Asset Management
Lecture 5. Machine Learning in Asset Management

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General information

1. **Overview**
   The objective of this course is to understand the theoretical and practical aspects of asset management
2. **Prerequisites**
   M1 Finance or equivalent
3. **ECTS**
   3
4. **Keywords**
   Finance, Asset Management, Optimization, Statistics
5. **Hours**
   Lectures: 24h, HomeWork: 30h
6. **Evaluation**
   Project + oral examination
7. **Course website**
   `http://www.thierry-roncalli.com/RiskBasedAM.html`

Objective of the course

The objective of the course is twofold:

1. having a financial culture on asset management
2. being proficient in quantitative portfolio management

Class schedule

Course sessions

- January 8 (6 hours, AM+PM)
- January 15 (6 hours, AM+PM)
- January 22 (6 hours, AM+PM)
- January 29 (6 hours, AM+PM)

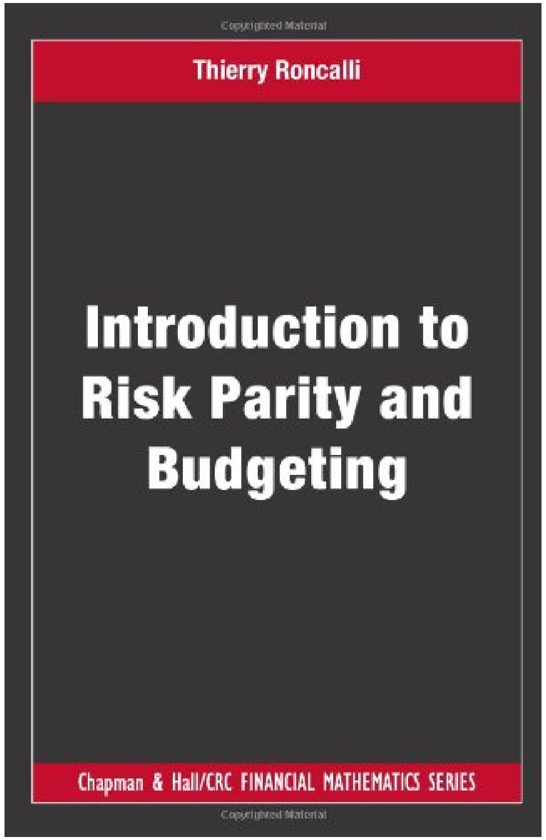
Class times: Fridays 9:00am-12:00pm, 1:00pm–4:00pm, University of Evry

Agenda

- Lecture 1: Portfolio Optimization
- Lecture 2: Risk Budgeting
- Lecture 3: Smart Beta, Factor Investing and Alternative Risk Premia
- Lecture 4: Green and Sustainable Finance, ESG Investing and Climate Risk
- Lecture 5: Machine Learning in Asset Management

Textbook

- Roncalli, T. (2013), *Introduction to Risk Parity and Budgeting*, Chapman & Hall/CRC Financial Mathematics Series.



Additional materials

- Slides, tutorial exercises and past exams can be downloaded at the following address:

  `http://www.thierry-roncalli.com/RiskBasedAM.html`

- Solutions of exercises can be found in the companion book, which can be downloaded at the following address:

  `http://www.thierry-roncalli.com/RiskParityBook.html`

Agenda

- Lecture 1: Portfolio Optimization
- Lecture 2: Risk Budgeting
- Lecture 3: Smart Beta, Factor Investing and Alternative Risk Premia
- Lecture 4: Green and Sustainable Finance, ESG Investing and Climate Risk
- **Lecture 5: Machine Learning in Asset Management**

Prologue

- Machine learning is a hot topic in asset management (and more generally in finance)
- Machine learning and data mining are two sides of the same coin

backtesting performance $\neq$ **live performance**

- Reaching for the stars: a complex/complicated process does not mean a good solution

Don't forget the 3 rules in asset management

1. It is difficult to make money
2. It is difficult to make money
3. It is difficult to make money

Prologue

- In this lecture, we focus on ML optimization algorithms, because they have proved their worth
- We have no time to study classical ML methods that can be used by quants to build investment strategies[1]

[1] Don't believe that they are always significantly better than standard statistical approaches!!!

Standard optimization algorithms

- Gradient descent methods
- Conjugate gradient (CG) methods (Fletcher–Reeves, Polak–Ribiere, etc.)
- Quasi-Newton (QN) methods (NR, BFGS, DFP, etc.)
- Quadratic programming (QP) methods
- Sequential QP methods
- Interior-point methods

Standard optimization algorithms

- We consider the following unconstrained minimization problem:

$$x^{\star} = \arg\min_{x} f(x) \tag{1}$$

  where $x \in \mathbb{R}^n$ and $f(x)$ is a continuous, smooth and convex function
- In order to find the solution $x^{\star}$, optimization algorithms use iterative algorithms:

$$\begin{aligned} x^{(k+1)} &= x^{(k)} + \Delta x^{(k)} \\ &= x^{(k)} - \eta^{(k)} D^{(k)} \end{aligned}$$

  where:
  - $x^{(0)}$ is the vector of starting values
  - $x^{(k)}$ is the approximated solution of Problem (1) at the k^{th} iteration
  - $\eta^{(k)} > 0$ is a scalar that determines the step size
  - $D^{(k)}$ is the direction

Standard optimization algorithms

- Gradient descent:

$$D^{(k)} = \nabla f\left(x^{(k)}\right) = \frac{\partial f\left(x^{(k)}\right)}{\partial x}$$

- Newton-Raphson method:

$$D^{(k)} = \left(\nabla^2 f\left(x^{(k)}\right)\right)^{-1} \nabla f\left(x^{(k)}\right) = \left(\frac{\partial^2 f\left(x^{(k)}\right)}{\partial x\, \partial x^\top}\right)^{-1} \frac{\partial f\left(x^{(k)}\right)}{\partial x}$$

- Quasi-Newton method:

$$D^{(k)} = H^{(k)} \nabla f\left(x^{(k)}\right)$$

where $H^{(k)}$ is an approximation of the inverse of the Hessian matrix

Standard optimization algorithms

What are the issues?

1. How to solve large-scale optimization problems?
2. How to solve optimization problems where there are multiple solutions?
3. How to just find an "*acceptable*" solution?

The case of neural networks and deep learning

$\Rightarrow$ Standard approaches are not well adapted

Machine learning optimization algorithms

Machine learning problems

- Non-smooth objective function
- Non-unique solution
- Large-scale dimension

Optimization in machine learning requires to reinvent numerical optimization

Machine learning optimization algorithms

We consider 4 methods:

- Cyclical coordinate descent (CCD)
- Alternative direction method of multipliers (ADMM)
- Proximal operators (PO)
- Dykstra's algorithm (DA)

Coordinate descent methods

The fall and the rise of the steepest descent method

In the 1980s:

- Conjugate gradient methods (Fletcher–Reeves, Polak–Ribiere, etc.)
- Quasi-Newton methods (NR, BFGS, DFP, etc.)

In the 1990s:

- Neural networks
- Learning rules: Descent, Momentum/Nesterov and Adaptive learning methods

In the 2000s:

- Gradient descent (by **observations**): Batch gradient descent (BGD), Stochatic gradient descent (SGD), Mini-batch gradient descent (MGD)
- Gradient descent (by **parameters**): Coordinate descent (CD), cyclical coordinate descent (CCD), Random coordinate descent (RCD)

Coordinate descent methods

Descent method

The descent algorithm is defined by the following rule:

$$x^{(k+1)} = x^{(k)} + \Delta x^{(k)} = x^{(k)} - \eta^{(k)} D^{(k)}$$

At the k^{th} Iteration, the current solution $x^{(k)}$ is updated by going in the opposite direction to $D^{(k)}$ (generally, we set $D^{(k)} = \partial_x f\left(x^{(k)}\right)$)

Coordinate descent method

Coordinate descent is a modification of the descent algorithm by minimizing the function along one coordinate at each step:

$$x_i^{(k+1)} = x_i^{(k)} + \Delta x_i^{(k)} = x_i^{(k)} - \eta^{(k)} D_i^{(k)}$$

$\Rightarrow$ The coordinate descent algorithm becomes a scalar problem

Coordinate descent methods

Choice of the variable i

1. Random coordinate descent (RCD)
   We assign a random number between 1 and n to the index i (Nesterov, 2012)
2. Cyclical coordinate descent (CCD)
   We cyclically iterate through the coordinates (Tseng, 2001):

$$x_i^{(k+1)} = \arg\min_x f\left(x_1^{(k+1)}, \ldots, x_{i-1}^{(k+1)}, x, x_{i+1}^{(k)}, \ldots, x_n^{(k)}\right)$$

Cyclical coordinate descent (CCD)

Example 1

We consider the following function:

$$f(x_1, x_2, x_3) = (x_1 - 1)^2 + x_2^2 - x_2 + (x_3 - 2)^4 e^{x_1 - x_2 + 3}$$

We have:

$$\begin{aligned} D_1 = \frac{\partial f(x_1, x_2, x_3)}{\partial x_1} &= 2(x_1 - 1) + (x_3 - 2)^4 e^{x_1 - x_2 + 3} \\ D_2 = \frac{\partial f(x_1, x_2, x_3)}{\partial x_2} &= 2x_2 - 1 - (x_3 - 2)^4 e^{x_1 - x_2 + 3} \\ D_3 = \frac{\partial f(x_1, x_2, x_3)}{\partial x_3} &= 4(x_3 - 2)^3 e^{x_1 - x_2 + 3} \end{aligned}$$

Cyclical coordinate descent (CCD)

The CCD algorithm is defined by the following iterations:

$$\left\{\begin{array}{l}
x_1^{(k+1)} = x_1^{(k)} - \eta^{(k)} \left(2\left(x_1^{(k)} - 1\right) + \left(x_3^{(k)} - 2\right)^4 e^{x_1^{(k)} - x_2^{(k)} + 3}\right) \\
x_2^{(k+1)} = x_2^{(k)} - \eta^{(k)} \left(2x_2^{(k)} - 1 - \left(x_3^{(k)} - 2\right)^4 e^{x_1^{(k+1)} - x_2^{(k)} + 3}\right) \\
x_3^{(k+1)} = x_3^{(k)} - \eta^{(k)} \left(4\left(x_3^{(k)} - 2\right)^3 e^{x_1^{(k+1)} - x_2^{(k+1)} + 3}\right)
\end{array}\right.$$

We have the following scheme:

$$\begin{array}{lcl}
\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) & \rightarrow & x_1^{(1)} \rightarrow \left(x_1^{(1)}, x_2^{(0)}, x_3^{(0)}\right) \rightarrow x_2^{(1)} \rightarrow \left(x_1^{(1)}, x_2^{(1)}, x_3^{(0)}\right) \rightarrow x_3^{(1)} \rightarrow \\
\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}\right) & \rightarrow & x_1^{(2)} \rightarrow \left(x_1^{(2)}, x_2^{(1)}, x_3^{(1)}\right) \rightarrow x_2^{(2)} \rightarrow \left(x_1^{(2)}, x_2^{(2)}, x_3^{(1)}\right) \rightarrow x_3^{(2)} \rightarrow \\
\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}\right) & \rightarrow & x_1^{(3)} \rightarrow \ldots
\end{array}$$

Cyclical coordinate descent (CCD)

Table 1: Solution obtained with the CCD algorithm ($\eta^{(k)} = 0.25$)

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$D_1^{(k)}$	$D_2^{(k)}$	$D_3^{(k)}$
0	1.0000	1.0000	1.0000			
1	-4.0214	0.7831	1.1646	20.0855	0.8675	-0.6582
2	-1.5307	0.8834	2.2121	-9.9626	-0.4013	-4.1902
3	-0.2663	0.6949	2.1388	-5.0578	0.7540	0.2932
4	0.3661	0.5988	2.0962	-2.5297	0.3845	0.1703
5	0.6827	0.5499	2.0758	-1.2663	0.1957	0.0818
6	0.8412	0.5252	2.0638	-0.6338	0.0989	0.0480
7	0.9205	0.5127	2.0560	-0.3172	0.0498	0.0314
8	0.9602	0.5064	2.0504	-0.1588	0.0251	0.0222
9	0.9800	0.5033	2.0463	-0.0795	0.0126	0.0166
∞	1.0000	0.5000	2.0000	0.0000	0.0000	0.0000

The lasso revolution

Least absolute shrinkage and selection operator (lasso)

The lasso method consists in adding a ℓ_1 penalty function to the least square problem:

$$\begin{aligned} \hat{\beta}^{\mathrm{lasso}}(\tau) &= \arg\min \frac{1}{2}(Y - X\beta)^\top (Y - X\beta) \\ \text{s.t.} & \quad \|\beta\|_1 = \sum_{j=1}^{m} |\beta_j| \leq \tau \end{aligned}$$

This problem is equivalent to:

$$\hat{\beta}^{\mathrm{lasso}}(\lambda) = \arg\min \frac{1}{2}(Y - X\beta)^\top (Y - X\beta) + \lambda \|\beta\|_1$$

We have:

$$\tau = \left\| \hat{\beta}^{\mathrm{lasso}}(\lambda) \right\|_1$$

Solving the lasso regression problem

We introduce the parametrization:

$$\beta = \begin{pmatrix} I_m & -I_m \end{pmatrix} \begin{pmatrix} \beta^+ \\ \beta^- \end{pmatrix} = \beta^+ - \beta^-$$

under the constraints $\beta^+ \geq \mathbf{0}_m$ and $\beta^- \geq \mathbf{0}_m$. We deduce that:

$$\|\beta\|_1 = \sum_{j=1}^m \left|\beta_j^+ - \beta_j^-\right| = \sum_{j=1}^m \left|\beta_j^+\right| + \sum_{j=1}^m \left|\beta_j^-\right| = \mathbf{1}_m^\top \beta^+ + \mathbf{1}_m^\top \beta^-$$

Solving the lasso regression problem

Augmented QP program of the lasso regression (λ-problem)

The augmented QP program is specified as follows:

$$\begin{aligned}\hat{\theta} &= \arg\min \frac{1}{2}\theta^\top Q\theta - \theta^\top R \\ &\text{s.t.} \quad \theta \geq \mathbf{0}_{2m}\end{aligned}$$

where $\theta = (\beta^+, \beta^-)$, $\tilde{X} = \begin{pmatrix} X & -X \end{pmatrix}$, $Q = \tilde{X}^\top \tilde{X}$ and $R = \tilde{X}^\top Y + \lambda \mathbf{1}_{2m}$. If we denote $T = \begin{pmatrix} I_m & -I_m \end{pmatrix}$, we obtain:

$$\hat{\beta}^{\text{lasso}}(\lambda) = T\hat{\theta}$$

Solving the lasso regression problem

Augmented QP program of the lasso regression (τ-problem)

If we consider the τ-problem, we obtain another augmented QP program:

$$\begin{aligned} \hat{\theta} &= \arg\min \frac{1}{2}\theta^\top Q\theta - \theta^\top R \\ & \text{s.t. } \left\{ \begin{array}{l} C\theta \leq D \\ \theta \geq \mathbf{0}_{2m} \end{array} \right. \end{aligned}$$

where $Q = \tilde{X}^\top \tilde{X}$, $R = \tilde{X}^\top Y$, $C = \mathbf{1}_{2m}^\top$ and $D = \tau$. Again, we have:

$$\hat{\beta}(\tau) = T\hat{\theta}$$

Solving the lasso regression problem

We consider the linear regression:

$$Y = X\beta + \varepsilon$$

where Y is a $n \times 1$ vector, X is a $n \times m$ matrix and β is a $m \times 1$ vector. The optimization problem is:

$$\hat{\beta} = \arg\min f(\beta) = \frac{1}{2}(Y - X\beta)^\top (Y - X\beta)$$

Since we have $\partial_\beta f(\beta) = -X^\top (Y - X\beta))$, we deduce that:

$$\begin{aligned}
\frac{\partial f(\beta)}{\partial \beta_j} &= x_j^\top (X\beta - Y) \\
&= x_j^\top \left(x_j\beta_j + X_{(-j)}\beta_{(-j)} - Y\right) \\
&= x_j^\top x_j\beta_j + x_j^\top X_{(-j)}\beta_{(-j)} - x_j^\top Y
\end{aligned}$$

where x_j is the $n \times 1$ vector corresponding to the j^{th} variable and $X_{(-j)}$ is the $n \times (m-1)$ matrix (without the j^{th} variable)

Solving the lasso regression problem

At the optimum, we have $\partial_{\beta_j} f(\beta) = 0$ or:

$$\beta_j = \frac{x_j^\top Y - x_j^\top X_{(-j)}\beta_{(-j)}}{x_j^\top x_j} = \frac{x_j^\top \left(Y - X_{(-j)}\beta_{(-j)}\right)}{x_j^\top x_j}$$

CCD algorithm for the linear regression

We have:

$$\beta_j^{(k+1)} = \frac{x_j^\top \left(Y - \sum_{j'=1}^{j-1} x_{j'}\beta_{j'}^{(k+1)} - \sum_{j'=j+1}^{m} x_{j'}\beta_{j'}^{(k)}\right)}{x_j^\top x_j}$$

$\Rightarrow$ Introducing pointwise constraints is straightforward

Solving the lasso regression problem

The objective function becomes:

$$\begin{aligned} f(\beta) &= \frac{1}{2}(Y - X\beta)^\top (Y - X\beta) + \lambda \|\beta\|_1 \\ &= f_{\mathrm{OLS}}(\beta) + \lambda \|\beta\|_1 \end{aligned}$$

Since the norm is separable — $\|\beta\|_1 = \sum_{j=1}^m |\beta_j|$, the first-order condition is:

$$\frac{\partial\, f_{\mathrm{OLS}}(\beta)}{\partial\, \beta_j} + \lambda \partial\, |\beta_j| = 0$$

or:

$$\underbrace{\left(x_j^\top x_j\right)}_{c}\beta_j - \underbrace{x_j^\top \left(Y - X_{(-j)}\beta_{(-j)}\right)}_{v} + \lambda \partial\, |\beta_j| = 0$$

