Solution of the Financial Risk Management Examination

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Remark 1 *The first five questions are corrected in TR-GDR*[1] *and in the document of exercise solutions, which is available in my web page*[2]*.*

1 The BCBS regulation

2 Market risk

3 Credit risk

4 Counterparty credit risk

5 Operational risk

6 Value-at-risk of an equity portfolio

The main reference for this exercise is TR-GDR (pages 61-63).

1. We note $P_A(t)$ (resp. $P_B(t)$) the value of the stock A (resp. B) at the date t. The portfolio value is:

$$P(t) = x_A \cdot P_A(t) + x_B \cdot P_B(t)$$

with x_A and x_B the number of stocks A and B. We deduce that the PnL between t and $t+1$ is:

$$\begin{aligned}
\text{PnL} &= P(t+1) - P(t) \\
&= x_A (P_A(t+1) - P_A(t)) + x_B (P_B(t+1) - P_B(t)) \\
&= x_A P_A(t) R_A + x_B P_B(t) R_B \\
&= W_A R_A + W_B R_B
\end{aligned}$$

where R_A and R_B are the asset returns of A and B between the dates t and $t+1$. We have also $W_A = x_A P_A(t)$ and $W_B = x_B P_B(t)$. Because $x_A = +2$ and $x_B = +1$, we have $W_A = 300$ and $W_B = 200$. It follows that:

$$\text{PnL} = 300 \times R_A + 200 \times R_B$$

We deduce that the yearly volatility of the PnL is:

$$\begin{aligned}
\sigma^2(\text{PnL}) &= 300^2 \times 0.2^2 + 200^2 \times 0.4^2 + \\
& \quad 2 \times 300 \times 200 \times 0.64 \times 0.2 \times 0.4
\end{aligned}$$

[1] Thierry Roncalli, *La Gestion des Risques Financiers*, Economica, deuxième édition, 2009.

[2] The direct link is `www.thierry-roncalli.com/download/gdr-correction.pdf`.

We obtain $\sigma(\text{PnL}) = 127.06$ euros. It follows that the Gaussian VaR for a one-week time horizon and a 99% confidence level:

$$\begin{aligned}
\text{VaR} &= \frac{\Phi^{-1}(\alpha) \times \sigma(\text{PnL})}{\sqrt{52}} \\
&= \frac{2.33 \times 127.06}{\sqrt{52}} \\
&= 41.05
\end{aligned}$$

The VaR is equal to 41.05 euros (or 8.21% of the portfolio value[3]).

2. The PnL becomes:

$$\text{PnL} = W_A R_A + W_B R_B - W_I R_I$$

(a) We have:

$$\begin{aligned}
\sigma^2(\text{PnL}) = {} & 300^2 \times 0.2^2 + 200^2 \times 0.4^2 + 500^2 \times 0.2^2 + \\
& 2 \times 300 \times 200 \times 0.64 \times 0.2 \times 0.4 - \\
& 2 \times 300 \times 500 \times 0.80 \times 0.2 \times 0.2 - \\
& 2 \times 200 \times 500 \times 0.80 \times 0.4 \times 0.2
\end{aligned}$$

We obtain $\sigma(\text{PnL}) = 61.19$ euros. It follows that the Gaussian VaR for a one-week time horizon and a 99% confidence level:

$$\begin{aligned}
\text{VaR} &= \frac{\Phi^{-1}(\alpha) \times \sigma(\text{PnL})}{\sqrt{52}} \\
&= \frac{2.33 \times 61.19}{\sqrt{52}} \\
&= 19.77
\end{aligned}$$

The VaR is equal to 19.77 euros (or 3.95% of the portfolio value).

(b) We have:

$$\begin{aligned}
\sigma^2(\text{PnL}) = {} & 300^2 \times 0.2^2 + 200^2 \times 0.4^2 + 560^2 \times 0.2^2 + \\
& 2 \times 300 \times 200 \times 0.64 \times 0.2 \times 0.4 - \\
& 2 \times 300 \times 560 \times 0.80 \times 0.2 \times 0.2 - \\
& 2 \times 200 \times 560 \times 0.80 \times 0.4 \times 0.2
\end{aligned}$$

We obtain $\sigma(\text{PnL}) = 60$ euros. It follows that the Gaussian VaR for a one-week time horizon and a 99% confidence level:

$$\begin{aligned}
\text{VaR} &= \frac{\Phi^{-1}(\alpha) \times \sigma(\text{PnL})}{\sqrt{52}} \\
&= \frac{2.33 \times 60}{\sqrt{52}} \\
&= 19.39
\end{aligned}$$

The VaR is equal to 19.39 euros (or 3.88% of the portfolio value).

