Portfolio Diversification & Asset Allocation
What Does It Mean?

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[1] The opinions expressed in this presentation are those of the authors and are not meant to represent the opinions or official positions of Amundi Asset Management.

Which method for diversifying?

- Portfolio optimization (Markowitz)
- Risk budgeting

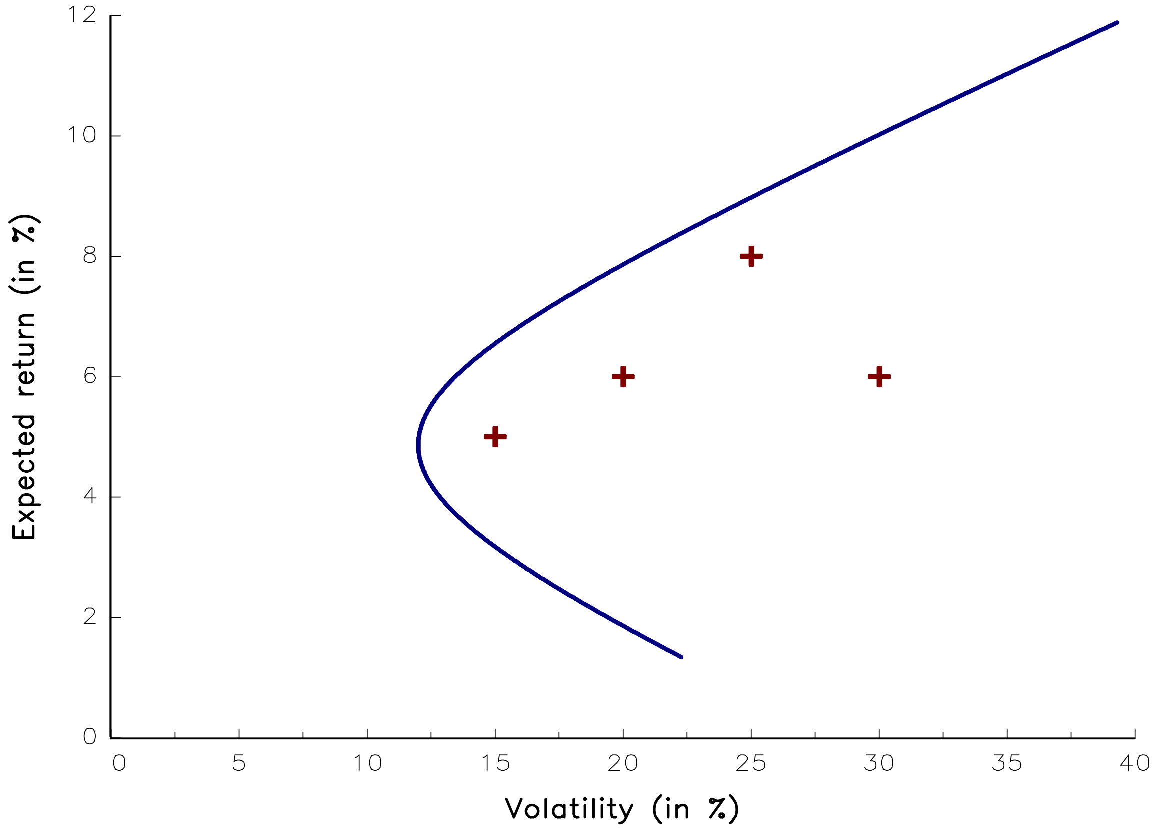
Mean-variance optimized portfolios

Let μ and Σ be the vector of expected returns and the covariance matrix of asset returns. The optimization problem is:

$$\begin{aligned} x^\star &= \arg\max x^\top \mu \\ \text{u.c.} &\quad \sqrt{x^\top \Sigma x} \leq \sigma^\star \end{aligned}$$

This problem is equivalent to the QP problem:

$$\begin{aligned} x^\star &= \arg\min \frac{1}{2} x^\top \Sigma x - \gamma x^\top \mu \\ &= \gamma \Sigma^{-1} \mu \end{aligned}$$



MVO portfolios are sensitive to arbitrage factors

MVO portfolios are of the following form: $x^\star \propto f\left(\Sigma^{-1}\right)$.

The important quantity is then the information matrix $\mathscr{I} = \Sigma^{-1}$.

We have $\Sigma = V \Lambda V^\top$ and $\Sigma^{-1} = \left(V \Lambda V^\top\right)^{-1} = V^{\top^{-1}} \Lambda^{-1} V^{-1} = V \Lambda^{-1} V^\top$.

If we consider the following example: $\sigma_1 = 20\%$, $\sigma_2 = 21\%$, $\sigma_3 = 10\%$ and $\rho_{i,j} = 80\%$, we obtain the following eigendecomposition:

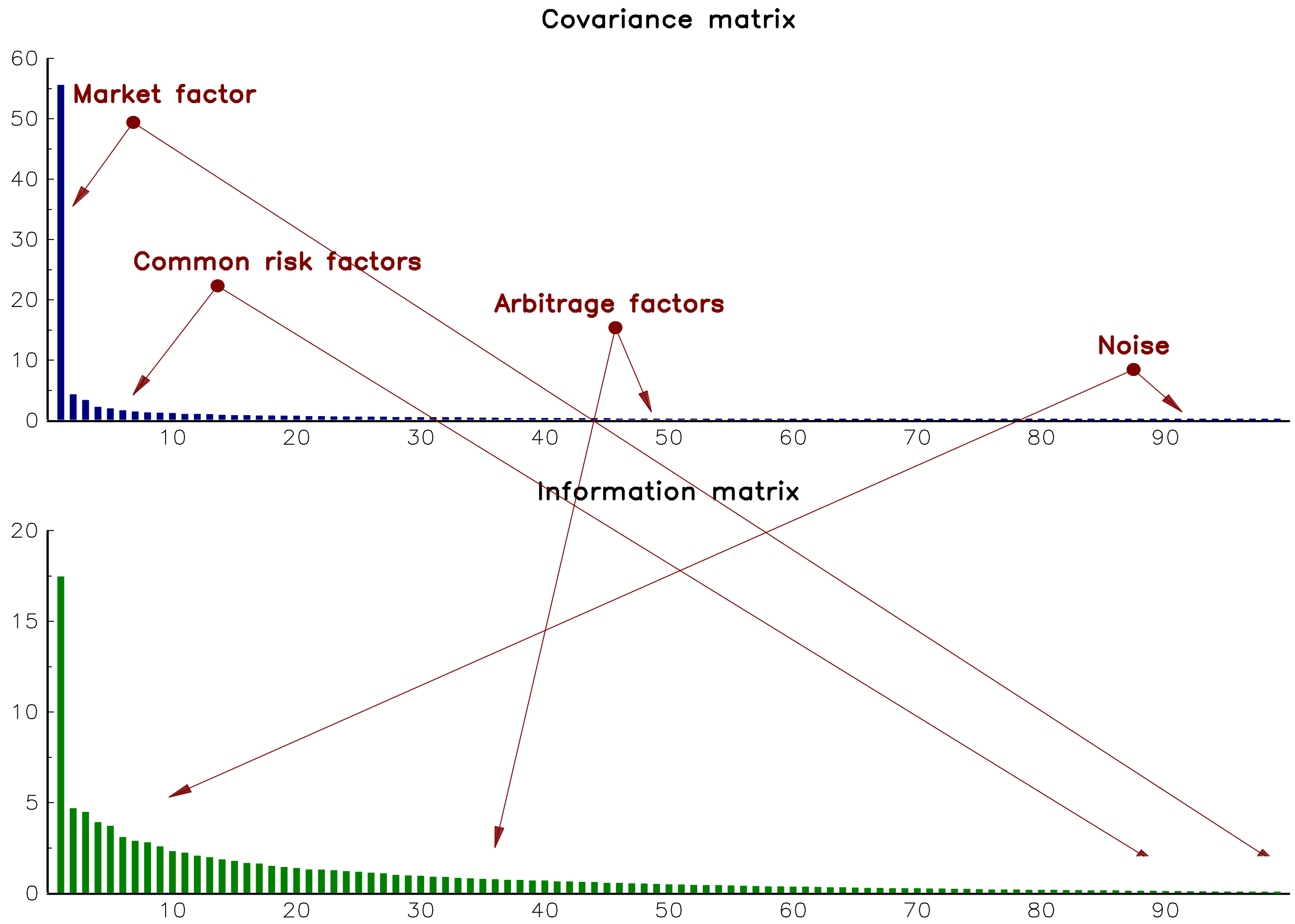
Asset / Factor	Covariance matrix Σ 1	2	3	Information matrix $\mathscr{I}$ 1	2	3
1	65.35%	−72.29%	−22.43%	−22.43%	−72.29%	65.35%
2	69.38%	69.06%	−20.43%	−20.43%	69.06%	69.38%
3	30.26%	−2.21%	95.29%	95.29%	−2.21%	30.26%
Eigenvalue	8.31%	0.84%	0.26%	379.97	119.18	12.04
% cumulated	88.29%	97.20%	100.00%	74.33%	97.65%	100.00%

$$12.04 = 1/8.31\%$$

Reverse order of eigenvectors

Common factors versus idiosyncratic factors

Figure: PCA applied to the stocks of the FTSE index (June 2012)



Why traditional shrinkage methods do not work?

Arbitrage factors and hedging portfolios

We consider the following regression model:

$$R_{i,t} = \beta_0 + \beta_i^\top R_t^{(-i)} + \varepsilon_{i,t}$$

- $R_t^{(-i)}$ denotes the vector of asset returns R_t excluding the i^{th} asset
- $\varepsilon_{i,t} \sim \mathcal{N}(0, s_i^2)$
- R_i^2 is the R-squared of the linear regression

Information matrix

Stevens (1998) shows that the information matrix is given by:

$$\mathcal{I}_{i,i} = \frac{1}{\hat{\sigma}_i^2 \left(1 - R_i^2\right)}$$

$$\mathcal{I}_{i,j} = -\frac{\hat{\beta}_{i,j}}{\hat{\sigma}_i^2 \left(1 - R_i^2\right)} = -\frac{\hat{\beta}_{j,i}}{\hat{\sigma}_j^2 \left(1 - R_j^2\right)}$$

Arbitrage factors and hedging portfolios

Table: Hedging portfolios (in %) at the end of 2006

	SPX	SX5E	TPX	RTY	EM	US HY	EMBI	EUR	JPY	GSCI
SPX		58.6	6.0	150.3	-30.8	-0.5	5.0	-7.3	15.3	-25.5
SX5E	9.0		-1.2	-1.3	35.2	0.8	3.2	-4.5	-5.0	-1.5
TPX	0.4	-0.6		-2.4	38.1	1.1	-3.5	-4.9	-0.8	-0.3
RTY	48.6	-2.7	-10.4		26.2	-0.6	1.9	0.2	-6.4	5.6
EM	-4.1	30.9	69.2	10.9		0.9	4.6	9.1	3.9	33.1
US HY	-5.0	53.5	160.0	-18.8	69.5		95.6	48.4	31.4	-211.7
EMBI	10.8	44.2	-102.1	12.3	73.4	19.4		-5.8	40.5	86.2
EUR	-3.6	-14.7	-33.4	0.3	33.8	2.3	-1.4		56.7	48.2
JPY	6.8	-14.5	-4.8	-8.8	12.7	1.3	8.4	50.4		-33.2
GSCI	-1.1	-0.4	-0.2	0.8	10.7	-0.9	1.8	4.2	-3.3	
$\hat{s}_i$	0.3	0.7	0.9	0.5	0.7	0.1	0.2	0.4	0.4	1.2
R_i^2	83.0	47.7	34.9	82.4	60.9	39.8	51.6	42.3	43.7	12.1

Arbitrage factors and hedging portfolios

We finally obtain:

$$x_i^{\star}(\gamma) = \gamma \frac{\mu_i - \hat{\beta}_i^{\top} \mu^{(-i)}}{\hat{s}_i^2}$$

From this equation, we deduce the following conclusions:

1. The better the hedge, the higher the exposure. This is why highly correlated assets produces unstable MVO portfolios.
2. The long-short position is defined by the sign of $\mu_i - \hat{\beta}_i^{\top} \mu^{(-i)}$. If the expected return of the asset is lower than the conditional expected return of the hedging portfolio, the weight is negative.

Markowitz diversification	$\neq$	**Diversification of risk factors**
	$=$	**Concentration on arbitrage factors**

Markowitz optimization and active management

The rules of the game

The mean-variance approach is one of the most aggressive active management models: it concentrates the portfolio on a small number of bets (idiosyncratic factors and arbitrage factors).

Traditional shrinkage approaches (RMT, Ledoit-Wolf, etc.) are not sufficient. This is why portfolio managers use discretionary constraints: $Cx \geq D$. Jagannathan and Ma (2003) showed that:

$$\tilde{\Sigma} = \Sigma - \left(C^\top \lambda \mathbf{1}^\top + \mathbf{1}^\top \lambda C^\top \right)$$

where λ is the vector of Lagrange coefficients associated to $Cx \geq D$.

