# Revisiting the dependence between financial markets with copulas


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**Abstract**

We consider the problem of modelling the dependence between financial markets. In financial economics, the classical tool is the Pearson (or linear correlation) to compare the dependence structure. We show that this coefficient does not give a precise information on the dependence structure. Instead, we propose a conceptual framework based on copulas. Two applications are proposed. The first one concerns the study of extreme dependence between international equity markets. The second one concerns the analysis of the East Asian crisis.


## 1 Introduction

This paper is related to some recurrent concerns in the economic and financial literatures. In the former one, it is linked to the notion of contagion between international markets, and in particular the propagation of financial crises between neighbouring countries and their repercussion on economic activity. Such concern has recently gained increased interest in view of the emerging markets crises in the nineties. In the latter one, it is linked to the notions of risk and portfolio diversification. In recent years, risk management methods such as *VaR* or the enlargement of financial integration to a broader scope of countries have renewed the interest in these topics.

Both literatures meet in their requirement to model the dependence between financial markets. With very few exceptions, the empirical treatment of this problem rests on standard statistical tools and/or standard econometric methods (regression methods, time series method, dichotomous models, quantile regressions). A commonly used tool is the standard (Pearson) correlation coefficient. One of the purpose of this paper is to argue that the use of this tool can lead to very misleading inference in this context. A well-known reason (see e.g. Boyer, Gibson, Loretan [1999] or Forbes and Rigobon [1999]) is that it is not robust to heteroskedascity (i.e. the fact that volatility is time-varying). As we shall below, other reasons can be invoked to be doubtful about analysis based on the Pearson correlation coefficient.

Recently, Straetmans [1999], Stărică [1999] and Longin and Solnik [2000] have proposed to rely on extreme value theory (EVT), and especially on multivariate tools associated with this literature, to analyse dependence between financial markets. As an extension, this paper relies on the notion of copulas which generalizes their approach and is rather new in this context (see however Embrechts, McNeil and Straumann [1999] or Bouyé, Durrleman, Nikeghbali, Riboulet and Roncalli [2000]). In broad terms, a copula can be defined as the dependence function between random variables. More precisely, it can be seen as a function that links univariate marginals to their full multivariate distribution. While they were introduced in 1959, the literature on copulas, and especially the study of their statistical properties and their applications, only

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developed in recent years. The main purpose of this paper is to introduce a conceptual framework based on copulas for the analysis of the dependence between financial markets. Furthermore, we propose two applications. The first ones concerns the study of extreme dependence between international equity markets. The second one concerns the analysis of the East-Asian crisis.

The layout of the paper is as follows. Section 2 operates a literature review on the dependence between financial markets with special emphasis on empirical methods. Section 3 offers introductions to copulas and multivariate extreme value theory and propose a new statistical framework based on copulas. Furthermore, it presents illustration of the associated tools with an application to the interactions between three major stock indexes (CAC 40, Dow Jones and Nikkei). Section 4 presents the application to the analysis of the Asian crisis. Section 5 concludes the article.

# 2 Literature review on the dependence between financial markets

The purpose of this section is to review the literature on the dependence between financial markets. As this literature is very large, it only recalls the main topics and results which emerge from it. The interested reader is invited to consult cited articles and references therein. Furthermore, we voluntarily ignore articles which rely on EVT as they are analyzed more fully in the next section.

As described in the introduction, the literature can be separated in two (but not really distinct) parts. The first one analyses the correlation between financial markets with a regard to the notions of portfolio diversification and financial integration. The second one is more economic in nature and relies on the notion of contagion during "balance of payments crises".

## 2.1 Correlation between financial markets

International financial market interactions have been widely studied. This literature is mainly empirical. Indeed, the foundations of the background theory have been laid down since the sixties and deal with portfolio diversification and/of international financial markets integration. This way, the empirical literature is originally concerned wth the long run relationships between markets or assets. Recently, however, empirical work has evolved towards the analysis of the short term interaction between financial markets on the basis of high frequency data (daily or intra-daily). Moreover, some studies tend to concentrate on certain periods and, in this case, on extremely volatile periods. This orientation has been motivated by studies such as Longin and Solnik [1995] which, on the basis of long term analysis, have noted that interactions between financial markets tend to be higher in volatile periods or, simply, on the observation that shocks in one market tend to propagate quicker when they are large (e.g. a stock market krach).

Most of this empirical literature[1] relies on standard econometric and statistical methods applied to multivariate returns or volatility measures (absolute or squared returns). For instance, the tools used in these studies fall in one of the following categories: linear regressions and/or lead-lag cross-correlations (Eun and Shim [1990], King and Wadhwani [1990], Becker et al. [1993], Susmel and Engle [1993], Lee and Kim [1993], Kofman and Martens [1997], Karolyi and Stulz [1996]), quantile regressions (Granger and Shin [2000]), Vector Autoregressions (VAR) (von Furstenberg and Jeon [1989]), GARCH or similar framework (Chou et al. [1994], Hamao et al. [1990], Lin et al. [1994]), cointegration methods (Longin and Solnik [1995], Chou et al. [1994], Cashin et al. [1995]). Previously cited studies concentrate on stock market interactions as does most of the literature, notable exceptions being Engle at al. [1990] on foreign exchange markets and Edwards [1998] on bonds markets. In some cases, special attention is given to crises (especially the 1987 stock market krach) but most of the times, such empirical work is dedicated to the analysis of relationships between markets irrespective of whether they are in turmoil or not.

[1] As we shall see below, two notable exceptions are Straetmans [1999] and Longin and Solnik [2000].

In this paper, we introduce a conceptual framework which wander from standard methods and allows to study differences in interactions between markets conditionaly on their state (volatile or not) in a natural way.

## 2.2 Contagion and "balance of payments" crisis

The recent financial crisis in emerging markets and, above all, the fact that they have hit several countries almost at the same time has led to an increased interest in the notion of contagion. The purpose of this section is to give an overview of theoretical and empirical literatures on "balance of payments" crises and contagion. More detailed descriptions can be found in recent articles (see e.g. Forbes and Rigobon [2000], Masson [1999, 2000], Claessens, Dornbusch and Park [2000]).

Obviously, this literature is naturally linked to the previous subsection one. However, it does depart from it as it clearly concentrates on crisis periods (and particularly exchange rate crises and capitals flights, so the term "balance of payments crises") and it is motivated by broad economic explanations of the phenomena (not just international diversification).

### 2.2.1 Theoretical literature

The notion of contagion shall be a quite precise one. However, in the literature, it is often confused with the broader term of international propagation of shocks. Forbes and Rigobon introduce an useful distinction between non-crisis-contingent theories and crisis-contingent theories[2].

Non-crisis-contingent theories refer to the international propagation of shocks without assuming that the transmission mechanisms have changed after the initial shock. In other terms, these theories refer to cases where the transmission is well founded on economic linkages between affected countries and/or on signal extraction from investors. This approach highlights four main channels: trade spillovers, financial linkages, common external factors and learning.

- Trade spillovers come from the fact that when a country faces a significant depreciation of its currency, other countries can suffer from a lost of competitiveness relative to the crisis country, both in bilateral trade and in third-country markets. Furthermore, if the exchange rate crash leads to a downturn in economic activity in the crisis country, the associated income effects further depress the exports of trade partners. This mechanism is formalized in Gerlach and Smets [1995] and some empirical support for it is found in Eichengreen, Rose and Wyplosz [1996] and Glick and Rose [1998].
- Financial linkages can induce propagation of shocks when investors are conducted to rebalance their portfolios after the initial shock. Investors are induced to liquid their position in other countries for risk management (if assets from both countries/zones are positively correlated) or liquidity (to cover losses or margin calls suffered in the first country) purposes. This approach has been explored by, inter alia, Baig and Goldfajn [1999], Bussière and Mulder [1999] or Valdès [1996].
- Common external factors ("moonsonal effects", Masson [2000]) are mainly defined as major economic shifts in industrial countries that trigger crises in emerging markets. For instance, such common shocks include a rise in world (or U.S.) interest rates, changes in bilateral exchange rates between the major world economics, slowdown in world aggregate demand. While mainly independent of crisis countries'policies, these shocks can affect asymmetrically these countries, depending on various factors such as exposure to foreign currency borrowing, size of the government debt or inefficiencies in the banking system. As examples of such common shocks, the increase of interest rates in the United States in the early eighties or the appreciation of the U.S. dollar against the yen have frequently be cited as major factors in the debt crisis or the Asian crisis respectively.

[2]In some respects, the distinction is based on the traditional opposition in the literature on balance of payments crises between first generation models (*à* la Krugman [1979]) which put emphasis on deteriorating fundamentals and second generation models (*à* la Obstfeld [1986]) which put emphasis on self-fulfilling expectations.

- Learning argues that a crisis in one country can serve as a "wake-up call" (Goldstein [1998]) which induces investors to reevaluate their sentiment and/or risk aversion towards countries with similar macroeconomic structures and policies. For example, if a country fails in reason of a weak banking system or financial vulnerabilities, investors could reevaluate their judgement of the banking and financial systems in other countries and adjust their expected probabilities of a crisis accordingly. While pre-shock behavior of investors seems to be irrational as they underestimate weaknesses of the country, the post-shock behavior appears rational as it is justified by a signal extraction approach (Kodres and Pritsker [1999]). Moreover, the pre-shock behavior can be induced by dissimulation of information which precludes realistic perception of the fundamentals.

Crisis-contingent theories refer to the international propagation of shocks with the assumption that the transmission mechanisms have changed after the initial shock. In other terms, these theories refer to cases where the transmission is not justified by economic and financial fundamentals or real linkages between markets. In this perspective, even countries with sound fundamentals or minor structural problems can be affected, only because of the self-fulfilling expectations of investors. This approach highlights three main explanations: herd behavior, multiple equilibria and endogenous liquidity shocks.

- Calvo [1996] presents a model in which global investors are subject to herd behavior, due to a lack of precise information seeking. Given that it is costly to develop precise monitoring and fundamental evaluation of each market, it is optimal (given that they are uniformed) to run away simultaneously from a group of markets when some problems appear in one of them.
- Multiple equilibria occurs when a crisis in one country is used as a sunspot for other countries (see Masson [2000] for a formalization). Small triggers in a country can act as a precipitating factor which induces a coordination of investor's expectations on the bad equilibrium for others countries. The shift from the good to the bad equilibrium is solely driven by a change in investor beliefs without any change in underlying fundamentals.
- Finally, Calvo [1999] proposes a model of endogenous liquidity shocks in presence of asymmetric information. When informed traders are hit by pure liquidity shocks, uninformed ones cannot distinguish whether sells come from liquidity shocks or bad signal arriving. In other words, they tend to over-interpret small (purely technical) price changes.

#### 2.2.2 Empirical literature

The analysis of currency crisis and their propagation has generated a lot of empirical literature. As one of the primary objective of this paper is to introduce a new statistical framework, we review the empirical literature from the methodological viewpoint rather than from the results themselves. Moreover, we exclude the empirical analysis of the Asian crisis which is specifically reviewed in the section 4. Basically we can consider three classes of methods: time series and regression methods, dichotomous models and standard statistical analysis of subsamples comparisons.

As noted before, empirical approach grounded on time series and regression methods are very often tests of market integration in the same time that they are tests of contagion. This approach has been surveyed in the paragraph 2.1.

Dichotomous models take the form of probit models of the probability of a crisis in one country or regression involving a discrete value endogenous variable (1 for crisis, 0 elsewhere). This kind of methodology (as also the standard statistical analysis) need to define a crisis period. This can be done in two main ways. The first solution is to "let the data speak themselves" and to adopt an algorithm to identify crisis. For example, we can define the beginning of the crisis as a time window inside which currencies or stock market returns (or a weighted average of returns and macroeconomic variables such as international reserves) are below a certain fractile of their own distribution. Second, we can choose these periods in an informal way, based on newspaper reports.

In general cases, dichotomous models try to infer the relationship between the evolution of fundamentals (real exchange appreciation, ratio of broad money to international reserves, ratio of current account deficit to GDP, etc) and the probability/occurrence of the crises (see e.g. Kaminsky et al. [1997], Berg and Patillo [1998]). They become more direct analysis of contagion when they integrate the possibility that the occurrence of the crisis in one country affect the probability of a crisis occurring in other countries (after controlling for fundamentals). Eichengreen, Rose and Wyplosz [1996] found such interrelationship in the case of the ERM crisis of 1992-1993. Using a larger panel, Kaminsky and Reinhart [1998] also found evidence of such interaction[3], especially when countries are located in the same geographical region, a phenomenon which is linked to trade linkages[4] by Glick and Rose [1998]. Ahluwalia [2000] shows the relevance for the emerging markets crises of the nineties (Mexico, Southeast Asia and Russia) of various contagion indicators, including one for the "wake-up call" hypothesis.

Standard statistical analysis of subsamples comparisons is usually done by comparing cross-correlations between a subsample defined as a crisis period and another subsample defined either as non-crisis period or as a total sample (i.e. including the crisis period). The underlying motivation is simple : if cross-correlations significantly increase in the crisis period, then it is evidence of contagion. However, such an approach can be misleading as (standard) correlation is not a robust statistics for the identification of dependence. A well-known reason (see Boyer, Gibson and Loretan [1999], Forbes and Rigobon [1999, 2000]) is that it is not robust to heteroskedasticity while such a phenomenon is obviously present in the data since a crisis period is by definition a more volatile one. Hence, Forbes and Rigobon [1999, 2000] propose an adjusted heteroskedastic consistent correlation formula. In this paper, we argue that there are deeper reasons to be careful with results based on correlation estimates. In this way, the adjustment proposed by Forbes and Rigobon can only marginally allow to obtain robust results. This point will be discussed more fully in the next section.

# 3 A new framework based on copulas

## 3.1 A brief introduction to copulas

Copulas have been introduced by Sklar [1959] to study probabilistic metric spaces. They have been rediscovered on several occasions by statisticians in the seventies (see Deheuvels [1978], Galambos [1978] and Kimeldorf and Sampson [1975]). However, the first statistical applications of copulas appear only in the middle of the eighties. In this paragraph, we adopt a simplified point of view to present copulas, and we invite the reader to consult the book of Nelsen [1998] to have a more rigourous presentation. Moreover, we restrict to the two-dimensional case, but generalization to higher dimensions is straightforward.

**Definition 1** *A 2-copula* $\mathbf{C}$ *is a bivariate uniform distribution.*

$\mathbf{C}$ is then a function from $[0,1]^2$ to $[0,1]$ with positive probability masses. Because all the margins are uniform, we have

$$\begin{aligned} \mathbf{C}(1,u_2) &= u_2 \quad (1) \\ \mathbf{C}(u_1,1) &= u_1 \quad (2) \end{aligned}$$

By construction, it comes that the function $\mathbf{F}$ defined by

$$\mathbf{F}(x_1,x_2) = \mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)) \quad (3)$$

[3]See also Caramazza et al. [2000].

[4]The fact that crises are mainly regional has been challenged by the transmission of the russian "virus" to countries like Brazil or to the U.S. corporate bonds market.

where $\mathbf{F}_n$ is a univariate distribution is a bivariate probability distribution. By definition, the margins of this joint distribution are $\mathbf{F}_1$ and $\mathbf{F}_2$. However, the main interest of copulas is summarized by the fundamental theorem of Sklar [1959].

**Theorem 2** *Let* $\mathbf{F}$ *be an* 2*-dimensional distribution function with margins* $\mathbf{F}_1$ *and* $\mathbf{F}_2$*. Then, there exist almost one copula such that*

$$\mathbf{F}(x_1, x_2) = \mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)) \tag{4}$$

*Moreover, if the margins are continuous, the copula function is unique.*

What is the probabilistic interpretation of the function $\mathbf{C}$? During 20 years, Schweizer and Sklar have explored intensively copulas in the framework of probabilistic metric spaces. They have seen that there are some connections between copulas and random variables (see Sklar [1973]). During a sabbatical leave in Italy, Schweizer find a paper of Rényi [1959] on measures of dependence. Schweizer has then the idea to connect copulas with measures of dependence. The work of his Ph.D. student Wolff is summarized in the article of Schweizer and Wolff [1981] published in *Annals of Statistics*. Here is a part of their abstract:

> *We show that the copula of a pair of random variables* $X, Y$ *is invariant under a.s. strictly increasing transformations of* $X$ *and* $Y$*, and that any property of the joint distribution function of* $X$ *and* $Y$ *which is invariant under such transformations is solely a function of their copula.*

For example, they show that the Kendall's tau and the Spearman's rho can be written in terms of copulas. Let $X_1$ and $X_2$ two continuous random variables with joint distribution $\mathbf{F}$. Let $(Y_1, Y_2)$ be a random vector with the same distribution $\mathbf{F}$ independent of the vector $(X_1, X_2)$. The kendall's tau is defined as the concordance probability minus the discordance probability for the two vectors $(X_1, X_2)$ and $(Y_1, Y_2)$ (Nelsen, [1998]):

$$\tau = \Pr\{(X_1 - Y_1)(X_2 - Y_2) > 0\} - \Pr\{(X_1 - Y_1)(X_2 - Y_2) < 0\} \tag{5}$$

Schweizer and Wolff [1981] show that another expression of the Kendall's tau is

$$\tau = 4 \iint_{[0,1]^2} \mathbf{C}(u_1, u_2)\, \mathrm{d}\mathbf{C}(u_1, u_2) - 1 \tag{6}$$

with $\mathbf{C}$ the associated copula of the joint distribution $\mathbf{F}$. Let $(Z_1, Z_2)$ be a random vector with distribution $\mathbf{F}$ independent of the vectors $(X_1, X_2)$ and $(Y_1, Y_2)$. The Spearman's rho is proportional to the concordance probability minus the discordance probability for the two vectors $(X_1, X_2)$ and $(Y_1, Z_2)$ (Nelsen [1998]). Like the Kendall's tau, the Spearman's rho admits a copula representation:

$$\varrho = 12 \iint_{\mathbf{I}^2} u_1 u_2 \, \mathrm{d}\mathbf{C}(u_1, u_2) - 3 \tag{7}$$

Nelsen [1998] show that these two measures satisfy the axiomes of a measure of concordance of Scarsini [1984].

What about the (standard) Pearson correlation measure? It is defined as

$$\rho(X_1, X_2) = \frac{\mathbb{E}[(X_1 - \mathbb{E}[X_1])(X_2 - \mathbb{E}[X_2])]}{\sqrt{\operatorname{var}[X_1]\operatorname{var}[X_2]}} \tag{8}$$

or

$$\rho(X_1, X_2) = \frac{1}{\sqrt{\operatorname{var}[X_1]\operatorname{var}[X_2]}} \iint_{[0,1]^2} [\mathbf{C}(u_1, u_2) - u_1 u_2]\, \mathrm{d}\mathbf{F}_1^{-1}(u_1)\, \mathrm{d}\mathbf{F}_2^{-1}(u_2) \tag{9}$$

The Pearson correlation can **not** be a dependence measure of a vector of random variables. As explained by Embrechts, McNeil and straumann [1999], it is a natural dependence measure only in the case of the multivariate normally and elliptically distributions.