(c) $W_I = 500$ corresponds to a fully hedged portfolio in terms of notional. Indeed, we have:

$$W_I = W_A + W_B$$

However, we observe that $W_I = 560$ is better to hedge the portfolio in terms of value-at-risk. This means that hedging the portfolio by considering the sum of nominal exposures is not optimal.

[3] The portfolio value is equal to 500 euros.

3. We deduce that the PnL of the hedged portfolio is:

$$\begin{aligned}
\text{PnL} &= W_A R_A + W_B R_B - W_I R_I \\
&= W_A \left(\beta_A R_I + \varepsilon_A\right) + W_B \left(\beta_B R_I + \varepsilon_B\right) - W_I R_I \\
&= \left(W_A \beta_A + W_B \beta_B - W_I\right) R_I + W_A \varepsilon_A + W_B \varepsilon_B \\
&= W^\star R_I + W_A \varepsilon_A + W_B \varepsilon_B
\end{aligned}$$

where $W^\star = W_A \beta_A + W_B \beta_B - W_I$ is the residual exposure on the index.

(a) If $W_I = 500$, we have:

$$\begin{aligned}
W^\star &= W_A \beta_A + W_B \beta_B - W_I \\
&= 300 \times 0.8 + 200 \times 1.6 - 500 \\
&= 60
\end{aligned}$$

If $W_I = 560$, we have:

$$\begin{aligned}
W^\star &= W_A \beta_A + W_B \beta_B - W_I \\
&= 300 \times 0.8 + 200 \times 1.6 - 500 \\
&= 0
\end{aligned}$$

We observe that the residual exposure is equal to zero if $W_I = 560$. This is not the case if $W_I = 500$. The reason is that the second asset has a high beta ($\beta_B = 1.6$), meaning that the sensitivity of the portfolio $\mathcal{P}_0$ is higher than its current value (500).

(b) We have:

$$\varepsilon_A = R_A - \beta_A R_I$$

We deduce that:

$$\begin{aligned}
\sigma\left(\varepsilon_A\right) &= \sqrt{\sigma_A^2 + \beta_A^2 \sigma_I^2 - 2\beta_A \rho_{A,I} \sigma_A \sigma_I} \\
&= \sqrt{0.2^2 + 0.8^2 \times 0.2^2 - 2 \times 0.8 \times 0.8 \times 0.2 \times 0.2} \\
&= 12\%
\end{aligned}$$

Because $R_A = \beta_A R_I + \varepsilon_A$, we have also:

$$\sigma_A^2 = \beta_A^2 \sigma_I^2 + \sigma^2\left(\varepsilon_A\right)$$

It follows that:

$$\begin{aligned}
\sigma\left(\varepsilon_A\right) &= \sqrt{\sigma_A^2 - \beta_A^2 \sigma_I^2} \\
&= \sqrt{0.2^2 - 0.8^2 \times 0.2^2}
\end{aligned}$$

In a same way, we have $\sigma\left(\varepsilon_B\right) = 24\%$.

(c) If $W_I = 560$, $W^\star = 0$. It follows that:

$$\text{PnL} = W_A \varepsilon_A + W_B \varepsilon_B$$

We deduce that:

$$\begin{aligned}
\sigma\left(\text{PnL}\right) &= \sqrt{300^2 \times 0.12^2 + 200^2 \times 0.24^2} \\
&= 60
\end{aligned}$$

We retrieve the result obtained in Question 2.(b):

$$\text{VaR} = \frac{2.33 \times 60}{\sqrt{52}} = 19.39$$

(d) We have:

$$\text{PnL} = W^{\star} R_I + W_A \varepsilon_A + W_B \varepsilon_B$$

It follows that:

$$\sigma(\text{PnL}) = \sqrt{W^{\star 2} \sigma_I^2 + W_A^2 \sigma^2(\varepsilon_A) + W_B^2 \sigma^2(\varepsilon_B)}$$

