Martin formula. For any Brownian motion B, we have (Revuz and Yor [1999], p. 445):

$$\mathbb{E}_x\left[\exp\left(-\frac{1}{2}\alpha^2 \int_0^t B_s^2 \,\mathrm{d}s\right)\right] = \left(\cosh(\alpha t)\right)^{-1/2} \exp\left(-\frac{x^2}{2}\alpha \tanh \alpha t\right) \tag{55}$$

Hence we get:

$$\begin{aligned}\mathbb{P}(\tau_1 > t_1, \tau_2 > t_2) &= \left(\cosh\left(\sqrt{2}(t_1 - t_2)\right)\right)^{-1/2} \times \\ &\quad \mathbb{E}\left[\exp\left(-\int_0^{t_2} \left(\left(W_s^1\right)^2 + \left(W_s^2\right)^2\right) \mathrm{d}s\right) - \frac{\left(W_{t_2}^1\right)^2}{2}\sqrt{2}\tanh\left(\sqrt{2}(t_1 - t_2)\right)\right]\end{aligned} \tag{56}$$

Now we diagonalize the correlation matrix ρ^W. This allows us to write $\mathbf{W} := \mathbf{Q}(\mathbf{D})^{\frac{1}{2}}\mathbf{Z}$ where $\rho^W = \mathbf{Q}\mathbf{D}\mathbf{Q}^\top$ with $\mathbf{D}$ the diagonal matrix of eigenvalues, $\mathbf{Q}$ an orthogonal matrix and $\mathbf{Z}$ a standard 2-dimensional Brownian motion. To be explicit, we write down $\mathbf{D}$ and $\mathbf{Q}$:

$$\mathbf{D} = \begin{pmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{pmatrix} \text{ and } \mathbf{Q} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

The changes of basis gives then

$$\mathbb{P}(\tau_1 > t_1, \tau_2 > t_2) = \mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^{t_2} \mathbf{Z}_s^\top (2\mathbf{D})\,\mathbf{Z}_s \,\mathrm{d}s - \frac{1}{2}\mathbf{Z}_{t_2}^\top \Psi(t_1, t_2)\,\mathbf{Z}_{t_2}\right)\right] \tag{57}$$

where $\Psi(t_1, t_2)$ stands for the following quadratic form:

$$\Psi(t_1, t_2) = \sqrt{2}\tanh\left(\sqrt{2}(t_1 - t_2)\right)\left(\mathbf{Q}\mathbf{D}^{1/2}\right)^\top \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{Q}\mathbf{D}^{1/2} \tag{58}$$

Finally the result stems from the following generalization of a formula of Pitman and Yor [1982]:

$$\mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^t \mathbf{Z}_s^\top \Gamma \mathbf{Z}_s \mathrm{d}s - \frac{1}{2}\mathbf{Z}_t^\top \Sigma\,\mathbf{Z}_t\right)\right] = \left(\det\left(\cosh\left(t\Gamma^{1/2}\right) + \Sigma\Gamma^{-1/2}\sinh\left(t\Gamma^{1/2}\right)\right)\right)^{-1/2} \tag{59}$$

The same method would apply to compute the joint distribution of the default times for I firms, but the calculations are really too cumbersome to be lead until the end. For example, we would find that, when $t_1 = \ldots = t_I = t$, the law of $\tau = \bigwedge_{i=1}^{I} \tau_i$ is given in the general case by:

$$\mathbf{S}_\tau(t) := \mathbb{P}(\tau > t) = \mathbb{P}(\tau_1 > t, ..., \tau_I > t) = \prod_{i=1}^{I} \cosh^{-1/2}\left(t\sqrt{2d_i}\right) \tag{60}$$

where we note $(d_1, \ldots, d_I)$ for the eigenvalues of the correlation matrix ρ^W.

B The Sloane copula

The Sloane copula is the survival copula of the random vector (τ_1, τ_2) in the model of Section 2.3:

$$\mathbf{C}(u_1, u_2; \rho) = \left(C_1 + \frac{\sqrt{1+\rho}}{2} C_2 + \frac{\sqrt{1-\rho}}{2} C_3\right)^{-\frac{1}{2}} \tag{61}$$

with

$$\begin{aligned} C_1 &= \cosh(\zeta)\cosh\left(\xi\sqrt{1+\rho}\right)\cosh\left(\xi\sqrt{1-\rho}\right) \\ C_2 &= \sinh(\zeta)\cosh\left(\xi\sqrt{1-\rho}\right)\sinh\left(\xi\sqrt{1+\rho}\right) \\ C_3 &= \sinh(\zeta)\cosh\left(\xi\sqrt{1+\rho}\right)\sinh\left(\xi\sqrt{1-\rho}\right) \end{aligned} \tag{62}$$

where $\xi = c_1 \wedge c_2$, $\zeta = |c_1 - c_2|$ and $c_1 = \operatorname{arccosh}\left(u_1^{-2}\right)$ and $c_2 = \operatorname{arccosh}\left(u_1^{-2}\right)$. It has several nice properties. For example, the copula is PQD, it is positively ordered with respect to the absolute value of ρ and it is the product copula if ρ is equal to 0. But it does not reach the upper Fréchet bound. Actually, it presents a limited range of dependence (see Figures 14 and 15).