While ancient in the probabilistic domain, the interprtation of copulas in statistical terms appeared later with DEHEUVELS [1978] and can be summarized as follows : **The copula of random variables is in fact the dependence function of these random variables.**

There are many copula functions (see JOE [1997] and NELSEN [1998]). We present here two copulas which can be of special interest in financial modelling. The first one is the Normal copula, the normal distribution being the most current one in finance (for example, in portfolio theory). Let $\Phi$ be the $\mathcal{N}(0,1)$ cumulative density function and $\Phi_\beta$ the bivariate gaussian cdf with correlation $\beta$. The bivariate Normal copula is defined by

$$\begin{aligned}\mathbf{C}(u_1,u_2) &= \Phi_\beta\left(\Phi^{-1}(u_1),\Phi^{-1}(u_2)\right) \\ &= \int_{-\infty}^{\Phi^{-1}(u_1)}\int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\beta^2}}\exp\left(-\frac{1}{2}\left(\frac{x^2-2\beta xy+y^2}{1-\beta^2}\right)\right)\mathrm{d}x\,\mathrm{d}y \qquad (10)\end{aligned}$$

Its corresponding density is[5]

$$c(u_1,u_2)=\frac{1}{\sqrt{1-\beta^2}}\exp\left(-\frac{1}{2}\left(\frac{\varsigma_1^2-2\beta\varsigma_1\varsigma_2+\varsigma_2^2}{1-\beta^2}-\left(\varsigma_1^2+\varsigma_2^2\right)\right)\right) \qquad (12)$$

with $\varsigma_n=\Phi^{-1}(u_n)$. By definition, a bivariate normal distribution corresponds to a Normal copula with gaussian margins. This copula is useful, because we could define other distributions with the same dependence structure as the normal distribution with non-gaussian margins. The second simple copula is the Gumbel one, the univariate Gumbel distribution being one of the three Generalized Extreme Value distribution. Another copula, which is extensively used in finance, is the Gumbel copula. Let $\beta$ be a scalar parameter with $\beta \geq 1$. We have

$$\mathbf{C}(u_1,u_2)=\exp\left(-\left[(-\ln u_1)^\beta+(-\ln u_2)^\beta\right]^{\frac{1}{\beta}}\right) \qquad (13)$$

and

$$c(u_1,u_2)=\frac{(-\ln u_1-\ln u_2)^{\beta-1}\exp\left(-\left[(-\ln u_1)^\beta+(-\ln u_2)^\beta\right]^{\frac{1}{\beta}}\right)\left(\left[(-\ln u_1)^\beta+(-\ln u_2)^\beta\right]^{\frac{1}{\beta}}+\beta-1\right)}{u_1u_2\left[(-\ln u_1)^\beta+(-\ln u_2)^\beta\right]^{2-\frac{1}{\beta}}} \qquad (14)$$

This copula is also known under the name of the logistic dependence function (TAWN [1988]) and it has recently been used by LONGIN and SOLNIK [1999] (while they do not explicitly refer to copulas).

Normality assumption is frequently done in finance, because of tractability aspects. However, copulas permit easily to leave the Gaussian world. In figure 1, we have plotted the density of the Normal and Gumbel copulas. The parameters have been choosen such that the Kendall's tau is equal to 70%. In the same graph, we have reported the density of the distribution of the random vector $(X_1,X_2)$ where $X_1\sim\mathcal{N}(0,1)$ and $X_2\sim\mathcal{N}(0,2)$. To see the difference between the two distributions, we have simulated 5000 random numbers of $(X_1,X_2)$ when the copula is respectively Normal and Gumbel (see figure 2). We see that the two bivariate series are very different. Normality assumption could then be wrong because the dependence structure is not Normal. But it could also be wrong because the margins are not gaussians. In figure 3, we have simulated four bivariate series $(Y_1,Y_2)$ with the same means and covariance matrix[6]. If we fit a bivariate gaussian distribution, we could

[5] We have

$$c(u_1,u_2)=\frac{\partial^2\,\mathbf{C}(u_1,u_2)}{\partial u_1^2\,\partial u_2^2} \qquad (11)$$

[6] Let $(X_1,X_2)$ be a random vector. $(Y_1,Y_2)$ is defined as follows:

$$\begin{aligned}Y_1 &= \frac{X_1-\mathbb{E}[X_1]}{\sigma[X_1]} \\ Y_2 &= \rho\frac{X_1-\mathbb{E}[X_1]}{\sigma[X_1]}+\sqrt{1-\rho^2}\frac{X_2-\mathbb{E}[X_2]}{\sigma[X_2]} \qquad (15)\end{aligned}$$

then consider that the four random vectors have the same joint distribution! To give an idea of the underlying dependence structure, we have plotted their empirical uniforms.

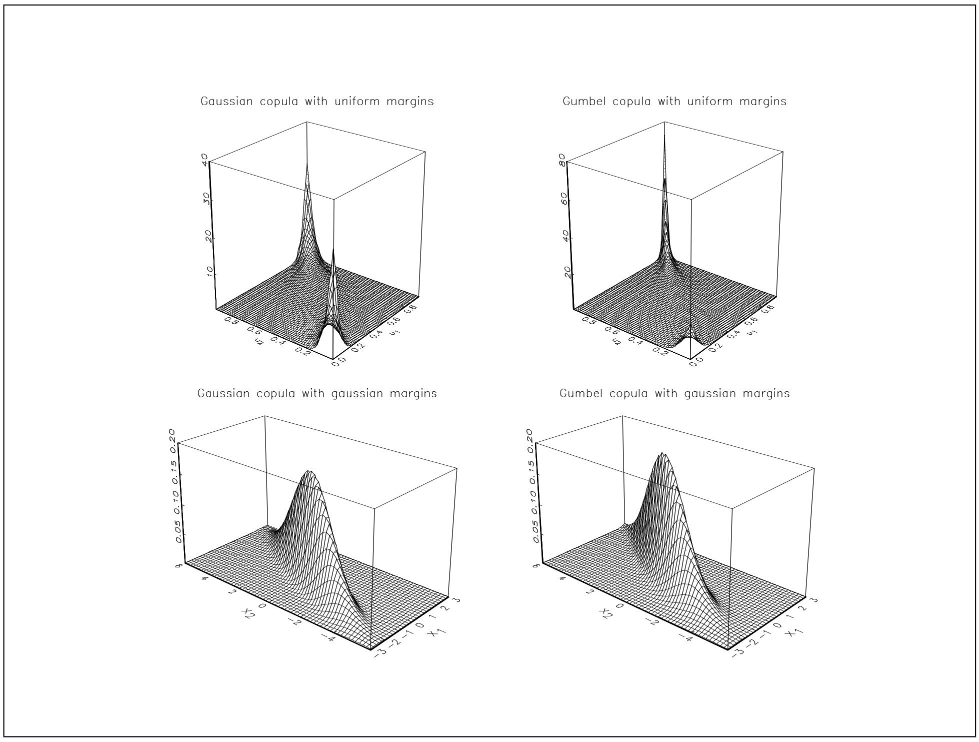


Figure 1: Plot of the probability density function

We end this paragraph with the comparison problem of copula functions. Three copulas play an important role

$$\begin{aligned}\mathbf{C}^{-}(u_1,u_2) &= \max(u_1+u_2-1,0)\\ \mathbf{C}^{\perp}(u_1,u_2) &= u_1u_2\\ \mathbf{C}^{+}(u_1,u_2) &= \min(u_1,u_2)\end{aligned} \tag{16}$$

We can prove that for any copula $\mathbf{C}$, we have

$$\mathbf{C}^{-}(u_1,u_2) \leq \mathbf{C}(u_1,u_2) \leq \mathbf{C}^{+}(u_1,u_2) \tag{17}$$

for all $(u_1,u_2)$ in $[0,1]^2$. Moreover, we say that $\mathbf{C}_1$ is smaller than $\mathbf{C}_2$ and write $\mathbf{C}_1 \prec \mathbf{C}_2$ if $\mathbf{C}_1(u_1,u_2) \leq \mathbf{C}_2(u_1,u_2)$ for all $(u_1,u_2)$ in $[0,1]^2$ (see section 2.8 of Nelsen [1998]). Let $X=(X_1,X_2)$ and $Y=(Y_1,Y_2)$ be two random vectors. We note $\mathbf{C}_X$ and $\mathbf{C}_Y$ the corresponding copulas. If $\mathbf{C}_X \prec \mathbf{C}_Y$, we will say that $X$ is less dependent that $Y$. $\prec$ defines a **partial** ordering relationship, because two copulas can not **always** be compared. But what is the interpretation of the three copulas $\mathbf{C}^{-}$, $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$? The answer is the following (see Wang [1999] and Mikusiński, Sherwood and Taylor [1997]):

- Two random variables $X_1$ and $X_2$ are **countermonotonic** — or $\mathbf{C}=\mathbf{C}^{-}$ — if there exists a random variable $X$ such that $X_1=f_1(X)$ and $X_2=f_2(X)$ with $f_1$ non-increasing and $f_2$ non-decreasing;

In this example, $\rho$ is equal to 0.7.

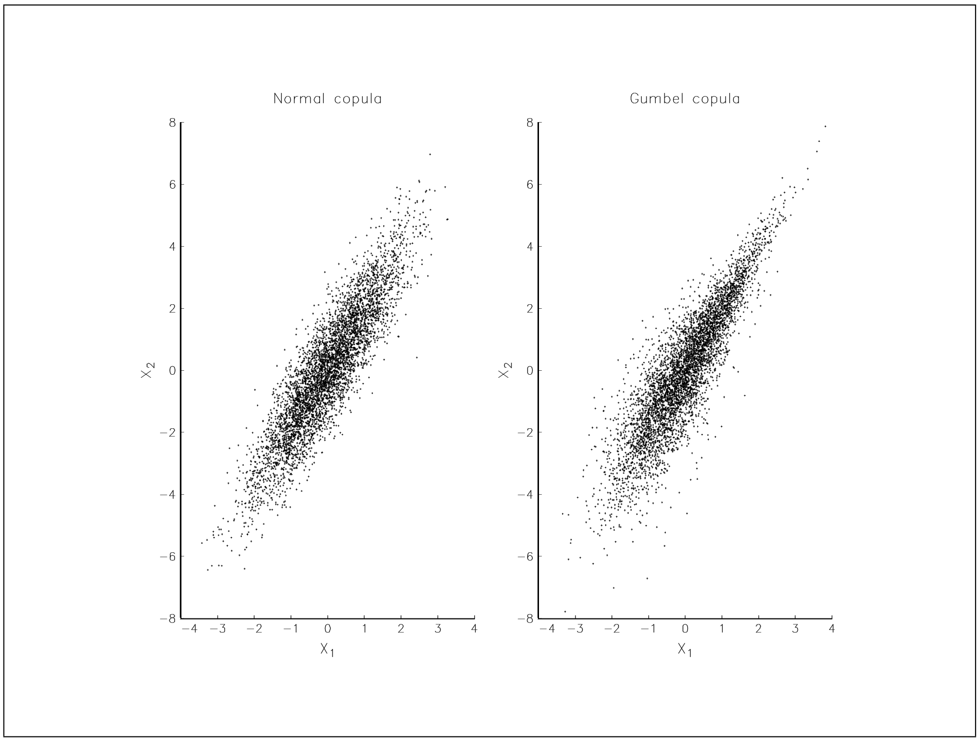


Figure 2: Random numbers of the joint distribution of the random variables $(X_1, X_2)$

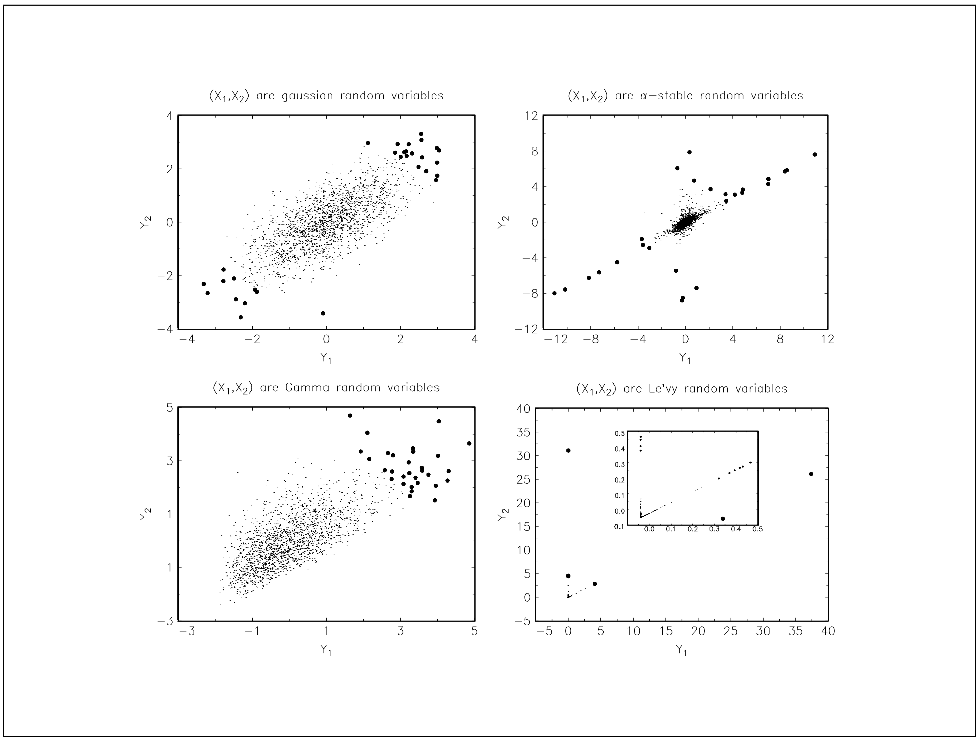


Figure 3: Four bivariate series with the same means and the same covariance matrix

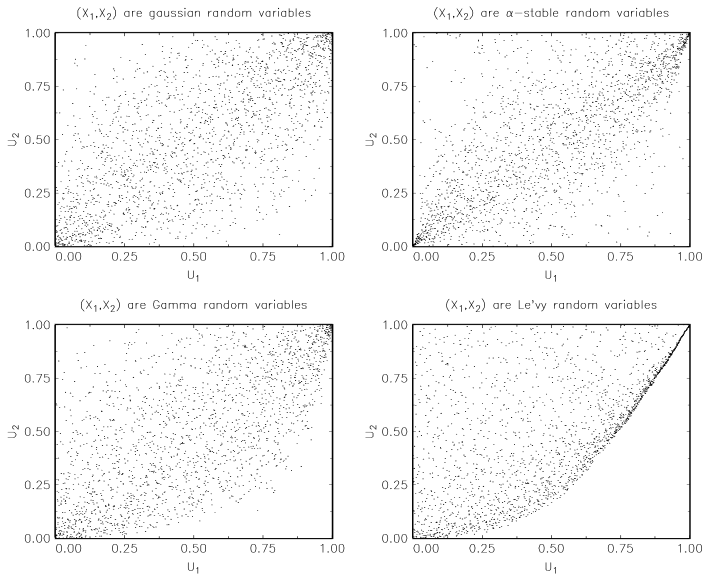


Figure 4: Empirical uniforms of the previous bivariate series

- Two random variables $X_1$ and $X_2$ are **independent** if the dependence structure is the product copula $\mathbf{C}^{\perp}$;
- Two random variables $X_1$ and $X_2$ are **comonotonic** — or $\mathbf{C} = \mathbf{C}^{+}$ — if there exists a random variable $X$ such that $X_1 = f_1(X)$ and $X_2 = f_2(X)$ where the functions $f_1$ and $f_2$ are non-decreasing;

Suppose that $\mathbf{C}_1$ and $\mathbf{C}_2$ are such that we verify

$$\mathbf{C}^{-} \prec \mathbf{C}_1 \prec \mathbf{C}^{\perp} \prec \mathbf{C}_2 \prec \mathbf{C}^{+} \tag{18}$$

We will say that $\mathbf{C}_1$ is a **negative** dependence structure and $\mathbf{C}_2$ is a **positive** dependence structure. For Normal copulas, we have

$$\mathbf{C}_{\beta=-1} = \mathbf{C}^{-} \prec \mathbf{C}_{\beta<0} \prec \mathbf{C}^{\perp} = \mathbf{C}_{\beta=0} \prec \mathbf{C}_{\beta>0} \prec \mathbf{C}^{+} = \mathbf{C}_{\beta=1} \tag{19}$$

In the case of the Gumbel copula, we verify that

$$\mathbf{C}^{\perp} = \mathbf{C}_{\beta=1} \prec \mathbf{C}_{\beta>1} \prec \mathbf{C}^{+} = \mathbf{C}_{\beta=+\infty} \tag{20}$$

Instead of the Normal copula, the Gumbel one does not allow negative dependence. Moreover, it is not possible to compare a Normal copula with a Gumbel copula (except for the trivial cases).

## 3.2 A brief introduction to extreme value theory

Because we are interested in crises, and so in the joint behaviour of extreme returns, it seems natural to use a framework based on extreme value theory. This is for example the point of view adopted by Straetmans [1999] and Longin and Solnik [1999]. We remind here some results, but we invite the reader to consult Joe [1997] for a more complete presentation of this theory.

We restrict to the bivariate case, but generalization to higher dimensions is straightforward (while notation is "heavier"). We consider two random variables $X_1$ and $X_2$ with joint distribution $\mathbf{F}$. Denote $X_{i,1}, \ldots, X_{i,m}$ a sequence of *iid* random variables. In extreme value theory, we are interested in the limit distribution $\mathbf{G}$ of the random vector $\chi_m^+$ defined as follows

$$\chi_m^+ = \begin{bmatrix} \max\left(X_{1,1}, \ldots, X_{1,m}\right) \\ \max\left(X_{2,1}, \ldots, X_{2,m}\right) \end{bmatrix} \tag{21}$$

More precisely, $\mathbf{G}$ is the limit distribution of the *normalized* extremes

$$\lim_{m \longrightarrow \infty} \Pr\left\{ \frac{\chi_{1,m}^+ - b_{1,m}}{a_{1,m}} \leq x_1, \frac{\chi_{2,m}^+ - b_{2,m}}{a_{2,m}} \leq x_2 \right\} = \mathbf{G}\left(x_1, x_2\right) \tag{22}$$

with $\{a_{i,m}\}$ and $\{b_{i,m}\}$ two appropriates scalars. If these scaling constants **and** the limit exist, we say that $\mathbf{G}$ is the bivariate extreme value distribution of $(X_1, X_2)$ and $\mathbf{F}$ is said to be in the *max domain attraction* of $\mathbf{G}$. Note that the expression (22) is equivalent to verify the following relation (Straetmans [1999])

$$\lim_{m \longrightarrow \infty} m\left[1 - \mathbf{F}\left(a_{1,m}x_1 + b_{1,m}, a_{2,m}x_2 + b_{2,m}\right)\right] = -\ln \mathbf{G}\left(x_1, x_2\right) \tag{23}$$

However, the previous expression is difficult to exploit and so it is hard to to characterize simply the extreme value distribution $\mathbf{G}$. Copulas is then a useful tool to do that. Let $\mathbf{C}$ be the copula function of $\mathbf{F}$. We have

$$\mathbf{F}\left(x_1, x_2\right) = \mathbf{C}\left(\mathbf{F}_1\left(x_1\right), \mathbf{F}_2\left(x_2\right)\right) \tag{24}$$

where $\mathbf{F}_1$ and $\mathbf{F}_2$ are the two margins. If $\mathbf{G}$ exists, there is almost one copula function $\mathbf{C}_\star$ such that

$$\mathbf{G}\left(x_1, x_2\right) = \mathbf{C}_\star\left(\mathbf{G}_1\left(x_1\right), \mathbf{G}_2\left(x_2\right)\right) \tag{25}$$

where $\mathbf{G}_1$ and $\mathbf{G}_2$ are the two margins of the distribution $\mathbf{G}$. It is not difficult to show that necessarily $\mathbf{G}_i$ is the extreme value distribution of $X_i$. So, the margins of $\mathbf{G}$ are characterized by the Fisher-Tippet theorem (see Embrechts, Klüppelberg and Mikosch (1997)). $\mathbf{G}_i$ is one of the three univariate extreme value distribution: Fréchet, Weibull or Gumbel. For the copula $\mathbf{C}_\star$, Deheuvels [1978] shows that

$$\mathbf{C}_\star\left(u_1, u_2\right) = \lim_{m \longrightarrow \infty} \mathbf{C}^m\left(u_1^{\frac{1}{m}}, u_2^{\frac{1}{m}}\right) \tag{26}$$

$\mathbf{C}_\star$ must then be an extreme value copula and satisfy

$$\mathbf{C}_\star\left(u_1^t, u_2^t\right) = \mathbf{C}_\star^t\left(u_1, u_2\right) \tag{27}$$

for all $t > 0$. We can verify that the Normal copula does not satisfy the relation (27). The Normal copula is also not an extreme value copula, and thus it can not be used for the modelling of extremes. This is not the case of the Gumbel copula, which is an extreme value copula.