In the case of Portfolio $\mathcal{P}_0$, we have:

$$W^{\star}(\mathcal{P}_0) = W_A \beta_A + W_B \beta_B = 560$$

In the case of Portfolio $\mathcal{P}_1$, we have:

$$W^{\star}(\mathcal{P}_1) = W_A \beta_A + W_B \beta_B - W_I = 560 - W_I$$

The VaR of the hedged portfolio $\mathcal{P}_1$ is lower than the VaR of the original portfolio $\mathcal{P}_0$ if[4]:

$$\begin{aligned} \sigma(\text{PnL}(\mathcal{P}_1)) \leq \sigma(\text{PnL}(\mathcal{P}_0)) & \Leftrightarrow (W^{\star}(\mathcal{P}_0))^2 \geq (W^{\star}(\mathcal{P}_1))^2 \\ & \Leftrightarrow 560^2 \geq (560 - W_I)^2 \\ & \Leftrightarrow W_I \leq 1120 \end{aligned}$$

7 Risk contribution in the Basle II model

1. We assume that (TR-GDR, page 179):

   (a) The loss given default LGD_i is independent of the default time τ_i;

   (b) the default times depend on common factors $X_1, \ldots, X_m$;

   (c) the portfolio is infinitely granular: there is no concentration on a specific credit, meaning that:

$$\text{EAD}_i \Big/ \sum_{i=1}^{I} \text{EAD}_i \simeq 0$$

2. Let $\mathcal{R}$ be the risk measure. The risk contribution of the credit i is the product of the exposure-at-default EAD_i and the marginal risk (TR-GDR, page 497):

$$\text{RC}_i = \text{EAD}_i \times \frac{\partial \mathcal{R}}{\partial \, \text{EAD}_i}$$

   In the case of a convex risk measure, we have:

$$\mathcal{R} = \sum_{i=1}^{I} \text{RC}_i$$

   The risk measure is then equal to the sum of the different risk contributions.

3. We have:

$$\begin{aligned} \text{EL} &= \mathbb{E}[L] \\ \text{UL} &= \mathbf{F}^{-1}(\alpha) - \text{EL} \end{aligned}$$

   where $\mathbf{F}$ is the cumulative distribution function of L. If the default times are independent, we obtain:

$$\mathbb{E}[L \mid X_1, \ldots, X_m] = \sum_{i=1}^{I} \text{EAD}_i \times \mathbb{E}[\text{LGD}_i] \times \text{PD}_i = \text{EL}$$

   L is then not random and we have $\Pr\{L = \text{EL}\} = 1$. It follows that $\mathbf{F}^{-1}(\alpha) = \text{EL}$ and $\text{UL} = 0$.

[4] By construction, we have $W_I > 0$.

4. We set (TR-GDR, page 180):

$$g(x) = \sum_{i=1}^{I} \mathrm{EAD}_i \times \mathbb{E}[\mathrm{LGD}_i] \times \mathrm{PD}_i(x)$$

By definition, we have:

$$\mathbb{E}[L \mid X = x] = g(x)$$

We deduce that:

$$\mathbf{F}(\ell) = \Pr\{L \leq \ell\} = \Pr\{g(X) \leq \ell\}$$

Because $\mathrm{EAD}_i \geq 0$ and $\mathbb{E}[\mathrm{LGD}_i] \geq 0$, $g(x)$ is an increasing (resp. decreasing) function if $\mathrm{PD}_i(x)$ is an increasing (resp. decreasing) function of x. In the case where $g(x)$ is an increasing function, we have:

$$\begin{array}{ll} & \Pr\{g(X) \leq \ell\} = \alpha \\ \Leftrightarrow & \Pr\left\{X \leq g^{-1}(\ell)\right\} = \alpha \\ \Leftrightarrow & \mathbf{H}\left(g^{-1}(\ell)\right) = \alpha \\ \Leftrightarrow & \ell = g\left(\mathbf{H}^{-1}(\alpha)\right) \end{array}$$

