Asset Management
Lecture 1. Portfolio Optimization

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General information

1. **Overview**
   The objective of this course is to understand the theoretical and practical aspects of asset management
2. **Prerequisites**
   M1 Finance or equivalent
3. **ECTS**
   3
4. **Keywords**
   Finance, Asset Management, Optimization, Statistics
5. **Hours**
   Lectures: 24h, HomeWork: 30h
6. **Evaluation**
   Project + oral examination
7. **Course website**
   `http://www.thierry-roncalli.com/RiskBasedAM.html`

Objective of the course

The objective of the course is twofold:

1. having a financial culture on asset management
2. being proficient in quantitative portfolio management

Class schedule

Course sessions

- January 8 (6 hours, AM+PM)
- January 15 (6 hours, AM+PM)
- January 22 (6 hours, AM+PM)
- January 29 (6 hours, AM+PM)

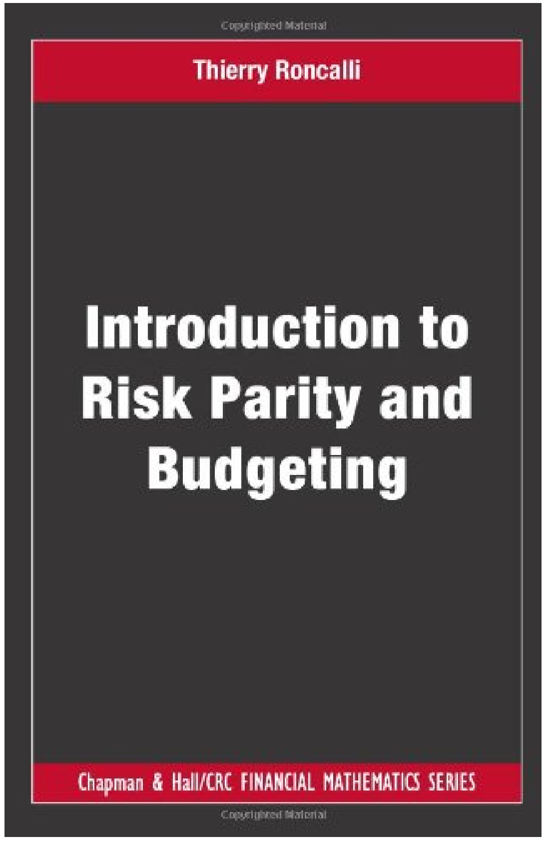
Class times: Fridays 9:00am-12:00pm, 1:00pm–4:00pm, University of Evry

Agenda

- Lecture 1: Portfolio Optimization
- Lecture 2: Risk Budgeting
- Lecture 3: Smart Beta, Factor Investing and Alternative Risk Premia
- Lecture 4: Green and Sustainable Finance, ESG Investing and Climate Risk
- Lecture 5: Machine Learning in Asset Management

Textbook

- Roncalli, T. (2013), *Introduction to Risk Parity and Budgeting*, Chapman & Hall/CRC Financial Mathematics Series.



Additional materials

- Slides, tutorial exercises and past exams can be downloaded at the following address:

  `http://www.thierry-roncalli.com/RiskBasedAM.html`

- Solutions of exercises can be found in the companion book, which can be downloaded at the following address:

  `http://www.thierry-roncalli.com/RiskParityBook.html`

Agenda

- **Lecture 1: Portfolio Optimization**
- Lecture 2: Risk Budgeting
- Lecture 3: Smart Beta, Factor Investing and Alternative Risk Premia
- Lecture 4: Green and Sustainable Finance, ESG Investing and Climate Risk
- Lecture 5: Machine Learning in Asset Management

Notations

- We consider a universe of n assets
- $x = (x_1, \ldots, x_n)$ is the vector of weights in the portfolio
- The portfolio is fully invested:

$$\sum_{i=1}^{n} x_i = \mathbf{1}_n^\top x = 1$$

- $R = (R_1, \ldots, R_n)$ is the vector of asset returns where R_i is the return of asset i
- The return of the portfolio is equal to:

$$R(x) = \sum_{i=1}^{n} x_i R_i = x^\top R$$

- $\mu = \mathbb{E}[R]$ and $\Sigma = \mathbb{E}\left[(R - \mu)(R - \mu)^\top\right]$ are the vector of expected returns and the covariance matrix of asset returns

Computation of the first two moments

The expected return of the portfolio is:

$$\mu(x) = \mathbb{E}[R(x)] = \mathbb{E}\left[x^\top R\right] = x^\top \mathbb{E}[R] = x^\top \mu$$

whereas its variance is equal to:

$$\begin{aligned}
\sigma^2(x) &= \mathbb{E}\left[(R(x) - \mu(x))(R(x) - \mu(x))^\top\right] \\
&= \mathbb{E}\left[\left(x^\top R - x^\top \mu\right)\left(x^\top R - x^\top \mu\right)^\top\right] \\
&= \mathbb{E}\left[x^\top (R - \mu)(R - \mu)^\top x\right] \\
&= x^\top \mathbb{E}\left[(R - \mu)(R - \mu)^\top\right] x \\
&= x^\top \Sigma x
\end{aligned}$$

Efficient frontier

Two equivalent optimization problems

1. Maximizing the expected return of the portfolio under a volatility constraint (σ-**problem**):

$$\max \mu(x) \quad \text{u.c.} \quad \sigma(x) \leq \sigma^{\star}$$

2. Or minimizing the volatility of the portfolio under a return constraint (μ-**problem**):

$$\min \sigma(x) \quad \text{u.c.} \quad \mu(x) \geq \mu^{\star}$$

Efficient frontier

Example 1

We consider four assets. Their expected returns are equal to 5%, 6%, 8% and 6% while their volatilities are equal to 15%, 20%, 25% and 30%. The correlation matrix of asset returns is given by the following matrix:

$$C = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.40 & 0.70 & 1.00 & \\ 0.50 & 0.40 & 0.80 & 1.00 \end{pmatrix}$$

Efficient frontier

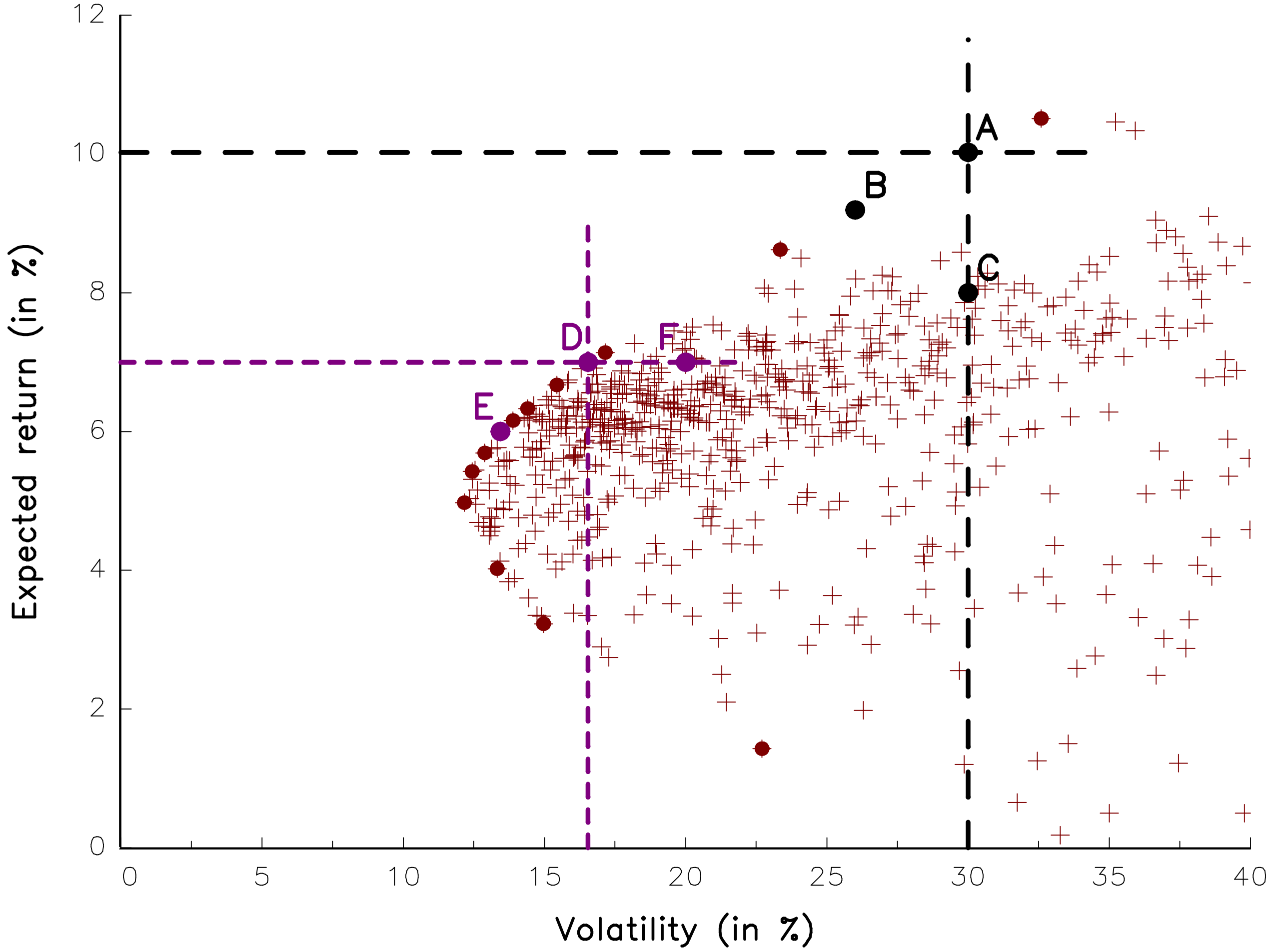


Figure 1: Optimized Markowitz portfolios (1 000 simulations)

Markowitz trick

Markowitz transforms the two original non-linear optimization problems into a quadratic optimization problem:

$$\begin{aligned} x^{\star}(\phi) &= \arg\max x^{\top}\mu - \frac{\phi}{2}x^{\top}\Sigma x \\ &\text{u.c.} \quad \mathbf{1}_n^{\top}x = 1 \end{aligned}$$

where ϕ is a risk-aversion parameter:

- $\phi = 0 \Rightarrow$ we have $\mu(x^{\star}(0)) = \mu^{+}$
- If $\phi = \infty$, the optimization problem becomes:

$$\begin{aligned} x^{\star}(\infty) &= \arg\min \frac{1}{2}x^{\top}\Sigma x \\ &\text{u.c.} \quad \mathbf{1}_n^{\top}x = 1 \end{aligned}$$

$\Rightarrow$ we have $\sigma(x^{\star}(\infty)) = \sigma^{-}$. This is the minimum variance (or MV) portfolio

The γ-problem

The previous problem can also be written as follows:

$$\begin{aligned} x^\star(\gamma) &= \arg\min \frac{1}{2} x^\top \Sigma x - \gamma x^\top \mu \\ \text{u.c.} & \quad \mathbf{1}_n^\top x = 1 \end{aligned}$$

with $\gamma = \phi^{-1}$

$\Rightarrow$ This is a standard QP problem

- The minimum variance portfolio corresponds to $\gamma = 0$
- Generally, we use the γ-problem, not the ϕ-problem

Quadratic programming problem

Definition

This is an optimization problem with a quadratic objective function and linear inequality constraints:

$$\begin{array}{rcl} x^{\star} & = & \arg\min \frac{1}{2} x^{\top} Q x - x^{\top} R \\ & \text{u.c.} & Sx \leq T \end{array}$$

where x is a $n \times 1$ vector, Q is a $n \times n$ matrix and R is a $n \times 1$ vector

$\Rightarrow Sx \leq T$ allows specifying linear equality constraints $Ax = B$ ($Ax \geq B$ and $Ax \leq B$) or weight constraints $x^{-} \leq x \leq x^{+}$

Quadratic programming problem

Mathematical softwares consider the following formulation:

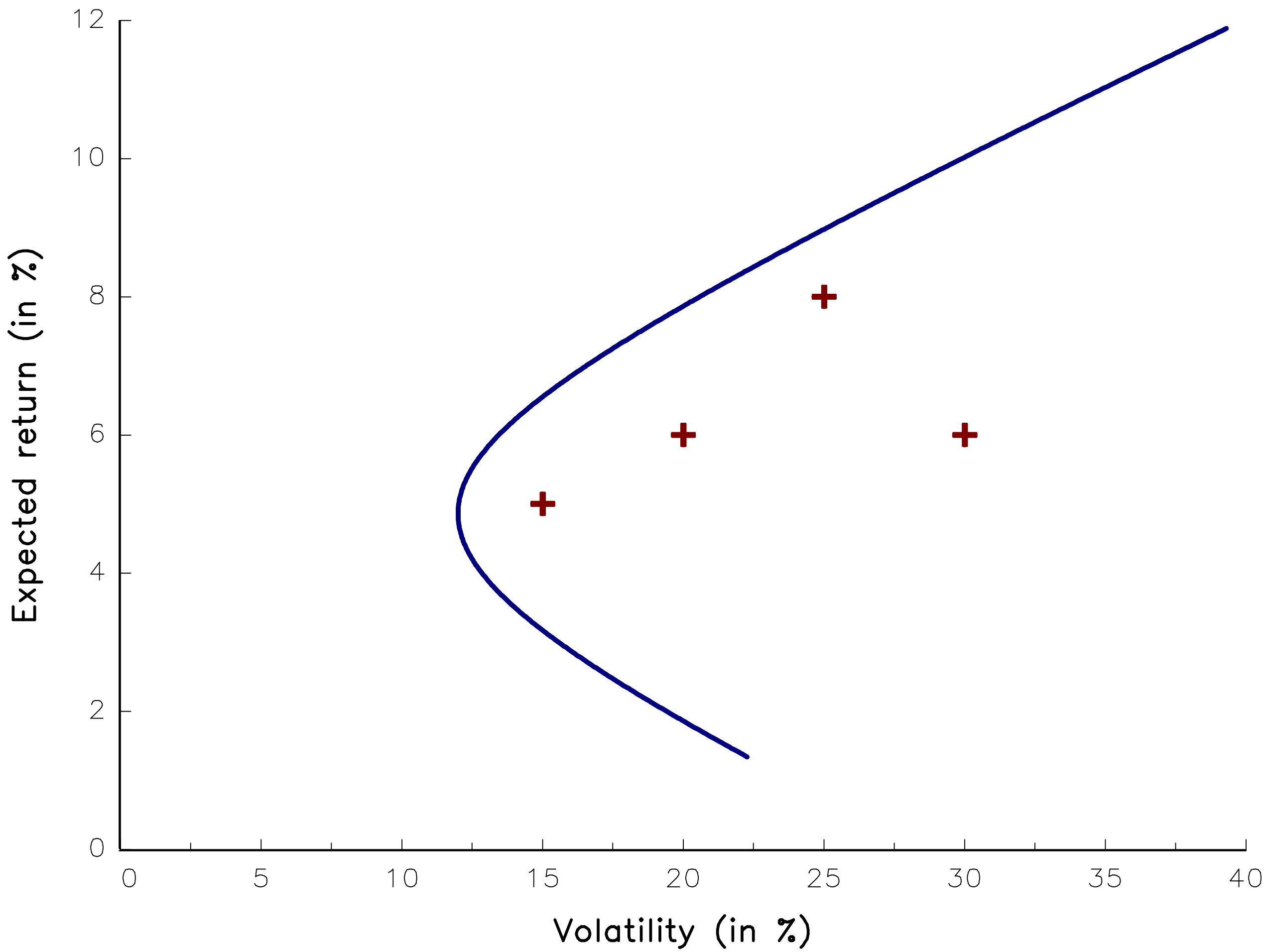
$$
\begin{aligned}
x^{\star} & = \arg\min \frac{1}{2} x^{\top} Q x - x^{\top} R \\
\text{u.c.} & \left\{
\begin{array}{l}
Ax = B \\
Cx \leq D \\
x^{-} \leq x \leq x^{+}
\end{array}
\right.
\end{aligned}
$$

because:

$$
Sx \leq T \Leftrightarrow \left[\begin{array}{c} -A \\ A \\ C \\ -I_n \\ I_n \end{array} \right] x \leq \left[\begin{array}{c} -B \\ B \\ D \\ -x^{-} \\ x^{+} \end{array} \right]
$$

Efficient frontier

The efficient frontier is the parametric function $\left(\sigma\left(x^{\star}\left(\phi\right)\right), \mu\left(x^{\star}\left(\phi\right)\right)\right)$ with $\phi \in \mathbb{R}_{+}$



Optimized portfolios

Table 1: Solving the ϕ-problem

ϕ	$+\infty$	5.00	2.00	1.00	0.50	0.20
$x_1^\star$	72.74	68.48	62.09	51.44	30.15	−33.75
$x_2^\star$	49.46	35.35	14.17	−21.13	−91.72	−303.49
$x_3^\star$	−20.45	12.61	62.21	144.88	310.22	806.22
$x_4^\star$	−1.75	−16.44	−38.48	−75.20	−148.65	−368.99
$\mu(x^\star)$	4.86	5.57	6.62	8.38	11.90	22.46
$\sigma(x^\star)$	12.00	12.57	15.23	22.27	39.39	94.57

Solving μ- and σ-problems

This is equivalent to finding the optimal value of γ such that:

$$\mu\left(x^{\star}\left(\gamma\right)\right)=\mu^{\star}$$

or:

$$\sigma\left(x^{\star}\left(\gamma\right)\right)=\sigma^{\star}$$

We know that:

- the functions $\mu\left(x^{\star}\left(\gamma\right)\right)$ and $\sigma\left(x^{\star}\left(\gamma\right)\right)$ are increasing with respect to γ
- the functions $\mu\left(x^{\star}\left(\gamma\right)\right)$ and $\sigma\left(x^{\star}\left(\gamma\right)\right)$ are bounded:

$$\begin{array}{rcl}\mu^{-} & \leq & \mu\left(x^{\star}\left(\gamma\right)\right)\leq\mu^{+}\\ \sigma^{-} & \leq & \sigma\left(x^{\star}\left(\gamma\right)\right)\leq\sigma^{+}\end{array}$$

$\Rightarrow$ The optimal value of γ can then be easily computed using the bisection algorithm

Solving μ- and σ-problems

We want to solve $f(\gamma) = c$ where:

- $f(\gamma) = \mu(x^\star(\gamma))$ and $c = \mu^\star$
- or $f(\gamma) = \sigma(x^\star(\gamma))$ and $c = \sigma^\star$

Bisection algorithm

1. We assume that $\gamma^\star \in [\gamma_1, \gamma_2]$
2. If $\gamma_2 - \gamma_1 \leq \varepsilon$, then stop
3. We compute:
$$\bar{\gamma} = \frac{\gamma_1 + \gamma_2}{2}$$
and $f(\bar{\gamma})$
4. We update γ_1 and γ_2 as follows:
    1. If $f(\bar{\gamma}) < c$, then $\gamma^\star \in [\gamma_c, \gamma_2]$ and $\gamma_1 \leftarrow \gamma_c$
    2. If $f(\bar{\gamma}) > c$, then $\gamma^\star \in [\gamma_1, \gamma_c]$ and $\gamma_2 \leftarrow \gamma_c$
5. Go to Step 2

Solving μ- and σ-problems

Table 2: Solving the unconstrained μ-problem

$\mu^\star$	5.00	6.00	7.00	8.00	9.00
$x_1^\star$	71.92	65.87	59.81	53.76	47.71
$x_2^\star$	46.73	26.67	6.62	−13.44	−33.50
$x_3^\star$	−14.04	32.93	79.91	126.88	173.86
$x_4^\star$	−4.60	−25.47	−46.34	−67.20	−88.07
$\sigma(x^\star)$	12.02	13.44	16.54	20.58	25.10
ϕ	25.79	3.10	1.65	1.12	0.85

Table 3: Solving the unconstrained σ-problem

$\sigma^\star$	15.00	20.00	25.00	30.00	35.00
$x_1^\star$	62.52	54.57	47.84	41.53	35.42
$x_2^\star$	15.58	−10.75	−33.07	−54.00	−74.25
$x_3^\star$	58.92	120.58	172.85	221.88	269.31
$x_4^\star$	−37.01	−64.41	−87.62	−109.40	−130.48
$\mu(x^\star)$	6.55	7.87	8.98	10.02	11.03
ϕ	2.08	1.17	0.86	0.68	0.57

Adding some constraints

We have:

$$\begin{aligned} x^{\star}(\gamma) &= \arg\min \frac{1}{2} x^{\top}\Sigma x - \gamma x^{\top}\mu \\ &\text{u.c.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^{\top} x = 1 \\ x \in \Omega \end{array} \right. \end{aligned}$$

where $x \in \Omega$ corresponds to the set of restrictions

Two classical constraints:

- no short-selling restriction

$$x_i \geq 0$$

- upper bound

$$x_i \leq c$$

Adding some constraints

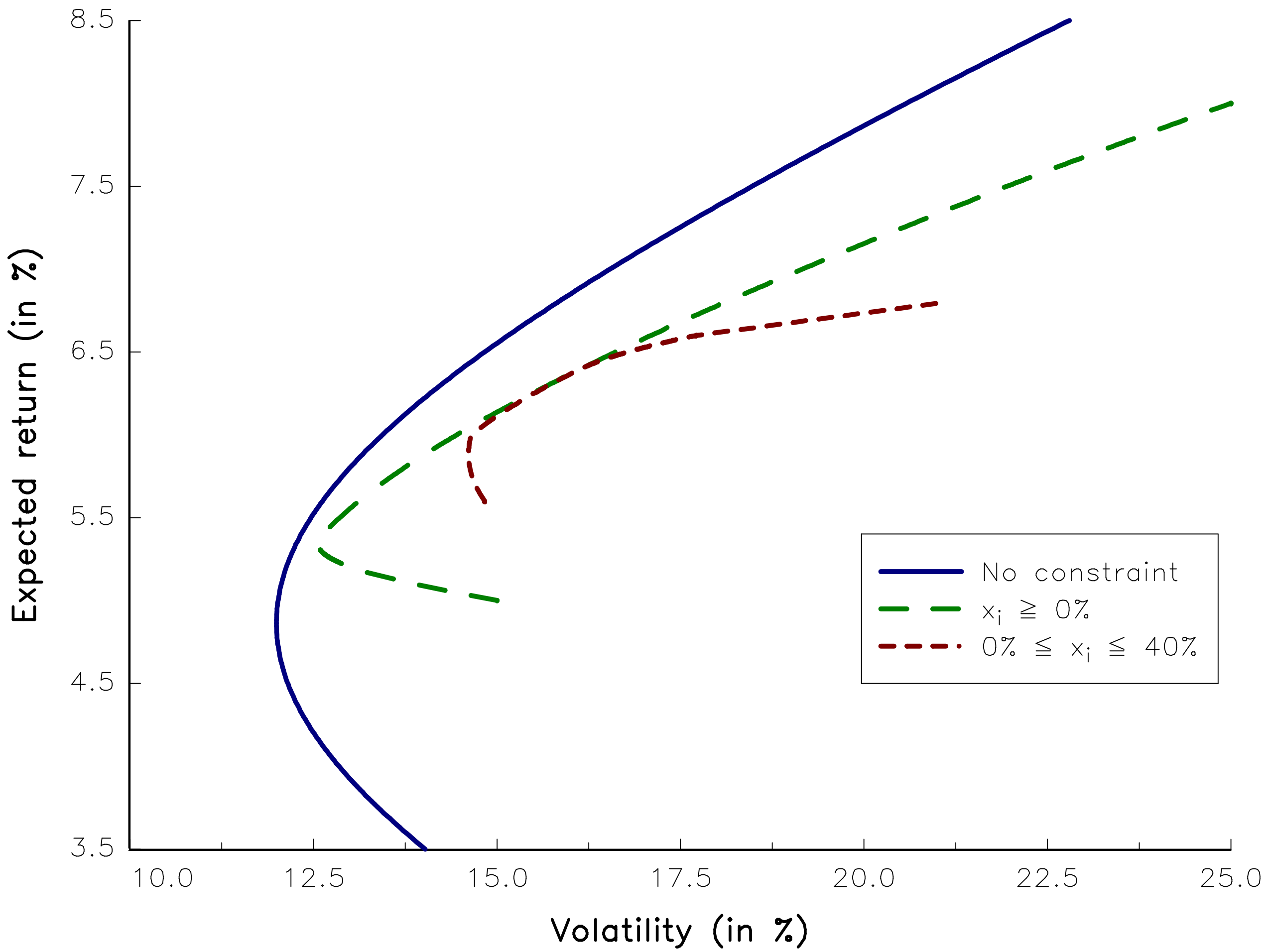


Figure 2: The efficient frontier with some weight constraints

Adding some constraints

Table 4: Solving the σ-problem with weight constraints

	$x_i \in \mathbb{R}$		$x_i \geq 0$		$0 \leq x_i \leq 40\%$	
$\sigma^\star$	15.00	20.00	15.00	20.00	15.00	20.00
$x_1^\star$	62.52	54.57	45.59	24.88	40.00	6.13
$x_2^\star$	15.58	−10.75	24.74	4.96	34.36	40.00
$x_3^\star$	58.92	120.58	29.67	70.15	25.64	40.00
$x_4^\star$	−37.01	−64.41	0.00	0.00	0.00	13.87
$\mu(x^\star)$	6.55	7.87	6.14	7.15	6.11	6.74
ϕ	2.08	1.17	1.61	0.91	1.97	0.28

Analytical solution

The Lagrange function is:

$$\mathcal{L}(x;\lambda_0) = x^\top \mu - \frac{\phi}{2} x^\top \Sigma x + \lambda_0 \left(\mathbf{1}_n^\top x - 1\right)$$

The first-order conditions are:

$$\left\{ \begin{array}{l} \partial_x \mathcal{L}(x;\lambda_0) = \mu - \phi \Sigma x + \lambda_0 \mathbf{1}_n = \mathbf{0}_n \\ \partial_{\lambda_0} \mathcal{L}(x;\lambda_0) = \mathbf{1}_n^\top x - 1 = 0 \end{array} \right.$$

We obtain:

$$x = \phi^{-1} \Sigma^{-1} \left(\mu + \lambda_0 \mathbf{1}_n\right)$$

Because $\mathbf{1}_n^\top x - 1 = 0$, we have:

$$\mathbf{1}_n^\top \phi^{-1} \Sigma^{-1} \mu + \lambda_0 \left(\mathbf{1}_n^\top \phi^{-1} \Sigma^{-1} \mathbf{1}_n\right) = 1$$