In place of the notation $\mathbf{F} \in \mathrm{MDA}\left(\mathbf{G}\right)$ — $\mathbf{F}$ is in the max domain attraction of $\mathbf{G}$ — we use $\left(\mathbf{C}, \mathbf{F}_1, \mathbf{F}_2\right) \in \mathrm{MDA}\left(\mathbf{C}_\star, \mathbf{G}_1, \mathbf{G}_2\right)$. We have then the following theorem:

**Theorem 3** $\left(\mathbf{C}, \mathbf{F}_1, \mathbf{F}_2\right) \in \mathrm{MDA}\left(\mathbf{C}_\star, \mathbf{G}_1, \mathbf{G}_2\right)$ *iff*

1. $\mathbf{F}_n \in \mathrm{MDA}\left(\mathbf{G}_n\right)$ *for* $n = 1, 2$*;*
2. $\mathbf{C} \in \mathrm{MDA}\left(\mathbf{C}_\star\right)$.

The characterization of the max domain attraction is a difficult[7], but a useful exercice. We give here some examples. Suppose that the two margins $\mathbf{F}_1$ and $\mathbf{F}_2$ are gaussians $\mathcal{N}(m_1, \sigma_1)$ and $\mathcal{N}(m_2, \sigma_2)$. In this case, we know that the corresponding univariate extreme value distribution is the Gumbel distribution $\mathbf{\Lambda}(x) = \exp(-e^{-x})$ (see EMBRECHTS, KLÜPPELBERG and MIKOSCH [1997]). If the dependence structure is the Normal copula (with $\beta < 1$), the corresponding extreme value copula is $\mathbf{C}^{\perp}$. The bivariate extreme value distribution is then as follows

$$\begin{aligned} \mathbf{G}(x_1, x_2) &= \mathbf{C}^{\perp}(\mathbf{\Lambda}(x_1), \mathbf{\Lambda}(x_2)) \\ &= \exp\left(-e^{-x_1}\right)\exp\left(-e^{-x_2}\right) \\ &= \exp\left(-e^{-x_1} - e^{-x_2}\right) \end{aligned} \tag{28}$$

Suppose now that the dependence structure is the Gumbel copula. We can show that the corresponding extreme value copula is a Gumbel copula with the same parameter. The bivariate extreme value distribution becomes

$$\mathbf{G}(x_1, x_2) = \exp\left(-\left[\left(e^{-x_1}\right)^{\beta} + \left(e^{-x2}\right)^{\beta}\right]^{\frac{1}{\beta}}\right) \tag{29}$$

In figure 5, we have represented the probability density function of the two distributions[8] and of the corresponding extreme value distributions. The parameter of the Normal copula is equal to $\sin\left(\frac{7\pi}{20}\right)$ whereas this of the Gumbel copula is set to $\frac{10}{3}$. These values have been choosen such that the Kendall's tau is equal to 70%. We verify that the two extreme value distributions are very different, whereas the original distributions seem very closed.

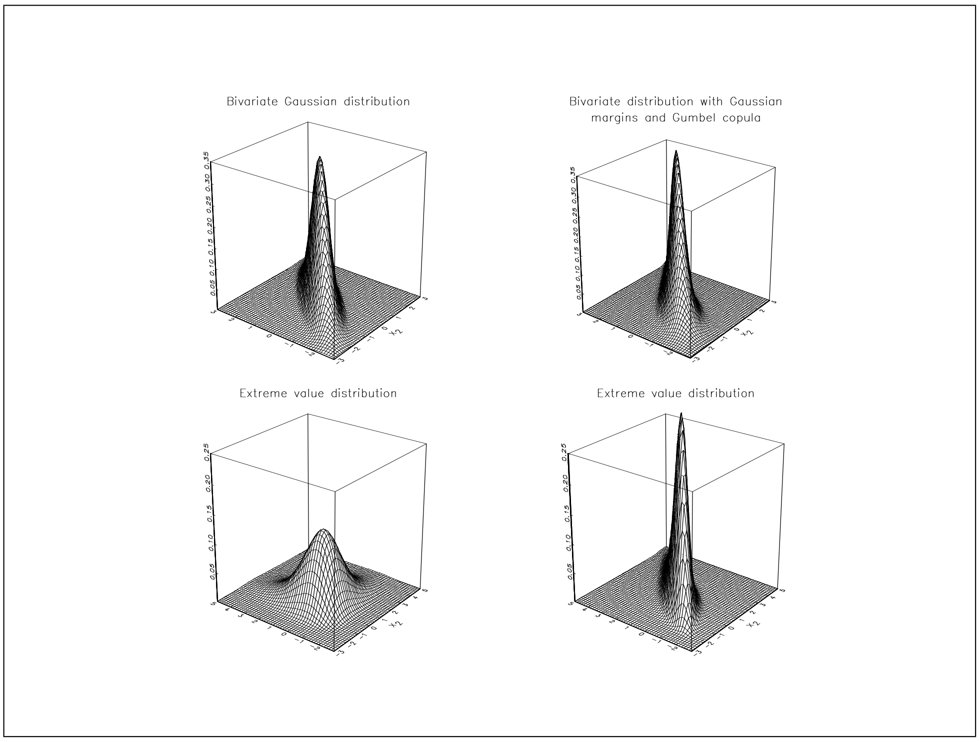


Figure 5: Probability density function of a distribution and its extreme value distribution

[7]see YUN [1997] for the examples of multivariate exponential, student, gamma and normal distributions.
[8]The Normal margins are centered with a unit standard error.

## 3.3 Correlation analysis caveats

As we have recalled before, contagion analysis is based most of the time on standard (Pearson) correlation coefficient. The purpose of this subsection is to present some of the caveats asociated with this measure.

### 3.3.1 A definition problem

A general definition of contagion can be the following : Contagion is defined as the fact that the occurrence of a currency or a financial crisis somewhere in the world increase the probability of a crisis in another country, independently of the latter's local economic and financial situation. Does an increase in the correlation of returns or residual returns (i.e. after controlling for fundamentals) be considered as the sign of such a contagion?

From a general formal point of view, the definition of a contagion (between two countries) can be expressed as follows: *There exists a phenoma of contagion from the market* 1 *to the market* 2 *if*[9]

$$\Pr\{X_2 > x_2 \mid X_1 > x_1\} > \Pr\{X_2 > x_2\} \tag{30}$$

*for a couple of well-chosen thresholds* $(x_1, x_2)$ *and* $(X_1, X_2)$ *a 2-dimensional vector of random variables.* In broad terms, the inequality (30) means that given that a crisis has occurred in market 1 ($X_1 > x_1$), the probability of a crisis in market 2 is higher that independently of what happened in market 1. With this definition, we can naturally compare the probability of a financial random variable to be above a certain threshold during "normal" period (all-sample) and "hectic" period (the sub-sample is defined as $X_1 > x_1$). We will see later that this definition means that $X_1$ and $X_2$ are *positive quadrant dependent* above a certain threshold. Moreover, we can note that this definition of contagion is symmetric in $X_1$ and $X_2$ since

$$\Pr\{X_2 > x_2 \mid X_1 > x_1\} \geq \Pr\{X_2 > x_2\} \text{ if and only if } \Pr\{X_1 > x_1 \mid X_2 > x_2\} \geq \Pr\{X_1 > x_1\} \tag{31}$$

This means that the measure of contagion is independent of the marginals (i.e., the ground-zero country). On the contrary, the use of a conditional probability alone leads to a measure of contagion dependent of the conditioning marginal (see STRAETMANS [1999]).

We can now give an example that shows that measuring correlation during these two periods cannot be a good indicator of this kind of phenomenas. This example is based on EMBRECHTS, MCNEIL and STRAUMANN [1999] who advance the theorical property that sustains it:

*Marginal distributions and correlation do not determine the joint distribution.*

Consider a bivariate distibution $(X_1, X_2)$ with Gamma marginals (with parameter equal to 3) but with a dependence structure that can change over time. Namely, if $X_1 > x_1$, the dependence function is a Gumbel copula and if $X_1 \leq x_1$, the dependence function is a Normal copula. The copula parameters are chosen so that the correlation is equal to 0.75 for the two copulas. As a result, a classical study based on correlations measure will conclude that there is no contagion and no change in the generating process between "hectic" — $X_1 > x_1$ — and "calm" — $X_1 \leq x_1$ — periods. However, if we take a threshold $x_1$ so that $\Pr\{X_1 > x_1\} = 1\%$ and choose $x_2$ equal to $x_1$, an empirical estimation gives[10] $\Pr\{X_2 > x_2 \mid X_1 > x_1\} = 67\%$. Given our definition of contagion, there is clearly contagion. This result is obtained in a case which is not only a purely matematical one: as a matter of fact, the idea of a change in the dependence structure during extremly volatile period — and especially a transformation from a Normal copula into a Gumbel copula — is now sustained by some empirical studies (see LONGIN and SOLNIK [2000]). This change in the dependence sructure can be interpreted as a change in the channels between the two respective markets of our financial series. Nevertheless, we see that a study of correlations is unable to detect this kind of change.

[9] In this definition, we restrict to the distribution of maxima (i.e. positive returns). The analysis of the distribution of minima (i.e. negative returns) is obviously the same when we restrict to extreme movements (it suffices to replace returns by minus returns).

[10] In this case, the parameters of the Gumbel and Normal copulas are respectively 2.3 and 0.80.

This example has used a simple comparison between correlations in "calm" and "hectic" periods wheras, as we have recalled ealier, one could take into account the selection bias used to define the two sub-samples. However, we could have imagined an example where the copulas had the same corrected correlation (like the one used in Boyer, Gibson and Loretan [1999]). It would not have changed our main point: for one stated parameter (like the corrected correlation) and two given marginals, you can find an infinity of dependence structure with various impacts on the probability of simultaneous crisis.

One could also argue that our results are, of course, dependent on the definition of contagion we use. As a matter of fact, if you define contagion as an increase in correlations, there is clearly no contagion in our example. However, we think that both risk managers and policy makers may be more interested in a contagion that focuses on the risk of a simultaneous crisis than in a contagion that focuses on changes in correlations in order to evaluate the modifications of the underlying process — especially when this kind of contagion is not always able to locate changes in the dependence structure of the process.

Given our definition of contagion — which needs a control of the fundamentals — it seems natural to think that the second kind of test, which is only based on *residual correlations,* may work better empirically. Moreover, with this approach, people do not have problems of selection bias or switching regimes since they only focus on crisis periods. But, by using again an example based on Normal and Gumbel copulas, we can show that correlation is not a well-adapted tool In this second example, we consider two random vectors $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ both with Gamma marginals (with parameter equal to 3) but the first one depends on a Normal copula and the second one on a Gumbel copula. In order to test for contagion, we measure correlations of $X$ and $Y$ during the periods defined as $X_1 > x_1$ and $y_1 > y_1$. Like in our first example, we take $\Pr\{X_1 > x_1\} = \Pr\{Y_1 > y_1\} = 1\%$ and the thresholds $x_2$ and $y_2$ are set to $x_1$ and $y_1$. We choose the parameters of our copulas so that in the crisis period the Pearson correlation is equal to 15% in both cases[11]. As a consequence, in both cases, a study based on correlations will conclude that contagion occurs. Moreover, because the two correlations are equal, people may think that contagion occurs at the same level. But if we give a closer look at the conditional probability, we find that $\Pr\{X_2 > x_2 \mid X_1 > x_1\} = 3\%$ whereas $\Pr\{Y_2 > y_2 \mid Y_1 > y_1\} = 20\%$. We see that contagion is clear only in the second case and that the probabilities of a simultaneous crisis are very different: the probability in the Gumbel case is seven times that of the Normal case.

### 3.3.2 Some misinterpretations of the correlation

We note $\rho(X_1, X_2)$ the Pearson correlation of the two random variables $X_1$ and $X_2$ with margins $\mathbf{F}_1$ and $\mathbf{F}_2$. We list here some "myths" about the correlation which are often used in financial economics:

1. the random variables $X_1$ and $X_2$ are independent if and only if $\rho(X_1, X_2) = 0$;
2. for given margins, the permissible range of $\rho(X_1, X_2)$ is $[-1, 1]$;
3. $\rho(X_1, X_2) > 0$ means that $X_1$ and $X_2$ have a positive dependence.

We consider the first evidence. We use the cubic copula introduced by Durrleman, Nikeghbali and Roncalli [2000] (see [24]) and defined as follows

$$\mathbf{C}(u_1, u_2) = u_1 u_2 + \alpha [u_1(u_1 - 1)(2u_1 - 1)] [u_2(u_2 - 1)(2u_2 - 1)] \tag{32}$$

with $\alpha \in [-1, 2]$. We assume that the margins $\mathbf{F}_1$ and $\mathbf{F}_2$ are continous and symmetric. In this case, the Person correlation is equal to zero. The proof of this result is straightforward if we remark that $\mathbf{C}(u_1, u_2) - u_1 u_2$ is symmetric about $\left(\mathbf{F}_1^{-1}\left(\frac{1}{2}\right), \mathbf{F}_2^{-1}\left(\frac{1}{2}\right)\right)$. Moreover, if $\alpha \neq 0$, the random variables $X_1$ and $X_2$ are not independent. **So, we have obtained a family of distributions such that $\rho(X_1, X_2) = 0$ and $X_1$ and**

[11] In this case, the parameters of the Gumbel and Normal copulas are respectively 1.18 and 0.20.

$X_2$ **are dependent**. To illustrate this fact, we have plotted in figure 6 the probability density function of distributions generated by the cubic copula[12]. Note that $\alpha = 0$ corresponds to the independent case, i.e. $\mathbf{C} = \mathbf{C}^{\perp}$.

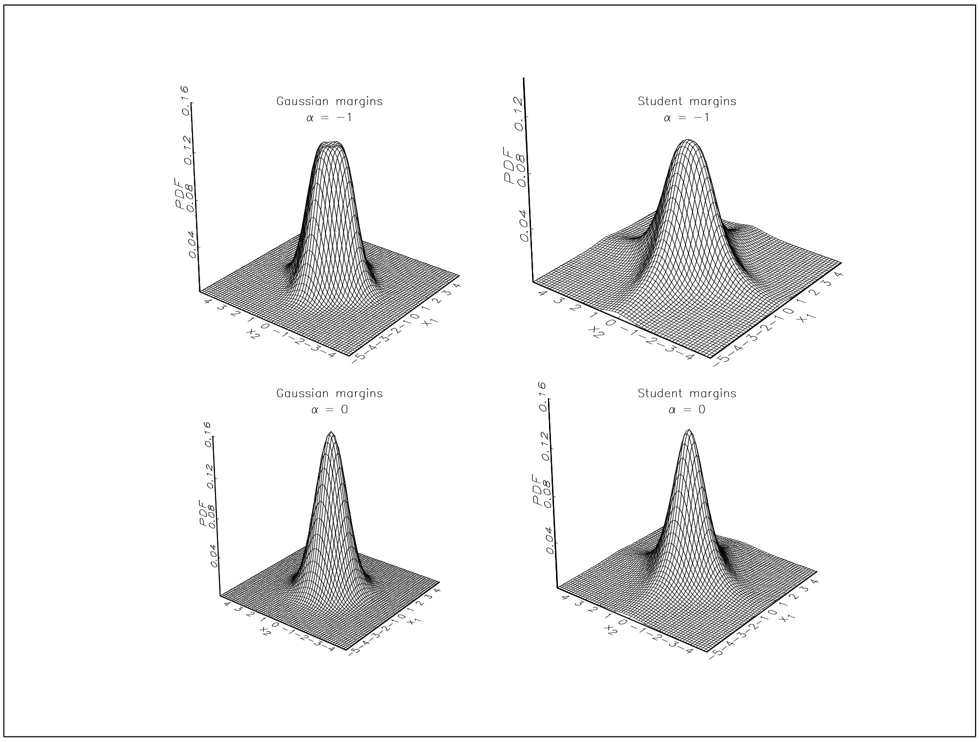


Figure 6: Probability density function of distributions constructed with the cubic copula

We know that $\rho(X_1, X_2) \in [-1, 1]$. $\rho(X_1, X_2)$ takes the value 1 (respectively $-1$) if and only if there exists a linear relationship $X_2 = aX_1 + b$ with $a > 0$ (respectively $a < 0$). In the case of non linear relationship between $X_1$ and $X_2$, $\rho(X_1, X_2) \in [\rho_-, \rho_+] \subset [-1, 1]$. We illustrate this second point with a simple (but beautiful) example due to Wang [1999]. We assume that $X_1 \sim \mathcal{LN}(\mu_1, \sigma_1)$ and $X_2 \sim \mathcal{LN}(\mu_2, \sigma_2)$. We can show that the minimum correlation $\rho_-$ is given when $X_2 = e^{\mu_2 + \frac{\sigma_2}{\sigma_1}\mu_1} X_1^{-\frac{\sigma_2}{\sigma_1}}$ and the maximum correlation $\rho_+$ is given when $X_2 = e^{\mu_2 - \frac{\sigma_2}{\sigma_1}\mu_1} X_1^{\frac{\sigma_2}{\sigma_1}}$. In this case, we have (see appendix A.4.2 of Wang [1999])

$$\begin{aligned}
\rho_- &= \frac{e^{-\sigma_1\sigma_2} - 1}{\sqrt{e^{\sigma_1^2} - 1}\sqrt{e^{\sigma_2^2} - 1}} \leq 0 \\
\rho_+ &= \frac{e^{\sigma_1\sigma_2} - 1}{\sqrt{e^{\sigma_1^2} - 1}\sqrt{e^{\sigma_2^2} - 1}} \geq 0
\end{aligned} \tag{33}$$

Note that $\rho_+$ is equal to 1 if and only if $\sigma_1 = \sigma_2$ and $\rho_-$ takes the value $-1$ if and only if

$$\lim_{\sigma_1 \longrightarrow 0^+} \sigma_1 = \lim_{\sigma_2 \longrightarrow 0^+} \sigma_2 = 0 \tag{34}$$

So, in the case where $\sigma_1 \neq \sigma_2$, the permissible range of $\rho(X_1, X_2)$ is not $[-1, 1]$ because $\rho_- > -1$ and $\rho_+ < 1$. Moreover, we have

$$\lim_{\sigma_1 \wedge \sigma_2 \longrightarrow \infty} \rho_- = 0 \tag{35}$$

[12] The margins are respectively gaussian ($X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$) and student ($X_1 \sim t_2$ and $X_2 \sim t_3$).

and

$$\lim_{|\sigma_1-\sigma_2|\longrightarrow\infty}\rho_+ = 0 \tag{36}$$

This is in contradiction with the belief in financial economics that

*Given marginal distributions $F_1$ and $F_2$ for $X_1$ and $X_2$, all Pearson correlation between $-1$ and $+1$ can be attained through a suitable specification of the joint distribution (Embrechts, McNeil and straumann [1999]).*

We have reported the range of $[\rho_-, \rho_+]$ for different values of $\sigma_1$ and $\sigma_2$. In some cases, we remark that $\rho_- \gg -1$ and $\rho_+ \ll 1$.