We deduce that:

$$\mathbf{F}^{-1}(\alpha) = \sum_{i=1}^{I} \mathrm{EAD}_i \times \mathbb{E}[\mathrm{LGD}_i] \times \mathrm{PD}_i\left(\mathbf{H}^{-1}(\alpha)\right)$$

In the case where $g(x)$ is a decreasing function, we have:

$$\begin{array}{ll} & \Pr\{g(X) \leq \ell\} = \alpha \\ \Leftrightarrow & \Pr\left\{X \geq g^{-1}(\ell)\right\} = \alpha \\ \Leftrightarrow & \mathbf{H}\left(g^{-1}(\ell)\right) = 1 - \alpha \\ \Leftrightarrow & \ell = g\left(\mathbf{H}^{-1}(1-\alpha)\right) \end{array}$$

We deduce that:

$$\mathbf{F}^{-1}(\alpha) = \sum_{i=1}^{I} \mathrm{EAD}_i \times \mathbb{E}[\mathrm{LGD}_i] \times \mathrm{PD}_i\left(\mathbf{H}^{-1}(1-\alpha)\right)$$

If at least one exposure EAD_i is negative, the function $g(x)$ is not monotonic. We don't check anymore $g(X) \leq \ell \Leftrightarrow X \leq g^{-1}(\ell)$ (increasing case) or $g(X) \leq \ell \Leftrightarrow X \geq g^{-1}(\ell)$ (decreasing case), implying that the expression $\mathbf{F}^{-1}(\alpha)$ is not valid. We can not use the Basle II model if the credit portfolio has one (or more) negative exposure. For the management of the credit portfolio, this implies that buying a CDS protection on the i^{th} credit can only be done to reduce the exposure-at-default on the counterparty, but not to be short on the counterparty.

5. We have (TR-GDR, pages 181-182):

$$\begin{aligned} \mathrm{PD}_i &= \Pr\{\tau_i \leq M_i\} \\ &= \Pr\{Z_i \leq B_i\} \\ &= \Phi(B_i) \end{aligned}$$

and $B_i = \Phi^{-1}(\mathrm{PD}_i)$. We have also:

$$\begin{aligned} \Pr\{\tau_i \leq M_i \mid X = x\} &= \Pr\{Z_i \leq B_i \mid X = x\} \\ &= \Pr\left\{\sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_i \leq B_i \mid X = x\right\} \\ &= \Pr\left\{\varepsilon_i \leq \frac{B_i - \sqrt{\rho}X}{\sqrt{1-\rho}} \mid X = x\right\} \\ &= \Phi\left(\frac{\Phi^{-1}(\mathrm{PD}_i) - \sqrt{\rho}x}{\sqrt{1-\rho}}\right) \end{aligned}$$

6. Because the conditional default probability is a decreasing function with respect to x, we have:

$$\begin{aligned}
\mathbf{F}^{-1}(\alpha) &= \sum_{i=1}^{I} \mathrm{EAD}_i \times \mathbb{E}[\mathrm{LGD}_i] \times \mathrm{PD}_i\left(\Phi^{-1}(1-\alpha)\right) \\
&= \sum_{i=1}^{I} \mathrm{EAD}_i \times \mathbb{E}[\mathrm{LGD}_i] \times \Phi\left(\frac{\Phi^{-1}(\mathrm{PD}_i) - \sqrt{\rho}\Phi^{-1}(1-\alpha)}{\sqrt{1-\rho}}\right) \\
&= \sum_{i=1}^{I} \mathrm{EAD}_i \times \mathbb{E}[\mathrm{LGD}_i] \times \Phi\left(\frac{\Phi^{-1}(\mathrm{PD}_i) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)
\end{aligned}$$

7. ρ is the constant correlation between assets:

$$\begin{aligned}
\mathrm{cor}(Z_i, Z_j) &= \mathbb{E}[Z_i Z_j] \\
&= \mathbb{E}\left[\rho X^2 + \sqrt{\rho(1-\rho)} X(\varepsilon_i + \varepsilon_j) + (1-\rho)\varepsilon_i\varepsilon_j\right] \\
&= \rho
\end{aligned}$$

8. The second pillar concerns the non-respect of the assumptions $(\mathcal{H})$. In particular, we have to understand the impact on the credit risk measure if the portfolio is not granular or if the asset correlation is not constant.