$\Rightarrow$ Using constraints is equivalent to shrink the covariance matrix (Ledoit-Wolf) or to introduce relative views (Black-Litterman)

Weight budgeting versus risk budgeting

Let $x = (x_1, \ldots, x_n)$ be the weights of n assets in the portfolio. Let $\mathscr{R}(x_1, \ldots, x_n)$ be a coherent and convex risk measure. We have:

$$\begin{aligned}\mathscr{R}(x_1, \ldots, x_n) &= \sum_{i=1}^{n} x_i \cdot \frac{\partial \mathscr{R}(x_1, \ldots, x_n)}{\partial x_i} \\ &= \sum_{i=1}^{n} \mathrm{RC}_i(x_1, \ldots, x_n)\end{aligned}$$

Let $b = (b_1, \ldots, b_n)$ be a vector of budgets such that $b_i \geq 0$ and $\sum_{i=1}^{n} b_i = 1$. We consider two allocation schemes:

1. Weight budgeting (WB)

$$x_i = b_i$$

2. Risk budgeting (RB)

$$\mathrm{RC}_i = b_i \cdot \mathscr{R}(x_1, \ldots, x_n)$$

Traditional risk parity with the volatility risk measure

Let Σ be the covariance matrix of the assets returns. We assume that the risk measure $\mathscr{R}(x)$ is the volatility of the portfolio $\sigma(x) = \sqrt{x^\top \Sigma x}$. We have:

$$
\begin{aligned}
\frac{\partial \mathscr{R}(x)}{\partial x} &= \frac{\Sigma x}{\sqrt{x^\top \Sigma x}} \\
\mathrm{RC}_i(x_1, \ldots, x_n) &= x_i \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} \\
\sum_{i=1}^{n} \mathrm{RC}_i(x_1, \ldots, x_n) &= \sum_{i=1}^{n} x_i \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}} = x^\top \frac{\Sigma x}{\sqrt{x^\top \Sigma x}} = \sigma(x)
\end{aligned}
$$

The risk budgeting portfolio is defined by this system of equations:

$$
\left\{
\begin{array}{l}
x_i \cdot (\Sigma x)_i = b_i \cdot \left(x^\top \Sigma x\right) \\
x_i \geq 0 \\
\sum_{i=1}^{n} x_i = 1
\end{array}
\right.
$$

An example

Illustration

- 3 assets
- Volatilities are equal to 30%, 20% and 15%
- Correlations are set to 80% between the 1st asset and the 2nd asset, 50% between the 1st asset and the 3rd asset and 30% between the 2nd asset and the 3rd asset
- Budgets are set to 50%, 20% and 30%
- For the ERC (Equal Risk Contribution) portfolio, all the assets have the same risk budget

Weight budgeting (or traditional) approach

Asset	Weight	Marginal Risk	Risk Contribution	
			Absolute	Relative
1	**50.00%**	29.40%	14.70%	70.43%
2	**20.00%**	16.63%	3.33%	15.93%
3	**30.00%**	9.49%	2.85%	13.64%
Volatility			20.87%	

Risk budgeting approach

Asset	Weight	Marginal Risk	Risk Contribution	
			Absolute	Relative
1	31.15%	28.08%	8.74%	**50.00%**
2	21.90%	15.97%	3.50%	**20.00%**
3	46.96%	11.17%	5.25%	**30.00%**
Volatility			17.49%	

ERC approach

Asset	Weight	Marginal Risk	Risk Contribution	
			Absolute	Relative
1	19.69%	27.31%	5.38%	**33.33%**
2	32.44%	16.57%	5.38%	**33.33%**
3	47.87%	11.23%	5.38%	**33.33%**
Volatility			16.13%	

The logarithmic barrier problem

Roncalli (2013) shows that:

$$x^{\star} = \arg\min \mathscr{R}(x) - \lambda \sum_{i=1}^{n} b_i \ln x_i$$

⇒ CCD algorithm (Griveau-Billion *et al.*, 2013).

- The RB portfolio is a combination of the MR and WB portfolios:

$$\begin{array}{ll} x_i / b_i = x_j / b_j & \text{(wb)} \\ \partial_{x_i} \mathscr{R}(x) = \partial_{x_j} \mathscr{R}(x) & \text{(mr)} \\ \mathrm{RC}_i / b_i = \mathrm{RC}_j / b_j & \text{(rb)} \end{array}$$

- The risk of the RB portfolio is between the risk of the MR portfolio and the risk of the WB portfolio:

$$\mathscr{R}(x_{\mathrm{mr}}) \leq \mathscr{R}(x_{\mathrm{rb}}) \leq \mathscr{R}(x_{\mathrm{wb}})$$

With risk budgeting, we always diminish the risk compared to the weight budgeting.

MVO portfolios versus RB portfolios

MVO portfolios

- Volatility optimization
- Marginal risk
- Sensitive to Σ^{-1}
- Arbitrage factors

RB portfolios

- Volatility diversification
- Risk contribution
- Sensitive to Σ
- Common risk factors

$\Rightarrow$ Risk parity is the right approach for managing the diversification of **long-only** diversified portfolios.

And in the long/short case?

Which assets (common risk factors) to diversify?

- Traditional assets or risk premia
  - Stocks
  - Bonds
- Equity risk factors
- Alternative risk premia
- Illiquid assets
  - Private equity
  - Private debt
  - Real estate
  - Infrastructure

What is the rationale for factor investing?

How to define risk factors?

Risk factors are common factors that explain the cross-section variance of expected returns

- 1964: Market or MKT (or BETA) factor
- 1972: Low beta or BAB factor
- 1981: Size or SMB factor
- 1985: Value or HML factor
- 1991: Low volatility or VOL factor
- 1993: Momentum or WML factor
- 2000: Quality or QMJ factor

Factor investing is a subset of smart (new) beta

What is the rationale for factor investing?

At the security level, there is a lot of idiosyncratic risk or alpha:

	Common Risk	Idiosyncratic Risk
GOOGLE	47%	53%
NETFLIX	24%	76%
MASTERCARD	50%	50%
NOKIA	32%	68%
TOTAL	89%	11%
AIRBUS	56%	44%

Carhart's model with 4 factors, 2010-2014
Source: Author's research

What is the rationale for factor investing?

- Jensen (1968) – **How to measure the performance of active management?**

$$R_t^F = \alpha + \beta R_t^{MKT} + \varepsilon_t$$

$\Rightarrow \bar{\alpha} = -\mathrm{fees}$

- Hendricks *et al.* (1993) – **Hot Hands in Mutual Funds**

$$\mathrm{cov}\left(\alpha_t^{\text{Jensen}}, \alpha_{t-1}^{\text{Jensen}}\right) > 0$$

where:

$$\alpha_t^{\text{Jensen}} = R_t^F - \beta^{\mathrm{MKT}} R_t^{MKT}$$

$\Rightarrow$ The persistence of the performance of active management is due to the **persistence of the alpha**

What is the rationale for factor investing?

- Grinblatt *et al.* (1995) – **Momentum investors versus Value investors**

  "*77% of mutual funds are momentum investors*"

- Carhart (1997):

$$\left\{ \begin{array}{l} \operatorname{cov}\left(\alpha_t^{\text{Jensen}}, \alpha_{t-1}^{\text{Jensen}}\right) > 0 \\ \operatorname{cov}\left(\alpha_t^{\text{Carhart}}, \alpha_{t-1}^{\text{Carhart}}\right) = 0 \end{array} \right.$$

where:

$$\alpha_t^{\text{Carhart}} = R_t^F - \beta^{\text{MKT}} R_t^{\text{MKT}} - \beta^{\text{SMB}} R_t^{\text{SMB}} - \beta^{\text{HML}} R_t^{\text{HML}} - \beta^{\text{WML}} R_t^{\text{WML}}$$

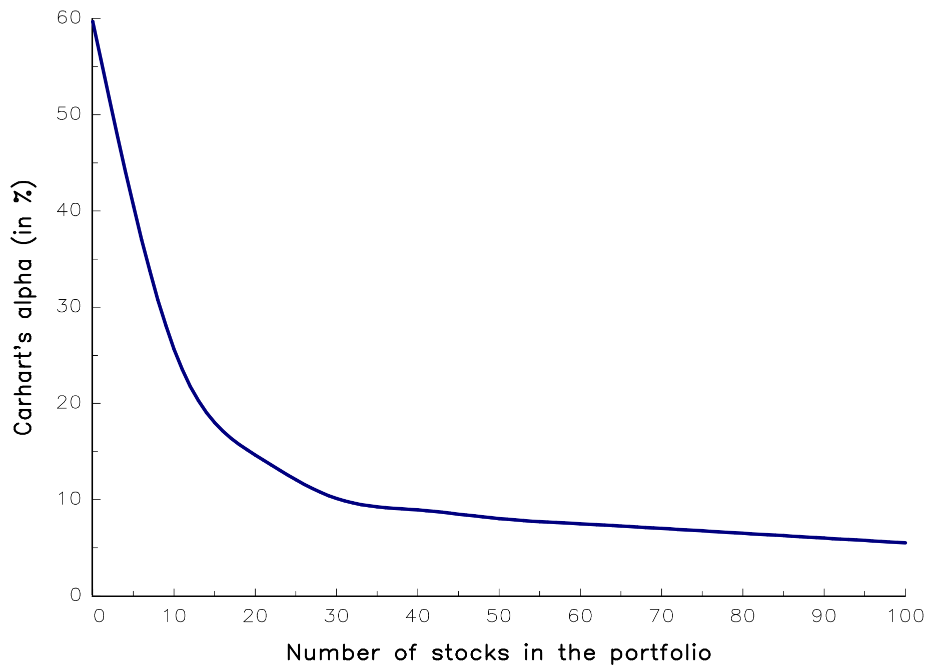
$\Rightarrow$ The (short-term) persistence of the performance of active management is due to the (short-term) **persistence of the performance of risk factors**

What is the rationale for factor investing?

David Swensen's rule for effective stock picking

Concentrated portfolio ⇒ No more than 20 bets?

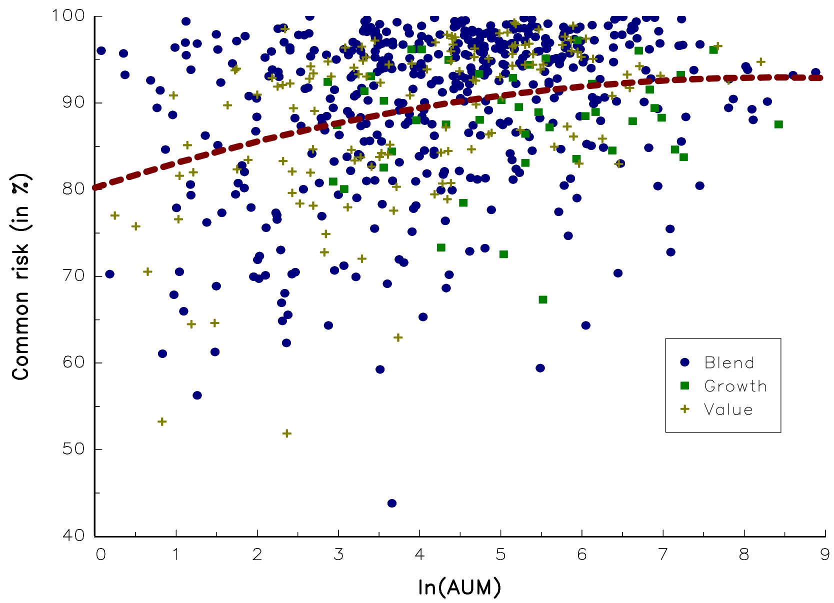
Figure: Carhart's alpha decreases with the number of holding assets



US equity markets, 2000-2014
Source: Author's research

What is the rationale for factor investing?

Figure: What proportion of return variance is explained?



Morningstar database, 880 mutual funds, European equities
Carhart's model with 4 factors, 2010-2014
Source: Author's research

How many bets are there in large portfolios of institutional investors?

1986 Less than 10% of institutional portfolio return is explained by security picking and market timing (Brinson *et al.*, 1986)

2009 Professors' Report on the Norwegian GPFG: Risk factors represent 99.1% of the fund return variation (Ang *et al.*, 2009)

What is the rationale for factor investing?

What lessons can we draw from this?

Idiosyncratic risks and specific bets disappear in (large) diversified portfolios. Performance of institutional investors is then exposed to risk factors.

Alpha is not scalable, but risk factors are scalable.

$\Rightarrow$ Risk factors are the only bets that are compatible with diversification.