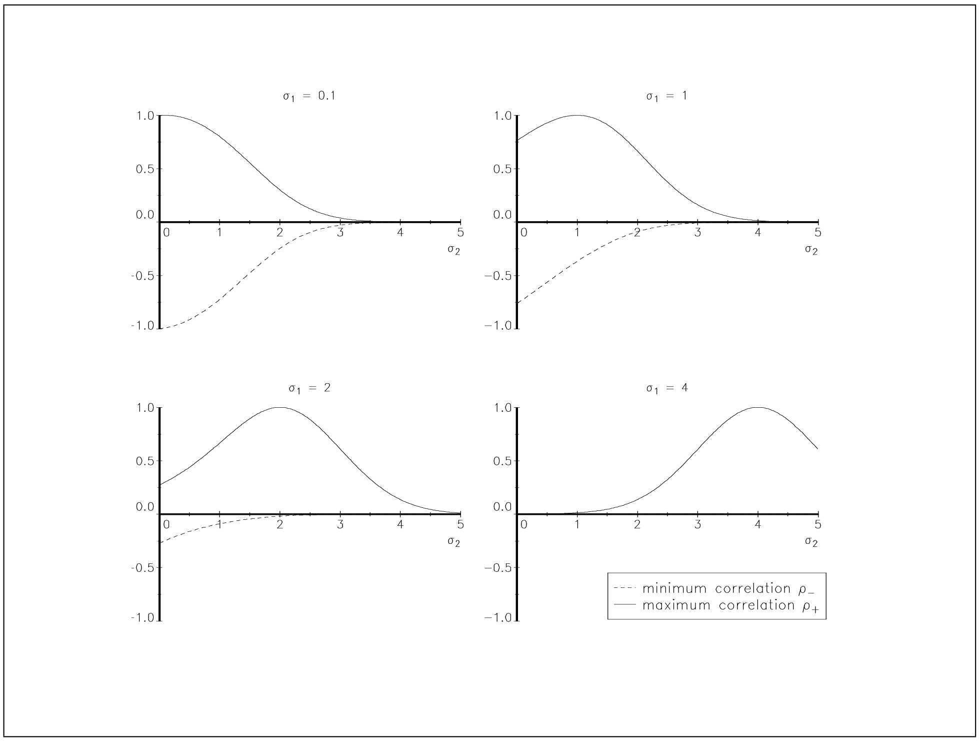


Figure 7: Permissible range of $\rho(X_1, X_2)$ when $X_1$ and $X_2$ are two log-normal random variables

We consider now the third point. We use the previous example. The linkage between the two random variables is done with the Gaussian copula. In table 1, we have reported the corresponding Pearson's correlation, Kendall's tau and Spearman's rho for $\sigma_1 = 1$ and $\sigma_2 = 3$. Because the permissible range of $\rho(X_1, X_2)$ is not large, we obtain some strange results. For example, in the case of the lower Fréchet bound $\mathbf{C}^-$, i.e. when $X_1$ and $X_2$ are countermonotonic and present the most negative dependence, the correlation is close to zero! This simple example shows us that **we can not use the Pearson correlation to measure the intensity of the dependence**. Kendall's tau and Spearman's rho are more appropriate.

Kendall's tau and Spearman's rho are two concordance measures (Nelsen [1998]). They satisfies different properties, in particular:

- if $X_1$ and $X_2$ are countermonotonic, the measure of concordance is equal to $-1$;
- if $X_1$ and $X_2$ are independent, the measure of concordance is equal to 0;

| Copula | $\rho(X_1, X_2)$ | $\tau(X_1, X_2)$ | $\varrho(X_1, X_2)$ |
|---|---|---|---|
| $\mathbf{C}^-$ | $-0.008$ | $-1$ | $-1$ |
| $\beta = -0.7$ | $\simeq 0$ | $-0.49$ | $-0.68$ |
| $\mathbf{C}^\perp$ | 0 | 0 | 0 |
| $\beta = 0.7$ | $\simeq 0.10$ | 0.49 | 0.68 |
| $\mathbf{C}^+$ | 0.16 | 1 | 1 |

Table 1: Value of the dependence measures

- if $X_1$ and $X_2$ are comonotonic, the measure of concordance is equal to 1;
- the measure of concordance between $X_1$ and $X_2$ is invariant under monotonic transformations of $X_1$ and $X_2$.

Kendall's tau is interpreted as the probability of concordant pairs minus the probability of discordant pairs whereas the Spearman's rho corresponds to the linear correlation coefficient of the ranks. Because of the previous properties, they can be used to compare dependence in a more coherent way than the Pearson correlation.

## 3.4 A new statistical framework to analyze contagion episodes and dependence between financial markets

We have seen in the previous paragraph that correlation is not a good tool for identifying the dependence function. In this subsection, we review different statistical methods related to copulas, which constitute a more appropriate tool to define the dependence function.

### 3.4.1 Modelling the dependence between extreme returns

Correlation is not a good measure for identifying contagion episodes. A more appropriate measure must entirely characterize the dependence structure.

**Definition of the upper tail dependence** In the rest of this paper, we will now use a more precise version of our first definition of contagion. We define contagion as follows:

**Definition 4** *Consider a two dimensional vector of random variables $(X_1, X_2)$ with distribution functions $\mathbf{F}_1$ and $\mathbf{F}_2$ and a copula functions $\mathbf{C}$, there exists a phenoma of contagion if*

$$\Pr\left\{X_2 > \mathbf{F}_2^{-1}(\alpha_2) \mid X_1 > \mathbf{F}_1^{-1}(\alpha_1)\right\} \geq \Pr\left\{X_2 > \mathbf{F}_2^{-1}(\alpha_2)\right\} \tag{37}$$

*for all $\alpha_1 \geq \alpha^\star$ and $\alpha_2 \geq \alpha^\star$.*

From a statistical point of view, this definition may be seen as a restricted property of *positive quadrant dependence* (PQD) above the threshold $\alpha^\star$. PQD means that the dependence function $\mathbf{C}$ is larger than the product copula $\mathbf{C}^\perp$ (see Joe [1997]). Note that the inequality (37) is equivalent to

$$\frac{\Pr\left\{X_2 > \mathbf{F}_2^{-1}(\alpha_2) \mid X_1 > \mathbf{F}_1^{-1}(\alpha_1)\right\}}{\Pr\left\{X_1 > \mathbf{F}_1^{-1}(\alpha_1)\right\}} \geq \Pr\left\{X_2 > \mathbf{F}_2^{-1}(\alpha_2)\right\} \tag{38}$$

It comes that

$$\Pr\left\{X_2 > \mathbf{F}_2^{-1}(\alpha_2) \mid X_1 > \mathbf{F}_1^{-1}(\alpha_1)\right\} \geq \Pr\left\{X_1 > \mathbf{F}_1^{-1}(\alpha_1)\right\} \Pr\left\{X_2 > \mathbf{F}_2^{-1}(\alpha_2)\right\} \tag{39}$$

When the margins are continous, this inequality is verified for all $(\alpha_1, \alpha_2) \in [0,1]^2$ if

$$\mathbf{C} \succ \mathbf{C}^\perp \tag{40}$$

One advantage of the previous definition is that it is expressed in terms of quantile. We can then associate directly $\alpha_1$ and $\alpha_2$ to small events, because they are probabilities. In practice, we will choose $\alpha_1 = \alpha_2 = \alpha$. Within this definition, we give a special attention to the asymptotic case

$$\begin{aligned}\lambda &= \lim_{\alpha \longrightarrow 1^-} \Pr\left\{X_2 > \mathbf{F}_2^{-1}(\alpha) \mid X_1 > \mathbf{F}_1^{-1}(\alpha)\right\} - \Pr\left\{X_2 > \mathbf{F}_2^{-1}(\alpha)\right\} \\ &= \lim_{\alpha \longrightarrow 1^-} \Pr\left\{X_2 > \mathbf{F}_2^{-1}(\alpha) \mid X_1 > \mathbf{F}_1^{-1}(\alpha)\right\}\end{aligned} \tag{41}$$

$\lambda$ is called the coefficient of *upper tail dependence*. In this case, a test for dependence between extreme returns is similar to the test of the null hypothesis $\lambda > 0$ against the alternative hypothesis $\lambda = 0$. In the Normal copula, $\lambda$ takes the value 0. This kind of test has already been used in LONGIN and SOLNIK [1999].

In the general case, we have to consider the quantile-dependent measure[13] $\lambda(\alpha)$ defined as follows

$$\begin{aligned}\lambda(\alpha) &= \Pr\left\{X_2 > \mathbf{F}_2^{-1}(\alpha) \mid X_1 > \mathbf{F}_1^{-1}(\alpha)\right\} \\ &= \frac{\Pr\left\{X_2 > \mathbf{F}_2^{-1}(\alpha_2), \mid X_1 > \mathbf{F}_1^{-1}(\alpha_1)\right\}}{\Pr\left\{X_1 > \mathbf{F}_1^{-1}(\alpha_1)\right\}} \\ &= \frac{1 - 2\alpha + \mathbf{C}(\alpha,\alpha)}{1-\alpha} \\ &= 2 - \frac{1 - \mathbf{C}(\alpha,\alpha)}{1-\alpha}\end{aligned} \tag{43}$$

Note that the equation (37) is equivalent to verify that $\lambda(\alpha) - (1-\alpha) \geq 0$. Since $\alpha \in [0,1]$, we just have to study the sign[14] of $\mathbf{C}(\alpha,\alpha) - \alpha^2$.

There are two problems with the modelisation of the upper tail dependence.

1. The first one is that $\lambda(\alpha)$ will be estimated from empirical observations, and "*estimates of $\lambda(\alpha)$ may appear constant and positive, even for asymptotically independent variables*" (COLES, CURRIE and TAWN [1999]). These authors have then considered a second dependence measure $\bar{\lambda}(\alpha)$ defined as

$$\bar{\lambda}(\alpha) = \frac{\ln(1-\alpha)^2}{\ln(1 - 2\alpha + \mathbf{C}(\alpha,\alpha))} - 1 \tag{44}$$

   We can show that the limit case $\bar{\lambda} = \lim_{\alpha \longrightarrow 1^-} \bar{\lambda}(\alpha)$ verifies $-1 < \bar{\lambda} \leq 1$. In this case[15], extremal dependence corresponds to $\bar{\lambda} = 1$. In figure 8, we compare the two measures $\lambda(\alpha)$ and $\bar{\lambda}(\alpha)$ when the copula is Normal. We remark that the convergence of $\lambda(\alpha)$ to zero can be very slow. As indicated by COLES, CURRIE and TAWN [1999], $\bar{\lambda}$ is equal to the correlation parameter $\beta$ of the Normal copula "*which provides a useful benchmark for interpreting the magnitude of $\bar{\lambda}$ in general models*".

[13] Sometimes, another expression is considered (COLES, CURRIE and TAWN [1999])

$$\lambda(\alpha) \simeq 2 - \frac{\ln \mathbf{C}(\alpha,\alpha)}{\ln \alpha} \tag{42}$$

[14] This result may be obtained directly by thinking contagion as a positive quadrant dependence on the diagonal.

[15] The authors motivate the introduction of this measure, because it is related to the coefficient of tail dependence $\eta$ of LEDFORD and TAWN [1996], which satisfies the condition

$$\Pr\{X_1 > x, X_2 > x\} \propto \mathcal{L}(x)\left[\Pr\{X_1 > x\}\right]^{\frac{1}{\eta}} \tag{45}$$

where the margins of $X_1$ and $X_2$ are two unit Fréchet and $\mathcal{L}(x)$ a slowly varying function. We have

$$\bar{\lambda} = 2\eta - 1 \tag{46}$$

2. The second problem concerns the relevance of the asymptotic case. Let us consider the example of the Normal copula with $\beta$ equal to 75%. We have $\lambda = 0$ but $\lambda(99.99\%) = 13.6\%$. In the case of the Student copula with 4 degrees of freedom and a parameter $\beta$ equal to $-50\%$, we have $\lambda = 1.2\%$ and $\lambda(99.99\%) \simeq \lambda$. This simple example shows us that the asymptotic case is not perhaps the most interesting case to study the dependence between extreme returns.

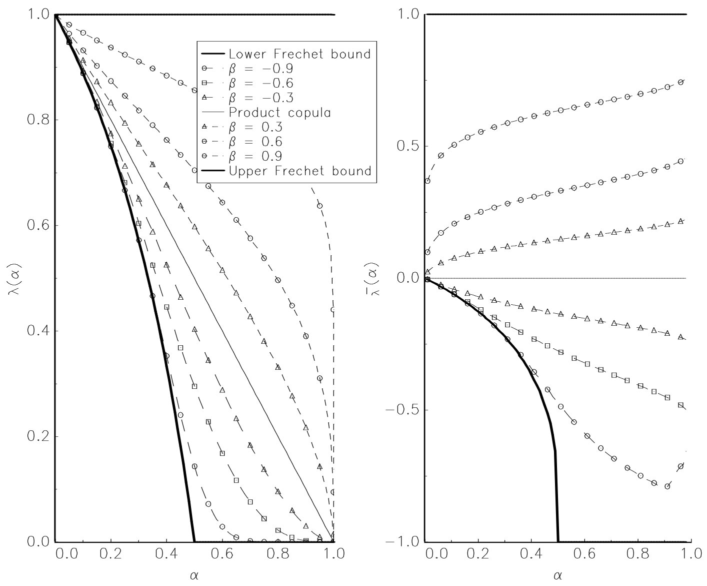


Figure 8: Comparison of the measures $\lambda(\alpha)$ and $\bar{\lambda}(\alpha)$ in the case of the Normal copula

**Estimation issues** As a matter of fact, if we want our definition of contagion to be a useful tool, we must be able to estimate expression (37). Namely, we must be able to estimate the conditional probability $\lambda(\alpha)$ and especially when we are in periods of crisis such as $\alpha$ is closed to one. In this perspective, we will present two approaches: the first one, which is non-parametric, is given in STRAETMANS [1999] and the second one, which is parametric, is given in LONGIN and SOLNIK [1999]. They are both interested in asymptotic measures but the first author focuses on a probability of spillover whereas the others estimate a coefficient of "extremal correlation". The two approaches can give an estimation of $\lambda(\alpha)$. In order to present these two approaches, we will first give the theoretical results that sustain both of them and then, their respective specificities. Finally, we will propose an extension to these methologies — based on extreme copulas — and apply it in the next paragraph in order to estimate $\lambda(\alpha)$ when $\alpha$ is close to one.

These two approaches first use the results given by EVT for the marginals $\mathbf{F}_1$ and $\mathbf{F}_2$ of two random variables $X_1$ and $X_2$. Namely, they use the fact that for fat-tailed distributions like those of financial returns, the distribution of well-normalized maxima tends to be Fréchet with parameter $\alpha_i$. From the properties of the max-domain of attraction, straetmans deduces that

$$1 - \mathbf{F}_i(x) \simeq a x^{-\alpha_i} \qquad \text{for } x \longrightarrow \infty \tag{47}$$

This representation allows him to estimate $\alpha_i$ and then the probability of $X_i$ to be above one given threshold solely by the use of the Hill estimator (see EMBRECHTS, KLÜPPELBERG and MIKOSCH [1997]). On the other hand, Longin and Solnik use the equivalence between belonging to the max-domain of attraction of a Fréchet and having a distribution of scaled excesses over high thresholds that tends to be a Generalised Pareto Distribution (GPD) which has the following expression

$$\mathsf{GPD}\,(x) = 1 - \left(1 + \frac{x - \mu}{\alpha\sigma}\right)^{-\alpha} \tag{48}$$

($\sigma$ is the scaling parameter and $\mu$ is the threshold). It is important to note that $1 - \mathsf{GPD}\,(x)$ is not equal to $\Pr\{X > x\}$ but to $\Pr\{X > x \mid X > \mu\}$ when $\mu \longrightarrow \infty$. In order to estimate the parameters of the GPD, they use maximum likelihood method whereas Straetmans's estimation is based on the Hill estimator. In summary, concerning the marginals, they both use theoretical results linked to the max-domain of attraction of the Fréchet distribution. However, whereas Straetmans uses semi-parametric method to estimate $\Pr\{X > x\}$ (when $x \longrightarrow \infty$), Longin and Solnik focus on a parametric estimation of $\Pr\{X > x \mid X > \mu\}$ (when $\mu \longrightarrow \infty$).

Concerning the dependence structure, we will show that the objects they use are very similar for high quantiles. However, in order to compare their two dependence structures, we must focus in both cases on the same probabilities. So, a first problem to solve is that straetmans used unconditional probabilities while Longin and Solnik focus on conditional probability. We choose to present what becomes Longin and Solnik's approach for $\mu = 0$ and $x \longrightarrow \infty$. Moreover, since we can no longer use the results based on the GPD for the marginals, we will maintain the assumption that the random variables are in the max-domain of attraction of a Fréchet distribution. Now, give a closer look at the two dependence functions introduced by the authors. In STRAETMANS [1999], this dependence structure is called the *stable tail dependence function* — $l_{\mathbf{F}}$ — and in LONGIN and SOLNIK [1999] the *dependence function* — $\mathfrak{D}$. We call $\mathbf{G}$ the extreme bivariate distribution of the two random variables $(X_1, X_2)$ with Fréchet distributions $\Upsilon_{\alpha_1}$ and $\Upsilon_{\alpha_2}$ and define $u_1$ and $u_2$ such as $u = 1 - \Upsilon_\alpha(x) \simeq x^{-\alpha}$. The *stable tail dependence function* is expressed as follows

$$l_{\mathbf{F}}(u_1, u_2) = -\ln \mathbf{G}\left(u_1^{-\frac{1}{\alpha_1}}, u_2^{-\frac{1}{\alpha_2}}\right) \tag{49}$$

whereas, in LONGIN and SOLNIK [1999], the *dependence function* verifies

$$\mathbf{G}\,(x_1, x_2) = \exp\left[-\mathfrak{D}\left(-\frac{1}{\ln \Upsilon_{\alpha_1}(x_1)}, -\frac{1}{\ln \Upsilon_{\alpha_1}(x_1)}\right)\right] \tag{50}$$

With limited development of order one, it can also be written as

$$\begin{aligned}
\mathbf{G}\,(x_1, x_2) &\simeq \exp\left[-\mathfrak{D}\left(\frac{1}{x_1^{\alpha_1}}, \frac{1}{x_2^{\alpha_2}}\right)\right] \\
&\simeq \exp\left[-\mathfrak{D}\,(u_1, u_2)\right]
\end{aligned} \tag{51}$$

For all $(u_1, u_2)$ close to zero, we finally obtain $-\ln \mathbf{G}\left(u_1^{-\frac{1}{\alpha_1}}, u_2^{-\frac{1}{\alpha_2}}\right) = \mathfrak{D}\,(u_1, u_2)$ and so $l_{\mathbf{F}}(u_1, u_2) = \mathfrak{D}\,(u_1, u_2)$. The restriction is not a problem, since we are interested in extremal events. Note that the existence of the two functions comes from the convergence of the point process $\mathbf{N}_T = \left\{T^{-1}X_i : i = 1, 2\right\}$ toward a non-homogeneous Poisson process whose intensity measure $\Lambda$ is constrained (COLES and TAWN [1991]). In this paper, the two authors show that the extreme bivariate distribution of componentwise maxima — with unit Fréchet marginals — can always be written as $\mathbf{G}\,(x_1, x_2) = \exp(-\Lambda(x_1, x_2))$. It is Pickands representation for multivariate extreme value distributions[16], where the function $\Lambda$ must be homogeneous of order $-1$. The link with extreme value copulas is straightforward:

[16]The assumption of having unit Fréchet margins is not restrictive since suitable transformations can be applied otherwise.

