Asset Management & Sustainable Finance
Final Examination

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5 February 2024

Deadline: 21 March 2024

Remark 1 *The final exam consists of 2 exercises. Please write your answers completely*[1]*. Be specific about the different concepts and different statistics you use. Define the optimization program associated with each portfolio. Provide also one Python program by exercise.*

- *Concerning risk decomposition*[2]*, present the results as follows:*

Asset	x_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
1				
2				
$\vdots$				
n				
$\mathcal{R}(x)$			✓	

- *The report is a zipped file whose filename is yourname.zip if you are doing the project alone or yourname1-yourname2.zip if you are doing the project in groups of two.*

- *The zipped file contains three files:*

  1. *The PDF document containing the answers to the two exercises and a cover sheet with your names;*
  2. *The Python program of each exercise with an explicit filename, e.g. exercise1.py.*

- *The project seems very long. However, once you understand how to solve a mean-variance optimization problem with a QP solver, you can duplicate your code for many questions. For example, Question 2.(c) is a duplication of Question 2.(b), as are Questions 3.(a), 3.(b) and 3.(c) in Exercise 1. The same is true for Questions 2.(b), 2.(c), 3.(a), 3.(b), 3.(c) and 3.(d) in Exercise 2.*

[1] Read the questions carefully and answer all elements of the questions. For example, when I say "Find the portfolio x and compute its volatility $\sigma(x)$", you must give the numeric values of x and $\sigma(x)$. If you just give the numeric value of $\sigma(x)$, the answer is wrong because I don't know what the portfolio's weights are.

[2] x_i is the weight (or the exposure) of the i^{th} asset in the portfolio, $\mathcal{MR}_i$ is the marginal risk, $\mathcal{RC}_i$ is the nominal risk contribution, $\mathcal{RC}_i^\star$ is the relative risk contribution and $\mathcal{R}(x)$ is the risk measure of the portfolio.

1 Portfolio optimization and risk budgeting

We consider an investment universe of $n = 5$ stocks. We assume that the expected returns μ_i are equal to 5%, 5%, 6%, 4%, and 7%, whereas the volatilities σ_i are equal to 20%, 22%, 25%, 18% and 45%. The correlation matrix is equal to:

$$\mathbb{C} = (\rho_{i,j}) = \begin{pmatrix} 100\% & & & & \\ 50\% & 100\% & & & \\ 30\% & 30\% & 100\% & & \\ 60\% & 60\% & 60\% & 100\% & \\ 40\% & 30\% & 70\% & 30\% & 100\% \end{pmatrix}$$

The risk free return is set to 2%.

1. (a) Compute the covariance matrix Σ of stock returns.
   (b) Compte the Sharpe ratio of each asset.

2. We consider **long/short MVO portfolios** such that $\sum_{i=1}^n x_i = 1$.

   (a) Give the QP formulation of the mean-variance optimization problem:

   $$\begin{array}{rcl} x^\star & = & \arg\min \frac{1}{2} x^\top \Sigma x - \gamma x^\top (\mu - r \mathbf{1}_5) \\ & & \text{s.t.} \quad \left\{ \begin{array}{l} \sum_{i=1}^n x_i = 1 \\ -10 \leq x_i \leq 10 \end{array} \right. \end{array}$$

   (b) Using the γ-problem, find the optimal solution[3] $x^\star(\gamma)$ when the coefficient γ is equal to 0, 0.10, 0.20, 0.50 and 1.00. For each optimized portfolio, compute its expected return $\mu(x^\star(\gamma))$, its volatility $\sigma(x^\star(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^\star(\gamma) \mid r)$.
   (c) Draw the efficient frontier by considering granular values[4] of γ.
   (d) Using a bisection algorithm, calculate the MVO portfolio if we target an ex-ante volatility of 16% and 20%. Give the corresponding values γ of the QP problem. For each optimized portfolio, compute its expected return $\mu(x^\star(\gamma))$, its volatility $\sigma(x^\star(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^\star(\gamma) \mid r)$.
   (e) Using the efficient frontier with a fine grid of γ, find the tangency portfolio. Compute its expected return $\mu(x^\star(\gamma))$, its volatility $\sigma(x^\star(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^\star(\gamma) \mid r)$.
   (f) Compare the previous brute force solution with the analytical solution. Comment on these results.

3. We consider **long-only MVO portfolios** such that $\sum_{i=1}^n x_i = 1$ and $0 \leq x_i \leq 1$.

   (a) Using the γ-problem, find the optimal solution[5] $x^\star(\gamma)$ when the coefficient γ is equal to 0, 0.10, 0.20, 0.50 and 1.00. For each optimized portfolio, compute its expected return $\mu(x^\star(\gamma))$, its volatility $\sigma(x^\star(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^\star(\gamma) \mid r)$.
   (b) Compare the efficient frontier by considering granular values of γ with the long/short efficient frontier obtained in Question 2.(c). Comment on these results.
   (c) Using a bisection algorithm, calculate the MVO portfolio if we target an ex-ante volatility of 16% and 20%. Give the corresponding values of γ of the QP problem. For each optimized portfolio, compute its expected return $\mu(x^\star(\gamma))$, its volatility $\sigma(x^\star(\gamma))$ and its Sharpe ratio $\mathrm{SR}(x^\star(\gamma) \mid r)$.