Alpha		Beta(s)
• Concentration • Portfolio optimization (e.g. MVO)	$\neq$	• Diversification • Risk-based allocation (e.g. RB)

A new opportunity for active managers

- Active management does not reduce to stock picking
- Understanding the diversification of equity portfolios
- New tactical products

Approaches of equity investing

1. Pure stock picking process (with a limited number of bets)
2. Factor-based stock picking process
3. Allocation between factor-based portfolios

A new opportunity for active managers

Figure: Heatmap of risk factors (before 2008) – MSCI Europe

2000	2001	2002	2003	2004	2005	2006	2007	2008
Value 25.5%	Value 6.2%	Momentum -3.3%	Value 66.9%	Low Beta 31.1%	Size 32.1%	Momentum 39.1%	Momentum 10.1%	Low Beta -40.9%
Size 23.9%	Momentum -1.7%	Low Beta -6.8%	Size 40.6%	Value 30.4%	Value 31.5%	Size 34.3%	Market 2.7%	Momentum -41.4%
Quality 9.5%	Low Beta -2.0%	Value -18.7%	Momentum 27.5%	Momentum 30.1%	Quality 27.9%	Low Beta 31.5%	Quality 1.8%	Market -43.6%
Low Beta 6.2%	Size -7.5%	Size -18.9%	Low Beta 23.9%	Quality 29.5%	Momentum 26.5%	Value 25.5%	Low Beta -1.0%	Size -49.0%
Market -2.2%	Quality -9.1%	Quality -26.0%	Quality 19.9%	Size 28.7%	Low Beta 26.1%	Quality 24.1%	Size -4.4%	Quality -53.9%
Momentum -2.3%	Market -15.5%	Market -30.7%	Market 15.3%	Market 12.2%	Market 26.1%	Market 19.6%	Value -9.0%	Value -63.6%

Source: Richard and Roncalli (2015)

A new opportunity for active managers

Figure: Heatmap of risk factors (after 2008) – MSCI Europe

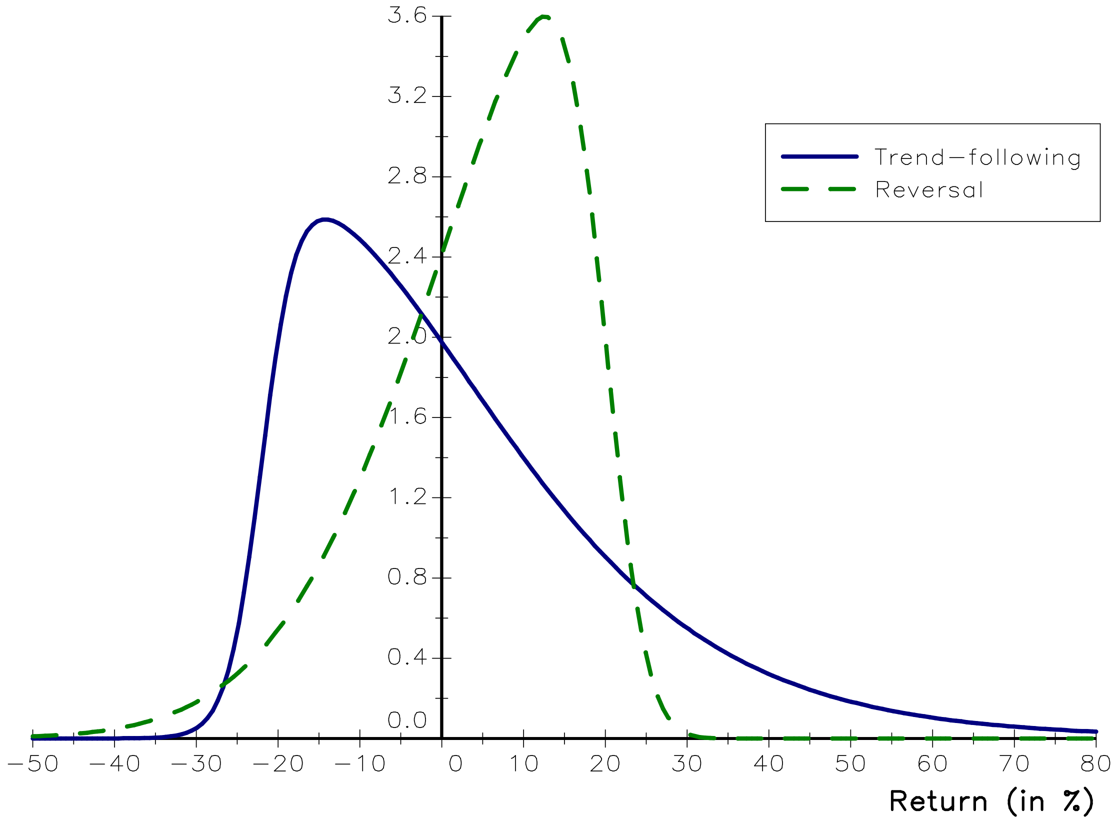
2008	2009	2010	2011	2012	2013	2014	2015	2016
Low Beta -40.9%	Value 65.7%	Quality 25.3%	Low Beta -2.2%	Quality 24.0%	Momentum 29.8%	Size 11.5%	Size 16.1%	Momentum -3.0%
Momentum -41.4%	Size 51.6%	Momentum 22.2%	Quality -3.2%	Momentum 24.0%	Value 28.4%	Value 10.8%	Quality 16.1%	Low Beta -7.1%
Market -43.6%	Quality 42.7%	Size 19.2%	Market -8.1%	Value 18.7%	Quality 21.0%	Quality 8.6%	Low Beta 15.7%	Market -7.2%
Size -49.0%	Market 31.6%	Low Beta 17.9%	Momentum -9.1%	Market 17.3%	Market 19.8%	Low Beta 8.1%	Momentum 12.3%	Quality -7.7%
Quality -53.9%	Momentum 22.3%	Market 11.1%	Size -25.0%	Low Beta 15.8%	Low Beta 17.0%	Market 6.8%	Market 8.2%	Size -12.1%
Value -63.6%	Low Beta 18.8%	Value 7.3%	Value -35.3%	Size 10.7%	Size 13.9%	Momentum 5.2%	Value -1.5%	Value -14.8%

Source: Richard and Roncalli (2015)

Risk premia & non-diversifiable risk

Consumption-based model (Lucas, 1978; Cochrane, 2001)

A risk premium is a compensation for accepting (systematic) risk in bad times.



Skewness risk premia & market anomalies

Characterization of alternative risk premia

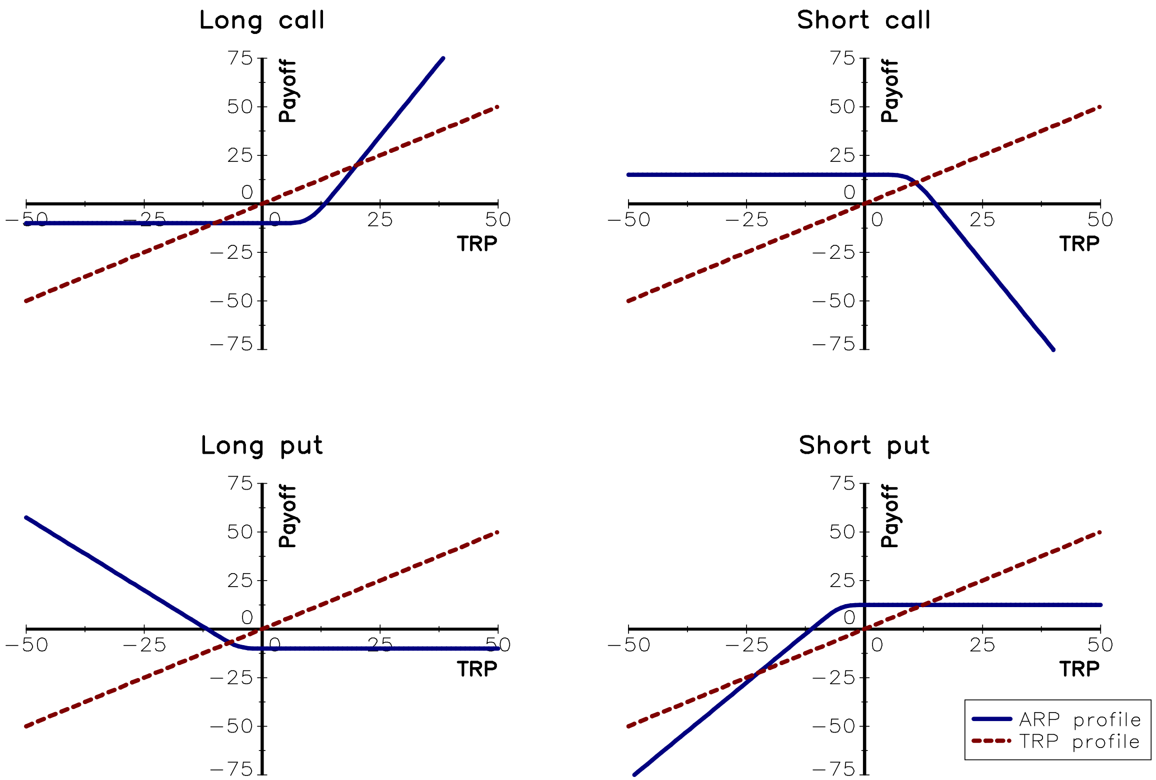
- An alternative risk premium (ARP) is a risk premium, which is not traditional
  - Traditional risk premia (TRP): equities, sovereign/corporate bonds
  - Currencies and commodities are not TRP
- The drawdown of an ARP must be positively correlated to bad times
  - Risk premia $\neq$ insurance against bad times
  - (SMB, HML) $\neq$ WML
- Risk premia are an increasing function of the volatility and a decreasing function of the skewness

In the market practice, alternative risk premia recovers:

1. Skewness risk premia (or pure risk premia), which present high negative skewness and potential large drawdown
2. Markets anomalies

Payoff function of alternative risk premia

Figure: Which option profile may be considered as a skewness risk premium?



- ~~Long call~~ (risk adverse)
- ~~Short call~~ (market anomaly)
- ~~Long put~~ (insurance)
- Short put

⇒ SMB, HML, ~~WML~~, ~~BAB~~, ~~QMJ~~

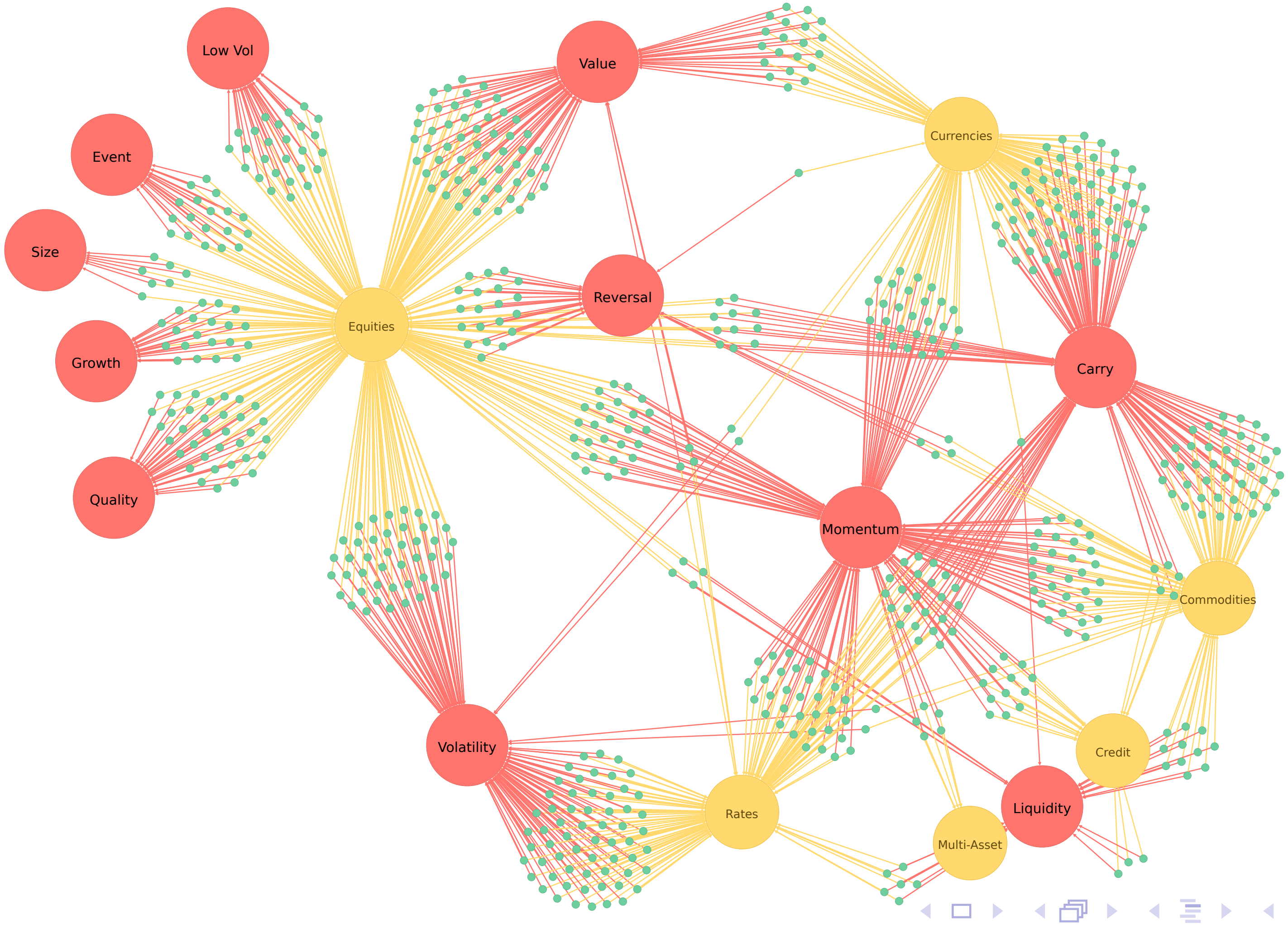
A myriad of alternative risk premia?

Figure: Mapping of ARP candidates

Risk Factor	Equities	Rates	Credit	Currencies	Commodities
Carry	Dividend Futures High Dividend Yield	FRB TSS CTS	FRB	FRB	FRB TSS CTS
Liquidity	Amihud liquidity	Turn-of-the-month	Turn-of-the-month		Turn-of-the-month
Momentum	Cross-section Time-series	Cross-section Time-series	Time-Series	Cross-section Time-series	Cross-section Time-series
Reversal	Time-series Variance	Time-series		Time-series	Time-series
Value	Value	Value	Value	PPP Economic model	Value
Volatility	Carry Term structure	Carry Term structure		Carry	Carry
Event	Buyback Merger arbitrage				
Growth	Growth				
Low volatility	Low volatility				
Quality	Quality				
Size	Size				

The example of bank's proprietary indices

Figure: Graph database of bank's proprietary indices



The identification problem

What is the problem?

- For traditional risk premia, the cross-correlation between several indices replicating the TRP is higher than 90%
- For alternative risk premia, the cross-correlation between several indices replicating the ARP is between -80% and 100%

Examples (2000-2015)

- In the case of the equities/US traditional risk premium, the cross-correlation between S&P 500, FTSE USA, MSCI USA, Russell 1000 and Russell 3000 indices is between 99.65% and 99.92%
- In the case of the equities/volatility/carry/US risk premium, the cross-correlation between the 14 short volatility indices is between -34.9% and 98.6% (mean $= 43.0\%$, $Q_3 - Q_1 > 35\%$)

The identification problem

Step 1 Define the set of relevant indices (qualitative due diligence).