Derivation of the soft-thresholding operator

We consider the following equation:

$$c\beta_j - v + \lambda \partial \left|\beta_j\right| \in \{0\}$$

where $c > 0$ and $\lambda > 0$. Since we have $\partial \left|\beta_j\right| = \operatorname{sign}\left(\beta_j\right)$, we deduce that:

$$\beta_j^{\star} = \begin{cases} c^{-1}\left(v + \lambda\right) & \text{if } \beta_j^{\star} < 0 \\ 0 & \text{if } \beta_j^{\star} = 0 \\ c^{-1}\left(v - \lambda\right) & \text{if } \beta_j^{\star} > 0 \end{cases}$$

If $\beta_j^{\star} < 0$ or $\beta_j^{\star} > 0$, then we have $v + \lambda < 0$ or $v - \lambda > 0$. This is equivalent to set $|v| > \lambda > 0$. The case $\beta_j^{\star} = 0$ implies that $|v| \leq \lambda$. We deduce that:

$$\beta_j^{\star} = c^{-1} \cdot \mathcal{S}\left(v; \lambda\right)$$

where $\mathcal{S}\left(v; \lambda\right)$ is the soft-thresholding operator:

$$\begin{aligned} \mathcal{S}\left(v; \lambda\right) &= \begin{cases} 0 & \text{if } |v| \leq \lambda \\ v - \lambda \operatorname{sign}\left(v\right) & \text{otherwise} \end{cases} \\ &= \operatorname{sign}\left(v\right) \cdot \left(|v| - \lambda\right)_+ \end{aligned}$$

Solving the lasso regression problem

CCD algorithm for the lasso regression

We have:

$$\beta_j^{(k+1)} = \frac{1}{x_j^\top x_j} \mathcal{S}\left(x_j^\top \left(Y - \sum_{j'=1}^{j-1} x_{j'} \beta_{j'}^{(k+1)} - \sum_{j'=j+1}^{m} x_{j'} \beta_{j'}^{(k)} \right) ; \lambda \right)$$

where $\mathcal{S}(v;\lambda)$ is the **soft-thresholding operator**:

$$\mathcal{S}(v;\lambda) = \operatorname{sign}(v) \cdot (|v| - \lambda)_+$$

Solving the lasso regression problem

Table 2: Matlab code

```
for k = 1:nIters
    for j = 1:m
        x_j = X(:,j);
        X_j = X;
        X_j(:,j) = zeros(n,1);
        if lambda > 0
            v = x_j'*(Y - X_j*beta);
            beta(j) = max(abs(v) - lambda,0) * sign(v) / (x_j'*x_j);
        else
            beta(j) = x_j'*(Y - X_j*beta) / (x_j'*x_j);
        end
    end
end
```

Solving the lasso regression problem

Example 2

We consider the following data:

i	y	x_1	x_2	x_3	x_4	x_5
1	3.1	2.8	4.3	0.3	2.2	3.5
2	24.9	5.9	3.6	3.2	0.7	6.4
3	27.3	6.0	9.6	7.6	9.5	0.9
4	25.4	8.4	5.4	1.8	1.0	7.1
5	46.1	5.2	7.6	8.3	0.6	4.5
6	45.7	6.0	7.0	9.6	0.6	0.6
7	47.4	6.1	1.0	8.5	9.6	8.6
8	−1.8	1.2	9.6	2.7	4.8	5.8
9	20.8	3.2	5.0	4.2	2.7	3.6
10	6.8	0.5	9.2	6.9	9.3	0.7
11	12.9	7.9	9.1	1.0	5.9	5.4
12	37.0	1.8	1.3	9.2	6.1	8.3
13	14.7	7.4	5.6	0.9	5.6	3.9
14	−3.2	2.3	6.6	0.0	3.6	6.4
15	44.3	7.7	2.2	6.5	1.3	0.7

Solving the lasso regression problem

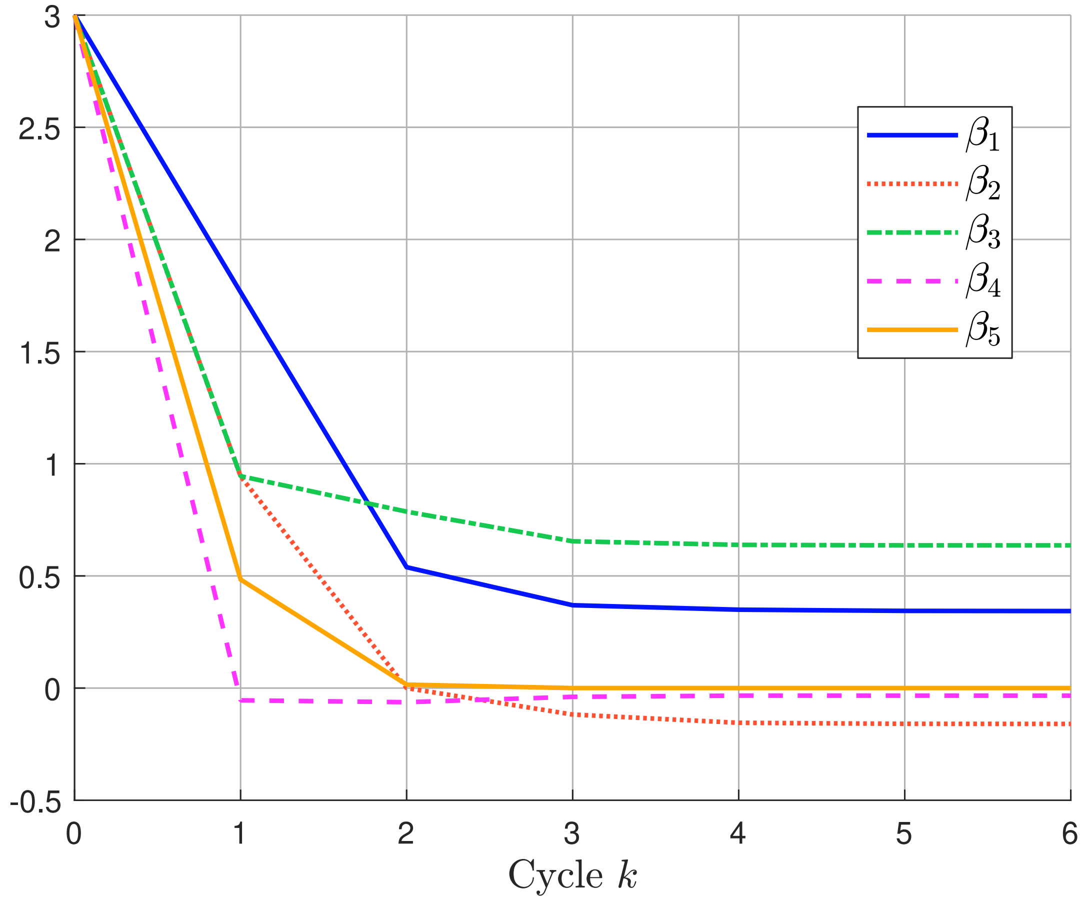


Figure 1: Convergence of the CCD algorithm (lasso regression, $\lambda = 2$)

Note: we start the CCD algorithm with $\beta_j^{(0)} = 0$ (don't forget to standardize the data!)

Solving the lasso regression problem

1. The dimension problem is $(2m, 2m)$ for QP and $(1, 0)$ for CCD!
2. CCD is faster for lasso regression than for linear regression (because of the soft-thresholding operator)!

Suppose $n = 50\,000$ and $m = 1\,000\,000$ (DNA sequence problem!)

Solving the lasso regression problem

Example 3

- We consider an experiment with $n = 100\,000$ observations and $m = 50$ variables.
- The design matrix X is built using the uniform distribution while the residuals are simulated using a Gaussian distribution and a standard deviation of 20%.
- The beta coefficients are distributed uniformly between -3 and $+3$ except four coefficients that take a larger value.
- We then standardize the data of X and Y.
- For initializing the coordinates, we use uniform random numbers between -1 and $+1$.

Solving the lasso regression problem

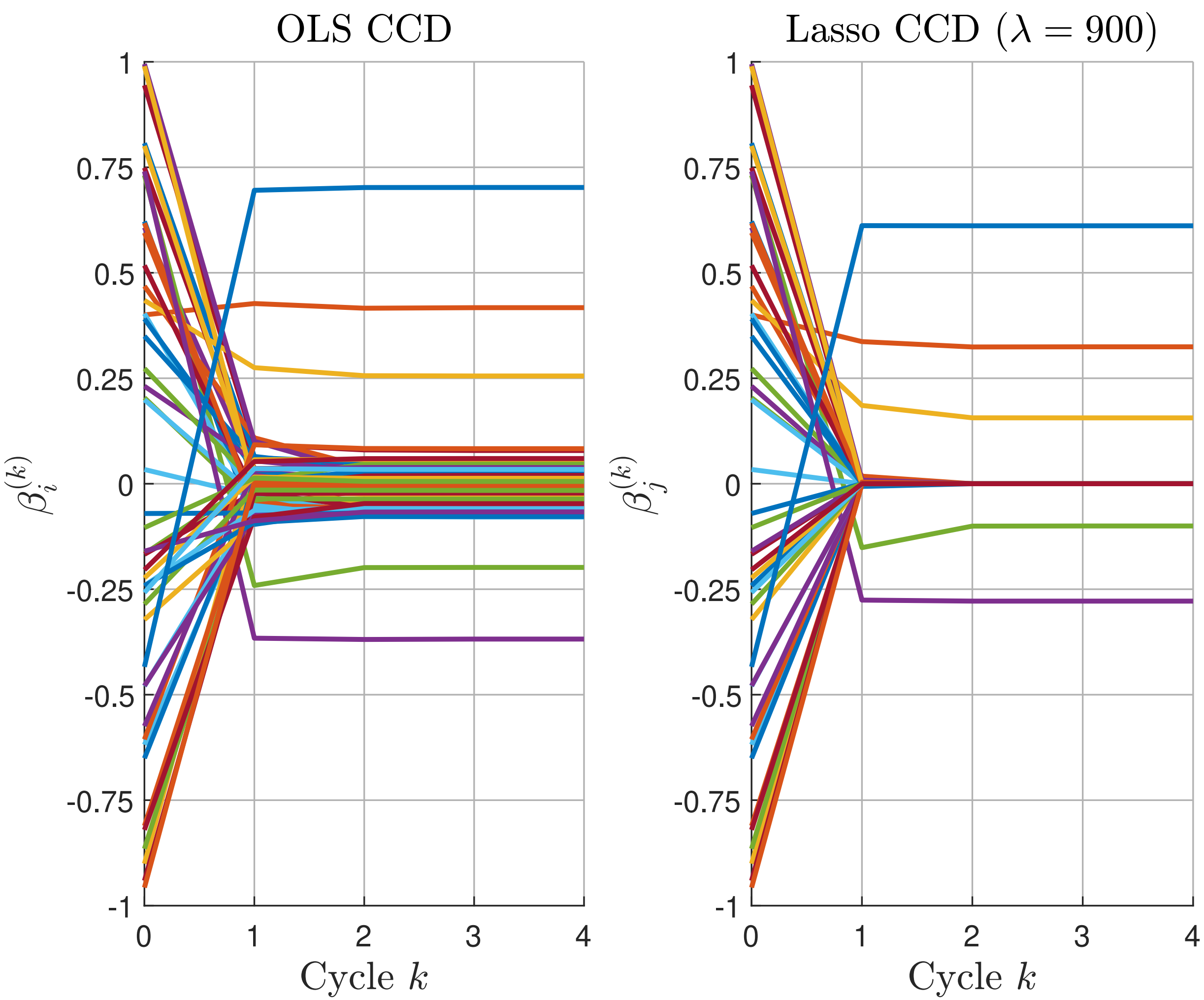


Figure 2: Convergence of the CCD algorithm (lasso vs linear regression)

Alternative direction method of multipliers

Definition

The alternating direction method of multipliers (ADMM) is an algorithm introduced by Gabay and Mercier (1976) to solve optimization problems which can be expressed as:

$$
\begin{aligned}
\{x^{\star}, y^{\star}\} & = \arg\min_{(x,y)} f_x(x) + f_y(y) \\
\text{s.t.} & \quad Ax + By = c
\end{aligned}
$$

The algorithm is:

$$
\begin{aligned}
x^{(k+1)} & = \arg\min_{x} \left\{ f_x(x) + \frac{\varphi}{2} \left\| Ax + By^{(k)} - c + u^{(k)} \right\|_2^2 \right\} \\
y^{(k+1)} & = \arg\min_{y} \left\{ f_y(y) + \frac{\varphi}{2} \left\| Ax^{(k+1)} + By - c + u^{(k)} \right\|_2^2 \right\} \\
u^{(k+1)} & = u^{(k)} + \left(Ax^{(k+1)} + By^{(k+1)} - c \right)
\end{aligned}
$$

Alternative direction method of multipliers

What is the underlying idea?

- Minimizing $f_x(x) + f_y(y)$ with respect to (x, y) is a difficult task
- Minimizing

$$g_x(x) = f_x(x) + \frac{\varphi}{2} \|Ax + By - c\|_2^2$$

with respect to x and minimizing

$$g_y(y) = f_y(y) + \frac{\varphi}{2} \|Ax + By - c\|_2^2$$

with respect to y is easier

Alternative direction method of multipliers

We use the following notations:

- $f_x^{(k+1)}(x)$ is the objective function of the x-update step:

$$f_x^{(k+1)}(x) = f_x(x) + \frac{\varphi}{2} \left\| Ax + By^{(k)} - c + u^{(k)} \right\|_2^2$$

- $f_y^{(k+1)}(y)$ is the objective function of the y-update step:

$$f_y^{(k+1)}(y) = f_y(y) + \frac{\varphi}{2} \left\| Ax^{(k+1)} + By - c + u^{(k)} \right\|_2^2$$

Alternative direction method of multipliers

When $A = I_n$ and $B = -I_n$, we have:

1. $$Ax + By^{(k)} - c + u^{(k)} = x - y^{(k)} - c + u^{(k)} = x - v_x^{(k+1)}$$
   where:
   $$v_x^{(k+1)} = y^{(k)} + c - u^{(k)}$$
2. $$Ax^{(k+1)} + By - c + u^{(k)} = x^{(k+1)} - y - c + u^{(k)} = v_y^{(k+1)} - y$$
   where:
   $$v_y^{(k+1)} = x^{(k+1)} - c + u^{(k)}$$
3. $$\begin{aligned} f_x^{(k+1)}(x) &= f_x(x) + \frac{\varphi}{2}\left\| x - v_x^{(k+1)} \right\|_2^2 \\ f_y^{(k+1)}(y) &= f_y(y) + \frac{\varphi}{2}\left\| y - v_y^{(k+1)} \right\|_2^2 \end{aligned}$$

Alternative direction method of multipliers

- We consider a problem of the form:

$$x^{\star} = \arg\min_{x} g(x)$$

The idea is then to write $g(x)$ as a separable function:

$$g(x) = g_1(x) + g_2(x)$$

and to consider the following equivalent ADMM problem:

$$\begin{aligned} \{x^{\star}, y^{\star}\} &= \arg\min_{(x,y)} f_x(x) + f_y(y) \\ \text{s.t.} & \quad x = y \end{aligned}$$

where $f_x(x) = g_1(x)$ and $f_y(y) = g_2(y)$

Alternative direction method of multipliers

- We consider a problem of the form:

$$\begin{aligned} x^{\star} &= \arg\min_{x} g(x) \\ \text{s.t.} &\quad x \in \Omega \end{aligned}$$

We have:

$$\begin{aligned} \{x^{\star}, y^{\star}\} &= \arg\min_{(x,y)} f_x(x) + f_y(y) \\ \text{s.t.} &\quad x = y \end{aligned}$$

where $f_x(x) = g(x)$, $f_y(y) = \mathbb{1}_{\Omega}(y)$ and:

$$\mathbb{1}_{\Omega}(y) = \begin{cases} 0 & \text{if} \quad y \in \Omega \\ +\infty & \text{if} \quad y \notin \Omega \end{cases}$$

Alternative direction method of multipliers

Special case

$$\Omega = \left\{x : x^- \leq x \leq x^+\right\}$$

By setting $\varphi = 1$, the y-step becomes:

$$\begin{aligned} y^{(k+1)} &= \arg\min \left\{ \mathbb{1}_\Omega \left(y\right) + \frac{1}{2} \left\| x^{(k+1)} - y + u^{(k)} \right\|_2^2 \right\} \\ &= \mathbf{prox}_{f_y} \left(x^{(k+1)} + u^{(k)}\right) \end{aligned}$$

where the proximal operator is the box projection or the truncation operator:

$$\begin{aligned} \mathbf{prox}_{f_y} \left(v\right) &= x^- \odot \mathbb{1}\left\{v < x^-\right\} + \\ & \quad v \odot \mathbb{1}\left\{x^- \leq v \leq x^+\right\} + \\ & \quad x^+ \odot \mathbb{1}\left\{v > x^+\right\} \\ &= \mathcal{T}\left(v; x^-, x^+\right) \end{aligned}$$

Alternative direction method of multipliers

Special case

$$\Omega = \left\{x : x^{-} \leq x \leq x^{+}\right\}$$

The ADMM algorithm is then:

$$\begin{aligned}
x^{(k+1)} &= \arg\min \left\{ g(x) + \frac{1}{2}\left\| x - y^{(k)} + u^{(k)} \right\|_2^2 \right\} \\
y^{(k+1)} &= \mathbf{prox}_{f_y}\left(x^{(k+1)} + u^{(k)} \right) \\
u^{(k+1)} &= u^{(k)} + \left(x^{(k+1)} - y^{(k+1)} \right)
\end{aligned}$$

$\Rightarrow$ Solving the constrained optimization problem consists in solving the unconstrained optimization problem, applying the box projection and iterating these steps until convergence

Alternative direction method of multipliers

Lasso regression

The λ-problem of the lasso regression has the following ADMM formulation:

$$\begin{aligned} \left\{\beta^{\star}, \bar{\beta}^{\star}\right\} &= \arg\min \frac{1}{2}(Y - X\beta)^{\top}(Y - X\beta) + \lambda\|\bar{\beta}\|_1 \\ \text{s.t.} & \quad \beta - \bar{\beta} = \mathbf{0}_m \end{aligned}$$