*If an extreme multivariate distribution* $\mathbf{G}$ *with unit Fréchet margins admits Pickands representation, then the copula* $\mathbf{C}$ *associated to the distribution is an extreme value copula.*

In summary, both of the two approaches uses extreme value theory for obtaining the form of the multivariate distribution. As a result, their dependence functions are equivalent for high quantiles. Nevertheless, these two methodologies have completely different point of views concerning the estimation of $l_{\mathbf{F}}$ and $\mathfrak{D}$. STRAETMANS [1999] uses the fact that $l_{\mathbf{F}}$ is homogeneous of order 1 — which comes from the homogeneity of $\Lambda$ — so that he can estimate asymptotic probabilities in two steps. In a first time, he uses the law of large numbers in a well chosen sample (with a sufficient number of observations) to estimate $l_{\mathbf{F}}$ and then consider more extreme levels thanks to the homogeneity property $l_{\mathbf{F}}(u_1, u_2) = a^{-1} l_{\mathbf{F}}(au_1, au_2)$. Note that the asymptotic case corresponds to $\lambda = 2 - l_{\mathbf{F}}(1,1)$. The crucial point of this methodology is the choice of the threshold $u_1 = u_2 = a^{-1}$. On the contrary, LONGIN and SOLNIK [1999] use a complete parametric method and set $\mathfrak{D}$ to be the logistic function, which is in fact the Gumbel copula. After having estimated $\beta$ by maximum likelihood method, $\lambda$ is given by $2 - 2^{\beta}$ (a well-know result for the Gumbel copula).

In our point of view, the difference between the two approaches concerns only the estimation method. We think that the copula framework is the natural (and the more general) approach for modelling extremes. In the next paragraph, we use a parametric methodology since we are interested in the dependence between returns in general not only in the asymptotic case. Our approach is to choose within the space of extreme value copulas the one which fits the best our data rather than to take arbitrarily one of them. In order to determine which copula is the right one, we use the methodology presented in DURRLEMAN, NIKEGHBALI and RONCALLI [2000]. This can be divided into three steps:

- we build a set of parametric extreme value copulas families;
- we define the vector of the componentwise maxima and estimate the parameters of the marginals (assuming they are GEV) and those of the copula functions by maximum likelihood method;
- we estimate the Deheuvels copula (see the paragraph 3.4.2) and compare the distance between the empirical copula and the different fitted copulas. We then choose the copula which minimizes the discrete $\ell_2$ norm.

The fitted copula can then be used to compute $\lambda(\alpha)$ for $\alpha$ close to one, and the asymptotic case $\lambda$.

**Illustration with the indices CAC40 and DowJones** In this paragraph, we give some results of our methodology with the indices CAC40 and DowJones. The set of extreme value copulas is the following:

| Copula | $\beta$ | $\mathbf{C}(u_1, u_2)$ |
|---|---|---|
| $\mathbf{C}^{\perp}$ | | $u_1 u_2$ |
| Gumbel | $[1, \infty)$ | $\exp\left[-\left(\tilde{u}_1^{\beta} + \tilde{u}_2^{\beta}\right)^{\frac{1}{\beta}}\right]$ |
| Galambos | $[0, \infty)$ | $u_1 u_2 \exp\left[\left(\tilde{u}_1^{-\beta} + \tilde{u}_2^{-\beta}\right)^{-\frac{1}{\beta}}\right]$ |
| Hüsler-Reiss | $[0, \infty)$ | $\exp\left[-\tilde{u}_1 \Phi(z_1) - \tilde{u}_2 \Phi(z_2)\right]$ |
| Marshall-Olkin | $[0, 1]^2$ | $u_1^{1-\beta_1} u_2^{1-\beta_2} \min\left(u_1^{\beta_1}, u_2^{\beta_2}\right)$ |
| $\mathbf{C}^{+}$ | | $\min(u_1, u_2)$ |
| Normal | $[-1, 1]$ | $\Phi_{\beta}(\Phi^{-1}(u_1), \Phi^{-1}(u_2))$ |

with $\tilde{u} = -\ln u$, $z_1 = \beta^{-1} + 0.5\beta \ln(\tilde{u}_1/\tilde{u}_2)$ and $z_2 = \beta^{-1} + 0.5\beta \ln(\tilde{u}_2/\tilde{u}_1)$.

It is important to note that the Normal copula is not an extreme value copula. But, it is retained in the set because of the importance of the gaussian structure in traditional empirical studies. We will see that according to the results of the last paragraph, the empirical copula is best fitted by extreme copulas than by the gaussian one. We also choose the Gumbel, Galambos and Hüsler-Reiss copulas because they are all extreme copulas and

| **Copula** | $\beta$ | Normalized $\ell_2$ norm |
|---|---|---|
| $\mathbf{C}^{\perp}$ | | $5.52 \times 10^{-2}$ |
| Gumbel | 1.57 | $1.04 \times 10^{-2}$ |
| Galambos | 0.85 | $1.02 \times 10^{-2}$ |
| Hüsler-Reiss | 1.26 | $1.06 \times 10^{-2}$ |
| Marshall-Olkin | $(0.51, 0.62)$ | $1.30 \times 10^{-2}$ |
| $\mathbf{C}^{+}$ | | $5.63 \times 10^{-2}$ |
| Normal | 0.28 | $3.02 \times 10^{-2}$ |

Table 2: Results with 25 trading days

| **Copula** | $\beta$ | Normalized $\ell_2$ norm |
|---|---|---|
| $\mathbf{C}^{\perp}$ | | $5.54 \times 10^{-2}$ |
| Gumbel | 1.53 | $1.48 \times 10^{-2}$ |
| Galambos | 0.82 | $1.46 \times 10^{-2}$ |
| Hüsler-Reiss | 1.28 | $1.43 \times 10^{-2}$ |
| Marshall-Olkin | $(0.52, 0.61)$ | $1.79 \times 10^{-2}$ |
| $\mathbf{C}^{+}$ | | $5.87 \times 10^{-2}$ |
| Normal | 0.28 | $3.21 \times 10^{-2}$ |

Table 3: Results with 50 trading days

take into account the cases of independence and total dependence. These three copulas are all symmetric in $u_1$ and $u_2$. Finally, we add the Marshall-Olkin copula which is also extreme but presents the advantage to to be asymmetric.

To choose the right copula, we apply the Durrleman, Nikeghbali and Roncalli [2000] methodology. Thus, we must create the vector of the componentwise maxima (here we will focus on bear markets and so consider only extreme negative returns). We use two sizes of blocks (which corresponds to 25 and 50 trading days). The results on the CAC40 and DowJones are given in the tables 2 and 3. We see on this example that the different copulas can be ranked by increasing efficiency into four groups:

1. the product and upper Fréchet copulas,
2. the Normal copula,
3. the Marshall-Olkin copula,
4. and the Gumbel, Galambos and Hüsler-Reiss copulas.

There are no significative differences between these last three copulas. The bad ranks of the product and upper Fréchet copulas results of course from the non-parametric nature of these two copulas. Finally, according to the extreme value theory, we can underline that the choice of the Normal copula is not optimal. We then choose the Galambos copula to estimate $\lambda(\alpha)$. In figure 9, we have plotted $\lambda(\alpha)$ for $\alpha \geq 0.8$ both with the central limit theorem (CLT) and the Galambos copula. Our methodology clearly shows that there is a phenomena of dependence between the two indices during extreme events (when the associated probability is less than 20%). Until very high quantiles (above 99%) are reached, one can rely on CLT since the number of observations remains important (always above 30). However, in case of big extremal events (namely, the thirty biggest variations of the last decade), there are not enough observations and so, we must use the results of extreme value theory. Here, these results show that dependence was even stronger during extreme periods. Whereas the unconditional probability tends to be null, the conditional probability $\lambda$ is around 40%.

One convenient way of visualizing this dependence during extreme crisis (high threshold) is to use failure areas (see Bouyé, Durrleman, Nikeghbali, Riboulet and Roncalli [2000] for a complete treatment of

| | | | Return time (in years) | | |
|---|---|---|---|---|---|
| Date | CAC40 | DowJones | EVT | GD (I) | GD (II) |
| 10/19/1987 | $-10.14\%$ | $-25.63\%$ | 105.79 | $1.44 \times 10^{14}$ | ✓ |
| 10/21/1987 | $+1.80\%$ | $+9.67\%$ | 18.14 | $2.88 \times 10^{14}$ | ✓ |
| 10/26/1987 | $-8.45\%$ | $-8.38\%$ | 9.18 | $1.80 \times 10^{13}$ | $2.70 \times 10^{13}$ |
| 11/09/1987 | $-11.65\%$ | $-3.10\%$ | 8.12 | $2.30 \times 10^{9}$ | $2.53 \times 10^{10}$ |
| 01/01/1992 | $+8.28\%$ | $+5.71\%$ | 6.85 | $1.66 \times 10^{8}$ | $1.69 \times 10^{8}$ |
| 01/02/1992 | $-9.18\%$ | $-5.59\%$ | 6.39 | $2.96 \times 10^{9}$ | $3.09 \times 10^{9}$ |
| 01/04/1992 | $+9.87\%$ | $+4.83\%$ | 7.06 | $2.05 \times 10^{9}$ | $2.06 \times 10^{9}$ |

Table 4: Return time of extreme returns

this topic). A failure area with a return time $\mathsf{t}$ is defined as the set $(x_1, x_2)$ such as $\Pr\{X_2 > x_2, X_1 > x_1\} < \frac{1}{\mathsf{t}}$. In figure 10, we set $\mathsf{t}$ equal to 5 years. It implies that the mean duration of the event $(X_2 > x_2, X_1 > x_1)$ is five years. We have represented the failure areas for both bear and bull markets obtained with the Galambos copula. To give an idea about the superiority of extreme value theory method on a methododology based on the multivariate gaussian distribution, we have computed the return time of the most extreme returns of the database. The results are reported in table 4. We can compare the obtained results (EVT) with these computed if we assume that the bivariate distribution is normal. Because we have rare events, the probabilities are very small in the gaussian case. We face some numerical problems about computing so small probabilities. So we we have reported two values, one based on the gaussian cdf (GD (I)), the other based on the logarithm gaussian cdf (GD (II))[17].

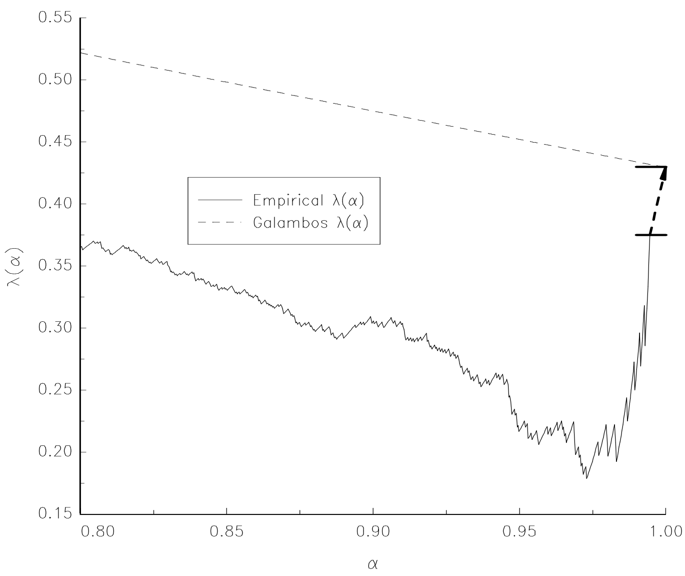


Figure 9: Comparison of $\lambda(\alpha)$ computed with the Galambos copula and the empirical method

[17] All the computation have been done with the GAUSS `cdfbvn` and `lncdfbvn` procedures.

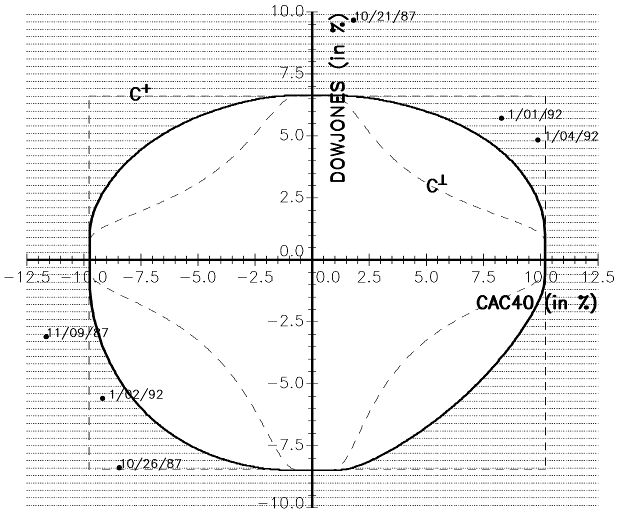


Figure 10: Failure area for a five years return time

#### 3.4.2 Comparing the dependence between asset returns

In the previous paragraph, we have concentrated on extreme returns. Sometimes, we have to compare the dependence structure between two periods or between two bivariate series for all of the distribution. We can then use two methods based on copulas. The first one is related to the concordance order, whereas the second one is based on quantile regression. Note that we can use these two methods in a parametric framework (by estimating a parametric copula) or in a non-parametric framework (using the empirical copula introduced by Deheuvels [1979]).

**The concordance order method** Suppose that we want to compare the dependence of a bivariate financial series between two periods, or the dependence of two bivariate financial series. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be the two underlying dependence functions. The most simple case is when we have

$$\mathbf{C}_1(u_1,u_2) \geq \mathbf{C}_2(u_1,u_2) \tag{52}$$

for all $(u_1,u_2) \in [0,1]$. In this case, we can conclude that the dependence in the first period (respectively of the first bivariate financial series) is stronger than the dependence in the second period (respectively of the second bivariate financial series). However, as we have said previously, we can not always compare two copula functions. In figures 11 and 12, we have represented the region where the relation $\mathbf{C}_1(u_1,u_2) \geq \mathbf{C}_2(u_1,u_2)$ does not work for different pairs of copulas. It does correspond to dark areas. In general, we remark that the region is clearly delimited. It could be localized in corners $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. Or it could be a more complicated area. This type of analysis indicates us the regions where we have locally a **stronger** dependence. It can be completed by a quantile-dependent measure. Previously, we have introduced the quantile-dependent measure of the upper tail

$$\lambda(u) = \Pr\{U_1 > u \mid U_2 > u\} = \frac{1-2u+\mathbf{C}(u,u)}{1-u} \tag{53}$$

But, we can define many other quantile-dependent measures, for example

$$\begin{aligned}
\lambda_{(0,0)}(u) &= \Pr\{U_1 < 1-u \mid U_2 < 1-u\} = \frac{\mathbf{C}(1-u,1-u)}{1-u} \\
\lambda_{(\frac{1}{2},0)}(u) &= \Pr\left\{U_1 < \frac{1}{2} \mid U_2 < 1-u\right\} = \frac{\mathbf{C}\left(\frac{1}{2},1-u\right)}{1-u} \\
\lambda_{(1,0)}(u) &= \Pr\{U_1 > u \mid U_2 < 1-u\} = 1 - \frac{\mathbf{C}(u,1-u)}{1-u} \\
\lambda_{(1,\frac{1}{2})}(u) &= \Pr\left\{U_1 > u \mid U_2 < \frac{1}{2}\right\} = 1 - 2\mathbf{C}\left(u,\frac{1}{2}\right) \\
\lambda_{(1,1)}(u) &= \Pr\{U_1 > u \mid U_2 > u\} = \frac{1-2u+\mathbf{C}(u,u)}{1-u} \\
\lambda_{(\frac{1}{2},1)}(u) &= \Pr\left\{U_1 < \frac{1}{2} \mid U_2 > u\right\} = \frac{\frac{1}{2}-\mathbf{C}\left(\frac{1}{2},u\right)}{1-u} \\
\lambda_{(0,1)}(u) &= \Pr\{U_1 < 1-u \mid U_2 > u\} = 1 - \frac{\mathbf{C}(1-u,u)}{1-u} \\
\lambda_{(0,\frac{1}{2})}(u) &= \Pr\left\{U_1 < 1-u \mid U_2 < \frac{1}{2}\right\} = 2\mathbf{C}\left(1-u,\frac{1}{2}\right)
\end{aligned} \tag{54}$$

$\lambda_{(a,b)}$ denotes the quantile-dependent measure when $U_1$ goes towards $a$ given that $U_2$ goes towards $b$. We have represented graphically the directions of these quantile dependent measures in figure 13. These measures can be used to verify the strength of the failure of the assumption (52). For example, we remark that even if the relationship (52) is not verified by the Normal copula with $\beta = 0.7$ and the Frank copula with $\beta = 0.5$ (see figure 11), the conditional probabilities of the failure region are very closed (see figure 14). We could then consider that the Normal copula is a more dependent function than the Frank copula.

**The quantile regression method** Koenker and Basset [1978] introduce quantile regression as an extension of the classical median regression. We note $x_2 = \mathbf{q}(x_1;\alpha)$ the quantile regression curve of $X_2$ on $X_1$, which is defined by

$$\Pr\{X_2 \le x_2 \mid X_1 = x_1\} = \alpha \tag{55}$$

Using the integral transforms $U_1 = \mathbf{F}_1(X_1)$ and $U_2 = \mathbf{F}_2(X_2)$, we have

$$\Pr\left\{X_2 \le \mathbf{F}_2^{-1}(u_2) \mid X_1 = \mathbf{F}_1^{-1}(u_1)\right\} = \alpha \tag{56}$$

with $x_1 = \mathbf{F}_1^{-1}(u_1)$ and $x_2 = \mathbf{F}_2^{-1}(u_2)$. It comes that the quantile regression curve of $X_2$ on $X_1$ is equivalent to solve this following problem

$$\Pr\{U_2 \le u_2 \mid U_1 = u_1\} = \alpha \tag{57}$$

or

$$\frac{\partial}{\partial u_1}\mathbf{C}(u_1,u_2) = \alpha \tag{58}$$

If we note $u_2 = \mathbf{q}^\star(u_1;\alpha)$ the solution of this equation, we have

$$x_2 = \mathbf{F}_2^{-1}\left(\mathbf{q}^\star\left(\mathbf{F}_1(x_1);\alpha\right)\right) \tag{59}$$

and so $\mathbf{q}(x;\alpha) = \mathbf{F}_2^{-1}\left(\mathbf{q}^\star\left(\mathbf{F}_1(x);\alpha\right)\right)$.

