Pricing multi-asset options and credit derivatives with copulas*

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*These slides may be downloaded from http://gro.creditlyonnais.fr/content/rd/wp.htm.

Agenda

1. Copulas and stochastic dependence functions
2. Modelling dependence for credit derivatives with copulas
3. Some remarks on two-asset options pricing and stochastic dependence of asset prices
4. Copulas, multivariate risk-neutral distributions and implied dependence functions

1 Copulas and dependence functions

- In finance, the tool for modelling the dependence between two random variables is the Pearson correlation $\Rightarrow$ Normality and Linearity assumptions.

- Indeed, the canonical measure of the dependence is the copula of the two random variables.

1.1 Pearson correlation and dependence

Pearson correlation ρ = linear dependence measure.

For two given asset prices processes $S_1(t)$ and $S_2(t)$ which are GBM, the range of the correlation is

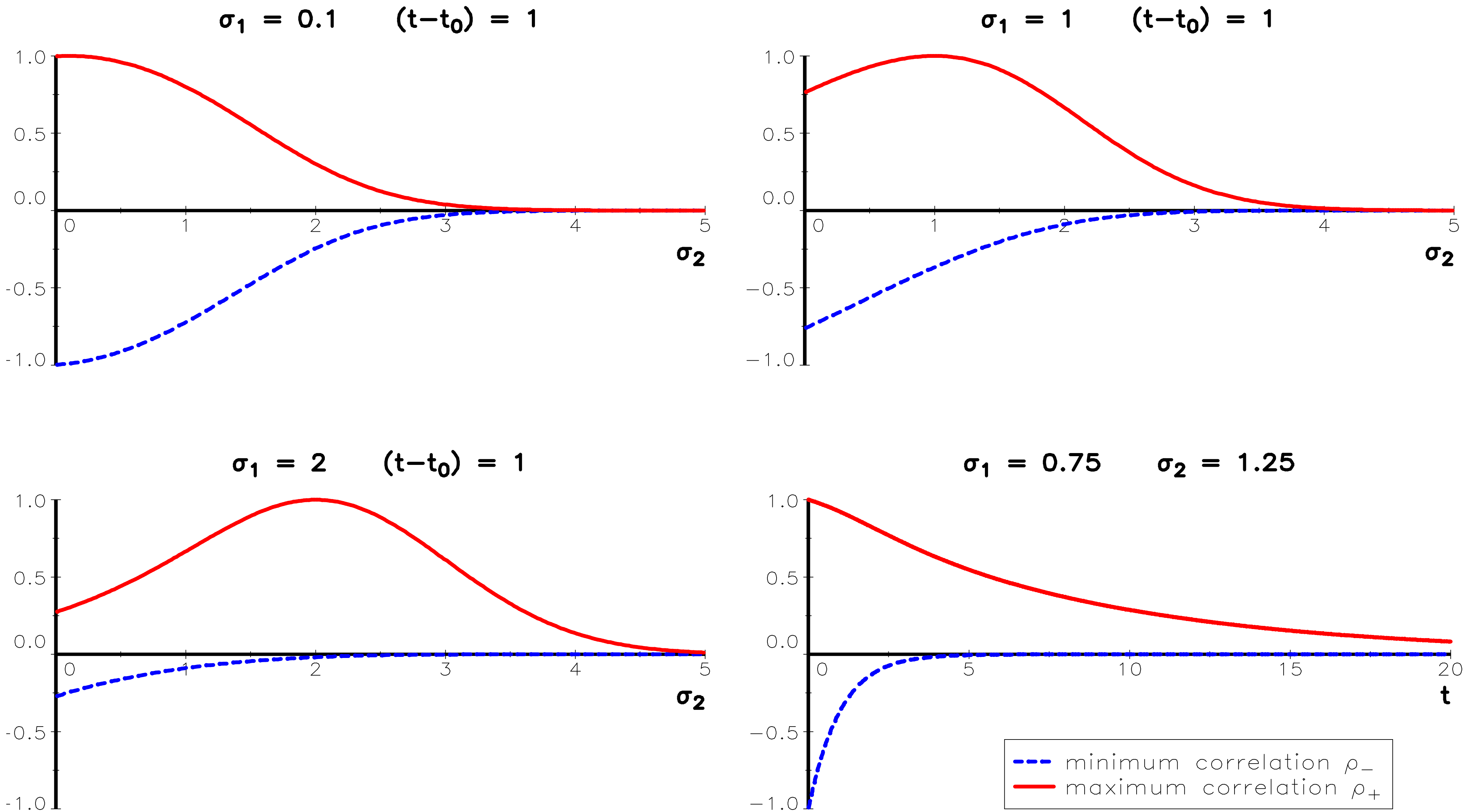
$$\rho^- \leq \rho(S_1(t), S_2(t)) \leq \rho^+$$

with

$$\rho^{\pm} = \frac{\exp(\pm\sigma_1\sigma_2(t-t_0)) - 1}{\sqrt{\exp\left(\sigma_1^2(t-t_0)\right) - 1} \cdot \sqrt{\exp\left(\sigma_2^2(t-t_0)\right) - 1}}$$

$\rho(S_1(t), S_2(t)) = \rho^-$ (resp. ρ^+) $\Leftrightarrow \mathbf{C}\langle S_1(t), S_2(t)\rangle = \mathbf{C}^-$ (resp. $\mathbf{C}^+$) $\Leftrightarrow S_2(t) = f(S_1(t))$ with f a decreasing (resp. increasing) function

$$\boxed{\text{Perfect dependence} \neq |\rho| = 1}$$



Permissible range of $\rho(S_1(t), S_2(t))$
when $S_1(t)$ and $S_2(t)$ are two GBM processes

1.2 Copulas in a nutshell

A copula function $\mathbf{C}$ is a multivariate probability distribution with uniform $[0, 1]$ margins.

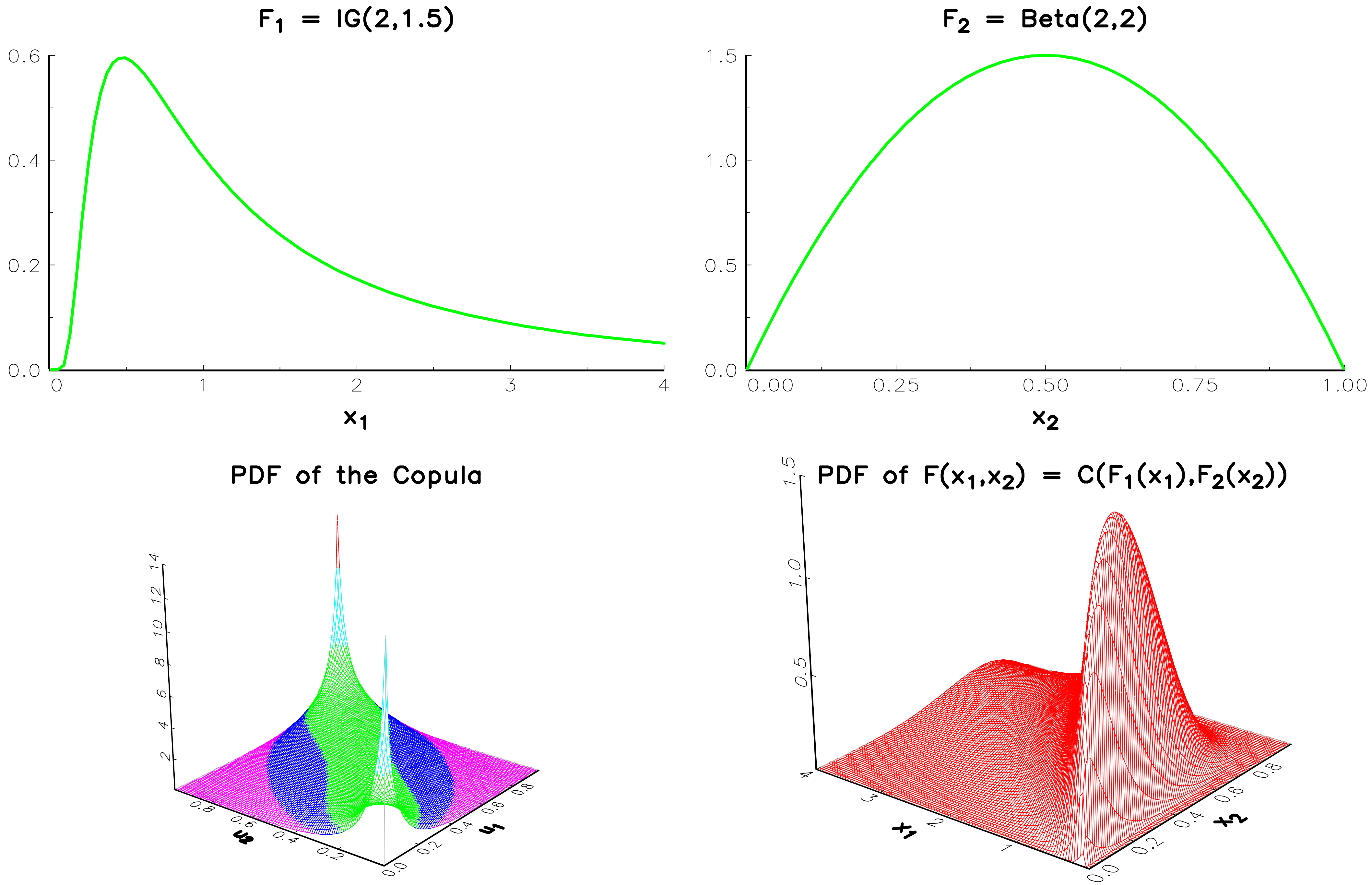
$\mathbf{C}\left(\mathbf{F}_1\left(x_1\right), \ldots, \mathbf{F}_N\left(x_N\right)\right)$ defines a multivariate cdf $\mathbf{F}$ with margins $\mathbf{F}_1, \ldots, \mathbf{F}_N \Rightarrow \mathbf{F}$ is a probability distribution with given marginals.

The copula function of the random variables $(X_1, \ldots, X_N)$ is **invariant** under strictly increasing transformations ($\partial_x h_n(x) > 0$):

$$\mathbf{C}\left\langle X_1, \ldots, X_N \right\rangle = \mathbf{C}\left\langle h_1\left(X_1\right), \ldots, h_N\left(X_N\right)\right\rangle$$

... the copula is invariant while the margins may be changed at will, it follows that is precisely the copula which captures those properties of the joint distribution which are invariant under a.s. strickly increasing transformations (Schweizer and Wolff [1981]).

$\Rightarrow$ **Copula = dependence function of r.v.** (Deheuvels [1978]).



Bivariate distribution with given marginals

1.3 Copula: a mathematical tool

Howard Sherwood in the AMS-IMS-SIAM Conference of 1993:

> *The subject matter of these conference proceedings comes in many guises. Some view it as the study of* **probability distributions with fixed marginals***; those coming to the subject from probabilistic geometry see it as the study of* **copulas***; experts in real analysis think of it as the study of* **doubly stochastic measures***; functional analysts think of it as the study of* **Markov operators***; and statisticians say it is the study of possible* **dependence** *relations between pairs of random variables. All are right since all these topics are isomorphic.*

- Probabilistic metric spaces and $t-$norms (Schweizer and Sklar [1960])
- Associative functions and Archimedean copulas (Ling [1965] $\longrightarrow$ Hilbert 13^{th} problem, Arnold-Kolmogorov, Aczél)
- Schweizer and Sklar [1974] $\longrightarrow$ Quasi-copulas
- Frank [1979] $\longrightarrow$ "Universal" fuzzy arithmetic
- Copulas and Markov processes (Darsow, Nguyen and Olsen [1982] $\longrightarrow$ Markov operators, Foguel [1969], Kantorovitch and Vulich [1937])
- (Infinite) doubly stochastic matrices
- Variational problems (Monge-Kantorovic, Entropy $\longrightarrow$ Dall'Aglio, Bertino [1968])

2 Modelling dependence for credit derivatives with copulas

- How to introduce dependence of default times into intensity models ?
- What is the influence of the dependence on Basket credit derivatives ?
- How to calibrate ?