[3] You have to give the composition of each optimized portfolios.
[4] For instance, you can consider that $\gamma = -0.5, -0.4, \ldots, -0.1, 0, 0.05, 0.10, \ldots, 0.95, 1, 2, \ldots, 10$.
[5] You have to give the composition of each optimized portfolios.

(d) Find the long-only tangency portfolio $x^{\star}_{\mathrm{MSR}}$. Compute its expected return $\mu\left(x^{\star}_{\mathrm{MSR}}\right)$, its volatility $\sigma\left(x^{\star}_{\mathrm{MSR}}\right)$ and its Sharpe ratio $\mathrm{SR}\left(x^{\star}_{\mathrm{MSR}} \mid r\right)$.

(e) Compute the beta coefficient β_i of each asset with respect to the long-only tangency portfolio $x^{\star}_{\mathrm{MSR}}$. Deduce the implied expected return μ_i that is priced in by the market[6], and the corresponding alpha coefficient α_i of each asset.

4. We consider risk-budgeting portfolios.

(a) Give the risk decomposition of the long-only tangency portfolio $x^{\star}_{\mathrm{MSR}}$ (MSR).

(b) Give the risk decomposition of the equally-weighted portfolio (EW).

(c) Give the risk decomposition of the long-only minimum variance portfolio (MV).

(d) Give the risk decomposition of the long-only most diversified portfolio (MDP).

(e) Compute the equal risk contribution portfolio using the CCD algorithm. Give its risk decomposition.

(f) Compute the beta $\beta\left(x \mid b\right)$ of the portfolios MSR, EW, MV, MDP and ERC with respect to the benchmark b when b is the long-only tangency portfolio $x^{\star}_{\mathrm{MSR}}$. Same question when b is the EW portfolio. Comment on these results.

[6] We assume that the market portfolio is the long-only tangency portfolio $x^{\star}_{\mathrm{MSR}}$.

2 Equity portfolio optimization with net zero objectives

We decompose an investment universe of several stocks using the GICS classification. Therefore, each sector is represented by a synthetic asset (*e.g.*, a futures contract) and we invest directly in these synthetic assets. We assume that the risk model is the CAPM:

$$R_i = r + \beta_i (R_m - r) + \varepsilon_i$$

where r is the return of the risk-free asset, $R_m \sim \mathcal{N}\left(\mu_m, \sigma_m^2\right)$ is the return of the market portfolio, β_i is the beta of sector i, and $\varepsilon_i \sim \mathcal{N}\left(0, \tilde{\sigma}_i^2\right)$ is the idiosyncratic risk of sector i. The numerical value of β_i and $\tilde{\sigma}_i$ are given in Table 1. For each sector, we also provide the carbon intensity $\mathcal{CI}_i$ and the carbon momentum $\mathcal{CM}_i$. There are two measures of carbon intensity and momentum, based on Scope 1 and 2 emissions, while the second is based on Scope 1, 2 and 3 upstream emissions. The green intensity $\mathcal{GI}_i$ is the green revenue share aligned with the EU taxonomy. We also assume that the benchmark b is the market portfolio and its composition is shown in the third column in Table 1. In the following, the volatility of the market portfolio is set to 20%, the return of the risk-free asset is 3%, and we assume that all portfolios w have a Sharpe ratio $\mathrm{SR}(w \mid r)$ of 0.25. We also assume that the portfolio-level climate risk measures are calculated as a weighted average of the sector-level climate risk measures. This means that $\mathcal{CI}(w) = \sum_{i=1}^{11} w_i \mathcal{CI}_i$, $\mathcal{CM}(w) = \sum_{i=1}^{11} w_i \mathcal{CM}_i$ and $\mathcal{GI}(w) = \sum_{i=1}^{11} w_i \mathcal{GI}_i$. In addition, investors are constrained and cannot short assets, which means that all optimized portfolios are long only.

Table 1: Metrics of the financial and climate risk models

#	Sector	b_i	β_i	$\tilde{\sigma}_i$	$\mathcal{CI}_i$ $\mathcal{SC}_{1-2}$	$\mathcal{CI}_i$ $\mathcal{SC}_{1-3}^{\mathrm{up}}$	$\mathcal{CM}_i$ $\mathcal{SC}_{1-2}$	$\mathcal{CM}_i$ $\mathcal{SC}_{1-3}^{\mathrm{up}}$	$\mathcal{GI}_i$
		%		%	tCO_2e/$ mn	tCO_2e/$ mn	%	%	%
1	Communication Services	8.20	0.95	26.2	24	78	−2.8	−0.8	0.0
2	Consumer Discretionary	12.30	1.05	32.9	54	203	−7.2	−1.6	1.5
3	Consumer Staples	6.90	0.45	21.1	47	392	−1.8	−0.1	0.0
4	Energy	3.10	1.40	33.8	434	803	−1.5	−0.2	0.7
5	Financials	13.20	1.15	23.1	19	55	−8.3	−1.9	0.0
6	Health Care	12.60	0.75	25.9	21	124	−7.8	−2.0	0.0
7	Industrials	10.20	1.00	26.5	105	283	−8.5	−2.5	2.4
8	Information Technology	23.00	1.20	27.1	23	123	−4.3	+2.1	0.2
9	Materials	4.50	1.10	30.1	559	892	−7.1	−3.6	0.8
10	Real Estate	2.80	0.80	27.4	89	135	−2.7	−0.8	1.4
11	Utilities	3.20	0.70	22.8	1 655	1 867	−9.9	−6.8	8.4