It follows that:

$$\lambda_0 = \frac{1 - \mathbf{1}_n^\top \phi^{-1} \Sigma^{-1} \mu}{\mathbf{1}_n^\top \phi^{-1} \Sigma^{-1} \mathbf{1}_n}$$

Analytical solution

The solution is then:

$$x^{\star}(\phi) = \frac{\Sigma^{-1}\mathbf{1}_n}{\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n} + \frac{1}{\phi}\cdot\frac{\left(\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n\right)\Sigma^{-1}\mu - \left(\mathbf{1}_n^{\top}\Sigma^{-1}\mu\right)\Sigma^{-1}\mathbf{1}_n}{\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n}$$

Remark

The global minimum variance portfolio is:

$$x_{\mathrm{mv}} = x^{\star}(\infty) = \frac{\Sigma^{-1}\mathbf{1}_n}{\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n}$$

Analytical solution

In the case of no short-selling, the Lagrange function becomes:

$$\mathcal{L}(x; \lambda_0, \lambda) = x^\top \mu - \frac{\phi}{2} x^\top \Sigma x + \lambda_0 \left(\mathbf{1}_n^\top x - 1 \right) + \lambda^\top x$$

where $\lambda = (\lambda_1, \ldots, \lambda_n) \geq \mathbf{0}_n$ is the vector of Lagrange coefficients associated with the constraints $x_i \geq 0$

- The first-order condition is:

$$\mu - \phi \Sigma x + \lambda_0 \mathbf{1} + \lambda = \mathbf{0}_n$$

- The Kuhn-Tucker conditions are:

$$\min(\lambda_i, x_i) = 0$$

The tangency portfolio

Markowitz

There are many optimized portfolios
$\Rightarrow$ there are many optimal portfolios

Tobin

One optimized portfolio dominates all the others if there is a risk-free asset

The tangency portfolio

We consider a combination of the risk-free asset and a portfolio x:

$$R(y) = (1 - \alpha) r + \alpha R(x)$$

where:

- r is the return of the risk-free asset
- $y = \begin{pmatrix} \alpha x \\ 1 - \alpha \end{pmatrix}$ is a vector of dimension $(n + 1)$
- $\alpha \geq 0$ is the proportion of the wealth invested in the risky portfolio

It follows that:

$$\mu(y) = (1 - \alpha) r + \alpha \mu(x) = r + \alpha (\mu(x) - r)$$

and:

$$\sigma^2(y) = \alpha^2 \sigma^2(x)$$

We deduce that:

$$\mu(y) = r + \frac{(\mu(x) - r)}{\sigma(x)} \sigma(y)$$

The tangency portfolio

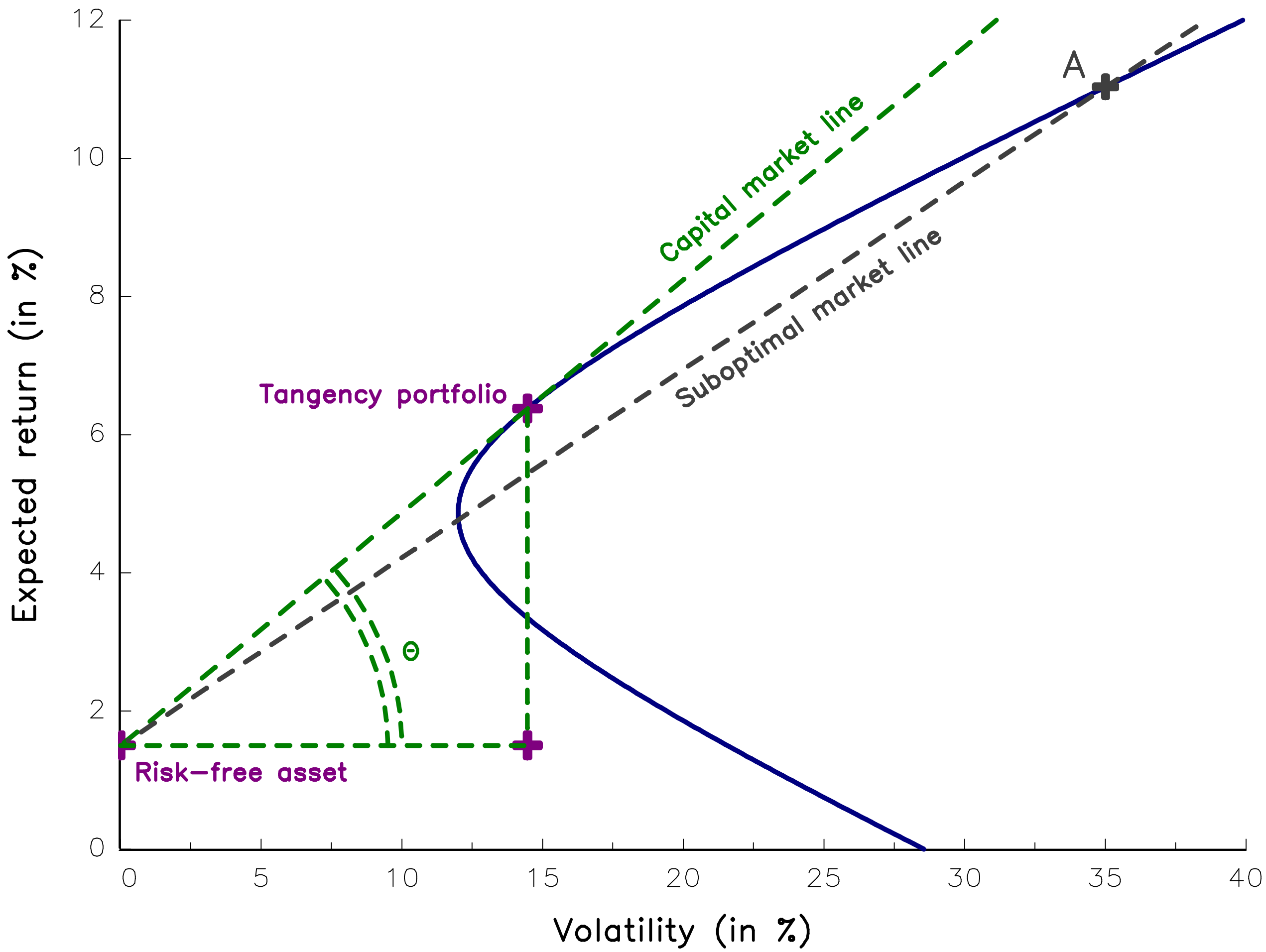


Figure 3: The capital market line ($r = 1.5\%$)

The tangency portfolio

Let $\mathrm{SR}(x \mid r)$ be the Sharpe ratio of portfolio x:

$$\mathrm{SR}(x \mid r) = \frac{\mu(x) - r}{\sigma(x)}$$

We obtain:

$$\frac{\mu(y) - r}{\sigma(y)} = \frac{\mu(x) - r}{\sigma(x)} \Leftrightarrow \mathrm{SR}(y \mid r) = \mathrm{SR}(x \mid r)$$

The tangency portfolio is the one that maximizes the angle θ or equivalently $\tan\theta$:

$$\tan\theta = \mathrm{SR}(x \mid r) = \frac{\mu(x) - r}{\sigma(x)}$$

The tangency portfolio is the risky portfolio corresponding to the maximum Sharpe ratio

The tangency portfolio

Example 2

We consider Example 1 and $r = 1.5\%$

The composition of the tangency portfolio $x^\star$ is:

$$x^\star = \begin{pmatrix} 63.63\% \\ 19.27\% \\ 50.28\% \\ -33.17\% \end{pmatrix}$$

We have:

$$\begin{aligned} \mu(x^\star) &= 6.37\% \\ \sigma(x^\star) &= 14.43\% \\ \mathrm{SR}(x^\star \mid r) &= 0.34 \\ \theta(x^\star) &= 18.64 \text{ degrees} \end{aligned}$$

The tangency portfolio

Let us consider a portfolio x of risky assets and a risk-free asset r. We denote by $\tilde{x}$ the augmented vector of dimension $n+1$ such that:

$$\tilde{x} = \begin{pmatrix} x \\ x_r \end{pmatrix} \quad \text{and} \quad \tilde{\Sigma} = \begin{pmatrix} \Sigma & \mathbf{0}_n \\ \mathbf{0}_n^\top & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\mu} = \begin{pmatrix} \mu \\ r \end{pmatrix}$$

If we include the risk-free asset, the Markowitz γ-problem becomes:

$$\begin{aligned} \tilde{x}^\star(\gamma) &= \arg\min \frac{1}{2}\tilde{x}^\top \tilde{\Sigma}\tilde{x} - \gamma \tilde{x}^\top \tilde{\mu} \\ \text{u.c.} \quad & \mathbf{1}_n^\top \tilde{x} = 1 \end{aligned}$$

Two-fund separation theorem

We can show that (RPB, pages 13-14):

$$\tilde{x}^\star = \underbrace{\alpha \cdot \begin{pmatrix} x_0^\star \\ 0 \end{pmatrix}}_{\text{risky assets}} + \underbrace{(1-\alpha) \cdot \begin{pmatrix} \mathbf{0}_n \\ 1 \end{pmatrix}}_{\text{risk-free asset}}$$

The tangency portfolio

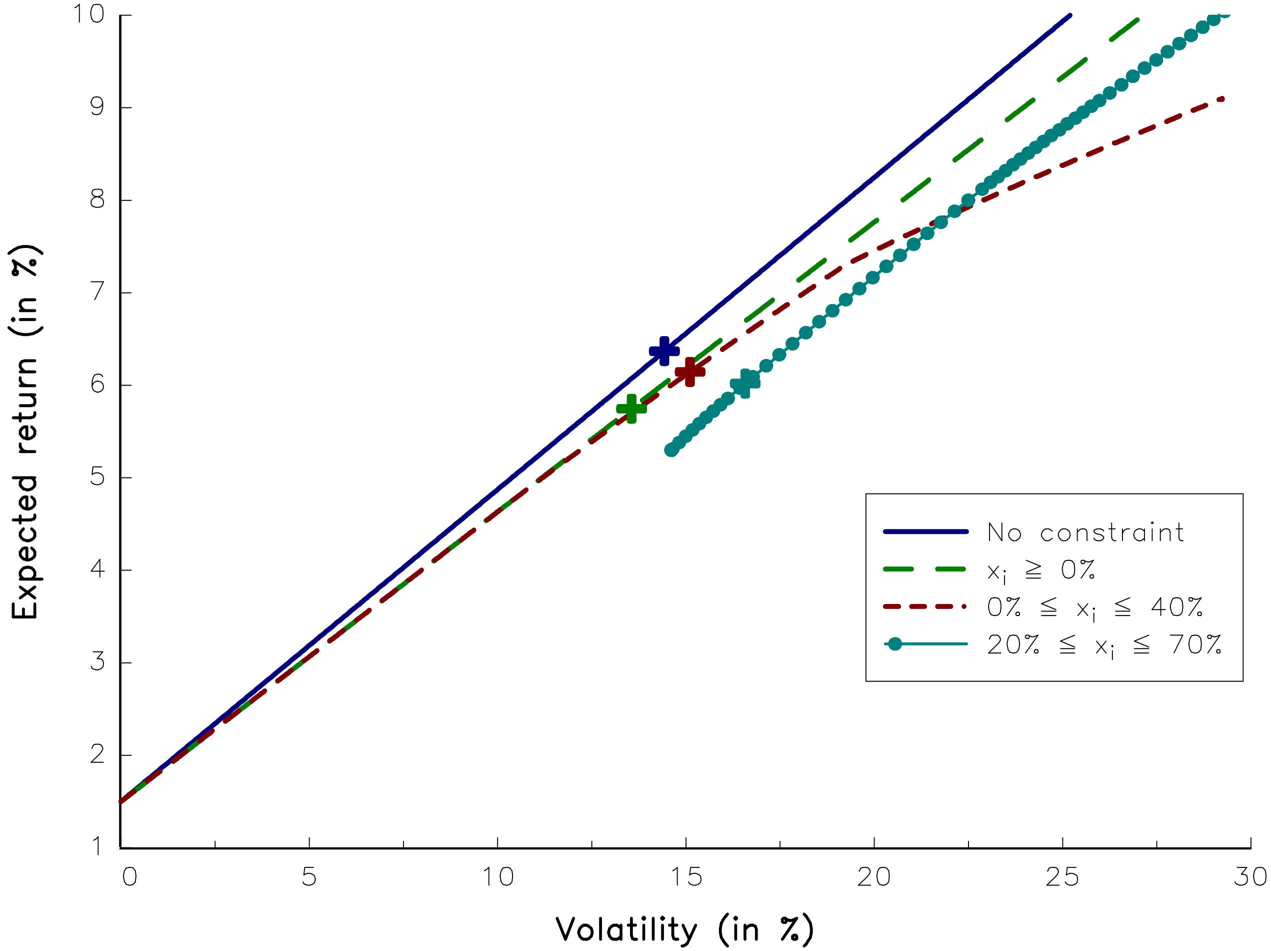


Figure 4: The efficient frontier with a risk-free asset

Market equilibrium and CAPM

- $x^{\star}$ is the tangency portfolio
- On the efficient frontier, we have:

$$\mu(y) = r + \frac{\sigma(y)}{\sigma(x^{\star})}\left(\mu(x^{\star}) - r\right)$$

- We consider a portfolio z with a proportion w invested in the asset i and a proportion $(1-w)$ invested in the tangency portfolio $x^{\star}$:

$$\begin{aligned}\mu(z) &= w\mu_i + (1-w)\,\mu(x^{\star}) \\ \sigma^2(z) &= w^2\sigma_i^2 + (1-w)^2\,\sigma^2(x^{\star}) + 2w(1-w)\,\rho(\mathbf{e}_i, x^{\star})\,\sigma_i\sigma(x^{\star})\end{aligned}$$

It follows that:

$$\frac{\partial\,\mu(z)}{\partial\,\sigma(z)} = \frac{\mu_i - \mu(x^{\star})}{\left(w\sigma_i^2 + (w-1)\,\sigma^2(x^{\star}) + (1-2w)\,\rho(\mathbf{e}_i, x^{\star})\,\sigma_i\sigma(x^{\star})\right)\sigma^{-1}(z)}$$

Market equilibrium and CAPM

1. When $w = 0$, we have:

$$\frac{\partial\, \mu(z)}{\partial\, \sigma(z)} = \frac{\mu_i - \mu(x^\star)}{\left(-\sigma^2(x^\star) + \rho(\mathbf{e}_i, x^\star)\,\sigma_i \sigma(x^\star)\right)\sigma^{-1}(x^\star)}$$

2. When $w = 0$, the portfolio z is the tangency portfolio $x^\star$ and the previous derivative is equal to the Sharpe ratio $\mathrm{SR}(x^\star \mid r)$

We deduce that:

$$\frac{\left(\mu_i - \mu(x^\star)\right)\sigma(x^\star)}{\rho(\mathbf{e}_i, x^\star)\,\sigma_i \sigma(x^\star) - \sigma^2(x^\star)} = \frac{\mu(x^\star) - r}{\sigma(x^\star)}$$

which is equivalent to:

$$\pi_i = \mu_i - r = \beta_i \left(\mu(x^\star) - r\right)$$

with π_i the risk premium of the asset i and:

$$\beta_i = \frac{\rho(\mathbf{e}_i, x^\star)\,\sigma_i}{\sigma(x^\star)} = \frac{\operatorname{cov}(R_i, R(x^\star))}{\operatorname{var}(R(x^\star))}$$

Market equilibrium and CAPM

CAPM

The risk premium of the asset i is equal to its beta times the excess return of the tangency portfolio

$\Rightarrow$ We can extend the previous result to the case of a portfolio x (and not only to the asset i):

$$z = wx + (1 - w) x^{\star}$$

In this case, we have:

$$\pi(x) = \mu(x) - r = \beta(x \mid x^{\star})(\mu(x^{\star}) - r)$$

Computation of the beta

The least squares method

- $R_{i,t}$ and $R_t(x)$ be the returns of asset i and portfolio x at time t
- β_i is estimated with the linear regression:

$$R_{i,t} = \alpha_i + \beta_i R_t(x) + \varepsilon_{i,t}$$

- For a portfolio y, we have:

$$R_t(y) = \alpha + \beta R_t(x) + \varepsilon_t$$

Computation of the beta

The covariance method

Another way to compute the beta of portfolio y is to use the following relationship:

$$\beta\left(y \mid x\right) = \frac{\sigma\left(y, x\right)}{\sigma^2\left(x\right)} = \frac{y^\top \Sigma x}{x^\top \Sigma x}$$

We deduce that the expression of the beta of asset i is also:

$$\beta_i = \beta\left(\mathbf{e}_i \mid x\right) = \frac{\mathbf{e}_i^\top \Sigma x}{x^\top \Sigma x} = \frac{\left(\Sigma x\right)_i}{x^\top \Sigma x}$$

The beta of a portfolio is the weighted average of the beta of the assets that compose the portfolio:

$$\beta\left(y \mid x\right) = \frac{y^\top \Sigma x}{x^\top \Sigma x} = y^\top \frac{\Sigma x}{x^\top \Sigma x} = \sum_{i=1}^{n} y_i \beta_i$$

Market equilibrium and CAPM

We have $x^{\star} = (63.63\%, 19.27\%, 50.28\%, -33.17\%)$ and $\mu(x^{\star}) = 6.37\%$

Table 5: Computation of the beta and the risk premium (Example 2)

Portfolio y	$\mu(y)$	$\mu(y) - r$	$\beta(y \mid x^{\star})$	$\pi(y \mid x^{\star})$
$\mathbf{e}_1$	5.00	3.50	0.72	3.50
$\mathbf{e}_2$	6.00	4.50	0.92	4.50
$\mathbf{e}_3$	8.00	6.50	1.33	6.50
$\mathbf{e}_4$	6.00	4.50	0.92	4.50
x_{ew}	6.25	4.75	0.98	4.75

Example 2

We consider four assets. Their expected returns are equal to 5%, 6%, 8% and 6% while their volatilities are equal to 15%, 20%, 25% and 30%. The correlation matrix of asset returns is given by the following matrix:

$$C = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.40 & 0.70 & 1.00 & \\ 0.50 & 0.40 & 0.80 & 1.00 \end{pmatrix}$$

The risk free rate is equal to $r = 1.5\%$

From active management to passive management

- Active management
- Sharpe (1964)

$$\pi(x) = \beta(x \mid x^\star)\,\pi(x^\star)$$

- Jensen (1969)

$$R_t(x) = \alpha + \beta R_t(b) + \varepsilon_t$$

where $R_t(x)$ is the fund return and $R_t(b)$ is the benchmark return

- Passive management (John McQuown, WFIA, 1971)

Active management = Alpha

Passive management = Beta

Impact of the constraints

If we impose a lower bound $x_i \geq 0$, the tangency portfolio becomes $x^\star = (53.64\%, 32.42\%, 13.93\%, 0.00\%)$ and we have $\mu(x^\star) = 5.74\%$

Table 6: Computation of the beta with a constrained tangency portfolio

Portfolio	$\mu(y) - r$	$\beta(y \mid x^\star)$	$\pi(y \mid x^\star)$
$\mathbf{e}_1$	3.50	0.83	3.50
$\mathbf{e}_2$	4.50	1.06	4.50
$\mathbf{e}_3$	6.50	1.53	6.50
$\mathbf{e}_4$	4.50	1.54	6.53
x_{ew}	4.75	1.24	5.26

$\Rightarrow \mu_4 - r = \beta_4 (\mu(x^\star) - r) + \pi_4^-$ where $\pi_4^- \leq 0$ represents a negative premium due to a lack of arbitrage on the fourth asset

Tracking error

- Portfolio $x = (x_1, \ldots, x_n)$
- Benchmark $b = (b_1, \ldots, b_n)$
- The tracking error between the active portfolio x and its benchmark b is the difference between the return of the portfolio and the return of the benchmark:

$$e = R(x) - R(b) = \sum_{i=1}^{n} x_i R_i - \sum_{i=1}^{n} b_i R_i = x^\top R - b^\top R = (x - b)^\top R$$

- The expected excess return is:

$$\mu(x \mid b) = \mathbb{E}[e] = (x - b)^\top \mu$$

- The volatility of the tracking error is:

$$\sigma(x \mid b) = \sigma(e) = \sqrt{(x - b)^\top \Sigma (x - b)}$$

Markowitz optimization problem

The expected return of the portfolio is replaced by the expected excess return and the volatility of the portfolio is replaced by the volatility of the tracking error

σ-problem

The objective of the investor is to maximize the expected tracking error with a constraint on the tracking error volatility:

$$
\begin{aligned}
x^{\star} &= \arg\max \mu\left(x \mid b\right) \\
\text{u.c.} &\quad \left\{
\begin{array}{l}
\mathbf{1}_n^{\top} x = 1 \\
\sigma\left(x \mid b\right) \leq \sigma^{\star}
\end{array}
\right.
\end{aligned}
$$

Equivalent QP problem

We transform the σ-problem into a γ-problem:

$$x^{\star}(\gamma) = \arg\min f(x \mid b)$$

with:

$$\begin{aligned} f(x \mid b) &= \frac{1}{2}(x-b)^{\top}\Sigma(x-b) - \gamma(x-b)^{\top}\mu \\ &= \frac{1}{2}x^{\top}\Sigma x - x^{\top}(\gamma\mu + \Sigma b) + \left(\frac{1}{2}b^{\top}\Sigma b + \gamma b^{\top}\mu\right) \\ &= \frac{1}{2}x^{\top}\Sigma x - x^{\top}(\gamma\mu + \Sigma b) + c \end{aligned}$$

where c is a constant which does not depend on Portfolio x

QP problem with $Q = \Sigma$ and $R = \gamma\mu + \Sigma b$

Remark

The efficient frontier is the parametric curve $(\sigma(x^{\star}(\gamma) \mid b), \mu(x^{\star}(\gamma) \mid b))$ with $\gamma \in \mathbb{R}_{+}$

Efficient frontier with a benchmark

Example 3

We consider four assets. Their expected returns are equal to 5%, 6%, 8% and 6% while their volatilities are equal to 15%, 20%, 25% and 30%. The correlation matrix of asset returns is given by the following matrix:

$$C = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.40 & 0.70 & 1.00 & \\ 0.50 & 0.40 & 0.80 & 1.00 \end{pmatrix}$$

The benchmark of the portfolio manager is equal to $b = (60\%, 40\%, 20\%, -20\%)$

- 1st case: No constraint
- 2nd case: $x_i^- \leq x_i$ with $x_i^- = -10\%$
- 3rd case: $x_i^- \leq x_i \leq x_i^+$ with $x_1^- = x_2^- = x_3^- = 0\%$, $x_4^- = -20\%$ and $x_i^+ = 50\%$

Efficient frontier with a benchmark

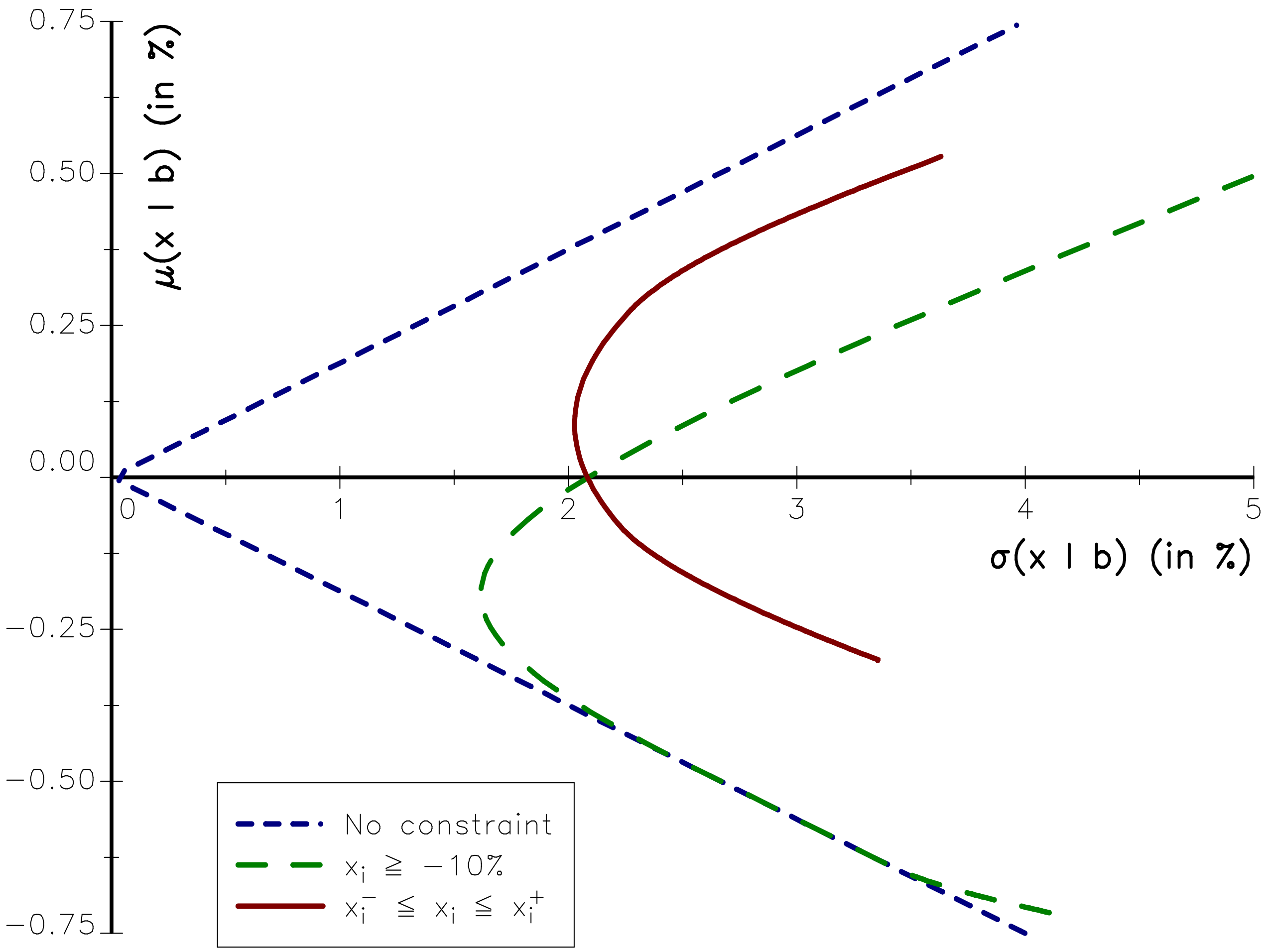


Figure 5: The efficient frontier with a benchmark (Example 3)