2.1 Correlate intensity processes ?

With Cox processes, the default time is often defined by

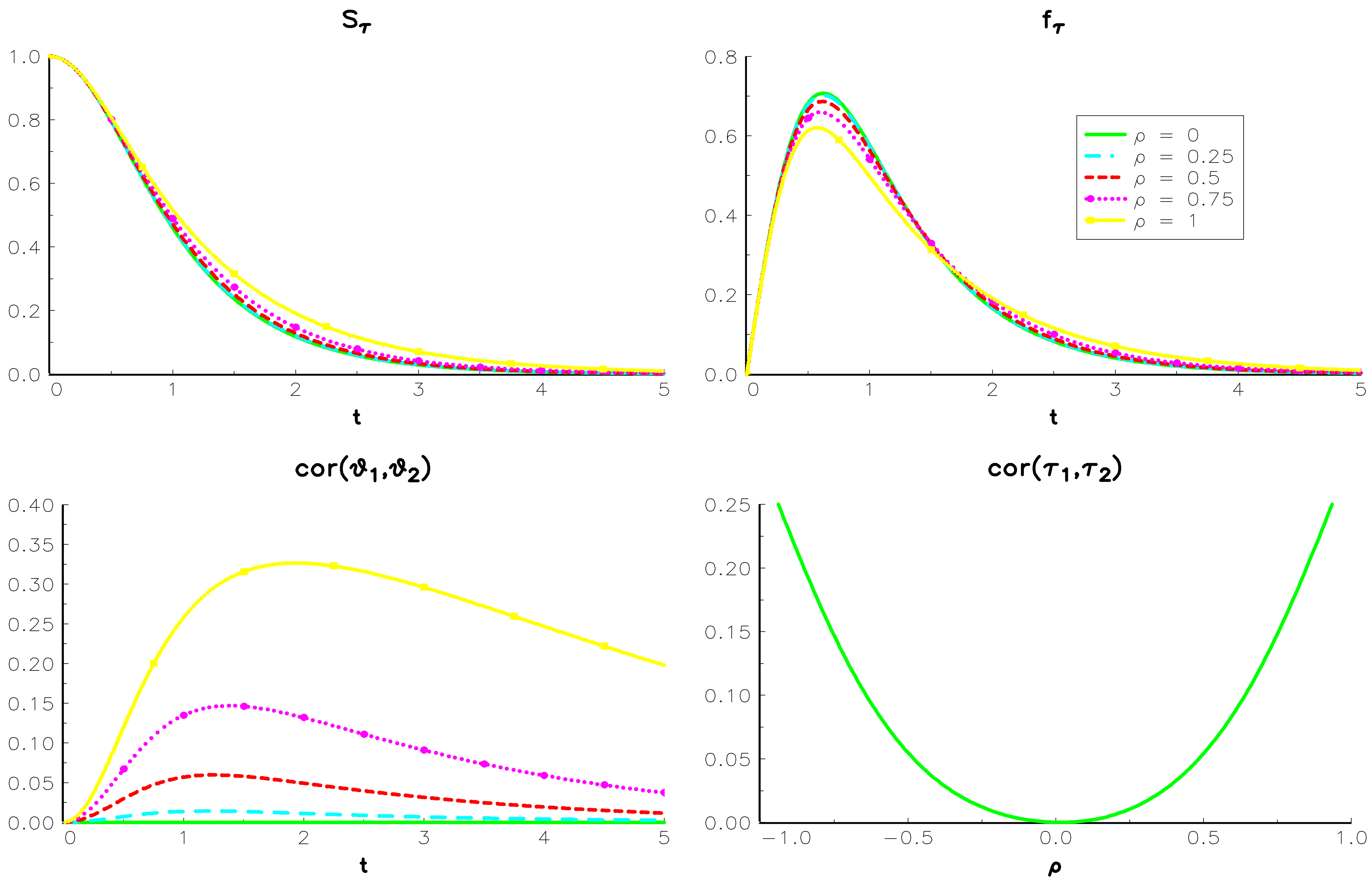
$$\tau := \inf \left\{ t : \int_0^t \lambda_s \, \mathrm{d}s \geq \theta \right\}$$

where θ is an exponential r.v. of parameter 1 and λ a nonnegative process called the intensity process.

$\Rightarrow$ Correlating intensity processes is not sufficient to obtain dependence of default times.

Quadratic intensities $\lambda_t^i := (W_t^i)^2$ where $\mathbf{W} = \left(W^1, W^2\right)$ is a vector of 2 correlated $(\mathcal{F}_t)$ Brownian motions with correlation ρ.

We may explicitely compute the survival function $\mathbf{S}(t_1, t_2) := \mathbb{P}(\tau_1 > t_1, \tau_2 > t_2)$ and its corresponding survival copula



Influence of the correlation parameter on the first default time

.

$$\mathbf{C}(u_1, u_2; \rho) = \left(C_1 + \frac{\sqrt{1+\rho}}{2} C_2 + \frac{\sqrt{1-\rho}}{2} C_3\right)^{-\frac{1}{2}}$$

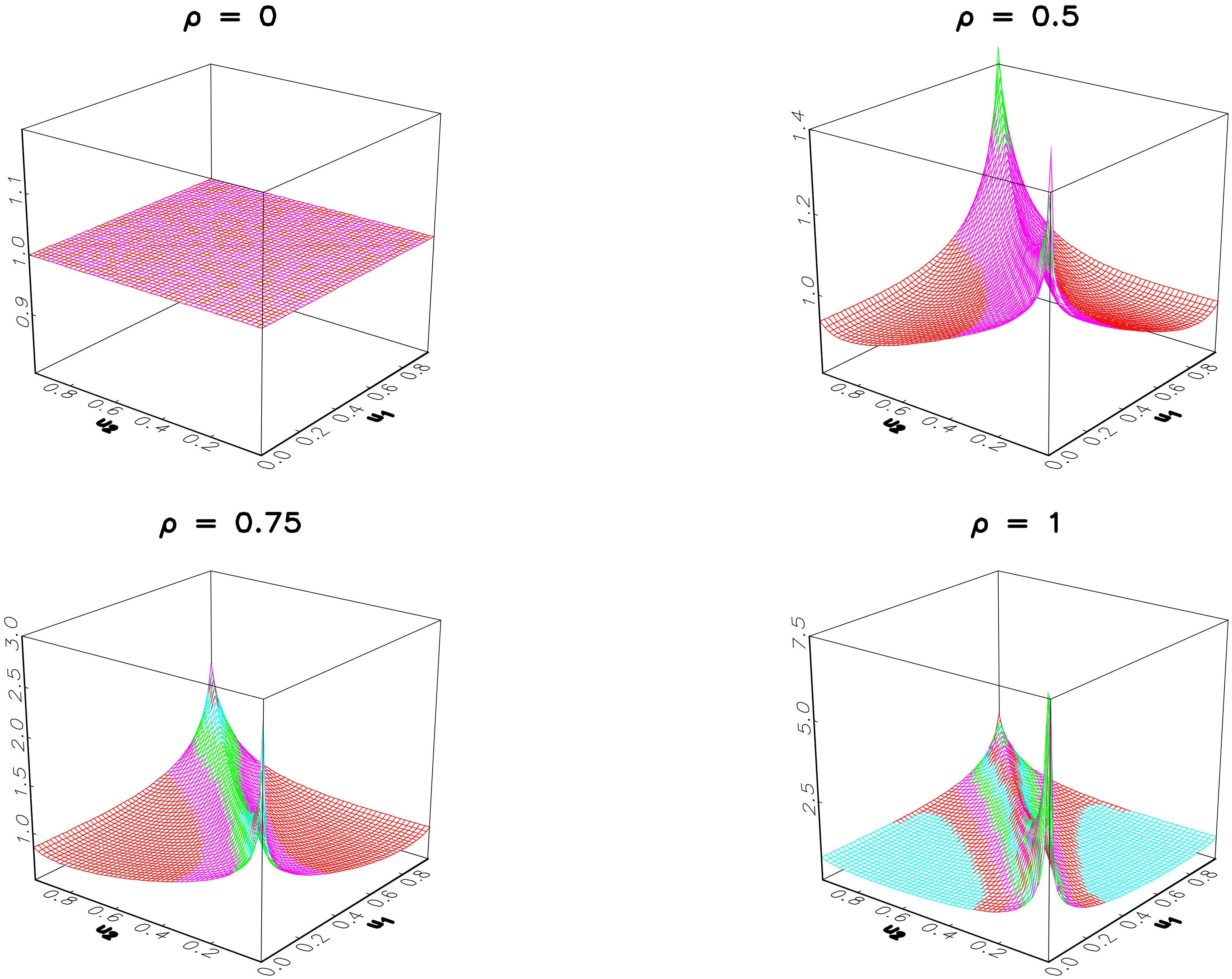
with

$$\begin{aligned} C_1 &= \cosh(\zeta) \cosh\left(\xi\sqrt{1+\rho}\right) \cosh\left(\xi\sqrt{1-\rho}\right) \\ C_2 &= \sinh(\zeta) \cosh\left(\xi\sqrt{1-\rho}\right) \sinh\left(\xi\sqrt{1+\rho}\right) \\ C_3 &= \sinh(\zeta) \cosh\left(\xi\sqrt{1+\rho}\right) \sinh\left(\xi\sqrt{1-\rho}\right) \end{aligned}$$

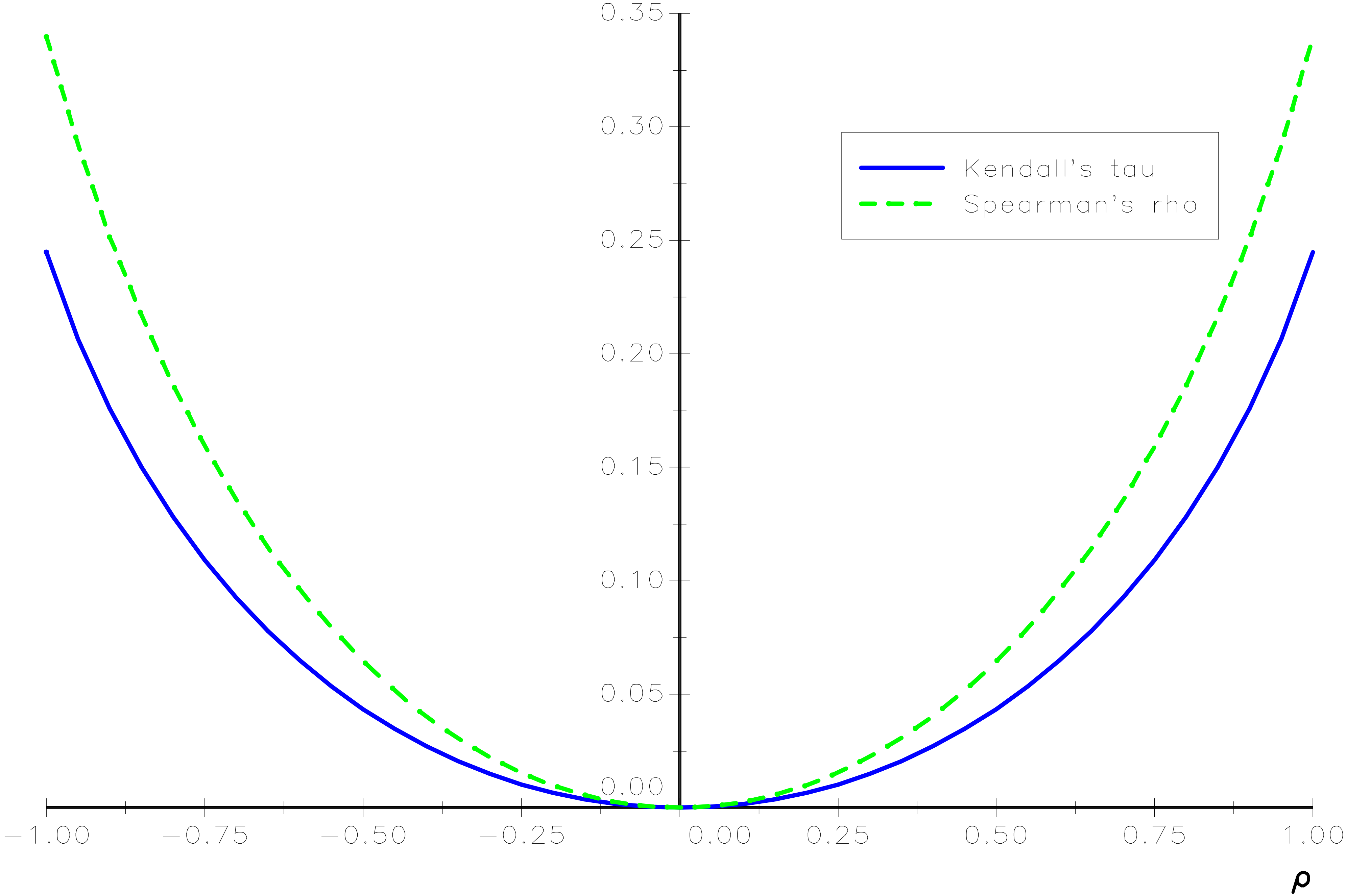
where $\xi = c_1 \wedge c_2$, $\zeta = |c_1 - c_2|$ and $c_1 = \operatorname{arccosh}\left(u_1^{-2}\right)$ and $c_2 = \operatorname{arccosh}\left(u_1^{-2}\right)$.