We have:

$$\begin{aligned} f_x(\beta) &= \frac{1}{2}(Y - X\beta)^{\top}(Y - X\beta) \\ &= \frac{1}{2}\beta^{\top}\left(X^{\top}X\right)\beta - \beta^{\top}\left(X^{\top}Y\right) + \frac{1}{2}Y^{\top}Y \end{aligned}$$

and:

$$f_y\left(\bar{\beta}\right) = \lambda\|\bar{\beta}\|_1$$

Alternative direction method of multipliers

The x-step is:

$$\beta^{(k+1)} = \arg\min_{\beta} \left\{ \frac{1}{2}\beta^\top \left(X^\top X\right)\beta - \beta^\top \left(X^\top Y\right) + \frac{\varphi}{2}\left\|\beta - \bar{\beta}^{(k)} + u^{(k)}\right\|_2^2 \right\}$$

Since we have:

$$\begin{aligned} \frac{\varphi}{2}\left\|\beta - \bar{\beta}^{(k)} + u^{(k)}\right\|_2^2 &= \frac{\varphi}{2}\beta^\top\beta - \varphi\beta^\top\left(\bar{\beta}^{(k)} - u^{(k)}\right) + \\ & \frac{\varphi}{2}\left(\bar{\beta}^{(k)} - u^{(k)}\right)^\top\left(\bar{\beta}^{(k)} - u^{(k)}\right) \end{aligned}$$

we deduce that the x-update is a standard QP problem where:

$$f_x^{(k+1)}(\beta) = \frac{1}{2}\beta^\top\left(X^\top X + \varphi I_m\right)\beta - \beta^\top\left(X^\top Y + \varphi\left(\bar{\beta}^{(k)} - u^{(k)}\right)\right)$$

It follows that the solution is:

$$\begin{aligned} \beta^{(k+1)} &= \arg\min f_x^{(k+1)}(\beta) \\ &= \left(X^\top X + \varphi I_m\right)^{-1}\left(X^\top Y + \varphi\left(\bar{\beta}^{(k)} - u^{(k)}\right)\right) \end{aligned}$$

Alternative direction method of multipliers

The y-step is:

$$\begin{aligned}\bar{\beta}^{(k+1)} &= \arg\min_{\bar{\beta}} \left\{ \lambda \|\bar{\beta}\|_1 + \frac{\varphi}{2} \left\| \beta^{(k+1)} - \bar{\beta} + u^{(k)} \right\|_2^2 \right\} \\ &= \arg\min \left\{ \frac{1}{2} \left\| \bar{\beta} - \left(\beta^{(k+1)} + u^{(k)} \right) \right\|_2^2 + \frac{\lambda}{\varphi} \|\bar{\beta}\|_1 \right\}\end{aligned}$$

We recognize the soft-thresholding problem with $v = \beta^{(k+1)} + u^{(k)}$. We have:

$$\bar{\beta}^{(k+1)} = \mathcal{S}\left(\beta^{(k+1)} + u^{(k)}; \varphi^{-1}\lambda \right)$$

where:

$$\mathcal{S}(v; \lambda) = \operatorname{sign}(v) \cdot (|v| - \lambda)_+$$

Alternative direction method of multipliers

ADMM-Lasso algorithm (Boyd *et al.*, 2011)

Finally, the ADMM algorithm is made up of the following steps:

$$\left\{ \begin{array}{l} \beta^{(k+1)} = \left(X^\top X + \varphi I_m\right)^{-1} \left(X^\top Y + \varphi \left(\bar{\beta}^{(k)} - u^{(k)}\right)\right) \\ \bar{\beta}^{(k+1)} = \mathcal{S}\left(\beta^{(k+1)} + u^{(k)}; \varphi^{-1}\lambda\right) \\ u^{(k+1)} = u^{(k)} + \left(\beta^{(k+1)} - \bar{\beta}^{(k+1)}\right) \end{array} \right.$$

Alternative direction method of multipliers

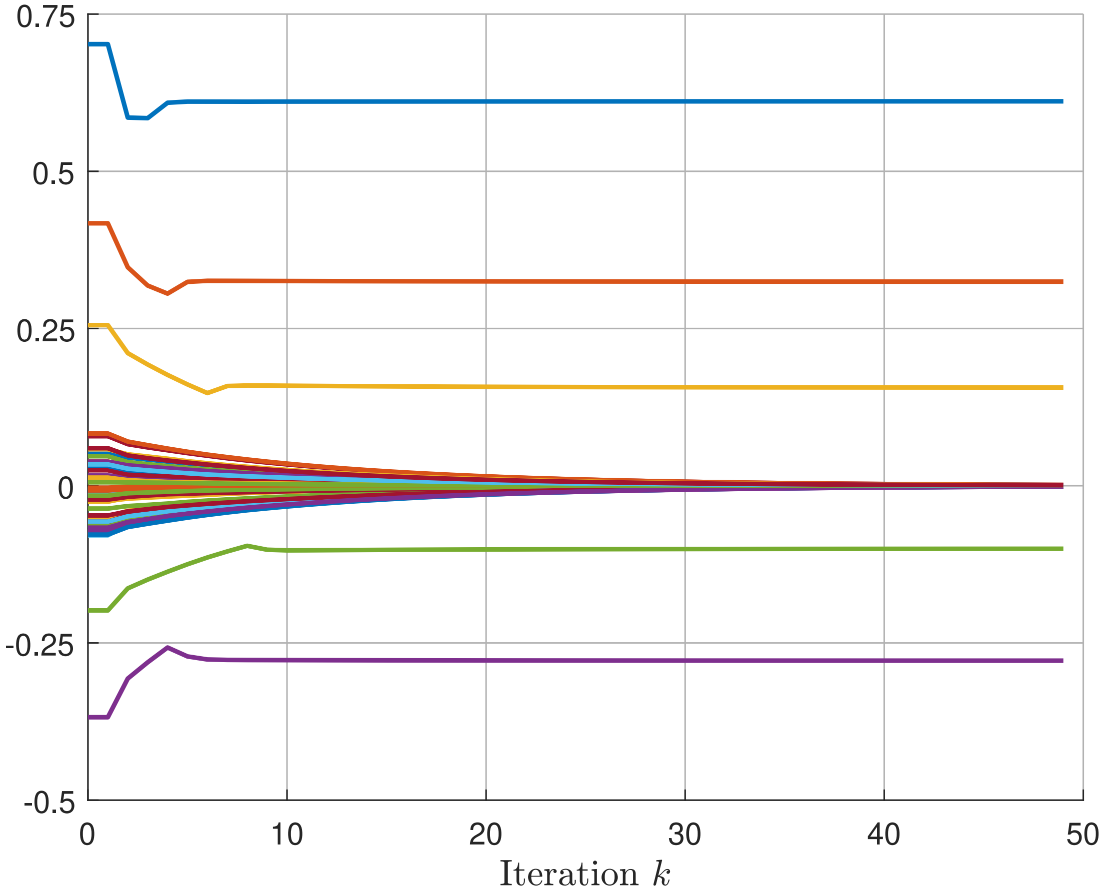


Figure 3: Convergence of the ADMM algorithm (Example 3, $\lambda = 900$)

Note: the initial values are the OLS estimates and we set $\varphi = \lambda$

Alternative direction method of multipliers

In practice, we use a time-varying parameter $\varphi^{(k)}$ (see Perrin and Roncalli, 2020).

Proximal operator

Definition

The proximal operator $\mathbf{prox}_f(v)$ of the function $f(x)$ is defined by:

$$\mathbf{prox}_f(v) = x^\star = \arg\min_x \left\{ f_v(x) = f(x) + \frac{1}{2}\|x - v\|_2^2 \right\}$$

Proximal operator

Example 4

We consider the scalar-valued logarithmic barrier function $f(x) = -\lambda \ln x$

Proximal operator

We have:

$$
\begin{aligned}
f_v(x) &= -\lambda \ln x + \frac{1}{2}(x-v)^2 \\
&= -\lambda \ln x + \frac{1}{2}x^2 - xv + \frac{1}{2}v^2
\end{aligned}
$$

The first-order condition is $-\lambda x^{-1} + x - v = 0$. We obtain two roots with opposite signs:

$$
x' = \frac{v - \sqrt{v^2 + 4\lambda}}{2} \text{ and } x'' = \frac{v + \sqrt{v^2 + 4\lambda}}{2}
$$

Since the logarithmic function is defined for $x > 0$, we deduce that:

$$
\mathbf{prox}_f(v) = \frac{v + \sqrt{v^2 + 4\lambda}}{2}
$$

Proximal operator

In the case where $f(x) = \mathbb{1}_{\Omega}(x)$, we have:

$$\begin{aligned}
\mathbf{prox}_f(v) &= \arg\min_{x} \left\{ \mathbb{1}_{\Omega}(x) + \frac{1}{2}\|x - v\|_2^2 \right\} \\
&= \arg\min_{x\in\Omega} \left\{ \|x - v\|_2^2 \right\} \\
&= \mathcal{P}_{\Omega}(v)
\end{aligned}$$

where $\mathcal{P}_{\Omega}(v)$ is the standard projection of v onto Ω

Proximal operator

Table 3: Projection for some simple polyhedra

Notation	Ω	$\mathcal{P}_{\Omega}(v)$
$\mathcal{A}_{ffineset}[A,B]$	$Ax = B$	$v - A^{\dagger}(Av - B)$
$\mathcal{H}_{yperplane}[a,b]$	$a^{\top}x = b$	$v - \dfrac{(a^{\top}v - b)}{\lVert a \rVert_2^2} a$
$\mathcal{H}_{alfspace}[c,d]$	$c^{\top}x \leq d$	$v - \dfrac{(c^{\top}v - d)_+}{\lVert c \rVert_2^2} c$
$\mathcal{B}_{ox}[x^-,x^+]$	$x^- \leq x \leq x^+$	$\mathcal{T}(v; x^-, x^+)$

Source: Parikh and Boyd (2014)

Note: $A^{\dagger}$ is the Moore-Penrose pseudo-inverse of A, and $\mathcal{T}(v; x^-, x^+)$ is the truncation operator

Remark: No analytical formula for the (multi-dimensional) inequality constraint $Cx \leq D \Rightarrow$ it may be solved using the Dykstra's algorithm

Proximal operator

Separable sum

If $f(x) = \sum_{i=1}^{n} f_i(x_i)$ is fully separable, then the proximal of $f(v)$ is the vector of the proximal operators applied to each scalar-valued function $f_i(x_i)$:

$$\mathbf{prox}_f(v) = \begin{pmatrix} \mathbf{prox}_{f_1}(v_1) \\ \vdots \\ \mathbf{prox}_{f_n}(v_n) \end{pmatrix}$$

Proximal operator

If $f(x) = -\lambda \ln x$, we have:

$$\mathbf{prox}_f(v) = \frac{v + \sqrt{v^2 + 4\lambda}}{2}$$

In the case of the vector-valued logarithmic barrier $f(x) = -\lambda \sum_{i=1}^{n} \ln x_i$, we deduce that:

$$\mathbf{prox}_f(v) = \frac{v + \sqrt{v \odot v + 4\lambda}}{2}$$

Proximal operator

Moreau decomposition theorem

We have:

$$\mathbf{prox}_f(v) + \mathbf{prox}_{f^*}(v) = v$$

where f^* is the convex conjugate of f.

Application

If $f(x)$ is a ℓ_q-norm function, then $f^*(x) = \mathbb{1}_{\mathcal{B}_p}(x)$ where $\mathcal{B}_p$ is the ℓ_p unit ball and $p^{-1} + q^{-1} = 1$. Since we have $\mathbf{prox}_{f^*}(v) = \mathcal{P}_{\mathcal{B}_p}(v)$, we deduce that:

$$\mathbf{prox}_f(v) + \mathcal{P}_{\mathcal{B}_p}(v) = v$$

The proximal of the ℓ_p-ball can be deduced from the proximal operator of the ℓ_q-norm function.

Proximal operator

Table 4: Proximal of the ℓ_p-norm function $f(x) = \|x\|_p$

p	$\mathbf{prox}_{\lambda f}(v)$
$p = 1$	$\mathcal{S}(v; \lambda) = \operatorname{sign}(v) \odot (\lvert v \rvert - \lambda \mathbf{1}_n)_+$
$p = 2$	$\left(1 - \dfrac{\lambda}{\max(\lambda, \|v\|_2)}\right) v$
$p = \infty$	$\operatorname{sign}(v) \odot \mathbf{prox}_{\lambda \max x}(\lvert v \rvert)$

We have:

$$\mathbf{prox}_{\lambda \max x}(v) = \min(v, s^\star)$$

where $s^\star$ is the solution of the following equation:

$$s^\star = \left\{ s \in \mathbb{R} : \sum_{i=1}^{n} (v_i - s)_+ = \lambda \right\}$$

Proximal operator

Table 5: Proximal of the ℓ_p-ball $\mathcal{B}_p(c,\lambda) = \left\{x \in \mathbb{R}^n : \|x - c\|_p \leq \lambda\right\}$ when c is equal to $\mathbf{0}_n$

p	$\mathcal{P}_{\mathcal{B}_p(\mathbf{0}_n,\lambda)}(v)$	q
$p = 1$	$v - \operatorname{sign}(v) \odot \mathbf{prox}_{\lambda \max x}(\lvert v \rvert)$	$q = \infty$
$p = 2$	$v - \mathbf{prox}_{\lambda \|x\|_2}(v)$	$q = 2$
$p = \infty$	$\mathcal{T}(v; -\lambda, \lambda)$	$q = 1$

Proximal operator

Scaling and translation

Let us define $g(x) = f(ax + b)$ where $a \neq 0$. We have:

$$\mathbf{prox}_{g}(v) = \frac{\mathbf{prox}_{a^2 f}(av + b) - b}{a}$$

Application

We can use this property when the center c of the ℓ_p ball is not equal to $\mathbf{0}_n$. Since we have $\mathbf{prox}_{g}(v) = \mathbf{prox}_{f}(v - c) + c$ where $g(x) = f(x - c)$ and the equivalence $\mathcal{B}_p(\mathbf{0}_n, \lambda) = \{x \in \mathbb{R}^n : f(x) \leq \lambda\}$ where $f(x) = \|x\|_p$, we deduce that:

$$\mathcal{P}_{\mathcal{B}_p(c,\lambda)}(v) = \mathcal{P}_{\mathcal{B}_p(\mathbf{0}_n,\lambda)}(v - c) + c$$

Application to the τ-problem of the lasso regression

We have:

$$\begin{aligned}\hat{\beta}(\tau) &= \arg\min_{\beta} \frac{1}{2}(Y - X\beta)^\top (Y - X\beta) \\ \text{s.t.} \quad & \|\beta\|_1 \leq \tau\end{aligned}$$

The ADMM formulation is:

$$\begin{aligned}\left\{\beta^\star, \bar{\beta}^\star\right\} &= \arg\min_{(\beta,\bar{\beta})} \frac{1}{2}(Y - X\beta)^\top (Y - X\beta) + \mathbb{1}_\Omega\left(\bar{\beta}\right) \\ \text{s.t.} \quad & \beta = \bar{\beta}\end{aligned}$$

where $\Omega = \mathcal{B}_1(\mathbf{0}_m, \tau)$ is the centered ℓ_1 ball with radius τ

Application to the τ-problem of the lasso regression

1. The x-update is:

$$\begin{aligned}\beta^{(k+1)} &= \arg\min_{\beta} \left\{ \frac{1}{2} (Y - X\beta)^{\top} (Y - X\beta) + \frac{\varphi}{2} \left\| \beta - \bar{\beta}^{(k)} + u^{(k)} \right\|_2^2 \right\} \\ &= \left(X^{\top} X + \varphi I_m \right)^{-1} \left(X^{\top} Y + \varphi \left(\bar{\beta}^{(k)} - u^{(k)} \right) \right)\end{aligned}$$

where $v_x^{(k+1)} = \bar{\beta}^{(k)} - u^{(k)}$

Application to the τ-problem of the lasso regression

2. The y-update is:

$$
\begin{aligned}
\bar{\beta}^{(k+1)} &= \arg\min_{\bar{\beta}} \left\{ \mathbb{1}_{\Omega}\left(\bar{\beta}\right) + \frac{\varphi}{2} \left\| \beta^{(k+1)} - \bar{\beta} + u^{(k)} \right\|_2^2 \right\} \\
&= \mathbf{prox}_{f_y}\left(\beta^{(k+1)} + u^{(k)}\right) \\
&= \mathcal{P}_{\Omega}\left(v_y^{(k+1)}\right) \\
&= v_y^{(k+1)} - \operatorname{sign}\left(v_y^{(k+1)}\right) \odot \mathbf{prox}_{\tau \max x}\left(\left|v_y^{(k+1)}\right|\right)
\end{aligned}
$$

where $v_y^{(k+1)} = \beta^{(k+1)} + u^{(k)}$

Application to the τ-problem of the lasso regression

3. The u-update is:

$$u^{(k+1)} = u^{(k)} + \beta^{(k+1)} - \bar{\beta}^{(k+1)}$$

Application to the τ-problem of the lasso regression

ADMM-Lasso algorithm

The ADMM algorithm is :

$$\left\{ \begin{array}{l} \beta^{(k+1)} = \left(X^\top X + \varphi I_m\right)^{-1} \left(X^\top Y + \varphi \left(\bar{\beta}^{(k)} - u^{(k)}\right)\right) \\ \bar{\beta}^{(k+1)} = \left\{ \begin{array}{ll} \mathcal{S}\left(\beta^{(k+1)} + u^{(k)}; \varphi^{-1}\lambda\right) & (\lambda\text{-problem}) \\ \mathcal{P}_{\mathcal{B}_1(\mathbf{0}_m,\tau)}\left(\beta^{(k+1)} + u^{(k)}\right) & (\tau\text{-problem}) \end{array} \right. \\ u^{(k+1)} = u^{(k)} + \left(\beta^{(k+1)} - \bar{\beta}^{(k+1)}\right) \end{array} \right.$$

Remark

The ADMM algorithm is similar for λ- and τ-problems since the only difference concerns the y-step. However, the τ-problem is easier to solve with the ADMM algorithm from a practical point of view, because the y-update of the τ-problem is independent of the penalization parameter φ.