8 Correlation and log-normal random variables

1. (a) The density of $X \sim \mathcal{N}(\mu, \sigma^2)$ is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

Let $Y \sim \mathcal{LN}(\mu, \sigma^2)$. We have:

$$Y = e^X$$

It comes that:

$$g(y) = f(x)\left|\frac{\mathrm{d}x}{\mathrm{d}y}\right|$$

with $x = \ln y$. We deduce that:

$$\begin{aligned}
g(y) &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \cdot \frac{1}{y} \\
&= \frac{1}{y\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln y-\mu}{\sigma}\right)^2\right)
\end{aligned}$$

(b) Let $m \geq 1$. We have:

$$\begin{aligned}
\mathbb{E}[Y^m] &= \int_0^\infty y^m g(y)\,\mathrm{d}y \\
&= \int_0^\infty \frac{y^m}{y\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln y-\mu}{\sigma}\right)^2\right)\mathrm{d}y
\end{aligned}$$

By considering the change of variable $z = \sigma^{-1}(\ln y - \mu)$, we have:

$$\begin{aligned}
\mathbb{E}[Y^m] &= \int_{-\infty}^{\infty} \frac{(e^{\mu+\sigma z})^m}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) \mathrm{d}z \\
&= e^{m\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2 + m\sigma z\right) \mathrm{d}z \\
&= e^{m\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(z - m\sigma)^2 + \frac{1}{2}m^2\sigma^2\right) \mathrm{d}z \\
&= e^{m\mu+\frac{1}{2}m^2\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(z - m\sigma)^2\right) \mathrm{d}z
\end{aligned}$$

By considering the change of variable $t = z - m\sigma$, we obtain:

$$\begin{aligned}
\mathbb{E}[Y^m] &= e^{m\mu+\frac{1}{2}m^2\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}t^2\right) \mathrm{d}t \\
&= e^{m\mu+\frac{1}{2}m^2\sigma^2} (\Phi(\infty) - \Phi(-\infty)) \\
&= e^{m\mu+\frac{1}{2}m^2\sigma^2}
\end{aligned}$$

We deduce that:

$$\mathbb{E}[Y] = \exp\left(\mu + \frac{1}{2}\sigma^2\right)$$

(c) We have shown that:

$$\mathbb{E}[Y^m] = \exp\left(m\mu + \frac{1}{2}m^2\sigma^2\right)$$

(d) It follows that:

$$\begin{aligned}
\operatorname{var}(Y) &= \mathbb{E}\left[Y^2\right] - \mathbb{E}^2[Y] \\
&= e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} \\
&= e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)
\end{aligned}$$

(e) See TR-GDR on page 239.

2. (a) $X_1 + X_2$ is a Gaussian random variable because it is a linear combination of the Gaussian vector (X_1, X_2). We have:

$$\mathbb{E}[X_1 + X_2] = \mu_1 + \mu_2$$

and:

$$\operatorname{var}(X_1 + X_2) = \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2$$

We deduce that:

$$X_1 + X_2 \sim \mathcal{N}\left(\mu_1 + \mu_2, \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2\right)$$

(b) We have:

$$\begin{aligned}
\operatorname{cov}(Y_1, Y_2) &= \mathbb{E}[Y_1 Y_2] - \mathbb{E}[Y_2]\,\mathbb{E}[Y_2] \\
&= \mathbb{E}\left[e^{X_1+X_2}\right] - \mathbb{E}[Y_2]\,\mathbb{E}[Y_2]
\end{aligned}$$

Using Question 2.a, we know that $e^{X_1+X_2}$ is a log-normal random variables. We deduce that:

$$\begin{aligned}
\mathbb{E}\left[e^{X_1+X_2}\right] &= e^{\mu_1+\mu_2+\frac{1}{2}\left(\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2\right)} \\
&= e^{\mu_1+\frac{1}{2}\sigma_1^2} e^{\mu_2+\frac{1}{2}\sigma_2^2} e^{\rho\sigma_1\sigma_2}
\end{aligned}$$

We finally obtain:

$$\operatorname{cov}(Y_1, Y_2) = e^{\mu_1+\frac{1}{2}\sigma_1^2} e^{\mu_2+\frac{1}{2}\sigma_2^2} \left(e^{\rho\sigma_1\sigma_2} - 1\right)$$