Step 2 Given an initial set of indices, the underlying idea is to find the subset, whose elements present very similar patterns. For that, we use the deletion algorithm using the $\mathbf{R}^2$ statistic:

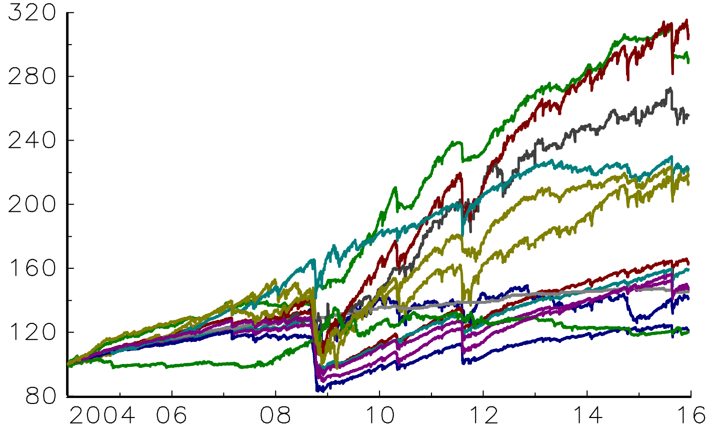
$$R_{k,t} = \alpha_k + \beta_k R_t^{(-k)} + \varepsilon_{k,t} \quad \Rightarrow \quad \mathbf{R}_k^2$$

Step 3 The algorithm stops when the similarity is larger than a given threshold for all the elements of the subset (e.g. $\mathbf{R}_k^2 > \mathbf{R}_{\min}^2 = 70\%$).

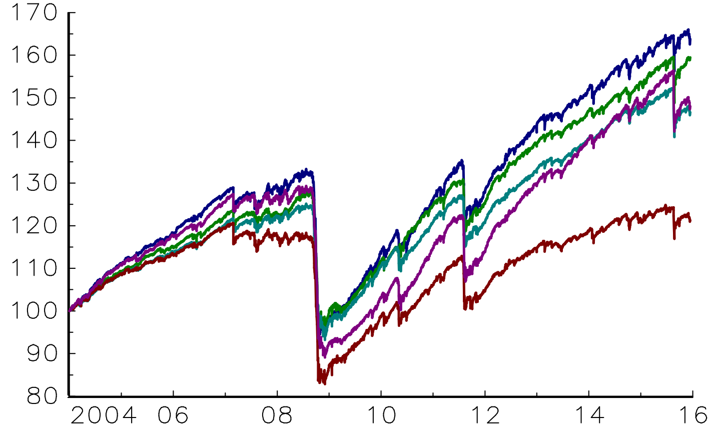
Step 4 The generic backtest of the ARP is the weighted average of the performance of the subset elements

Illustration with the volatility carry risk premium

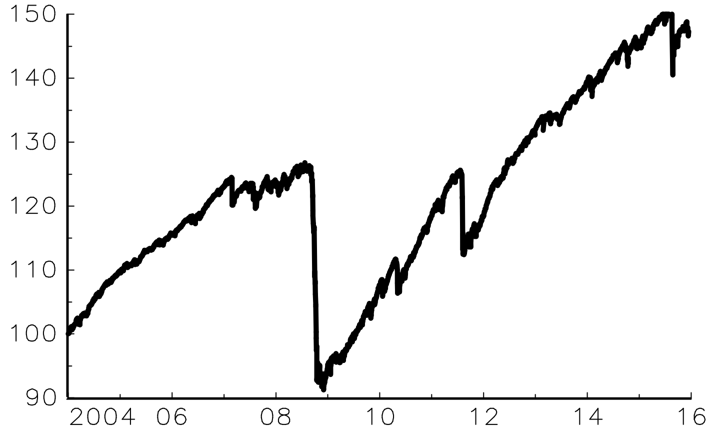
Indices after the 1st step



Selected indices after the 3rd step



Generic cumulative return



- Barclays (BXIISVUE) 90.2%
- Citi (CIISEVCU) 92.4%
- Citi (CIISEVWU) 97.0%
- JP Morgan (AIJPSV1U) 93.4%
- SG (SGIXVPUX) 94.9%

~~Value~~Carry and momentum everywhere

Figure: Mapping of **relevant** ARP[2]

Risk Factor	Equities	Rates	Credit	Currencies	Commodities
Carry	~~Dividend Futures~~ High Dividend Yield	FRB TSS ~~CTS~~	~~FRB~~	FRB	FRB TSS ~~CTS~~
Liquidity	~~Amihud liquidity~~	~~Turn-of-the-month~~	~~Turn-of-the-month~~		Turn-of-the-month
Momentum	Cross-section Time-series	~~Cross-section~~ Time-series	~~Time-Series~~	~~Cross-section~~ Time-series	Cross-section Time-series
Reversal	Time-series ~~Variance~~	~~Time-series~~		~~Time-series~~	~~Time-series~~
Value	Value	~~Value~~	~~Value~~	PPP Economic model	~~Value~~
Volatility	Carry Term structure	Carry ~~Term structure~~		~~Carry~~	~~Carry~~
Event	~~Buyback~~ Merger arbitrage				
Growth	Growth				
Low volatility	Low volatility				
Quality	Quality				
Size	Size				

[2] Based on bank's proprietary indices.

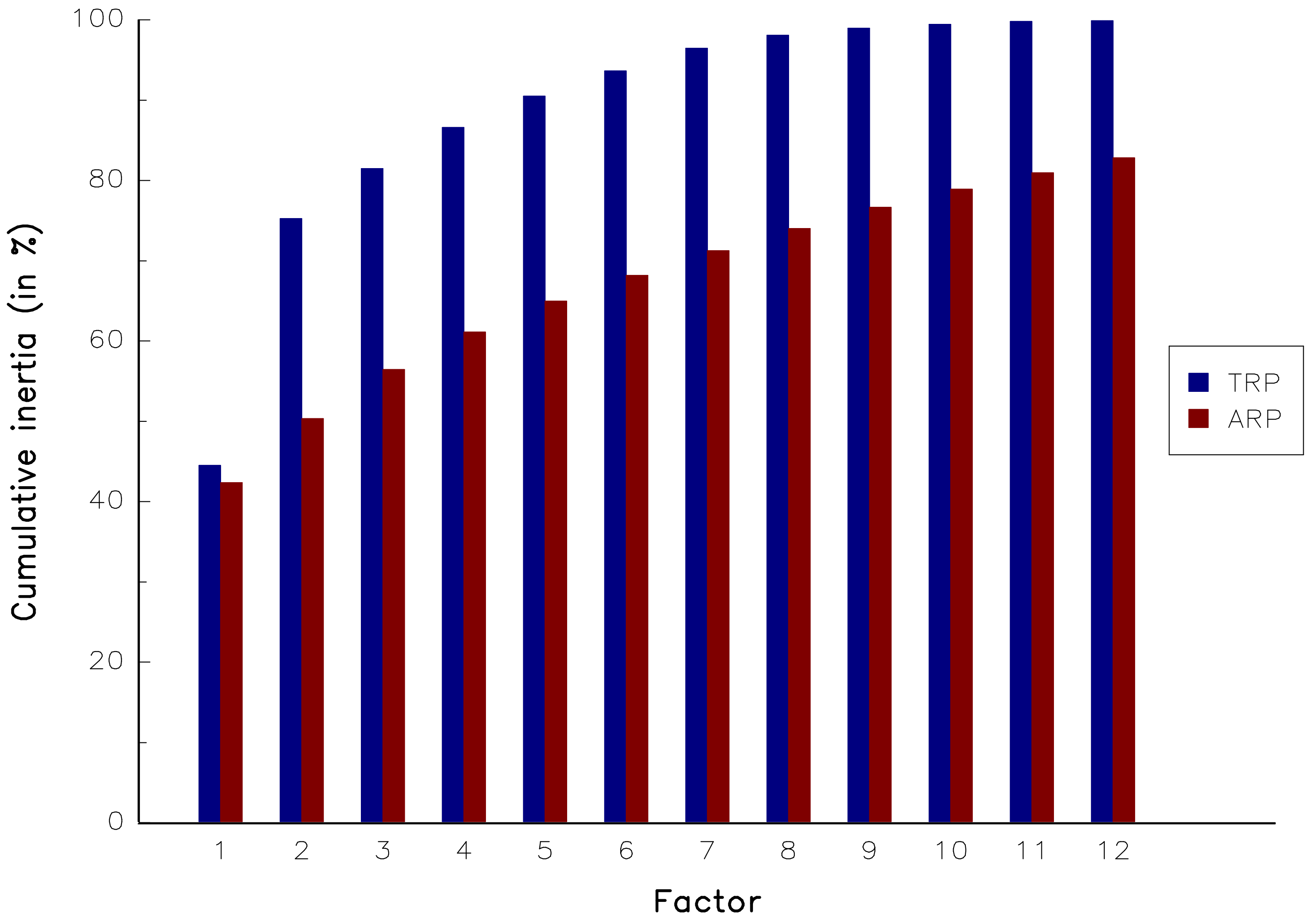
~~Value~~Carry and momentum everywhere

- ~~Value~~ Carry and momentum everywhere
- Some ARP candidates are not relevant (e.g. liquidity premium in equities, rates and currencies; reversal premium using variance swaps; value premium in rates and commodities; dividend premium; volatility premium in currencies and commodities; correlation premium; seasonality premium.)
- Hierarchy of ARP

Equities	value, carry, low volatility, volatility/carry, momentum, quality, growth, size, event, reversal
Rates	volatility/carry, momentum, carry
Currencies	carry, momentum, value
Commodities	carry, momentum, liquidity

- Carry recovers different notions: FRB (Forward Rate Bias), TSS (Term Structure Slope) and CTS (Cross Term Structure).

Volatility diversification



How to diversify (common risk factors)?

- Volatility diversification
- Skewness diversification
- Liquidity diversification

Correlation and diversification

Consider a portfolio with 2 assets: $R(x) = x_1 R_1 + x_2 R_2$. We have:

$$\operatorname{var}(R(x)) = x_1^2 \sigma_1^2 + x_2^2 \sigma_2^2 + 2 x_1 x_2 \rho \sigma_1 \sigma_2$$

Best solution in terms of volatility diversification

- Long-only portfolios: $\rho = -1$
- Long/short portfolios: $\rho = 0$

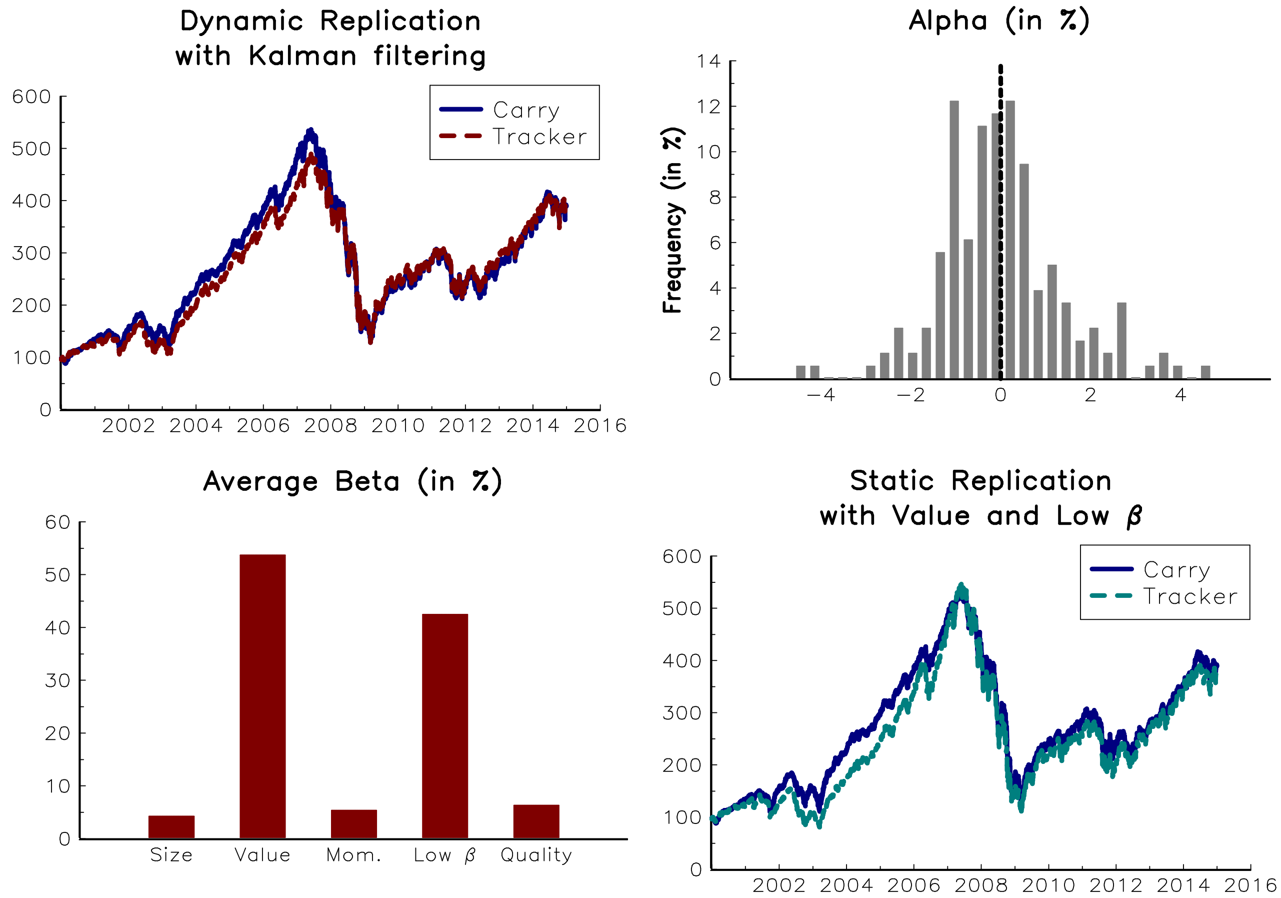
In long-only portfolios, volatility diversification consists in finding assets with negative correlations. In long/short portfolios, volatility diversification consists in finding assets with zero correlations.