Derivation of the soft-thresholding operator

We consider the following equation:

$$cx - v + \lambda \partial \left| x \right| \in 0$$

where $c > 0$ and $\lambda > 0$. Since we have $\partial \left| x \right| = \operatorname{sign}(x)$, we deduce that:

$$x^{\star} = \begin{cases} c^{-1}(v + \lambda) & \text{if } x^{\star} < 0 \\ 0 & \text{if } x^{\star} = 0 \\ c^{-1}(v - \lambda) & \text{if } x^{\star} > 0 \end{cases}$$

If $x^{\star} < 0$ or $x^{\star} > 0$, then we have $v + \lambda < 0$ or $v - \lambda > 0$. This is equivalent to set $\left| v \right| > \lambda > 0$. The case $x^{\star} = 0$ implies that $\left| v \right| \leq \lambda$. We deduce that:

$$x^{\star} = c^{-1} \cdot \mathcal{S}(v; \lambda)$$

where $\mathcal{S}(v; \lambda)$ is the soft-thresholding operator:

$$\begin{aligned} \mathcal{S}(v; \lambda) &= \begin{cases} 0 & \text{if } \left| v \right| \leq \lambda \\ v - \lambda \operatorname{sign}(v) & \text{otherwise} \end{cases} \\ &= \operatorname{sign}(v) \cdot (\left| v \right| - \lambda)_{+} \end{aligned}$$

Derivation of the soft-thresholding operator

We use the result on the separable sum

Remark

If $f(x) = \lambda \|x\|_1$, we have $f(x) = \lambda \sum_{i=1}^{n} |x_i|$ and $f_i(x_i) = \lambda |x_i|$. We deduce that the proximal operator of $f(x)$ is the vector formulation of the soft-thresholding operator:

$$\mathbf{prox}_{\lambda \|x\|_1}(v) = \begin{pmatrix} \operatorname{sign}(v_1) \cdot (|v_1| - \lambda)_+ \\ \vdots \\ \operatorname{sign}(v_n) \cdot (|v_n| - \lambda)_+ \end{pmatrix} = \operatorname{sign}(v) \odot (|v| - \lambda \mathbf{1}_n)_+$$

The soft-thresholding operator is the proximal operator of the ℓ_1-norm $f(x) = \|x\|_1$. Indeed, we have $\mathbf{prox}_f(v) = \mathcal{S}(v; 1)$ and $\mathbf{prox}_{\lambda f}(v) = \mathcal{S}(v; \lambda)$.

Dykstra's algorithm

We consider the following optimization problem:

$$\begin{aligned} x^{\star} &= \arg\min f_x(x) \\ \text{s.t.} &\quad x \in \Omega \end{aligned}$$

where Ω is a complex set of constraints:

$$\Omega = \Omega_1 \cap \Omega_2 \cap \cdots \Omega_m$$

We set $y = x$ and $f_y(y) = \mathbb{1}_{\Omega}(y)$. The ADMM algorithm becomes

$$\begin{aligned} x^{(k+1)} &= \arg\min \left\{ f_x(x) + \frac{\varphi}{2} \left\| x - y^{(k)} + u^{(k)} \right\|_2^2 \right\} \\ v^{(k)} &= x^{(k+1)} + u^{(k)} \\ y^{(k+1)} &= \mathcal{P}_{\Omega}\left(v^{(k)} \right) \\ u^{(k+1)} &= u^{(k)} + \left(x^{(k+1)} - y^{(k+1)} \right) \end{aligned}$$

How to compute $\mathcal{P}_{\Omega}(v)$?

Dykstra's algorithm

More generally, we consider the proximal optimization problem where the function $f(x)$ is the convex sum of basic functions $f_j(x)$:

$$x^{\star} = \arg\min_{x} \left\{ \sum_{j=1}^{m} f_j(x) + \frac{1}{2} \left\| x - v \right\|_2^2 \right\}$$

and the proximal of each basic function is known.

How to find the solution $x^{\star}$?

Dykstra's algorithm

The case $m = 2$

- We know the proximal solution of the ℓ_1-norm function $f_1(x) = \lambda_1 \|x\|_1$
- We know the proximal solution of the logarithmic barrier function $f_2(x) = \lambda_2 \sum_{i=1}^{n} \ln x_i$
- We don't know how to compute the proximal operator of $f(x) = f_1(x) + f_2(x)$:

$$\begin{aligned} x^\star &= \arg\min_x f_1(x) + f_2(x) + \frac{1}{2}\|x - v\|_2^2 \\ &= \mathbf{prox}_f(v) \end{aligned}$$

Dykstra's algorithm

The case $m = 2$

The Dykstra's algorithm consists in the following iterations:

$$\left\{ \begin{array}{l} x^{(k+1)} = \mathbf{prox}_{f_1}\left(y^{(k)} + p^{(k)}\right) \\ p^{(k+1)} = y^{(k)} + p^{(k)} - x^{(k+1)} \\ y^{(k+1)} = \mathbf{prox}_{f_2}\left(x^{(k+1)} + q^{(k)}\right) \\ q^{(k+1)} = x^{(k+1)} + q^{(k)} - y^{(k+1)} \end{array} \right.$$

where $x^{(0)} = y^{(0)} = v$ and $p^{(0)} = q^{(0)} = \mathbf{0}_n$

Dykstra's algorithm

The case $m = 2$

This algorithm is related to the Douglas-Rachford splitting framework:

$$\left\{\begin{array}{l} x^{\left(k+\frac{1}{2}\right)} = \mathbf{prox}_{f_1}\left(x^{(k)} + p^{(k)}\right) \\ p^{(k+1)} = p^{(k)} - \Delta_{1/2} x^{\left(k+\frac{1}{2}\right)} \\ x^{(k+1)} = \mathbf{prox}_{f_2}\left(x^{\left(k+\frac{1}{2}\right)} + q^{(k)}\right) \\ q^{(k+1)} = q^{(k)} - \Delta_{1/2} x^{(k+1)} \end{array}\right.$$

where $\Delta_h x^{(k)} = x^{(k)} - x^{(k-h)}$

Dykstra's algorithm

The case $m = 2$

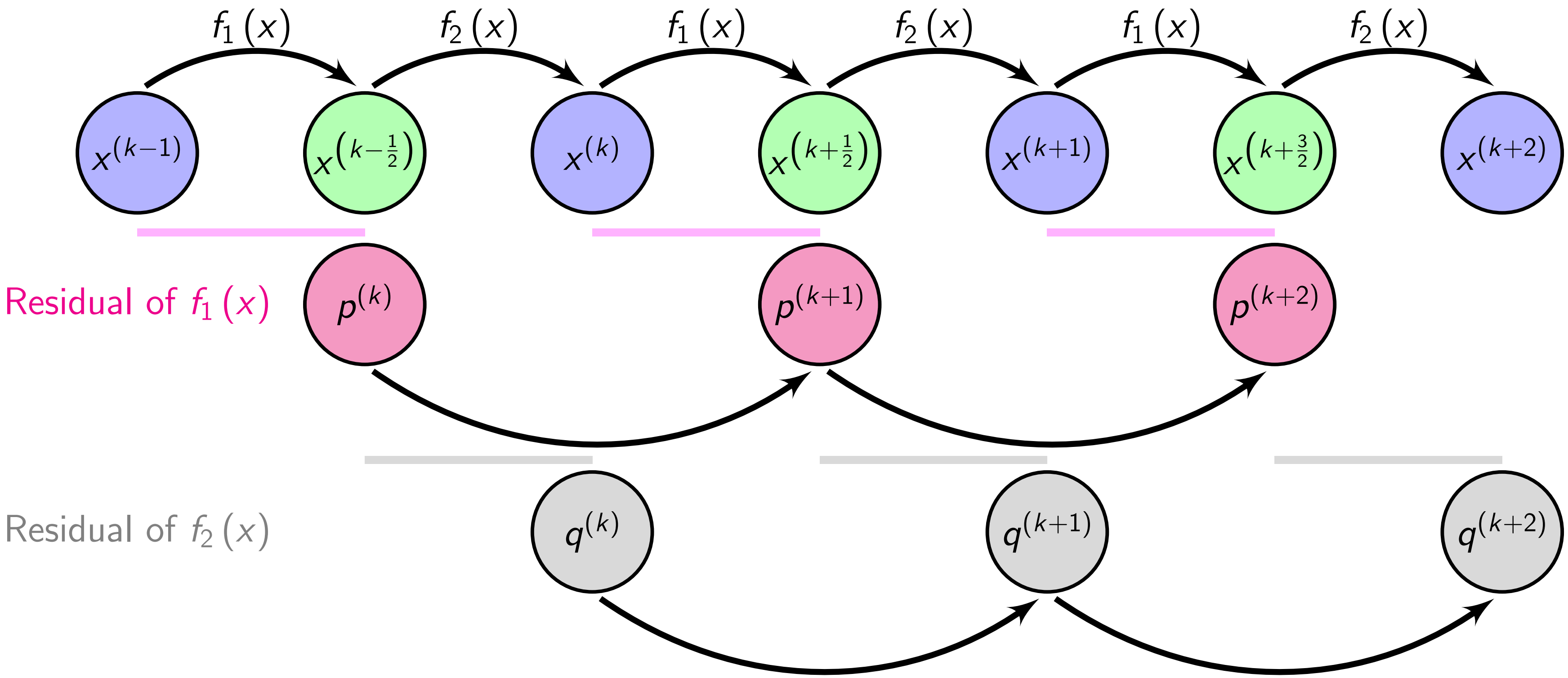


Figure 4: Splitting method of the Dykstra's algorithm

Dykstra's algorithm

The case $m > 2$

The case $m > 2$ is a generalization of the previous algorithm by considering m residuals:

1. The x-update is:

$$x^{(k+1)} = \mathbf{prox}_{f_{j(k)}} \left(x^{(k)} + z^{(k+1-m)} \right)$$

2. The z-update is:

$$z^{(k+1)} = x^{(k)} + z^{(k+1-m)} - x^{(k+1)}$$

where $x^{(0)} = v$, $z^{(k)} = \mathbf{0}_n$ for $k < 0$ and $j(k) = \operatorname{mod}(k+1, m)$ denotes the modulo operator taking values in $\{1, \ldots, m\}$

Remark

The variable $x^{(k)}$ is updated at each iteration while the residual $z^{(k)}$ is updated every m iterations. This implies that the basic function $f_j(x)$ is related to the residuals $z^{(j)}$, $z^{(j+m)}$, $z^{(j+2m)}$, etc.

Dykstra's algorithm

The case $m > 2$

Tibshirani (2017) proposes to write the Dykstra's algorithm by using two iteration indices k and j. The main index k refers to the cycle, whereas the sub-index j refers to the constraint number

The Dykstra's algorithm becomes:

1. The x-update is:

$$x^{(k+1,j)} = \mathbf{prox}_{f_j}\left(x^{(k+1,j-1)} + z^{(k,j)}\right)$$

2. The z-update is:

$$z^{(k+1,j)} = x^{(k+1,j-1)} + z^{(k,j)} - x^{(k+1,j)}$$

where $x^{(1,0)} = v$, $z^{(k,j)} = \mathbf{0}_n$ for $k = 0$ and $x^{(k+1,0)} = x^{(k,m)}$

Dykstra's algorithm

The case $m > 2$

The Dykstra's algorithm is particularly efficient when we consider the projection problem:

$$x^{\star} = \mathcal{P}_{\Omega}(v)$$

where:

$$\Omega = \Omega_1 \cap \Omega_2 \cap \cdots \cap \Omega_m$$

Indeed, the Dykstra's algorithm becomes:

1. The x-update is:

$$x^{(k+1,j)} = \mathbf{prox}_{f_j}\left(x^{(k+1,j-1)} + z^{(k,j)}\right) = \mathcal{P}_{\Omega_j}\left(x^{(k+1,j-1)} + z^{(k,j)}\right)$$

2. The z-update is:

$$z^{(k+1,j)} = x^{(k+1,j-1)} + z^{(k,j)} - x^{(k+1,j)}$$

where $x^{(1,0)} = v$, $z^{(k,j)} = \mathbf{0}_n$ for $k = 0$ and $x^{(k+1,0)} = x^{(k,m)}$

Dykstra's algorithm

Successive projections of $\mathcal{P}_{\Omega_j}\left(x^{(k+1,j-1)}\right)$ do not work!

Successive projections of $\mathcal{P}_{\Omega_j}\left(x^{(k+1,j-1)} + z^{(k,j)}\right)$ do work!

Dykstra's algorithm

Table 6: Solving the proximal problem with linear inequality constraints

The goal is to compute the solution $x^{\star} = \mathbf{prox}_f(v)$ where $f(x) = \mathbb{1}_{\Omega}(x)$ and $\Omega = \{x \in \mathbb{R}^n : Cx \leq D\}$
We initialize $x^{(0,m)} \leftarrow v$
We set $z^{(0,1)} \leftarrow \mathbf{0}_n, \ldots, z^{(0,m)} \leftarrow \mathbf{0}_n$
$k \leftarrow 0$
repeat
  $x^{(k+1,0)} \leftarrow x^{(k,m)}$
  for $j = 1 : m$ **do**
    The x-update is:

$$x^{(k+1,j)} = x^{(k+1,j-1)} + z^{(k,j)} - \frac{\left(c_{(j)}^{\top} x^{(k+1;j-1)} + c_{(j)}^{\top} z^{(k,j)} - d_{(j)}\right)_{+}}{\left\| c_{(j)} \right\|_2^2} c_{(j)}$$

    The z-update is:

$$z^{(k+1,j)} = x^{(k+1,j-1)} + z^{(k,j)} - x^{(k+1,j)}$$

  end for
  $k \leftarrow k + 1$
until Convergence
return $x^{\star} \leftarrow x^{(k,m)}$

Dykstra's algorithm

Table 7: Solving the proximal problem with general linear constraints

The goal is to compute the solution $x^{\star} = \mathbf{prox}_f(v)$ where $f(x) = \mathbb{1}_{\Omega}(x)$, $\Omega = \Omega_1 \cap \Omega_2 \cap \Omega_3$, $\Omega_1 = \{x \in \mathbb{R}^n : Ax = B\}$, $\Omega_2 = \{x \in \mathbb{R}^n : Cx \leq D\}$ and $\Omega_3 = \{x \in \mathbb{R}^n : x^- \leq x \leq x^+\}$
We initialize $x_m^{(0)} \leftarrow v$
We set $z_1^{(0)} \leftarrow \mathbf{0}_n$, $z_2^{(0)} \leftarrow \mathbf{0}_n$ and $z_3^{(0)} \leftarrow \mathbf{0}_n$
$k \leftarrow 0$
repeat
$x_0^{(k+1)} \leftarrow x_m^{(k)}$
$x_1^{(k+1)} \leftarrow x_0^{(k+1)} + z_1^{(k)} - A^{\dagger}\left(Ax_0^{(k+1)} + Az_1^{(k)} - B\right)$
$z_1^{(k+1)} \leftarrow x_0^{(k+1)} + z_1^{(k)} - x_1^{(k+1)}$
$x_2^{(k+1)} \leftarrow \mathcal{P}_{\Omega_2}\left(x_1^{(k+1)} + z_2^{(k)}\right)$
$z_2^{(k+1)} \leftarrow x_1^{(k+1)} + z_2^{(k)} - x_2^{(k+1)}$
$x_3^{(k+1)} \leftarrow \mathcal{T}\left(x_2^{(k+1)} + z_3^{(k)}; x^-, x^+\right)$
$z_3^{(k+1)} \leftarrow x_2^{(k+1)} + z_3^{(k)} - x_3^{(k+1)}$
$k \leftarrow k + 1$
until Convergence
return $x^{\star} \leftarrow x_3^{(k)}$

▶ Previous algorithm

Dykstra's algorithm

Remark

Since we have:

$$\frac{1}{2}\left\|x-v\right\|_2^2=\frac{1}{2}x^\top x-x^\top v+\frac{1}{2}v^\top v$$

the two previous problems can be cast into a QP problem:

$$\begin{aligned} x^\star &= \arg\min_x \frac{1}{2}x^\top I_n x - x^\top v \\ \text{s.t.} &\quad x\in\Omega \end{aligned}$$

Dykstra's algorithm

Dykstra's algorithm versus QP algorithm

- The vector v is defined by the elements $v_i = \ln\left(1 + i^2\right)$
- The set of constraints is:

$$\Omega = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \leq \frac{1}{2}, \sum_{i=1}^{n} e^{-i} x_i \geq 0 \right\}$$

- Using a Matlab implementation, we find that the computational time of the Dykstra's algorithm when n is equal to 10 million is equal to the QP algorithm when n is equal to 12 500!
- The QP algorithm requires to store the matrix I_n — impossible when $n > 10^5$. For instance, the size of I_n is equal to 7450.6 GB when $n = 10^6$

Application to portfolio allocation

Table 8: Some objective functions used in portfolio optimization

Item	Portfolio	$f(x)$	Reference
(1)	MVO	$\frac{1}{2}x^\top \Sigma x - \gamma x^\top \mu$	Markowitz (1952)
(2)	GMV	$\frac{1}{2}x^\top \Sigma x$	Jagganathan and Ma (2003)
(3)	MDP	$\ln\left(\sqrt{x^\top \Sigma x}\right) - \ln\left(x^\top \sigma\right)$	Choueifaty and Coignard (2008)
(4)	KL	$\sum_{i=1}^n x_i \ln\left(x_i / \tilde{x}_i\right)$	Bera and Park (2008)
(5)	ERC	$\frac{1}{2}x^\top \Sigma x - \lambda \sum_{i=1}^n \ln x_i$	Maillard *et al.* (2010)
(6)	RB	$\mathcal{R}(x) - \lambda \sum_{i=1}^n \mathcal{RB}_i \cdot \ln x_i$	Roncalli (2015)
(7)	RQE	$\frac{1}{2}x^\top D x$	Carmichael *et al.* (2018)

Application to portfolio allocation

Table 9: Some regularization penalties used in portfolio optimization

Item	Regularization	$\mathfrak{R}(x)$	Reference
(8)	Ridge	$\lambda \left\Vert x - \tilde{x} \right\Vert_2^2$	DeMiguel *et al.* (2009)
(9)	Lasso	$\lambda \left\Vert x - \tilde{x} \right\Vert_1$	Brodie *at al.* (2009)
(10)	Log-barrier	$-\sum_{i=1}^{n} \lambda_i \ln x_i$	Roncalli (2013)
(11)	Shannon's entropy	$\lambda \sum_{i=1}^{n} x_i \ln x_i$	Yu *et al.* (2014)