Information ratio

Definition

The information ratio is defined as follows:

$$\mathrm{IR}(x \mid b) = \frac{\mu(x \mid b)}{\sigma(x \mid b)} = \frac{(x-b)^\top \mu}{\sqrt{(x-b)^\top \Sigma (x-b)}}$$

Information ratio

If we consider a combination of the benchmark b and the active portfolio x, the composition of the portfolio is:

$$y = (1 - \alpha) b + \alpha x$$

with $\alpha \geq 0$ the proportion of wealth invested in the portfolio x. It follows that:

$$\mu (y \mid b) = (y - b)^\top \mu = \alpha \mu (x \mid b)$$

and:

$$\sigma^2 (y \mid b) = (y - b)^\top \Sigma (y - b) = \alpha^2 \sigma^2 (x \mid b)$$

We deduce that:

$$\mu (y \mid b) = \mathrm{IR} (x \mid b) \cdot \sigma (y \mid b)$$

The efficient frontier is a straight line

Tangency portfolio

If we add some constraints, the portfolio optimization problem becomes:

$$
\begin{aligned}
x^{\star}(\gamma) &= \arg\min \frac{1}{2} x^{\top} \Sigma x - x^{\top} (\gamma \mu + \Sigma b) \\
\text{u.c.} &\quad \left\{ \begin{array}{l} \mathbf{1}_n^{\top} x = 1 \\ x \in \Omega \end{array} \right.
\end{aligned}
$$

The efficient frontier is no longer a straight line

Tangency portfolio

One optimized portfolio dominates all the other portfolios. It is the portfolio which belongs to the efficient frontier and the straight line which is tangent to the efficient frontier. It is also the portfolio which maximizes the information ratio

Constrained efficient frontier with a benchmark

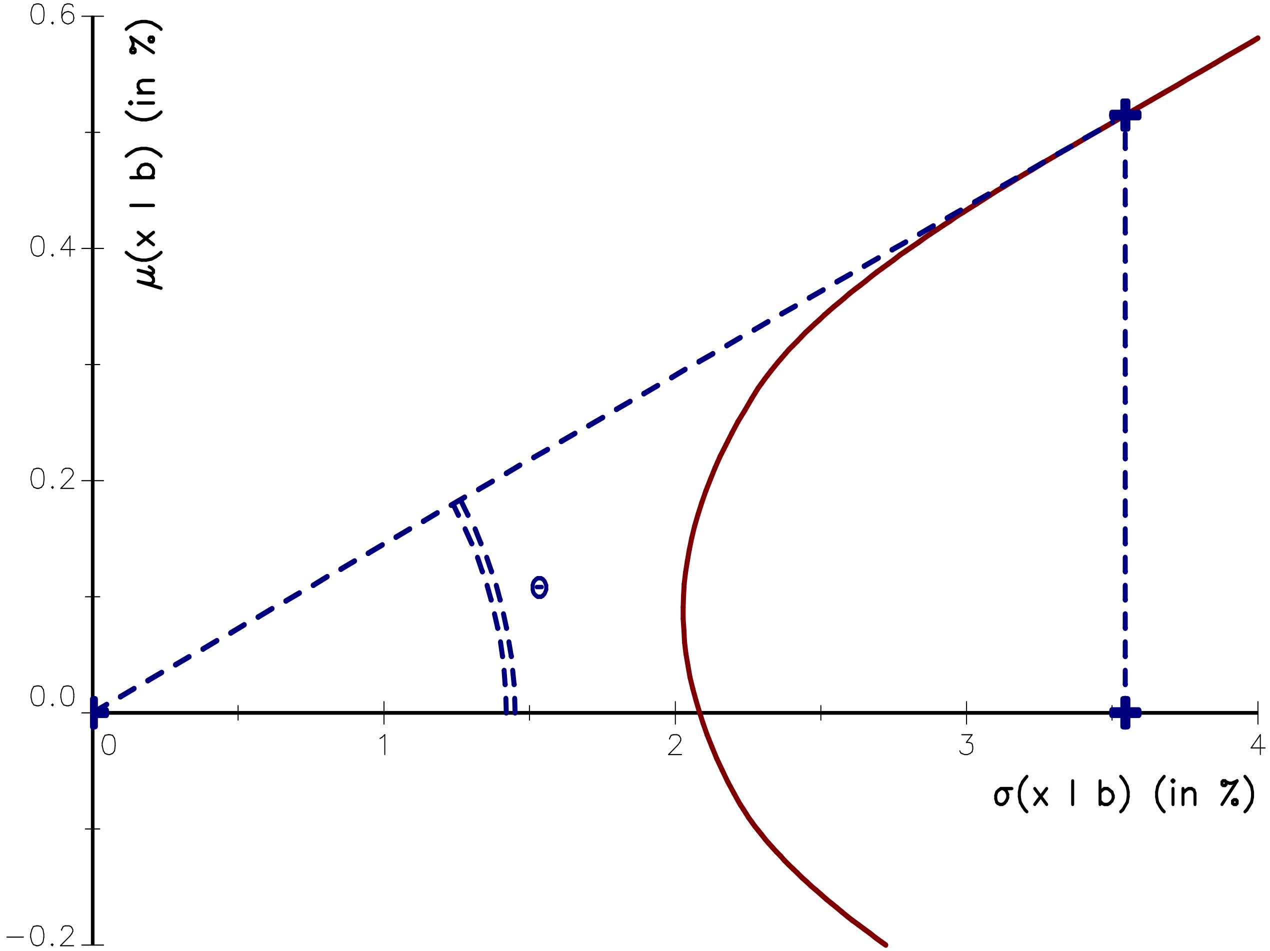


Figure 6: The tangency portfolio with respect to a benchmark (Example 3, $3^{\rm rd}$ case)

Tangency portfolio

If $x_i^- \leq x_i \leq x_i^+$ with $x_1^- = x_2^- = x_3^- = 0\%$, $x_4^- = -20\%$ and $x_i^+ = 50\%$, the tangency portfolio is equal to:

$$x^\star = \begin{pmatrix} 49.51\% \\ 29.99\% \\ 40.50\% \\ -20.00\% \end{pmatrix}$$

If $r = 1.5\%$, we recall that the MSR (maximum Sharpe ratio) portfolio is equal to:

$$x^\star = \begin{pmatrix} 63.63\% \\ 19.27\% \\ 50.28\% \\ -33.17\% \end{pmatrix}$$

When the benchmark is the risk-free rate

The Markowitz-Tobin-Sharpe approach is obtained when the benchmark is the risk-free asset r. We have:

$$\tilde{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{b} = \begin{pmatrix} \mathbf{0}_n \\ 1 \end{pmatrix}$$

It follows that:

$$\tilde{\Sigma} = \begin{pmatrix} \Sigma & \mathbf{0}_n \\ \mathbf{0}_n^\top & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\mu} = \begin{pmatrix} \mu \\ r \end{pmatrix}$$

When the benchmark is the risk-free rate

The objective function is then defined as follows:

$$\begin{aligned}
f\left(\tilde{x} \mid \tilde{b}\right) &= \frac{1}{2}\left(\tilde{x}-\tilde{b}\right)^\top \Sigma\left(\tilde{x}-\tilde{b}\right) - \gamma\left(\tilde{x}-\tilde{b}\right)^\top \mu \\
&= \frac{1}{2}\tilde{x}^\top \tilde{\Sigma}\tilde{x} - \tilde{x}^\top\left(\gamma\tilde{\mu} + \tilde{\Sigma}\tilde{b}\right) + \left(\frac{1}{2}\tilde{b}^\top\tilde{\Sigma}\tilde{b} + \gamma\tilde{b}^\top\tilde{\mu}\right) \\
&= \frac{1}{2}x^\top \Sigma x - \gamma\left(x^\top \mu - r\right) \\
&= \frac{1}{2}x^\top \Sigma x - \gamma x^\top\left(\mu - r\mathbf{1}_n\right)
\end{aligned}$$

When the benchmark is the risk-free rate

The solution of the QP problem $\tilde{x}^{\star}(\gamma) = \arg\min f\left(\tilde{x} \mid \tilde{b}\right)$ is related to the solution $x^{\star}(\gamma)$ of the Markowitz γ-problem in the following way:

$$\tilde{x}^{\star}(\gamma) = \begin{pmatrix} x^{\star}(\gamma) \\ 0 \end{pmatrix}$$

We have $\sigma\left(\tilde{x}^{\star}(\gamma) \mid \tilde{b}\right) = \sigma\left(x^{\star}(\phi)\right)$

Remark

$\Rightarrow$ *The MSR portfolio is obtained by replacing the vector μ of expected returns by the vector $\mu - r\mathbf{1}_n$ of expected excess returns. We have:*

$$\mathrm{SR}\left(x^{\star}(\gamma) \mid r\right) = \mathrm{IR}\left(\tilde{x}^{\star}(\gamma) \mid \tilde{b}\right)$$

Black-Litterman model

Tactical asset allocation (TAA) model

How to incorporate portfolio manager's views in a strategic asset allocation (SAA)?

Two-step approach:

1. Initial allocation $\Rightarrow$ implied risk premia (Sharpe)
2. Portfolio optimization $\Rightarrow$ coherent with the bets of the portfolio manager (Markowitz)

Implied risk premium

$$
\begin{aligned}
x^{\star} & = \arg\min \frac{1}{2} x^{\top} \Sigma x - \gamma x^{\top} \left(\mu - r \mathbf{1}_n \right) \\
\text{u.c.} & \quad \left\{ \begin{array}{l} \mathbf{1}_n^{\top} x = 1 \\ x \in \Omega \end{array} \right.
\end{aligned}
$$

If the constraints are satisfied, the first-order condition is:

$$\Sigma x - \gamma \left(\mu - r \mathbf{1}_n \right) = \mathbf{0}_n$$

The solution is:

$$x^{\star} = \gamma \Sigma^{-1} \left(\mu - r \mathbf{1}_n \right)$$

- In the Markowitz model, the unknown variable is the vector x
- If the initial allocation x_0 is given, it must be optimal for the investor, implying that:

$$\tilde{\mu} = r \mathbf{1}_n + \frac{1}{\gamma} \Sigma x_0$$

- $\tilde{\mu}$ is the vector of expected returns which is coherent with x_0

Implied risk premium

We deduce that:

$$\begin{aligned} \tilde{\pi} &= \tilde{\mu} - r \\ &= \frac{1}{\gamma} \Sigma x_0 \end{aligned}$$

The variable $\tilde{\pi}$ is:

- the *risk premium priced* by the portfolio manager
- the '*implied risk premium*' of the portfolio manager
- the '*market risk premium*' when x_0 is the market portfolio

Implied risk aversion

The computation of $\tilde{\mu}$ needs to the value of the parameter γ or the risk aversion $\phi = \gamma^{-1}$

Since we have $\Sigma x_0 - \gamma \left(\tilde{\mu} - r\mathbf{1}_n\right) = \mathbf{0}_n$, we deduce that:

$$
\begin{aligned}
(*) \quad & \Leftrightarrow \quad \gamma \left(\tilde{\mu} - r\mathbf{1}_n\right) = \Sigma x_0 \\
& \Leftrightarrow \quad \gamma \left(x_0^\top \tilde{\mu} - r x_0^\top \mathbf{1}_n\right) = x_0^\top \Sigma x_0 \\
& \Leftrightarrow \quad \gamma \left(x_0^\top \tilde{\mu} - r\right) = x_0^\top \Sigma x_0 \\
& \Leftrightarrow \quad \gamma = \frac{x_0^\top \Sigma x_0}{x_0^\top \tilde{\mu} - r}
\end{aligned}
$$

It follows that

$$
\phi = \frac{x_0^\top \tilde{\mu} - r}{x_0^\top \Sigma x_0} = \frac{\mathrm{SR}\left(x_0 \mid r\right)}{\sqrt{x_0^\top \Sigma x_0}} = \frac{\mathrm{SR}\left(x_0 \mid r\right)}{\sigma\left(x_0\right)}
$$

where $\mathrm{SR}\left(x_0 \mid r\right)$ is the portfolio's expected Sharpe ratio

Implied risk aversion

We have:

$$\tilde{\mu} = r + \mathrm{SR}\left(x_0 \mid r\right) \frac{\Sigma x_0}{\sqrt{x_0^\top \Sigma x_0}}$$

and:

$$\tilde{\pi} = \mathrm{SR}\left(x_0 \mid r\right) \frac{\Sigma x_0}{\sqrt{x_0^\top \Sigma x_0}}$$

Implied risk premium

Example 4

We consider Example 1 and we suppose that the initial allocation x_0 is $(40\%, 30\%, 20\%, 10\%)$

- The volatility of the portfolio is equal to:

$$\sigma(x_0) = 15.35\%$$

- The objective of the portfolio manager is to target a Sharpe ratio equal to 0.25
- We obtain $\phi = 1.63$
- If $r = 3\%$, the implied expected returns are:

$$\tilde{\mu} = \begin{pmatrix} 5.47\% \\ 6.68\% \\ 8.70\% \\ 9.06\% \end{pmatrix}$$

Specification of the bets

Black and Litterman assume that μ is a Gaussian vector with expected returns $\tilde{\mu}$ and covariance matrix Γ:

$$\mu \sim \mathcal{N}\left(\tilde{\mu}, \Gamma\right)$$

The portfolio manager's views are given by this relationship:

$$P\mu = Q + \varepsilon$$

where P is a $(k \times n)$ matrix, Q is a $(k \times 1)$ vector and $\varepsilon \sim \mathcal{N}\left(0, \Omega\right)$ is a Gaussian vector of dimension k

- If the portfolio manager has two views, the matrix P has two rows $\Rightarrow$ k is then the number of views
- Ω is the covariance matrix of $P\mu - Q$, therefore it measures the uncertainty of the views

Absolute views

- We consider the three-asset case:

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$$

- The portfolio manager has an absolute view on the expected return of the first asset:

$$\mu_1 = q_1 + \varepsilon_1$$

We have:

$$P = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \; Q = q_1, \; \varepsilon = \varepsilon_1 \text{ and } \Omega = \omega_1^2$$

If $\omega_1 = 0$, the portfolio manager has a very high level of confidence. If $\omega_1 \neq 0$, his view is uncertain

Absolute views

- The portfolio manager has an absolute view on the expected return of the second asset:

$$\mu_2 = q_2 + \varepsilon_2$$

We have:

$$P = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix},\ Q = q_2,\ \varepsilon = \varepsilon_2 \text{ and } \Omega = \omega_2^2$$

- The portfolio manager has two absolute views:

$$\begin{aligned} \mu_1 &= q_1 + \varepsilon_1 \\ \mu_2 &= q_2 + \varepsilon_2 \end{aligned}$$

We have:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ Q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},\ \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \text{ and } \Omega = \begin{pmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{pmatrix}$$

Relative views

- The portfolio manager thinks that the outperformance of the first asset with respect to the second asset is q:

$$\mu_1 - \mu_2 = q_{1|2} + \varepsilon_{1|2}$$

We have:

$$P = \left(\begin{array}{ccc} 1 & -1 & 0 \end{array} \right), Q = q_{1|2}, \varepsilon = \varepsilon_{1|2} \text{ and } \Omega = \omega_{1|2}^2$$

Portfolio optimization

The Markowitz optimization problem becomes:

$$\begin{aligned} x^\star(\gamma) &= \arg\min \frac{1}{2} x^\top \Sigma x - \gamma x^\top (\bar{\mu} - r \mathbf{1}_n) \\ \text{u.c.} &\quad \mathbf{1}_n^\top x = 1 \end{aligned}$$

where $\bar{\mu}$ is the vector of expected returns conditional to the views:

$$\begin{aligned} \bar{\mu} &= \mathbb{E}\left[\mu \mid \text{views}\right] \\ &= \mathbb{E}\left[\mu \mid P\mu = Q + \varepsilon\right] \\ &= \mathbb{E}\left[\mu \mid P\mu - \varepsilon = Q\right] \end{aligned}$$

To compute $\bar{\mu}$, we consider the random vector:

$$\left(\begin{array}{c} \mu \\ \nu = P\mu - \varepsilon \end{array} \right) \sim \mathcal{N}\left(\left(\begin{array}{c} \tilde{\mu} \\ P\tilde{\mu} \end{array} \right), \left(\begin{array}{cc} \Gamma & \Gamma P^\top \\ P\Gamma & P\Gamma P^\top + \Omega \end{array} \right) \right)$$

Conditional distribution in the case of the normal distribution

Let us consider a Gaussian random vector defined as follows:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \Sigma_{x,x} & \Sigma_{x,y} \\ \Sigma_{y,x} & \Sigma_{y,y} \end{pmatrix}\right)$$

We have:

$$Y \mid X = x \sim \mathcal{N}\left(\mu_{y|x}, \Sigma_{y,y|x}\right)$$

where:

$$\mu_{y|x} = \mathbb{E}\left[Y \mid X = x\right] = \mu_y + \Sigma_{y,x}\Sigma_{x,x}^{-1}\left(x - \mu_x\right)$$

and:

$$\Sigma_{y,y|x} = \operatorname{cov}\left(Y \mid X = x\right) = \Sigma_{y,y} - \Sigma_{y,x}\Sigma_{x,x}^{-1}\Sigma_{x,y}$$

Computation of the conditional expectation

We apply the conditional expectation formula:

$$\begin{aligned}\bar{\mu} &= \mathbb{E}\left[\mu \mid \nu = Q\right] \\ &= \mathbb{E}\left[\mu\right] + \operatorname{cov}\left(\mu, v\right) \operatorname{var}\left(v\right)^{-1} \left(Q - \mathbb{E}\left[v\right]\right) \\ &= \tilde{\mu} + \Gamma P^\top \left(P \Gamma P^\top + \Omega\right)^{-1} \left(Q - P\tilde{\mu}\right)\end{aligned}$$

The conditional expectation $\bar{\mu}$ has two components:

1. The first component corresponds to the vector of implied expected returns $\tilde{\mu}$
2. The second component is a correction term which takes into account the *disequilibrium* $(Q - P\tilde{\mu})$ between the manager views and the market views

Computation of the conditional covariance matrix

The condition covariance matrix is equal to:

$$\begin{aligned}\bar{\Sigma} &= \operatorname{var}\left(\mu \mid \nu = Q\right) \\ &= \Gamma - \Gamma P^\top \left(P\Gamma P^\top + \Omega\right)^{-1} P\Gamma\end{aligned}$$

Another expression is:

$$\begin{aligned}\bar{\Sigma} &= \left(I_n + \Gamma P^\top \Omega^{-1} P\right)^{-1} \Gamma \\ &= \left(\Gamma^{-1} + P^\top \Omega^{-1} P\right)^{-1}\end{aligned}$$

The conditional covariance matrix is a weighted average of the covariance matrix Γ and the covariance matrix Ω of the manager views.

Choice of covariance matrices

Choice of Σ

From a theoretical point of view, we have:

$$\Sigma = \bar{\Sigma} = \left(\Gamma^{-1} + P^\top \Omega^{-1} P\right)^{-1}$$

In practice, we use:

$$\Sigma = \hat{\Sigma}$$

Choice of Γ

We assume that:

$$\Gamma = \tau \Sigma$$

We can also target a tracking error volatility and deduce τ

Numerical implementation of the model

The five-step approach to implement the Black-Litterman model is:

1. We estimate the empirical covariance matrix $\hat{\Sigma}$ and set $\Sigma = \hat{\Sigma}$
2. Given the current portfolio, we compute the implied risk aversion $\phi = \gamma^{-1}$ and we deduce the vector $\tilde{\mu}$ of implied expected returns
3. We specify the views by defining the P, Q and Ω matrices
4. Given a matrix Γ, we compute the conditional expectation $\bar{\mu}$
5. We finally perform the portfolio optimization with $\hat{\Sigma}$, $\bar{\mu}$ and γ

Illustration

- We use Example 4 and impose that the optimized weights are positive
- The portfolio manager has an absolute view on the first asset and a relative view on the second and third assets:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \; Q = \begin{pmatrix} q_1 \\ q_{2-3} \end{pmatrix} \quad \text{and} \quad \Omega = \begin{pmatrix} \varpi_1^2 & 0 \\ 0 & \varpi_{2-3}^2 \end{pmatrix}$$

- $q_1 = 4\%$, $q_{2-3} = -1\%$, $\varpi_1 = 10\%$ and $\varpi_{2-3} = 5\%$

Illustration

- Case #1: $\tau = 1$
- Case #2: $\tau = 1$ and $q_1 = 7\%$
- Case #3: $\tau = 1$ and $\varpi_1 = \varpi_{2-3} = 20\%$
- Case #4: $\tau = 10\%$
- Case #5: $\tau = 1\%$

Illustration

Table 7: Black-Litterman portfolios

	#0	#1	#2	#3	#4	#5
$x_1^\star$	40.00	33.41	51.16	36.41	38.25	39.77
$x_2^\star$	30.00	51.56	39.91	42.97	42.72	32.60
$x_3^\star$	20.00	5.46	0.00	10.85	9.14	17.65
$x_4^\star$	10.00	9.58	8.93	9.77	9.89	9.98
$\sigma\left(x^\star \mid x_0\right)$	0.00	3.65	3.67	2.19	2.18	0.45

Illustration

To calibrate the parameter τ, we could target a tracking error volatility $\sigma^{\star}$:

- If $\sigma^{\star} = 2\%$, the optimized portfolio is between portfolios #4 ($\sigma(x^{\star} \mid x_0) = 2.18\%$) and #5 ($\sigma(x^{\star} \mid x_0) = 0.45\%$)
- The optimal value of τ is between 10% and 1%
- Using a bisection algorithm, we obtain $\tau = 5.2\%$

The optimal portfolio is:

$$x^{\star} = \begin{pmatrix} 36.80\% \\ 41.83\% \\ 11.58\% \\ 9.79\% \end{pmatrix}$$

Empirical estimator

We have:

$$\hat{\Sigma} = \frac{1}{T} \sum_{t=1}^{T} \left(R_t - \bar{R}\right) \left(R_t - \bar{R}\right)^\top$$

Asynchronous markets

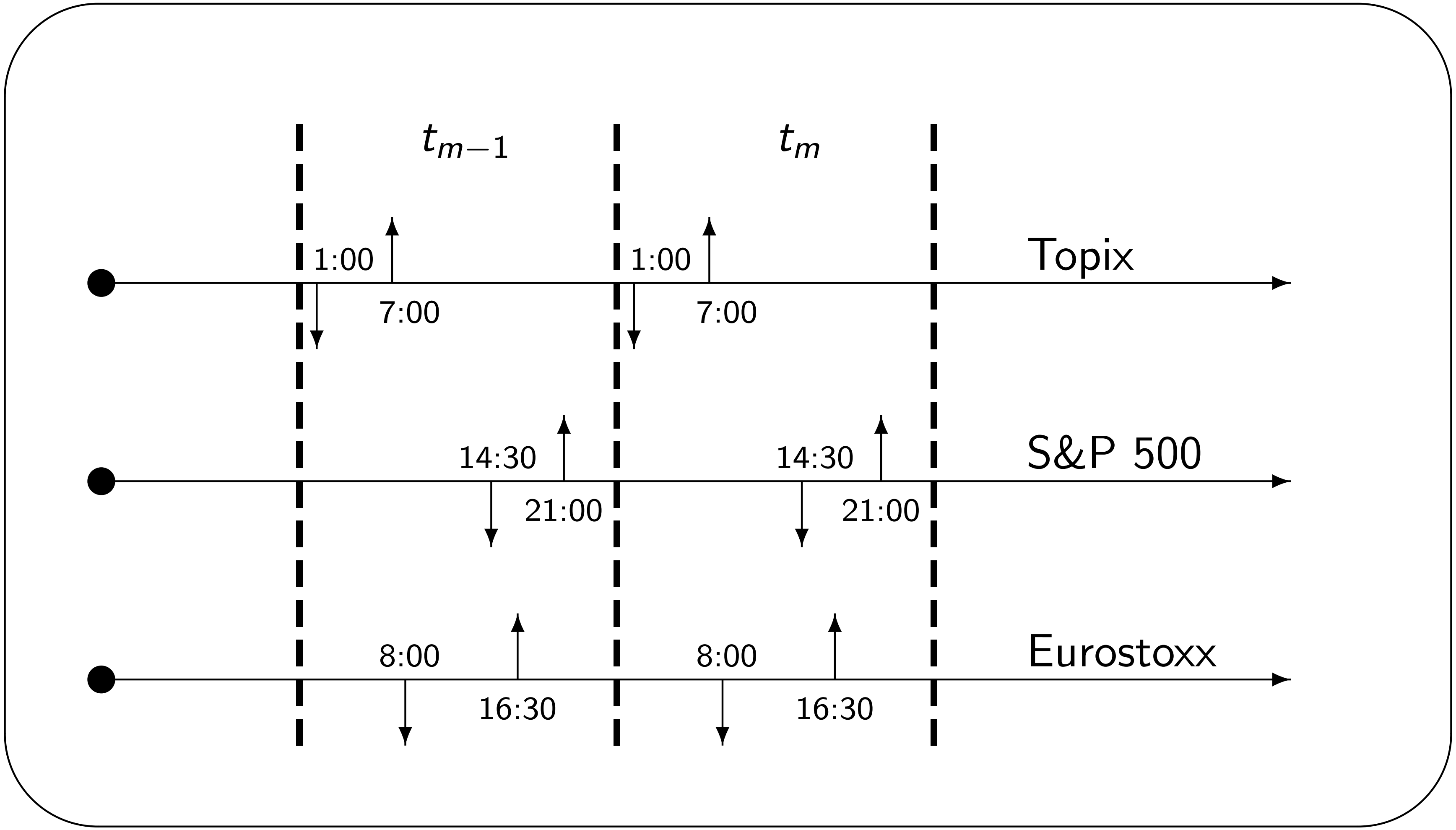


Figure 7: Trading hours of asynchronous markets (UTC time)