$\Rightarrow \mathbf{C}(u_1, u_2; \rho) = \mathbf{C}(u_1, u_2; -\rho)$

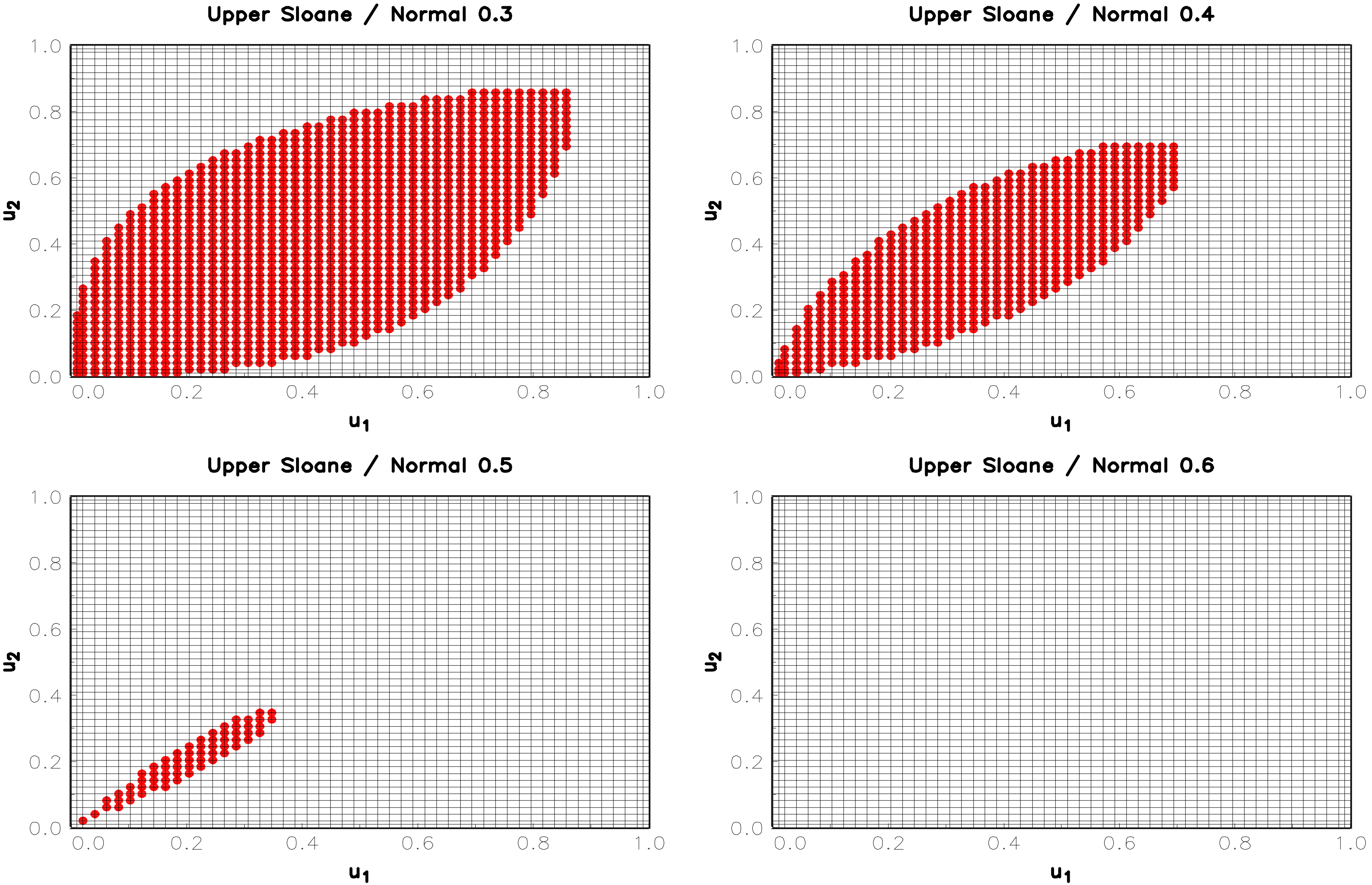
$\Rightarrow \mathbf{C} \succ \mathbf{C}^{\perp}$



Density of the Sloane copula



Concordance measures of the Sloane copula



Pointwise concordance order

2.2 The survival approach (Li [2000])

We consider a family of I exponential random variables θ_i and λ_t^i a family of intensity process. We define $\tau = (\tau_1, \ldots, \tau_I)$ be the random vector of default times with

$$\tau_i := \inf \left\{ t \geq 0 : \int_0^t \lambda_s^i \, \mathrm{d}s \geq \theta_i \right\}$$

The joint survival function $\mathbf{S}$ of the random vector τ corresponds to

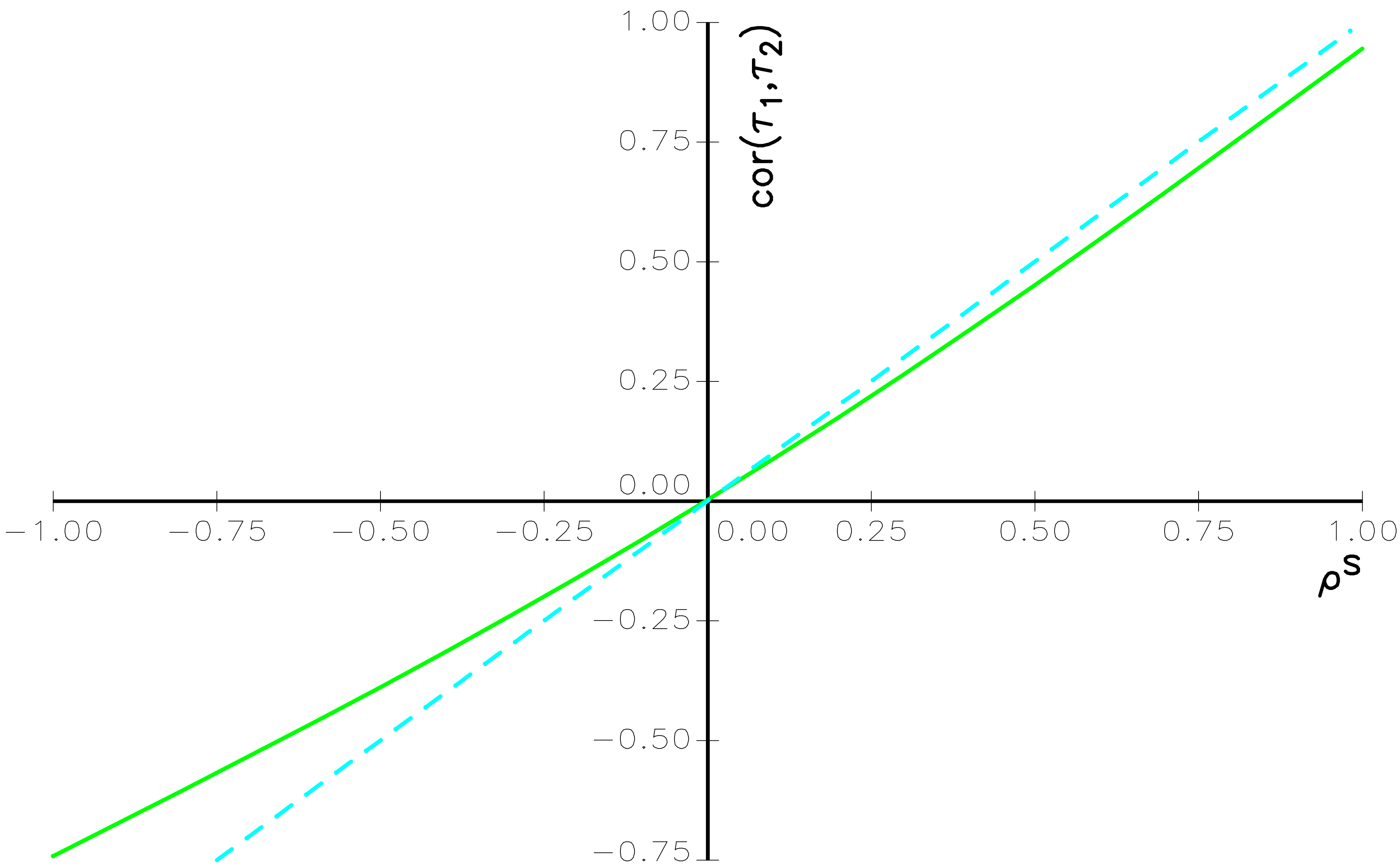
$$\mathbf{S}(t_1, \ldots, t_I) = \mathbb{P}(\tau_1 > t_1, \ldots, \tau_I > t_I)$$

Using Sklar's theorem, $\mathbf{S}$ has a copula representation

$$\mathbf{S}(t_1, \ldots, t_I) = \breve{\mathbf{C}}(\mathbf{S}_1(t_1), \ldots, \mathbf{S}_I(t_I))$$

In this case, we put directly a copula function on the random variables τ_i.

$\Rightarrow$ Computing the price of a contingent claim is done using Monte Carlo simulations (with the *empirical survival functions* $\mathbb{S}_i$).



Relationship between the Normal parameter ρ^S
and the survival time correlation $cor(\tau_1,\tau_2)$

2.3 The threshold approach (Gesiecke [2001], Schönbucher and Schubert [2001]

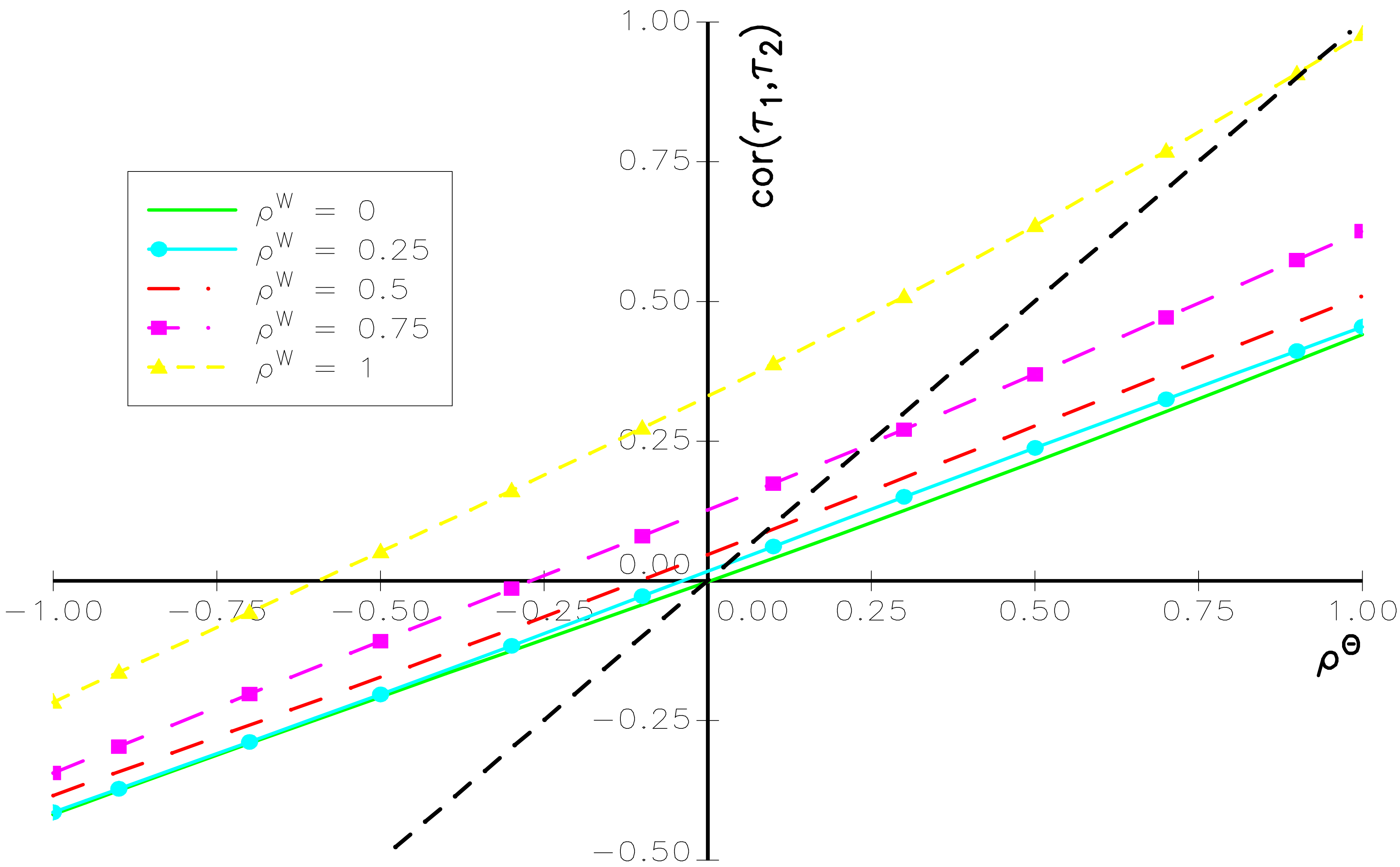
Here, we put $\mathbf{C}^{\theta} = \mathbf{C}_{\langle\theta_1,\ldots,\theta_I\rangle}$ directly on the random variables θ_i.

⇒ Computing the price of a contingent claim is done using Monte Carlo simulations or with closed-form formula.