(c) We have:

$$\begin{aligned}\rho(Y_1,Y_2) &= \frac{e^{\mu_1+\frac{1}{2}\sigma_1^2}e^{\mu_2+\frac{1}{2}\sigma_2^2}\left(e^{\rho\sigma_1\sigma_2}-1\right)}{\sqrt{e^{2\mu_1+\sigma_1^2}\left(e^{\sigma_1^2}-1\right)}\sqrt{e^{2\mu_2+\sigma_2^2}\left(e^{\sigma_2^2}-1\right)}} \\ &= \frac{e^{\rho\sigma_1\sigma_2}-1}{\sqrt{e^{\sigma_1^2}-1}\sqrt{e^{\sigma_2^2}-1}}\end{aligned}$$

$\rho(Y_1,Y_2)$ is an increasing function with respect to ρ. We deduce that:

$$\rho(Y_1,Y_2)=1\Leftrightarrow\left\{\begin{array}{c}\rho=1\\ \sigma_1=\sigma_2\end{array}\right.$$

The lower bound is reached if $\rho=-1$. In this case, we have:

$$\rho(Y_1,Y_2)=\frac{e^{-\sigma_1\sigma_2}-1}{\sqrt{e^{\sigma_1^2}-1}\sqrt{e^{\sigma_2^2}-1}}>-1$$

It follows that $\rho(Y_1,Y_2)\neq -1$.

(d) A concordance measure reached the lower bound -1 (resp. the upper bound $+1$) if the random variables are countermonotonic (resp. comonotonic). It is not the case with the linear correlation. For instance, if $\rho=-1$, the dependence function is $\mathbf{C}^-$ and $\rho(Y_1,Y_2)\neq -1$. if $\rho=1$, the dependence function is $\mathbf{C}^+$. In this case, $\rho(Y_1,Y_2)=1$ if and only if $\ln Y_1$ and $\ln Y_2$ have the same variance.

9 Credit spreads

1. We have (TR-GDR, page 427) :

$$\begin{aligned}\mathbf{F}(t) &= 1-e^{-\lambda t}\\ \mathbf{S}(t) &= e^{-\lambda t}\\ f(t) &= \lambda e^{-\lambda t}\end{aligned}$$

Let $U=\mathbf{S}(\tau)$. We have $U\in[0,1]$ and:

$$\begin{aligned}\Pr\{U\leq u\} &= \Pr\{\mathbf{S}(\tau)\leq u\}\\ &= \Pr\left\{\tau\leq \mathbf{S}^{-1}(u)\right\}\\ &= \mathbf{S}\left(\mathbf{S}^{-1}(u)\right)\\ &= u\end{aligned}$$

We deduce that $\mathbf{S}(\tau)\sim\mathcal{U}_{[0,1]}$ (TR-GDR, page 428). It comes that $\tau=\mathbf{S}^{-1}(U)$ with $U\sim\mathcal{U}_{[0,1]}$. Let u be a uniform random variate. Simulating τ is equivalent to transform u into t:

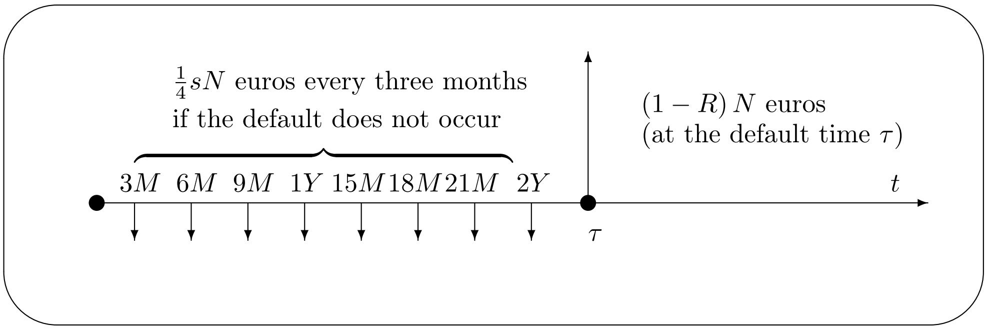
$$t=-\frac{1}{\lambda}\ln u$$

2. We have (TR-GDR, pages 409-411) :

$$\begin{aligned}P_- &= \frac{1}{4}\cdot s\cdot N\\ P_+ &= (1-R)\cdot N\end{aligned}$$

with s the spread, N the notional, P_- the premium leg and P_+ the protection leg. The quarterly payment of the premium leg explains the factor $1/4$ in the formula of P_-. We deduce the flow chart given in Figure 1.