Asynchronous markets

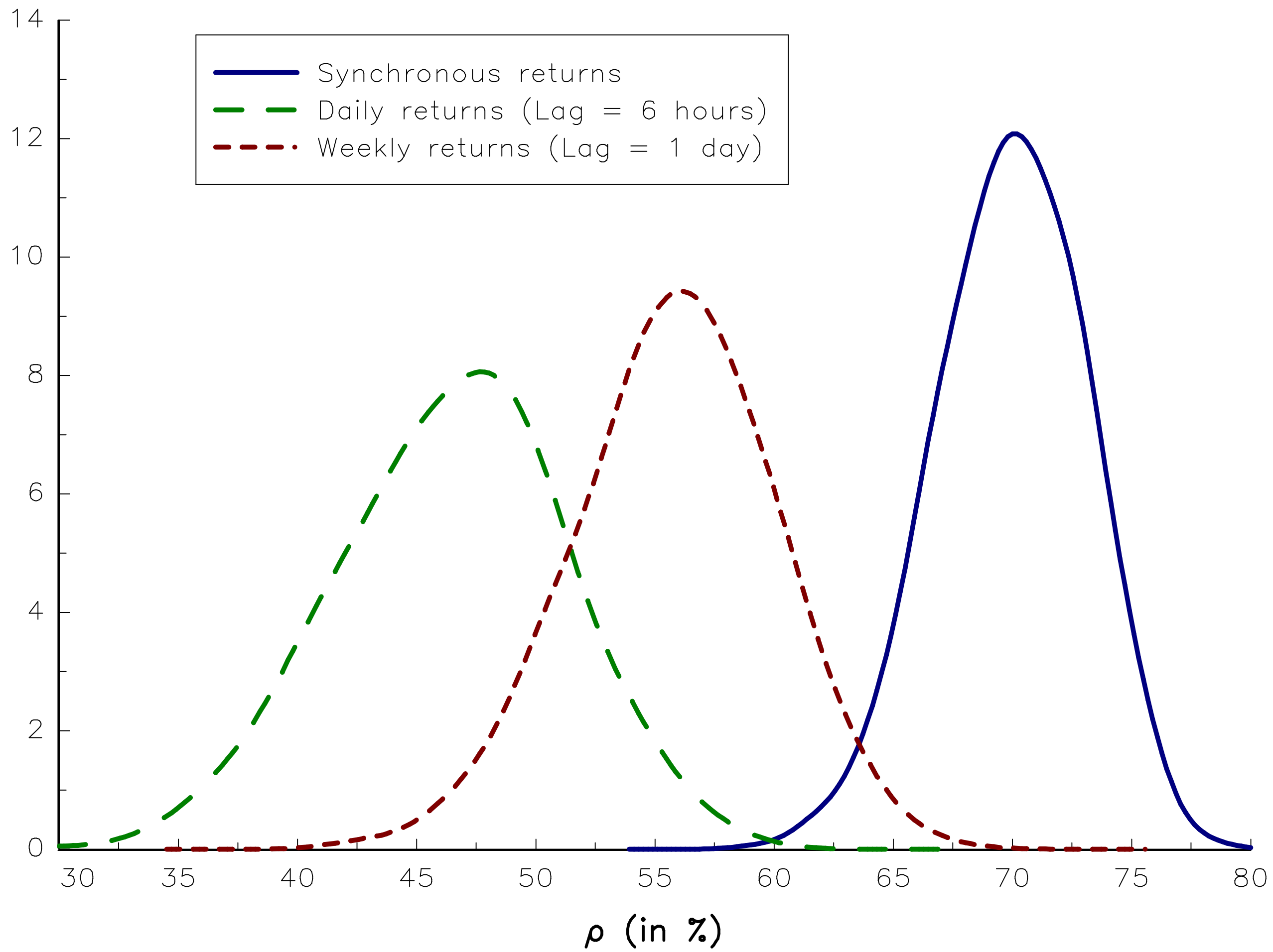


Figure 8: Density of the estimator $\hat{\rho}$ with asynchronous returns ($\rho = 70\%$)

Asynchronous markets

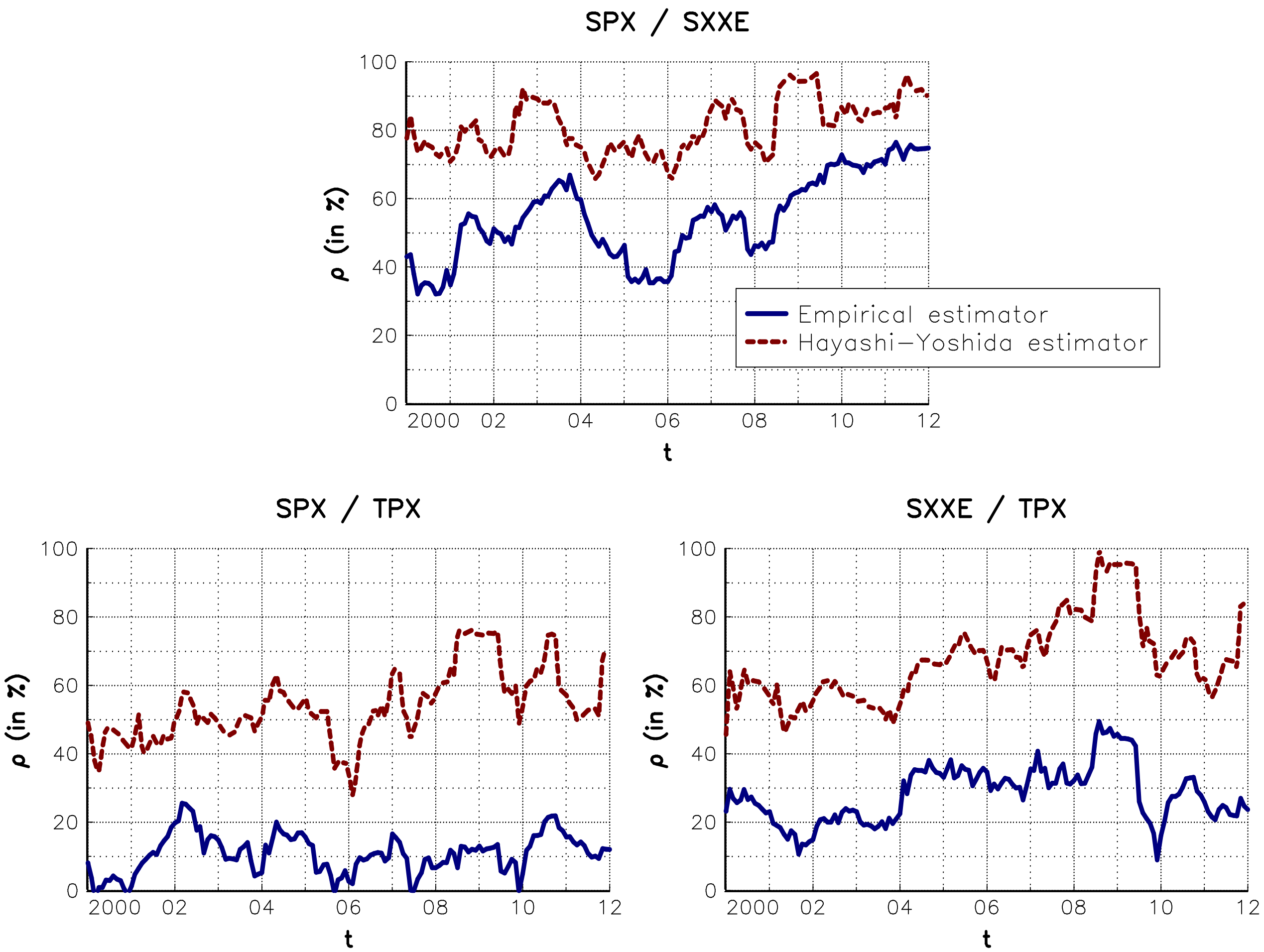


Figure 9: Hayashi-Yoshida estimator

Hayashi-Yoshida estimator

We have:

$$\tilde{\Sigma}_{i,j} = \frac{1}{T} \sum_{t=1}^{T} \left(R_{i,t} - \bar{R}_i\right) \left(R_{j,t} - \bar{R}_j\right) + \frac{1}{T} \sum_{t=1}^{T} \left(R_{i,t} - \bar{R}_i\right) \left(R_{j,t-1} - \bar{R}_j\right)$$

where j is the equity index which has a closing time after the equity index i. In our case, j is necessarily the S&P 500 index whereas i can be the Topix index or the Eurostoxx index. This estimator has two components:

1. The first component is the classical covariance estimator $\hat{\Sigma}_{i,j}$
2. The second component is a correction to take into account the lag between the two closing times

Other statistical methods

- EWMA methods
- GARCH models
- Factor models
  - Uniform correlation
  $$\rho_{i,j} = \rho$$
  - Sector approach (inter-correlation and intra-correlation)
  - Linear factor models:
  $$R_{i,t} = A_i^\top \mathcal{F}_t + \varepsilon_{i,t}$$

Economic/econometric approach

- Market timing (MT)
- Tactical asset allocation (TAA)
- Strategic asset allocation (SAA)

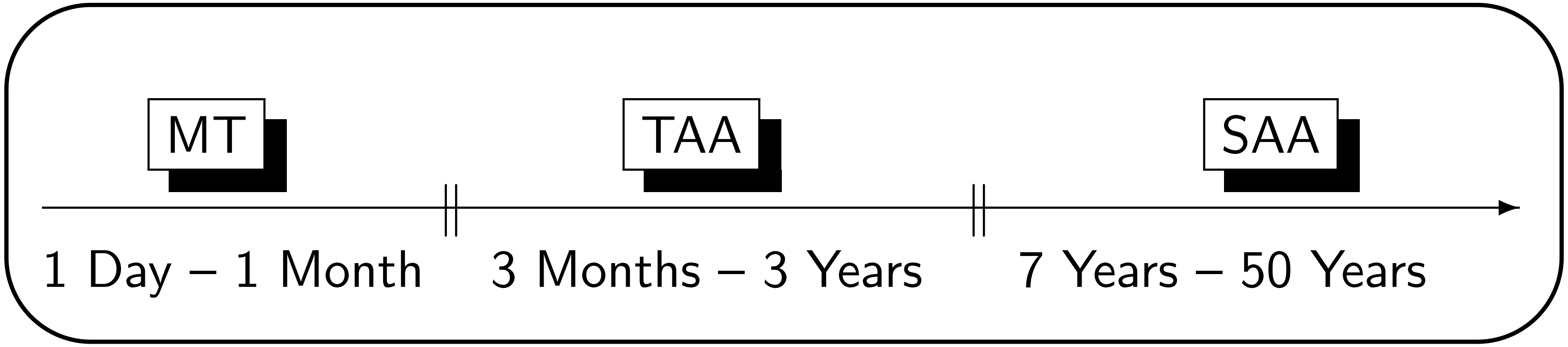


Figure 10: Time horizon of MT, TAA and SAA

Statistical/scoring approach

- Stock picking models: fundamental scoring, value, quality, sector analysis, etc.
- Bond picking models: fundamental scoring, structural model, credit arbitrage model, etc.
- Statistical models: mean-reverting, trend-following, cointegration, etc.
- Machine learning: return forecasting, scoring model, etc.

Stability issues

Example 5

We consider a universe of 3 assets. The parameters are: $\mu_1 = \mu_2 = 8\%$, $\mu_3 = 5\%$, $\sigma_1 = 20\%$, $\sigma_2 = 21\%$, $\sigma_3 = 10\%$ and $\rho_{i,j} = 80\%$. The objective is to maximize the expected return for a 15% volatility target. The optimal portfolio is $(38.3\%, 20.2\%, 41.5\%)$.

Table 8: Sensitivity of the MVO portfolio to input parameters

ρ		70%	90%		90%	
σ_2				18%	18%	
μ_1						9%
x_1	38.3	38.3	44.6	13.7	-8.0	60.6
x_2	20.2	25.9	8.9	56.1	74.1	-5.4
x_3	41.5	35.8	46.5	30.2	34.0	44.8

Solutions

In order to stabilize the optimal portfolio, we have to introduce some regularization techniques:

- Resampling techniques
- Factor analysis
- Shrinkage methods
- Random matrix theory
- Norm penalization
- Etc.

Resampling techniques

- Jacknife
- Cross validation
  - Hold-out
  - K-fold
- Bootstrap
  - Resubstitution
  - Out of the bag
  - .632

Resampling techniques

Example 6

We consider a universe of four assets. The expected returns are $\hat{\mu}_1 = 5\%$, $\hat{\mu}_2 = 9\%$, $\hat{\mu}_3 = 7\%$ and $\hat{\mu}_4 = 6\%$ whereas the volatilities are equal to $\hat{\sigma}_1 = 4\%$, $\hat{\sigma}_2 = 15\%$, $\hat{\sigma}_3 = 5\%$ and $\hat{\sigma}_4 = 10\%$. The correlation matrix is the following:

$$\hat{C} = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.40 & 0.20 & 1.00 & \\ -0.10 & -0.10 & -0.20 & 1.00 \end{pmatrix}$$

Resampling techniques

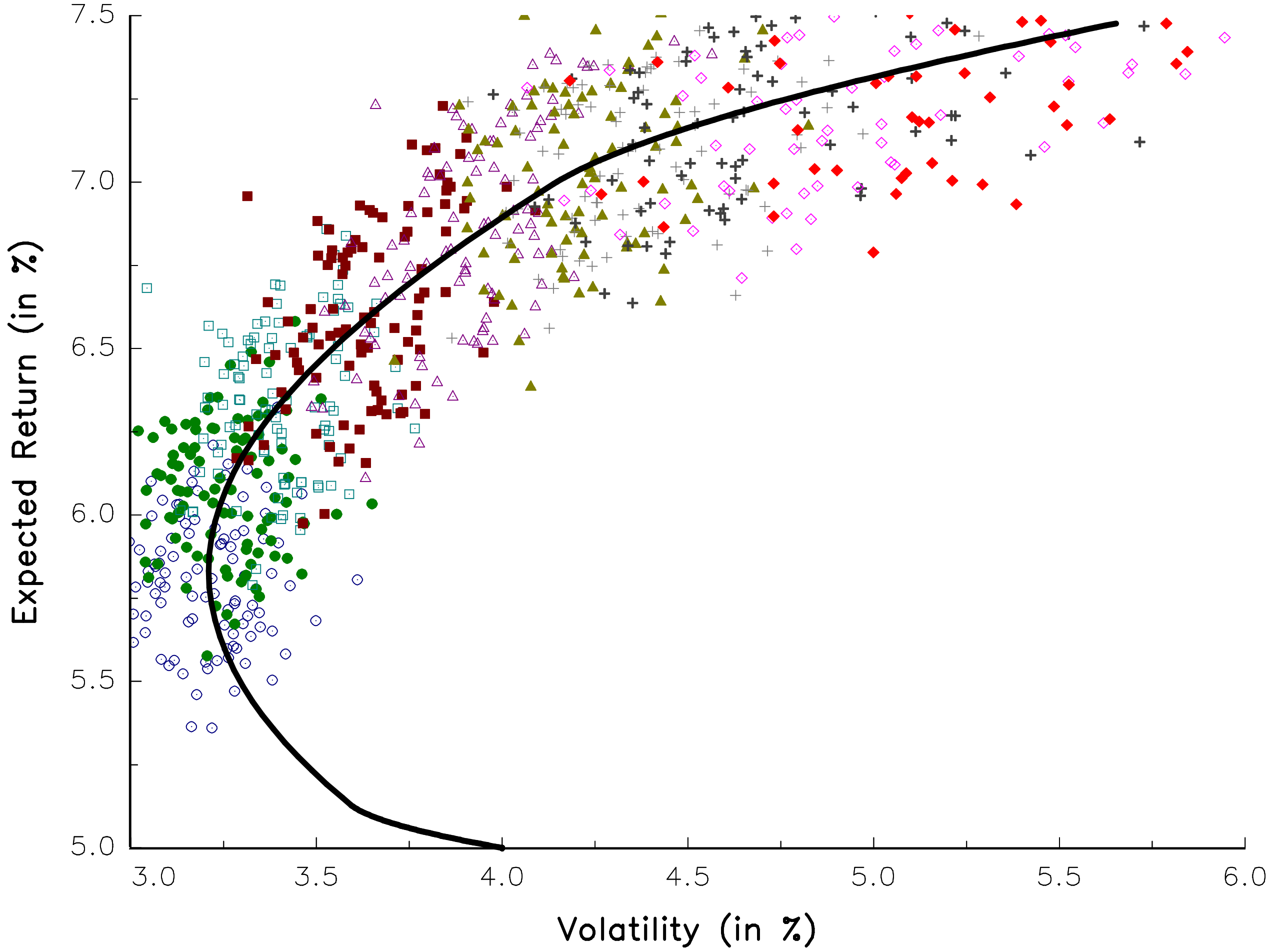


Figure 11: Uncertainty of the efficient frontier

Resampling techniques

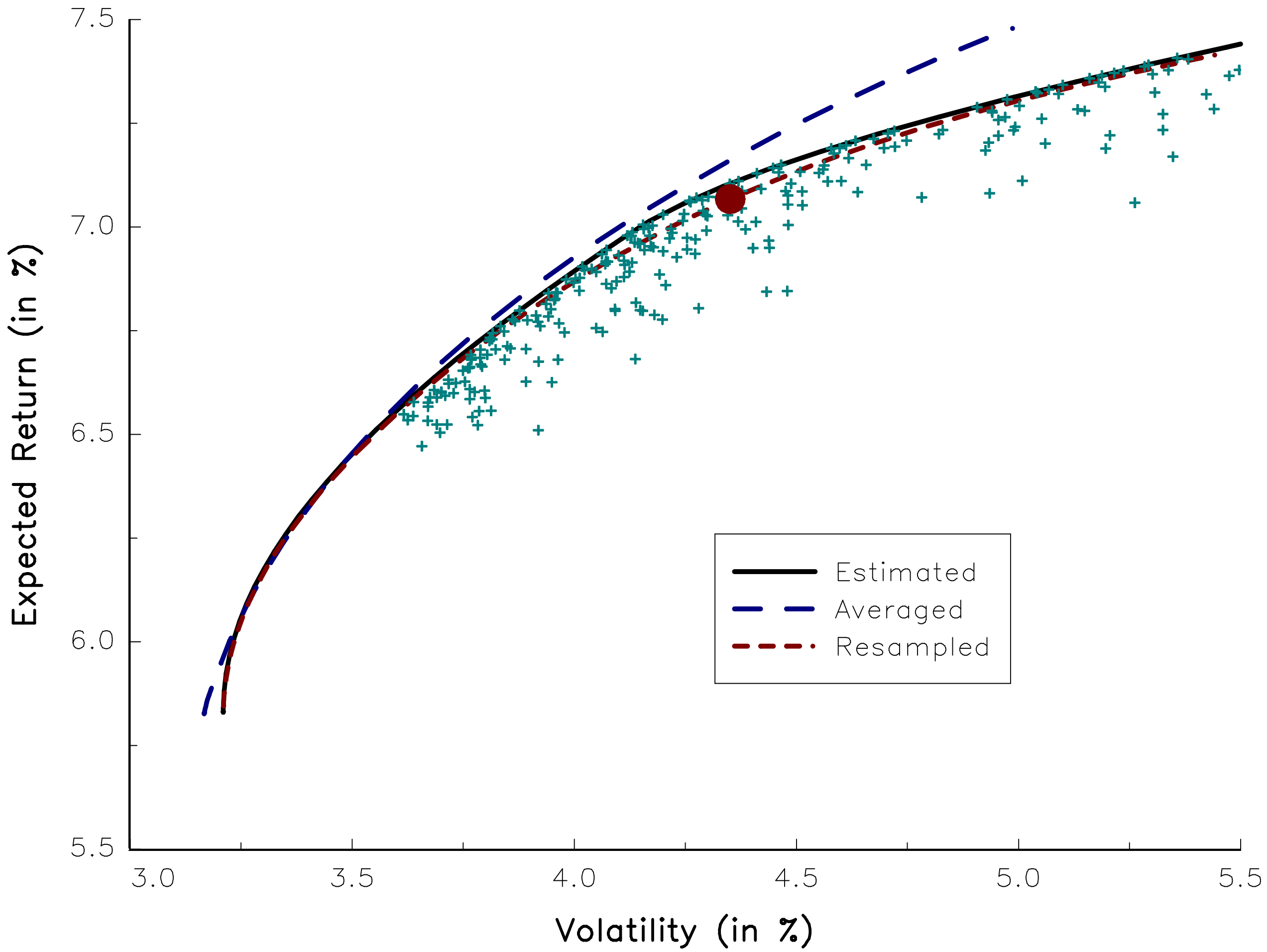


Figure 12: Resampled efficient frontier

Resampling techniques

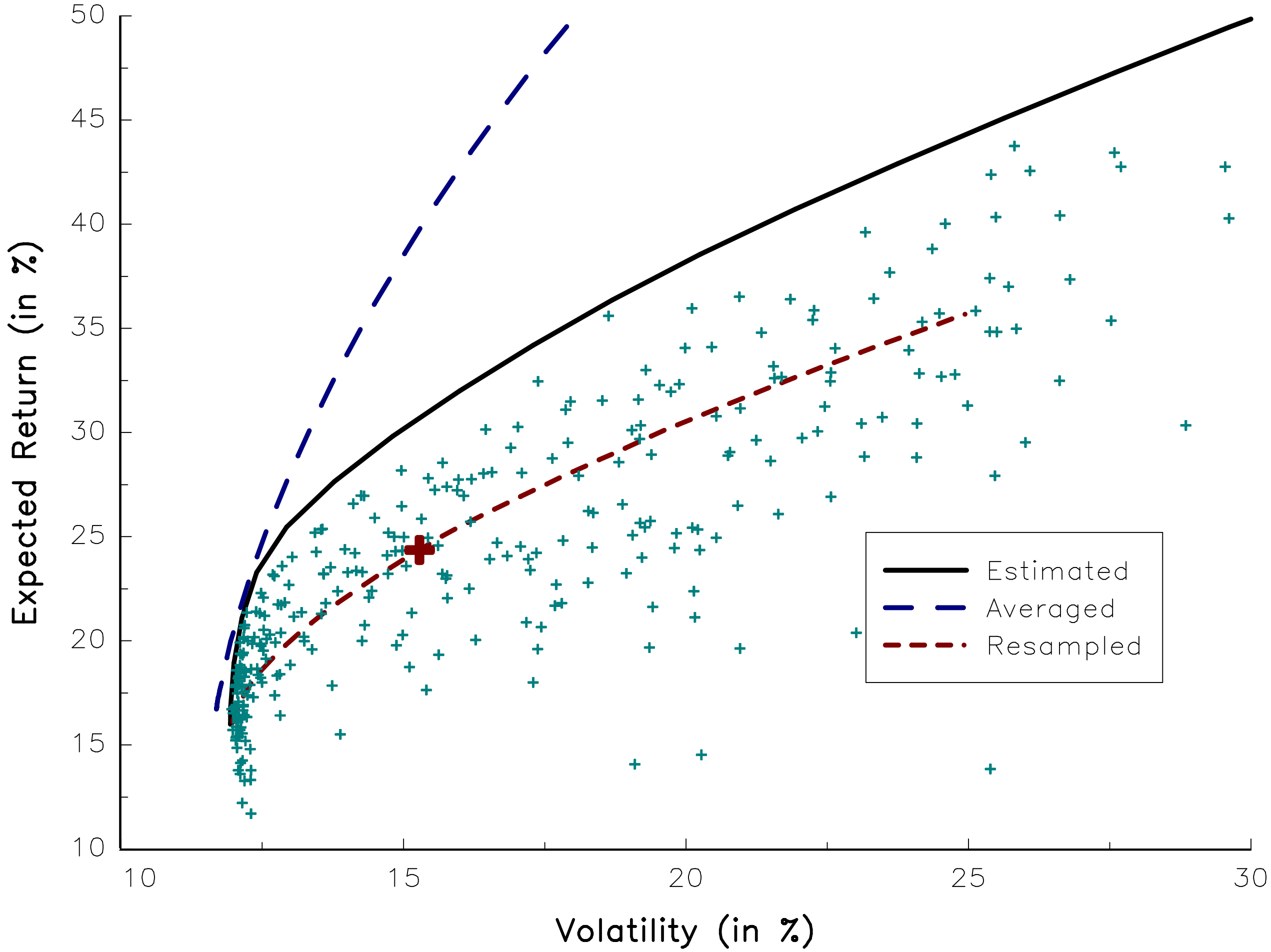


Figure 13: S&P 100 resampled efficient frontier (Bootstrap approach)

Source: Bruder *et al.* (2013)

How to denoise the covariance matrix?

1. Factor analysis by imposing a correlation structure (MSCI Barra)
2. Factor analysis by filtering the correlation structure (APT)
3. Principal component analysis
4. Random matrix theory
5. Shrinkage methods

How to denoise the covariance matrix?

- The eigendecomposition $\hat{\Sigma}$ of is

$$\hat{\Sigma} = V \Lambda V^{\top}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix of eigenvalues with $\lambda_1 > \lambda_2 > \ldots > \lambda_n$ and V is an orthonormal matrix

- The endogenous factors are $\mathcal{F}_t = \Lambda^{-1/2} V^{\top} R_t$
- By considering only the m first components, we can build an estimation of Σ with less noise

How to denoise the covariance matrix?

Choice of m

1. We keep factors that explain more than 1/n of asset variance:

$$m = \sup \left\{ i : \lambda_i \geq \left(\lambda_1 + \ldots + \lambda_n \right) / n \right\}$$

2. Laloux *et al.* (1999) propose to use the random matrix theory (RMT)
   1. The maximum eigenvalue of a random matrix M is equal to:

   $$\lambda_{\max} = \sigma^2 \left(1 + n/T + 2\sqrt{n/T} \right)$$

   where T is the sample size
   2. We keep the first m factors such that:

   $$m = \sup \left\{ i : \lambda_i > \lambda_{\max} \right\}$$

How to denoise the covariance matrix?