The relationship between $\breve{\mathbf{C}}^{\tau}$ and $\breve{\mathbf{C}}^{\theta}$ is given by

$$\breve{\mathbf{C}}^{\tau}\left(\mathbf{S}_1\left(t_1\right),\ldots,\mathbf{S}_I\left(t_I\right)\right) = \mathbb{E}\left[\breve{\mathbf{C}}^{\theta}\left(\exp\left(-\int_0^{t_1}\lambda_s^1\,\mathrm{d}s\right),\ldots,\exp\left(-\int_0^{t_I}\lambda_s^I\,\mathrm{d}s\right)\right)\right]$$

⇒ **We notice that if the intensities are all deterministic and constant, the two copulas are the same — $\breve{\mathbf{C}}^{\tau} = \breve{\mathbf{C}}^{\theta}$ and $\mathbf{C}^{\tau} = \mathbf{C}^{\theta}$.**



Relationship between the Normal parameter ρ^Θ and the survival time correlation cor(τ_1,τ_2)

2.4 Pricing the first-to-default

1. Payoff at maturity

$$\mathbb{E}\left[\frac{B_t}{B_T}\mathbf{1}_{\{\tau>T\}} \mid \mathcal{G}_t\right] = \mathbb{E}\left[\frac{B_t}{B_T}\frac{\breve{\mathbf{C}}^{\theta}\left(\exp\left(-\int_0^T \lambda_s^1\,\mathrm{d}s\right),\ldots,\exp\left(-\int_0^T \lambda_s^I\,\mathrm{d}s\right)\right)}{\breve{\mathbf{C}}^{\theta}\left(\exp\left(-\int_0^t \lambda_s^1\,\mathrm{d}s\right),\ldots,\exp\left(-\int_0^t \lambda_s^I\,\mathrm{d}s\right)\right)} \mid \mathcal{F}_t\right]$$

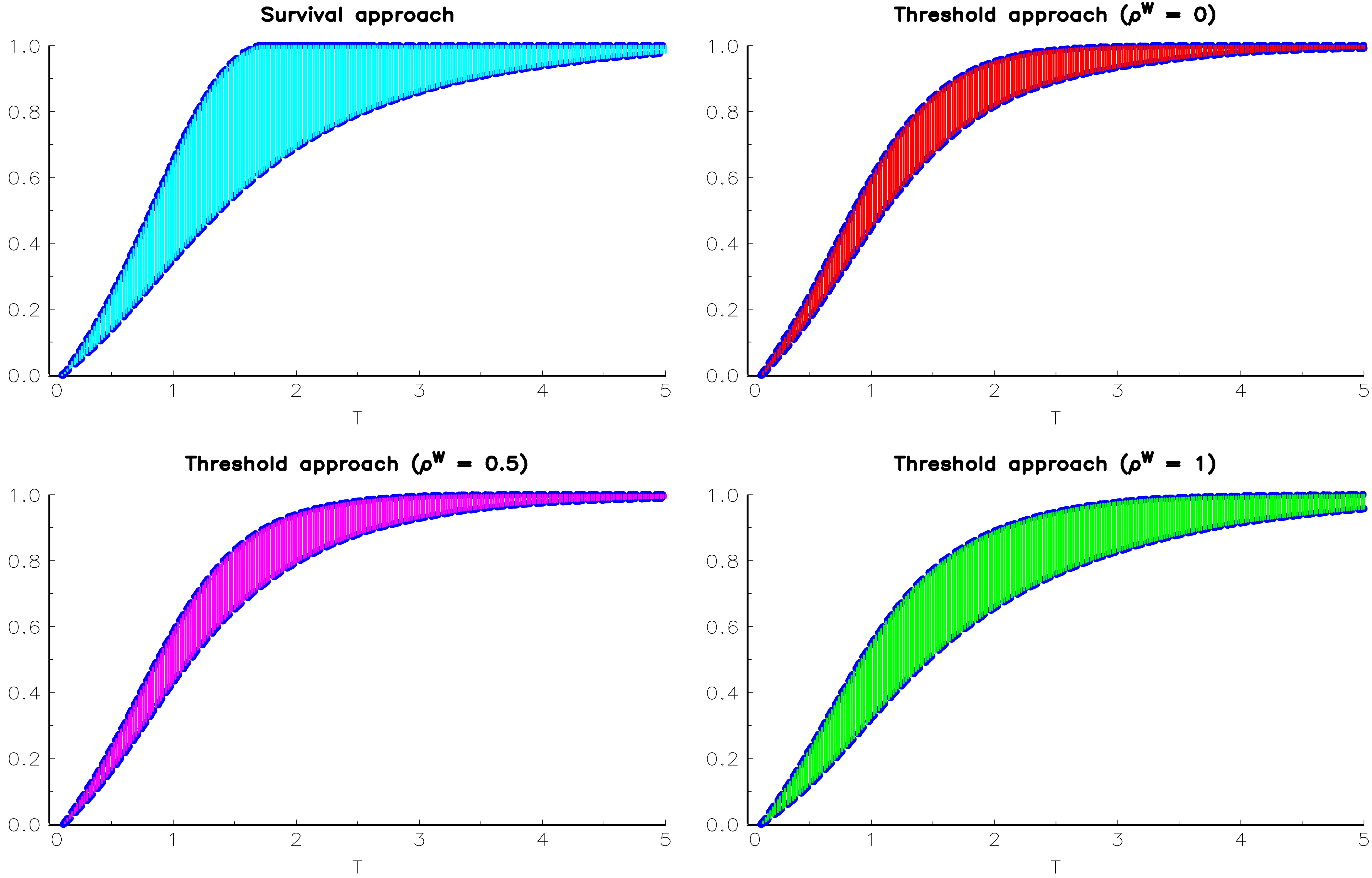
2. Payoff at default

$$\mathbb{E}\left[\exp\left(-\int_t^{\tau} r_s\,\mathrm{d}s\right)\mathbf{1}_{\{\tau\leqslant T\}} \mid \mathcal{G}_t\right] = \mathbb{E}\left[\int_t^T \zeta_t(\mathrm{d}\upsilon)\exp\left(-\int_t^{\upsilon} r_s\,\mathrm{d}s\right) \mid \mathcal{F}_t\right]$$

with

$$\zeta_t(\mathrm{d}\upsilon) = -\frac{\partial}{\partial\upsilon}\frac{\breve{\mathbf{C}}^{\theta}\left(\exp\left(-\int_0^{\upsilon} \lambda_s^1\,\mathrm{d}s\right),\ldots,\exp\left(-\int_0^{\upsilon} \lambda_s^I\,\mathrm{d}s\right)\right)}{\breve{\mathbf{C}}^{\theta}\left(\exp\left(-\int_0^t \lambda_s^1\,\mathrm{d}s\right),\ldots,\exp\left(-\int_0^t \lambda_s^I\,\mathrm{d}s\right)\right)}\,\mathrm{d}\upsilon$$

$\Rightarrow$ Importance of the dependence.



Bounds of the first-to-default price (payoff at maturity)

2.5 Calibration issues

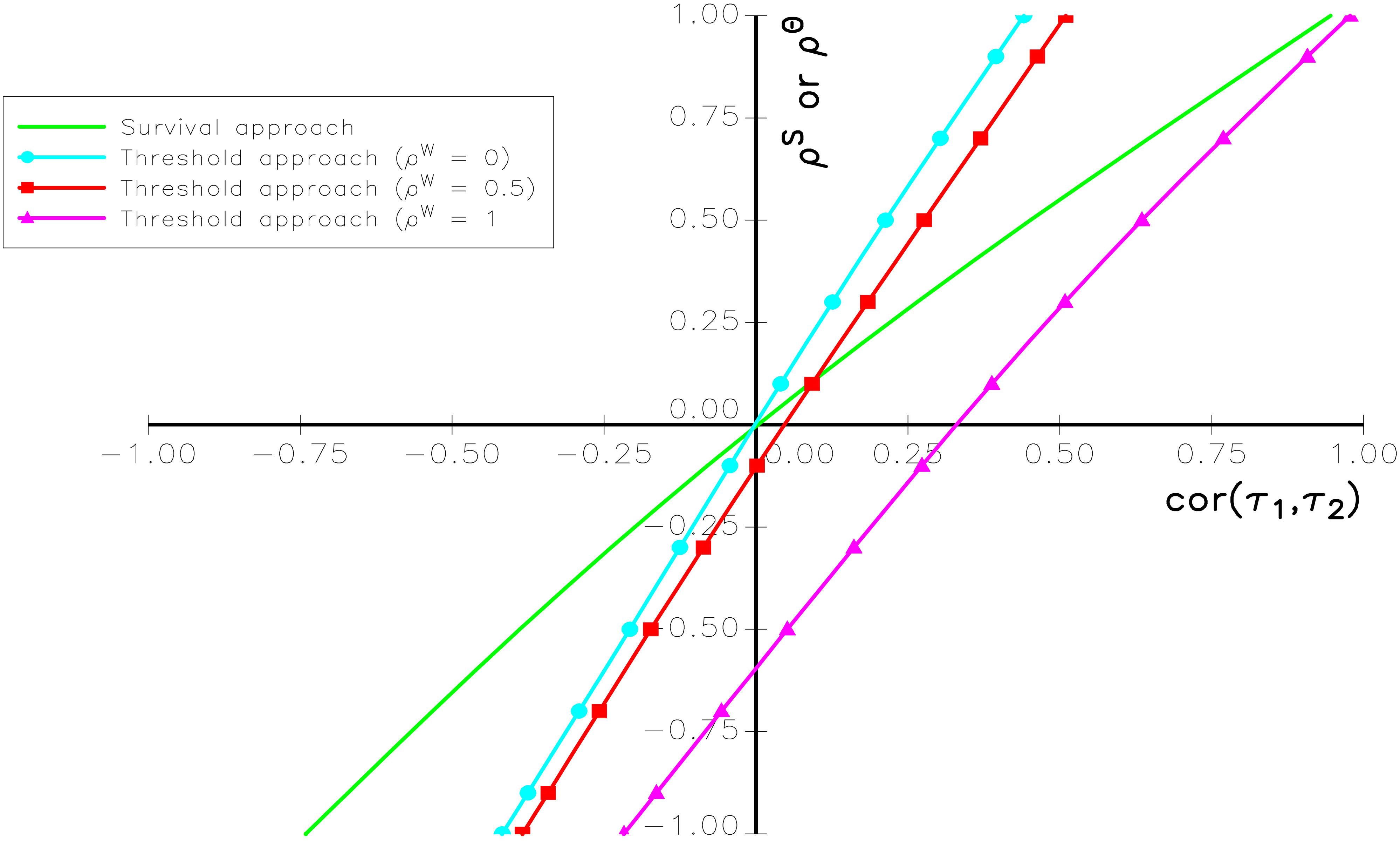
Using correlations

- Discrete default correlations cor $\left(\mathbf{1}_{\{\tau_i>T\}}, \mathbf{1}_{\{\tau_j>T\}}\right)$
- or survival time correlations cor (τ_1, τ_2) ?

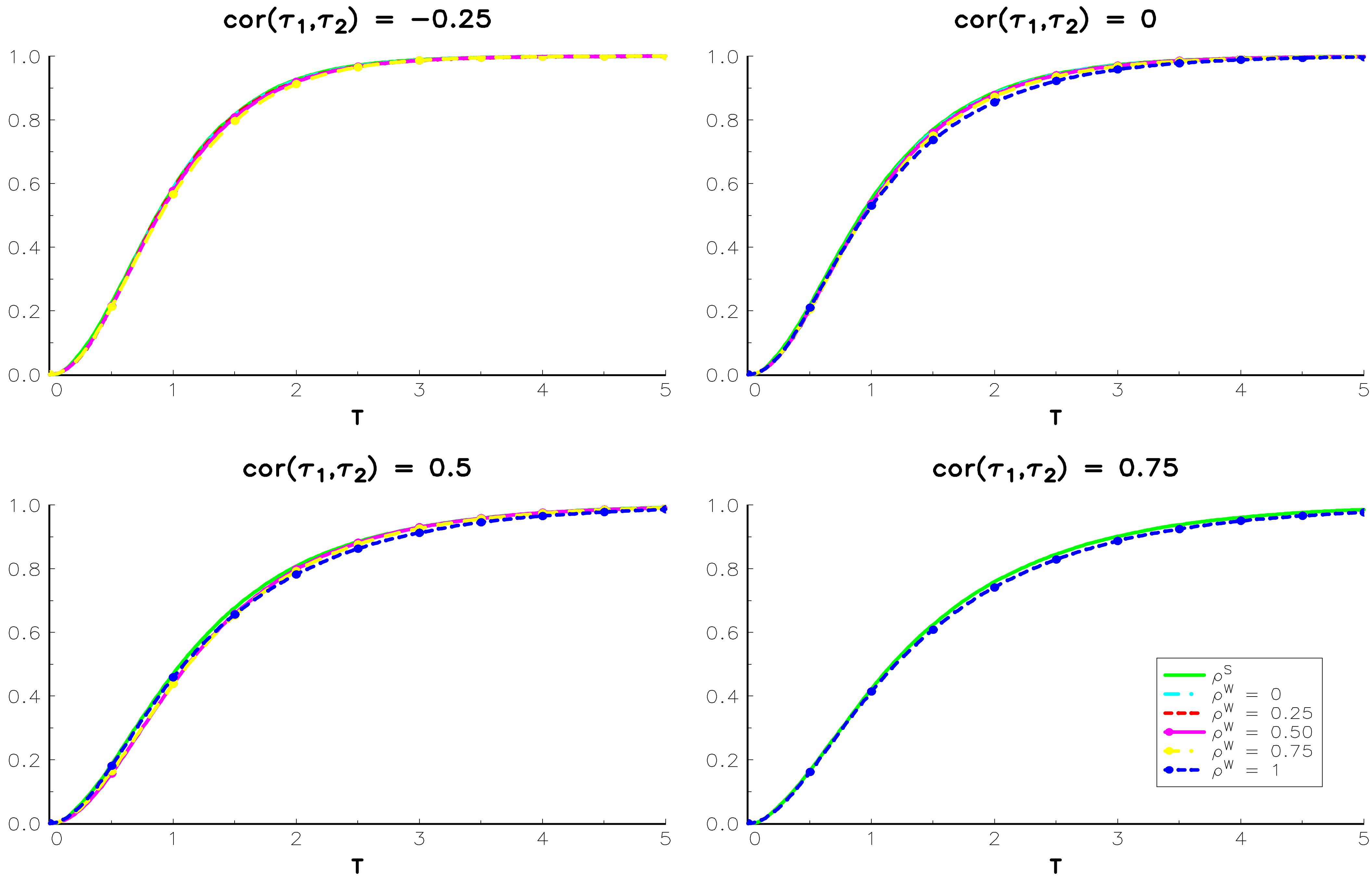
Using Moody's diversity score The distribution of the number of defaults among I risky issuers of the same sector could be summarized using only D independant issuers