Application to portfolio allocation

Table 10: Some constraints used in portfolio optimization

Item	Constraint	Ω
(12)	No cash and leverage	$\sum_{i=1}^{n} x_i = 1$
(13)	No short selling	$x_i \geq 0$
(14)	Weight bounds	$x_i^- \leq x_i \leq x_i^+$
(15)	Asset class limits	$c_j^- \leq \sum_{i \in \mathcal{C}_j} x_i \leq c_j^+$
(16)	Turnover	$\sum_{i=1}^{n} \lvert x_i - \tilde{x}_i \rvert \leq \boldsymbol{\tau}^+$
(17)	Transaction costs	$\sum_{i=1}^{n} \left(c_i^- \left(\tilde{x}_i - x_i \right)_+ + c_i^+ \left(x_i - \tilde{x}_i \right)_+ \right) \leq \boldsymbol{c}^+$
(18)	Leverage limit	$\sum_{i=1}^{n} \lvert x_i \rvert \leq \mathcal{L}^+$
(19)	Long/short exposure	$-\mathcal{LS}^- \leq \sum_{i=1}^{n} x_i \leq \mathcal{LS}^+$
(20)	Benchmarking	$\sqrt{\left(x - \tilde{x} \right)^\top \Sigma \left(x - \tilde{x} \right)} \leq \sigma^+$
(21)	Tracking error floor	$\sqrt{\left(x - \tilde{x} \right)^\top \Sigma \left(x - \tilde{x} \right)} \geq \sigma^-$
(22)	Active share floor	$\frac{1}{2} \sum_{i=1}^{n} \lvert x_i - \tilde{x}_i \rvert \geq \mathcal{AS}^-$
(23)	Number of active bets	$\left(x^\top x \right)^{-1} \geq \mathcal{N}^-$

Application to portfolio allocation

Most of portfolio optimization problems are a combination of:

1. an objective function (Table 8)
2. one or two regularization penalty functions (Table 9)
3. some constraints (Table 10)

Perrin and Roncalli (2020) solve **all these problems** using CCD, ADMM, Dykstra and the appropriate proximal functions. For that, they derive:

- the semi-analytical solution of the x-step for all objective functions
- the proximal solution of the y-step for all regularization penalty functions and constraints

Herfindahl-MV optimization

Formulation of the mathematical problem

- The second generation of minimum variance strategies uses a global diversification constraint
- The most popular solution is based on the Herfindahl index:

$$\mathcal{H}(x) = \sum_{i=1}^{n} x_i^2$$

- The effective number of bets is the inverse of the Herfindahl index:

$$\mathcal{N}(x) = \mathcal{H}(x)^{-1}$$

- The optimization program is:

$$\begin{aligned} x^{\star} &= \arg\min_{x} \frac{1}{2} x^{\top} \Sigma x \\ \text{s.t.} & \quad \left\{ \begin{array}{l} \mathbf{1}_n^{\top} x = 1 \\ \mathbf{0}_n \leq x \leq x^{+} \\ \mathcal{N}(x) \geq \mathcal{N}^{-} \end{array} \right. \end{aligned}$$

where $\mathcal{N}^{-}$ is the minimum number of effective bets.

Herfindahl-MV optimization

The QP solution

- The Herfindhal constraint is equivalent to:

$$\begin{aligned}\mathcal{N}(x) \geq \mathcal{N}^{-} &\Leftrightarrow \left(x^\top x\right)^{-1} \geq \mathcal{N}^{-} \\ &\Leftrightarrow x^\top x \leq \frac{1}{\mathcal{N}^{-}}\end{aligned}$$

- The QP problem is:

$$\begin{aligned}x^{\star}(\lambda) &= \arg\min_x \frac{1}{2} x^\top \Sigma x + \lambda x^\top x = \frac{1}{2} x^\top \left(\Sigma + 2\lambda I_n\right) x \\ &\text{s.t.} \left\{\begin{array}{l}\mathbf{1}_n^\top x = 1 \\ \mathbf{0}_n \leq x \leq x^{+}\end{array}\right.\end{aligned}$$

  where $\lambda \geq 0$ is a scalar
- We have $\mathcal{N}(x) \in \left[\mathcal{N}\left(x^{\star}(0)\right), n\right]$
- The optimal value $\lambda^{\star}$ is found using the bi-section algorithm such that $\mathcal{N}\left(x^{\star}(\lambda)\right) = \mathcal{N}^{-}$

Herfindahl-MV optimization

The ADMM solution (first version)

- The ADMM form is:

$$\begin{aligned} \{x^\star, y^\star\} &= \arg\min_{(x,y)} \frac{1}{2} x^\top \Sigma x + \mathbb{1}_{\Omega_1}(x) + \mathbb{1}_{\Omega_2}(y) \\ \text{s.t.} \quad & x = y \end{aligned}$$

where $\Omega_1 = \left\{x \in \mathbb{R}^n : \mathbf{1}_n^\top x = 1, \mathbf{0}_n \leq x \leq x^+\right\}$ and $\Omega_2 = \mathcal{B}_2\left(\mathbf{0}_n, \sqrt{\frac{1}{\mathcal{N}^-}}\right)$

- The x-update is a QP problem:

$$x^{(k+1)} = \arg\min_x \left\{\frac{1}{2} x^\top \left(\Sigma + \varphi I_n\right) x - \varphi x^\top \left(y^{(k)} - u^{(k)}\right) + \mathbb{1}_{\Omega_1}(x)\right\}$$

- The y-update is:

$$y^{(k+1)} = \frac{x^{(k+1)} + u^{(k)}}{\max\left(1, \sqrt{\mathcal{N}^-} \left\| x^{(k+1)} + u^{(k)} \right\|_2\right)}$$

Herfindahl-MV optimization

The ADMM solution (second version)

- A better approach is to write the problem as follows:

$$\begin{aligned} \{x^\star, y^\star\} &= \arg\min_{(x,y)} \frac{1}{2} x^\top \Sigma x + \mathbb{1}_{\Omega_3}(x) + \mathbb{1}_{\Omega_4}(y) \\ \text{s.t. } & x = y \end{aligned}$$

  where $\Omega_3 = \mathcal{H}_{yperplane}[\mathbf{1}_n, 1]$ and $\Omega_4 = \mathcal{B}_{ox}[\mathbf{0}_n, x^+] \cap \mathcal{B}_2\left(\mathbf{0}_n, \sqrt{\frac{1}{\mathcal{N}^-}}\right)$

- The x-update is:

$$x^{(k+1)} = (\Sigma + \varphi I_n)^{-1} \left(\varphi \left(y^{(k)} - u^{(k)} \right) + \frac{1 - \mathbf{1}_n^\top (\Sigma + \varphi I_n)^{-1} \varphi \left(y^{(k)} - u^{(k)} \right)}{\mathbf{1}_n^\top (\Sigma + \varphi I_n)^{-1} \mathbf{1}_n} \mathbf{1}_n \right)$$

- The y-update is:

$$y^{(k+1)} = \mathcal{P}_{\mathcal{B}\text{ox}-\mathcal{B}\text{all}}\left(x^{(k+1)} + u^{(k)}; \mathbf{0}_n, x^+, \mathbf{0}_n, \sqrt{\frac{1}{\mathcal{N}^-}} \right)$$

  where $\mathcal{P}_{\mathcal{B}\text{ox}-\mathcal{B}\text{all}}$ corresponds to the Dykstra's algorithm given by Perrin and Roncalli (2020)

Herfindahl-MV optimization

Remark

If we compare the computational time of the three approaches, we observe that the best method is the second version of the ADMM algorithm:

$$\mathcal{CT}\left(\mathrm{QP}; n = 1000\right) = 50 \times \mathcal{CT}\left(\mathrm{ADMM}_2; n = 1000\right)$$
$$\mathcal{CT}\left(\mathrm{ADMM}_1; n = 1000\right) = 400 \times \mathcal{CT}\left(\mathrm{ADMM}_2; n = 1000\right)$$

Herfindahl-MV optimization

The QP solution

Example 5

We consider an investment universe of eight stocks. We assume that their volatilities are 21%, 20%, 40%, 18%, 35%, 23%, **7%** and 29%. The correlation matrix is defined as follows:

$$\rho = \left(\begin{array}{rrrrrrrr} 100\% & & & & & & & \\ 80\% & 100\% & & & & & & \\ 70\% & 75\% & 100\% & & & & & \\ 60\% & 65\% & 90\% & 100\% & & & & \\ 70\% & 50\% & 70\% & 85\% & 100\% & & & \\ 50\% & 60\% & 70\% & 80\% & 60\% & 100\% & & \\ 70\% & 50\% & 70\% & 75\% & 80\% & 50\% & 100\% & \\ 60\% & 65\% & 70\% & 75\% & 65\% & 70\% & 80\% & 100\% \end{array} \right)$$

Herfindahl-MV optimization

Table 11: Minimum variance portfolios (in %)

$\mathcal{N}^-$	1.00	2.00	3.00	4.00	5.00	6.00	6.50	7.00	7.50	8.00
$x_1^\star$	0.00	3.22	9.60	13.83	15.18	15.05	14.69	14.27	13.75	12.50
$x_2^\star$	0.00	12.75	14.14	15.85	16.19	15.89	15.39	14.82	14.13	12.50
$x_3^\star$	0.00	0.00	0.00	0.00	0.00	0.07	2.05	4.21	6.79	12.50
$x_4^\star$	0.00	10.13	15.01	17.38	17.21	16.09	15.40	14.72	13.97	12.50
$x_5^\star$	0.00	0.00	0.00	0.00	0.71	5.10	6.33	7.64	9.17	12.50
$x_6^\star$	0.00	5.36	8.95	12.42	13.68	14.01	13.80	13.56	13.25	12.50
$x_7^\star$	100.00	68.53	52.31	40.01	31.52	25.13	22.92	20.63	18.00	12.50
$x_8^\star$	0.00	0.00	0.00	0.50	5.51	8.66	9.41	10.14	10.95	12.50
$\lambda^\star$ (in %)	0.00	1.59	3.10	5.90	10.38	18.31	23.45	31.73	49.79	∞

Note: the upper bound x^+ is set to $\mathbf{1}_n$. The solutions are those found by the ADMM algorithm. We also report the value of $\lambda^\star$ found by the bi-section algorithm when we use the QP algorithm.

ERC portfolio optimization

We recall that:

$$x^\star = \arg\min_x \frac{1}{2} x^\top \Sigma x - \lambda \sum_{i=1}^{n} \ln x_i$$

and:

$$x_{\mathrm{erc}} = \frac{x^\star}{\mathbf{1}_n^\top x^\star}$$

ERC portfolio optimization

The CCD solution

- The first-order condition $(\Sigma x)_i - \lambda x_i^{-1} = 0$ implies that:

$$x_i^2 \sigma_i^2 + x_i \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j - \lambda = 0$$

- The CCD algorithm is:

$$x_i^{(k+1)} = \frac{-v_i^{(k+1)} + \sqrt{\left(v_i^{(k+1)}\right)^2 + 4\lambda\sigma_i^2}}{2\sigma_i^2}$$

where:

$$v_i^{(k+1)} = \sigma_i \sum_{j<i} x_j^{(k+1)} \rho_{i,j} \sigma_j + \sigma_i \sum_{j>i} x_j^{(k)} \rho_{i,j} \sigma_j$$

ERC portfolio optimization

The ADMM solution

- In the case of the ADMM algorithm, we set:

$$\begin{aligned} f_x(x) &= \frac{1}{2} x^\top \Sigma x \\ f_y(y) &= -\lambda \sum_{i=1}^{n} \ln y_i \\ x &= y \end{aligned}$$

- The x-update step is:

$$x^{(k+1)} = (\Sigma + \varphi I_n)^{-1} \varphi \left(y^{(k)} - u^{(k)} \right)$$

- The y-update step is:

$$y_i^{(k+1)} = \frac{1}{2} \left(\left(x_i^{(k+1)} + u_i^{(k)} \right) + \sqrt{\left(x_i^{(k+1)} + u_i^{(k)} \right)^2 + 4\lambda\varphi^{-1}} \right)$$

RB portfolio optimization

The RB portfolio is equal to:

$$x_{\mathrm{rb}} = \frac{x^{\star}}{\mathbf{1}_n^\top x^{\star}}$$

where $x^{\star}$ is the solution of the logarithmic barrier problem:

$$x^{\star} = \arg\min_{x} \mathcal{R}(x) - \lambda \sum_{i=1}^{n} \mathcal{RB}_i \cdot \ln x_i$$

λ is any positive scalar and $\mathcal{RB}_i$ is the risk budget allocated to Asset i

RB portfolio optimization

The CCD solution (SD risk measure)

- In the case of the standard deviation-based risk measure:

$$\mathcal{R}(x) = -x^\top (\mu - r) + \xi \sqrt{x^\top \Sigma x}$$

  the first-order condition for defining the CCD algorithm is:

$$-(\mu_i - r) + \xi \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} - \lambda \frac{\mathcal{RB}_i}{x_i} = 0$$

- It follows that $\xi x_i (\Sigma x)_i - (\mu_i - r) x_i \sigma(x) - \lambda \sigma(x) \cdot \mathcal{RB}_i = 0$ or equivalently:

$$\alpha_i x_i^2 + \beta_i x_i + \gamma_i = 0$$

  where $\alpha_i = \xi \sigma_i^2$, $\beta_i = \xi \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j - (\mu_i - r) \sigma(x)$ and $\gamma_i = -\lambda \sigma(x) \cdot \mathcal{RB}_i$

RB portfolio optimization

The CCD solution (SD risk measure)

- The CCD algorithm is:

$$x_i^{(k+1)} = \frac{-\beta_i^{(k+1)} + \sqrt{\left(\beta_i^{(k+1)}\right)^2 - 4\alpha_i^{(k+1)}\gamma_i^{(k+1)}}}{2\alpha_i^{(k+1)}}$$

where:

$$\begin{cases} \alpha_i^{(k+1)} = \xi\sigma_i^2 \\ \beta_i^{(k+1)} = \xi\sigma_i\left(\sum_{j<i} x_j^{(k+1)}\rho_{i,j}\sigma_j + \sum_{j>i} x_j^{(k)}\rho_{i,j}\sigma_j\right) - (\mu_i - r)\,\sigma_i^{(k+1)}(x) \\ \gamma_i^{(k+1)} = -\lambda\sigma_i^{(k+1)}(x)\cdot\mathcal{RB}_i \\ \sigma_i^{(k+1)}(x) = \sqrt{\chi^\top\Sigma\chi} \\ \chi = \left(x_1^{(k+1)},\ldots,x_{i-1}^{(k+1)},x_i^{(k)},x_{i+1}^{(k)}\ldots,x_n^{(k)}\right) \end{cases}$$

RB portfolio optimization

The ADMM solution (convex risk measure)

- We have:

$$\begin{aligned}\{x^\star, y^\star\} &= \arg\min_{x,y} \mathcal{R}(x) - \lambda \sum_{i=1}^{n} \mathcal{RB}_i \cdot \ln y_i \\ \text{s.t.} &\quad x = y\end{aligned}$$

- The ADMM algorithm is:

$$\left\{\begin{array}{l} x^{(k+1)} = \mathbf{prox}_{\varphi^{-1}\mathcal{R}(x)}\left(y^{(k)} - u^{(k)}\right) \\ v_y^{(k+1)} = x^{(k+1)} + u^{(k)} \\ y^{(k+1)} = \frac{1}{2}\left(v_y^{(k+1)} + \sqrt{v_y^{(k+1)} \odot v_y^{(k+1)} + 4\lambda\varphi^{-1} \cdot \mathcal{RB}}\right) \\ u^{(k+1)} = u^{(k)} + x^{(k+1)} - y^{(k+1)} \end{array}\right.$$

Tips and tricks of portfolio optimization

- Full allocation — $\sum_{i=1}^{n} x_i = 1$:

$$\Omega = \mathcal{H}_{yperplane}\left[\mathbf{1}_n, 1\right]$$

We have:

$$\mathcal{P}_{\Omega}\left(v\right) = v - \left(\frac{\mathbf{1}_n^\top v - 1}{n}\right)\mathbf{1}_n$$

- Cash neutral — $\sum_{i=1}^{n} x_i = 0$:

$$\Omega = \mathcal{H}_{yperplane}\left[\mathbf{1}_n, 0\right]$$

We have:

$$\mathcal{P}_{\Omega}\left(v\right) = v - \left(\frac{\mathbf{1}_n^\top v}{n}\right)\mathbf{1}_n$$

Tips and tricks of portfolio optimization

- No short selling — $x \geq \mathbf{0}_n$:

$$\Omega = \mathcal{B}_{ox}\left[\mathbf{0}_n, \infty\right]$$

  We have:

$$\mathcal{P}_{\Omega}(v) = \mathcal{T}\left(v; \mathbf{0}_n, \infty\right)$$

- Weight bounds — $x^- \leq x \leq x^+$:

$$\Omega = \mathcal{B}_{ox}\left[x^-, x^+\right]$$

  We have:

$$\mathcal{P}_{\Omega}(v) = \mathcal{T}\left(v; x^-, x^+\right)$$

Tips and tricks of portfolio optimization

- μ-problem — $\mu(x) \geq \mu^{\star}$:

$$\Omega = \mathcal{H}_{alfspace}\left[-\mu, -\mu^{\star}\right]$$

We have:

$$\mathcal{P}_{\Omega}(v) = v + \frac{\left(\mu^{\star} - \mu^{\top} v\right)_{+}}{\left\|\mu\right\|_{2}^{2}}\mu$$

Tips and tricks of portfolio optimization

- σ-problem — $\sigma(x) \leq \sigma^{\star}$:

$$\Omega = \left\{ x : \sqrt{x^\top \Sigma x} \leq \sigma^{\star} \right\}$$

We have:

$$\begin{aligned} \sqrt{x^\top \Sigma x} \leq \sigma^{\star} & \Leftrightarrow & \sqrt{x^\top \left(LL^\top\right) x} \leq \sigma^{\star} \\ & \Leftrightarrow & \left\| y^\top y \right\|_2 \leq \sigma^{\star} \\ & \Leftrightarrow & y \in \mathcal{B}_2\left(\mathbf{0}_n, \sigma^{\star}\right) \end{aligned}$$

where $y = L^\top x$ and L is the Cholesky decomposition of Σ. It follows that the proximal of the y-update is the projection onto the ℓ_2 ball $\mathcal{B}_2\left(\mathbf{0}_n, \sigma^{\star}\right)$:

$$\begin{aligned} \mathcal{P}_{\Omega}(v) & = & v - \mathbf{prox}_{\sigma^{\star}\|x\|_2}(v) \\ & = & v - \left(1 - \frac{\sigma^{\star}}{\max\left(\sigma^{\star}, \|v\|_2\right)}\right) v \end{aligned}$$

Tips and tricks of portfolio optimization

- Leverage management — $\sum_{i=1}^{n} |x_i| \leq \mathcal{L}^+$:

$$\begin{aligned} \Omega &= \left\{x : \|x\|_1 \leq \mathcal{L}^+\right\} \\ &= \mathcal{B}_1\left(\mathbf{0}_n, \mathcal{L}^+\right) \end{aligned}$$

The proximal of the y-update is the projection onto the ℓ_1 ball $\mathcal{B}_1\left(\mathbf{0}_n, \mathcal{L}^+\right)$:

$$\mathcal{P}_\Omega\left(v\right) = v - \operatorname{sign}\left(v\right) \odot \mathbf{prox}_{\mathcal{L}^+ \max x}\left(\left|v\right|\right)$$

Tips and tricks of portfolio optimization

- Leverage management — $\mathcal{LS}^{-} \leq \sum_{i=1}^{n} x_i \leq \mathcal{LS}^{+}$:

$$\Omega = \mathcal{H}_{alfspace}\left[\mathbf{1}_n, \mathcal{LS}^{+}\right] \cap \mathcal{H}_{alfspace}\left[-\mathbf{1}_n, -\mathcal{LS}^{-}\right]$$

  The proximal of the y-update is obtained with the Dykstra's algorithm by combining the two half-space projections.