Shrinkage methods

- $\hat{\Sigma}$ is an unbiased estimator, but its convergence is very slow
- $\hat{\Phi}$ is a biased estimator that converges more quickly

Ledoit and Wolf (2003) propose to combine $\hat{\Sigma}$ and $\hat{\Phi}$:

$$\hat{\Sigma}_{\alpha} = \alpha \hat{\Phi} + (1 - \alpha) \hat{\Sigma}$$

The value of α is estimated by minimizing a quadratic loss:

$$\alpha^{\star} = \arg\min \mathbb{E} \left[\left\| \alpha \hat{\Phi} + (1 - \alpha) \hat{\Sigma} - \Sigma \right\|^2 \right]$$

They find an analytical expression of $\alpha^{\star}$ when:

- $\hat{\Phi}$ has a constant correlation structure
- $\hat{\Phi}$ corresponds to a factor model or is deduced from PCA

How to denoise the covariance matrix?

Example 7 (equity correlation matrix)

We consider a universe with eight equity indices: S&P 500, Eurostoxx, FTSE 100, Topix, Bovespa, RTS, Nifty and HSI. The study period is January 2005–December 2011 and we use weekly returns.

The empirical correlation matrix is:

$$\hat{C} = \left(\begin{array}{cccc|cccc}
1.00 & & & & & & & \\
0.88 & 1.00 & & & & & & \\
0.88 & 0.94 & 1.00 & & & & & \\
0.64 & 0.68 & 0.65 & 1.00 & & & & \\
\hline
0.77 & 0.76 & 0.78 & 0.61 & 1.00 & & & \\
0.56 & 0.61 & 0.61 & 0.50 & 0.64 & 1.00 & & \\
0.53 & 0.61 & 0.57 & 0.53 & 0.60 & 0.57 & 1.00 & \\
0.64 & 0.68 & 0.67 & 0.68 & 0.68 & 0.60 & 0.66 & 1.00
\end{array}\right)$$

How to denoise the covariance matrix?

- Uniform correlation

$$\hat{\rho} = 66.24\%$$

- One common factor + two specific factors

$$\hat{C} = \left(\begin{array}{cccc|cccc}
1.00 & & & & & & & \\
0.77 & 1.00 & & & & & & \\
0.77 & 0.77 & 1.00 & & & & & \\
0.77 & 0.77 & 0.77 & 1.00 & & & & \\
\hline
0.50 & 0.50 & 0.50 & 0.50 & 1.00 & & & \\
0.50 & 0.50 & 0.50 & 0.50 & 0.59 & 1.00 & & \\
0.50 & 0.50 & 0.50 & 0.50 & 0.59 & 0.59 & 1.00 & \\
0.50 & 0.50 & 0.50 & 0.50 & 0.59 & 0.59 & 0.59 & 1.00
\end{array}\right)$$

How to denoise the covariance matrix?

- Two-linear factor model

$$\hat{C} = \left(\begin{array}{cccc|cccc}
1.00 & & & & & & & \\
0.88 & 1.00 & & & & & & \\
0.88 & 0.94 & 1.00 & & & & & \\
0.63 & 0.67 & 0.66 & 1.00 & & & & \\
\hline
0.73 & 0.78 & 0.78 & 0.63 & 1.00 & & & \\
0.58 & 0.62 & 0.60 & 0.54 & 0.59 & 1.00 & & \\
0.56 & 0.59 & 0.58 & 0.56 & 0.60 & 0.54 & 1.00 & \\
0.64 & 0.68 & 0.66 & 0.65 & 0.69 & 0.62 & 0.67 & 1.00
\end{array}\right)$$

How to denoise the covariance matrix?

- RMT estimation

$$\hat{C} = \left(\begin{array}{cccc|cccc}
1.00 & & & & & & & \\
0.73 & 1.00 & & & & & & \\
0.72 & 0.76 & 1.00 & & & & & \\
0.61 & 0.64 & 0.64 & 1.00 & & & & \\
\hline
0.72 & 0.76 & 0.75 & 0.64 & 1.00 & & & \\
0.71 & 0.75 & 0.74 & 0.63 & 0.74 & 1.00 & & \\
0.63 & 0.66 & 0.65 & 0.56 & 0.66 & 0.65 & 1.00 & \\
0.68 & 0.72 & 0.71 & 0.60 & 0.71 & 0.70 & 0.62 & 1.00
\end{array}\right)$$

How to denoise the covariance matrix?

- Ledoit-Wolf shrinkage estimation (constant correlation matrix)

$$\hat{C} = \left(\begin{array}{cccc|cccc} 1.00 & & & & & & & \\ 0.77 & 1.00 & & & & & & \\ 0.77 & 0.80 & 1.00 & & & & & \\ 0.65 & 0.67 & 0.65 & 1.00 & & & & \\ \hline 0.72 & 0.71 & 0.72 & 0.63 & 1.00 & & & \\ 0.61 & 0.64 & 0.63 & 0.58 & 0.65 & 1.00 & & \\ 0.60 & 0.64 & 0.62 & 0.60 & 0.63 & 0.62 & 1.00 & \\ 0.65 & 0.67 & 0.67 & 0.67 & 0.67 & 0.63 & 0.66 & 1.00 \end{array}\right)$$

- We obtain:

$$\alpha^{\star} = 51.2\%$$

- What does this result become in the case of a multi-asset-class universe?

$$\alpha^{\star} \simeq 0$$

Why standard regularization techniques are not sufficient

Optimized portfolios are solutions of the following quadratic program:

$$\begin{aligned} x^{\star}(\gamma) &= \arg\min \frac{1}{2} x^{\top}\Sigma x - \gamma x^{\top}\mu \\ &\text{u.c.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^{\top} x = 1 \\ x \in \mathbb{R}^n \end{array} \right. \end{aligned}$$

We have:

$$x^{\star}(\gamma) = \frac{\Sigma^{-1}\mathbf{1}_n}{\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n} + \gamma \cdot \frac{\left(\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n\right)\Sigma^{-1}\mu - \left(\mathbf{1}_n^{\top}\Sigma^{-1}\mu\right)\Sigma^{-1}\mathbf{1}_n}{\mathbf{1}_n^{\top}\Sigma^{-1}\mathbf{1}_n}$$

Why standard regularization techniques are not sufficient

Optimal solutions are of the following form:

$$x^{\star} \propto f\left(\Sigma^{-1}\right)$$

The important quantity is then the precision matrix $\mathcal{I} = \Sigma^{-1}$, not the covariance matrix Σ

Why standard regularization techniques are not sufficient

- For the covariance matrix Σ, we have:

$$\Sigma = V \Lambda V^{\top}$$

where $V^{-1} = V^{\top}$ and $\Lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \ldots \geq \lambda_n$ the ordered eigenvalues

- The decomposition for the precisions matrix is

$$\mathcal{I} = U \Delta U^{\top}$$

- We have:

$$\begin{aligned} \Sigma^{-1} &= \left(V \Lambda V^{\top}\right)^{-1} \\ &= \left(V^{\top}\right)^{-1} \Lambda^{-1} V^{-1} \\ &= V \Lambda^{-1} V^{\top} \end{aligned}$$

- We deduce that $U = V$ and $\delta_i = 1/\lambda_{n-i+1}$

Why standard regularization techniques are not sufficient

Remark

The eigenvectors of the precision matrix are the same as those of the covariance matrix, but the eigenvalues of the precision matrix are the inverse of the eigenvalues of the covariance matrix. This means that the risk factors are the same, but they are in the reverse order

Why standard regularization techniques are not sufficient

Example 8

We consider a universe of 3 assets, where $\mu_1 = \mu_2 = 8\%$, $\mu_3 = 5\%$, $\sigma_1 = 20\%$, $\sigma_2 = 21\%$, $\sigma_3 = 10\%$ and $\rho_{i,j} = 80\%$.

The eigendecomposition of the covariance and precision matrices is:

	Covariance matrix Σ			Information matrix $\mathcal{I}$		
Asset / Factor	1	2	3	1	2	3
1	65.35%	−72.29%	−22.43%	−22.43%	−72.29%	65.35%
2	69.38%	69.06%	−20.43%	−20.43%	69.06%	69.38%
3	30.26%	−2.21%	95.29%	95.29%	−2.21%	30.26%
Eigenvalue	8.31%	0.84%	0.26%	379.97	119.18	12.04
% cumulated	88.29%	97.20%	100.00%	74.33%	97.65%	100.00%

⇒ It means that the first factor of the information matrix corresponds to the last factor of the covariance matrix and that the last factor of the information matrix corresponds to the first factor.

⇒ Optimization on arbitrage risk factors, idiosyncratic risk factors and (certainly) noise factors!

Why standard regularization techniques are not sufficient

Example 9

We consider a universe of 6 assets. The volatilities are respectively equal to 20%, 21%, 17%, 24%, 20% and 16%. For the correlation matrix, we have:

$$\rho = \begin{pmatrix} 1.00 & & & & & \\ 0.40 & 1.00 & & & & \\ 0.40 & 0.40 & 1.00 & & & \\ 0.50 & 0.50 & 0.50 & 1.00 & & \\ 0.50 & 0.50 & 0.50 & 0.60 & 1.00 & \\ 0.50 & 0.50 & 0.50 & 0.60 & 0.60 & 1.00 \end{pmatrix}$$

$\Rightarrow$ We compute the minimum variance (MV) portfolio with a shortsale constraint

Why standard regularization techniques are not sufficient

Table 9: Effect of deleting a PCA factor

$x^\star$	MV	$\lambda_1 = 0$	$\lambda_2 = 0$	$\lambda_3 = 0$	$\lambda_4 = 0$	$\lambda_5 = 0$	$\lambda_6 = 0$
$x_1^\star$	15.29	15.77	20.79	27.98	0.00	13.40	0.00
$x_2^\star$	10.98	16.92	1.46	12.31	0.00	8.86	0.00
$x_3^\star$	34.40	12.68	35.76	28.24	52.73	53.38	2.58
$x_4^\star$	0.00	22.88	0.00	0.00	0.00	0.00	0.00
$x_5^\star$	1.01	17.99	2.42	0.00	15.93	0.00	0.00
$x_6^\star$	38.32	13.76	39.57	31.48	31.34	24.36	97.42

Why standard regularization techniques are not sufficient

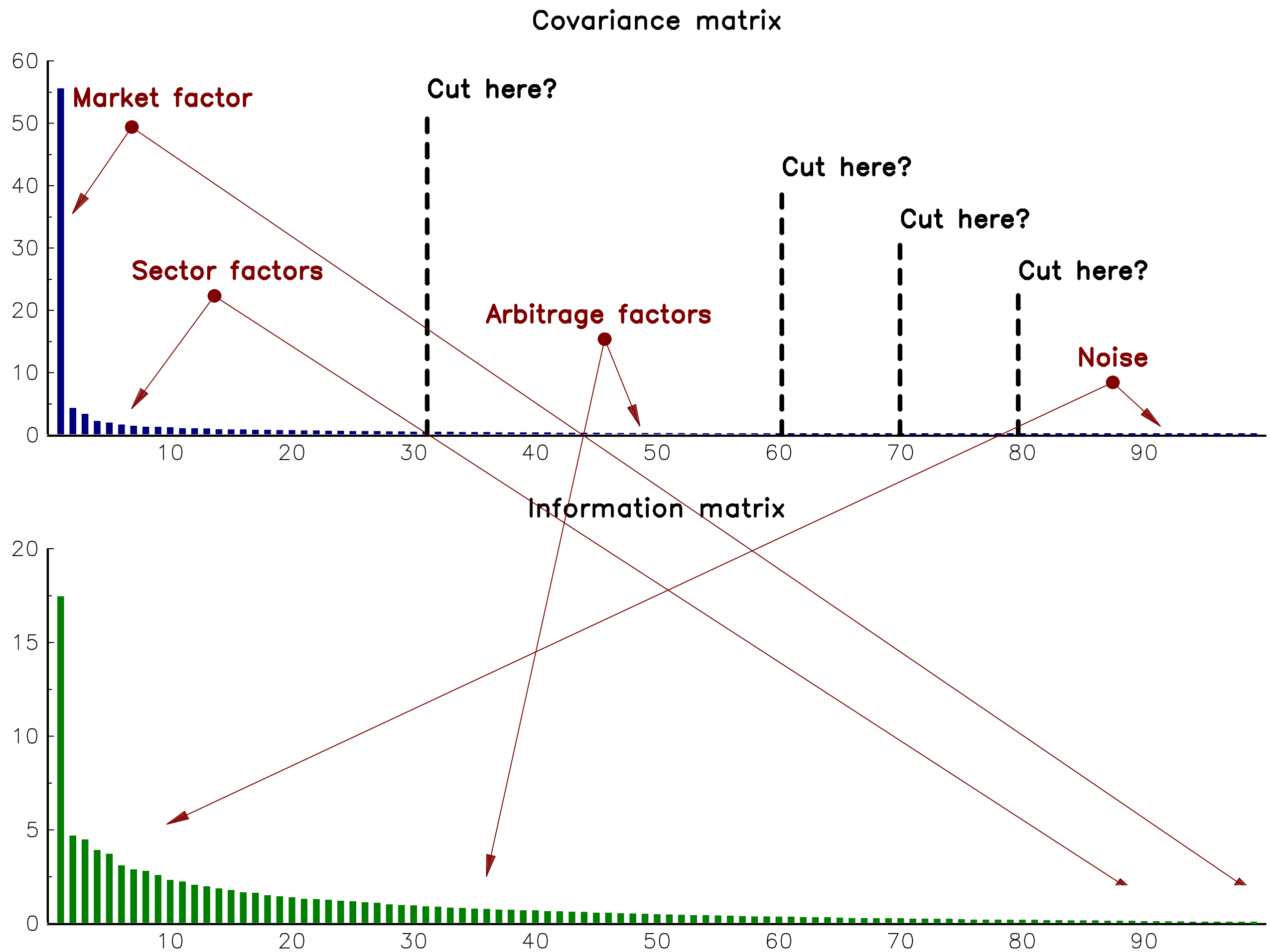


Figure 14: PCA applied to the stocks of the FTSE index (June 2012)

Arbitrage factors, hedging factors or risk factors

We consider the following linear regression model:

$$R_{i,t} = \beta_0 + \beta_i^\top R_t^{(-i)} + \varepsilon_{i,t}$$

- $R_t^{(-i)}$ denotes the vector of asset returns R_t excluding the i^{th} asset
- $\varepsilon_{i,t} \sim \mathcal{N}(0, s_i^2)$
- $\mathcal{R}_i^2$ is the R-squared of the linear regression

Precision matrix

Stevens (1998) shows that the precision matrix is given by:

$$\mathcal{I}_{i,i} = \frac{1}{\hat{\sigma}_i^2 (1 - \mathcal{R}_i^2)} \text{ and } \mathcal{I}_{i,j} = -\frac{\hat{\beta}_{i,j}}{\hat{\sigma}_i^2 (1 - \mathcal{R}_i^2)} = -\frac{\hat{\beta}_{j,i}}{\hat{\sigma}_j^2 \left(1 - \mathcal{R}_j^2\right)}$$

Arbitrage factors, hedging factors or risk factors

Example 10

We consider a universe of four assets. The expected returns are $\hat{\mu}_1 = 7\%$, $\hat{\mu}_2 = 8\%$, $\hat{\mu}_3 = 9\%$ and $\hat{\mu}_4 = 10\%$ whereas the volatilities are equal to $\hat{\sigma}_1 = 15\%$, $\hat{\sigma}_2 = 18\%$, $\hat{\sigma}_3 = 20\%$ and $\hat{\sigma}_4 = 25\%$. The correlation matrix is the following:

$$\hat{C} = \begin{pmatrix} 1.00 & & & \\ 0.50 & 1.00 & & \\ 0.50 & 0.50 & 1.00 & \\ 0.60 & 0.50 & 0.40 & 1.00 \end{pmatrix}$$

We do not impose that the sum of weights are equal to 100%

Arbitrage factors, hedging factors or risk factors

Table 10: Hedging portfolios when $\rho_{3,4} = 40\%$

Asset	$\hat{\beta}_i$				$\mathcal{R}_i^2$	$\hat{s}_i$	$\bar{\mu}_i$	$x^\star$
1		0.139	0.187	0.250	45.83%	11.04%	1.70%	69.80%
2	0.230		0.268	0.191	37.77%	14.20%	2.06%	51.18%
3	0.409	0.354		0.045	33.52%	16.31%	2.85%	53.66%
4	0.750	0.347	0.063		41.50%	19.12%	1.41%	19.28%

Table 11: Hedging portfolios when $\rho_{3,4} = 95\%$

Asset	$\hat{\beta}_i$				$\mathcal{R}_i^2$	$\hat{s}_i$	$\bar{\mu}_i$	$x^\star$
1		0.244	−0.595	0.724	47.41%	10.88%	3.16%	133.45%
2	0.443		0.470	−0.157	33.70%	14.66%	2.23%	52.01%
3	−0.174	0.076		0.795	91.34%	5.89%	1.66%	239.34%
4	0.292	−0.035	1.094		92.38%	6.90%	−1.61%	−168.67%

Arbitrage factors, hedging factors or risk factors

Table 12: Hedging portfolios (in %) at the end of 2006

	SPX	SX5E	TPX	RTY	EM	US HY	EMBI	EUR	JPY	GSCI
SPX		58.6	6.0	150.3	-30.8	-0.5	5.0	-7.3	15.3	-25.5
SX5E	9.0		-1.2	-1.3	35.2	0.8	3.2	-4.5	-5.0	-1.5
TPX	0.4	-0.6		-2.4	38.1	1.1	-3.5	-4.9	-0.8	-0.3
RTY	48.6	-2.7	-10.4		26.2	-0.6	1.9	0.2	-6.4	5.6
EM	-4.1	30.9	69.2	10.9		0.9	4.6	9.1	3.9	33.1
US HY	-5.0	53.5	160.0	-18.8	69.5		95.6	48.4	31.4	-211.7
EMBI	10.8	44.2	-102.1	12.3	73.4	19.4		-5.8	40.5	86.2
EUR	-3.6	-14.7	-33.4	0.3	33.8	2.3	-1.4		56.7	48.2
JPY	6.8	-14.5	-4.8	-8.8	12.7	1.3	8.4	50.4		-33.2
GSCI	-1.1	-0.4	-0.2	0.8	10.7	-0.9	1.8	4.2	-3.3	
$\hat{s}_i$	0.3	0.7	0.9	0.5	0.7	0.1	0.2	0.4	0.4	1.2
$\mathcal{R}_i^2$	83.0	47.7	34.9	82.4	60.9	39.8	51.6	42.3	43.7	12.1

Source: Bruder *et al.* (2013)

Arbitrage factors, hedging factors or risk factors

We finally obtain:

$$x_i^\star(\gamma) = \gamma \frac{\mu_i - \hat{\beta}_i^\top \mu^{(-i)}}{\hat{s}_i^2}$$

From this equation, we deduce the following conclusions:

1. The better the hedge, the higher the exposure. This is why highly correlated assets produces unstable MVO portfolios
2. The long/short position is defined by the sign of $\mu_i - \hat{\beta}_i^\top \mu^{(-i)}$. If the expected return of the asset is lower than the conditional expected return of the hedging portfolio, the weight is negative

Markowitz diversification	$\neq$	**Diversification of risk factors**
	$=$	**Concentration on arbitrage factors**

QP problem

We use the following formulation of the QP problem:

$$\begin{aligned} x^{\star} &= \arg\min \frac{1}{2} x^{\top} Q x - x^{\top} R \\ &\text{u.c.} \quad \left\{ \begin{array}{l} Ax = B \\ Cx \leq D \\ x^{-} \leq x \leq x^{+} \end{array} \right. \end{aligned}$$

Standard constraints

- γ-problem

$$\arg\min \frac{1}{2} x^\top \Sigma x - \gamma x^\top (\mu - r \mathbf{1}_n) \Rightarrow \left\{ \begin{array}{l} Q = \Sigma \\ R = \gamma \mu \end{array} \right.$$

- Full allocation

$$\mathbf{1}_n^\top x = 1 \Rightarrow \left\{ \begin{array}{l} A = \mathbf{1}_n^\top \\ B = 1 \end{array} \right.$$

- No short selling

$$x_i \geq 0 \Rightarrow x^- = \mathbf{0}_n$$

- Cash neutral (and portfolio optimization with unfunded strategies)

$$\mathbf{1}_n^\top x = 0 \Rightarrow \left\{ \begin{array}{l} A = \mathbf{1}_n^\top \\ B = 0 \end{array} \right.$$

Asset class constraints

Example 11

We consider a multi-asset universe of eight asset classes represented by the following indices:

- four equity indices: S&P 500, Eurostoxx, Topix, MSCI EM
- two bond indices: EGBI, US BIG
- two alternatives indices: GSCI, EPRA

The portfolio manager wants the following exposures:

- at least 50% bonds
- less than 10% commodities
- Emerging market equities cannot represent more than one third of the total exposure on equities

Asset class constraints

The constraints are then expressed as follows:

$$\begin{cases} x_5 + x_6 \geq 50\% \\ x_7 \leq 10\% \\ x_4 \leq \frac{1}{3}\left(x_1 + x_2 + x_3 + x_4\right) \end{cases}$$

The corresponding formulation $Cx \leq D$ of the QP problem is:

$$\left(\begin{array}{cccccccc} 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1/3 & -1/3 & -1/3 & 2/3 & 0 & 0 & 0 & 0 \end{array}\right)\left(\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{array}\right) \leq \left(\begin{array}{c} -0.50 \\ 0.10 \\ 0.00 \end{array}\right)$$

Non-standard constraints (turnover management)

- We want to limit the turnover of the long-only optimized portfolio with respect to a current portfolio x^0:

$$\Omega = \left\{ x \in [0,1]^n : \sum_{i=1}^{n} \left| x_i - x_i^0 \right| \leq \tau^+ \right\}$$

  where τ^+ is the maximum turnover
- Scherer (2007) proposes to introduce some additional variables x_i^- and x_i^+ such that:

$$x_i = x_i^0 + \Delta x_i^+ - \Delta x_i^-$$

  with $\Delta x_i^- \geq 0$ and $\Delta x_i^+ \geq 0$
- Δx_i^+ indicates a positive weight change with respect to the initial weight x_i^0
- Δx_i^- indicates a negative weight change with respect to the initial weight x_i^0

Non-standard constraints (turnover management)

- The expression of the turnover becomes:

$$\sum_{i=1}^{n} \left| x_i - x_i^0 \right| = \sum_{i=1}^{n} \left| \Delta x_i^+ - \Delta x_i^- \right| = \sum_{i=1}^{n} \Delta x_i^+ + \sum_{i=1}^{n} \Delta x_i^-$$

- We obtain the following γ-problem:

$$\begin{aligned} x^\star & = & \arg\min \frac{1}{2} x^\top \Sigma x - \gamma x^\top \mu \\ & \text{u.c.} & \left\{ \begin{array}{l} \sum_{i=1}^n x_i = 1 \\ x_i = x_i^0 + \Delta x_i^+ - \Delta x_i^- \\ \sum_{i=1}^n \Delta x_i^+ + \sum_{i=1}^n \Delta x_i^- \leq \tau^+ \\ 0 \leq x_i \leq 1 \\ 0 \leq \Delta x_i^- \leq 1 \\ 0 \leq \Delta x_i^+ \leq 1 \end{array} \right. \end{aligned}$$

Non-standard constraints (turnover management)

We obtain an augmented QP problem of dimension $3n$ instead of n:

$$\begin{aligned} X^{\star} &= \arg\min \frac{1}{2} X^{\top} Q X - X^{\top} R \\ & \text{u.c.} \quad \left\{ \begin{array}{l} AX = B \\ CX \leq D \\ \mathbf{0}_{3n} \leq X \leq \mathbf{1}_{3n} \end{array} \right. \end{aligned}$$

where X is a $3n \times 1$ vector:

$$X = \left(x_1, \ldots, x_n, \Delta x_1^{-}, \ldots, \Delta x_n^{-}, \Delta x_1^{+}, \ldots, \Delta x_n^{+} \right)$$

Non-standard constraints (turnover management)

The augmented QP matrices are:

$$Q_{3n\times 3n} = \begin{pmatrix} \Sigma & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{pmatrix}, \quad R_{3n\times 1} = \begin{pmatrix} \gamma\mu \\ \mathbf{0}_n \\ \mathbf{0}_n \end{pmatrix},$$

$$A_{(n+1)\times 3n} = \begin{pmatrix} \mathbf{1}_n^\top & \mathbf{0}_n^\top & \mathbf{0}_n^\top \\ I_n & I_n & -I_n \end{pmatrix}, \quad B_{(n+1)\times 1} = \begin{pmatrix} 1 \\ x^0 \end{pmatrix},$$

$$C_{1\times 3n} = \begin{pmatrix} \mathbf{0}_n^\top & \mathbf{1}_n^\top & \mathbf{1}_n^\top \end{pmatrix} \quad \text{and} \quad D_{1\times 1} = \tau^+$$

Non-standard constraints (turnover management)

Example 12

We consider four assets. Their expected returns are equal to 5%, 6%, 8% and 6% while their volatilities are equal to 15%, 20%, 25% and 30%. The correlation matrix of asset returns is given by the following matrix:

$$\rho = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.40 & 0.70 & 1.00 & \\ 0.50 & 0.40 & 0.80 & 1.00 \end{pmatrix}$$

We impose that the weights are positive

- The optimal portfolio $x^\star$ for a 15% volatility target is $(45.59\%, 24.74\%, 29.67\%, 0.00\%)$
- We assume that the current portfolio x^0 is $(30\%, 45\%, 15\%, 10\%)$
- If we move directly from portfolio x^0 to portfolio $x^\star$, the turnover is equal to 60.53%

Non-standard constraints (turnover management)

Table 13: Limiting the turnover of MVO portfolios

τ^{+}	5.00	10.00	25.00	50.00	75.00	x^{0}
$x_1^{\star}$		35.00	36.40	42.34	45.59	30.00
$x_2^{\star}$		45.00	42.50	30.00	24.74	45.00
$x_3^{\star}$		15.00	21.10	27.66	29.67	15.00
$x_4^{\star}$		5.00	0.00	0.00	0.00	10.00
$\mu(x^{\star})$		5.95	6.06	6.13	6.14	6.00
$\sigma(x^{\star})$		15.00	15.00	15.00	15.00	15.69
$\tau(x^{\star} \mid x^{0})$		10.00	25.00	50.00	60.53	

Non-standard constraints (transaction cost management)