Remark

In long/short portfolios, a correlation of $-\rho$ is equivalent to a correlation of $+\rho$.

Dependence between risk factors

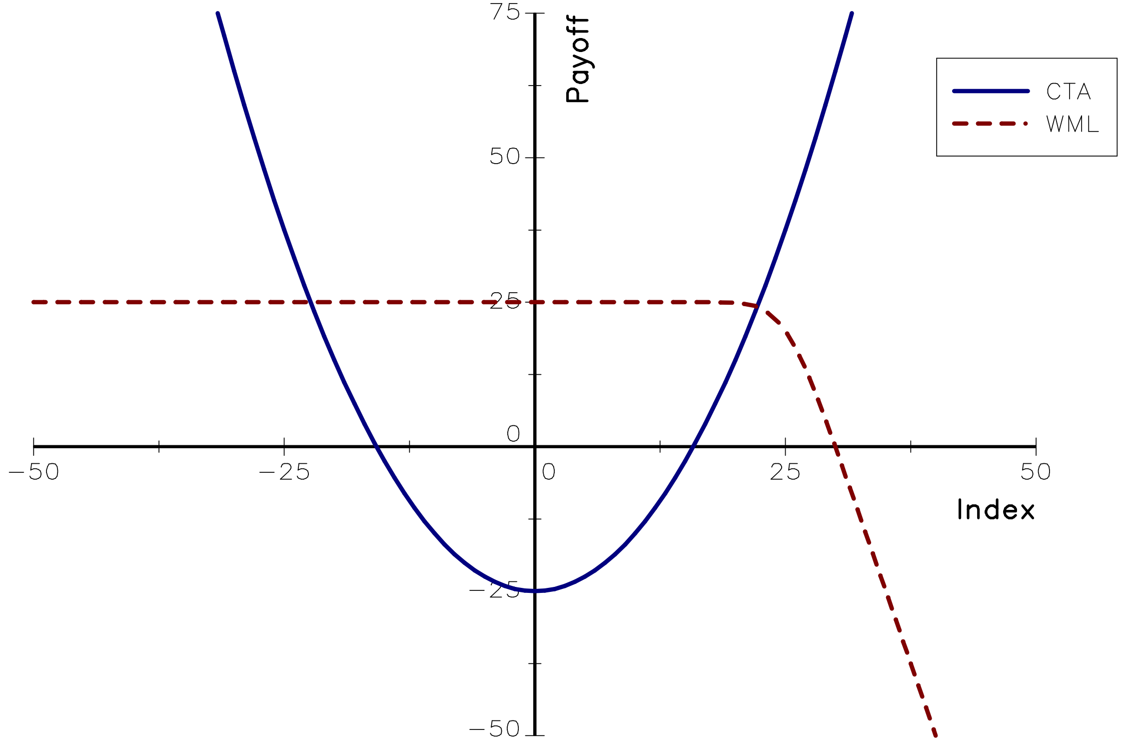
Figure: Value, low beta and carry are not orthogonal risk factors



Source: Author's calculation.

TRP and non-linear payoff functions

Figure: WML does not exhibit a CTA option profile

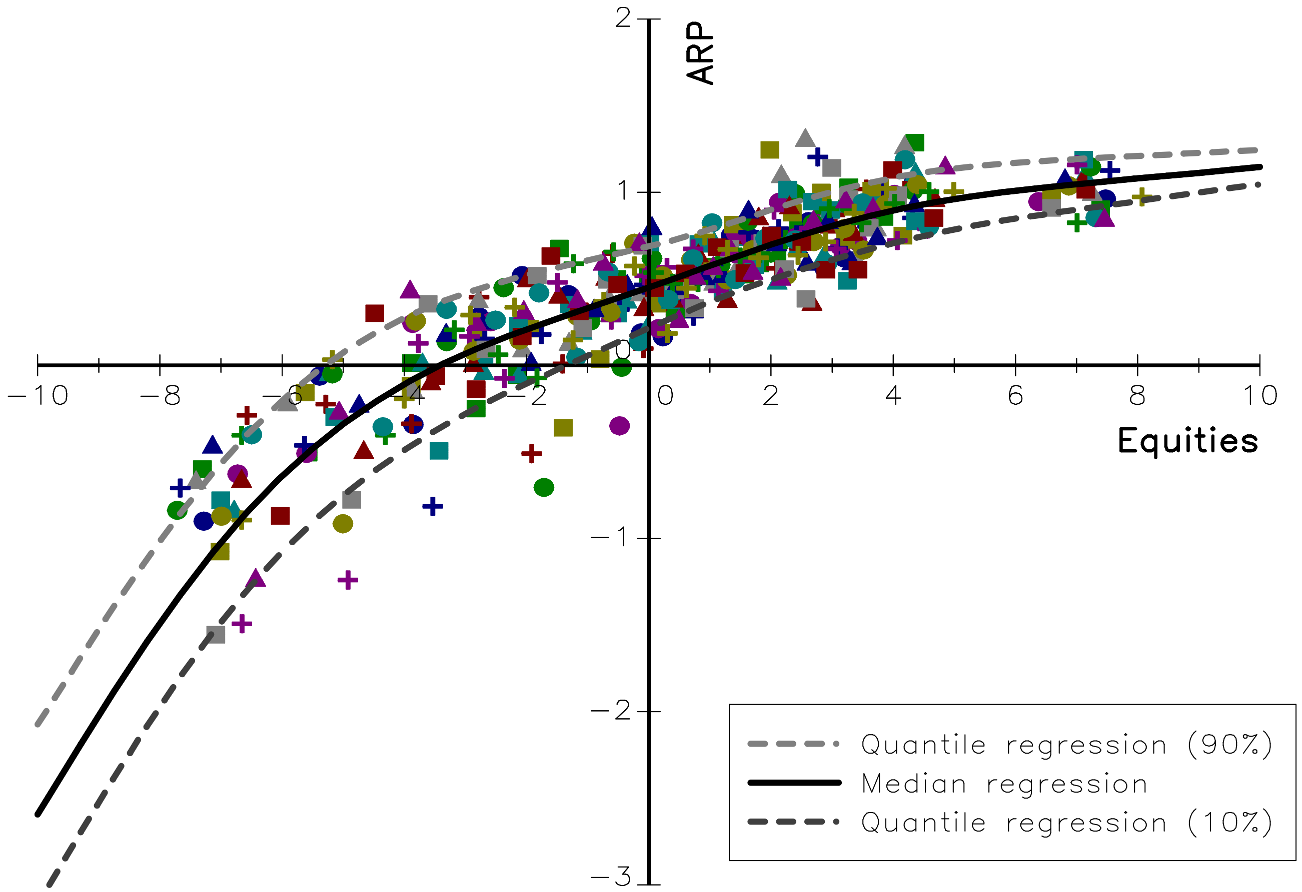


Source: Cazalet and Roncalli (2014)

- Cross-section momentum $\neq$ Time-series momentum
- Long-only momentum $\neq$ Long/short momentum

TRP and non-linear payoff functions

Figure: Payoff function of the US short volatility strategy



The skewness puzzle

- ARP are not all-weather strategies:
  - Extreme risks of ARP are high and may be correlated
  - Aggregation of skewness is not straightforward

Skewness aggregation $\neq$ volatility aggregation

When we accumulate long/short skewness risk premia in a portfolio, the volatility of this portfolio decreases dramatically, but its skewness risk generally increases!

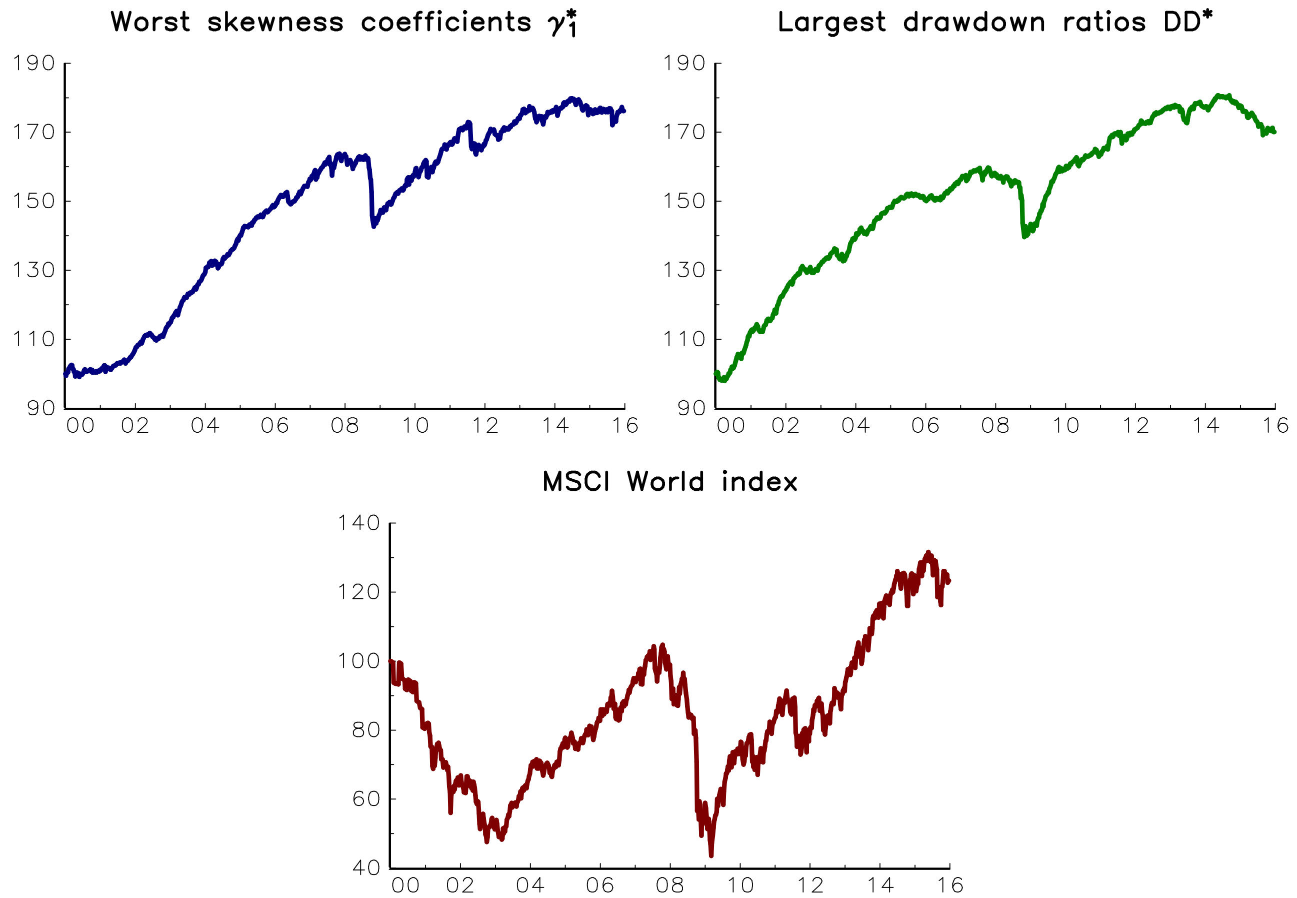
- Skewness diversification $\neq$ volatility diversification

$$\begin{aligned} \sigma(X+Y) &\leq \sigma(X)+\sigma(X) \\ \gamma_1(X+Y) &\nleq \gamma_1(X)+\gamma_1(Y) \end{aligned}$$

$\Rightarrow$ Skewness is not a convex risk measure

The skewness puzzle

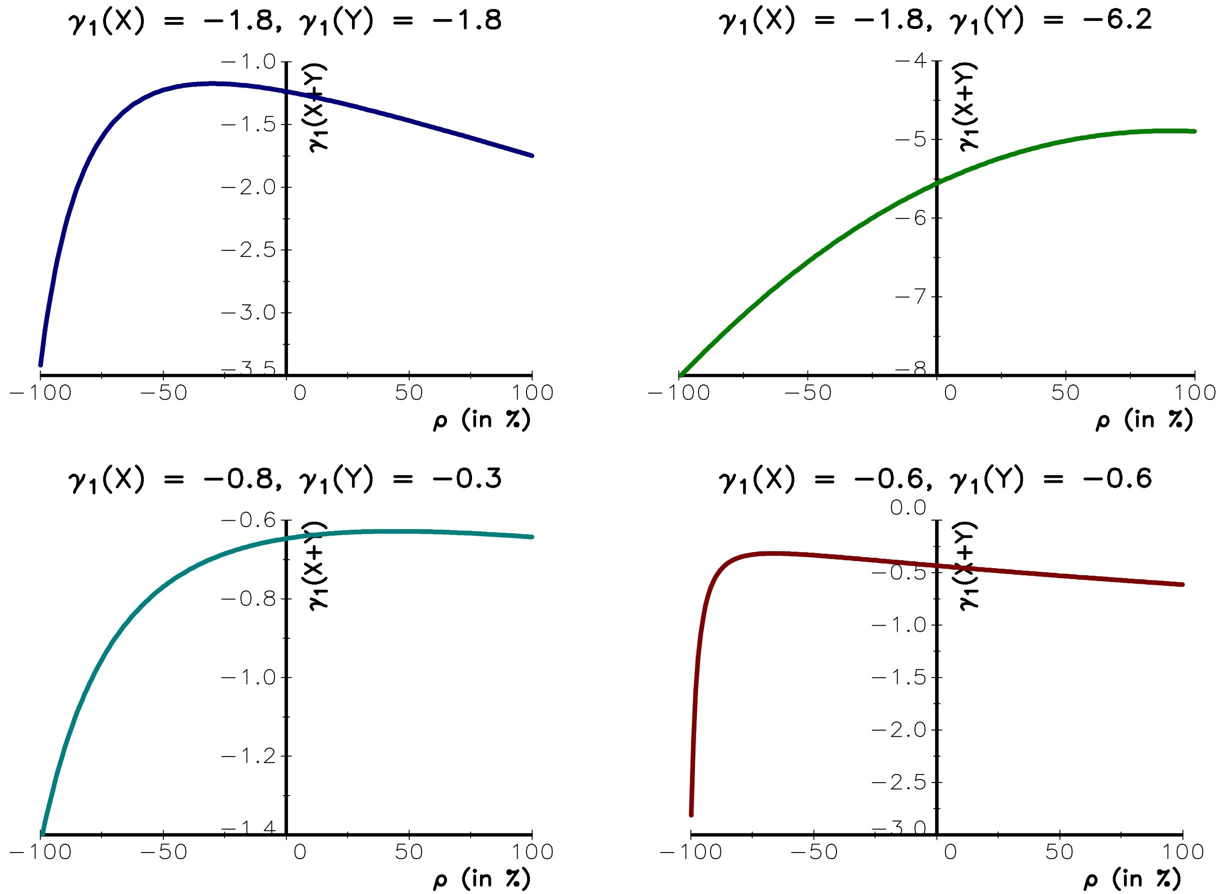
Figure: Skewness aggregation of L/S alternative risk premia



Source: HPRZ (2016)

The skewness puzzle

Figure: Skewness aggregation in the case of the bivariate log-normal distribution



Source: HPRZ (2016)

The skewness puzzle

Why?