- Leverage management — $\left|\sum_{i=1}^{n} x_i\right| \leq \mathcal{L}^{+}$:

$$\Omega = \left\{x : \left|\mathbf{1}_n^\top x\right| \leq \mathcal{L}^{+}\right\}$$

  This is a special case of the previous result where $\mathcal{LS}^{+} = \mathcal{L}^{+}$ and $\mathcal{LS}^{-} = -\mathcal{L}^{+}$:

$$\Omega = \mathcal{H}_{alfspace}\left[\mathbf{1}_n, \mathcal{L}^{+}\right] \cap \mathcal{H}_{alfspace}\left[-\mathbf{1}_n, \mathcal{L}^{+}\right]$$

Tips and tricks of portfolio optimization

- Concentration management[2]
  Portfolio managers can also use another constraint concerning the sum of the k largest values:

$$f(x) = \sum_{i=n-k+1}^{n} x_{(i:n)} = x_{(n:n)} + \ldots + x_{(n-k+1:n)}$$

where $x_{(i:n)}$ is the order statistics of x: $x_{(1:n)} \leq x_{(2:n)} \leq \cdots \leq x_{(n:n)}$.
Beck (2017) shows that:

$$\mathbf{prox}_{\lambda f(x)}(v) = v - \lambda \mathcal{P}_{\Omega}\left(\frac{v}{\lambda}\right)$$

where:

$$\Omega = \left\{x \in [0,1]^n : \mathbf{1}_n^\top x = k\right\} = \mathcal{B}_{ox}\left[\mathbf{0}_n, \mathbf{1}_n\right] \cap \mathcal{H}_{yperlane}\left[\mathbf{1}_n, k\right]$$

[2] An example is the 5/10/40 UCITS rule: A UCITS fund may invest no more than 10% of its net assets in transferable securities or money market instruments issued by the same body, with a further aggregate limitation of 40% of net assets on exposures of greater than 5% to single issuers.

Tips and tricks of portfolio optimization

- Entropy portfolio management
  Bera and Park (2008) propose using a cross-entropy measure as the objective function:

$$
\begin{aligned}
x^{\star} & = & \arg\min_{x} \mathrm{KL}\left(x \mid \tilde{x}\right) \\
& & \text{s.t.} \quad \left\{
\begin{array}{l}
\mathbf{1}_n^{\top} x = 1 \\
\mathbf{0}_n \leq x \leq \mathbf{1}_n \\
\mu\left(x\right) \geq \mu^{\star}, \sigma\left(x\right) \leq \sigma^{\star}
\end{array}
\right.
\end{aligned}
$$

  where $\mathrm{KL}\left(x \mid \tilde{x}\right)$ is the Kullback-Leibler measure:

$$
\mathrm{KL}\left(x \mid \tilde{x}\right) = \sum_{i=1}^{n} x_i \ln\left(x_i / \tilde{x}_i\right)
$$

  and $\tilde{x}$ is a reference portfolio

Tips and tricks of portfolio optimization

- Entropy portfolio management
  We have:

$$\mathbf{prox}_{\lambda \,\mathrm{KL}(v|\tilde{x})}(v) = \lambda \begin{pmatrix} W\left(\lambda^{-1}\tilde{x}_1 e^{\lambda^{-1}v_1 - \tilde{x}_1^{-1}}\right) \\ \vdots \\ W\left(\lambda^{-1}\tilde{x}_n e^{\lambda^{-1}v_n - \tilde{x}_n^{-1}}\right) \end{pmatrix}$$

  where $W(x)$ is the Lambert W function

Tips and tricks of portfolio optimization

Remark

Since the Shannon's entropy is equal to $\mathrm{SE}(x) = -\mathrm{KL}(x \mid \mathbf{1}_n)$, we deduce that:

$$\mathbf{prox}_{\lambda\,\mathrm{SE}(x)}(v) = \lambda \begin{pmatrix} W\left(\lambda^{-1} e^{\lambda^{-1} v_1 - 1}\right) \\ \vdots \\ W\left(\lambda^{-1} e^{\lambda^{-1} v_n - 1}\right) \end{pmatrix}$$

Tips and tricks of portfolio optimization

- Active share constraint — $\mathcal{AS}(x \mid \tilde{x}) \geq \mathcal{AS}^{-}$:

$$\mathcal{AS}(x \mid \tilde{x}) = \frac{1}{2}\sum_{i=1}^{n} |x_i - \tilde{x}_i| \geq \mathcal{AS}^{-}$$

  We use the projection onto the complement $\bar{\mathcal{B}}_1(c, r)$ of the ℓ_1 ball and we obtain:

$$\mathcal{P}_{\Omega}(v) = v + \operatorname{sign}(v - \tilde{x}) \odot \frac{\max\left(2\mathcal{AS}^{-} - \|v - \tilde{x}\|_1, 0\right)}{n}$$

Tips and tricks of portfolio optimization

- Tracking error volatility — $\sigma(x \mid \tilde{x}) \leq \sigma^\star$:

$$\begin{aligned}\sigma(x \mid \tilde{x}) \leq \sigma^\star & \Leftrightarrow \sqrt{(x-\tilde{x})^\top \Sigma (x-\tilde{x})} \leq \sigma^\star \\ & \Leftrightarrow \|y\|_2 \leq \sigma^\star \\ & \Leftrightarrow y \in \mathcal{B}_2(\mathbf{0}_n, \sigma^\star)\end{aligned}$$

where $y = L^\top x - L^\top \tilde{x}$. It follows that $Ax + By = c$ where $A = L^\top$, $B = -I_n$ and $c = L^\top \tilde{x}$. It follows that the proximal of the y-update is the projection onto the ℓ_2 ball $\mathcal{B}_2(\mathbf{0}_n, \sigma^\star)$:

$$\begin{aligned}\mathcal{P}_\Omega(v) &= v - \mathbf{prox}_{\sigma^\star \|x\|_2}(v) \\ &= v - \left(1 - \frac{\sigma^\star}{\max(\sigma^\star, \|v\|_2)}\right) v\end{aligned}$$

Tips and tricks of portfolio optimization

- Bid-ask transaction cost management:

$$\boldsymbol{c}\left(x \mid x_0\right) = \lambda \sum_{i=1}^{n} \left(c_i^- \left(x_{0,i} - x_i\right)_+ + c_i^+ \left(x_i - x_{0,i}\right)_+\right)$$

where c_i^- and c_i^+ are the bid and ask transaction costs. We have:

$$\mathbf{prox}_{\boldsymbol{c}(x|x_0)}\left(v\right) = x_0 + \mathcal{S}\left(v - x_0; \lambda c^-, \lambda c^+\right)$$

where $\mathcal{S}\left(v; \lambda_-, \lambda_+\right) = \left(v - \lambda_+\right)_+ - \left(v + \lambda_-\right)_-$ is the two-sided soft-thresholding operator.

Tips and tricks of portfolio optimization

- Turnover management:

$$\Omega = \left\{ x \in \mathbb{R}^n : \left\| x - x_0 \right\|_1 \leq \boldsymbol{\tau}^+ \right\}$$

The proximal operator is:

$$\mathcal{P}_{\Omega}(v) = v - \operatorname{sign}(v - x_0) \odot \min\left(\left| v - x_0 \right|, s^\star \right)$$

where $s^\star = \left\{ s \in \mathbb{R} : \sum_{i=1}^n \left(\left| v_i - x_{0,i} \right| - s \right)_+ = \boldsymbol{\tau}^+ \right\}$.

Pattern learning and self-automated strategies

Table 12: What works / What doesn't

	Bond Scoring	Stock Picking	Trend Filtering	Mean Reverting	Index Tracking	HF Tracking	Stock Classification	Technical Analysis
Lasso		☺	☺	☺	☹	☺		
NMF							☺	☹
Boosting		☺				☺		
Bagging		☺				☺		
Random forests	☺			☹				☹
Neural nets	☺					☹		
SVM	☺	☹	☹				☹	
Sparse Kalman					☹	☺		
K-NN	☹							
K-means	☺						☺	
Testing protocols[3]	☺	☺	☺	☺		☺		

Source: Roncalli (2014), Big Data in Asset Management, ESMA/CEMA/GEA meeting, Madrid.

[3] Cross-validation, training/test/probe sets, K-fold, etc.

Pattern learning and self-automated strategies

$$2021 \neq 2014$$

The evolution of machine learning in finance is fast, very fast!

Pattern learning and self-automated strategies

Some examples

- Natural Language Processing (NLP)
- Deep learning (DL)
- Reinforcement learning (RL)
- Gaussian process (GP) and Bayesian optimization (BO)
- Learning to rank (MLR)
- Etc.

Some applications

- Robo-advisory
- Stock classification
- $Q_1 - Q_5$ long/short strategy
- Trend-following strategies
- Mean-reverting strategies
- Scoring models
- Sentiment and news analysis
- Etc.

Market generators

- The underlying idea is to simulate artificial multi-dimensional financial time series, whose statistical properties are the same as those observed in the financial markets

  $\approx$ **Monte Carlo simulation of the financial market**

- 3 main approaches:
  1. Restricted Boltzmann machines (RBM)
  2. Generative adversarial networks (GAN)
  3. Convolutional Wasserstein models (W-GAN)

- The goal is to:
  - improve the the risk management of quantitative investment strategies
  - avoid the over-fitting bias of backtesting

The current research shows that results are disappointed until now

Portfolio optimization with CCD and ADMM algorithms

Question 1

We consider the following optimization program:

$$x^{\star} = \arg\min \frac{1}{2} x^{\top} \Sigma x - \lambda \sum_{i=1}^{n} b_i \ln x_i$$

where Σ is the covariance matrix, b is a vector of positive budgets and x is the vector of portfolio weights.

Portfolio optimization with CCD and ADMM algorithms

Question 1.a

Write the first-order condition with respect to the coordinate x_i and show that the solution $x^{\star}$ corresponds to a risk-budgeting portfolio.

Portfolio optimization with CCD and ADMM algorithms

We have:

$$\mathcal{L}(x;\lambda) = \arg\min \frac{1}{2} x^\top \Sigma x - \lambda \sum_{i=1}^{n} b_i \ln x_i$$

The first-order condition is:

$$\frac{\partial\, \mathcal{L}(x;\lambda)}{\partial\, x_i} = (\Sigma x)_i - \lambda \frac{b_i}{x_i} = 0$$

or:

$$x_i \cdot (\Sigma x)_i = \lambda b_i$$

Portfolio optimization with CCD and ADMM algorithms

If we assume that the risk measure is the portfolio volatility:

$$\mathcal{R}(x) = \sqrt{x^\top \Sigma x}$$

the risk contribution of Asset i is equal to:

$$\mathcal{RC}_i(x) = \frac{x_i \cdot (\Sigma x)_i}{\sqrt{x^\top \Sigma x}}$$

We deduce that the optimization problem defines a risk budgeting portfolio:

$$\frac{x_i \cdot (\Sigma x)_i}{b_i} = \frac{x_j \cdot (\Sigma x)_j}{b_j} = \lambda \Leftrightarrow \frac{\mathcal{RC}_i(x)}{b_i} = \frac{\mathcal{RC}_j(x)}{b_j}$$

where the risk measure is the portfolio volatility and the risk budgets are $(b_1, \ldots, b_n)$.

Portfolio optimization with CCD and ADMM algorithms

Question 1.b

Find the optimal value $x_i^\star$ when we consider the other coordinates $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ as fixed.

Portfolio optimization with CCD and ADMM algorithms

The first-order condition is equivalent to:

$$x_i \cdot (\Sigma x)_i - \lambda b_i = 0$$

We have:

$$(\Sigma x)_i = x_i \sigma_i^2 + \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j$$

It follows that:

$$x_i^2 \sigma_i^2 + x_i \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j - \lambda b_i = 0$$

Portfolio optimization with CCD and ADMM algorithms

We obtain a second-degree equation:

$$\alpha_i x_i^2 + \beta_i x_i + \gamma_i = 0$$

where:

$$\begin{cases} \alpha_i = \sigma_i^2 \\ \beta_i = \sigma_i \sum_{j \neq i} x_j \rho_{i,j} \sigma_j \\ \gamma_i = -\lambda b_i \end{cases}$$

1. The polynomial function is convex because we have $\alpha_i = \sigma_i^2 > 0$
2. The product of the roots is negative:

$$x_i' x_i'' = \frac{\gamma_i}{\alpha_i} = -\frac{\lambda b_i}{\sigma_i^2} < 0$$

3. The discriminant is positive:

$$\Delta = \beta_i^2 - 4\alpha_i\gamma_i = \left(\sigma_i \sum_{j \neq i} \rho_{i,j}\sigma_j y_j\right)^2 + 4\lambda b_i \sigma_i^2 > 0$$

Portfolio optimization with CCD and ADMM algorithms

We always have two solutions with opposite signs. We deduce that the solution is the positive root of the second-degree equation:

$$
\begin{aligned}
x_i^\star &= x_i'' = \frac{-\beta_i + \sqrt{\beta_i^2 - 4\alpha_i\gamma_i}}{2\alpha_i} \\
&= \frac{-\sigma_i \sum_{j\neq i} x_j \rho_{i,j}\sigma_j + \sqrt{\sigma_i^2 \left(\sum_{j\neq i} x_j \rho_{i,j}\sigma_j\right)^2 + 4\lambda b_i \sigma_i^2}}{2\sigma_i^2}
\end{aligned}
$$

Portfolio optimization with CCD and ADMM algorithms

Question 1.c

We note $x_i^{(k)}$ the value of the i^{th} coordinate at the k^{th} iteration. Deduce the corresponding CCD algorithm. How to find the RB portfolio x_{rb}?

Portfolio optimization with CCD and ADMM algorithms

The CCD algorithm consists in iterating the following formula:

$$x_i^{(k)} = \frac{-\beta_i^{(k)} + \sqrt{\left(\beta_i^{(k)}\right)^2 - 4\alpha_i^{(k)}\gamma_i^{(k)}}}{2\alpha_i^{(k)}}$$

where:

$$\left\{\begin{array}{l} \alpha_i^{(k)} = \sigma_i^2 \\ \beta_i^{(k)} = \sigma_i \left(\sum_{j<i} \rho_{i,j}\sigma_j x_j^{(k)} + \sum_{j>i} \rho_{i,j}\sigma_j x_j^{(k-1)}\right) \\ \gamma_i^{(k)} = -\lambda b_i \end{array}\right.$$

The RB portfolio is the scaled solution:

$$x_{\mathrm{rb}} = \frac{x^\star}{\sum_{i=1}^n x_i^\star}$$

Portfolio optimization with CCD and ADMM algorithms

Question 1.d

We consider a universe of three assets, whose volatilities are equal to 20%, 25% and 30%. The correlation matrix is equal to:

$$\rho = \begin{pmatrix} 100\% & & \\ 50\% & 100\% & \\ 60\% & 70\% & 100\% \end{pmatrix}$$

We would like to compute the ERC portfolio[a] using the CCD algorithm. We initialize the CCD algorithm with the following starting values $x^{(0)} = (33.3\%, 33.3\%, 33.3\%)$. We assume that $\lambda = 1$.

[a] This means that:

$$b_i = \frac{1}{3}$$

Portfolio optimization with CCD and ADMM algorithms

Question 1.d.i

Starting from $x^{(0)}$, find the optimal coordinate $x_1^{(1)}$ for the first asset.

Portfolio optimization with CCD and ADMM algorithms

We have:

$$\left\{\begin{array}{l}\alpha_1^{(1)} = 0.2^2 = 4\% \\ \beta_1^{(1)} = 0.02033 \\ \gamma_i^{(1)} = -0.333\%\end{array}\right.$$

We obtain:

$$x_1^{(1)} = 2.64375$$

Portfolio optimization with CCD and ADMM algorithms

Question 1.d.ii

Compute then the optimal coordinate $x_2^{(1)}$ for the second asset.

Portfolio optimization with CCD and ADMM algorithms

We have:

$$\begin{cases} \alpha_2^{(1)} = 0.25^2 = 6.25\% \\ \beta_2^{(1)} = 0.08359 \\ \gamma_2^{(1)} = -0.333\% \end{cases}$$

We obtain:

$$x_2^{(1)} = 1.73553$$

Portfolio optimization with CCD and ADMM algorithms

Question 1.d.iii

Compute then the optimal coordinate $x_3^{(1)}$ for the third asset.