1. We consider the benchmark b.
   (a) Compute the covariance matrix Σ, the correlation matrix ρ, and the vector σ of sector volatilities.
   (b) Compute the volatility $\sigma(b)$ of the benchmark. Why do we have $\sigma(b) \neq \sigma_m$?
   (c) Compute the weighted average $\beta(b) = \sum_{i=1}^{11} b_i \beta_i$. How do you explain this result?
   (d) Compute the vector $\tilde{\pi}$ of implied risk premia and derive the expected return $\tilde{\mu}_i$ of each sector as priced by the market.
   (e) Compute $\mathcal{CI}(b)$, $\mathcal{CM}(b)$ and $\mathcal{GI}(b)$.

2. The investor's decarbonization pathway follows the CTB trajectory, meaning that the carbon intensity of the investor's portfolio at time t must be less than a threshold $\mathcal{CI}^\star(t)$:

$$\mathcal{CI}(t, w) \leq \mathcal{CI}^\star(t) := (1 - 30\%)(1 - 7\%)^t \, \mathcal{CI}(b) \tag{1}$$

The investor's objective is to minimize the volatility of the tracking error relative to the benchmark and to meet the decarbonization constraint based on Scope 1 and 2 emissions.

(a) What is the optimization problem? Deduce the QP form.

(b) Compute the optimized portfolio $w^{\star}(t)$ for $t \in \{0, 1, 2, 5, 10\}$. For each optimized portfolio, compute the tracking error volatility, the carbon intensity, the carbon momentum, the green intensity and the reduction rate:

$$\mathcal{R}(t, w) = 1 - \frac{\mathcal{CI}(t, w)}{\mathcal{CI}(b)}$$

Comment on these results.

(c) Same question if we take into account Scope 3 emissions.

(d) Let $\tilde{\mu}_i(w)$ be the implied expected return of sector i that the investor has priced in, assuming that portfolio w is optimal. For each solution given in Questions 2.(b) and 2.(c), give the implied bet $\tilde{\mu}_i(w^{\star}(t)) - \tilde{\mu}_i$. Comment on these results.

(e) At time $t = 0$, the investor implements the solution given in Question 2.(b). What level of carbon intensity do we expect just before the rebalancing time $t = 1$? Comment on these results.

3. In addition to the decarbonization scenario, the investor wants to add new constraints. In this question, we focus on Scope 1+2 emissions.

(a) The investor imposes the following constraint:

$$w_i \geq \frac{b_i}{2} \tag{2}$$

What is the reason for this constraint? Write the QP form of the problem to be optimized. Find the optimal portfolio $w^{\star}(t)$ for $t = 0$. Compute the tracking error volatility, the carbon intensity, the reduction rate, the carbon momentum and the green intensity.

(b) The investor imposes the following constraint:

$$\mathcal{CM}(t, w) := \sum_{i=1}^{11} w_i \mathcal{CM}_i \leq \mathcal{CM}^{\star} \tag{3}$$

What is the reason for this constraint? Write the QP form of the problem to be optimized. Find the optimal portfolio $w^{\star}(t)$ for $t = 0$ and $\mathcal{CM}^{\star} \in \{-5\%, -6\%, -7\%, -8\%\}$. For each optimized portfolio, compute the tracking error volatility, the carbon intensity, the reduction rate, the carbon momentum and the green intensity.

(c) The investor imposes the following constraint:

$$\mathcal{GI}(t, w) := \sum_{i=1}^{11} w_i \mathcal{GI}_i \geq (1 + \mathcal{G})\, \mathcal{GI}(b) \tag{4}$$

where $\mathcal{G} \geq 1$. How do you interpret this constraint? Write the QP form of the problem to be optimized. Find the optimal portfolio $w^{\star}(t)$ for $t = 0$ and $\mathcal{G} \in \{0, 0.5, 1, 2\}$. For each optimized portfolio, compute the tracking error volatility, the carbon intensity, the reduction rate, the carbon momentum and the green intensity.

(d) The investor imposes the three constraints (2), (3) and (4). Write the QP form of the problem to be optimized. Find the optimal portfolio $w^{\star}(t)$ for $t = 0$ and $(\mathcal{CM}^{\star}, \mathcal{G}) \in \{(-6\%, 0.25), (0, 0.50), (-7\%, 0), (-7\%, 0.25)\}$. For each optimized portfolio, compute the tracking error volatility, the carbon intensity, the reduction rate, the carbon momentum and the green intensity.

(e) Comment on the previous results. What solutions are realistic?