$$\sum_{i=1}^{I} \mathbf{1}_{\{\tau_i \leq t\}} \overset{law}{=} \frac{I}{D} \sum_{d=1}^{D} \mathbf{1}_{\{\bar{\tau}_i \leq t\}}$$

where $\left(\mathbf{1}_{\{\tau_1 \leq t\}}, \ldots, \mathbf{1}_{\{\tau_I \leq t\}}\right)$ are dependent Bernoulli random variables with parameters p_i $(i \in \{1, \ldots, I\})$ and $\left(\mathbf{1}_{\{\bar{\tau}_1 \leq t\}}, \ldots, \mathbf{1}_{\{\bar{\tau}_D \leq t\}}\right)$ are *i.i.d.* Bernoulli random variables with parameter p.



Calibration of the survival time correlations



Impact on the first-to-default price

$\Rightarrow D$ measures the dependence risk ($D \in [1, I]$).

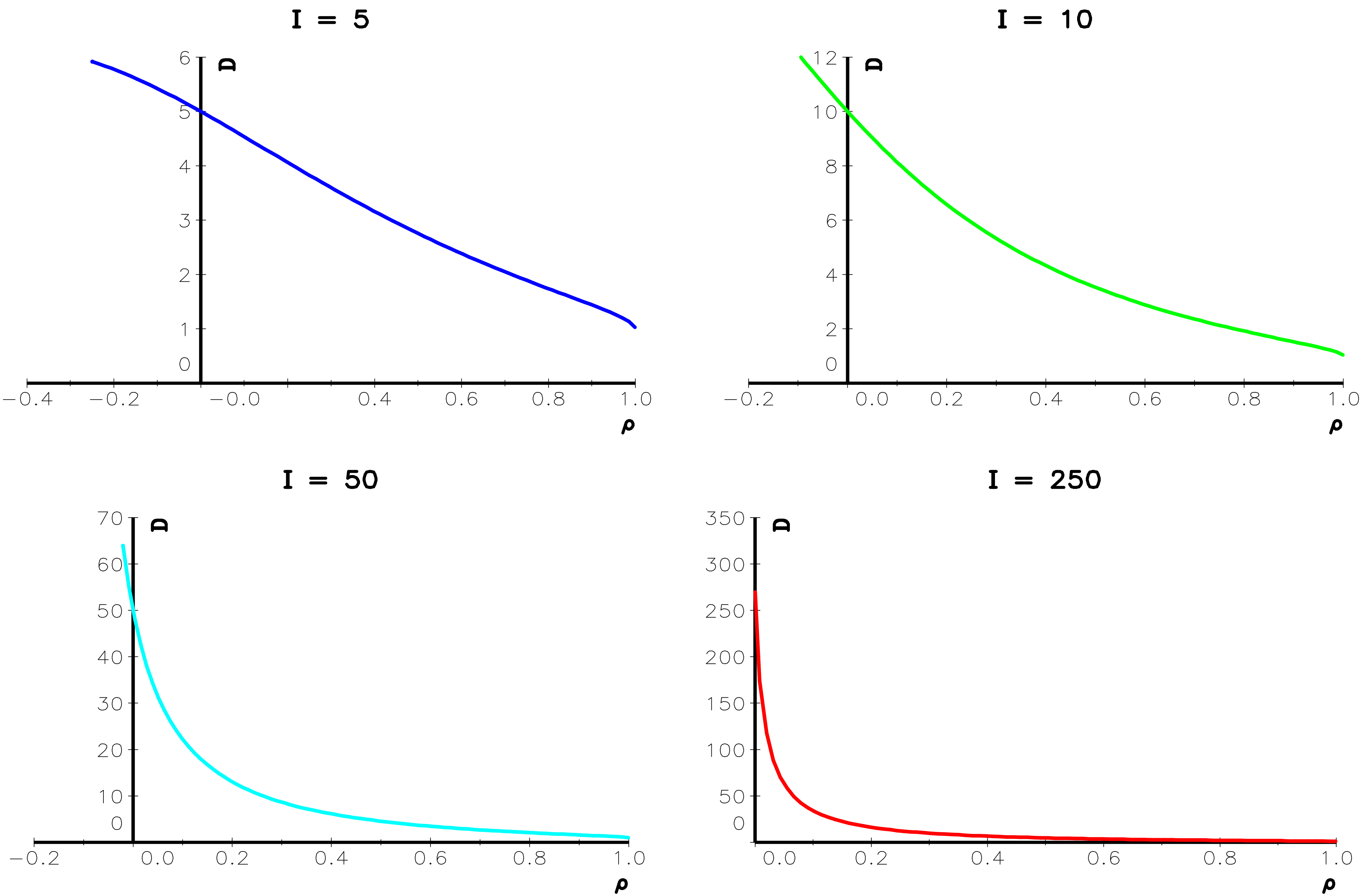
- $D = 1 \Rightarrow$ Perfect dependence.
- $D = I \Rightarrow$ Independence.

Conditioning the expectations on $\mathcal{F}_\infty$, the two first moments calibration procedure induces the following equalities

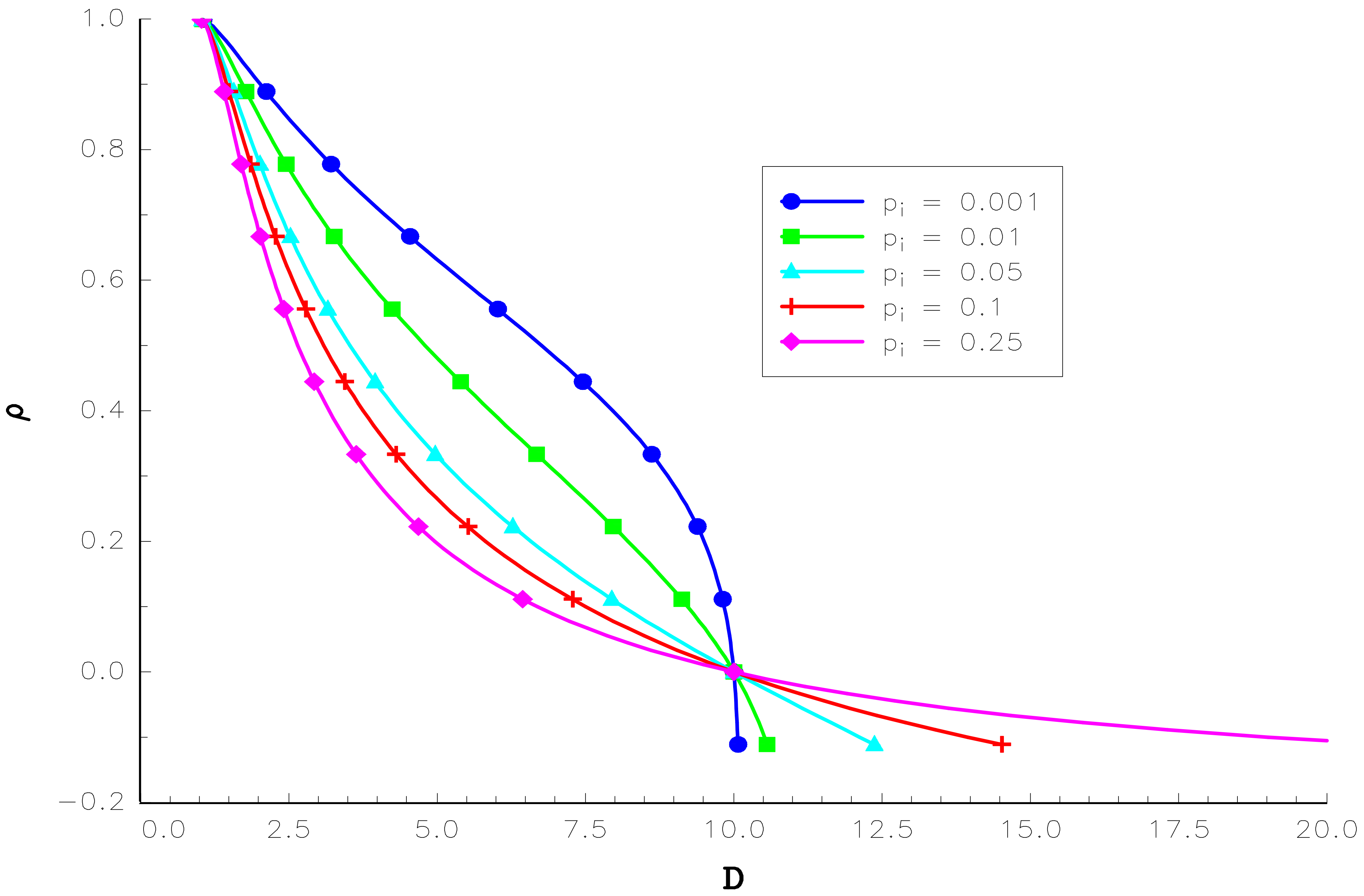
$$Ip = \mathbb{E}\left[\sum_{i=1}^{I} \mathbf{1}_{\{\tau_i \le t\}} \mid \mathcal{F}_\infty\right] = \sum_{i=1}^{I} p_i$$