In some cases, we can find the analytical expression of $\mathbf{q}^\star$. For example, Frees and Valdez [1998] give the solution for the Frank copula. It is defined as

$$\mathbf{C}(u_1,u_2) = -\frac{1}{\beta}\ln\left(1+\frac{\left(e^{-\beta u_1}-1\right)\left(e^{-\beta u_2}-1\right)}{e^{-\beta}-1}\right) \tag{60}$$

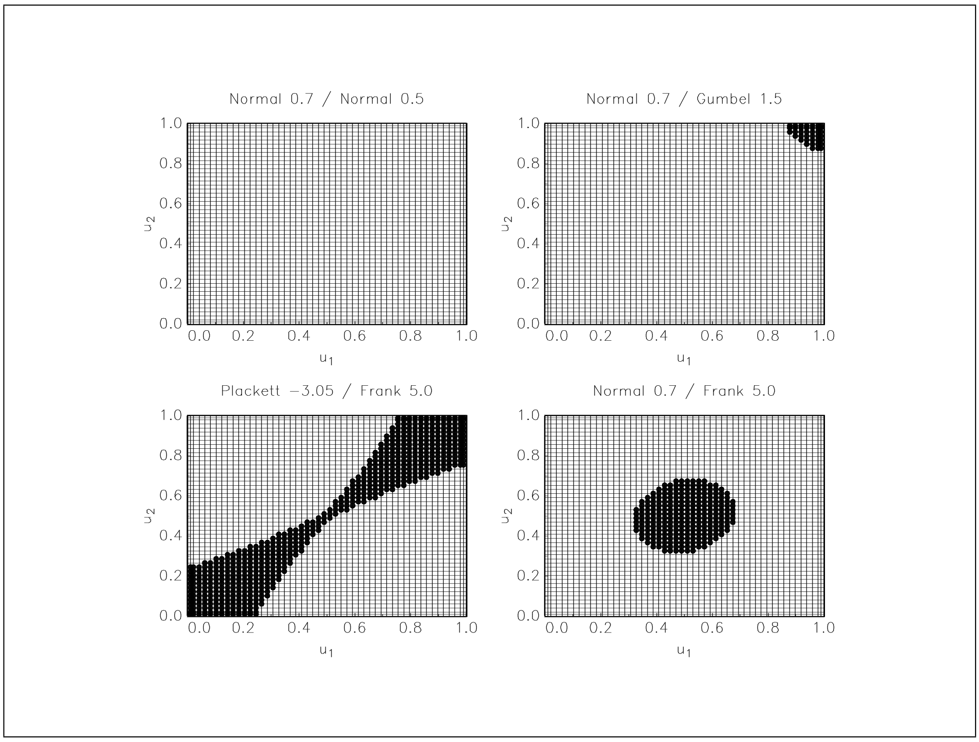


Figure 11: Comparison of dependence functions (I)

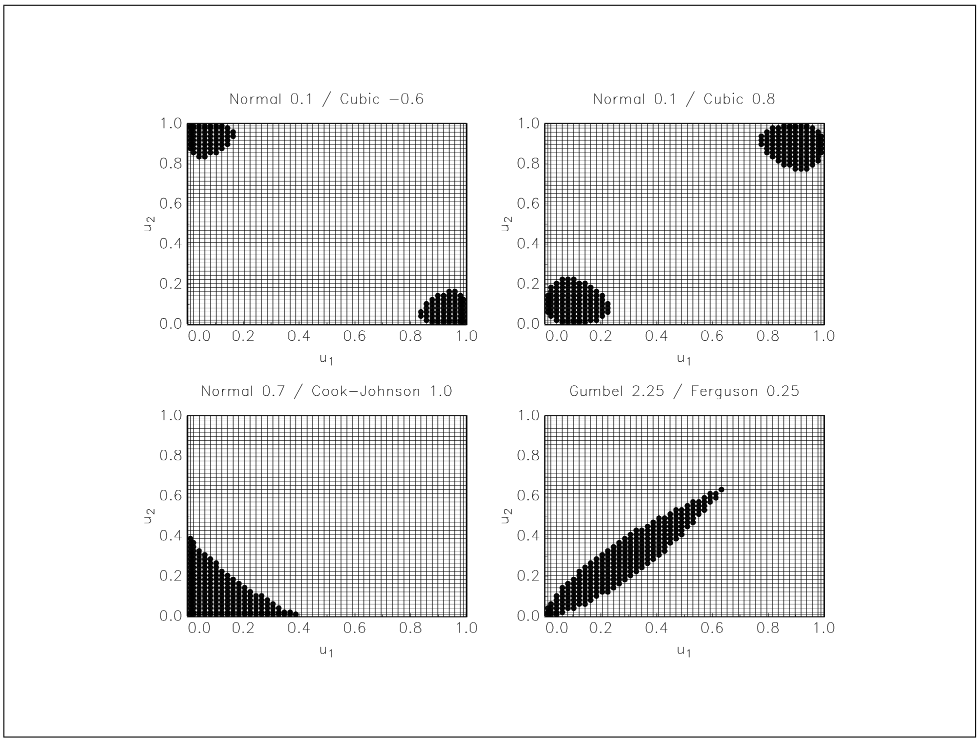


Figure 12: Comparison of dependence functions (II)

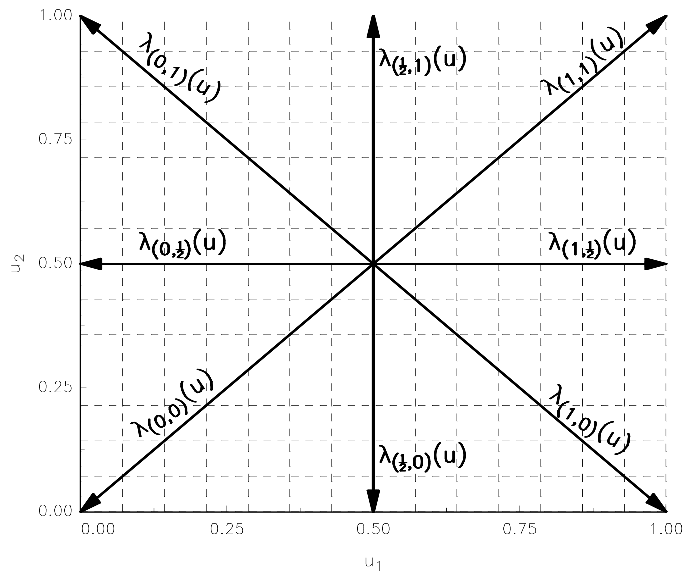


Figure 13: Directions of the quantile dependent measures

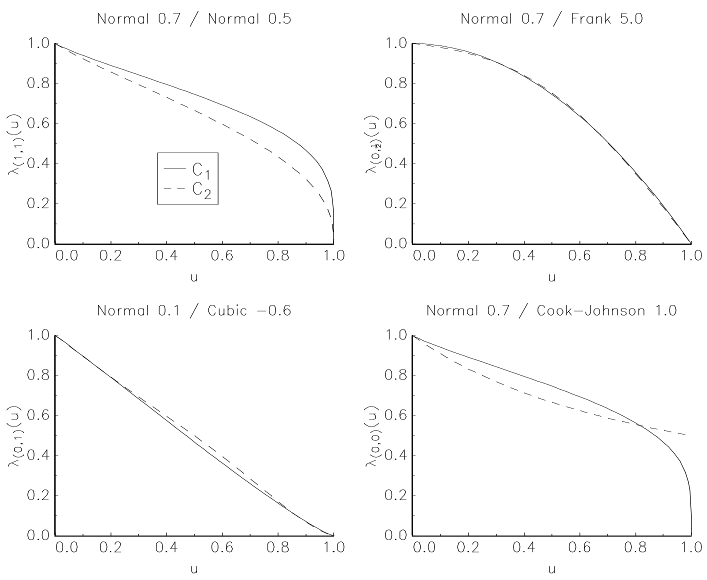


Figure 14: Exemples of quantile dependent measures

It comes that

$$\frac{\partial}{\partial u_1}\mathbf{C}(u_1,u_2)=\frac{e^{-\beta u_1}\left(e^{-\beta u_2}-1\right)}{\left(e^{-\beta}-1\right)+\left(e^{-\beta u_1}-1\right)\left(e^{-\beta u_2}-1\right)} \tag{61}$$

The quantile regression function $u_2=\mathbf{q}^{\star}(u_1;\alpha)$ is then

$$u_2=-\frac{1}{\beta}\ln\left(1+\frac{\alpha\left(e^{-\beta}-1\right)}{\alpha+e^{-\beta u_1}(1-\alpha)}\right) \tag{62}$$

In the case of the Normal copula, Bouyé, Durrleman, Nikeghbali, Riboulet and Roncalli [2000] show that

$$\frac{\partial}{\partial u_1}\mathbf{C}(u_1,u_2)=\Phi(\varsigma) \tag{63}$$

with

$$\varsigma=\frac{\mathbf{\Phi}^{-1}(u_2)-\beta\mathbf{\Phi}^{-1}(u_1)}{\sqrt{1-\beta^2}} \tag{64}$$

The expression of the function $u_2=\mathbf{q}^{\star}(u_1;\alpha)$ is also

$$u_2=\Phi\left(\beta\mathbf{\Phi}^{-1}(u_1)+\sqrt{1-\beta^2}\mathbf{\Phi}^{-1}(\alpha)\right) \tag{65}$$

Note that if the margins are gaussians, we obtain the well-known regression curve

$$X_2=\left[\mu_2-\beta\frac{\sigma_2}{\sigma_1}\mu_1+\sqrt{1-\beta^2}\mathbf{\Phi}^{-1}(\alpha)\right]+\beta\frac{\sigma_2}{\sigma_1}X_1 \tag{66}$$

We remark that the relationship is **linear** and could be written as $X_2=a+bX_1$. When the margins are not gaussians, the relationship is linear in transformed random variables $Y_1=\left(\Phi^{-1}\circ\mathbf{F}_1\right)(X_1)$ and $Y_2=\left(\Phi^{-1}\circ\mathbf{F}_2\right)(X_2)$.

In figure 15, we have reported the quantile regression curves for different copulas (Frank, Normal and Gumbel) and different margins (uniform, gaussian and student). For the Gumbel copula, an analytical expression is not available and so we use a root finding procedure. The parameters of the three copulas have been choosen such that they have the same Spearman's rho, which is equal to 75%. We remark the influence of the margins on the curves. In fact, it appears that the interpretation of $\mathbf{q}^{\star}$ is more easy than this of $\mathbf{q}$, because the values $u_1$ and $u_2$ are the quantiles of $X_1$ and $X_2$.

We finish this paragraph by some remarks about two methods for estimating quantile regression. The first one (RQ) is based on the algorithm of Portnoy and Koenker [1997]. This estimating method is well known in Econometrics. It assumes that the relationship between $X_1$ and $X_2$ is linear. The second one is based on local regressions of Yu and Jones [1998], which can be linear (LLR) or quadratic (QLR). We consider the case where the copula is Normal ($\beta=0.5$) and the margins are gaussians. We have reported in figure 16 the results given by these methods on a simulation with 5000 observations. We remark that the two methods give good results. In figure 17, the copula is the same, whereas the margins of $X_1$ and $X_2$ are respectively Student with parameter equal to 2 and Gamma with parameter equal to 1.5. In this case, we remark that the RQ method is not appropriate, because the relationship between $X_1$ and $X_2$ is not linear. On the contrary, local regressions give some good results.

**Remark 5** *If we assume that the dependence function is Normal, we can use the Portnoy-Koenker algorithm with the transformed variables $Y_i=\left(\Phi^{-1}\circ\mathbf{F}_i\right)(X_i)$. Let $\hat{a}$ and $\hat{b}$ be the estimates of the linear quantile regression*

$$\left\{\begin{array}{l} Y_2=a+bY_1+u \\ \Pr\{Y_2\leq bY_1\}=\alpha \end{array}\right. \tag{67}$$

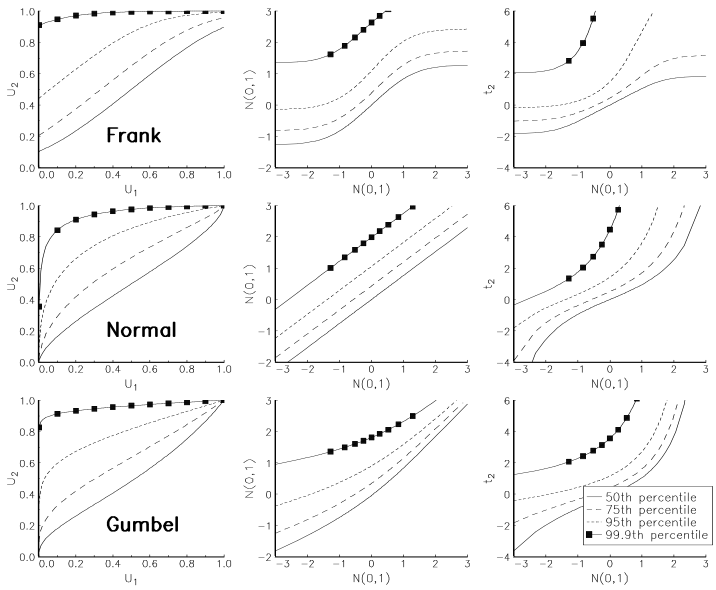


Figure 15: Quantile regression with different copula functions

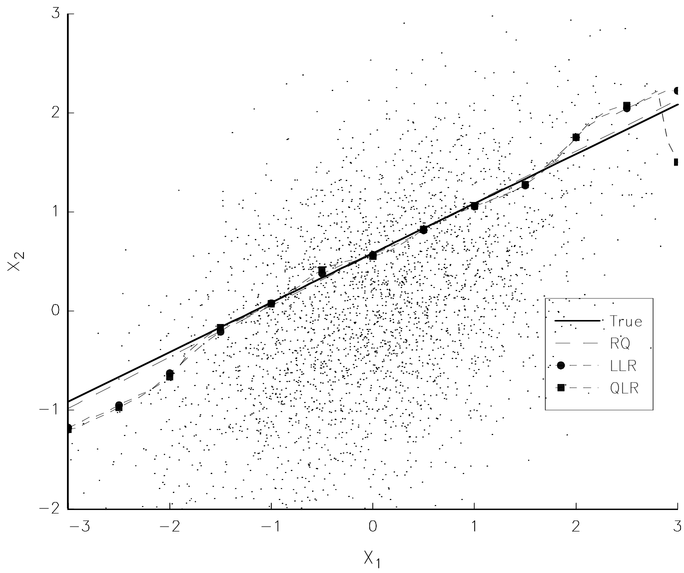


Figure 16: Estimating quantile regressions (I)

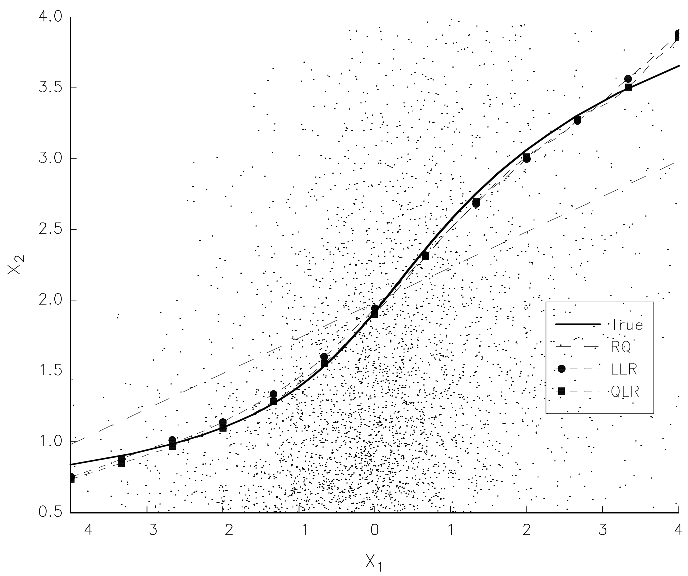


Figure 17: Estimating quantile regressions (II)

*The quantile regression curve of $X_2$ on $X_1$ is then obtained as follows*

$$X_2 = \mathbf{F}_2^{-1}\left(\Phi\left(\hat{a} + \hat{b}\Phi^{-1}\left(\mathbf{F}_1\left(X_1\right)\right)\right)\right) \tag{68}$$

*If the margins are not known, we can estimate them or we can use the empirical distributions. We have implemented this transformation with the previous example, and it gives very good result (see figure 18).*

**The Deheuvels copula** Let $\mathcal{X} = \left\{\left(x_1^t, x_2^t\right)\right\}_{t=1}^T$ be a sample of the random vector $(X_1, X_2)$. The empirical (or Deheuvels) copula is given by

$$\hat{\mathbf{C}}_{(T)}\left(\frac{t_1}{T}, \frac{t_2}{T}\right) = \frac{1}{T}\sum_{t=1}^{T} \mathbf{1}_{\left[x_1^t \leq x_1^{(t_1)}, x_2^t \leq x_2^{(t_2)}\right]} \tag{69}$$

where $x_n^{(t)}$ is the order statistics. The estimation of the Deheuvels copula is then easy thank to a rank procedure. Note that the empirical copula is only defined on lattice $\mathfrak{L} = \left\{\left(\frac{t_1}{T}, \frac{t_2}{T}\right) : t_n = 0, \ldots, T\right\}$. Deheuvels [1979] shows two fundamentals properties:

1. The empirical copula $\hat{\mathbf{C}}$ defined on $\mathfrak{L}$ is in distribution independent of the margins $\mathbf{F}_1$ and $\mathbf{F}_2$.
2. If $\hat{\mathbf{C}}_{(T)}$ is any empirical copula of order $T$, then $\hat{\mathbf{C}}_{(T)} \rightarrow \mathbf{C}$.