- Volatility diversification works very well with L/S risk premia:

$$\sigma(R(x)) \approx \frac{\bar{\sigma}}{\sqrt{n}}$$

- Drawdown diversification don't work very well because bad times are correlated and are difficult to hedge:

$$\mathrm{DD}(x) \approx \overline{\mathrm{DD}}$$

The jump-diffusion representation

- n risky assets represented by the vector of prices $S_t = (S_{1,t}, \ldots, S_{n,t})$ with:

$$\left\{ \begin{array}{l} \mathrm{d}S_t = \operatorname{diag}(S_t)\,\mathrm{d}L_t \\ \mathrm{d}L_t = \mu\,\mathrm{d}t + \Sigma^{1/2}\,\mathrm{d}W_t + \mathrm{d}Z_t \end{array} \right.$$

  where Z_t is a pure n-dimensional jump process.

- We assume that the jump process Z_t is a compound Poisson process:

$$Z_t = \sum_{i=1}^{N_t} Z_i$$

  where $N_t \sim \mathscr{P}(\lambda)$ and $Z_i \sim \mathscr{N}\left(\tilde{\mu}, \tilde{\Sigma}\right)$.

The characteristic function of asset returns $R_t = (R_{1,t}, \ldots, R_{n,t})$ for the holding period $\mathrm{d}t$ may be approximated by:

$$\mathbb{E}\left[e^{-iu.R_t}\right] \approx (1 - \lambda\,\mathrm{d}t) \cdot e^{\left(iu^\top \mu - \frac{1}{2} u^\top \Sigma u\right)\mathrm{d}t} + (\lambda\,\mathrm{d}t) \cdot e^{iu^\top (\mu\,\mathrm{d}t + \tilde{\mu}) - \frac{1}{2} u^\top \left(\Sigma\,\mathrm{d}t + \tilde{\Sigma}\right) u}$$

The Gaussian mixture representation

We consider a Gaussian mixture model with two regimes to define R_t:

1. The continuous component, which has the probability $(1-\lambda\,\mathrm{d}t)$ to occur, is driven by the Gaussian distribution $\mathcal{N}(\mu\,\mathrm{d}t, \Sigma\,\mathrm{d}t)$;
2. The jump component, which has the probability $\lambda\,\mathrm{d}t$ to occur, is driven by the Gaussian distribution $\mathcal{N}\left(\tilde{\mu}, \tilde{\Sigma}\right)$.

The multivariate density function of R_t is:

$$
\begin{aligned}
f(y) \quad = \quad & \frac{1-\lambda\,\mathrm{d}t}{(2\pi)^{n/2}\left|\Sigma\,\mathrm{d}t\right|^{1/2}} e^{-\frac{1}{2}(y-\mu\,\mathrm{d}t)^\top(\Sigma\,\mathrm{d}t)^{-1}(y-\mu\,\mathrm{d}t)} + \\
& \frac{\lambda\,\mathrm{d}t}{(2\pi)^{n/2}\left|\Sigma\,\mathrm{d}t+\tilde{\Sigma}\right|^{1/2}} e^{-\frac{1}{2}(y-(\mu\,\mathrm{d}t+\tilde{\mu}))^\top\left(\Sigma\,\mathrm{d}t+\tilde{\Sigma}\right)^{-1}(y-(\mu\,\mathrm{d}t+\tilde{\mu}))}
\end{aligned}
$$

The characteristic function of R_t is equal to:

$$
\mathbb{E}\left[e^{-iu.R_t}\right] = (1-\lambda\,\mathrm{d}t)\cdot e^{\left(iu^\top\mu-\frac{1}{2}u^\top\Sigma u\right)\mathrm{d}t} + (\lambda\,\mathrm{d}t)\cdot e^{iu^\top(\mu\,\mathrm{d}t+\tilde{\mu})-\frac{1}{2}u^\top\left(\Sigma\,\mathrm{d}t+\tilde{\Sigma}\right)u}
$$

Distribution function of the portfolio's return

Let $x = (x_1, \ldots, x_n)$ be the vector of weights in the portfolio. We have:

$$R(x) = Y = B_1 \cdot Y_1 + B_2 \cdot Y_2$$

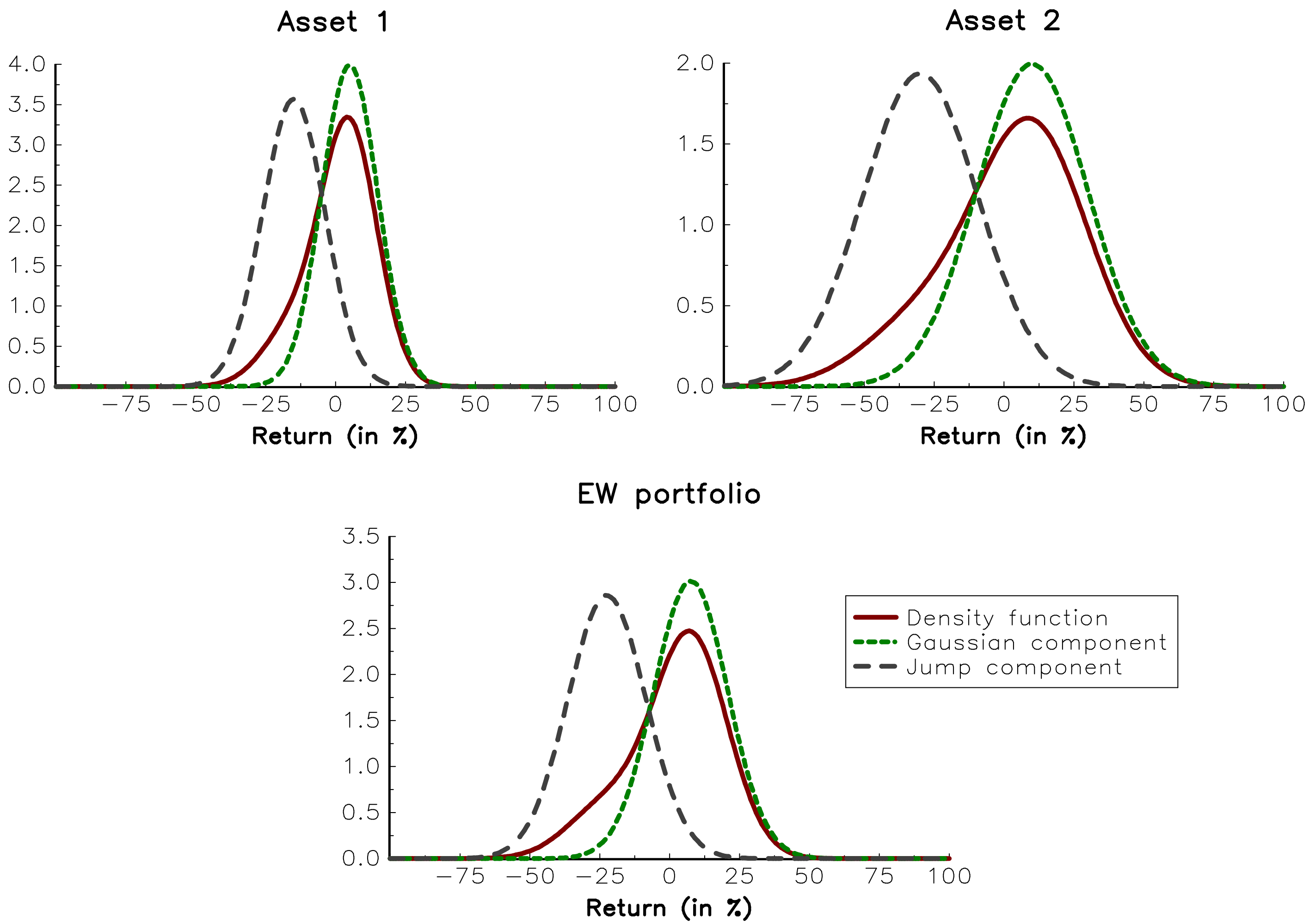
where:

- $B_1 \sim \mathcal{B}(\pi_1)$, $B_2 = 1 - B_1 \sim \mathcal{B}(\pi_2)$, $\pi_1 = 1 - \lambda$ and $\pi_2 = \lambda$ ($\underline{\mathscr{H} : \mathrm{d}t = 1}$);
- $Y_1 \sim \mathcal{N}\left(\mu_1(x), \sigma_1^2(x)\right)$, $\mu_1(x) = x^\top \mu$ and $\sigma_1^2(x) = x^\top \Sigma x$;
- $Y_2 \sim \mathcal{N}\left(\mu_2(x), \sigma_2^2(x)\right)$, $\mu_2(x) = x^\top(\mu + \tilde{\mu})$ and $\sigma_2^2(x) = x^\top \left(\Sigma + \tilde{\Sigma}\right) x$.

$\Rightarrow$ The portfolio's return $R(x)$ has the following density function:

$$\begin{aligned} f(y) &= \pi_1 f_1(y) + \pi_2 f_2(y) \\ &= (1-\lambda)\frac{1}{\sigma_1(x)}\phi\left(\frac{y-\mu_1(x)}{\sigma_1(x)}\right) + \lambda\frac{1}{\sigma_2(x)}\phi\left(\frac{y-\mu_2(x)}{\sigma_2(x)}\right) \end{aligned}$$

Distribution function of the portfolio's return



Parameters: $\mu_1 = 5\%$, $\sigma_1 = 10\%$, $\tilde{\mu}_1 = -20\%$, $\tilde{\sigma}_1 = 5\%$, $\mu_2 = 10\%$, $\sigma_2 = 20\%$, $\tilde{\mu}_2 = -40\%$, $\tilde{\sigma}_2 = 5\%$, $\rho = 50\%$, $\tilde{\rho} = 60\%$ and $\lambda = 0.20$.

Relationship between jump risk and skewness risk

The skewness of $R(x)$ is equal to:

$$\gamma_1 = \frac{(\lambda - \lambda^2)\left((1-2\lambda)(x^\top \tilde{\mu})^3 + 3(x^\top \tilde{\mu})\left(x^\top \tilde{\Sigma} x\right)\right)}{\left(x^\top \Sigma x + \lambda x^\top \tilde{\Sigma} x + (\lambda - \lambda^2)(x^\top \tilde{\mu})^2\right)^{3/2}}$$

The portfolio exhibits skewness, except for some limit cases:

$$\gamma_1 = 0 \Leftrightarrow x^\top \tilde{\mu} = 0 \text{ or } \lambda = 0 \text{ or } \lambda = 1$$