Let c_i^- and c_i^+ be the bid and ask transactions costs. The net expected return is equal to:

$$\mu(x) = \sum_{i=1}^n x_i \mu_i - \sum_{i=1}^n \Delta x_i^- c_i^- - \sum_{i=1}^n \Delta x_i^+ c_i^+$$

The γ-problem becomes:

$$x^\star = \arg\min \frac{1}{2} x^\top \Sigma x - \gamma \left(\sum_{i=1}^n x_i \mu_i - \sum_{i=1}^n \Delta x_i^- c_i^- - \sum_{i=1}^n \Delta x_i^+ c_i^+ \right)$$

$$\text{u.c.} \quad \begin{cases} \sum_{i=1}^n \left(x_i + \Delta x_i^- c_i^- + \Delta x_i^+ c_i^+ \right) = 1 \\ x_i = x_i^0 + \Delta x_i^+ - \Delta x_i^- \\ 0 \leq x_i \leq 1 \\ 0 \leq \Delta x_i^- \leq 1 \\ 0 \leq \Delta x_i^+ \leq 1 \end{cases}$$

Non-standard constraints (transaction cost management)

The augmented QP problem becomes:

$$\begin{aligned} X^{\star} &= \arg\min \frac{1}{2} X^{\top} Q X - X^{\top} R \\ \text{u.c.} & \left\{ \begin{array}{l} AX = B \\ \mathbf{0}_{3n} \leq X \leq \mathbf{1}_{3n} \end{array} \right. \end{aligned}$$

where X is a $3n \times 1$ vector:

$$X = \left(x_1, \ldots, x_n, \Delta x_1^-, \ldots, \Delta x_n^-, \Delta x_1^+, \ldots, \Delta x_n^+\right)$$

and:

$$\begin{aligned} Q_{3n\times 3n} &= \begin{pmatrix} \Sigma & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{pmatrix}, \quad R_{3n\times 1} = \begin{pmatrix} \gamma\mu \\ -c^- \\ -c^+ \end{pmatrix}, \\ A_{(n+1)\times 3n} &= \begin{pmatrix} \mathbf{1}_n^{\top} & \left(c^-\right)^{\top} & \left(c^+\right)^{\top} \\ I_n & I_n & -I_n \end{pmatrix} \quad \text{and} \quad B_{(n+1)\times 1} = \begin{pmatrix} 1 \\ x^0 \end{pmatrix} \end{aligned}$$

Index sampling

Index sampling

The underlying idea is to replicate an index b with n stocks by a portfolio x with n_x stocks and $n_x \ll n$

From a mathematical point of view, index sampling can be written as a portfolio optimization problem with a benchmark:

$$
\begin{aligned}
x^{\star} &= \arg\min \frac{1}{2} (x - b)^{\top} \Sigma (x - b) \\
\text{u.c.} & \quad \left\{ \begin{array}{l} \mathbf{1}_n^{\top} x = 1 \\ x \geq \mathbf{0}_n \\ \sum_{i=1}^{n} \mathbb{1} \{x_i > 0\} \leq n_x \end{array} \right.
\end{aligned}
$$

where b is the vector of index weights

We obtain a mixed integer non-linear optimization problem

Index sampling

Three stepwise algorithms:

1. The backward elimination algorithm starts with all the stocks, computes the optimized portfolio, deletes the stock which presents the highest tracking error variance, and repeats this process until the number of stocks in the optimized portfolio reaches the target value n_x
2. The forward selection algorithm starts with no stocks in the portfolio, adds the stock which presents the smallest tracking error variance, and repeats this process until the number of stocks in the optimized portfolio reaches the target value n_x
3. The heuristic algorithm is a variant of the backward elimination algorithm, but the elimination process of the heuristic algorithm uses the criterion of the smallest weight

Heuristic algorithm

1. The algorithm is initialized with $\mathcal{N}_{(0)} = \emptyset$ and $x^{\star}_{(0)} = b$.
2. At the iteration k, we define a set $\mathcal{I}_{(k)}$ of stocks having the smallest positive weights in the portfolio $x^{\star}_{(k-1)}$. We then update the set $\mathcal{N}_{(k)}$ with $\mathcal{N}_{(k)} = \mathcal{N}_{(k-1)} \cup \mathcal{I}_{(k)}$ and define the upper bounds $x^{+}_{(k)}$:

$$x^{+}_{(k),i} = \left\{ \begin{array}{lll} 0 & \text{if} & i \in \mathcal{N}_{(k)} \\ 1 & \text{if} & i \notin \mathcal{N}_{(k)} \end{array} \right.$$

3. We solve the QP problem by using the new upper bounds $x^{+}_{(k)}$:

$$\begin{array}{rcl} x^{\star}_{(k)} & = & \arg\min \dfrac{1}{2} \left(x_{(k)} - b\right)^{\top} \Sigma \left(x_{(k)} - b\right) \\ & \text{u.c.} & \left\{ \begin{array}{l} \mathbf{1}_n^{\top} x_{(k)} = 1 \\ \mathbf{0}_n \leq x_{(k)} \leq x^{+}_{(k)} \end{array} \right. \end{array}$$

4. We iterate steps 2 and 3 until the convergence criterion:

$$\sum_{i=1}^{n} \mathbb{1}\left\{ x^{*}_{(k),i} > 0 \right\} \leq n_x$$

Complexity of the three numerical algorithms

The number of solved QP problems is respectively equal to:

- $n_b - n_x$ for the heuristic algorithm
- $(n_b - n_x)(n_b + n_x + 1)/2$ for the backward elimination algorithm
- $n_x(2n_b - n_x + 1)/2$ for the forward selection algorithm

		Number of solved QP problems		
n_b	n_x	Heuristic	Backward	Forward
50	10	40	1 220	455
	40	10	455	1 220
500	50	450	123 975	23 775
	450	50	23 775	123 975
1 500	100	1 400	1 120 700	145 050
	1 000	500	625 250	1 000 500

Index sampling (Eurostoxx 50, June 2012)

Table 14: Sampling the SX5E index with the heuristic algorithm

k	Stock	b_i	$\sigma\left(x_{(k)} \mid b\right)$
1	Nokia	0.45	0.18
2	Carrefour	0.60	0.23
3	Repsol	0.71	0.28
4	Unibail-Rodamco	0.99	0.30
5	Muenchener Rueckver	1.34	0.32
6	RWE	1.18	0.36
7	Koninklijke Philips	1.07	0.41
8	Generali	1.06	0.45
9	CRH	0.82	0.51
10	Volkswagen	1.34	0.55
42	LVMH	2.39	3.67
43	Telefonica	3.08	3.81
44	Bayer	3.51	4.33
45	Vinci	1.46	5.02
46	BBVA	2.13	6.53
47	Sanofi	5.38	7.26
48	Allianz	2.67	10.76
49	Total	5.89	12.83
50	Siemens	4.36	30.33

Index sampling (Eurostoxx 50, June 2012)

Table 15: Sampling the SX5E index with the backward elimination algorithm

k	Stock	b_i	$\sigma\left(x_{(k)} \mid b\right)$
1	Iberdrola	1.05	0.11
2	France Telecom	1.48	0.18
3	Carrefour	0.60	0.22
4	Muenchener Rueckver	1.34	0.26
5	Repsol	0.71	0.30
6	BMW	1.37	0.34
7	Generali	1.06	0.37
8	RWE	1.18	0.41
9	Koninklijke Philips	1.07	0.44
10	Air Liquide	2.10	0.48
42	GDF Suez	1.92	3.49
43	Bayer	3.51	3.88
44	BNP Paribas	2.26	4.42
45	Total	5.89	4.99
46	LVMH	2.39	5.74
47	Allianz	2.67	7.15
48	Sanofi	5.38	8.90
49	BBVA	2.13	12.83
50	Siemens	4.36	30.33

Index sampling (Eurostoxx 50, June 2012)

Table 16: Sampling the SX5E index with the forward selection algorithm

k	Stock	b_i	$\sigma\left(x_{(k)} \mid b\right)$
1	Siemens	4.36	12.83
2	Banco Santander	3.65	8.86
3	Bayer	3.51	6.92
4	Eni	3.32	5.98
5	Allianz	2.67	5.11
6	LVMH	2.39	4.55
7	France Telecom	1.48	3.93
8	Carrefour	0.60	3.62
9	BMW	1.37	3.35
41	Société Générale	1.07	0.50
42	CRH	0.82	0.45
43	Air Liquide	2.10	0.41
44	RWE	1.18	0.37
45	Nokia	0.45	0.33
46	Unibail-Rodamco	0.99	0.28
47	Repsol	0.71	0.24
48	Essilor	1.17	0.18
49	Muenchener Rueckver	1.34	0.11
50	Iberdrola	1.05	0.00

Index sampling

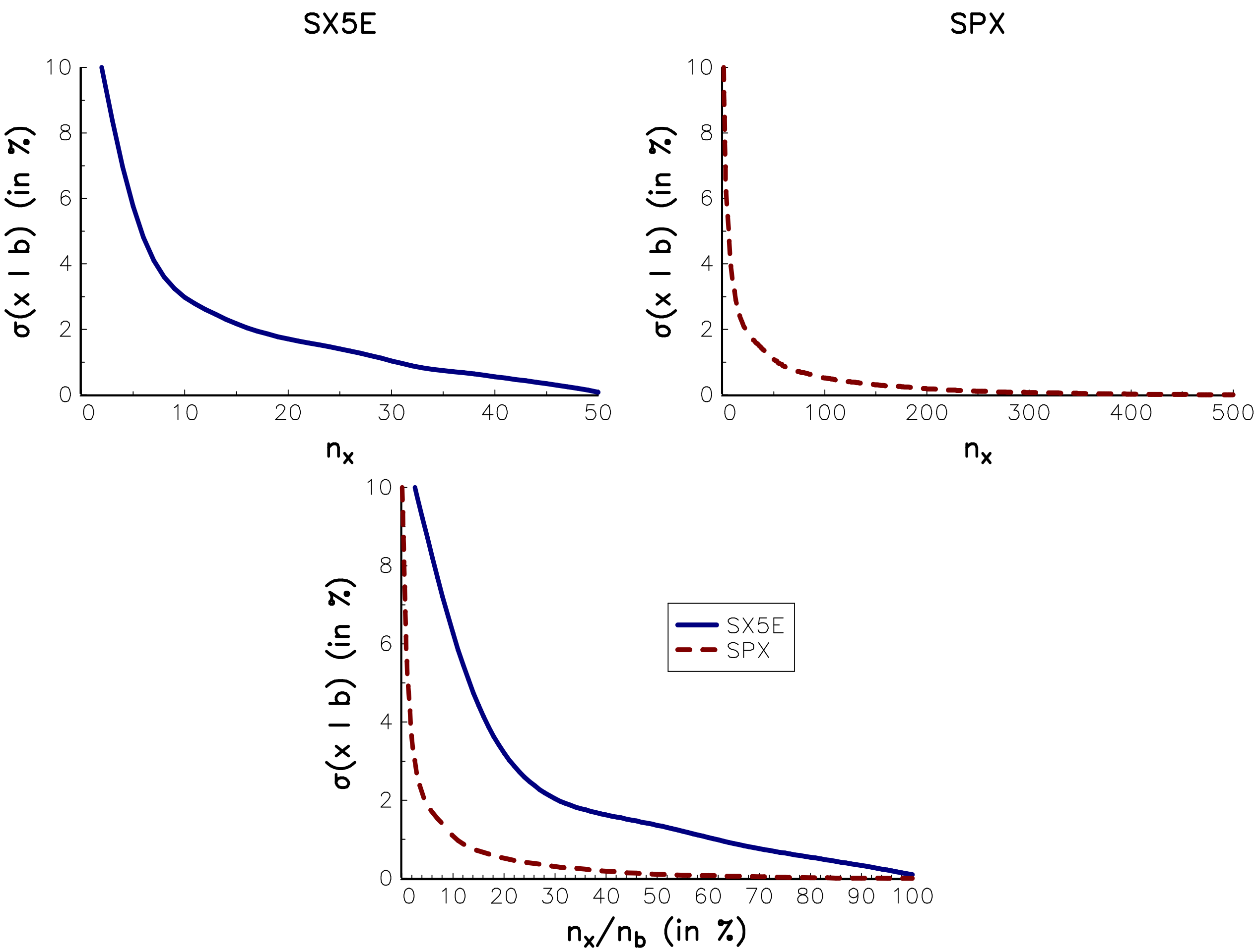


Figure 15: Sampling the SX5E and SPX indices (June 2012)

The impact of weight constraints

We specify the optimization problem as follows:

$$\min \frac{1}{2} x^\top \Sigma x$$
$$\text{u.c.} \left\{ \begin{array}{l} \mathbf{1}_n^\top x = 1 \\ \mu^\top x \geq \mu^\star \\ x \in \mathcal{C} \end{array} \right.$$

where $\mathcal{C}$ is the set of weights constraints. We define:

- the unconstrained portfolio $x^\star$ or $x^\star(\mu, \Sigma)$:

$$\mathcal{C} = \mathbb{R}^n$$

- the constrained portfolio $\tilde{x}$:

$$\mathcal{C}\left(x^-, x^+\right) = \left\{ x \in \mathbb{R}^n : x_i^- \leq x_i \leq x_i^+ \right\}$$

The impact of weight constraints

Theorem

Jagannathan and Ma (2003) show that the constrained portfolio is the solution of the unconstrained problem:

$$\tilde{x} = x^\star \left(\tilde{\mu}, \tilde{\Sigma}\right)$$

with:

$$\left\{ \begin{array}{l} \tilde{\mu} = \mu \\ \tilde{\Sigma} = \Sigma + \left(\lambda^+ - \lambda^-\right) \mathbf{1}_n^\top + \mathbf{1}_n \left(\lambda^+ - \lambda^-\right)^\top \end{array} \right.$$

where λ^- and λ^+ are the Lagrange coefficients vectors associated to the lower and upper bounds.

$\Rightarrow$ Introducing weights constraints is equivalent to introduce a shrinkage method or to introduce some relative views (similar to the Black-Litterman approach).

The impact of weight constraints

Proof (step 1)

Without weight constraints, the expression of the Lagrangian is:

$$\mathcal{L}(x; \lambda_0, \lambda_1) = \frac{1}{2} x^\top \Sigma x - \lambda_0 \left(\mathbf{1}_n^\top x - 1\right) - \lambda_1 \left(\mu^\top x - \mu^\star\right)$$

with $\lambda_0 \geq 0$ and $\lambda_1 \geq 0$. The first-order conditions are:

$$\left\{ \begin{array}{l} \Sigma x - \lambda_0 \mathbf{1}_n - \lambda_1 \mu = \mathbf{0}_n \\ \mathbf{1}_n^\top x - 1 = 0 \\ \mu^\top x - \mu^\star = 0 \end{array} \right.$$

We deduce that the solution $x^\star$ depends on the vector of expected return μ and the covariance matrix Σ and we note $x^\star = x^\star(\mu, \Sigma)$

The impact of weight constraints

Proof (step 2)

If we impose now the weight constraints $\mathcal{C}\left(x^{-},x^{+}\right)$, we have:

$$\begin{aligned}\mathcal{L}\left(x;\lambda_0,\lambda_1,\lambda^{-},\lambda^{+}\right) &= \frac{1}{2}x^\top \Sigma x-\lambda_0\left(\mathbf{1}_n^\top x-1\right)-\lambda_1\left(\mu^\top x-\mu^\star\right)- \\ & \quad \lambda^{-\top}\left(x-x^{-}\right)-\lambda^{+\top}\left(x^{+}-x\right)\end{aligned}$$

with $\lambda_0 \geq 0$, $\lambda_1 \geq 0$, $\lambda_i^{-} \geq 0$ and $\lambda_i^{+} \geq 0$. In this case, the Kuhn-Tucker conditions are:

$$\left\{\begin{array}{l}\Sigma x-\lambda_0\mathbf{1}_n-\lambda_1\mu-\lambda^{-}+\lambda^{+}=\mathbf{0}_n \\ \mathbf{1}_n^\top x-1=0 \\ \mu^\top x-\mu^\star=0 \\ \min\left(\lambda_i^{-},x_i-x_i^{-}\right)=0 \\ \min\left(\lambda_i^{+},x_i^{+}-x_i\right)=0\end{array}\right.$$

The impact of weight constraints

Proof (step 3)

Given a constrained portfolio $\tilde{x}$, it is possible to find a covariance matrix $\tilde{\Sigma}$ such that $\tilde{x}$ is the solution of unconstrained mean-variance portfolio. Let $\mathcal{E} = \left\{\tilde{\Sigma} > 0 : \tilde{x} = x^{\star}\left(\mu, \tilde{\Sigma}\right)\right\}$ denote the corresponding set:

$$\mathcal{E} = \left\{\tilde{\Sigma} > 0 : \tilde{\Sigma}\tilde{x} - \lambda_0 \mathbf{1}_n - \lambda_1 \mu = \mathbf{0}_n\right\}$$

Of course, the set $\mathcal{E}$ contains several solutions. From a financial point of view, we are interested in covariance matrices $\tilde{\Sigma}$ that are close to Σ. Jagannathan and Ma note that the matrix $\tilde{\Sigma}$ defined by:

$$\tilde{\Sigma} = \Sigma + \left(\lambda^{+} - \lambda^{-}\right)\mathbf{1}_n^{\top} + \mathbf{1}_n\left(\lambda^{+} - \lambda^{-}\right)^{\top}$$

is a solution of $\mathcal{E}$

The impact of weight constraints

Proof (step 4)

Indeed, we have:

$$\begin{aligned}
\tilde{\Sigma}\tilde{x} &= \Sigma\tilde{x} + \left(\lambda^{+} - \lambda^{-}\right)\mathbf{1}_n^\top \tilde{x} + \mathbf{1}_n\left(\lambda^{+} - \lambda^{-}\right)^\top \tilde{x} \\
&= \Sigma\tilde{x} + \left(\lambda^{+} - \lambda^{-}\right) + \mathbf{1}_n\left(\lambda^{+} - \lambda^{-}\right)^\top \tilde{x} \\
&= \lambda_0\mathbf{1}_n + \lambda_1\mu + \mathbf{1}_n\left(\lambda_0\mathbf{1}_n + \lambda_1\mu - \Sigma\tilde{x}\right)^\top \tilde{x} \\
&= \lambda_0\mathbf{1}_n + \lambda_1\mu + \mathbf{1}_n\left(\lambda_0 + \lambda_1\mu^\star - \tilde{x}^\top\Sigma\tilde{x}\right) \\
&= \left(2\lambda_0 - \tilde{x}^\top\Sigma\tilde{x} + \lambda_1\mu^\star\right)\mathbf{1}_n + \lambda_1\mu
\end{aligned}$$

It proves that $\tilde{x}$ is the solution of the unconstrained optimization problem. The Lagrange coefficients $\lambda_0^\star$ and $\lambda_1^\star$ for the unconstrained problem are respectively equal to $2\tilde{\lambda}_0 - \tilde{x}^\top\Sigma\tilde{x} + \tilde{\lambda}_1\mu^\star$ and $\tilde{\lambda}_1$ where $\tilde{\lambda}_0$ and $\tilde{\lambda}_1$ are the Lagrange coefficient for the constrained problem. Moreover, $\tilde{\Sigma}$ is generally a positive definite matrix

The impact of weight constraints

Example 13

We consider four assets. Their expected returns are equal to 5%, 6%, 8% and 6% while their volatilities are equal to 15%, 20%, 25% and 30%. The correlation matrix of asset returns is given by the following matrix:

$$C = \begin{pmatrix} 1.00 & & & \\ 0.10 & 1.00 & & \\ 0.40 & 0.70 & 1.00 & \\ 0.50 & 0.40 & 0.80 & 1.00 \end{pmatrix}$$

Given these parameters, the global minimum variance portfolio is equal to:

$$x^{\star} = \begin{pmatrix} 72.742\% \\ 49.464\% \\ -20.454\% \\ -1.753\% \end{pmatrix}$$

The impact of weight constraints

Table 17: Minimum variance portfolio when $x_i \geq 10\%$

$x_i^\star$	$\tilde{x}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
72.742	56.195	0.000	0.000	15.00	100.00			
49.464	23.805	0.000	0.000	20.00	10.00	100.00		
−20.454	10.000	1.190	0.000	19.67	10.50	58.71	100.00	
−1.753	10.000	1.625	0.000	23.98	17.38	16.16	67.52	100.00

Table 18: Minimum variance portfolio when $10\% \leq x_i \leq 40\%$

$x_i^\star$	$\tilde{x}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
72.742	40.000	0.000	0.915	20.20	100.00			
49.464	40.000	0.000	0.000	20.00	30.08	100.00		
−20.454	10.000	0.915	0.000	21.02	35.32	61.48	100.00	
−1.753	10.000	1.050	0.000	26.27	39.86	25.70	73.06	100.00

The impact of weight constraints

Table 19: Mean-variance portfolio when $10\% \leq x_i \leq 40\%$ and $\mu^\star = 6\%$

$x_i^\star$	$\tilde{x}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
65.866	40.000	0.000	0.125	15.81	100.00			
26.670	30.000	0.000	0.000	20.00	13.44	100.00		
32.933	20.000	0.000	0.000	25.00	41.11	70.00	100.00	
−25.470	10.000	1.460	0.000	24.66	23.47	19.06	73.65	100.00

Table 20: MSR portfolio when $10\% \leq x_i \leq 40\%$

$x_i^\star$	$\tilde{x}_i$	λ_i^-	λ_i^+	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$			
51.197	40.000	0.000	0.342	17.13	100.00			
50.784	39.377	0.000	0.000	20.00	18.75	100.00		
−21.800	10.000	0.390	0.000	23.39	36.25	66.49	100.00	
19.818	10.623	0.000	0.000	30.00	50.44	40.00	79.96	100.00

Variations on the efficient frontier

Exercise

We consider an investment universe of four assets. We assume that their expected returns are equal to 5%, 6%, 8% and 6%, and their volatilities are equal to 15%, 20%, 25% and 30%. The correlation matrix is:

$$\rho = \begin{pmatrix} 100\% & & & \\ 10\% & 100\% & & \\ 40\% & 70\% & 100\% & \\ 50\% & 40\% & 80\% & 100\% \end{pmatrix}$$

We note x_i the weight of the i^{th} asset in the portfolio. We only impose that the sum of the weights is equal to 100%.

Variations on the efficient frontier

Question 1

Represent the efficient frontier by considering the following values of γ: -1, -0.5, -0.25, 0, 0.25, 0.5, 1 and 2.

Variations on the efficient frontier

We deduce that the covariance matrix is:

$$\Sigma = \begin{pmatrix} 2.250 & 0.300 & 1.500 & 2.250 \\ 0.300 & 4.000 & 3.500 & 2.400 \\ 1.500 & 3.500 & 6.250 & 6.000 \\ 2.250 & 2.400 & 6.000 & 9.000 \end{pmatrix} \times 10^{-2}$$

We then have to solve the γ-formulation of the Markowitz problem:

$$\begin{aligned} x^{\star}(\gamma) &= \arg\min \frac{1}{2} x^{\top} \Sigma x - \gamma x^{\top} \mu \\ \text{u.c.} \quad & \mathbf{1}_n^{\top} x = 1 \end{aligned}$$

We obtain the results[1] given in Table 21. We represent the efficient frontier in Figure 16.

[1] The weights, expected returns and volatilities are expressed in %.

Variations on the efficient frontier

Table 21: Solution of Question 1

γ	-1.00	-0.50	-0.25	0.00	0.25	0.50	1.00	2.00
$x_1^\star$	94.04	83.39	78.07	72.74	67.42	62.09	51.44	30.15
$x_2^\star$	120.05	84.76	67.11	49.46	31.82	14.17	-21.13	-91.72
$x_3^\star$	-185.79	-103.12	-61.79	-20.45	20.88	62.21	144.88	310.22
$x_4^\star$	71.69	34.97	16.61	-1.75	-20.12	-38.48	-75.20	-148.65
$\mu(x^\star)$	1.34	3.10	3.98	4.86	5.74	6.62	8.38	11.90
$\sigma(x^\star)$	22.27	15.23	12.88	12.00	12.88	15.23	22.27	39.39

Variations on the efficient frontier

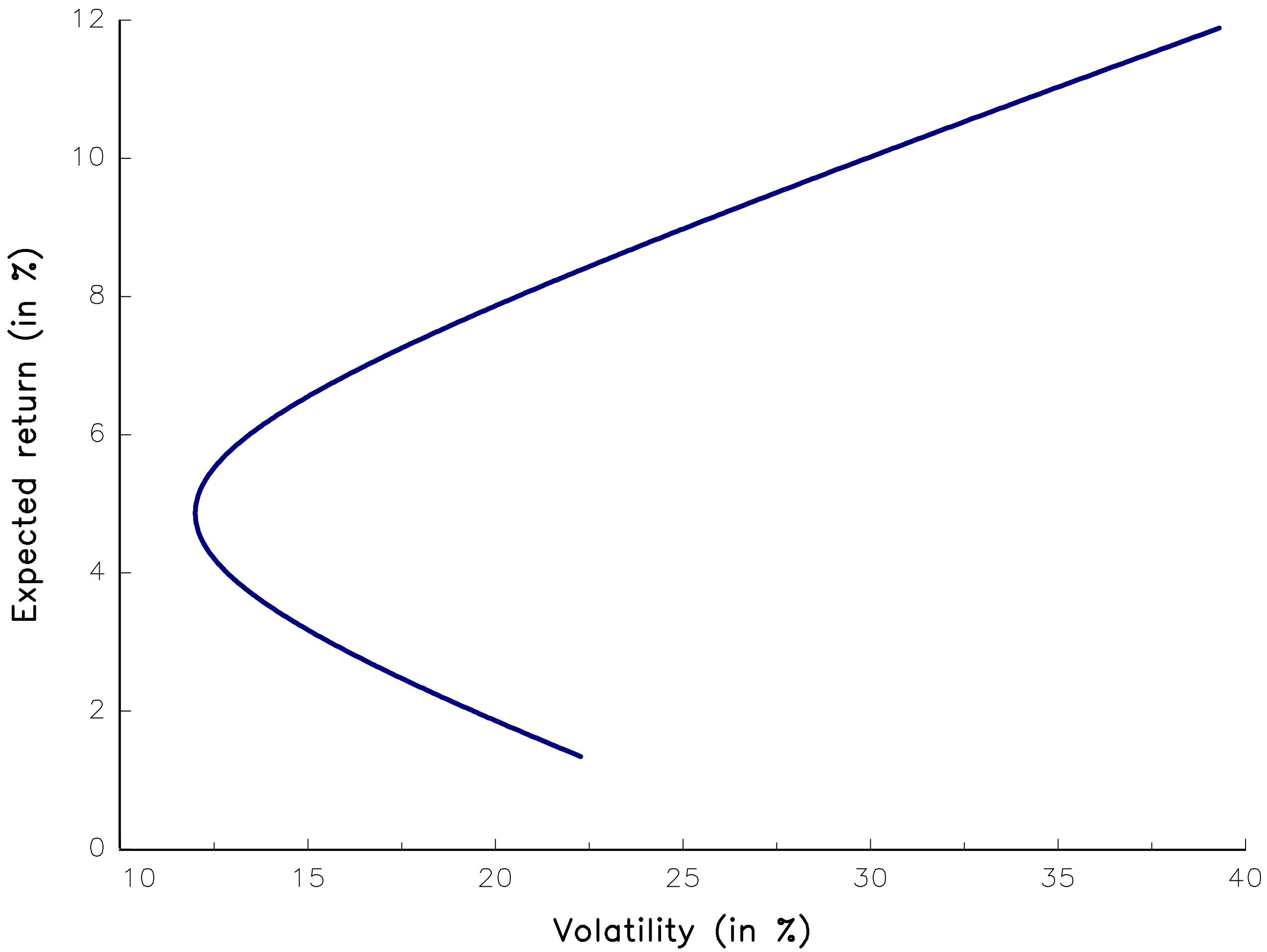


Figure 16: Markowitz efficient frontier

Variations on the efficient frontier

Question 2

Calculate the minimum variance portfolio. What are its expected return and its volatility?