Portfolio optimization with CCD and ADMM algorithms

We have:

$$\left\{\begin{array}{l} \alpha_3^{(1)} = 0.3^2 = 9\% \\ \beta_3^{(1)} = 0.18629 \\ \gamma_3^{(1)} = -0.333\% \end{array}\right.$$

We obtain:

$$x_3^{(1)} = 1.15019$$

Portfolio optimization with CCD and ADMM algorithms

Question 1.d.iv

Give the CCD coordinates $x_i^{(k)}$ for $k = 1, \ldots, 10$.

Portfolio optimization with CCD and ADMM algorithms

Table 13: CCD coordinates ($k = 1, \ldots, 5$)

k	i	$\alpha_i^{(k)}$	$\beta_i^{(k)}$	$\gamma_i^{(k)}$	$x_i^{(k)}$	CCD coordinates		
						x_1	x_2	x_3
0						0.33333	0.33333	0.33333
1	1	0.04000	0.02033	−0.33333	2.64375	2.64375	0.33333	0.33333
1	2	0.06250	0.08359	−0.33333	1.73553	2.64375	1.73553	0.33333
1	3	0.09000	0.18629	−0.33333	1.15019	2.64375	1.73553	1.15019
2	1	0.04000	0.08480	−0.33333	2.01525	2.01525	1.73553	1.15019
2	2	0.06250	0.11077	−0.33333	1.58744	2.01525	1.58744	1.15019
2	3	0.09000	0.15589	−0.33333	1.24434	2.01525	1.58744	1.24434
3	1	0.04000	0.08448	−0.33333	2.01782	2.01782	1.58744	1.24434
3	2	0.06250	0.11577	−0.33333	1.56202	2.01782	1.56202	1.24434
3	3	0.09000	0.15465	−0.33333	1.24842	2.01782	1.56202	1.24842
4	1	0.04000	0.08399	−0.33333	2.02183	2.02183	1.56202	1.24842
4	2	0.06250	0.11609	−0.33333	1.56044	2.02183	1.56044	1.24842
4	3	0.09000	0.15471	−0.33333	1.24821	2.02183	1.56044	1.24821
5	1	0.04000	0.08395	−0.33333	2.02222	2.02222	1.56044	1.24821
5	2	0.06250	0.11609	−0.33333	1.56044	2.02222	1.56044	1.24821
5	3	0.09000	0.15472	−0.33333	1.24817	2.02222	1.56044	1.24817

Portfolio optimization with CCD and ADMM algorithms

Table 14: CCD coordinates ($k = 6, \ldots, 10$)

k	i	$\alpha_i^{(k)}$	$\beta_i^{(k)}$	$\gamma_i^{(k)}$	$x_i^{(k)}$	CCD coordinates		
						x_1	x_2	x_3
0						0.33333	0.33333	0.33333
6	1	0.04000	0.08395	−0.33333	2.02223	2.02223	1.56044	1.24817
6	2	0.06250	0.11608	−0.33333	1.56045	2.02223	1.56045	1.24817
6	3	0.09000	0.15472	−0.33333	1.24816	2.02223	1.56045	1.24816
7	1	0.04000	0.08395	−0.33333	2.02223	2.02223	1.56045	1.24816
7	2	0.06250	0.11608	−0.33333	1.56046	2.02223	1.56046	1.24816
7	3	0.09000	0.15472	−0.33333	1.24816	2.02223	1.56046	1.24816
8	1	0.04000	0.08395	−0.33333	2.02223	2.02223	1.56046	1.24816
8	2	0.06250	0.11608	−0.33333	1.56046	2.02223	1.56046	1.24816
8	3	0.09000	0.15472	−0.33333	1.24816	2.02223	1.56046	1.24816
9	1	0.04000	0.08395	−0.33333	2.02223	2.02223	1.56046	1.24816
9	2	0.06250	0.11608	−0.33333	1.56046	2.02223	1.56046	1.24816
9	3	0.09000	0.15472	−0.33333	1.24816	2.02223	1.56046	1.24816
10	1	0.04000	0.08395	−0.33333	2.02223	2.02223	1.56046	1.24816
10	2	0.06250	0.11608	−0.33333	1.56046	2.02223	1.56046	1.24816
10	3	0.09000	0.15472	−0.33333	1.24816	2.02223	1.56046	1.24816

Portfolio optimization with CCD and ADMM algorithms

Question 1.d.v

Deduce the ERC portfolio.

Portfolio optimization with CCD and ADMM algorithms

The CCD algorithm has converged to the following solution:

$$x^{\star} = \begin{pmatrix} 2.02223 \\ 1.56046 \\ 1.24816 \end{pmatrix}$$

Since $\sum_{i=1}^{3} x_i^{\star} = 4.83085$, we deduce that:

$$x_{\mathrm{erc}} = \frac{1}{4.83085} \begin{pmatrix} 2.02223 \\ 1.56046 \\ 1.24816 \end{pmatrix} = \begin{pmatrix} 41.86076\% \\ 32.30189\% \\ 25.83736\% \end{pmatrix}$$

Portfolio optimization with CCD and ADMM algorithms

Question 1.d.vi

Compute the variance of the previous CCD solution. What do you notice? Explain this result.

Portfolio optimization with CCD and ADMM algorithms

We remind that the CCD solution is:

$$x^{\star} = \begin{pmatrix} 2.02223 \\ 1.56046 \\ 1.24816 \end{pmatrix}$$

We have:

$$\sigma^2 \left(x^{\star} \right) = x^{\star \top} \Sigma x^{\star} = 1$$

We notice that:

$$\sigma^2 \left(x^{\star} \right) = \lambda$$

Portfolio optimization with CCD and ADMM algorithms

At the optimum, we remind that:

$$\lambda = \frac{x_i^\star \cdot (\Sigma x^\star)_i}{b_i} = \frac{x_i^\star \cdot (\Sigma x^\star)_i}{n^{-1}}$$

We deduce that:

$$\begin{aligned}
\lambda &= \frac{1}{n} \sum_{i=1}^{n} \frac{x_i^\star \cdot (\Sigma x^\star)_i}{n^{-1}} \\
&= \sum_{i=1}^{n} x_i^\star \cdot (\Sigma x^\star)_i \\
&= x^{\star\top} \Sigma x^\star \\
&= \sigma^2 (x^\star)
\end{aligned}$$

It follows that the portfolio variance of the CCD solution is exactly equal to λ.

Portfolio optimization with CCD and ADMM algorithms

Question 1.d.vii

Verify that the CCD solution converges faster to the ERC portfolio when we assume that $\lambda = x_{\mathrm{erc}}^{\top} \Sigma x_{\mathrm{erc}}$.

Portfolio optimization with CCD and ADMM algorithms

We have:

$$\sigma\left(x_{\mathrm{erc}}\right) = \sqrt{x_{\mathrm{erc}}^{\top} \Sigma x_{\mathrm{erc}}} = 20.70029\%$$

and:

$$\sigma^{2}\left(x_{\mathrm{erc}}\right) = 4.28502\%$$

We obtain the results given in Table 15 when $\lambda = 4.28502\%$. If we compare with those given in Tables 13 and 14, it is obvious that the convergence is faster in the present case.

Portfolio optimization with CCD and ADMM algorithms

Table 15: CCD coordinates ($k = 1, \ldots, 5$)

k	i	$\alpha_i^{(k)}$	$\beta_i^{(k)}$	$\gamma_i^{(k)}$	$x_i^{(k)}$	CCD coordinates		
						x_1	x_2	x_3
0						0.33333	0.33333	0.33333
1	1	0.04000	0.02033	−0.01428	0.39521	0.39521	0.33333	0.33333
1	2	0.06250	0.02738	−0.01428	0.30680	0.39521	0.30680	0.33333
1	3	0.09000	0.03033	−0.01428	0.26403	0.39521	0.30680	0.26403
2	1	0.04000	0.01718	−0.01428	0.42027	0.42027	0.30680	0.26403
2	2	0.06250	0.02437	−0.01428	0.32133	0.42027	0.32133	0.26403
2	3	0.09000	0.03200	−0.01428	0.25847	0.42027	0.32133	0.25847
3	1	0.04000	0.01734	−0.01428	0.41893	0.41893	0.32133	0.25847
3	2	0.06250	0.02404	−0.01428	0.32295	0.41893	0.32295	0.25847
3	3	0.09000	0.03204	−0.01428	0.25835	0.41893	0.32295	0.25835
4	1	0.04000	0.01737	−0.01428	0.41863	0.41863	0.32295	0.25835
4	2	0.06250	0.02403	−0.01428	0.32302	0.41863	0.32302	0.25835
4	3	0.09000	0.03203	−0.01428	0.25837	0.41863	0.32302	0.25837
5	1	0.04000	0.01738	−0.01428	0.41861	0.41861	0.32302	0.25837
5	2	0.06250	0.02403	−0.01428	0.32302	0.41861	0.32302	0.25837
5	3	0.09000	0.03203	−0.01428	0.25837	0.41861	0.32302	0.25837

Portfolio optimization with CCD and ADMM algorithms

Question 2

We recall that the ADMM algorithm is based on the following optimization problem:

$$\begin{aligned} \{x^\star, y^\star\} &= \arg\min f_x(x) + f_y(y) \\ \text{s.t.} &\quad Ax + By = c \end{aligned}$$

Portfolio optimization with CCD and ADMM algorithms

Question 2.a

Describe the ADMM algorithm.

Portfolio optimization with CCD and ADMM algorithms

The ADMM algorithm consists in the following iterations:

$$\left\{ \begin{array}{l} x^{(k+1)} = \arg\min_x \left\{ f_x(x) + \dfrac{\varphi}{2} \left\| Ax + By^{(k)} - c + u^{(k)} \right\|_2^2 \right\} \\ y^{(k+1)} = \arg\min_y \left\{ f_y(y) + \dfrac{\varphi}{2} \left\| Ax^{(k+1)} + By - c + u^{(k)} \right\|_2^2 \right\} \\ u^{(k+1)} = u^{(k)} + \left(Ax^{(k+1)} + By^{(k+1)} - c \right) \end{array} \right.$$

Portfolio optimization with CCD and ADMM algorithms

Question 2.b

We consider the following optimization problem:

$$\begin{aligned} w^{\star}(\gamma) &= \arg\min \frac{1}{2}(w-b)^{\top}\Sigma(w-b)-\gamma(w-b)^{\top}\mu \\ & \text{s.t.} \quad \left\{\begin{array}{l} \mathbf{1}_n^{\top} w = 1 \\ \sum_{i=1}^{n}\left|w_i-b_i\right| \leq \boldsymbol{\tau}^{+} \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array}\right. \end{aligned}$$

Portfolio optimization with CCD and ADMM algorithms

Question 2.b.i

Give the meaning of the symbols w, b, Σ, and μ. What is the goal of this optimization program? What is the meaning of the constraint $\sum_{i=1}^{n} |w_i - b_i| \leq \boldsymbol{\tau}^+$?

Portfolio optimization with CCD and ADMM algorithms

- w is the vector of portfolio weights:

$$w = (w_1, \ldots, w_n)$$

- b is the vector of benchmark weights:

$$b = (b_1, \ldots, b_n)$$

- Σ is the covariance matrix of asset returns
- μ is the vector of expected returns

Portfolio optimization with CCD and ADMM algorithms

The goal of the optimization problem is to tilt a benchmark portfolio by controlling the volatility of the tracking error:

$$\sigma(w \mid b) = \sqrt{(w - b)^\top \Sigma (w - b)}$$

and improving the expected excess return:

$$\mu(w \mid b) = (w - b)^\top \mu$$

This is a typical γ-problem when there is a benchmark

Portfolio optimization with CCD and ADMM algorithms

We remind that the turnover between the benchmark b and the portfolio w is equal to:

$$\boldsymbol{\tau}(w \mid b) = \sum_{i=1}^{n} |w_i - b_i|$$

Therefore, we impose that the turnover is less than an upper limit:

$$\boldsymbol{\tau}(w \mid b) \leq \boldsymbol{\tau}^{+}$$

Portfolio optimization with CCD and ADMM algorithms

Question 2.b.ii

What is the best way to specify $f_x(x)$ and $f_y(y)$ in order to find numerically the solution. Justify your choice.

Portfolio optimization with CCD and ADMM algorithms

The best way to specify $f_x(x)$ and $f_y(y)$ is to split the QP problem and the turnover constraint:

$$\begin{aligned} \{x^\star, y^\star\} &= \arg\min_{x,y} f_x(x) + f_y(y) \\ \text{s.t.} &\quad x - y = \mathbf{0}_n \end{aligned}$$

where:

$$\begin{aligned} f_x(x) &= \frac{1}{2}(x-b)^\top \Sigma (x-b) - \gamma (x-b)^\top \mu + \mathbb{1}_{\Omega_1}(x) + \mathbb{1}_{\Omega_3}(x) \\ f_y(y) &= \mathbb{1}_{\Omega_2}(y) \\ \Omega_1(x) &= \left\{x : \mathbf{1}_n^\top x = 1\right\} \\ \Omega_2(y) &= \left\{y : \sum_{i=1}^n |y_i - b_i| \leq \boldsymbol{\tau}^+\right\} \\ \Omega_3(x) &= \{x : \mathbf{0}_n \leq x \leq \mathbf{1}_n\} \end{aligned}$$

Indeed, the x-update step is a standard QP problem whereas the y-update step is the projection onto the ℓ_1-ball $\mathcal{B}_1(b, \boldsymbol{\tau}^+)$.

Portfolio optimization with CCD and ADMM algorithms

Question 2.b.iii

Give the corresponding ADMM algorithm.

Portfolio optimization with CCD and ADMM algorithms

We have:

$$\begin{aligned}(*) &= \frac{1}{2}(x-b)^\top \Sigma (x-b) - \gamma (x-b)^\top \mu \\ &= \frac{1}{2}x^\top \Sigma x - x^\top \Sigma b + \frac{1}{2}b^\top \Sigma b - \gamma x^\top \mu + \gamma b^\top \mu \\ &= \frac{1}{2}x^\top \Sigma x - x^\top (\Sigma b + \gamma \mu) + \underbrace{\left(\gamma b^\top \mu + \frac{1}{2}b^\top \Sigma b\right)}_{\text{constant}}\end{aligned}$$

Portfolio optimization with CCD and ADMM algorithms

If we note $v_x^{(k+1)} = y^{(k)} - u^{(k)}$, we have:

$$\begin{aligned}
\left\| x - y^{(k)} + u^{(k)} \right\|_2^2 &= \left\| x - v_x^{(k+1)} \right\|_2^2 \\
&= \left(x - v_x^{(k+1)} \right)^\top \left(x - v_x^{(k+1)} \right) \\
&= x^\top I_n x - 2x^\top v_x^{(k+1)} + \underbrace{\left(v_x^{(k+1)} \right)^\top v_x^{(k+1)}}_{\text{constant}}
\end{aligned}$$

Portfolio optimization with CCD and ADMM algorithms

It follows that:

$$\begin{aligned}
f_x^{(k+1)}(x) &= f_x(x) + \frac{\varphi}{2}\left\| x - y^{(k)} + u^{(k)} \right\|_2^2 \\
&= \frac{1}{2}(x-b)^\top \Sigma (x-b) - \gamma (x-b)^\top \mu + \\
&\quad \mathbb{1}_{\Omega_1}(x) + \mathbb{1}_{\Omega_3}(x) + \frac{\varphi}{2}\left\| x - y^{(k)} + u^{(k)} \right\|_2^2 \\
&= \frac{1}{2}x^\top (\Sigma + \varphi I_n) x - x^\top \left(\Sigma b + \gamma\mu + \varphi v_x^{(k+1)}\right) + \\
&\quad \mathbb{1}_{\Omega_1}(x) + \mathbb{1}_{\Omega_3}(x) + \text{constant}
\end{aligned}$$

Portfolio optimization with CCD and ADMM algorithms

We have:

$$\begin{aligned} f_y^{(k+1)}(y) &= \mathbb{1}_{\Omega_2}(y) + \frac{\varphi}{2}\left\| x^{(k+1)} - y + u^{(k)} \right\|_2^2 \\ &= \mathbb{1}_{\Omega_2}(y) + \frac{\varphi}{2}\left\| y - v_y^{(k+1)} \right\|_2^2 \end{aligned}$$

where $v_y^{(k+1)} = x^{(k+1)} + u^{(k)}$. We deduce that:

$$\begin{aligned} y^{(k+1)} &= \arg\min_y f_y^{(k+1)}(y) \\ &= \mathcal{P}_{\Omega_2}\left(v_y^{(k+1)}\right) \end{aligned}$$

where:

$$\Omega_2 = \mathcal{B}_1\left(b, \tau^+\right)$$

Portfolio optimization with CCD and ADMM algorithms

We remind that:

$$\begin{aligned}
\mathcal{P}_{\mathcal{B}_1(c,\lambda)}(v) &= \mathcal{P}_{\mathcal{B}_1(\mathbf{0}_n,\lambda)}(v-c)+c \\
\mathcal{P}_{\mathcal{B}_1(\mathbf{0}_n,\lambda)}(v) &= v-\operatorname{sign}(v)\odot \mathbf{prox}_{\lambda \max x}(|v|) \\
\mathbf{prox}_{\lambda \max x}(v) &= \min(v,s^{\star})
\end{aligned}$$

where $s^{\star}$ is the solution of the following equation:

$$s^{\star}=\left\{s\in\mathbb{R}:\sum_{i=1}^{n}(v_i-s)_{+}=\lambda\right\}$$

Portfolio optimization with CCD and ADMM algorithms

We deduce that:

$$\begin{aligned}
\mathcal{P}_{\Omega_2}\left(v_y^{(k+1)}\right) &= \mathcal{P}_{\mathcal{B}_1(b,\boldsymbol{\tau}^+)}\left(v_y^{(k+1)}\right) \\
&= \mathcal{P}_{\mathcal{B}_1(\mathbf{0}_n,\boldsymbol{\tau}^+)}\left(v_y^{(k+1)} - b\right) + b \\
&= v_y^{(k+1)} - \operatorname{sign}\left(v_y^{(k+1)} - b\right) \odot \mathbf{prox}_{\boldsymbol{\tau}^+ \max x}\left(\left|v_y^{(k+1)} - b\right|\right) \\
&= v_y^{(k+1)} - \operatorname{sign}\left(v_y^{(k+1)} - b\right) \odot \min\left(\left|v_y^{(k+1)} - b\right|, s^\star\right)
\end{aligned}$$

where $s^\star$ is the solution of the following equation:

$$s^\star = \left\{ s \in \mathbb{R} : \sum_{i=1}^{n} \left(\left|v_{y,i}^{(k+1)} - b_i\right| - s\right)_+ = \boldsymbol{\tau}^+ \right\}$$