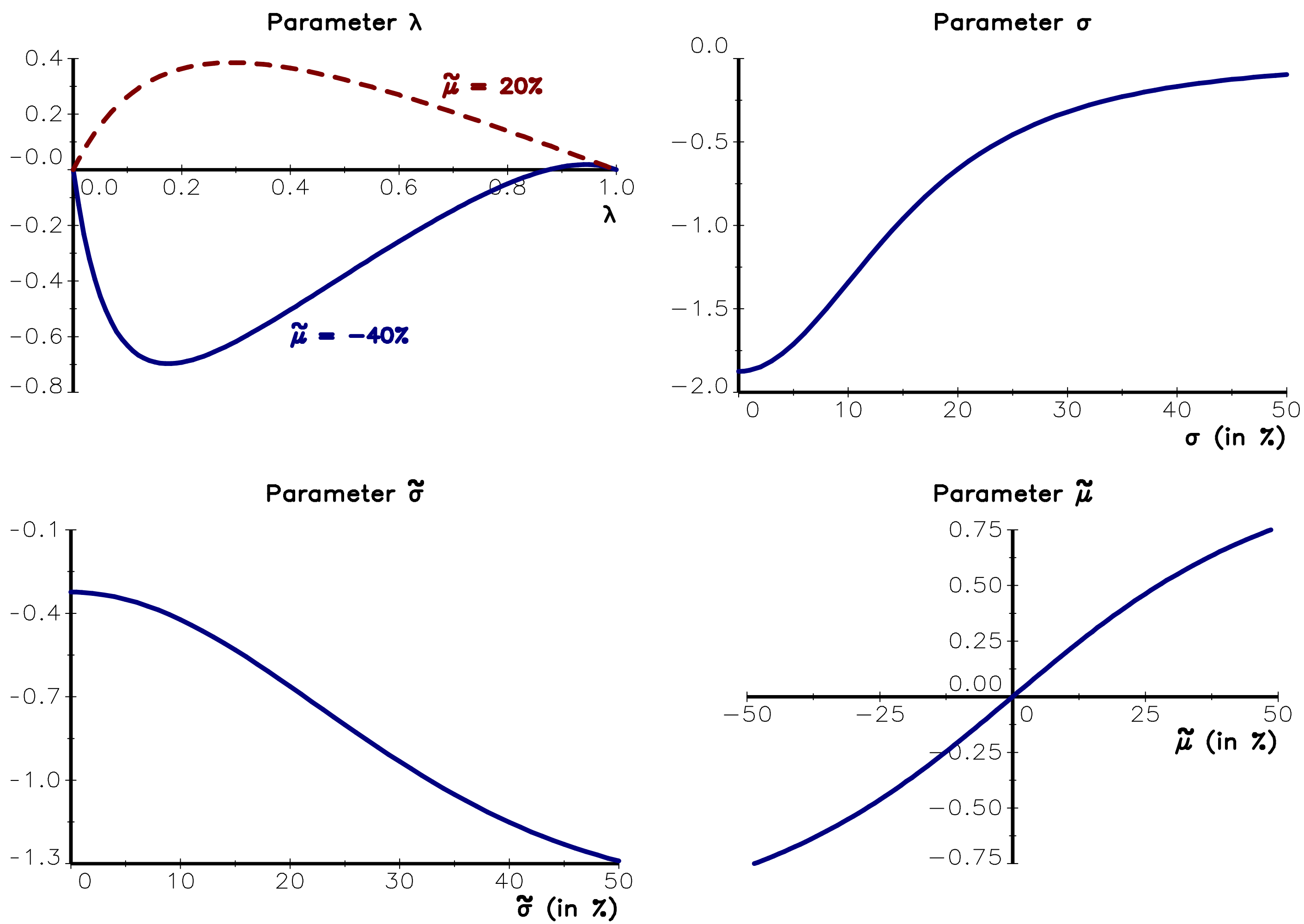
We have:

- If $x^\top \tilde{\mu} > 0$, then $\gamma_1 > 0$;
- If $x^\top \tilde{\mu} < 0$, then $\gamma_1 < 0$ in most cases.

$\Rightarrow$ We retrieve the result of Hamdan *et al.* (2016):

Skewness risk is maximum when volatility risk is minimum

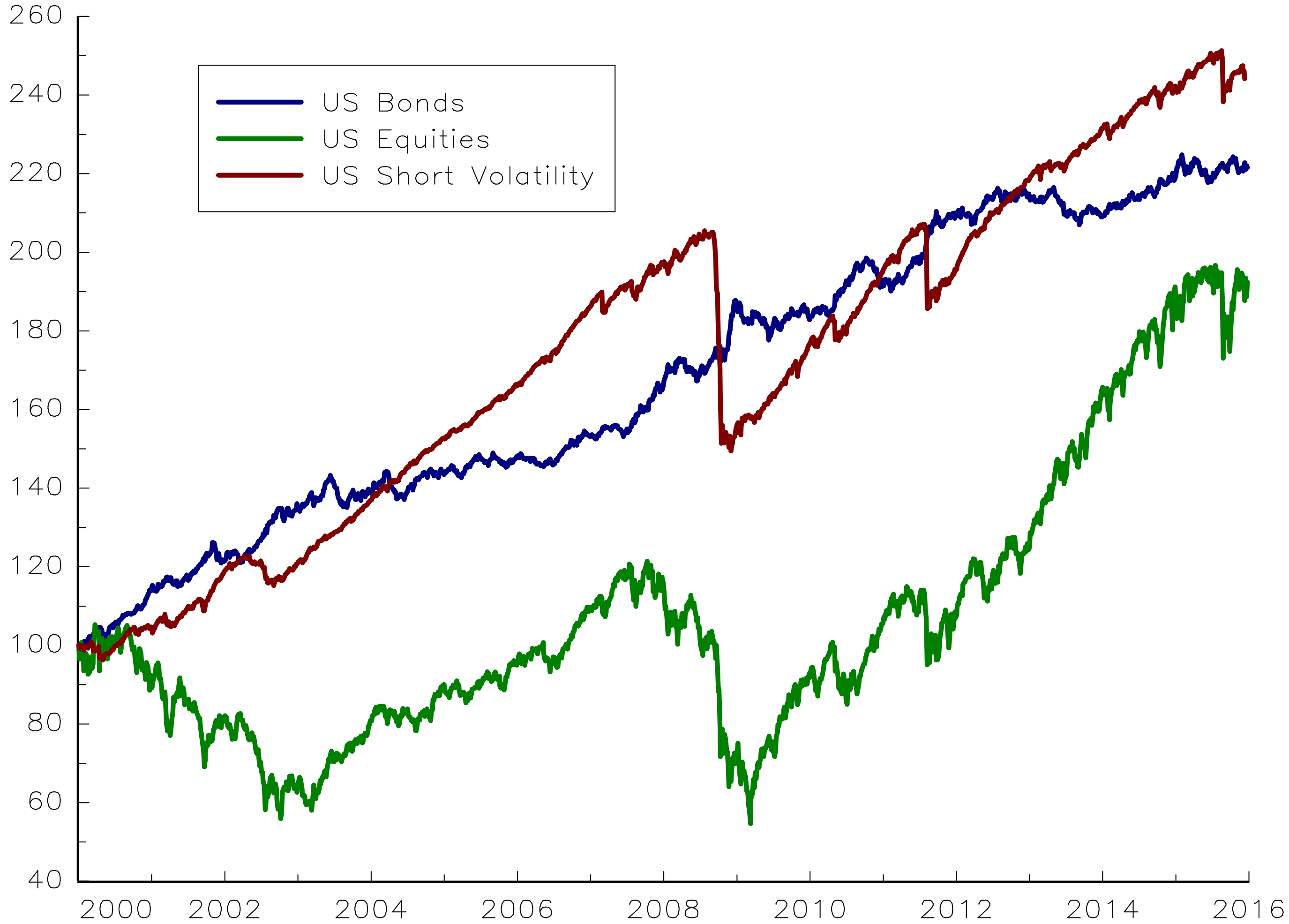
Relationship between jump risk and skewness risk



Parameters: $\sigma = 20\%$, $\tilde{\mu} = -40\%$, $\tilde{\sigma} = 20\%$ and $\lambda = 25\%$.

The Equity/Bond/Volatility asset mix policy

Figure: Cumulative performance of US bonds, US equities and US short volatility



Statistics

Table: Worst returns (in %)

Asset	Daily	Weekly	Monthly	Annually	Maximum
Bonds	−1.67	−2.81	−4.40	−3.41	−6.03
Equities	−9.03	−18.29	−29.67	−49.69	−55.25
Carry	−6.82	−11.04	−23.43	−23.37	−27.30

Table: Skewness coefficients

Asset	Daily	Weekly	Monthly	Annually	Volatility
Bonds	−0.12	−0.17	0.07	0.22	4.17
Equities	0.01	−0.44	−0.81	−0.57	18.38
Carry	−7.24	−5.77	−6.32	−2.23	5.50

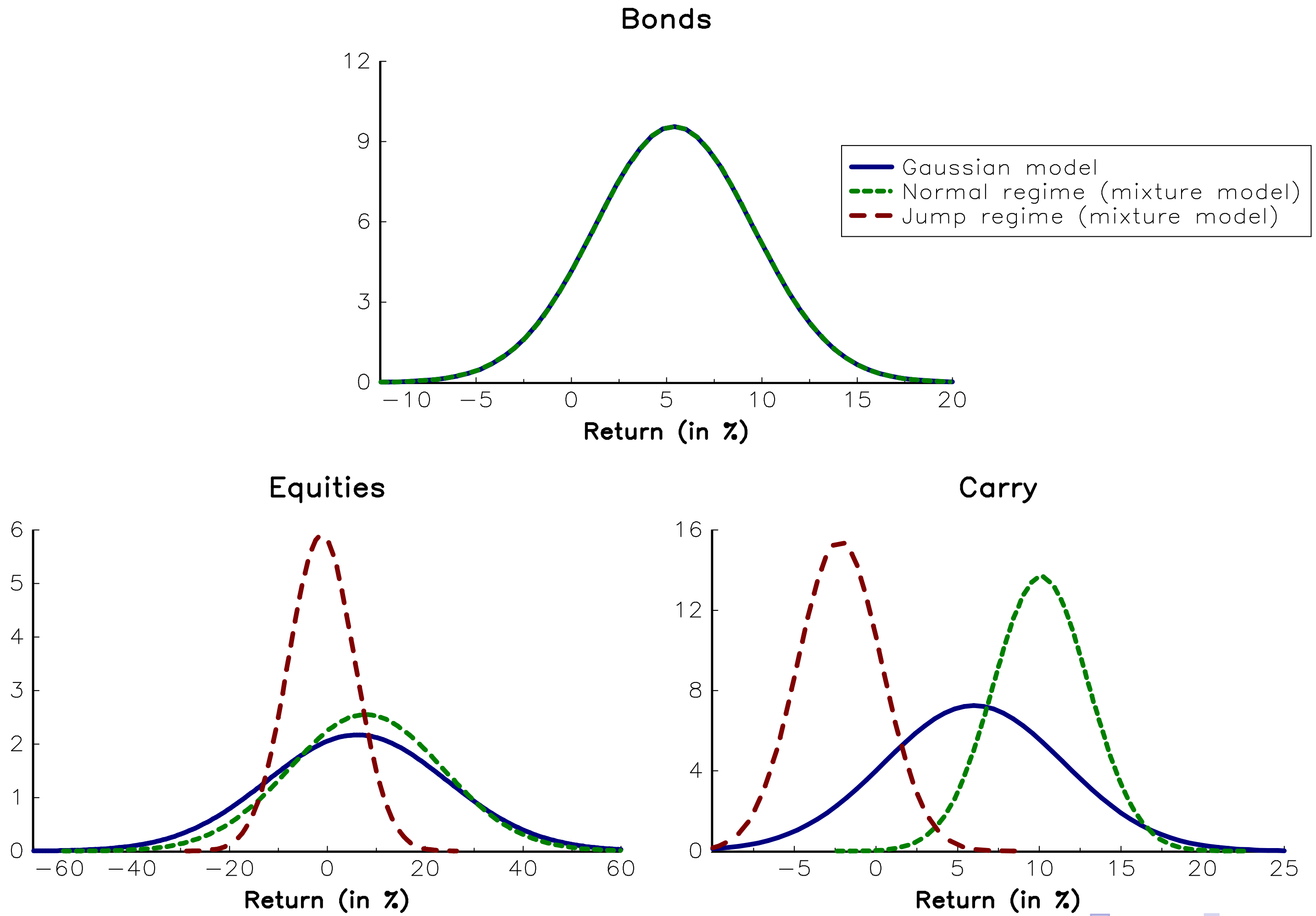
Calibration of the model

Table: Estimation of the mixture model when $\lambda \mathrm{d}t = 0.5\%$ (weekly model)

Regime	Asset	μ_i	σ_i	$\rho_{i,j}$		
Normal	Bonds	5.38	4.17	100.00		
	Equities	7.89	15.64	−36.80	100.00	
	Carry	10.10	2.91	−25.17	57.43	100.00
Regime	**Asset**	$\tilde{\mu}_i$	$\tilde{\sigma}_i$	$\tilde{\rho}_{i,j}$		
Jump	Bonds	0.00	0.00	100.00		
	Equities	−1.20	6.76	0.00	100.00	
	Carry	−2.23	2.57	0.00	60.45	100.00

Impact of the jump component

Figure: PDF of asset returns (weekly model)



The expected shortfall risk measure

Definition of the expected shortfall

$$\mathrm{ES}_{\alpha}(x) = \mathbb{E}\left[L(x) \mid L(x) \geq \mathrm{VaR}_{\alpha}(x)\right]$$

where $L(x) = -R(x)$ is the portfolio's loss.

We obtain:

$$\mathrm{ES}_{\alpha}(x) = (1-\lambda)\cdot\varphi\left(\mathrm{VaR}_{\alpha}(x), \mu_1(x), \sigma_1(x)\right) + \lambda\cdot\varphi\left(\mathrm{VaR}_{\alpha}(x), \mu_2(x), \sigma_2(x)\right)$$

where the function $\varphi(a,b,c)$ is defined by:

$$\varphi(a,b,c) = \frac{c}{1-\alpha}\phi\left(\frac{a+b}{c}\right) - \frac{b}{1-\alpha}\Phi\left(-\frac{a+b}{c}\right)$$

Here, the value-at-risk $\mathrm{VaR}_{\alpha}(x)$ is the root of the following equation:

$$(1-\lambda)\cdot\Phi\left(\frac{\mathrm{VaR}_{\alpha}(x)+\mu_1(x)}{\sigma_1(x)}\right) + \lambda\cdot\Phi\left(\frac{\mathrm{VaR}_{\alpha}(x)+\mu_2(x)}{\sigma_2(x)}\right) = \alpha$$

Analytical expression of risk contributions

We obtain a complicated expression of the risk contribution:

$$\mathscr{RC}_i(x) = x_i \frac{\partial \, \mathrm{ES}_\alpha(x)}{\partial \, x_i} = \ldots$$

But it is an analytical formula!

$\Rightarrow$ **No numerical issues for implementing the model**

Euler decomposition

We have:

$$\sum_{i=1}^{n} \mathscr{RC}_i(x) = \mathrm{ES}_\alpha(x)$$

$\Rightarrow$ Comparison with the value-at-risk based on the Cornish-Fisher expansion

Risk budgeting portfolios

The RB portfolio is defined by the following non-linear system:

$$\begin{cases} \mathscr{RC}_i(x) = b_i \mathscr{R}(x) \\ b_i > 0 \\ x_i \geq 0 \\ \sum_{i=1}^n b_i = 1 \\ \sum_{i=1}^n x_i = 1 \end{cases}$$

where b_i is the ex-ante risk budget of asset i expressed in relative terms.