and

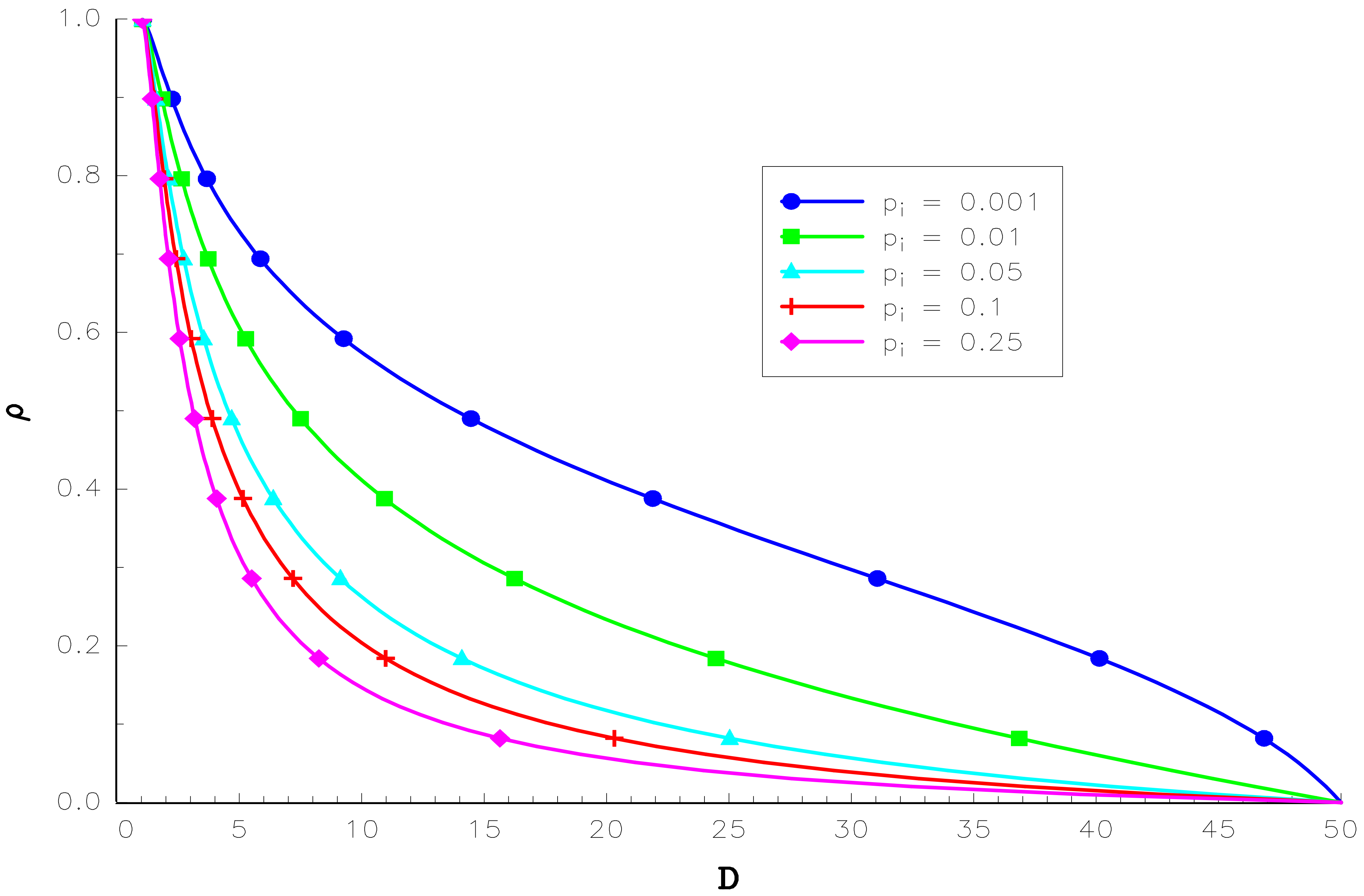
$$\begin{aligned} \frac{I^2 p(1 + (D-1)\,p)}{D} &= \mathbb{E}\left[\left(\sum_{i=1}^{I} \mathbf{1}_{\{\tau_i \le t\}}\right)^2 \mid \mathcal{F}_\infty\right] \\ &= \sum_{i=1}^{I} p_i + 2\sum_{i<j} \mathbf{C}^\theta\left(1, \ldots, p_i, \ldots, p_j, \ldots, 1\right) \end{aligned}$$



Relationship between ρ and D (p_i = 5%)



Implied parameter ρ to diversity score D (I = 10)



Implied parameter ρ to diversity score D (I = 50)

Moody's diversity score

$$D_{\text{Moody's}} = \frac{-1 + \sqrt{1 + 8I}}{2}$$

Tail dependence and copula calibration

Let $\lambda_L^{\theta}(u) = \mathbf{C}^{\theta}(u,u)/u$. With same default probabilities, we have

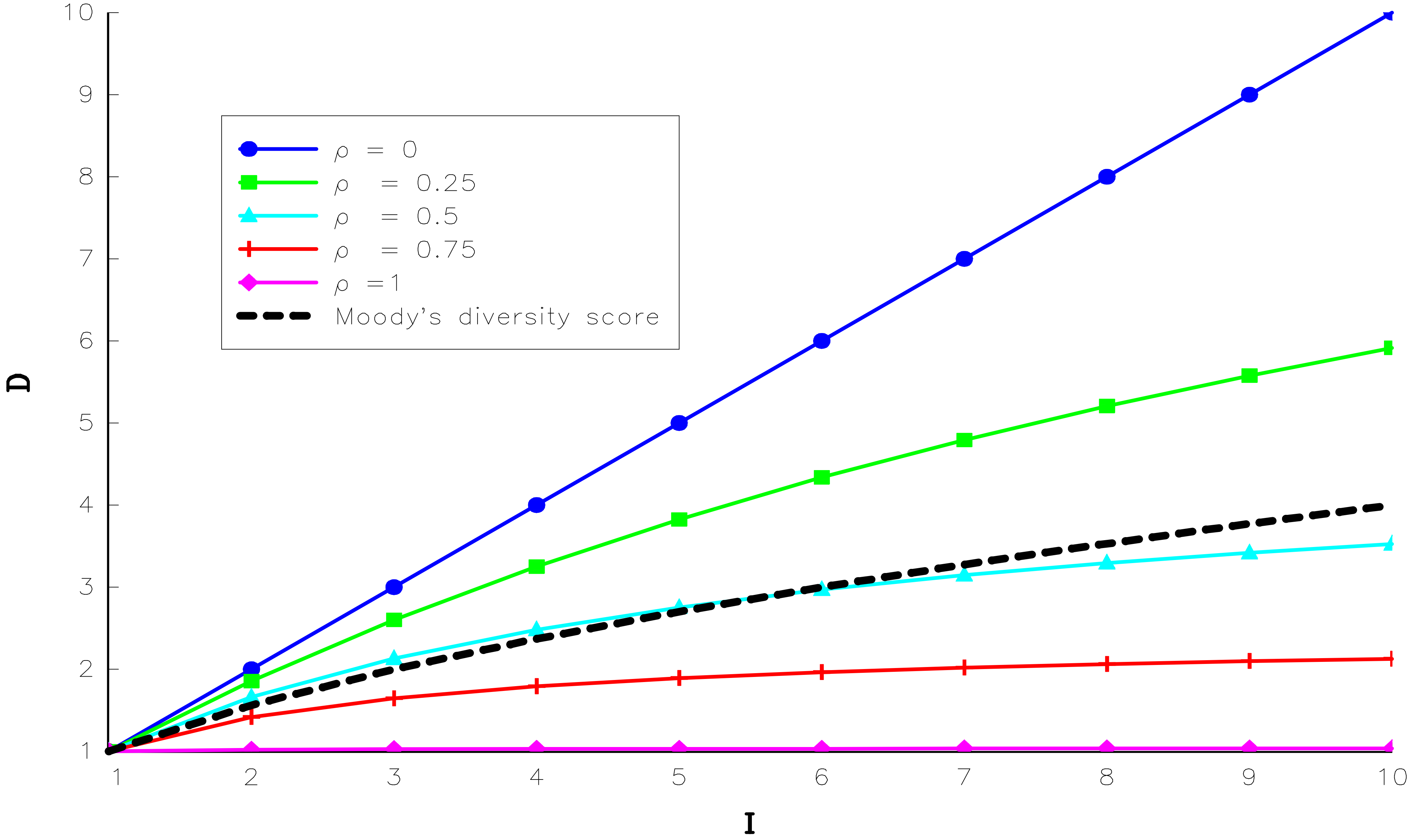
$$D = \frac{I(1 - p_i)}{(1 - Ip_i) + (I - 1)\lambda_L^{\theta}(p_i)}$$

The limit case $(p_i \rightarrow 0)$ is then

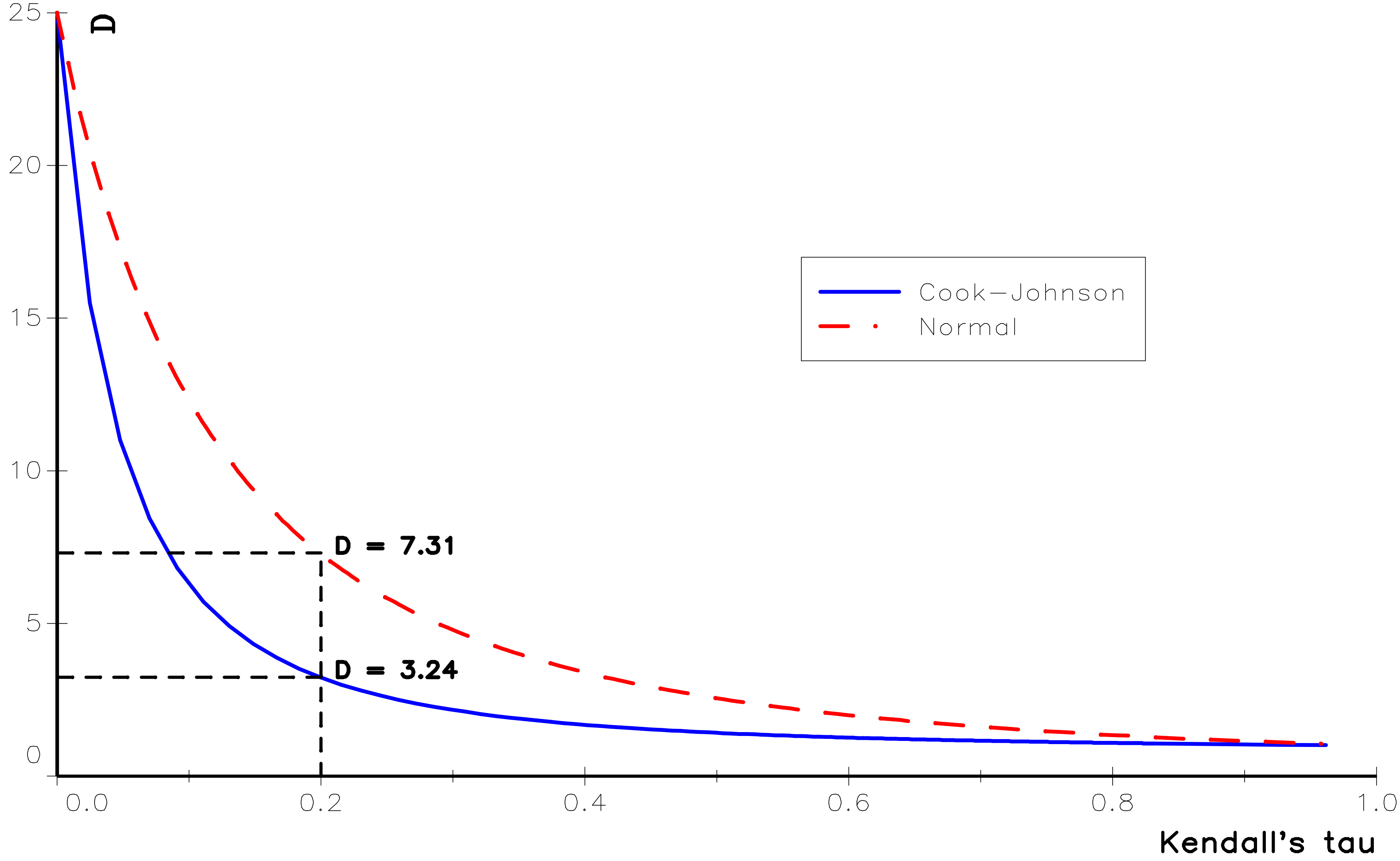
$$D = \frac{I}{1 + (I - 1)\lambda_L^{\theta}}$$

If the size of the credit portfolio is large $(I \rightarrow \infty)$, D is equal to $1/\lambda_L^{\theta}$.

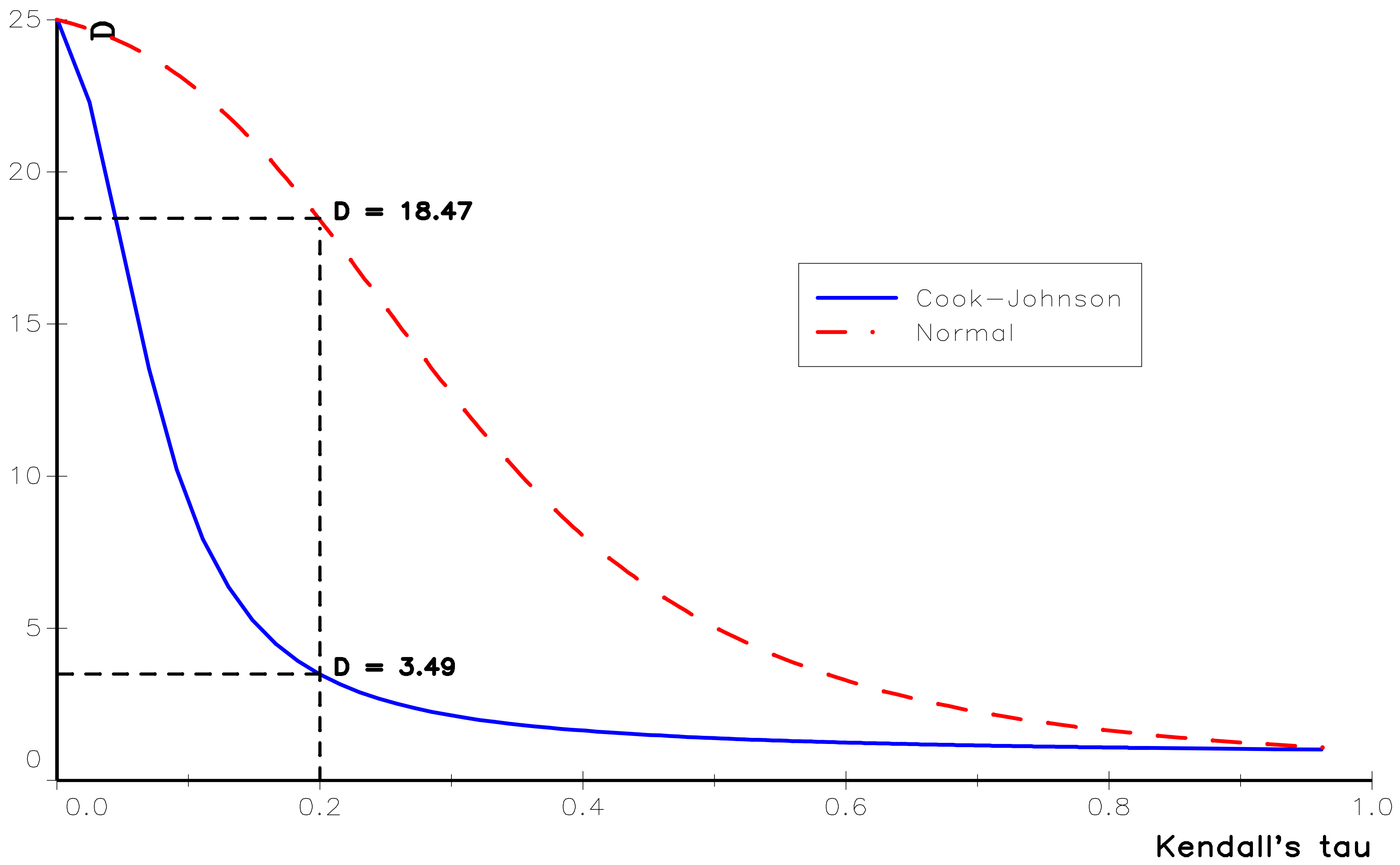
$\Rightarrow$ Big impact of the choice of the copula on the calibration issues for rare credit events (for example, for bonds which are rated AAA or AA).