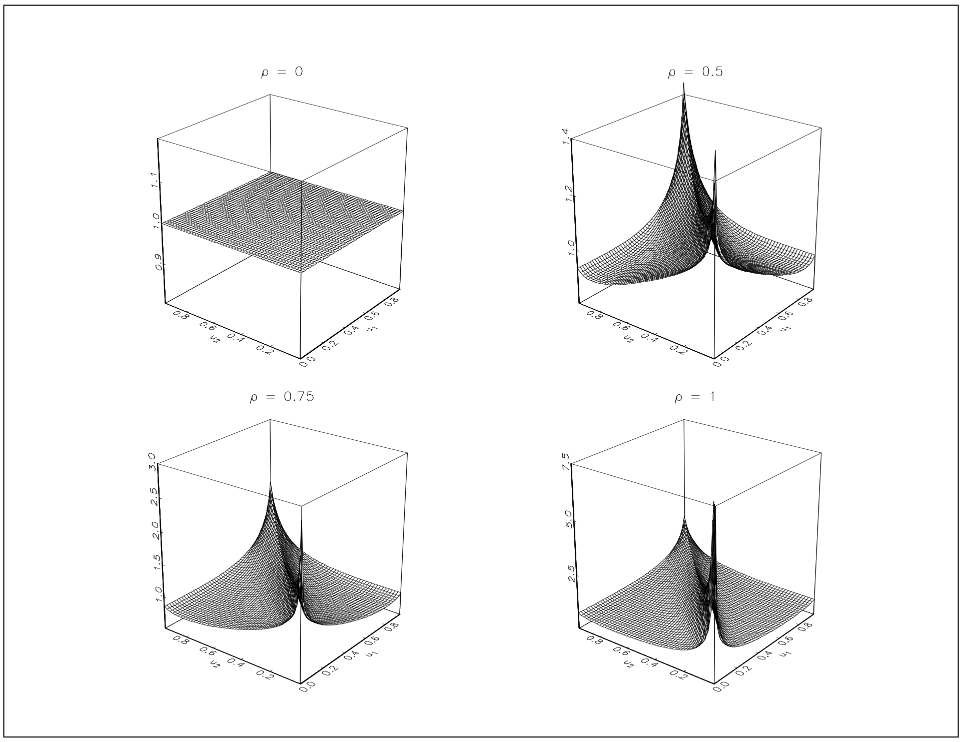

Figure 14: Density of the Sloane copula

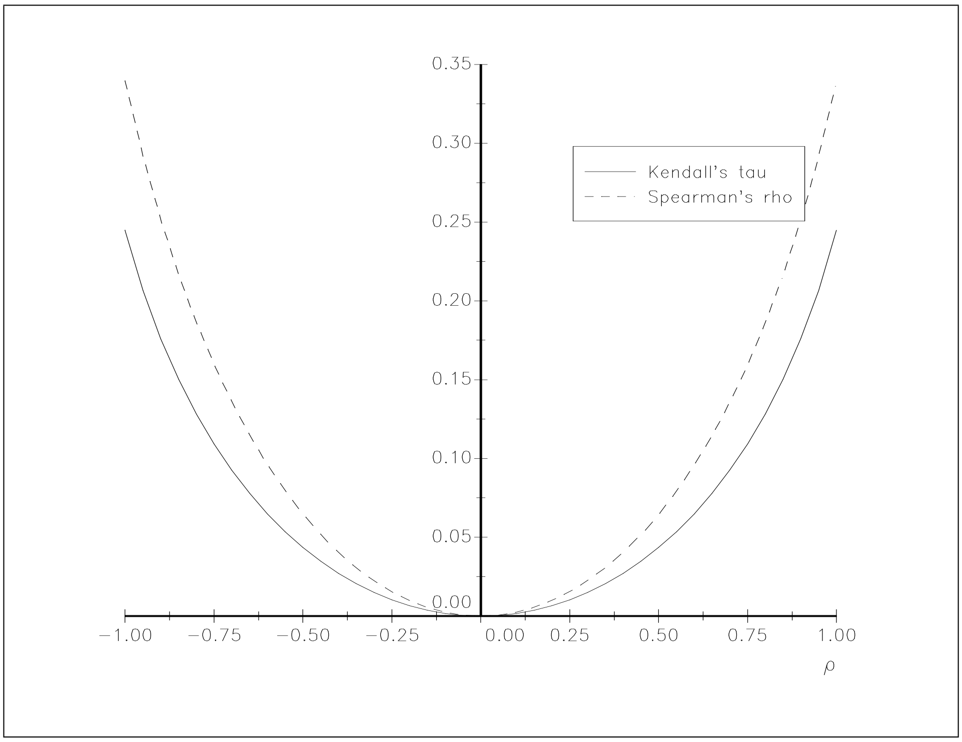

Figure 15: Kendall's tau and Spearman's rho of the Sloane copula