Numerical solution of the RB portfolio

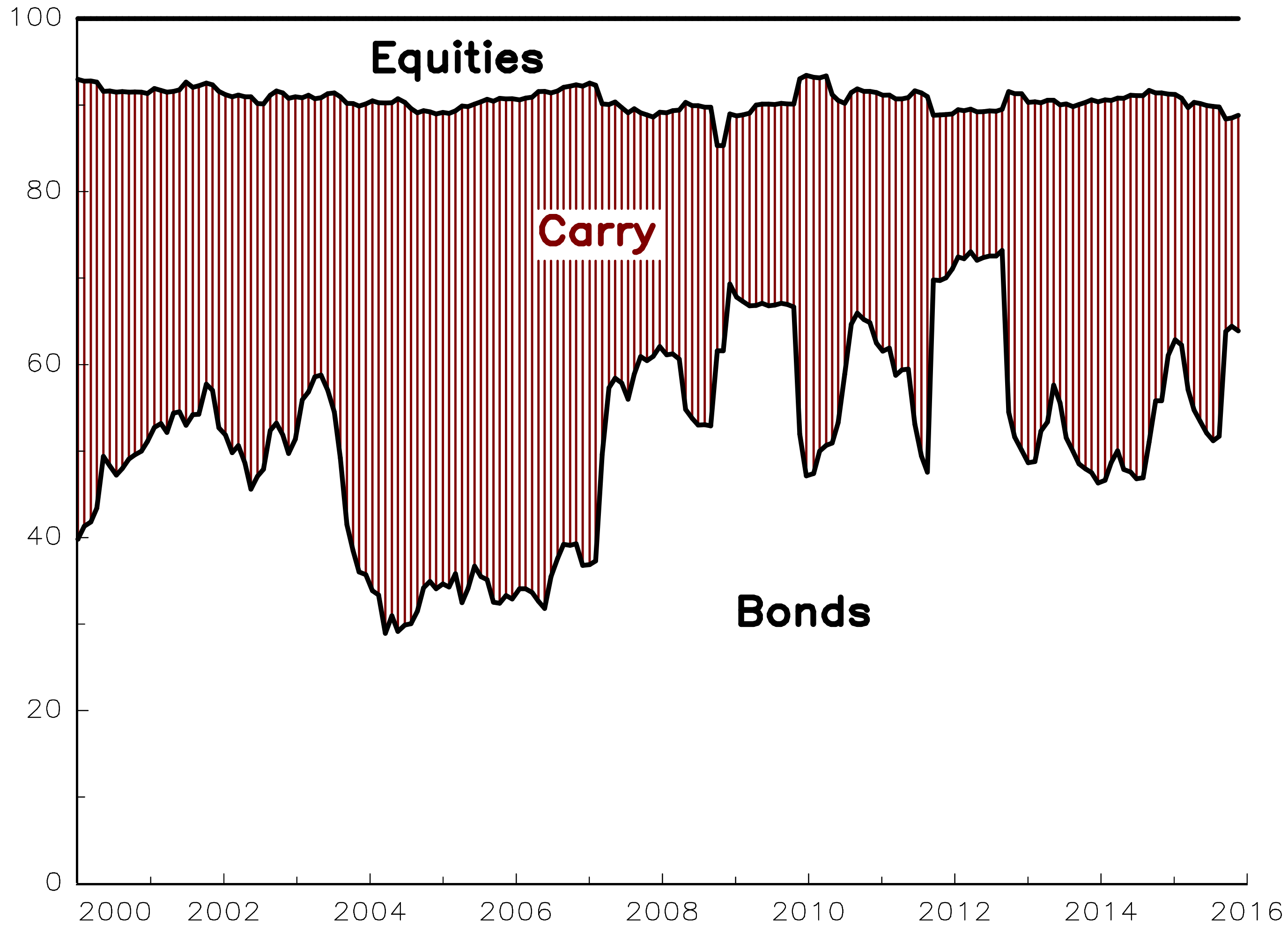
$$y^\star = \arg\min \mathrm{ES}_\alpha(y) - \sum_{i=1}^n b_i \ln y_i \quad \text{u.c.} \quad y \geq \mathbf{0}$$

The RB portfolio corresponds to the normalized portfolio:

$$x_i^\star = \frac{y_i^\star}{\sum_{j=1}^n y_j^\star}$$

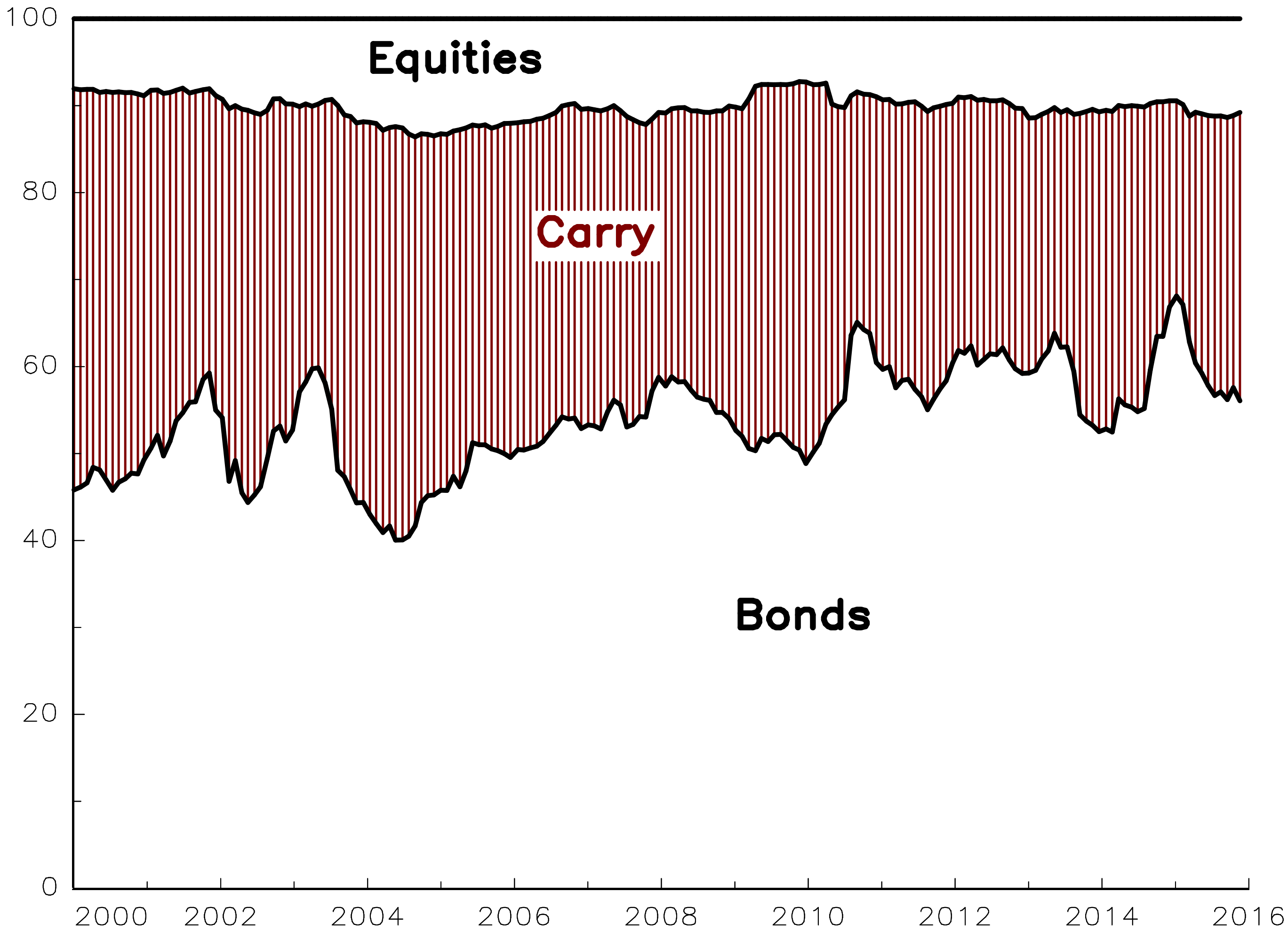
Risk allocation

Figure: Volatility-based ERC portfolio



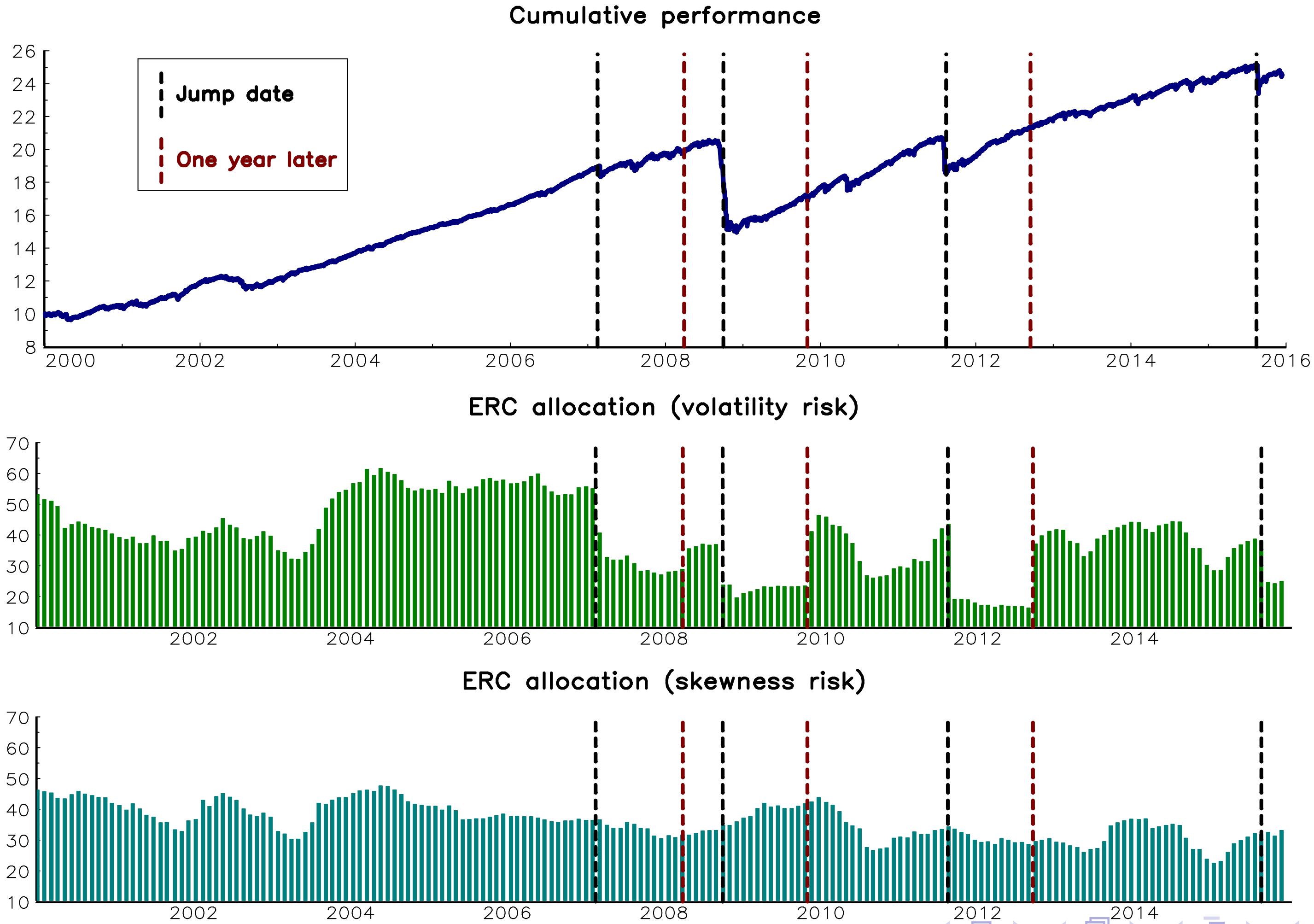
Risk allocation

Figure: Skewness-based ERC portfolio



Risk allocation

Figure: Comparison of the carry allocation



Volatility hedging versus skewness hedging

Table: Volatility and skewness risks of risk-based portfolios (weekly model)

Portfolio	MV	MV	ERC	MES
Model	Gaussian (full sample)	Jump model Normal	Jump model Mixture	Jump model Mixture
Bonds	63.26%	36.05%	52.71%	100.00%
Equities	2.23%	0.00%	10.36%	0.00%
Carry	34.51%	63.95%	36.93%	0.00%
$\sigma(x)$	2.62%	2.33%	2.75%	4.17%
γ_1	-2.75	-19.81	-6.17	0.00

Source: BKR (2016)

The arithmetics of skewness

$$-(36.05\% \times 0.17 + 0\% \times 0.44 + 63.95\% \times 5.77) = -19.81$$

Conclusion

- The portfolio management of alpha and beta must be different
  - Portfolio optimization (MVO) is suitable for managing the concentration of active bets
  - Risk-based allocation (RB) is suitable for managing the diversification of risk premia or risk factors
- Volatility diversification $\neq$ skewness diversification
  - Volatility hedging $\neq$ skewness hedging
  - Skewness risk = main driver of strategic asset allocation (SAA)
  - Volatility risk = main driver of tactical asset allocation (TAA)
- Long-only diversification vs long/short diversification
- Liquidity issue?

Skewness risk = a strategic allocation decision

Volatility risk = a tactical allocation decision

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