Variations on the efficient frontier

We solve the γ-problem with $\gamma = 0$. The minimum variance portfolio is then $x_1^\star = 72.74\%$, $x_2^\star = 49.46\%$, $x_3^\star = -20.45\%$ and $x_4^\star = -1.75\%$. We deduce that $\mu(x^\star) = 4.86\%$ and $\sigma(x^\star) = 12.00\%$.

Variations on the efficient frontier

Question 3

Calculate the optimal portfolio which has an ex-ante volatility $\sigma^{\star}$ equal to 10%. Same question if $\sigma^{\star} = 15\%$ and $\sigma^{\star} = 20\%$.

Variations on the efficient frontier

There is no solution when the target volatility $\sigma^\star$ is equal to 10% because the minimum variance portfolio has a volatility larger than 10%. Finding the optimized portfolio for $\sigma^\star = 15\%$ or $\sigma^\star = 20\%$ is equivalent to solving a σ-problem. If $\sigma^\star = 15\%$ (resp. $\sigma^\star = 20\%$), we obtain an implied value of γ equal to 0.48 (resp. 0.85). Results are given in the following Table:

$\sigma^\star$	15.00	20.00
$x_1^\star$	62.52	54.57
$x_2^\star$	15.58	−10.75
$x_3^\star$	58.92	120.58
$x_4^\star$	−37.01	−64.41
$\mu(x^\star)$	6.55	7.87
γ	0.48	0.85

Variations on the efficient frontier

Question 4

We note $x^{(1)}$ the minimum variance portfolio and $x^{(2)}$ the optimal portfolio with $\sigma^{\star} = 20\%$. We consider the set of portfolios $x^{(\alpha)}$ defined by the relationship:

$$x^{(\alpha)} = (1 - \alpha)\, x^{(1)} + \alpha x^{(2)}$$

In the previous efficient frontier, place the portfolios $x^{(\alpha)}$ when α is equal to -0.5, -0.25, 0, 0.1, 0.2, 0.5, 0.7 and 1. What do you observe? Comment on this result.

Variations on the efficient frontier

Let $x^{(\alpha)}$ be the portfolio defined by the relationship $x^{(\alpha)} = (1-\alpha)\, x^{(1)} + \alpha x^{(2)}$ where $x^{(1)}$ is the minium variance portfolio and $x^{(2)}$ is the optimized portfolio with a 20% ex-ante volatility. We obtain the following results:

α	$\sigma\left(x^{(\alpha)}\right)$	$\mu\left(x^{(\alpha)}\right)$
-0.50	14.42	3.36
-0.25	12.64	4.11
0.00	12.00	4.86
0.10	12.10	5.16
0.20	12.41	5.46
0.50	14.42	6.36
0.70	16.41	6.97
1.00	20.00	7.87

We have reported these portfolios in Figure 17. We notice that they are located on the efficient frontier. This is perfectly normal because we know that a combination of two optimal portfolios corresponds to another optimal portfolio.

Variations on the efficient frontier

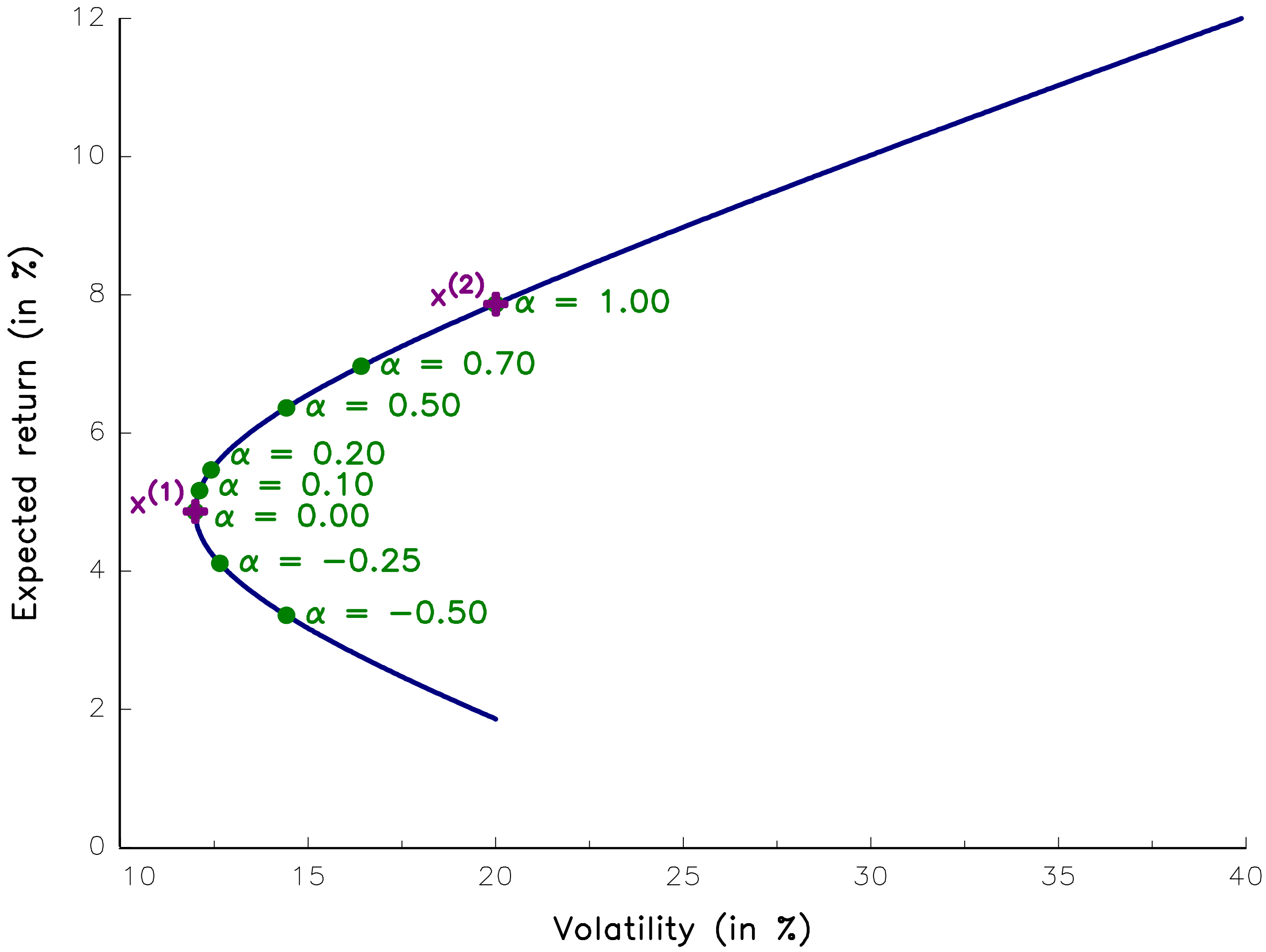


Figure 17: Mean-variance diagram of portfolios $x^{(\alpha)}$

Variations on the efficient frontier

Question 5

Repeat Questions 3 and 4 by considering the constraint $0 \leq x_i \leq 1$.
Explain why we do not retrieve the same observation.

Variations on the efficient frontier

If we consider the constraint $0 \leq x_i \leq 1$, the γ-formulation of the Markowitz problem becomes:

$$\begin{aligned} x^\star(\gamma) &= \arg\min \frac{1}{2} x^\top \Sigma x - \gamma x^\top \mu \\ &\text{u.c.} \quad \left\{ \begin{array}{l} \mathbf{1}_n^\top x = 1 \\ \mathbf{0}_n \leq x \leq \mathbf{1}_n \end{array} \right. \end{aligned}$$

Variations on the efficient frontier

We obtain the following results:

$\sigma^\star$	MV	12.00	15.00	20.00
$x_1^\star$	65.49	✓	45.59	24.88
$x_2^\star$	34.51	✓	24.74	4.96
$x_3^\star$	0.00	✓	29.67	70.15
$x_4^\star$	0.00	✓	0.00	0.00
$\mu(x^\star)$	5.35	✓	6.14	7.15
$\sigma(x^\star)$	12.56	✓	15.00	20.00
γ	0.00	✓	0.62	1.10

We observe that we cannot target a volatility $\sigma^\star = 10\%$. Moreover, the expected return $\mu(x^\star)$ of the optimal portfolios are reduced due to the additional constraints.

Variations on the efficient frontier

Question 6

We now include in the investment universe a fifth asset corresponding to the risk-free asset. Its return is equal to 3%.

Variations on the efficient frontier

Question 6.a

Define the expected return vector and the covariance matrix of asset returns.

Variations on the efficient frontier

We have:

$$\mu = \begin{pmatrix} 5.0 \\ 6.0 \\ 8.0 \\ 6.0 \\ 3.0 \end{pmatrix} \times 10^{-2}$$

and:

$$\Sigma = \begin{pmatrix} 2.250 & 0.300 & 1.500 & 2.250 & 0.000 \\ 0.300 & 4.000 & 3.500 & 2.400 & 0.000 \\ 1.500 & 3.500 & 6.250 & 6.000 & 0.000 \\ 2.250 & 2.400 & 6.000 & 9.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{pmatrix} \times 10^{-2}$$

Variations on the efficient frontier

Question 6.b

Deduce the efficient frontier by solving directly the quadratic problem.

Variations on the efficient frontier

We solve the γ-problem and obtain the efficient frontier given in Figure 18.

Variations on the efficient frontier

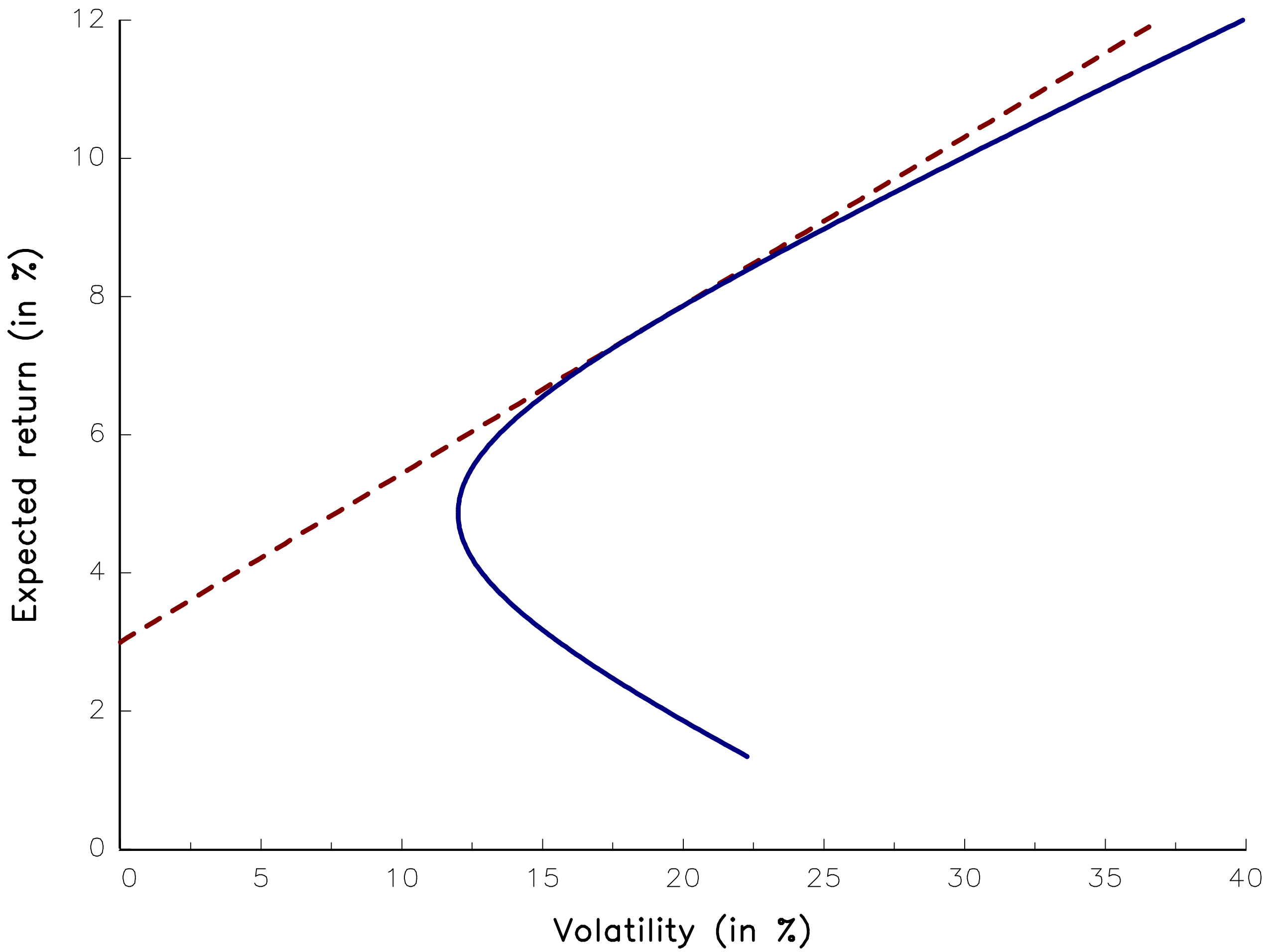


Figure 18: Efficient frontier when the risk-free asset is introduced

Variations on the efficient frontier

Question 6.c

What is the shape of the efficient frontier? Comment on this result.

Variations on the efficient frontier

This efficient frontier is a straight line. This line passes through the risk-free asset and is tangent to the efficient frontier of Figure 16. This question is a direct application of the *Separation Theorem* of Tobin.

Variations on the efficient frontier

Question 6.d

Choose two arbitrary portfolios $x^{(1)}$ and $x^{(2)}$ of this efficient frontier. Deduce the Sharpe ratio of the tangency portfolio.

Variations on the efficient frontier

We consider two optimized portfolios of this efficient frontier. They corresponds to $\gamma = 0.25$ and $\gamma = 0.50$. We obtain the following results:

γ	0.25	0.50
$x_1^\star$	18.23	36.46
$x_2^\star$	-1.63	-3.26
$x_3^\star$	34.71	69.42
$x_4^\star$	-18.93	-37.86
$x_5^\star$	67.62	35.24
$\mu(x^\star)$	4.48	5.97
$\sigma(x^\star)$	6.09	12.18

Variations on the efficient frontier

The first portfolio has an expected return equal to 4.48% and a volatility equal to 6.09%. The weight of the risk-free asset is 67.62%. This explains the low volatility of this portfolio. For the second portfolio, the weight of the risk-free asset is lower and equal to 35.24%. The expected return and the volatility are then equal to 5.97% and 12.18%. We note $x^{(1)}$ and $x^{(2)}$ these two portfolios. By definition, the Sharpe ratio of the market portfolio $x^{\star}$ is the tangency of the line. We deduce that:

$$\begin{aligned}\mathrm{SR}\left(x^{\star} \mid r\right) &= \frac{\mu\left(x^{(2)}\right)-\mu\left(x^{(1)}\right)}{\sigma\left(x^{(2)}\right)-\sigma\left(x^{(1)}\right)} \\ &= \frac{5.97-4.48}{12.18-6.09} \\ &= 0.2436\end{aligned}$$

The Sharpe ratio of the market portfolio $x^{\star}$ is then equal to 0.2436.

Variations on the efficient frontier

Question 6.e

Calculate then the composition of the tangency portfolio from $x^{(1)}$ and $x^{(2)}$.

Variations on the efficient frontier

By construction, every portfolio $x^{(\alpha)}$ which belongs to the tangency line is a linear combination of two portfolios $x^{(1)}$ and $x^{(2)}$ of this efficient frontier:

$$x^{(\alpha)} = (1 - \alpha) x^{(1)} + \alpha x^{(2)}$$

The market portfolio $x^{\star}$ is the portfolio $x^{(\alpha)}$ which has a zero weight in the risk-free asset. We deduce that the value $\alpha^{\star}$ which corresponds to the market portfolio satisfies the following relationship:

$$(1 - \alpha^{\star}) x_5^{(1)} + \alpha^{\star} x_5^{(2)} = 0$$

because the risk-free asset is the fifth asset of the portfolio.

Variations on the efficient frontier

It follows that:

$$\begin{aligned}\alpha^{\star} &= \frac{x_5^{(1)}}{x_5^{(1)} - x_5^{(2)}} \\ &= \frac{67.62}{67.62 - 35.24} \\ &= 2.09\end{aligned}$$

We deduce that the market portfolio is:

$$x^{\star} = (1 - 2.09) \cdot \begin{pmatrix} 18.23 \\ -1.63 \\ 34.71 \\ -18.93 \\ 67.62 \end{pmatrix} + 2.09 \cdot \begin{pmatrix} 36.46 \\ -3.26 \\ 69.42 \\ -37.86 \\ 35.24 \end{pmatrix} = \begin{pmatrix} 56.30 \\ -5.04 \\ 107.21 \\ -58.46 \\ 0.00 \end{pmatrix}$$

We check that the Sharpe ratio of this portfolio is 0.2436.

Variations on the efficient frontier

Question 7

We consider the general framework with n risky assets whose vector of expected returns is μ and the covariance matrix of asset returns is Σ while the return of the risk-free asset is r. We note $\tilde{x}$ the portfolio invested in the $n+1$ assets. We have:

$$\tilde{x} = \begin{pmatrix} x \\ x_r \end{pmatrix}$$

with x the weight vector of risky assets and x_r the weight of the risk-free asset. We impose the following constraint:

$$\sum_{i=1}^{n} \tilde{x}_i = \sum_{i=1}^{n} x_i = 1$$

Variations on the efficient frontier

Question 7.a

Define $\tilde{\mu}$ and $\tilde{\Sigma}$ the vector of expected returns and the covariance matrix of asset returns associated with the $n+1$ assets.

Variations on the efficient frontier

We have:

$$\tilde{\mu} = \begin{pmatrix} \mu \\ r \end{pmatrix}$$

and:

$$\tilde{\Sigma} = \begin{pmatrix} \Sigma & \mathbf{0}_n \\ \mathbf{0}_n^\top & 0 \end{pmatrix}$$

Variations on the efficient frontier

Question 7.b

By using the Markowitz ϕ-problem, retrieve the *Separation Theorem* of Tobin.

Variations on the efficient frontier

If we include the risk-free asset, the Markowitz ϕ-problem becomes:

$$\begin{aligned} \tilde{x}^{\star}(\phi) &= \arg\max \tilde{x}^{\top}\tilde{\mu} - \frac{\phi}{2}\tilde{x}^{\top}\tilde{\Sigma}\tilde{x} \\ & \text{u.c.} \quad \mathbf{1}_n^{\top}\tilde{x} = 1 \end{aligned}$$

We note that the objective function can be written as follows:

$$\begin{aligned} f(\tilde{x}) &= \tilde{x}^{\top}\tilde{\mu} - \frac{\phi}{2}\tilde{x}^{\top}\tilde{\Sigma}\tilde{x} \\ &= x^{\top}\mu + x_r r - \frac{\phi}{2}x^{\top}\Sigma x \\ &= g(x, x_r) \end{aligned}$$

The constraint becomes $\mathbf{1}_n^{\top}x + x_r = 1$. We deduce that the Lagrange function is:

$$\mathcal{L}(x, x_r; \lambda_0) = x^{\top}\mu + x_r r - \frac{\phi}{2}x^{\top}\Sigma x - \lambda_0\left(\mathbf{1}_n^{\top}x + x_r - 1\right)$$

Variations on the efficient frontier

The first-order conditions are:

$$\left\{ \begin{array}{l} \partial_x \mathcal{L}(x, x_r; \lambda_0) = \mu - \phi \Sigma x - \lambda_0 \mathbf{1}_n = \mathbf{0}_n \\ \partial_{x_r} \mathcal{L}(x, x_r; \lambda_0) = r - \lambda_0 = 0 \\ \partial_{\lambda_0} \mathcal{L}(x, x_r; \lambda_0) = \mathbf{1}_n^\top x + x_r - 1 = 0 \end{array} \right.$$

The solution of the optimization problem is then:

$$\left\{ \begin{array}{l} x^\star = \phi^{-1} \Sigma^{-1} (\mu - r \mathbf{1}_n) \\ \lambda_0^\star = r \\ x_r^\star = 1 - \phi^{-1} \mathbf{1}_n^\top \Sigma^{-1} (\mu - r \mathbf{1}_n) \end{array} \right.$$

Let $x_0^\star$ be the following portfolio:

$$x_0^\star = \frac{\Sigma^{-1} (\mu - r \mathbf{1}_n)}{\mathbf{1}_n^\top \Sigma^{-1} (\mu - r \mathbf{1}_n)}$$

Variations on the efficient frontier

We can then write the solution of the optimization problem in the following way:

$$\begin{cases} x^\star = \alpha x_0^\star \\ \lambda_0^\star = r \\ x_r^\star = 1 - \alpha \\ \alpha = \phi^{-1} \mathbf{1}_n^\top \Sigma^{-1} (\mu - r\mathbf{1}_n) \end{cases}$$

The first equation indicates that the relative proportions of risky assets in the optimized portfolio remain constant. If $\phi = \phi_0 = \mathbf{1}_n^\top \Sigma^{-1} (\mu - r\mathbf{1}_n)$, then $x^\star = x_0^\star$ and $x_r^\star = 0$. We deduce that $x_0^\star$ is the tangency portfolio. If $\phi \neq \phi_0$, $x^\star$ is proportional to $x_0^\star$ and the wealth invested in the risk-free asset is the complement $(1 - \alpha)$ to obtain a total exposure equal to 100%. We retrieve then the separation theorem:

$$\tilde{x}^\star = \underbrace{\alpha \cdot \begin{pmatrix} x_0^\star \\ 0 \end{pmatrix}}_{\text{risky assets}} + \underbrace{(1-\alpha) \cdot \begin{pmatrix} \mathbf{0}_n \\ 1 \end{pmatrix}}_{\text{risk-free asset}}$$

Beta coefficient

Question 1

We consider an investment universe of n assets with:

$$R = \begin{pmatrix} R_1 \\ \vdots \\ R_n \end{pmatrix} \sim \mathcal{N}(\mu, \Sigma)$$

The weights of the market portfolio (or the benchmark) are $b = (b_1, \ldots, b_n)$.

Beta coefficient

Question 1.a

Define the beta β_i of asset i with respect to the market portfolio.

Beta coefficient

The beta of an asset is the ratio between its covariance with the market portfolio return and the variance of the market portfolio return. In the CAPM theory, we have:

$$\mathbb{E}\left[R_i\right] = r + \beta_i \left(\mathbb{E}\left[R\left(b\right)\right] - r\right)$$

where R_i is the return of asset i, $R\left(b\right)$ is the return of the market portfolio and r is the risk-free rate. The beta β_i of asset i is:

$$\beta_i = \frac{\operatorname{cov}\left(R_i, R\left(b\right)\right)}{\operatorname{var}\left(R\left(b\right)\right)}$$

Let Σ be the covariance matrix of asset returns. We have $\operatorname{cov}\left(R, R\left(b\right)\right) = \Sigma b$ and $\operatorname{var}\left(R\left(b\right)\right) = b^\top \Sigma b$. We deduce that:

$$\beta_i = \frac{\left(\Sigma b\right)_i}{b^\top \Sigma b}$$

Beta coefficient

Question 1.b

Let X_1, X_2 and X_3 be three random variables. Show that:

$$\operatorname{cov}\left(c_1 X_1 + c_2 X_2, X_3\right) = c_1 \operatorname{cov}\left(X_1, X_3\right) + c_2 \operatorname{cov}\left(X_2, X_3\right)$$

Beta coefficient

We recall that the mathematical operator $\mathbb{E}$ is bilinear. Let c be the covariance $\text{cov}\left(c_1 X_1 + c_2 X_2, X_3\right)$. We then have:

$$
\begin{aligned}
c &= \mathbb{E}\left[\left(c_1 X_1 + c_2 X_2 - \mathbb{E}\left[c_1 X_1 + c_2 X_2\right]\right)\left(X_3 - \mathbb{E}\left[X_3\right]\right)\right] \\
&= \mathbb{E}\left[\left(c_1\left(X_1 - \mathbb{E}\left[X_1\right]\right) + c_2\left(X_2 - \mathbb{E}\left[X_2\right]\right)\right)\left(X_3 - \mathbb{E}\left[X_3\right]\right)\right] \\
&= c_1 \mathbb{E}\left[\left(X_1 - \mathbb{E}\left[X_1\right]\right)\left(X_3 - \mathbb{E}\left[X_3\right]\right)\right] + c_2 \mathbb{E}\left[\left(X_2 - \mathbb{E}\left[X_2\right]\right)\left(X_3 - \mathbb{E}\left[X_3\right]\right)\right] \\
&= c_1 \text{cov}\left(X_1, X_3\right) + c_2 \text{cov}\left(X_2, X_3\right)
\end{aligned}
$$

Beta coefficient

Question 1.c

We consider the asset portfolio $x = (x_1, \ldots, x_n)$ such that $\sum_{i=1}^{n} x_i = 1$. What is the relationship between the beta $\beta(x \mid b)$ of the portfolio and the betas β_i of the assets?