Figure 1: Flow chart from the viewpoint of the protection buyer



3. The ATM margin (or spread) is the value of s such that the CDS price is zero (TR-GDR, page 410):

$$P(t) = \mathbb{E}\left[\left(\sum P_-\right) - P_+\right] = 0$$

We have the following triangle relationship (TR-GDR, page 410):

$$s \simeq \lambda \times (1-R)$$

4. Let PD be the annual default probability. We have

$$\begin{aligned} \text{PD} &= 1 - \mathbf{S}(1) \\ &= 1 - e^{-\lambda} \\ &\simeq 1 - (1-\lambda) \\ &\simeq \lambda \end{aligned}$$

because λ is generally small ($\lambda \leq 10\%$). We deduce that:

$$\text{PD} \simeq \frac{s}{1-R}$$

We have:

$$\text{PD} = \frac{2\%}{1-25\%} = 267 \text{ bps}$$

10 Extreme value theory and stress-testing

1. See TR-GDR, page 121-129.

2. We have (TR-GDR, pages 131-133):

$$\begin{aligned} \mathbf{G}_n(x) &= \Pr\{\max(X_1,\ldots,X_n) \leq x\} \\ &= \Pr\{X_1 \leq x, \ldots, X_n \leq x\} \\ &= \prod_{i=1}^{n} \Pr\{X_i \leq x\} \\ &= \Phi\left(\frac{x-\mu}{\sigma}\right)^n \end{aligned}$$

3. See TR-GDR, page 139.

4. (a) An extreme value (EV) copula $\mathbf{C}$ satisfies the following relationship:

$$\mathbf{C}\left(u_1^t, u_2^t\right) = \mathbf{C}^t\left(u_1, u_2\right)$$

for all $t > 0$.

(b) The product copula is an EV copula because:

$$\begin{aligned}\mathbf{C}^{\perp}\left(u_1^t, u_2^t\right) &= u_1^t u_2^t \\ &= \left(u_1 u_2\right)^t \\ &= \left[\mathbf{C}^{\perp}\left(u_1, u_2\right)\right]^t\end{aligned}$$

(c) We have:

$$\begin{aligned}\mathbf{C}\left(u_1^t, u_2^t\right) &= \exp\left(-\left[\left(-\ln u_1^t\right)^\theta + \left(-\ln u_2^t\right)^\theta\right]^{1/\theta}\right) \\ &= \exp\left(-\left[\left(-t\ln u_1\right)^\theta + \left(-t\ln u_2\right)^\theta\right]^{1/\theta}\right) \\ &= \exp\left(-t\left[\left(-\ln u_1\right)^\theta + \left(-\ln u_2\right)^\theta\right]^{1/\theta}\right) \\ &= \left[e^{-\left[(-\ln u_1)^\theta + (-\ln u_2)^\theta\right]^{1/\theta}}\right]^t \\ &= \mathbf{C}^t\left(u_1, u_2\right)\end{aligned}$$

(d) The upper tail dependence λ is defined as follows:

$$\lambda = \lim_{u \to 1^+} \frac{1 - 2u + \mathbf{C}\left(u_1, u_2\right)}{1 - u}$$

It indicates the probability to have an extreme in one direction knowing that we have already an extreme in the other direction. If $\lambda = 0$, extremes are independent and the copula of extreme values is the product copula $\mathbf{C}^{\perp}$. If $\lambda = 1$, extremes are comonotonic and the copula of extreme values is the upper Fréchet copula $\mathbf{C}^{+}$. Moreover, the upper tail dependence of the copula between the random variables is equal to the upper tail dependence of the copula between the extremes.