The Deheuvels copula is very important, because it can be viewed as the non parametric copula of the data. We can use it in conjunction with the previous methods. However, we can face some problems. For example, if we would like to compare the two Deheuvels copula $\hat{\mathbf{C}}_1$ and $\hat{\mathbf{C}}_2$, we will check the relationship

$$\hat{\mathbf{C}}_1\left(u_1, u_2\right) \geq \hat{\mathbf{C}}_2\left(u_1, u_2\right) \tag{70}$$

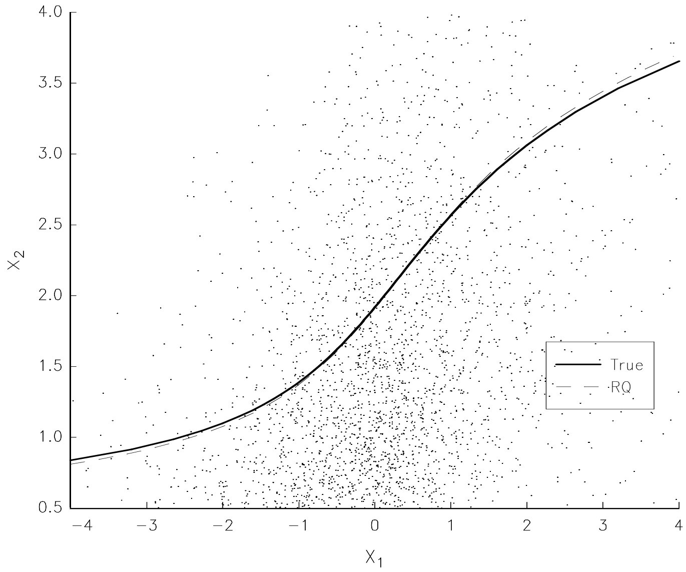


Figure 18: Estimating quantile regressions (III)

for all $(u_1, u_2) \in \mathfrak{L}_1 \bigcap \mathfrak{L}_2$. When $\mathfrak{L}_1 = \mathfrak{L}_2$ (this is the case when we want to compare two bivariate asset returns on the same period) or $\mathfrak{L}_1 \subset \mathfrak{L}_2$, we have no difficulty to perform the comparison. In the other cases, the lattices can not be compatible. We can then use approximation of copulas, like the Bernstein or Checkerboard methods (Li, Mikusiński, Sherwood and Taylor [1997]). In this case, the approximated copula is defined for all $(u_1, u_2) \in [0,1]^2$. Moreover, these approximations are useful in the case of quantile regressions, because Durrleman, Nikeghbali and T. Roncalli [2000] provide analytical expressions of the conditional probabilities (see [23]).

**Illustration with the indices CAC40, DowJones and NIKKEI** We consider the returns of the indices CAC40, DowJones and NIKKEI from 1/4/1988 to 10/9/2000. In figure 19, we indicate in black the region where the relationship $\hat{\mathbf{C}}_{(\mathsf{CAC40},\mathsf{DowJones})} \succ \hat{\mathbf{C}}_{(\mathsf{NIKKEI},\mathsf{DowJones})}$ does not hold. We remark that this region is tiny. We can then say that the dependence function between the indices CAC40 and DowJones is stronger than the one between the indices NIKKEI and DowJones. Moreover, we have reported the quantile-dependent measure $\lambda_{(0,0)}(u)$ and $\lambda_{(1,1)}(u)$ to compare the behaviour of the bear and bull markets. It is obvious that extreme returns are more dependent between the indices CAC40 and DowJones than the indices NIKKEI and DowJones.

We restrict now the study to the period before 1/1/1990. If we estimate the correlation matrix, we obtain

$$\hat{\rho} = \begin{bmatrix} 1 & 0.158 & 0.175 \\ & 1 & 0.0589 \\ & & 1 \end{bmatrix} \tag{71}$$

If we assume that the random vector of the trivariate asset returns is normal, it means that

$$\mathbf{C}_{(\mathsf{CAC40},\mathsf{NIKKEI})} \succ \mathbf{C}_{(\mathsf{CAC40},\mathsf{DowJones})} \succ \mathbf{C}_{(\mathsf{NIKKEI},\mathsf{DowJones})}$$

If now, we only suppose that the dependence is Normal (without assumptions on margins), we can estimate the parameter matrix $\beta$ of the 3-dimensional Normal copula. We obtain

$$\hat{\beta} = \begin{bmatrix} 1 & 0.207 & 0.157 \\ & 1 & 0.0962 \\ & & 1 \end{bmatrix}$$

In this case, we verify that

$$\mathbf{C}_{(\mathsf{CAC40,DowJones})} \succ \mathbf{C}_{(\mathsf{CAC40,NIKKEI})} \succ \mathbf{C}_{(\mathsf{NIKKEI,DowJones})}$$

The order of the dependence has changed. This little example shows that assuming a normal distribution can lead to big errors of interpreting the dependence functions. In figure 20,we have represented the region where the Deheuvels copula is stronger than the dependence function induced from the normal distribution. We remark that the region is very large. It indicates obviously that the dependence function has been underestimated by the gaussian assumption.

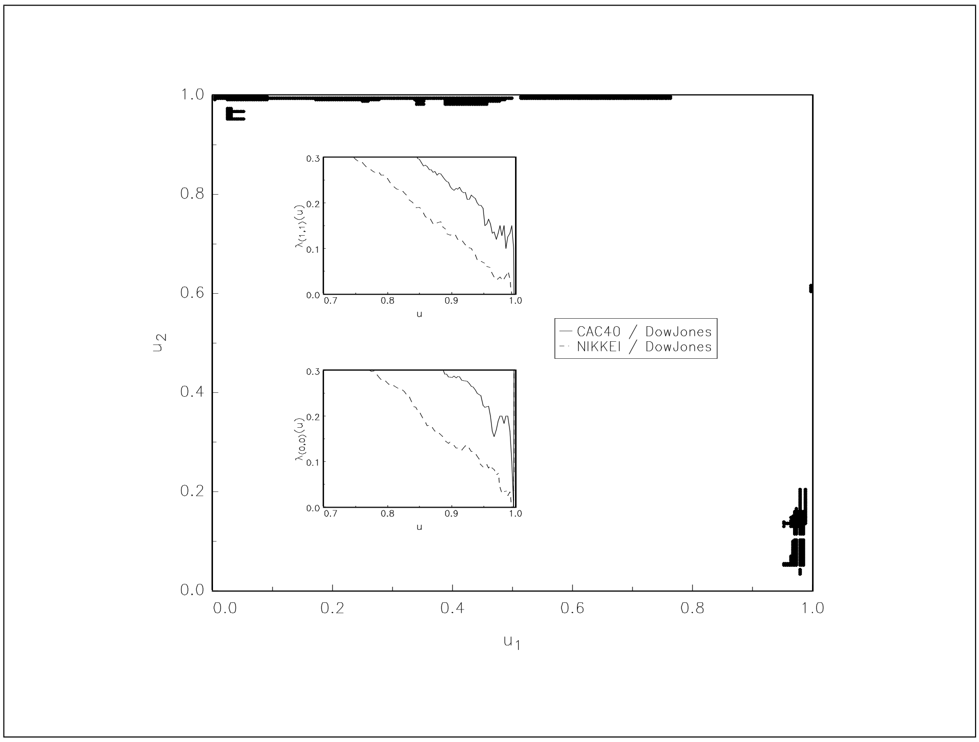


Figure 19: Comparison of dependence functions $\hat{\mathbf{C}}_{(\mathsf{CAC40,DowJones})}$ and $\hat{\mathbf{C}}_{(\mathsf{NIKKEI,DowJones})}$

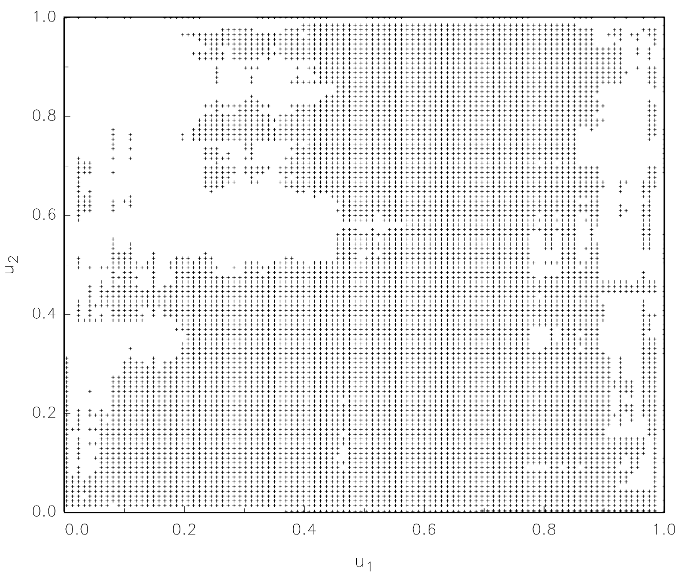


Figure 20: Comparison of dependence functions $\hat{\mathbf{C}}_{(\mathsf{CAC40},\mathsf{DowJones})}$ and the copula induced by the normal assumption

# 4 An application: the Asian crisis revisited

## 4.1 The Asian crisis: previous evidence

In this paragraph, we beginning by a short recall of the chronology of the Asian crisis. More detailed descriptions can be found in Kaminsky and Schmukler [1999] or Radelet and Sachs [1998] and especially in Roubini [1998]. Then we review existing empirical studies.

### 4.1.1 A short chronology of the Asian crisis

The beginning of the Asian crisis is generally associated with the devaluation of the Thai bath on July 2, 1997 after successive speculative attacks and the resignation of the Thailand's Finance Minister Virava — a great supporter of the peg to the dollar.

In the following days, various neighbouring countries were attacked leading to abandonments of pegs or devaluations (Malaysian ringitt, Philippine peso, Singapore dollar). On August 14, Indonesia let the rupiah float. The following day, speculators attack Kong Kong dollar. To defend its currency, the central bank is obliged to increase its overnight interest rates up by 150 basis points. On October 14, the Taiwan dollar is devaluated, creating doubts about the sustainability of the Hong Kong dollar peg. The following week, the Hang Seng Index (Hong Kong) lost around a quarter of its value in four days. On October 27, the crisis spreads all over the world with, for example, trading suspended in Wall Street and biggest single-day losses in Brazil, Argentina and Mexico. During the days following the announcement of an agreement between Indonesia and IMF for a financial support package, Asian stock markets started to recover sharply.

On November 7 and the days aftermath, attention began to focus on South Korea which heavily intervened to sustain its currency. Asian and Latin American stock markets suffered from substantial losses. On Novembre 20, the South Korean won lost around 10% of its value, followed by most other regional currencies (albeit in smallest falls). On December 3, the won, rupiah, baht and ringgit crashed to all-time lows against the dollar. On December 9, rumors that Indonesia's President Suharto is gravely ill propagated. The won attained a new record low. In the end of the year, the announcements of financial support to Korea from leading financial institutions and speedy disbursements of the IMF aid package permitted to recover. On January, 8, 1998, attention turned to Indonesia where the currency and stock market fell by 26% and 12% respectively. On the following day, the announcement of a commitment of President Suharto to implement economic reforms attached to the financial aid package of the IMF helped the rupiah to recover sharply. By the mid of the month, the investors adopted more optimistic views about Southeast Asia with the exception of the rupiah. From February to the end of April, markets alternated periods of falls and periods of recover, but in a context of very volatile markets. May was marked by political events and riots in Indonesia and associated sharp falls in the rupiah.

The end of the Asian crisis is generally associated with the resignation of President Suharto on May 21 and the beginning of the Russian crisis.

#### 4.1.2 A (selective) review of empirical work

In this subsection, we offer a selective review of empirical work on the Asian crisis. In particular, we do not consider the broader literature which has analyzed the Asian crisis in part of a larger sample including other crises (Mexico, Russia, ERM, 1982 debt crisis) ; see the first section for references. Rather we concentrate on the literature which have focused on the Asian crisis. There are quite sensible differences with other studies.

First, in this case, the crisis period is always defined on a priori grounds. Baig and Goldfjan [1999] defines the crisis period as beginning from the day of the bath devaluation and extending up to end May, 1998. Kaminsky and Schmukler [1999] retain the larger period January 1997 – May 1998. Nagayasu [2000] select the period 11/15/1996 – 12/31/1998. Quite surprisingly, Forbes and Rigobon [2000] defines the turmoil period as the month following the Hong Kong stock market crash (October 17, 1997). This choice is motivated by their impression that occidental newspapers did not pay attention to the Southeast Asia events before this date[18]. In this paper, we choose, in line with the previous chronology, to define the crisis period in the same way as Baig and Goldfjan.

Second, in difference with probit models, these studies do not introduce macroeconomic variables in the analysis. This choice presents the advantage that it becomes possible to use higher frequency data (daily rather than monthly). In this way, it is possible to identify more precisely the relationship between different markets since, in general, information proceeds very fast and contagion operates in hours rather than in quarters. On the other way round, this choice presents the disadvantage that it can be misleading to analyze stock markets or exchange rates returns without paying attention to the evolution of fundamentals. Indeed, it becomes difficult to distinguish if simultaneous price movements come from fundamental reasons or from pure contagion. Faced with this dilemma, Kaminsky and Schmukler [1998] and Baig and Goldfajn [1999] propose to approximate movements in fundamentals by constructing dummy variables from news reported in the press or in continuous time information agencies (such as Reuters or Bloomberg). These dummies are classified in "bad" or "good" news and distinguished as "local" news or "international" news. In this paper, while our primary objective is methodological, we apply our methods to raw returns and to "news filtered" returns.

We can now summarize the earlier empirical findings. Kaminsky and Schmukler [1998] show, via regression methods, that some of the market jitters cannot be explained by any apparent substantial news but seem to be driven by herd behavior. Moreover investors tend to overreact to "bad" news. The same kind of conclusion is drawn by Baig and Goldfajn [1999] using cross-country correlations or Cerra and Saxena [2000] using a Markov switching model. More generally, the fact that comovements in Asian financial markets during the crisis

[18] However, Forbes and Rigobon [2000] proceeds to sensitivity analysis by modifying periods definitions.

cannot entirely attributed to "fundamentals" has been underlined by numbers of researchers. For example, some have noted that trade linkages cannot have a big impact in explaining spillovers effects since the trade linkages between Southeast Asian countries were only modest[19]. In the same way, the appreciation of the dollar against the yen predates the crisis by at least a year (MASSON [2000]) and some estimates (see e.g. CHINN [1997]) contradict the idea that the concerned currencies (expect Thailand) were overvaluated. More detailed results are furnished by NAGAYSU [2000] who confirms the evidence that the upward pressure on exchange rates was essentially caused by sectoral indices (and particularly those of banking and financial sectors) or by BAIG and GOLDFAJN [1999] who show that contagion operated more sensibly in currency markets than in stock markets.

Nearer to our concerns in this paper, FORBES and RIGOBON [2000] challenge the result that cross-country correlations have significantly increase during the Southeast Asian turmoil and conclude that there was no contagion. In their opinion, it is a fallacious result which disappear once heteroskedasticity is taken into account. On the other way round, BAIG and GOLDFAJN [1999] present results favorables to the hypothesis of contagion based on cross-correlations adjusted in the way recommended by Forbes and Rigobon (see below) . Clearly, the question that there was pure contagion remains open. In the rest of this paper, we try to offer an answer on such a question, relying on a robust (and new in this context) statistical framework.

## 4.2 Empirical results

In this subsection, we propose an application of copulas to the analysis of the Asian crisis. It is based on a reassessement of BAIG and GOLDFJAN [1998] results, particularly those comparing a crisis period defined between 07/01/1997 and 05/18/1998 and a normal period defined between 01/01/1995 and 12/31/1996. Their analysis is based on the daily returns of stock indexes, exchange rates, interest rates and sovereign spreads for the five countries which were more severely by the crisis: Indonesia, Korea, Malaysia, Philippines and Thailand. Furthermore, they investigate whether their results are sensible to conditionning on fundamentals which are measured as bad or good news (on the basis of Bloomberg reports)[20].

We begin with a short recall of Baig and Goldfjan results. They present correlation matrices for whole returns where the correlation coefficient $\rho$ is computed taking into account the change in volatility (i.e. heteroskedasticity). While they refer to the adjustement proposed by FORBES and RIGOBON [1999], they do not explicitly use it. The adjustement proposed by Forbes and Rigobon gives the following estimate of the correlation coefficient

$$\tilde{\rho} = \frac{\rho}{\sqrt{1 + \delta \left(1 - \rho^2\right)}} \tag{72}$$

where $\rho$ is the the standard correlation coefficient and $\delta$ is the relative increase in the conditional variance in the crisis country[21]. On the contrary, Baig and Goldfjan use a Fisher transformation of their data but it is not clear why such a transformation allows them to adjust for the heteroskedasticity bias. As we show below, the choice of the adjustement has sensible consequences for the results. Moreover, their method does not allow to estimate an adjusted correlation coefficient, only to compute an heteroscedastic consistent $t$-test. In the following tables, we also report $t$-tests but based on the Forbes and Rigobon adjusted coefficients.

The results[22] are reported in tables 5 to 10. Three general remarks can be given.

[19] The same argument can be applied to indirect competition since the Southeast Asian countries do not share very similar third-country export profiles.

[20] The database can be downloaded from the IMF website with the article. The news chronology can be found in an appendix of the working paper version of the article (also downloadable from the IMF website).

[21] This adjustement needs to pre-determine the crisis country. In the following tables where we compute pairwise correlations, we assume that the crisis country is the one in which the increase in relative volatility is the greatest. However, the main conclusions are not sensible to this choice.

[22] While not reported here to save space, we have done the same exercice for *residual* correlations (i.e. residuals of a regression of returns on news dummies, yen/dollar exchange rate returns and US stock exchange returns; see Baig and Goldfjan). The conclusions are identically the sames.

| | Indonesia | Korea | Malaysia | Philippines | Thailand |
|---|---|---|---|---|---|
| Indonesia | 1 | −0.033 | 0.026 | 0.064 | 0.066 |
| Korea | | 1 | 0.044 | 0.015 | −0.012 |
| Malaysia | | | 1 | −0.043 | 0.116 |
| Philippines | | | | 1 | −0.026 |
| Thailand | | | | | 1 |

Table 5: Exchange rate returns (non-crisis period)

| | Indonesia | Korea | Malaysia | Philippines | Thailand |
|---|---|---|---|---|---|
| Indonesia | | 0.243*** | 0.326*** | 0.104 | 0.292*** |
| Korea | 0.009 | | 0.049 | 0.048 | 0.136*** |
| Malaysia | 0.013 | 0.004 | | 0.277*** | 0.379*** |
| Philippines | 0.004 | 0.004 | 0.032*** | | 0.094*** |
| Thailand | 0.011 | 0.008 | 0.024 | 0.006 | |
| $\delta$ | 706.5 | 146.0 | 80.1 | 23.9 | 296.5 |

Table 6: Exchange rate returns (crisis period)

| | Indonesia | Korea | Malaysia | Philippines | Thailand |
|---|---|---|---|---|---|
| Indonesia | 1 | −0.014 | 0.338 | 0.424 | 0.314 |
| Korea | | 1 | 0.049 | 0.039 | 0.023 |
| Malaysia | | | 1 | 0.386 | 0.403 |
| Philippines | | | | 1 | 0.304 |
| Thailand | | | | | 1 |

Table 7: Stock exchange returns (non-crisis period)

| | Indonesia | Korea | Malaysia | Philippines | Thailand |
|---|---|---|---|---|---|
| Indonesia | | 0.108*** | 0.469*** | 0.428 | 0.375* |
| Korea | 0.035 | | 0.265*** | 0.134** | 0.208*** |
| Malaysia | 0.168 | 0.089 | | 0.426 | 0.410 |
| Philippines | 0.150 | 0.050 | 0.151 | | 0.394*** |
| Thailand | 0.128 | 0.078* | 0.145 | 0.181 | |
| $\delta$ | 8.75 | 6.34 | 8.46 | 4.45 | 3.32 |

Table 8: Stock exchange returns (crisis period)

| | Indonesia | Korea | Malaysia | Philippines | Thailand |
|---|---|---|---|---|---|
| Indonesia | 1 | −0.161 | −0.166 | 0.091 | −0.071 |
| Korea | | 1 | −0.124 | −0.107 | 0.374 |
| Malaysia | | | 1 | 0.062 | −0.240 |
| Philippines | | | | 1 | 0.135 |
| Thailand | | | | | 1 |

Table 9: Interest rates (non-crisis period)

| | Indonesia | Korea | Malaysia | Philippines | Thailand |
|---|---|---|---|---|---|
| Indonesia | | 0.392 | 0.089 | −0.195 | 0.350 |
| Korea | 0.061 | | 0.452 | −0.336 | 0.421 |
| Malaysia | 0.013 | 0.161 | | −0.081 | 0.274 |
| Philippines | −0.077 | −0.147 | −0.031 | | −0.251 |
| Thailand | 0.053 | 0.190 | 0.091 | −0.263 | |
| $\delta$ | 47.86 | 4.78 | 8.68 | 5.70 | −0.10 |

Table 10: Interest rates (crisis period)

- First, it appears that the increase in correlation coefficients has been more sensible in exchange rate returns and in stock exchange returns than in interest rates.
- Second, we can observe that the increase in volatility (see $\delta$) is impressive in the case of the exchange rates (vis-à-vis the USD) where the one day volatility has been multiplied by a factor between 20 (for Philippines) and 700 (for Indonesia)! This result is not surprising since before the crisis, most of the concerned countries had adopted a peg based on the dollar. This increase is also sensible for stock exchange returns and interest rate but in lower proportions.
- Third, taking account the two previous facts, we can deduce that the results with correlation coefficients really depend whether we adjust for heteroskedasticity or not[23]. If we don't adjust as in the upper part of tables 6, 8 and 10, we deduce that there was a significant increase (at the 10% level) in correlation (and thus contagion) in 70% of cases for exchange rate and stock exchange returns. If we adjust (lower part of tables), we conclude there was contagion in only one case among ten for exchange rate (Philippines/Malaysia) and stock exchange (Korea/Thailand) returns. These results are at variance with those of Baig and Goldfjan who conclude that contagion significantly occurred for all pairs for currency markets and in 60% of all pairs for stock exchange markets.