Beta coefficient

We have:

$$\begin{aligned}
\beta(x \mid b) &= \frac{\operatorname{cov}(R(x), R(b))}{\operatorname{var}(R(b))} = \frac{\operatorname{cov}\left(x^\top R, b^\top R\right)}{\operatorname{var}\left(b^\top R\right)} \\
&= \frac{x^\top \mathbb{E}\left[(R-\mu)(R-\mu)^\top\right] b}{b^\top \mathbb{E}\left[(R-\mu)(R-\mu)^\top\right] b} \\
&= \frac{x^\top \Sigma b}{b^\top \Sigma b} = x^\top \frac{\Sigma b}{b^\top \Sigma b} = x^\top \beta = \sum_{i=1}^n x_i \beta_i
\end{aligned}$$

with $\beta = (\beta_1, \ldots, \beta_n)$. The beta of portfolio x is then the weighted mean of asset betas. Another way to show this result is to exploit the result of Question 1.b. We have:

$$\beta(x \mid b) = \frac{\operatorname{cov}\left(\sum_{i=1}^n x_i R_i, R(b)\right)}{\operatorname{var}(R(b))} = \sum_{i=1}^n x_i \frac{\operatorname{cov}(R_i, R(b))}{\operatorname{var}(R(b))} = \sum_{i=1}^n x_i \beta_i$$

Beta coefficient

Question 1.d

Calculate the beta of the portfolios $x^{(1)}$ and $x^{(2)}$ with the following data:

i	1	2	3	4	5
β_i	0.7	0.9	1.1	1.3	1.5
$x_i^{(1)}$	0.5	0.5			
$x_i^{(2)}$	0.25	0.25	0.5	0.5	-0.5

Beta coefficient

We obtain $\beta\left(x^{(1)} \mid b\right) = 0.80$ and $\beta\left(x^{(2)} \mid b\right) = 0.85$.

Beta coefficient

Question 2

We assume that the market portfolio is the equally weighted portfolio[a].

[a] We have $b_i = n^{-1}$.

Beta coefficient

Question 2.a

Show that $\sum_{i=1}^{n} \beta_i = n$.

Beta coefficient

The weights of the market portfolio are then $b = n^{-1}\mathbf{1}_n$. We have:

$$\beta = \frac{\operatorname{cov}\left(R, R\left(b\right)\right)}{\operatorname{var}\left(R\left(b\right)\right)} = \frac{\Sigma b}{b^\top \Sigma b} = \frac{n^{-1}\Sigma \mathbf{1}_n}{n^{-2}\left(\mathbf{1}_n^\top \Sigma \mathbf{1}_n\right)} = n\frac{\Sigma \mathbf{1}_n}{\left(\mathbf{1}_n^\top \Sigma \mathbf{1}_n\right)}$$

We deduce that:

$$\sum_{i=1}^n \beta_i = \mathbf{1}_n^\top \beta = \mathbf{1}_n^\top n\frac{\Sigma \mathbf{1}_n}{\left(\mathbf{1}_n^\top \Sigma \mathbf{1}_n\right)} = n\frac{\mathbf{1}_n^\top \Sigma \mathbf{1}_n}{\left(\mathbf{1}_n^\top \Sigma \mathbf{1}_n\right)} = n$$

Beta coefficient

Question 2.b

We consider the case $n = 3$. Show that $\beta_1 \geq \beta_2 \geq \beta_3$ implies $\sigma_1 \geq \sigma_2 \geq \sigma_3$ if $\rho_{i,j} = 0$.

Beta coefficient

If $\rho_{i,j} = 0$, we have:

$$\beta_i = n \frac{\sigma_i^2}{\sum_{j=1}^n \sigma_j^2}$$

We deduce that:

$$\begin{aligned} \beta_1 \geq \beta_2 \geq \beta_3 & \Rightarrow & n \frac{\sigma_1^2}{\sum_{j=1}^3 \sigma_j^2} \geq n \frac{\sigma_2^2}{\sum_{j=1}^3 \sigma_j^2} \geq n \frac{\sigma_3^2}{\sum_{j=1}^3 \sigma_j^2} \\ & \Rightarrow & \sigma_1^2 \geq \sigma_2^2 \geq \sigma_3^2 \\ & \Rightarrow & \sigma_1 \geq \sigma_2 \geq \sigma_3 \end{aligned}$$

Beta coefficient

Question 2.c

What is the result if the correlation is uniform $\rho_{i,j} = \rho$?

Beta coefficient

If $\rho_{i,j} = \rho$, it follows that:

$$\begin{aligned}
\beta_i & \propto \sigma_i^2 + \sum_{j \neq i} \rho \sigma_i \sigma_j \\
& = \sigma_i^2 + \rho \sigma_i \sum_{j \neq i} \sigma_j + \rho \sigma_i^2 - \rho \sigma_i^2 \\
& = (1 - \rho) \sigma_i^2 + \rho \sigma_i \sum_{j=1}^{n} \sigma_j \\
& = f(\sigma_i)
\end{aligned}$$

with:

$$f(z) = (1 - \rho) z^2 + \rho z \sum_{j=1}^{n} \sigma_j$$

Beta coefficient

The first derivative of $f(z)$ is:

$$f'(z) = 2(1-\rho) z + \rho \sum_{j=1}^{n} \sigma_j$$

If $\rho \geq 0$, then $f(z)$ is an increasing function for $z \geq 0$ because $(1-\rho) \geq 0$ and $\rho \sum_{j=1}^{n} \sigma_j \geq 0$. This explains why the previous result remains valid:

$$\beta_1 \geq \beta_2 \geq \beta_3 \Rightarrow \sigma_1 \geq \sigma_2 \geq \sigma_3 \quad \text{if} \quad \rho_{i,j} = \rho \geq 0$$

If $-(n-1)^{-1} \leq \rho < 0$, then f' is decreasing if $z < -2^{-1}\rho(1-\rho)^{-1} \sum_{j=1}^{n} \sigma_j$ and increasing otherwise. We then have:

$$\beta_1 \geq \beta_2 \geq \beta_3 \nRightarrow \sigma_1 \geq \sigma_2 \geq \sigma_3 \quad \text{if} \quad \rho_{i,j} = \rho < 0$$

In fact, the result remains valid in most cases. To obtain a counter-example, we must have large differences between the volatilities and a correlation close to $-(n-1)^{-1}$. For example, if $\sigma_1 = 5\%$, $\sigma_2 = 6\%$, $\sigma_3 = 80\%$ and $\rho = -49\%$, we have $\beta_1 = -0.100$, $\beta_2 = -0.115$ and $\beta_3 = 3.215$.

Beta coefficient

Question 2.d

Find a general example such that $\beta_1 > \beta_2 > \beta_3$ and $\sigma_1 < \sigma_2 < \sigma_3$.

Beta coefficient

We assume that $\sigma_1 = 15\%$, $\sigma_2 = 20\%$, $\sigma_3 = 22\%$, $\rho_{1,2} = 70\%$, $\rho_{1,3} = 20\%$ and $\rho_{2,3} = -50\%$. It follows that $\beta_1 = 1.231$, $\beta_2 = 0.958$ and $\beta_3 = 0.811$. We thus have found an example such that $\beta_1 > \beta_2 > \beta_3$ and $\sigma_1 < \sigma_2 < \sigma_3$.

Beta coefficient

Question 2.e

Do we have $\sum_{i=1}^{n} \beta_i < n$ or $\sum_{i=1}^{n} \beta_i > n$ if the market portfolio is not equally weighted?

Beta coefficient

There is no reason that we have either $\sum_{i=1}^{n} \beta_i < n$ or $\sum_{i=1}^{n} \beta_i > n$. Let us consider the previous numerical example. If $b = (5\%, 25\%, 70\%)$, we obtain $\sum_{i=1}^{3} \beta_i = 1.808$ whereas if $b = (20\%, 40\%, 40\%)$, we have $\sum_{i=1}^{3} \beta_i = 3.126$.

Beta coefficient

Question 3

We search a market portfolio $b \in \mathbb{R}^n$ such that the betas are the same for all the assets: $\beta_i = \beta_j = \beta$.

Beta coefficient

Question 3.a

Show that there is an obvious solution which satisfies $\beta = 1$.

Beta coefficient

We have:

$$\begin{aligned}\sum_{i=1}^{n} b_i \beta_i &= \sum_{i=1}^{n} b_i \frac{(\Sigma b)_i}{b^\top \Sigma b} \\ &= b^\top \frac{\Sigma b}{b^\top \Sigma b} \\ &= 1\end{aligned}$$

If $\beta_i = \beta_j = \beta$, then $\beta = 1$ is an obvious solution because the previous relationship is satisfied:

$$\sum_{i=1}^{n} b_i \beta_i = \sum_{i=1}^{n} b_i = 1$$

Beta coefficient

Question 3.b

Show that this solution is unique and corresponds to the minimum variance portfolio.

Beta coefficient

If $\beta_i = \beta_j = \beta$, then we have:

$$\sum_{i=1}^{n} b_i \beta = 1 \Leftrightarrow \beta = \frac{1}{\sum_{i=1}^{n} b_i} = 1$$

β can only take one value, the solution is then unique. We know that the marginal volatilities are the same in the case of the minimum variance portfolio x (TR-RPB, page 173):

$$\frac{\partial\, \sigma\,(x)}{\partial\, x_i} = \frac{\partial\, \sigma\,(x)}{\partial\, x_j}$$

with $\sigma\,(x) = \sqrt{x^\top \Sigma x}$ the volatility of the portfolio x.

Beta coefficient

It follows that:

$$\frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} = \frac{(\Sigma x)_j}{\sqrt{x^\top \Sigma x}}$$

By dividing the two terms by $\sqrt{x^\top \Sigma x}$, we obtain:

$$\frac{(\Sigma x)_i}{x^\top \Sigma x} = \frac{(\Sigma x)_j}{x^\top \Sigma x}$$

The asset betas are then the same in the minimum variance portfolio. Because we have:

$$\left\{ \begin{array}{l} \beta_i = \beta_j \\ \sum_{i=1}^n x_i \beta_i = 1 \end{array} \right.$$

we deduce that:

$$\beta_i = 1$$

Beta coefficient

Question 4

We assume that $b \in [0,1]^n$.

Beta coefficient

Question 4.a

Show that if one asset has a beta greater than one, there exists another asset which has a beta smaller than one.

Beta coefficient

We have:

$$
\begin{aligned}
& \sum_{i=1}^{n} b_i \beta_i = 1 \\
\Leftrightarrow \quad & \sum_{i=1}^{n} b_i \beta_i = \sum_{i=1}^{n} b_i \\
\Leftrightarrow \quad & \sum_{i=1}^{n} b_i \beta_i - \sum_{i=1}^{n} b_i = 0 \\
\Leftrightarrow \quad & \sum_{i=1}^{n} b_i \left(\beta_i - 1 \right) = 0
\end{aligned}
$$

Beta coefficient

We obtain the following system of equations:

$$\left\{ \begin{array}{l} \sum_{i=1}^{n} b_i \left(\beta_i - 1 \right) = 0 \\ b_i \geq 0 \end{array} \right.$$

Let us assume that the asset j has a beta greater than 1. We then have:

$$\left\{ \begin{array}{l} b_j \left(\beta_j - 1 \right) + \sum_{i \neq j} b_i \left(\beta_i - 1 \right) = 0 \\ b_i \geq 0 \end{array} \right.$$

It follows that $b_j \left(\beta_j - 1 \right) > 0$ because $b_j > 0$ (otherwise the beta is zero). We must therefore have $\sum_{i \neq j} x_i \left(\beta_i - 1 \right) < 0$. Because $b_i \geq 0$, it is necessary that at least one asset has a beta smaller than 1.

Beta coefficient

Question 4.b

We consider the case $n = 3$. Find a covariance matrix Σ and a market portfolio b such that one asset has a negative beta.

Beta coefficient

We use standard notations to represent Σ. We seek a portfolio such that $\beta_1 > 0$, $\beta_2 > 0$ and $\beta_3 < 0$. To simplify this problem, we assume that the three assets have the same volatility. We also obtain the following system of inequalities:

$$\left\{ \begin{array}{l} b_1 + b_2\rho_{1,2} + b_3\rho_{1,3} > 0 \\ b_1\rho_{1,2} + b_2 + b_3\rho_{2,3} > 0 \\ b_1\rho_{1,3} + b_2\rho_{2,3} + b_3 < 0 \end{array} \right.$$

It is sufficient that $b_1\rho_{1,3} + b_2\rho_{2,3}$ is negative and b_3 is small. For example, we may consider $b_1 = 50\%$, $b_2 = 45\%$, $b_3 = 5\%$, $\rho_{1,2} = 50\%$, $\rho_{1,3} = 0\%$ and $\rho_{2,3} = -50\%$. We obtain $\beta_1 = 1.10$, $\beta_2 = 1.03$ and $\beta_3 = -0.27$.

Beta coefficient

Question 5

We report the return $R_{i,t}$ and $R_t(b)$ of asset i and market portfolio b at different dates:

t	1	2	3	4	5	6
$R_{i,t}$	−22	−11	−10	−8	13	11
$R_t(b)$	−26	−9	−10	−10	16	14
t	7	8	9	10	11	12
$R_{i,t}$	21	13	−30	−6	−5	−5
$R_t(b)$	14	15	−22	−7	−11	2
t	13	14	15	16	17	18
$R_{i,t}$	19	−17	2	−24	25	−7
$R_t(b)$	15	−15	−1	−23	15	−6

Beta coefficient

Question 5.a

Estimate the beta of the asset.

Beta coefficient

We perform the linear regression $R_{i,t} = \alpha_i + \beta_i R_t(b) + \varepsilon_{i,t}$ and we obtain $\hat{\beta}_i = 1.06$.

Beta coefficient

Question 5.b

What is the proportion of the asset volatility explained by the market?

Beta coefficient

We deduce that the contribution c_i of the market factor is (TR-RPB, page 16):

$$c_i = \frac{\beta_i^2 \operatorname{var}(R(b))}{\operatorname{var}(R_i)} = 90.62\%$$

Black-Litterman model

Exercise

We consider a universe of three assets. Their volatilities are 20%, 20% and 15%. The correlation matrix of asset returns is:

$$\rho = \begin{pmatrix} 1.00 & & \\ 0.50 & 1.00 & \\ 0.20 & 0.60 & 1.00 \end{pmatrix}$$

We would like to implement a trend-following strategy. For that, we estimate the trend of each asset and the volatility of the trend. We obtain the following results:

Asset	1	2	3
$\hat{\mu}$	10%	−5%	15%
$\sigma(\hat{\mu})$	4%	2%	10%

We assume that the neutral portfolio is the equally weighted portfolio.

Black-Litterman model

Question 1

Find the optimal portfolio if the constraint of the tracking error volatility is set to 1%, 2%, 3%, 4% and 5%.

Black-Litterman model

We consider the portfolio optimization problem in the presence of a benchmark (TR-RPB, page 17). We obtain the following results (expressed in %):

$\sigma(x^\star \mid b)$	1.00	2.00	3.00	4.00	5.00
$x_1^\star$	35.15	36.97	38.78	40.60	42.42
$x_2^\star$	26.32	19.30	12.28	5.26	−1.76
$x_3^\star$	38.53	43.74	48.94	54.14	59.34
$\mu(x^\star \mid b)$	1.31	2.63	3.94	5.25	6.56

Black-Litterman model

Question 2

In order to tilt the neutral portfolio, we now consider the Black-Litterman model. The risk-free rate is set to 0.

Black-Litterman model

Question 2.a

Find the implied risk premium of the assets if we target a Sharpe ratio equal to 0.50. What is the value of ϕ?

Black-Litterman model

Let b be the benchmark (that is the equally weighted portfolio). We recall that the implied risk aversion parameter is:

$$\phi = \frac{\mathrm{SR}\left(b \mid r\right)}{\sqrt{b^{\top}\Sigma b}}$$

and the implied risk premium is:

$$\tilde{\mu} = r + \mathrm{SR}\left(b \mid r\right) \frac{\Sigma b}{\sqrt{b^{\top}\Sigma b}}$$

We obtain $\phi = 3.4367$ and:

$$\tilde{\mu} = \left(\begin{array}{c} \tilde{\mu}_1 \\ \tilde{\mu}_2 \\ \tilde{\mu}_3 \end{array}\right) = \left(\begin{array}{c} 7.56\% \\ 8.94\% \\ 5.33\% \end{array}\right)$$

Black-Litterman model

Question 2.b

How does one incorporate a trend-following strategy in the Black-Litterman model? Give the P, Q and Ω matrices.

Black-Litterman model

In this case, the views of the portfolio manager corresponds to the trends observed in the market. We then have[2]:

$$\begin{aligned} P &= I_3 \\ Q &= \hat{\mu} \\ \Omega &= \operatorname{diag}\left(\sigma^2\left(\hat{\mu}_1\right), \ldots, \sigma^2\left(\hat{\mu}_n\right)\right) \end{aligned}$$

The views $P\mu = Q + \varepsilon$ become:

$$\mu = \hat{\mu} + \varepsilon$$

with $\varepsilon \sim \mathcal{N}\left(\mathbf{0}_3, \Omega\right)$.

[2] If we suppose that the trends are not correlated.

Black-Litterman model

Question 2.c

Calculate the conditional expectation $\bar{\mu} = \mathbb{E}\left[\mu \mid P\mu = Q + \varepsilon\right]$ if we assume that $\Gamma = \tau\Sigma$ and $\tau = 0.01$.

Black-Litterman model

We have (TR-RPB, page 25):

$$\begin{aligned}\bar{\mu} &= E\left[\mu \mid P\mu = Q + \varepsilon\right] \\ &= \tilde{\mu} + \Gamma P^\top \left(P \Gamma P^\top + \Omega\right)^{-1} \left(Q - P\tilde{\mu}\right) \\ &= \tilde{\mu} + \tau\Sigma \left(\tau\Sigma + \Omega\right)^{-1} \left(\hat{\mu} - \tilde{\mu}\right)\end{aligned}$$

We obtain:

$$\bar{\mu} = \begin{pmatrix} \bar{\mu}_1 \\ \bar{\mu}_2 \\ \bar{\mu}_3 \end{pmatrix} = \begin{pmatrix} 5.16\% \\ 2.38\% \\ 2.47\% \end{pmatrix}$$

Black-Litterman model

Question 2.d

Find the Black-Litterman optimized portfolio.

Black-Litterman model

We optimize the quadratic utility function with $\phi = 3.4367$. The Black-Litterman portfolio is then:

$$x^{\star} = \begin{pmatrix} x_1^{\star} \\ x_2^{\star} \\ x_3^{\star} \end{pmatrix} = \begin{pmatrix} 56.81\% \\ -23.61\% \\ 66.80\% \end{pmatrix}$$

Its volatility tracking error is $\sigma\left(x^{\star} \mid b\right) = 8.02\%$ and its alpha is $\mu\left(x^{\star} \mid b\right) = 10.21\%$.

Black-Litterman model

Question 3

We would like to compute the Black-Litterman optimized portfolio, corresponding to a 3% tracking error volatility.

Black-Litterman model

Question 3.a

What is the Black-Litterman portfolio when $\tau = 0$ and $\tau = +\infty$?

Black-Litterman model

- If $\tau = 0$, $\bar{\mu} = \tilde{\mu}$. The BL portfolio x is then equal to the neutral portfolio b.
- We also have:

$$\begin{aligned} \lim_{\tau \to \infty} \bar{\mu} &= \tilde{\mu} + \lim_{\tau \to \infty} \tau \Sigma^{\top} \left(\tau \Sigma + \Omega \right)^{-1} \left(\hat{\mu} - \tilde{\mu} \right) \\ &= \tilde{\mu} + \left(\hat{\mu} - \tilde{\mu} \right) \\ &= \hat{\mu} \end{aligned}$$

In this case, $\bar{\mu}$ is independent from the implied risk premium $\hat{\mu}$ and is exactly equal to the estimated trends $\hat{\mu}$. The BL portfolio x is then the Markowitz optimized portfolio with the given value of ϕ.

Black-Litterman model

Question 3.b

Using the previous results, apply the bisection algorithm and find the Black-Litterman optimized portfolio, which corresponds to a 3% tracking error volatility.

Black-Litterman model

We would like to find the BL portfolio such that $\sigma(x \mid b) = 3\%$. We know that $\sigma(x \mid b) = 0$ if $\tau = 0$. Thanks to Question 2.d, we also know that $\sigma(x \mid b) = 8.02\%$ if $\tau = 1\%$. It implies that the optimal portfolio corresponds to a specific value of τ which is between 0 and 1%. If we apply the bi-section algorithm, we find that:

$$\tau^{\star} = 0.242\%$$

. The composition of the optimal portfolio is then

$$x^{\star} = \begin{pmatrix} x_1^{\star} \\ x_2^{\star} \\ x_3^{\star} \end{pmatrix} = \begin{pmatrix} 41.18\% \\ 11.96\% \\ 46.85\% \end{pmatrix}$$

We obtain an alpha equal to 3.88%, which is a little bit smaller than the alpha of 3.94% obtained for the TE portfolio.

Black-Litterman model

Question 3.c

Compare the relationship between $\sigma(x \mid b)$ and $\mu(x \mid b)$ of the Black-Litterman model with the one of the tracking error model. Comment on these results.

Black-Litterman model

We have reported the relationship between $\sigma(x \mid b)$ and $\mu(x \mid b)$ in Figure 19. We notice that the information ratio of BL portfolios is very close to the information ratio of TE portfolios. We may explain that because of the homogeneity of the estimated trends $\hat{\mu}_i$ and the volatilities $\sigma(\hat{\mu}_i)$. If we suppose that $\sigma(\hat{\mu}_1) = 1\%$, $\sigma(\hat{\mu}_2) = 5\%$ and $\sigma(\hat{\mu}_3) = 15\%$, we obtain the relationship $\#2$. In this case, the BL model produces a smaller information ratio than the TE model. We explain this because $\bar{\mu}$ is the right measure of expected return for the BL model whereas it is $\hat{\mu}$ for the TE model. We deduce that the ratios $\bar{\mu}_i/\hat{\mu}_i$ are more volatile for the parameter set $\#2$, in particular when τ is small.

Black-Litterman model

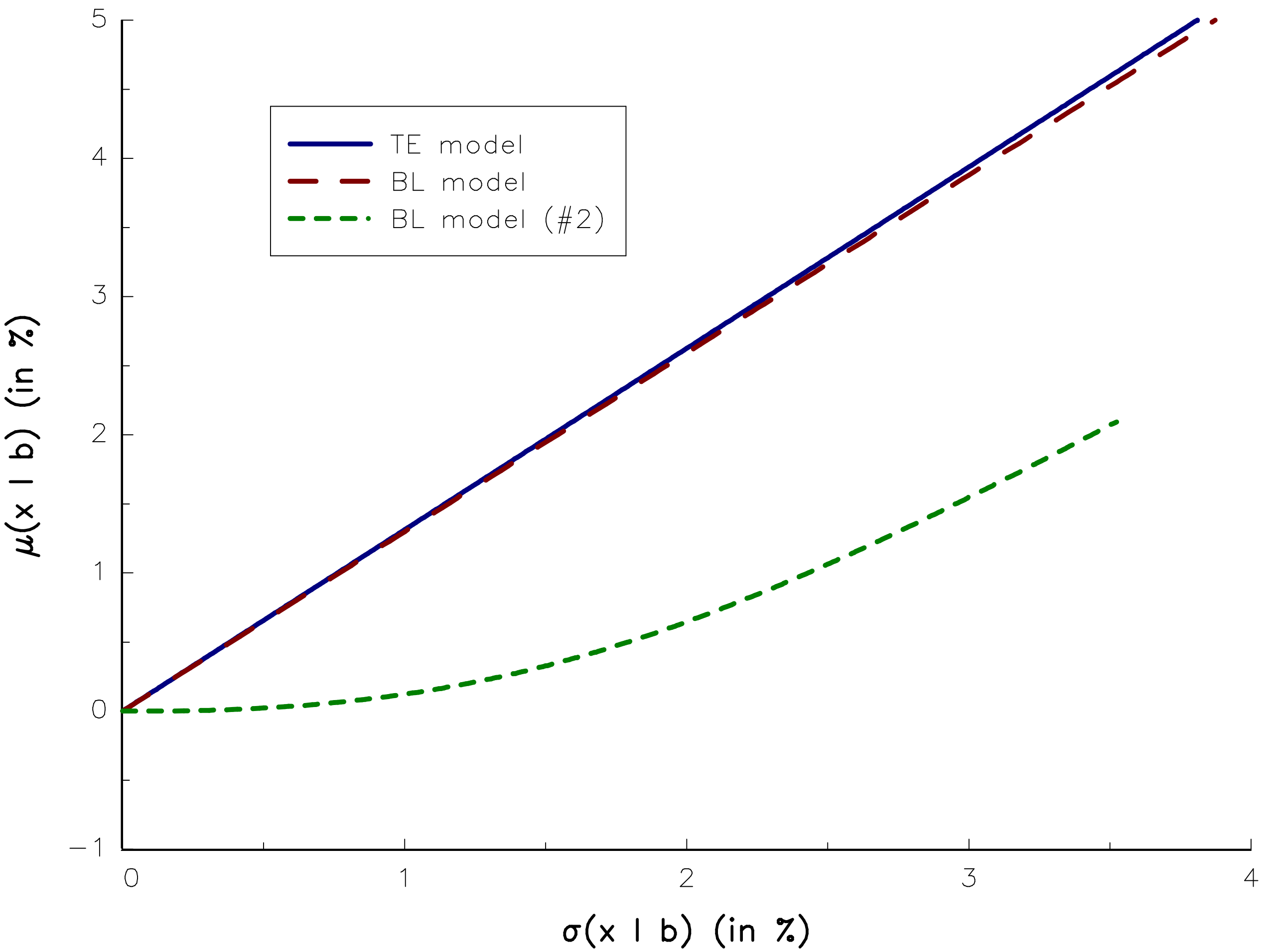


Figure 19: Efficient frontier of TE and BL portfolios

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