(e) We obtain using L'Hospital's rule:

$$\begin{aligned}\lambda &= \lim_{u \to 1^+} \frac{1 - 2u + e^{-\left[(-\ln u)^\theta + (-\ln u)^\theta\right]^{1/\theta}}}{1 - u} \\ &= \lim_{u \to 1^+} \frac{1 - 2u + e^{-\left[2(-\ln u)^\theta\right]^{1/\theta}}}{1 - u} \\ &= \lim_{u \to 1^+} \frac{1 - 2u + u^{2^{1/\theta}}}{1 - u} \\ &= \lim_{u \to 1^+} \frac{0 - 2 + 2^{1/\theta} u^{2^{1/\theta} - 1}}{-1} \\ &= \lim_{u \to 1^+} 2 - 2^{1/\theta} u^{2^{1/\theta} - 1} \\ &= 2 - 2^{1/\theta}\end{aligned}$$

(f) If $\theta = 1$, $\lambda = 0$. It comes that the copula of the extremes is the product copula. Extremes are then not correlated. This result is not surprising because the Gumbel-Houggard copula is equal to the product copula:

$$e^{-\left[(-\ln u_1)^1 + (-\ln u_2)^1\right]^1} = u_1 u_2 = \mathbf{C}^{\perp}\left(u_1, u_2\right)$$

5. (a) i. We have:

$$\begin{aligned}\mathbf{G}_{\infty}(x_1,x_2) &= \mathbf{C}^{\perp}\left(\mathbf{G}_{\infty}(x_1),\mathbf{G}_{\infty}(x_2)\right)\\ &= \mathbf{\Lambda}(x_1)\,\mathbf{\Psi}_1(x_2-1)\end{aligned}$$

ii. We have:

$$\begin{aligned}\mathbf{G}_{\infty}(x_1,x_2) &= \mathbf{C}^{\perp}\left(\mathbf{G}_{\infty}(x_1),\mathbf{G}_{\infty}(x_2)\right)\\ &= \mathbf{\Lambda}(x_1)\,\mathbf{\Phi}_{\alpha}\left(1+\frac{x_2}{\alpha}\right)\end{aligned}$$

(b) We know that the upper tail dependence is equal to zero for the Gaussian copula if $\rho < 1$. We then obtain exactly the same results as previously:

i.

$$\mathbf{G}_{\infty}(x_1,x_2) = \exp\left(-e^{-x_1}+x_2-1\right)$$

ii.

$$\mathbf{G}_{\infty}(x_1,x_2) = \exp\left(-e^{-x_1}-\left(1+\frac{x_2}{\alpha}\right)^{-\alpha}\right)$$

(c) If $\rho = 1$, the Gaussian copula is the upper Fréchet copula $\mathbf{C}^{+}$, which is an EV copula. We deduce that:

i.

$$\mathbf{G}_{\infty}(x_1,x_2) = \min\left(\mathbf{\Lambda}(x_1),\mathbf{\Psi}_1(x_2-1)\right)$$

ii.

$$\mathbf{G}_{\infty}(x_1,x_2) = \min\left(\mathbf{\Lambda}(x_1),\mathbf{\Phi}_{\alpha}\left(1+\frac{x_2}{\alpha}\right)\right)$$

(d) We have shown that the Gumbel-Houggard copula is an EV copula. We deduce that:

i.

$$\begin{aligned}\mathbf{G}_{\infty}(x_1,x_2) &= e^{-\left[(-\ln \mathbf{\Lambda}(x_1))^{\theta}+(-\ln \mathbf{\Psi}_1(x_2-1))^{\theta}\right]^{1/\theta}}\\ &= \exp\left(-\left[e^{-\theta x_1}+(1-x_2)^{\theta}\right]^{1/\theta}\right)\end{aligned}$$

ii.

$$\begin{aligned}\mathbf{G}_{\infty}(x_1,x_2) &= e^{-\left[(-\ln \mathbf{\Lambda}(x_1))^{\theta}+\left(-\ln \mathbf{\Phi}_{\alpha}\left(1+\frac{x_2}{\alpha}\right)\right)^{\theta}\right]^{1/\theta}}\\ &= \exp\left(-\left[e^{-\theta x_1}+\left(1+\frac{x_2}{\alpha}\right)^{-\alpha\theta}\right]^{1/\theta}\right)\end{aligned}$$