While unadjusted results would seem too optimistic for the contagion hypothesis, we can simultaneously be quite skeptical with the ajusted results. Indeed, the correction proposed by Forbes and Rigobon seems derived under too specific hypothesis to be applicable to all problems. In face of this dilemma, it seems interesting to analyze the Asian crisis on the basis of a more general framework.

If we compare the empirical distributions during the non-crisis period and the crisis period, it is clear that they are different. For example, we have reported[24] these of the exchange rate returns for Indonesia and Korea in figure 21. It is obvious that the two bivariate distributions can not be compared, and it is then very dangerous to compare the dependence using the full distributions. Let $\mathbf{F}^{(1)}(x_1,x_2) = \mathbf{C}^{(1)}\left(\mathbf{F}_1^{(1)}(x_1),\mathbf{F}_2^{(1)}(x_2)\right)$ and $\mathbf{F}^{(2)}(x_1,x_2) = \mathbf{C}^{(2)}\left(\mathbf{F}_1^{(2)}(x_1),\mathbf{F}_2^{(2)}(x_2)\right)$ be the bivariate distributions for the non-crisis and crisis periods. In figure 21, we remark that the marginals have changed dramatically between the two periods. So, we can observe a modification of the correlation without necessarily a modification of the copula.

Figure 22 represents pointwise concordance comparison of the dependence functions of exchange rate returns for Indonesia and Korea between the two periods. In most cases, we do not observe that one copula is much larger than the other. We can then use a synthetic measure of concordance, like the Spearman's rho which is more appropriate than the Pearson correlation to compare the dependence functions. We recall that its definition is

$$\varrho = 12 \iint_{\mathbf{I}^2} u_1 u_2 \,\mathrm{d}\mathbf{C}(u_1,u_2) - 3 \tag{73}$$

[23]Unadjusted correlation coefficients are in the upper part of the matrix. Adjusted correlation coefficients (via Forbes and Rigobon [2000]) are in the lower part of the matrix. *, ** and *** denote rejection of the null hypothesis of no-change (**or** decrease) in correlation coefficients at the 10%, 5% and 1% levels respectively. $\delta$ is the relative increase in variance of the returns.

[24]The empirical distributions have been estimated using a Kernel method with an Epanechnikov window.

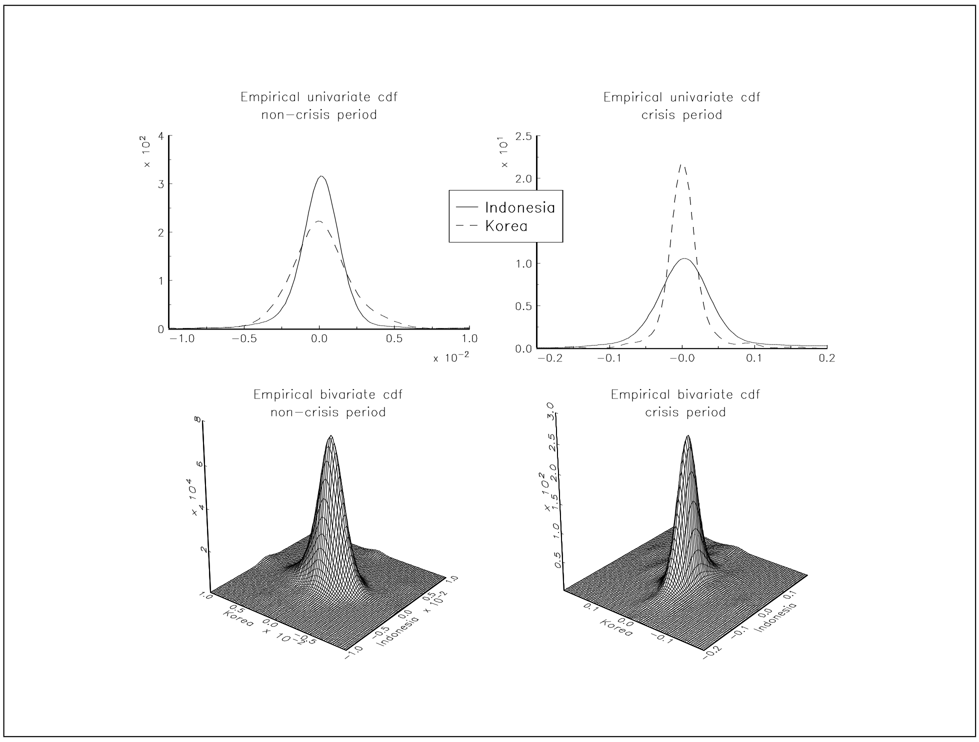


Figure 21: Empirical distributions of exchange rate returns

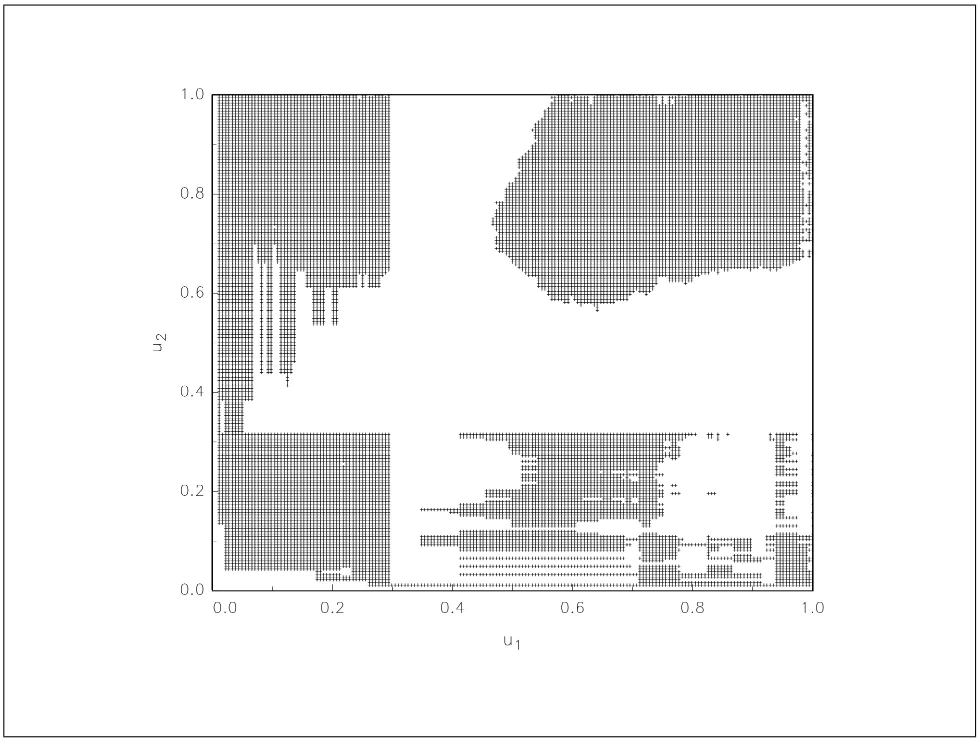


Figure 22: Comparison of empirical dependence of exchange rate returns

| | Indonesia | Korea | Malaysia | Philippines | Thailand |
|---|---|---|---|---|---|
| Indonesia | | −0.044 | 0.051 | 0.009 | 0.082 |
| Korea | 0.219*** | | 0.028 | 0.066 | 0.024 |
| Malaysia | 0.451*** | 0.117 | | −0.058 | 0.149 |
| Philippines | 0.090 | 0.199* | 0.185*** | | 0.005 |
| Thailand | 0.314*** | 0.118 | 0.364*** | 0.057 | |

Table 11: Exchange rate returns

| | Indonesia | Korea | Malaysia | Philippines | Thailand |
|---|---|---|---|---|---|
| Indonesia | | −0.028 | 0.345 | 0.405 | 0.328 |
| Korea | 0.094* | | 0.027 | 0.026 | 0.024 |
| Malaysia | 0.374 | 0.219*** | | 0.336 | 0.337 |
| Philippines | 0.394 | 0.105 | 0.414 | | 0.275 |
| Thailand | 0.331 | 0.246*** | 0.329 | 0.233 | |

Table 12: Stock exchange returns

As explained by Nelsen [1998], Spearman's rho is identical to the correlation coefficient for the integral transforms $U_1 = \mathbf{F}_1(X_1)$ and $U_2 = \mathbf{F}_2(X_2)$. We have reported the estimates in tables 11 to 13. The upper part corresponds to the non-crisis period whereas the lower part represents the crisis period. Let us denote $\varrho^{(1)}$ and $\varrho^{(2)}$ the Spearman's rho of the non-crisis period and the crisis period. We have tested the null hypothesis $\varrho^{(1)} = \varrho^{(2)}$ versus the alternative hypothesis $\varrho^{(1)} \neq \varrho^{(2)}$. Results are reported on the lower part of the tables[25]. They are very different of these obtained previously with the Pearson correlation. We observe significantly an increase in 60% of the cases for the exchange rates and in 30% of the cases for the stock exchange. These results give more evidence for the contagion hypothesis than these of Baig and Goldfjan [1999].

Figure 23 represents the quantile regressions for exchange rate returns for Indonesia and Korea between the two periods. The bottom graphs give quantile regressions estimated in the usual way which are implicity taking into account the marginals. On the contrary, the top graphs are only based on the dependence function (assuming that a Normal copula) and, thus, are not affected by the marginals. We can see that there exists, for a given quantile, some notable differences between the two methods. However, broadly speaking, both methods conclude there was an increase in the "dependence" of the exchange rate returns between the two periods (the slopes of the curves increase), and thus contagion. The case of Indonesia and Malaysia is represented in figure 24. We remark that conclusion is not obvious if we take account into marginals. In this case, the quantile regressions in the unit square give more evident results. To finish this application, note that we can observe of course some bias if we assume linearity between random variables. For example, we have reported in figure 25 the results obtained with the RQ algorithm (top graphs correspond to the case Indonesia/Korea whereas bottom graphs correspond to the case Indonesia/Malaysia). They are very different from the ones of the bottom graphs in figures 23 and 24.

[25] *, ** and *** denote rejection of the null hypothesis of no-change (**or** decrease) in correlation coefficients at the 10%, 5% and 1% levels respectively.

| | Indonesia | Korea | Malaysia | Philippines | Thailand |
|---|---|---|---|---|---|
| Indonesia | | −0.179 | −0.144 | 0.007 | −0.041 |
| Korea | 0.540*** | | −0.133 | −0.230 | 0.413 |
| Malaysia | 0.389*** | 0.737*** | | 0.331 | −0.272 |
| Philippines | −0.090 | −0.377 | −0.098 | | −0.017 |
| Thailand | 0.462*** | 0.376 | 0.373*** | −0.259 | |

Table 13: Interest rates

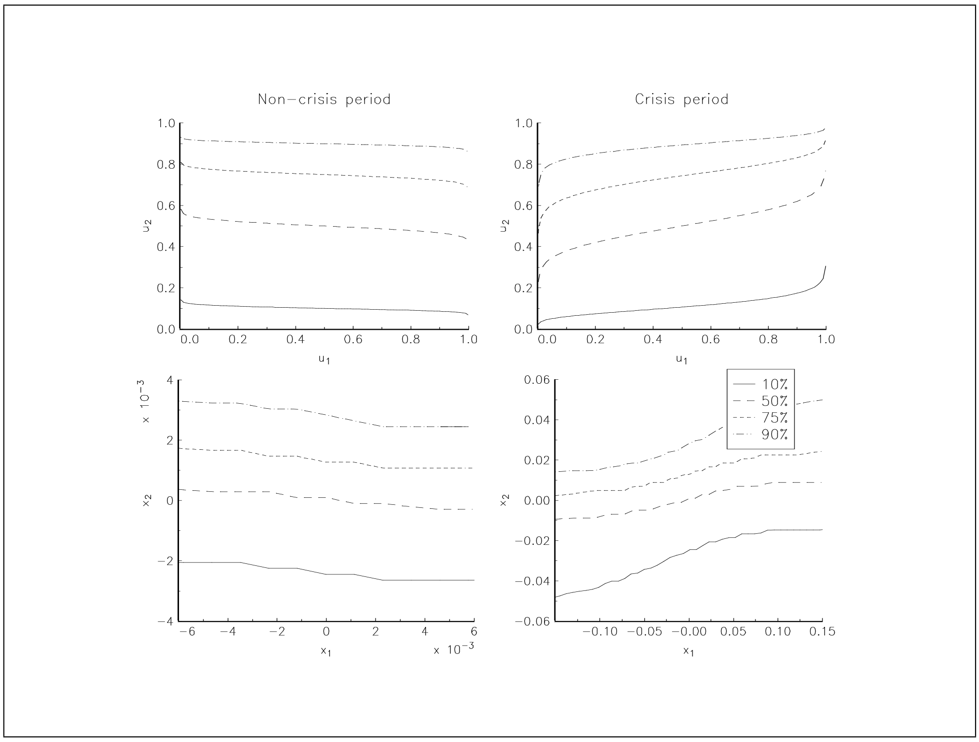


Figure 23: Quantile regressions of exchange rate returns for Indonesia/Korea

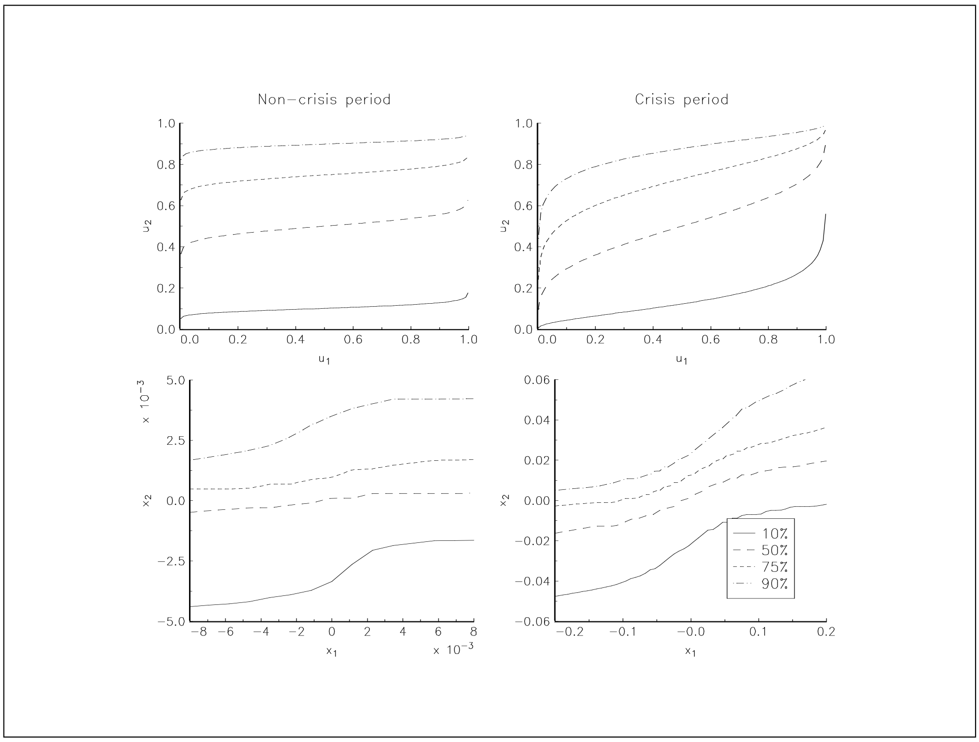


Figure 24: Quantile regressions of exchange rate returns for Indonesia/Malaysia

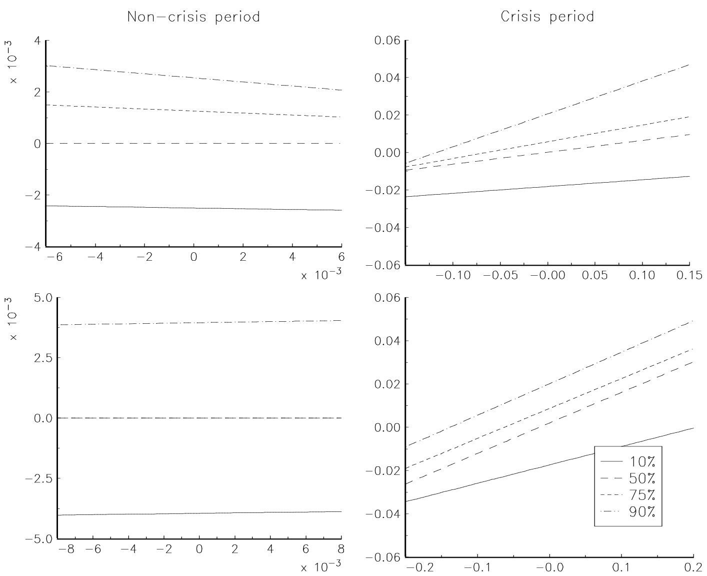


Figure 25: Quantile regressions of exchange rate returns with the RQ algorithm

# 5 Conclusion

In recent years, the interest in dependence has increased in the economic and financial literatures. As we have recalled, most of the statistical inference in this subject is based on standard (Pearson) correlation coefficients (or similar measures). However, we have shown that these tools are not reliable in this context as they give too much importance to the marginal distributions rather than concentrating on the dependence function. As an alternative, this paper, as a few others before, has proposed to resort on copulas. In this perspective, we have presented several tools to analyze the dependence between financial markets and we have proposed two applications.

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