Relationship between Moody's D and the number of firms I (p_i = 5%)



Difference between the Normal copula and the Cook–Johnson copula (p_i = 5%, I = 25)



Difference between the Normal copula and the Cook–Johnson copula (p_i = 0.1%, I = 25)

2.6 A remark on credit derivatives pricing

CD = Credit risk mitigation in **BASLE II**.

Using CD, risk weighted assets are calculated using the following formula:

$$RWA = r^{\star} \times E = r \times (E - G_A) + [(r \times w) + g\,(1 - w)]\,G_A$$

where r is the risk weight of the obligor, w is the residual risk factor, g is the risk weight of the the protection provider, G_A is the nominal amount of the cover and E is the value of the exposure.

CD implies lower capital consumption, which return is

$$ROE \times \frac{(r - r^{\star}) \times E}{12.5}$$

$\Rightarrow$ This is (almost) the implied regulatory price of the CD.

3 Some remarks on two-asset options pricing and stochastic dependence of asset prices

Vanilla options
/
Multi-asset options

Volatility = Risk Neutral Distribution
/
Correlation = Dependence function

Given the margins of the multivariate risk-neutral distribution or given the volatility smiles, what is the influence of the stochastic dependence on the price ?

$\Rightarrow$ What is a conservative price ?

3.1 Black-scholes model and two-asset options

The model The dynamics of the asset prices are

$$\begin{cases} \mathrm{d}S_1(t) = \mu_1 S_1(t)\,\mathrm{d}t + \sigma_1 S_1(t)\,\mathrm{d}W_1(t) \\ \mathrm{d}S_2(t) = \mu_2 S_2(t)\,\mathrm{d}t + \sigma_2 S_2(t)\,\mathrm{d}W_2(t) \end{cases}$$

where $\mathbb{E}\left[W_1(t)\,W_2(t)\right] = \rho t$.

The price $P(S_1, S_2, t)$ of the European two-asset option with the payoff function $G(S_1, S_2)$ is the solution of the following parabolic PDE

$$\begin{cases} \frac{1}{2}\sigma_1^2 S_1^2 \partial_{1,1}^2 P + \rho\sigma_1\sigma_2 S_1 S_2 \partial_{1,2}^2 P + \frac{1}{2}\sigma_2^2 S_2^2 \partial_{2,2}^2 P + \\ b_1 S_1 \partial_1 P + b_2 S_2 \partial_2 P - rP + \partial_t P = 0 \\ P(S_1, S_2, T) = G(S_1, S_2) \end{cases}$$

where b_1 and b_2 are the cost-of-carry parameters and r is the instantaneous constant interest rate.

Main result

Using a PDE maximum principle, we obtain the following result on the monotonicity of the price:

> *Let G be the payoff function. If $\partial^2_{1,2} G$ is a nonpositive (resp. nonnegative) measure then the option price is nonincreasing (resp. nondecreasing) with respect to ρ.*

Proof

1. Change of variables $\tilde{S}_1 = \ln S_1$ and $\tilde{S}_2 = \ln S_2$ (example of the Spread option payoff)

$$\begin{cases} \mathcal{L}_\rho P = 0 \\ P\left(\tilde{S}_1, \tilde{S}_2, T\right) = \left(\exp \tilde{S}_2 - \exp \tilde{S}_1 - K\right)^+ \end{cases}$$

The operator $\mathcal{L}_\rho u = \frac{1}{2}\sigma_1^2\partial_{1,1}^2 u + \rho\sigma_1\sigma_2\partial_{1,2}^2 u + \frac{1}{2}\sigma_2^2\partial_{2,2}^2 u + \left(b_1 + \frac{1}{2}\sigma_1^2\right)\partial_1 u + \left(b_2 + \frac{1}{2}\sigma_2^2\right)\partial_2 u - ru + \partial_t u$ is also elliptic for $\rho \in \,]-1, 1[$.

2. We first verify the exponential growth condition.
3. Let $\Delta\left(\tilde{S}_1, \tilde{S}_2, t\right) = P_{\rho_1}\left(\tilde{S}_1, \tilde{S}_2, t\right) - P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, t\right)$ with $\rho_1 < \rho_2$. Δ is the solution of the PDE

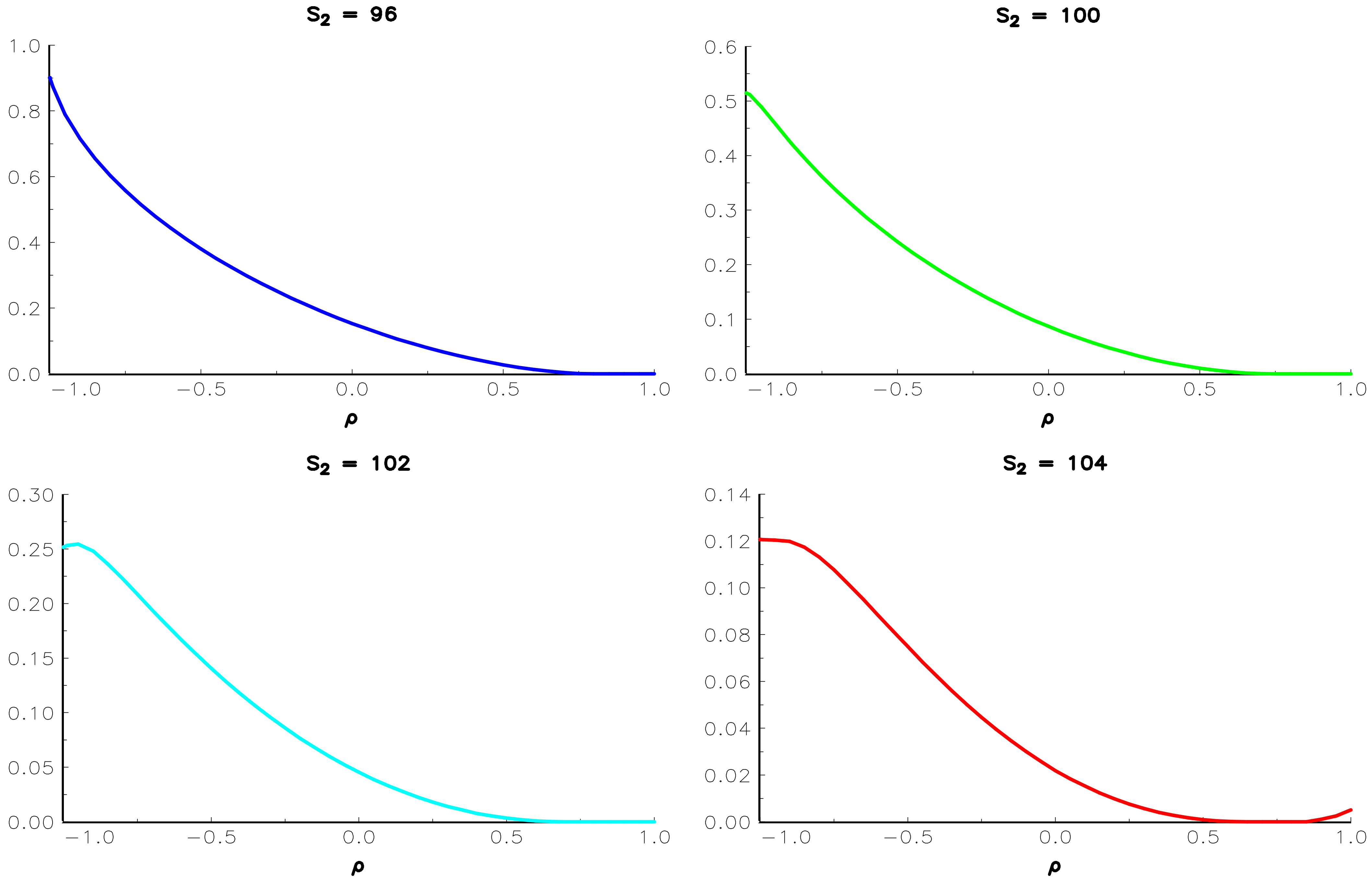
$$\begin{cases} \mathcal{L}_{\rho_1}\Delta\left(\tilde{S}_1, \tilde{S}_2, t\right) = \left(\rho_2 - \rho_1\right)\sigma_1\sigma_2\partial_{1,2}^2 P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, t\right) \\ \Delta\left(\tilde{S}_1, \tilde{S}_2, T\right) = 0 \end{cases}$$

4. In order to apply the maximum principle to $\Delta\left(\tilde{S}_1, \tilde{S}_2, t\right)$, we would like to show that $\partial_{1,2}^2 P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, t\right) \leq 0$. We show this by using again the maximum principle to $\partial_{1,2}^2 P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, t\right)$.
5. It comes finally that $\Delta\left(\tilde{S}_1, \tilde{S}_2, t\right) \geq 0$. So, we conclude that

$$\rho_1 < \rho_2 \Rightarrow P_{\rho_1}\left(\tilde{S}_1, \tilde{S}_2, t\right) \geq P_{\rho_2}\left(\tilde{S}_1, \tilde{S}_2, t\right)$$

Relationship between option prices and the correlation parameter ρ

Option type	Payoff	increasing	decreasing
Spread	$(S_2 - S_1 - K)^+$		✓
Basket	$(\alpha_1 S_1 + \alpha_2 S_2 - K)^+$	$\alpha_1\alpha_2 > 0$	$\alpha_1\alpha_2 < 0$
Max	$(\max(S_1, S_2) - K)^+$		✓
Min	$(\min(S_1, S_2) - K)^+$	✓	
BestOf call/call	$\max\left((S_1 - K_1)^+, (S_2 - K_2)^+\right)$		✓
BestOf put/put	$\max\left((K_1 - S_1)^+, (K_2 - S_2)^+\right)$		✓
Worst call/call	$\min\left((S_1 - K_1)^+, (S_2 - K_2)^+\right)$	✓	
Worst put/put	$\min\left((K_1 - S_1)^+, (K_2 - S_2)^+\right)$	✓	



Relationship between the WorstOf call/put price and ρ ($S_1 = 100$)

3.2 The general case

The European prices of two-asset options are given by

$$P(S_1, S_2, t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[G(S_1(T), S_2(T)) \middle| \mathcal{F}_t\right]$$

The multivariate RND at maturity has a copula representation

$$\mathbb{Q}(S_1(T), S_2(T)) = \mathbf{C}(\mathbb{Q}_1(S_1(T)), \mathbb{Q}_2(S_2(T)))$$

where $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are the two univariate RND at maturity.

3.2.1 Supermodular order

The function f is supermodular if and only if

$$f(x_1+\varepsilon_1, x_2+\varepsilon_2) - f(x_1+\varepsilon_1, x_2) - f(x_1, x_2+\varepsilon_2) + f(x_1, x_2) \geq 0$$

for all $(x_1, x_2) \in \mathbb{R}^2$ and $(\varepsilon_1, \varepsilon_2) \in \mathbb{R}_+^2$.