Portfolio optimization with CCD and ADMM algorithms

The ADMM algorithm becomes:

$$\left\{\begin{array}{l} v_x^{(k+1)} = y^{(k)} - u^{(k)} \\ Q^{(k+1)} = \Sigma + \varphi I_n \\ R^{(k+1)} = \Sigma b + \gamma\mu + \varphi v_x^{(k+1)} \\ x^{(k+1)} = \arg\min_x \left\{\frac{1}{2}x^\top Q^{(k+1)}x - x^\top R^{(k+1)} + \mathbb{1}_{\Omega_1}(x) + \mathbb{1}_{\Omega_3}(x)\right\} \\ v_y^{(k+1)} = x^{(k+1)} + u^{(k)} \\ s^\star = \left\{s \in \mathbb{R} : \sum_{i=1}^n \left(\left|v_{y,i}^{(k+1)} - b_i\right| - s\right)_+ = \tau^+\right\} \\ y^{(k+1)} = v_y^{(k+1)} - \operatorname{sign}\left(v_y^{(k+1)} - b\right) \odot \min\left(\left|v_y^{(k+1)} - b\right|, s^\star\right) \\ u^{(k+1)} = u^{(k)} + x^{(k+1)} - y^{(k+1)} \end{array}\right.$$

Portfolio optimization with CCD and ADMM algorithms

Question 2.c

We consider the following optimization problem:

$$
\begin{aligned}
w^{\star} & = \arg\min \left\| w - \tilde{w} \right\|_1 \\
\text{s.t.} & \quad \left\{
\begin{array}{l}
\mathbf{1}_n^\top w = 1 \\
\sqrt{\left(w - b\right)^\top \Sigma \left(w - b\right)} \leq \sigma^{+} \\
\mathbf{0}_n \leq w \leq \mathbf{1}_n
\end{array}
\right.
\end{aligned}
$$

Portfolio optimization with CCD and ADMM algorithms

Question 2.c.i

What is the meaning of the objective function $\left\| w - \tilde{w} \right\|_1$? What is the meaning of the constraint $\sqrt{(w - b)^\top \Sigma (w - b)} \leq \sigma^+$?

Portfolio optimization with CCD and ADMM algorithms

The objective function $\left\| w - \tilde{w} \right\|_1$ is the turnover between a given portfolio $\tilde{w}$ and the optimized portfolio w

The constraint $\sqrt{(w - b)^\top \Sigma (w - b)} \leq \sigma^+$ is a tracking error limit with respect to a benchmark b

Portfolio optimization with CCD and ADMM algorithms

Question 2.c.ii

Propose an equivalent optimization problem such that $f_x(x)$ is a QP problem. How to solve the y-update?

Portfolio optimization with CCD and ADMM algorithms

The optimization problem is equivalent to solve the following program:

$$\begin{aligned} w^{\star} &= \arg\min \frac{1}{2} (w - b)^{\top} \Sigma (w - b) + \lambda \| w - \tilde{w} \|_1 \\ \text{s.t.} & \left\{ \begin{array}{l} \mathbf{1}_n^{\top} w = 1 \\ \mathbf{0}_n \leq w \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

Portfolio optimization with CCD and ADMM algorithms

We deduce that:

$$f_x(x) = \frac{1}{2}(x-b)^\top \Sigma (x-b) + \mathbb{1}_{\Omega_1}(x) + \mathbb{1}_{\Omega_2}(x)$$

where:

$$\Omega_1(x) = \left\{x : \mathbf{1}_n^\top x = 1\right\}$$

and:

$$\Omega_2(x) = \left\{x : \mathbf{0}_n \leq x \leq \mathbf{1}_n\right\}$$

Portfolio optimization with CCD and ADMM algorithms

We have:

$$f_y(y) = \lambda \|w - \tilde{w}\|_1$$

We remind that:

$$\mathbf{prox}_{\lambda\|x\|_1}(v) = \mathcal{S}(v;\lambda) = \mathrm{sign}(v) \odot (|v| - \lambda \mathbf{1}_n)_+$$

and:

$$\mathbf{prox}_{f(x+b)}(v) = \mathbf{prox}_f(v+b) - b$$

The y-update step is then equal to:

$$\begin{aligned} y^{(k+1)} &= \mathbf{prox}_{\lambda\|w-\tilde{w}\|_1}\left(x^{(k+1)} + u^{(k)}\right) \\ &= \tilde{w} + \mathrm{sign}\left(x^{(k+1)} + u^{(k)} - \tilde{w}\right) \odot \left(\left|x^{(k+1)} + u^{(k)} - \tilde{w}\right| - \lambda \mathbf{1}_n\right)_+ \end{aligned}$$

because $f_y(y)$ is fully separable[4]

[4]Otherwise the scaling property does not work!

Regularized portfolio optimization

Exercise

We consider an investment universe with 6 assets. We assume that their expected returns are 4%, 6%, 7%, 8%, 10% and 10%,, and their volatilities are 6%, 10%, 11%, 15%, 15% and 20%. The correlation matrix is given by:

$$\rho = \begin{pmatrix} 100\% & & & & & \\ 50\% & 100\% & & & & \\ 20\% & 20\% & 100\% & & & \\ 50\% & 50\% & 80\% & 100\% & & \\ 0\% & -20\% & -50\% & -30\% & 100\% & \\ 0\% & 20\% & 30\% & 0\% & 0\% & 100\% \end{pmatrix}$$

Regularized portfolio optimization

Question 1

We restrict the analysis to long-only portfolios meaning that $\sum_{i=1}^{n} x_i = 1$ and $x_i \geq 0$.

Regularized portfolio optimization

Question 1.a

We consider the Herfindahl index $\mathcal{H}(x) = \sum_{i=1}^{n} x_i^2$. What are the two limit cases of $\mathcal{H}(x)$? What is the interpretation of the statistic $\mathcal{N}(x) = \mathcal{H}^{-1}(x)$?

Regularized portfolio optimization

We consider the following optimization problem:

$$\begin{aligned} x^{\star} &= \arg\min \mathcal{H}(x) \\ &\text{s.t. } \sum_{i=1}^{n} x_i = 1 \end{aligned}$$

We deduce that the Lagrange function is:

$$\begin{aligned} \mathcal{L}(x;\lambda) &= \mathcal{H}(x) - \lambda \left(\sum_{i=1}^{n} x_i = 1 \right) \\ &= x^{\top} x - \lambda \left(\mathbf{1}_n^{\top} x - 1 \right) \end{aligned}$$

Regularized portfolio optimization

The first-order condition is:

$$\frac{\partial \mathcal{L}(x;\lambda)}{\partial x} = x - \lambda \mathbf{1}_n = \mathbf{0}_n$$

Since we have $\mathbf{1}_n^\top x - 1 = 0$, we deduce that:

$$\lambda = \frac{1}{\mathbf{1}_n^\top \mathbf{1}_n} = \frac{1}{n}$$

We conclude that the lower bound is reached for the equally-weighted portfolio:

$$x_{\mathrm{ew}} = \frac{1}{n} \cdot \mathbf{1}_n$$

and we have:

$$\mathcal{H}(x_{\mathrm{ew}}) = \frac{1}{n^2} \cdot \mathbf{1}_n^\top \mathbf{1}_n = \frac{1}{n}$$

Regularized portfolio optimization

Since the weights are positive, we have:

$$\begin{aligned}\mathcal{H}(x) &= \sum_{i=1}^{n} x_i^2 \\ &\leq \left(\sum_{i=1}^{n} x_i\right)^2 \\ &\leq 1\end{aligned}$$

The upper bound is reached when the portfolio is concentrated on one asset:

$$\exists i : x_i = 1$$

Regularized portfolio optimization

We conclude that:

$$\frac{1}{n} \leq \mathcal{H}(x) \leq 1$$

The statistic $\mathcal{N}(x) = \mathcal{H}^{-1}(x)$ is the effective number of assets

Regularized portfolio optimization

Question 1.b

We consider the following optimization problem $(\mathcal{P}_1)$:

$$
\begin{aligned}
x^{\star}(\lambda) & = \arg\min \frac{1}{2} x^{\top} \Sigma x + \lambda x^{\top} x \\
& \text{s.t.} \quad \left\{ \begin{array}{l} \sum_{i=1}^{n} x_i = 1 \\ x_i \geq 0 \end{array} \right.
\end{aligned}
$$

What is the link between this constrained optimization program and the weight diversification based on the Herfindahl index?

Regularized portfolio optimization

The optimization problem $(\mathcal{P}_1)$ is equivalent to:

$$\begin{aligned} x^\star\left(\mathcal{H}^+\right) &= \arg\min \frac{1}{2} x^\top \Sigma x \\ \text{s.t.} & \left\{ \begin{array}{l} \sum_{i=1}^n x_i = 1 \\ x_i \geq 0 \\ x^\top x \leq \mathcal{H}^+ \end{array} \right. \end{aligned}$$

We obtain a long-only minimum variance portfolio with a diversification constraint based on the Herfindahl index:

$$\mathcal{H}(x) \leq \mathcal{H}^+$$

We have the following correspondance:

$$\mathcal{H}^+ = \mathcal{H}\left(x^\star(\lambda)\right) = x^\star(\lambda)^\top x^\star(\lambda)$$

Given a value of λ, we can then compute the implicit constraint $\mathcal{H}(x) \leq \mathcal{H}^+$.

Regularized portfolio optimization

Question 1.c

Solve Program $(\mathcal{P}_1)$ when λ is equal to respectively 0, 0.001, 0.01, 0.05, 0.10 and 10. Compute the statistic $\mathcal{N}(x)$. Comment on these results.

Regularized portfolio optimization

Table 16: Solution of the optimization problem $(\mathcal{P}_1)$

λ	0.000	0.001	0.010	0.050	0.100	10.000
$x_1^\star(\lambda)$ (in %)	44.60	35.66	23.97	18.71	17.76	16.68
$x_2^\star(\lambda)$ (in %)	9.12	14.60	18.10	17.08	16.89	16.67
$x_3^\star(\lambda)$ (in %)	25.46	26.57	19.96	16.89	16.71	16.67
$x_4^\star(\lambda)$ (in %)	0.00	0.00	7.64	14.46	15.52	16.65
$x_5^\star(\lambda)$ (in %)	20.40	22.11	22.38	19.31	18.21	16.69
$x_6^\star(\lambda)$ (in %)	0.43	1.07	7.94	13.55	14.92	16.65
$\mathcal{H}(x^\star(\lambda))$	0.3137	0.2680	0.1923	0.1693	0.1675	0.1667
$\mathcal{N}(x^\star(\lambda))$	3.19	3.73	5.20	5.91	5.97	6.00

Regularized portfolio optimization

Question 1.d

Using the bisection algorithm, find the optimal value of $\lambda^{\star}$ that satisfies:

$$\mathcal{N}\left(x^{\star}\left(\lambda^{\star}\right)\right) = 4$$

Give the composition of $x^{\star}\left(\lambda^{\star}\right)$. What is the interpretation of $x^{\star}\left(\lambda^{\star}\right)$?

Regularized portfolio optimization

The optimal solution is:

$$\lambda^{\star} = 0.002301$$

The optimal weights (in %) are equal to:

$$x^{\star} = \begin{pmatrix} 31.62\% \\ 17.24\% \\ 26.18\% \\ 0.00\% \\ 22.63\% \\ 2.33\% \end{pmatrix}$$

The effective number of bets $\mathcal{N}\left(x^{\star}\right)$ is equal to 4

Regularized portfolio optimization

Question 2

We consider long/short portfolios and the following optimization problem $(\mathcal{P}_2)$:

$$\begin{aligned} x^{\star}(\lambda) &= \arg\min \frac{1}{2} x^{\top} \Sigma x + \lambda \sum_{i=1}^{n} |x_i| \\ \text{s.t.} & \sum_{i=1}^{n} x_i = 1 \end{aligned}$$

Regularized portfolio optimization

Question 2.a

Solve Program $(\mathcal{P}_2)$ when λ is equal to respectively 0, 0.0001, 0.001, 0.01, 0.05, 0.10 and 10. Comment on these results.

Regularized portfolio optimization

Table 17: Solution of the optimization problem $(\mathcal{P}_2)$

λ	0.000	0.0001	0.001	0.010	0.050	0.100	10.000
$x_1^\star(\lambda)$ (in %)	35.82	37.17	44.50	44.60	44.60	44.60	44.60
$x_2^\star(\lambda)$ (in %)	33.08	30.26	11.48	9.12	9.12	9.12	9.12
$x_3^\star(\lambda)$ (in %)	77.62	71.77	31.28	25.46	25.46	25.46	25.46
$x_4^\star(\lambda)$ (in %)	−53.48	−47.97	−7.16	0.00	0.00	0.00	0.00
$x_5^\star(\lambda)$ (in %)	20.83	20.56	19.90	20.40	20.40	20.40	20.40
$x_6^\star(\lambda)$ (in %)	−13.87	−11.78	0.00	0.43	0.43	0.43	0.43
$\mathcal{L}(x)$ (in %)	234.69	219.50	114.33	100.00	100.00	100.00	100.00

Regularized portfolio optimization

Question 2.b

For each optimized portfolio, calculate the following statistic:

$$\mathcal{L}(x) = \sum_{i=1}^{n} |x_i|$$

What is the interpretation of $\mathcal{L}(x)$? What is the impact of Lasso regularization?

Regularized portfolio optimization

$\mathcal{L}(x) = \sum_{i=1}^{n} |x_i|$ is the leverage ratio. Their values are reported in Table 17.

Regularized portfolio optimization

Question 3

We assume that the investor holds an initial portfolio $x^{(0)}$ defined as follows:

$$x^{(0)} = \begin{pmatrix} 10\% \\ 15\% \\ 20\% \\ 25\% \\ 30\% \\ 0\% \end{pmatrix}$$

We consider the optimization problem $(\mathcal{P}_3)$:

$$\begin{aligned} x^{\star}(\lambda) &= \arg\min \frac{1}{2} x^{\top} \Sigma x + \lambda \sum_{i=1}^{n} \left| x_i - x_i^{(0)} \right| \\ & \text{s.t.} \quad \sum_{i=1}^{n} x_i = 1 \end{aligned}$$

Regularized portfolio optimization

Question 3.a

Solve Program $(\mathcal{P}_3)$ when λ is equal respectively to 0, 0.0001, 0.001, 0.0015 and 0.01. Compute the turnover of each optimized portfolio. Comment on these results.

Regularized portfolio optimization

Table 18: Solution of the optimization problem $(\mathcal{P}_3)$

λ	0.000	0.000	0.001	0.002	0.010
$x_1^\star(\lambda)$ (in %)	35.82	35.55	27.90	24.28	10.00
$x_2^\star(\lambda)$ (in %)	33.08	30.61	15.00	15.00	15.00
$x_3^\star(\lambda)$ (in %)	77.62	72.35	33.36	22.86	20.00
$x_4^\star(\lambda)$ (in %)	−53.48	−48.00	−5.20	7.87	25.00
$x_5^\star(\lambda)$ (in %)	20.83	21.51	28.94	30.00	30.00
$x_6^\star(\lambda)$ (in %)	−13.87	−12.02	0.00	0.00	0.00
$\tau\left(x^\star(\lambda) \mid x^{(0)}\right)$ (in %)	203.04	187.02	62.51	34.27	0.00

Regularized portfolio optimization

Question 3.b

Using the bisection algorithm, find the optimal value of $\lambda^{\star}$ such that the two-way turnover is equal to 60%. Give the composition of $x^{\star}(\lambda^{\star})$.

Regularized portfolio optimization

The optimal solution is:

$$\lambda^{\star} = 0.00103$$

The optimal weights (in %) are equal to:

$$x^{\star} = \begin{pmatrix} 27.23\% \\ 15.00\% \\ 32.77\% \\ -4.30\% \\ 29.30\% \\ 0.00\% \end{pmatrix}$$

The turnover $\tau\left(x^{\star} \mid x^{(0)}\right)$ is equal to 60%

Regularized portfolio optimization

Question 3.c

Same question when the two-way turnover is equal to 50%.

Regularized portfolio optimization

The optimal solution is:

$$\lambda^{\star} = 0.00119$$

The optimal weights (in %) are equal to:

$$x^{\star} = \begin{pmatrix} 25.53\% \\ 15.00\% \\ 29.47\% \\ 0.00\% \\ 30.00\% \\ 0.00\% \end{pmatrix}$$

The turnover $\tau\left(x^{\star} \mid x^{(0)}\right)$ is equal to 50%

Regularized portfolio optimization

Question 3.d

What becomes the portfolio $x^{\star}(\lambda)$ when $\lambda \rightarrow \infty$? How do you explain this result?

Regularized portfolio optimization

We notice that:

$$\lim_{\lambda \to \infty} x^{\star}(\lambda) = x^{(0)}$$

This is normal since we have:

$$\begin{aligned} x^{\star}(\lambda) &= \arg\min \frac{1}{2} x^{\top} \Sigma x + \lambda \sum_{i=1}^{n} \left| x_i - x_i^{(0)} \right| \\ & \text{s.t.} \quad \sum_{i=1}^{n} x_i = 1 \end{aligned}$$

We deduce that:

$$\begin{aligned} x^{\star}(\infty) &= \arg\min \sum_{i=1}^{n} \left| x_i - x_i^{(0)} \right| \\ & \text{s.t.} \quad \sum_{i=1}^{n} x_i = 1 \end{aligned}$$

The solution is $x^{\star}(\infty) = x^{(0)}$

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