The relationship between the concordance order and supermodular functions is given by the following theorem (Tchen [1980]):

> *Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be the probability distribution functions of X_1 and X_2. Let $\mathbb{E}_{\mathbf{C}}[f(X_1, X_2)]$ denote the expectation of the function $f(X_1, X_2)$ when the copula of the random vector (X_1, X_2) is $\mathbf{C}$. If $\mathbf{C}_1 \prec \mathbf{C}_2$, then $\mathbb{E}_{\mathbf{C}_1}[f(X_1, X_2)] \leq \mathbb{E}_{\mathbf{C}_2}[f(X_1, X_2)]$ for all supermodular functions f such that the expectations exists.*

3.2.2 Generalisation of the BS result

Main result

Let G be a continuous payoff function. If the distribution $\partial^2_{1,2}G$ is a nonnegative (resp. nonpositive) measure then the option price is nondecreasing (resp. nonincreasing) with respect to the concordance order.

Special case

In the case of the Normal copula, we know that the parametric family $\mathbf{C}_\rho$ is positively ordered

$$\rho_1 < \rho_2 \Rightarrow \mathbf{C}_{\rho_1} \prec \mathbf{C}_{\rho_2}$$

$\Rightarrow$ result of the Black-Scholes model $=$ special case.

3.3 Bounds of two-asset options prices

Let $P^{-}(S_1, S_2, t)$ and $P^{+}(S_1, S_2, t)$ be respectively the lower and upper bounds:

$$P^{-}(S_1, S_2, t) \leq P_{\mathbf{C}}(S_1, S_2, t) \leq P^{+}(S_1, S_2, t)$$

We have the following proposition:

If $\partial^2_{1,2} G$ is a nonpositive (resp. nonnegative) measure then $P^{-}(S_1, S_2, t)$ and $P^{+}(S_1, S_2, t)$ correspond to the cases $\mathbf{C} = \mathbf{C}^{+}$ (resp. $\mathbf{C} = \mathbf{C}^{-}$) and $\mathbf{C} = \mathbf{C}^{-}$ (resp. $\mathbf{C} = \mathbf{C}^{+}$).

$\Rightarrow$ Explicit bounds can then be derived.

3.4 The multivariate case

BS model

- If we fix all the correlations $\rho_{i,j}$ except one, we retrieve the same condition as in the two-assets options case.
- If $\rho_{i,j} = \rho$, we have the following result: *Assume that G is continuous. If $\sum_{i<j} \sigma_i \sigma_j \partial^2_{i,j} G$ is a nonnegative (resp. nonpositive) measure, then the price is nondecreasing (resp. nonincreasing) with respect to ρ.* As an example, with the three-asset option $G\left(S_1, S_{2,} S_3\right) = (S_1 + S_2 - S_3 - K)^+$, we have $\sum_{i<j} \sigma_i \sigma_j \partial^2_{i,j} G = (\sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3)\, \delta_{(S_1+S_2-S_3-K=0)}$. Hence, if $\sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3 > 0$, the price nondecreases with ρ, and if $\sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3 < 0$, the price nonincreases with ρ.

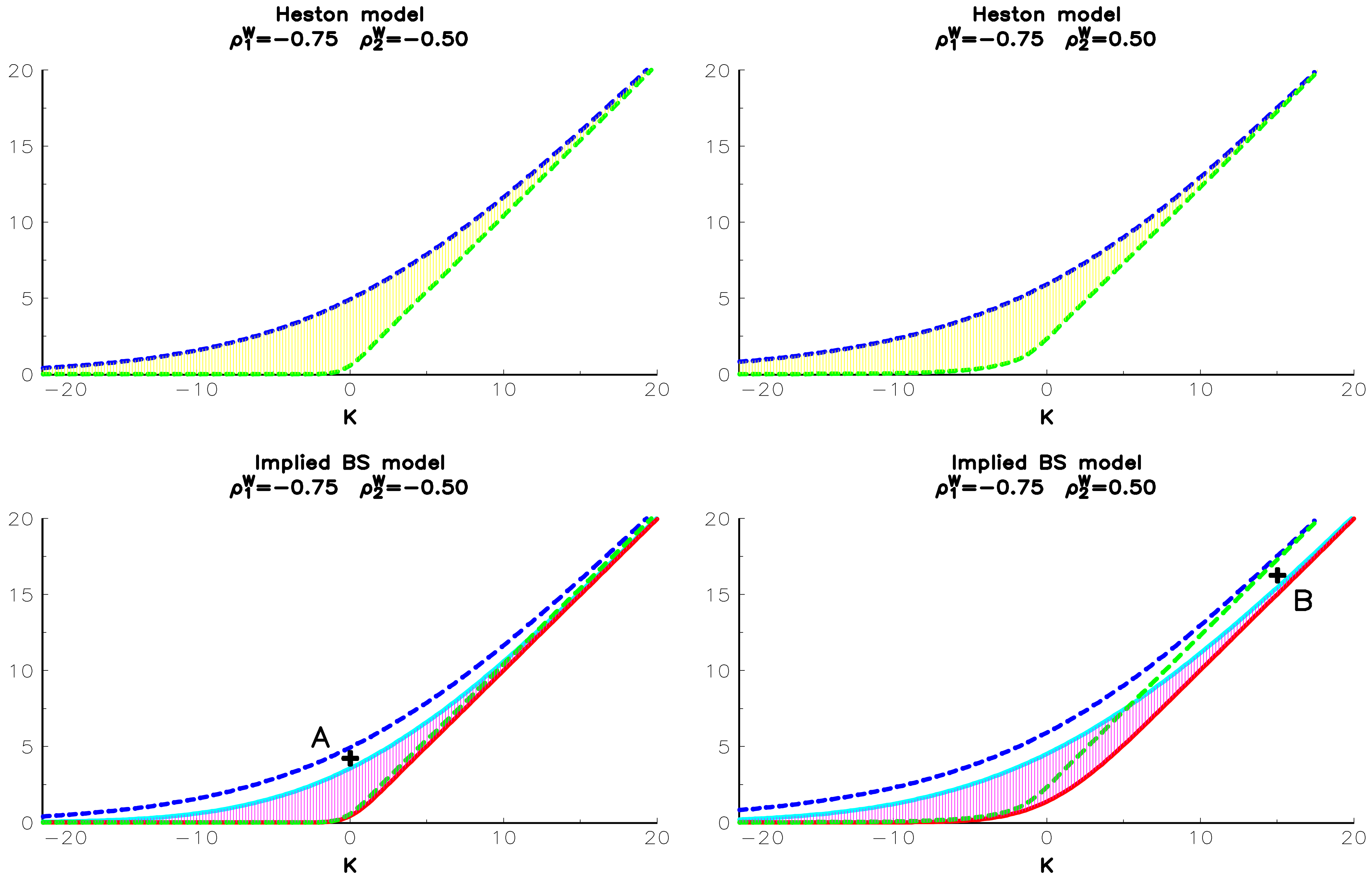
General case Open problem because the supermodular order is strictly stronger than the concordance order for dimension bigger than three.

4 Copulas, multivariate risk-neutral distributions and implied dependence functions

Suppose that the bank uses two models: a model $\mathcal{M}$ for one-asset options and the Black-Scholes model for multi-asset options. In this case, the marginals of the multivariate RND are not the univariate RND.

What could be the problems?

1. Arbitrage opportunities
2. Hedging strategies
3. Pricing coherence



Bounds of the Spread put option

4.1 From the multivariate RND to the risk-neutral copula

The margins of the risk-neutral distribution $\mathbb{Q}$ are necessarily the univariate risk-neutral distributions $\mathbb{Q}_n$.

Using Sklar's theorem, it comes that the RND $\mathbb{Q}^t$ at time t has the following canonical representation:

$$\mathbb{Q}^t(x_1, \ldots, x_N) = \mathbf{C}_t^{\mathbb{Q}}\left(\mathbb{Q}_1^t(x_1), \ldots, \mathbb{Q}_N^t(x_N)\right)$$

$\mathbf{C}_t^{\mathbb{Q}}$ is called the risk-neutral copula (RNC) (Rosenberg [2000]). It is the dependence function between the risk-neutral random variables.

4.2 Pricing formulas

Following Breeden and Litzenberger [1978], the main idea is to differentiate the price with respect to the strike.

Then, the price of the spread option is given by

$$\begin{aligned} P^{c}(t_0) &= S_2(t_0) - S_1(t_0) - K e^{-r(T-t_0)} + \\ & e^{-r(T-t_0)} \int_0^{+\infty} \int_{-x}^{K} f_1(x) \partial_1 \mathbf{C}^{\mathbb{Q}}(\mathbf{F}_1(x), \mathbf{F}_2(x+y)) \, \mathrm{d}x \, \mathrm{d}y \end{aligned}$$

The price of the put on the minimum of two assets is

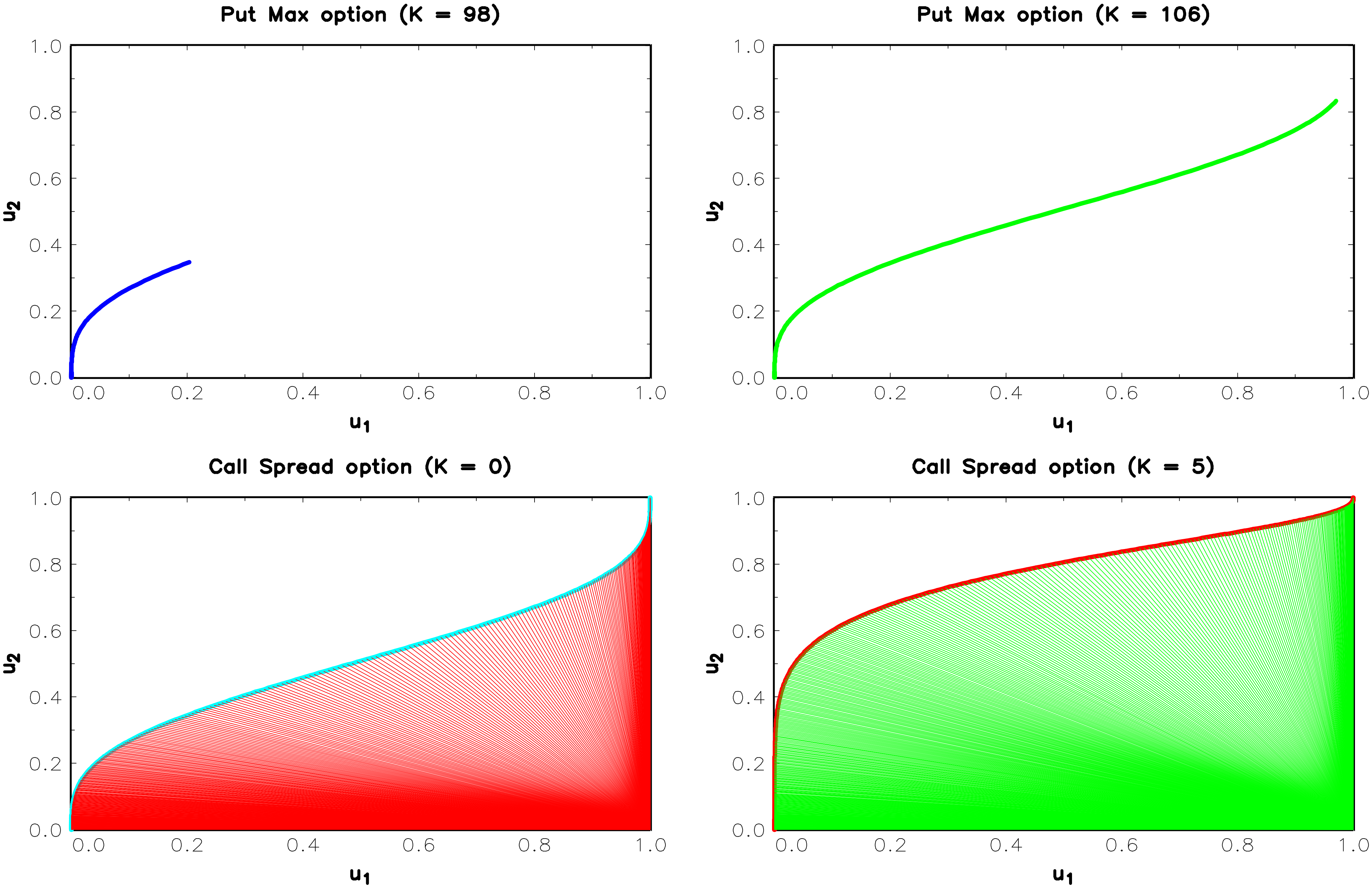
$$P^{p}_{\max}(t_0) = e^{-r(T-t_0)} \int_0^{K} \mathbf{C}^{\mathbb{Q}}(\mathbf{F}_1(y), \mathbf{F}_2(y)) \, \mathrm{d}y$$

4.3 Implied dependence functions of asset returns

Bikos [2000] suggests then the following method:

1. Estimate the univariate RND $\hat{\mathbb{Q}}_n$ using Vanilla options;
2. Estimate the copula $\hat{\mathbf{C}}$ using multi-asset options by imposing that $\mathbb{Q}_n = \hat{\mathbb{Q}}_n$;
3. Derive "forward-looking" indicators directly from $\hat{\mathbf{C}}$.

It seems more relevant to calibrate copulas rather with Spread options than Max options.



Information on the copula used by multi-asset options

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